9.1 Understanding the concept of Relations

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Cornell Notes	**Topic:** 9.1 Understanding the concept of Relations	Course: BSc Computer Science Class: Discrete Mathematics-Lecture Date: January 08, 2020

Essential Question:

What are relations & their properties. Also, how can they be represented?

Questions/Cues:

- What is a Relation?
- What is the Cartesian product?
- What is a binary relation from A to B?
- What are relations on the same set?
- How can relations be represented by using matrices?
- How do we combine relations?
- How are relations represented using digraphs?
- What is reflexivity in relations?
- What is the digraph/matrix of a reflective relation?
- What is symmetry in relations?
- What is the digraph/matrix of a symmetric relation?
- What is anti-symmetry in relations?
- What is the digraph/matrix of a anti-symmetric relation?
- What is transitivity in relations?
- What is the transitive closure of a relation?

Notes

What is a relation?

- A relation can be defined between elements of a set A and elements of another set B

- A relation can be defined between elements of the same set.

- We always use the the letter R to refer to a relation.

What is a relation?

- Let A and B be sets

- Let R be a relation **linking** elements of A to elements of B

- Let $x \in A$ and $y \in B$,

  - We say that **x is related** to y with respect to the relation R and we write x R y

What is a relation?

- A relation is a **link** between two elements of a set
  - For example:
    - *A person x is a SON OF' a person y*
    - *SON OF is a relation that **links** x to y*

- We usually use the letter R to refer to a relation:
  - In this case R= 'SON OF'
  - If x is a SON OF y we write x R y
  - If y is Not a SON OF x we write $y \not R x$

Example

- Let A be the **students** in a the CS program
  - A = {Sofia, Samir, Sarah }
- Let B be the **courses** the department offers
  - B = { Mathematics, Java, Databases, Art}
- Let R be a relation linking students in the set A to classes they are enrolled in : A student is **related** to a course if the student is **enrolled** on this course

Examples

- Sofia is enrolled in Mathematics and Java,
- Samir is enrolled in Java and Databases,
- Sarah is enrolled in Mathematics and Art
- Sofia is not enrolled in Art

Notations

- Sofia R Mathematics
- Sofia R Java
- Samir R Java
- Samir R Databases
- Sarah R Mathematics
- Sarah R Art
- Sofia $\not R$ Art

Cartesian product

- Let A and B be two sets
- The **Cartesian product** A x B is defined by a **set of pairs** (x,y) such that $x \in A$ and $y \in B$

$$A \times B = \{(x,y) : x \in A \text{ and } y \in B\}$$

For example:

- $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3\}$
- $A \times B = \{(a_1, b_1), (a_1, b_2), (a_1, b_3), (a_2, b_1), (a_2, b_2), (a_2, b_3)\}$

Definition of a relation

Let A and B be two sets.
A **binary relation** from A to B is a **subset** of a **Cartesian product** A x B

- $R \subseteq A \times B$ means R is a set of ordered pairs of the form (x,y) where $x \in A$ and $y \in B$.
- $(x,y) \in R$ means x R y (x is related to y)

For example:

- A={a,b,c} and B={1,2,3 }
- The following is a **relation** defined from A to B:
  R={(a,1),(b,2), (c,3)}
- This means that: a R 1, b R 2 and c R 3

Relations on a set

- When A= B
- A relation **R on the set** *A* is a relation from *A* to *A*
  - R $\subseteq A \times A$
- We will generally be studying relations of this type.

Example

- A ={1,2,3,4}
- Let R be relation **on the set A** :
  - x,y $\in A$, x R y **if and only if** x < y
- We have: 1 R 2 , 1 R 3, 1 R 4, 2 R 3, 2 R 4, 3 R 4
- R= {(1,2)(1,3), (1,4), (2,3), (2,4),(3,4)}

Relations using matrices

- Given a **relation** R from a set **A** to set **B**.
- List the elements of sets *A* and *B* in a particular order
- Let n_a =|A| and n_b=|B|

- The **matrix of R** is a $n_a \times n_b$ matrix.

$$M_r = [m_{ij}]n_{a \times} n_b$$

$$m_{ij=} \begin{cases} 1. \; if \; (a_i, bj) \in R \\ 0 \; \; if \; (a_i, bj) \notin R \end{cases}$$

Example 1

- Let A = { Sofia, Samir, Sarah }
- Let B = { CS100, CS101, CS102, CS103 }

- Consider the relation of who is **enrolled** in which **class**
- R = { (*a*,*b*) | person *a* is enrolled in course *b* }

	CS100	CS101	CS102
Sofia	X	X	
Samir		X	X
Sarah	X		x

$$M_r = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Example 2

- Let A = { 1, 2, 3, 4, 5 }
- Consider a relation: < (x,y) ∈ R **if and only if** x < y
- Every element is *not* related to itself as x ≮ x

$$\mathbf{M}_{\leq} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 3

- Let A = { 1, 2, 3, 4, 5 }
- Consider a relation : ≤ (x,y) ∈ R **if and only if** x ≤ y

$$\mathbf{M}_{\leq} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Combining relations

Union

- The union of two relations is a new set that contains all of the pairs of elements that are in at least one of the two relations.
- The union is written as R∪S or "R or S"
- R∪S = { (a,b): (a,b) ∈ R or (a,b) ∈ S }

Intersection

- The intersection of two relations is a new set that contains all of the pairs that are in both sets.
- The intersection is written as R∩S or "R and S".
- R ∩ S = { (a,b): (a,b) ∈ R and (a,b) ∈ S }

Combining relations: via Boolean operators

Let $\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $\mathbf{M}_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Join $\mathbf{M}_{R \cup S} = \mathbf{M}_R \vee \mathbf{M}_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Meet $\mathbf{M}_{R \cap S} = \mathbf{M}_R \wedge \mathbf{M}_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Representing relations using directed graphs

Definition :

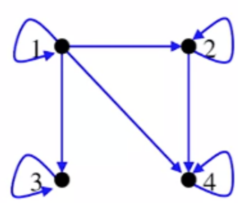
- When a relation is defined on a set, it can be represented by a digraph

Building the digraph :

- First, the elements of A are written down,
- When (a,b) ∈ R arrows are drawn from a to b.

