Pensées 0: The λ Manifesto

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Definition

$$\lambda = \frac{\tau}{4} = \frac{\pi}{2}$$

Note

I no longer believe a lot of these points about λ and τ and π, you can check Pensées 1 for updates and more stuff about this topic, this is still interesting though.

Volume and Surface Area of n-balls

An n-ball is a ball in n dimensions, for the cases of $n = 2$ and $n = 3$ being the points inside a circle and a sphere, respectively.

$$\mathbb{B}_n(r) = \{x \in \mathbb{R}^n \mid |x| \leq r\}$$

It is known that this expression evaluates to the volume of an n-ball. For example, for $n = 2$ and $n = 3$ the area in a circle and the volume of a sphere.

$$V_n(r) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)} r^n$$

There are 2^n orthants (e.g. 4 quadrants in the case of $n = 2$) in a n dimensional vector space. For a volume with equal volume in each of these octants, it is natural to multiply 2^n by the volume of each octant. λ is the angle between any two axes and therefore naturally characterizes the octant, which is a relationship with n axes, each having an angle of λ between them.

$$V_n(r) = \frac{2^{\frac{n}{2}} \lambda^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)} r^n = 2^n \frac{2^{-\frac{n}{2}} \lambda^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)} r^n$$

Through an identity for the double factorial this may be simplified to this expression, much easier by a large margin to evaluate for an odd (or even) integer n than the formula involving π or the continous case of λ.

$$n \in \mathbb{Z} \Rightarrow V_n(r) = 2^n \frac{\lambda^{\lfloor \frac{n}{2} \rfloor}}{n!!} r^n$$

The αth fractional derivative of the volume expression is thus. This value is not generally necessary to know, and is also extremely easy to find by simply differentiating with respect to r.

$$\frac{\partial^\alpha V_n(r)}{\partial r^\alpha} = 2^n \frac{2^{-\frac{n}{2}} \lambda^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)} \frac{\Gamma(n + 1)}{\Gamma(n - \alpha + 1)} r^{n-\alpha}$$

The case for integer n and α may appear more familiar, and is generally more useful.

$$n, \alpha \in \mathbb{Z} \Rightarrow \frac{\partial^\alpha V_n(r)}{\partial r^\alpha} = 2^n \frac{\lambda^{\lfloor \frac{n}{2} \rfloor}}{n!!} \frac{n!}{(n-\alpha)!} r^n$$

The surface area is clearly the case of the derivative for which $\alpha = 1$. In the case of $n = 2$, this is $\frac{\partial}{\partial r}\left(\frac{1}{2}\tau r^2\right) = \tau r$.

$$A_{n-1}(r) = \frac{\partial V_n(r)}{\partial r} = 2^n \frac{2^{-\frac{n}{2}} \lambda^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)} n r^{n-1}$$

The integer case of the surface area is also, in the λ formulation, extremely convenient to evaluate manually

$$n, \alpha \in \mathbb{Z} \Rightarrow A_{n-1}(r) = 2^n \frac{\lambda^{\lfloor \frac{n}{2} \rfloor}}{n!!} n r^{n-1} = 2^n \frac{\lambda^{\lfloor \frac{n}{2} \rfloor}}{(n-2)!!} r^{n-1}$$

The place of τ is as a member of the family of surface area constants, defined thus.

$$\tau_{n-1} = \frac{A_{n-1}(r)}{r^{n-1}} = 2^n \frac{2^{-\frac{n}{2}} \lambda^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right)} n$$

The integer case of the surface area constants is similarly facile to evaluate.

$$n \in \mathbb{Z} \Rightarrow \tau_{n-1} = 2^n \frac{\lambda^{\lfloor \frac{n}{2} \rfloor}}{n!!} n = 2^n \frac{\lambda^{\lfloor \frac{n}{2} \rfloor}}{(n-2)!!}$$

The relationship between the surface area and the corresponding surface area constant is obvious from its definition.

$$A_{n-1} = \tau_{n-1} r^{n-1}$$

In a clear parallel to the case of τ_1, $\frac{1}{2}\tau r^2$, this is the general form for the relationship between the nth n-ball volume and the nth surface constant.

$$V_n = \int A_{n-1}(r) \, \mathrm{d}r = \frac{1}{n} \tau_{n-1} r^n$$

Thus, τ is a single case of a form most logically defined using λ, clearly the ideal choice for almost all two-dimensional expressions (the largest proportion of usage of circle constants by far), but its logicality for higher-dimensional geometry is lacking. The usage of π is clearly suboptimal except, arguably, for cases directly involving the geometric meaning of half a turn. In addition, for cases in which the space does not have 2^n orthants, for example with the condition $x_n \in [0, \infty]$, λ is clearly the logical and optimal representation of the space's geometry.

