Pensées 3: Change of Basis

`0x10af`

Change of Basis of Linear Map

A manifold is parameterized by a position function ξ and coordinates x^α and x'^α. The covariant basis (the one most people care about) is the derivative of the position function with respect to the coordinates and the contravariant basis is the differential of the coordinates. This is relatively intuitive but geometric explanations can be found elsewhere.

$$e'_\alpha = \frac{\partial \xi}{\partial x'^\alpha} = \frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial \xi}{\partial x^\gamma} = \frac{\partial x^\gamma}{\partial x'^\alpha} e_\gamma$$

$$e'^\alpha = \mathrm{d}x'^\alpha = \frac{\partial x'^\alpha}{\partial x^\gamma}\,\mathrm{d}x^\gamma = \frac{\partial x'^\alpha}{\partial x^\gamma} e^\gamma$$

$$e_\gamma = \frac{\partial x'^\alpha}{\partial x^\gamma} e'_\alpha$$

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From this you can easily derive the transformation rules of V and V^*, their inverses, and the invariance of this representation.

$$v'^\alpha = \frac{\partial x'^\alpha}{\partial x^\gamma} v^\gamma$$

$$v'_\alpha = \frac{\partial x^\gamma}{\partial x'^\alpha} v_\gamma$$

$$v'^\alpha e'_\alpha = \frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x'^\alpha} v^\gamma e_\gamma = v^\gamma e_\gamma$$

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$$v^\gamma = \frac{\partial x^\gamma}{\partial x'^\alpha} v'^\alpha$$

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The linear algebra representation of this is much more opaque but sometimes necessary. From the contraction with the basis transformations the transformation rule for a general tensor is seen to be the application of the jacobian or inverse jacobian of the coordinate transformation to each index based on its variance. Note that here $\wedge$ does not signify the wedge product but logical conjunction ("and").

$$v' = T^{-1}v \to T = \frac{\partial x^\gamma}{\partial x'^\alpha} \wedge T^{-1} = \frac{\partial x'^\alpha}{\partial x^\gamma} \wedge v = Tv'$$

$$A'^\alpha_\beta = \frac{\partial x'^\alpha}{\partial x^\gamma} A^\gamma_\mu \frac{\partial x^\mu}{\partial x'^\beta} \to A' = T^{-1}AT$$

$$A^\gamma_\mu = \frac{\partial x^\gamma}{\partial x'^\alpha} A'^\alpha_\beta \frac{\partial x'^\beta}{\partial x^\mu} \to A = TA'T^{-1}$$

If we look at the transformation T as a map between vector spaces we can see an intuitive explanation of the arrangement of the transformations used to transform the matrix representation of a linear map. I wrote all the mappings backwards so they would be in the same order as the notation.

$$X \overset{T}{\leftarrow} X'$$

$$X' \overset{T^{-1}}{\leftarrow} X$$

$$X \overset{A}{\leftarrow} X$$

$$X' \overset{A'}{\leftarrow} X'$$

$$T^{-1}AT : X' \overset{T^{-1}}{\leftarrow} X \overset{A}{\leftarrow} X \overset{T}{\leftarrow} X'$$

$$TA'T^{-1} : X \overset{T}{\leftarrow} X' \overset{A'}{\leftarrow} X' \overset{T^{-1}}{\leftarrow} X$$

We know the effect of A on the bases of X but not X', thus we may convert the vectors of X' to vectors in X and then we know the action upon them. Our result in then a vector in X, and we must convert it back to X'.

Change of Basis in Integral

The volume form is the scaling of the volume in a local basis on the manifold, and thus is necessary in integration. The properties of the wedge product and the transformation allow facile derivation of the scaling of the volume form. A proof of the second fact and be found somewhere, it's rather intuitively obvious though. ω is the volume form of an orthonormal coordinate system, ω' is not necessarily.

$$\omega = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$$
$$T\,\mathrm{d}x^1 \wedge \ldots \wedge T\,\mathrm{d}x^n = \det(T)\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$$
$$\rightarrow \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n = \det(T^{-1})T\,\mathrm{d}x^1 \wedge \ldots \wedge T\,\mathrm{d}x^n$$
$$\omega' = \frac{\partial x^\gamma}{\partial x'^1}\,\mathrm{d}x'^1 \wedge \ldots \wedge \frac{\partial x^\nu}{\partial x'^n}\,\mathrm{d}x'^n = \det\left(\frac{\partial x^\mu}{\partial x'^\kappa}\right)\mathrm{d}x'^1 \wedge \ldots \wedge \mathrm{d}x'^n$$
$$= \det\left(\frac{\partial x^\lambda}{\partial x'^\rho}\right)\frac{\partial x'^\xi}{\partial x^1}\,\mathrm{d}x^1 \wedge \ldots \wedge \frac{\partial x'^\tau}{\partial x^n}\,\mathrm{d}x^n$$
$$= \det\left(\frac{\partial x^\lambda}{\partial x'^\rho}\right)\det\left(\frac{\partial x'^\xi}{\partial x^1}\right)\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$$

The integral is thus:

$$\int_X f(x)\omega = \int_{X'} f(x'^\alpha)\omega' = \int_{X'} f\left(x'^\beta\right)\det\left(\frac{\partial x^\gamma}{\partial x'^\nu}\right)\mathrm{d}x'^1 \wedge \ldots \wedge \mathrm{d}x'^n$$

Expressing that jacobian in terms of the transformation of the metric tensor relative to an orthonormal basis we find an expression for the volume element which requires only the metric tensor.

$$g_{\alpha\beta} := \begin{cases} 1 \mid \alpha = \beta \\ 0 \mid \alpha \neq \beta \end{cases}$$
$$g'_{\alpha\beta} = \frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}g_{\gamma\nu}$$
$$\det\left(g_{\alpha\beta}\right) = 1$$
$$\det\left(g'_{\alpha\beta}\right) = \det\left(\frac{\partial x^\gamma}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}g_{\gamma\nu}\right) = \det\left(\frac{\partial x^\mu}{\partial x'^\rho}\right)^2$$
$$\det\left(\frac{\partial x^\mu}{\partial x'^\rho}\right) = \delta\sqrt{\left|\det\left(g_{\alpha_\beta}\right)\right|}$$

δ represents the orientation (or handedness) of the coordinate system at the point, it can be pulled out of the integral if the orientation is constant. This is what is done usually because it it necessary if using the metric tensor to compute it, and the orientation of the coordinate system is kind of ignored in "normal" integration $\mathrm{d}x\,\mathrm{d}y = \mathrm{d}y\,\mathrm{d}x$ as opposed to $\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x$.

$$\omega' = \delta\sqrt{\left|\det\left(g_{\alpha\beta}\right)\right|}\,\mathrm{d}x'^0 \wedge \ldots \wedge \mathrm{d}x'^n$$
$$\mathrm{d}\delta = 0 \rightarrow \int_{X'} f\left(x'^\beta\right)\omega' = \delta\int_{X'} f\left(x'^\beta\right)\sqrt{\left|\det\left(g_{\alpha\beta}\right)\right|}\,\mathrm{d}x'^0 \wedge \ldots \wedge \mathrm{d}x'^n$$