Pensées 6

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0.1 Proof of Convolution Theorem

$$\begin{aligned}
(f * g)(x) &= \int_{-\infty}^{\infty} f(\tau) g(x-\tau)\,\mathrm{d}\tau \\
\mathcal{F}\{f\}(\xi) &= \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x}\,\mathrm{d}x \\
(f * g)(x) &= \mathcal{F}^{-1}\{\mathcal{F}\{f\} \cdot \mathcal{F}\{g\}\}(x) \\
&\Leftarrow \mathcal{F}\{f * g\}(\xi) = \mathcal{F}\{f\}(\xi) \cdot \mathcal{F}\{g\}(\xi) \\
\mathcal{F}\{f * g\}(\xi) &= \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} f(\tau) g(x-\tau) e^{-i2\pi\xi x}\,\mathrm{d}\tau\,\mathrm{d}x \\
&= \int\limits_{-\infty}^{\infty} f(\tau) \left(\int\limits_{-\infty}^{\infty} g(x-\tau) e^{-i2\pi\xi x}\,\mathrm{d}x \right) \mathrm{d}\tau \\
\int\limits_{-\infty}^{\infty} g(x-\tau) e^{-i2\pi\xi x}\,\mathrm{d}x &= e^{-i2\pi\xi\tau} \int\limits_{-\infty}^{\infty} g(\phi) e^{-i2\pi\xi\phi}\,\mathrm{d}\phi \\
&= \left(\int\limits_{-\infty}^{\infty} f(\tau) e^{-i2\pi\xi\tau}\,\mathrm{d}\tau \right) \left(\int\limits_{-\infty}^{\infty} g(\phi) e^{-i2\pi\xi\phi}\,\mathrm{d}\phi \right) \\
&= \mathcal{F}\{f\}(\xi) \cdot \mathcal{F}\{f\}(\xi)
\end{aligned}$$

0.2 Distribution of Sum of Normal Distributions

$$\begin{aligned}
\mathcal{F}\{\mathcal{N}(x; \mu, \sigma^2)\} &= \exp(-i\xi\mu - \frac{1}{2}\xi^2\sigma^2) \\
A_i &= \mathcal{N}(\mu_i, \sigma_i^2) \\
A_1 * ... * A_n &= \mathcal{F}^{-1}\{\prod_{i=1}^{n} \mathcal{F}\{A_i\}\} \\
&= \mathcal{F}^{-1}\{\prod_{i=1}^{n} \exp(-i\xi\mu_i - \frac{1}{2}\xi^2\sigma_i^2)\} \\
&= \mathcal{F}^{-1}\{\exp(-i\xi \sum_{i=1}^{n} \mu_i - \frac{1}{2}\xi^2 \sum_{i=1}^{n} \sigma_i^2)\} \\
&= \mathcal{N}(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2)
\end{aligned}$$

0.3 Cumulative Distribution Function of Normal Distribution

$$
\begin{aligned}
\operatorname{erf} x &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,\mathrm{d}t \\
\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}\,\mathrm{d}x \\
t = \frac{x-\mu}{\sqrt{2}\sigma} &\Rightarrow \mathrm{d}x = \sqrt{2}\sigma\,\mathrm{d}t \\
\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2}\sqrt{2}\sigma\,\mathrm{d}t \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{z} e^{-t^2}\,\mathrm{d}t \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{0} e^{-t^2}\,\mathrm{d}t + \frac{1}{\sqrt{\pi}} \int_{0}^{z} e^{-t^2}\,\mathrm{d}t \\
&= \frac{1}{2} + \frac{\operatorname{erf} t}{2} = \frac{1}{2} + \frac{\operatorname{erf}(\frac{x-\mu}{\sqrt{2}\sigma})}{2} \\
&= \frac{1}{2}\left(1 + \operatorname{erf}(\frac{\sqrt{2}}{2}\frac{x-\mu}{\sigma})\right)
\end{aligned}
$$

0.4 Mean and Variance of Scaled Distribution

$$
\begin{aligned}
E\left[\sum_{i=1}^{n} a_i X_i\right] &= \sum_{i=1} a_i E[X_i] \\
\operatorname{Var}(\alpha X) &= E[(\alpha X - E[\alpha X])^2] \\
&= \alpha^2 E[(X - E[X])^2] \\
&= \alpha^2 Var(X)
\end{aligned}
$$

