Decompositions with Orthogonality w.r.t. Arbitrary Inner Product

We want to replace the condition $A^*A = I$ with $A^*GA = I$ where $G = LL^*$ is a hermitian positive definite matrix representing an arbitrary inner product.
Some other decompositions not listed are done in similar ways

QR Decomposition

Let $Q' := (L^*)^{-1}Q$ and $R' = R$ and decompose $L^*A = QR$

$$Q'R' = (L^*)^{-1}L^*QR = A$$

$$\begin{aligned} Q'^*GQ' &= \left((L^*)^{-1}Q\right)^*G(L^*)^{-1}Q = Q^*L^{-1}G(L^*)^{-1}Q \\ &= Q^*L^{-1}LL^*(L^*)^{-1}Q = I \end{aligned}$$

Credit to 21-242 (this was inspired by and solves a homework problem)

SVD

Let $U' := (L^*)^{-1}U$ and $V' := (L^*)^{-1}V$ and decompose $L^*AL = U\Sigma V^*$

$$U'\Sigma'V'^* = (L^*)^{-1}U\Sigma V^*L^{-1} = (L^*)^{-1}L^*ALL^{-1} = A$$

Same as with the QR

$$\begin{aligned} U'^*GU' &= I \\ V'^*GV' &= I \end{aligned}$$

EVD

We only consider $A^* = A$
I don't think there is any solution C_0 to the problem of this form:

$$\begin{aligned} Q' &:= C_0Q \\ C_0^{-1}AC_0 &= C_0^*A(C_0^*)^{-1} = Q\Lambda Q^{-1} \\ Q'\Lambda'Q'^{-1} &= C_0Q\Lambda Q^{-1}C_0^{-1} = A \end{aligned}$$

$$Q'^*GQ = Q^*C_0^*GC_0Q = I$$

I can't see any way for hermiticity of $C_0^{-1}AC_0$ to hold unless C_0 is hermitian, in which case

$$\begin{aligned} Q'^*GQ &= Q^*C_0^{-1}GC_0Q = I \\ \Rightarrow C_0^{-1}GC_0 &= (Q^*)^{-1}Q^{-1} = I \\ \Rightarrow G &= C_0C_0^{-1} = I \end{aligned}$$