Properties in Relation to Vectors

The dot product is equivalent to the product of the magnitudes of the vectors scaled by the closeness of their directions

$$a \cdot b = a^T b = b^T a = |a|\ |b| \cos(\theta)$$

$\theta = \lambda$ implies they face in perπndicular directions, an extremely important property to all of linear algebra, geometry, and other fields. The symbol $\perp$ denotes orthogonality in the first expression and perpendicularity in the second.

$$\theta = \lambda \Rightarrow a \cdot b = |a|\ |b| \cos(\theta) = 0 \Leftrightarrow a \perp b$$

$$\lambda = \cos^{-1}\left(\frac{a \cdot b}{|a|\ |b|}\right) \Rightarrow a \perp b$$

Complex Properties

The geometric interpretation of λ, $e^{i\theta}$, and i^{θ} clearly demonstrates that i is a turn by λ in the complex plane.

$$e^{i\lambda} = i$$
$$\ln i = i\lambda$$
$$\lambda = \frac{\ln i}{i}$$
$$e^{i\lambda\theta} = i^{\theta}$$

The function i^{θ}, given $\theta \in \mathbb{R}$, can be interpreted as a rotation in the complex plane at a certain speed, and this speed $|\frac{\mathrm{d}i^{\theta}}{\mathrm{d}\theta}|$ can be found to be λ. The $i\lambda$ in the derivative expression demonstrates that the derivative vector is the position rotated and scaled by λ.

$$\frac{\mathrm{d}i^{\theta}}{\mathrm{d}\theta} = i^{\theta} \ln i = i\lambda i^{\theta} = \lambda i^{\theta+1}$$
$$|\lambda i^{\theta+1}| = |\lambda|\ |i^{\theta+1}|$$
$$|i^{\theta+1}| = 1$$
$$|\frac{\mathrm{d}i^{\theta}}{\mathrm{d}\theta}| = |i\lambda i^{\theta}| = \lambda$$

An obvious property of the hyperoperations of i:

$$i^i = e^{-\lambda}$$
$$i \uparrow^{\beta} \alpha = e^{i\lambda} \uparrow^{\beta} \alpha$$

The αth derivative of i^{θ}

$$(i\lambda)^{\alpha} i^{\theta}$$

Trigonometric functions extended to $\mathbb{C}$ demonstrate

$$\cos(\lambda\theta) = \frac{e^{i\lambda\theta} + e^{-i\lambda\theta}}{2} = \frac{i^{\theta} + i^{-\theta}}{2}$$
$$\sin(\lambda\theta) = \frac{e^{i\lambda\theta} - e^{-i\lambda\theta}}{2i} = \frac{i^{\theta} - i^{-\theta}}{2i}$$
$$\tan(\lambda\theta) = -i\frac{e^{i\lambda\theta} - e^{-i\lambda\theta}}{e^{i\lambda\theta} + e^{-i\lambda\theta}} = -i\frac{i^{\theta} - i^{-\theta}}{i^{\theta} + i^{-\theta}}$$

I cannot see any use for this except proving trigonometric identities through substituting $\theta = \varphi\lambda$.

Trigonometric properties

The value of λ in terms of trigonometry, when compared to τ, is dubious, given that all trigonometric functions are, for $\theta \in \mathbb{R}$, two dimensional ($\mathbb{C}^2$ is not 4-dimensional). Nevertheless, here are various properties involving λ (the first not being specific to λ). The values of trigonometric functions may be interpreted as ratios of sides of a right (with an angle λ) triangle, though.

$$\theta = \varphi + n\lambda \Rightarrow \varphi \in [0, \lambda] \land n \in \mathbb{Z}$$

The values of the three most common special angles have a (very likely coincidental) correspondence to fractions of λ, one and two thirds including $\sqrt{3}$ and a half including $\sqrt{2}$.

$$\theta \to \cos(\theta), \sin(\theta), \tan(\theta)$$
$$\frac{\lambda}{3} \to \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{3}}$$
$$\frac{\lambda}{2} \to \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1$$
$$\frac{2\lambda}{3} \to \frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}$$

Multiplying the angle in trigonometric expressions by λ results in a potentially useful mapπng given $\theta \in \mathbb{I}$. For example, a function which desired a transition of equivalent temporal character from α to β could be defined as such, over time δ.

$$\alpha + (\beta - \alpha)\sin\left(\lambda\frac{t}{\delta}\right)$$

These trigonometric properties are the weakest argument presented for λ, properties of similar intuitiveness, relevance, and applicability could be found for τ, π, or λ, as they are solely based on their geometric significance. This may be taken as an argument for τ, in that that significance is most logically interpreted as a fraction of a turn, $\frac{\tau}{n}$, which I would agree with.

$$\cos(\theta \pm \lambda) = \mp\sin(\theta)$$
$$\sin(\theta \pm \lambda) = \pm\cos(\theta)$$
$$\tan(\theta \pm \lambda) = -\cot(\theta)$$

$$\cos(\lambda - x) = \sin(\theta)$$
$$\sin(\lambda - x) = \cos(\theta)$$
$$\tan(\pm\lambda - \theta) = \cot(\theta)$$

Conclusion

λ is an interesting and, I argue, the most fundamental circle constant. Its value in most 2-dimensonal cases is less than τ, except for its relationship with i. I do not argue that λ should replace τ in all, or even a significant portion of, circumstances, but instead that it is the most useful descriptor of τ and the attributes of arbitrary-dimensional shapes such as n-balls, hyperspheres, etc. and their volume and surface area. τ, τ_1, or $A_1(1)$ appears in many fields of mathematics which are generally thought of as unrelated to geometry, demonstrating that it has a more widely applicable aspect, which λ has to a lesser degree.