0.5 The Immediate Convergence of the Central Limit Theorem when Applied to Normal Distributions

$$
\begin{aligned}
A &= \mathcal{N}(\mu, \sigma^2) \\
\frac{1}{n}\overbrace{A * ... * A}^{n} &= \frac{1}{n}\mathcal{F}^{-1}\{\mathcal{F}\{A\}^n\} \\
&= \frac{1}{n}\mathcal{F}^{-1}\{\exp(-i\xi n\mu - \frac{1}{2}\xi^2 n\sigma^2)\} \\
&= \frac{1}{n}\mathcal{N}(n\mu, n\sigma^2) = \mathcal{N}(\mu, \sigma^2/n)
\end{aligned}
$$

0.6 Mean and Variance of Sum of Random Variables

$$
\begin{aligned}
\mu = E\left[\sum_{i=1}^{n} X_i\right] &= \sum_{i=1}^{n} E[X_i] \\
\sigma^2 = \operatorname{Var}(X+Y) &= E[((X+Y) - E[X+Y])^2] \\
&= E[((X - E[X]) + (Y - E[Y]))^2] \\
&= E[(X - E[X])^2] + E[(Y - E[Y])^2] + 2E[(X - E[X])(Y - E[Y])] \\
&= \operatorname{Var}(X) + \operatorname{Var}(Y) + 2\operatorname{Cov}(X, Y)
\end{aligned}
$$

0.7 Linear Maps, the Transpose, and the Pseudoinverse

$$\begin{tikzcd}
X^* \arrow[d, leftrightarrow] & Y^* \arrow[l, "A^*"] \arrow[d, leftrightarrow] \\
X \arrow[r, "A"] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
X^* \arrow[d, leftrightarrow] & Y^* \arrow[l, two heads, "A^*"] \arrow[d, leftrightarrow] & X^* \arrow[d, leftrightarrow] & Y^* \arrow[l, hook, "A^*"] \arrow[d, leftrightarrow] \\
X \arrow[r, hook, "A"] & Y & X \arrow[r, two heads, "A"] & Y
\end{tikzcd}$$

$$\begin{tikzcd}
X^* \arrow[d, leftrightarrow] & Y^* \arrow[l, hook, two heads, "A^*"] \arrow[d, leftrightarrow] \\
X \arrow[r, hook, two heads, "A"] & Y
\end{tikzcd}$$

Assuming that the matrix is unitary, $U^*U = UU^* = U^{-1}U = UU^{-1} = I$, the induced bases are orthogonal and the effect of the maps between the vector spaces and their duals can be ignored, maintaining the coefficients and only changing the bases, thus A^{-1} has equivalent coefficients to A^*.

$$\begin{tikzcd}
\operatorname{im} A^* \arrow[d, leftrightarrow] & \operatorname{im} A^{*+} \arrow[l, hook, two heads, "A^*"] \arrow[d, leftrightarrow] \\
\operatorname{im} A^+ \arrow[r, hook, two heads, "A"] \arrow[d, dash, "\perp"] & \operatorname{im} A \arrow[d, dash, "\perp"] \\
\ker A & \ker A^+ \arrow[d, leftrightarrow] \\
 & \ker A^*
\end{tikzcd}$$

For a surjection there exists a right inverse: $AA^+ = I$, corresponding to $Y = \operatorname{im} A$ and $\ker A^+ = \{0\}$. For an injection there exists a left inverse: $A^+A = I$, corresponding to $\ker A = \{0\}$ and $X = \operatorname{im} A^+$. For bijections both are the case, as is seen with invertible matrices, where $A^+ = A^{-1}$, and thus $AA^+ = I$ and $A^+A = I$.
If we have a matrix for which the source and induced bases are orthogonal but not of the same dimension then the transpose is a left inverse if the map is injective and a right inverse if the map is surjective.

0.8 Green's Theorem, Stokes's Theorem, and the Divergence Theorem in the Language of Differential Forms

We can derive all these theorems from the generalized stokes theorem and the practical definition of the exterior derivative (differential), with multi-index I.

$$\begin{aligned}
\partial_i &= \frac{\partial}{\partial x^i} \\
\mathrm{d}x^i \wedge \mathrm{d}x^j &= 0 \Leftarrow i = j \\
\mathrm{d}x^i \wedge \mathrm{d}x^j &= -\,\mathrm{d}x^j \wedge \mathrm{d}x^i \\
\int_{\partial\Omega} \omega &= \int_{\Omega} \mathrm{d}\omega \\
\omega &= f_I \,\mathrm{d}x^I \\
\mathrm{d}\omega &= \partial_i f_I \,\mathrm{d}x^i \wedge \mathrm{d}x^I
\end{aligned}$$

Green's Theorem involves integrating a 2-form on the surface and a 1-form on the boundary of the surface in 2-dimensional space.

$$\begin{aligned}
\omega &= f_i \,\mathrm{d}x^i \\
\mathrm{d}\omega &= \partial_j f_i \,\mathrm{d}x^j \wedge \mathrm{d}x^i \\
\int_{\partial\Omega} f_i \,\mathrm{d}x^i &= \int_{\Omega} \partial_j f_i \,\mathrm{d}x^j \wedge \mathrm{d}x^i \\
&= \int_{\Omega} (\partial_0 f_1 - \partial_1 f_0) \,\mathrm{d}x^0 \wedge \mathrm{d}x^1
\end{aligned}$$

Stokes's Theorem involves the same situation happening in 3-dimensional space.

$$
\begin{aligned}
\omega &= f_i \,\mathrm{d}x^i \\
\mathrm{d}\omega &= \partial_j f_i \,\mathrm{d}x^j \wedge \mathrm{d}x^i \\
\int_{\partial\Omega} f_i \,\mathrm{d}x^i &= \int_{\Omega} \partial_j f_i \,\mathrm{d}x^j \wedge \mathrm{d}x^i \\
&= \int_{\Omega} (\partial_1 f_2 - \partial_2 f_1) \,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \\
&\quad +(\partial_2 f_0 - \partial_0 f_2) \,\mathrm{d}x^2 \wedge \mathrm{d}x^0 \\
&\quad +(\partial_0 f_1 - \partial_1 f_0) \,\mathrm{d}x^0 \wedge \mathrm{d}x^1
\end{aligned}
$$

The 1-form corresponding to curl is its hodge dual, this is kind of like taking the normal to a plane.

$$
\begin{aligned}
\star\,\mathrm{d}\omega &= (\partial_1 f_2 - \partial_2 f_1) \,\mathrm{d}x^0 \\
&\quad +(\partial_2 f_0 - \partial_0 f_2) \,\mathrm{d}x^1 \\
&\quad +(\partial_0 f_1 - \partial_1 f_0) \,\mathrm{d}x^2
\end{aligned}
$$

The Divergence Theorem involves the integration of a 3-form on a volume in 3-dimensional space and the integration of a 2-form on its boundary. Usually this is presented as the integration of a vector over a surface, but we should only integrate n-forms over manifolds of dimension n, thus we have to take the hodge dual. Nothing becomes negative in the exterior derivative because all the bases are two switches away (even parity) from what we want.

$$
\begin{aligned}
\alpha &= f_i \,\mathrm{d}x^i \\
\omega = \star\alpha &= f_0 \,\mathrm{d}x^1 \wedge \mathrm{d}x^2 + f_1 \,\mathrm{d}x^2 \wedge \mathrm{d}x^0 + f_2 \,\mathrm{d}x^0 \wedge \mathrm{d}x^1 \\
\mathrm{d}\omega = \mathrm{d}\star\alpha &= \partial_0 f_0 \,\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\
&\quad +\partial_1 f_1 \,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^0 \\
&\quad +\partial_2 f_2 \,\mathrm{d}x^2 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^0 \\
&= (\partial_0 f_0 + \partial_1 f_1 + \partial_2 f_2) \,\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \\
\int_{\partial\Omega} f_0 \,\mathrm{d}x^1 \wedge \mathrm{d}x^2 + f_1 \,\mathrm{d}x^2 \wedge \mathrm{d}x^0 + f_2 \,\mathrm{d}x^0 \wedge \mathrm{d}x^1 &= \int_{\Omega} (\partial_0 f_0 + \partial_1 f_1 + \partial_2 f_2) \,\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2
\end{aligned}
$$