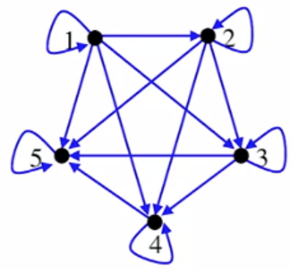
Example 1

- A = {1,2,3,4}
- Let R be relation on A defined as follows:
  - *R* = { (*x*,*y*) | *x* ***divides*** *y* }
- *R* can be represented by the following **digraph**



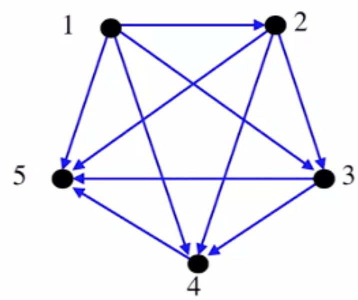
Example 2

- Let A = { 1, 2, 3, 4, 5 }
- Consider a relation : ≤ (x,y) ∈ R **if and only if** x ≤ y



Example 3

- Let A = { 1, 2, 3, 4, 5 }
- Consider a relation: < (x,y) ∈ R **if and only if** x < y



- Every element is *not* related to itself

Definition of reflexivity

A relation **R** in a **set S** is said to be **reflexive** if and only if

x R x, $\forall$ x $\in$ S

(x,x) $\in$ R , $\forall$ x $\in$ S

Example 1

- Let **R** be a **relation** of elements in **Z**:
  R = { (a,b) $\in Z^2$ | a≤b }
- For all x elements of **Z**, we have **x** ≤ **x**, hence **x R x**
- This implies that **R is reflexive**.

Example 2

- Let **R** be a relation of elements in Z:
  $R = \{ (a,b) \in Z^2 \mid a<b \}$
- For all x elements of **Z**, we have $\mathbf{x \not< x}$ hence $\mathbf{x \not R x}$
- For example $1 \in$ **Z**, however, 1 is not strictly less than 1
- Hence, 1 is not related to 1
- This implies that R is **not reflexive**.

Exercise: reflexive relation

Which of the following relations is reflexive?

- $R_1 = \{ (a,b) \mid a,b \in Z, a \bmod 2 = b \bmod 2 \}$
- $R_2 = \{ (a,b) \mid a,b \in Z, a - b = 2 \}$

- R_1 is reflexive since for every $a \in Z$, $a \bmod 2 = a \bmod 2$
- R_2 is not reflexive since $a - a = 0 \neq 2$ for every $a \in Z$
- i.e. $1 \not R_2 1$ as $1 - 1 = 0 \neq 2$.

Digraph of reflexive relation

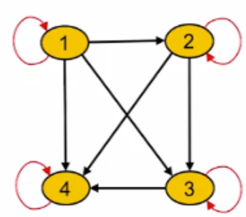
The **digraph** of a **reflexive relation** on elements in a set S contains **a loop** on every element of S.

Digraph of reflexive relation

Example

Let S = {1, 2, 3, 4} and let **R** be a relation of elements in S

$\mathbf{R} = \{ (a,b) \in S^2 \mid a \leq b \}$



Matrix of reflexive relation

Let $\mathbf{M_r}$ be the matrix of a **reflexive** relation, in which all the values of the diagonal of $\mathbf{M_r}$ are equal to 1.

Matrix of reflexive relation

Example

Let S = {1, 2, 3, 4} and let **R** be a relation of elements in S

$\mathbf{R} = \{ (a,b) \in S^2 | a \leq b \}$

$$\mathbf{M}_{\leq} = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{matrix}$$

Definition of symmetry

A relation R on a set S is said to be symmetric
if and only if
$\forall$ a,b $\in$ S, **if a R b** then **b R a**.

Example

- Let **R** be a relation of elements in Z:

  $\mathbf{R} = \{ (a,b) \in Z^2 \mid a \bmod 2 = b \bmod 2 \}$

•Proof: let a, b $\in$ Z with a **R** b:
- a mod 2 = b mod 2
- b mod 2 = a mod 2
- b R a

•R is symmetric.

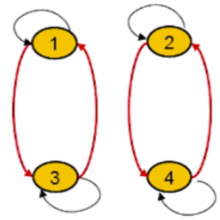
Digraph of a symmetric relation = The digraph of a symmetric relation on elements in a set S contains a symmetric pair of arcs for every edge of S. This means that there exist no single arrow between any two different vertices of the digraph graph. There are either no edges or two parallel edges showing the relation is both ways.

Digraph of a symmetric relation

Example

Let S = {1, 2, 3, 4} and **R** be relation of elements in S

R = { (a,b) $\in$ S^2 | a mod 2 = b mod 2 }



Matrix of symmetric relation

The adjacency matrix, $\mathbf{M_r}$, of a **symmetric** relation is **symmetric**.

Matrix of symmetric relation

Example

Let S = {1, 2, 3, 4} and let **R** be relation of elements in S

R = { (a,b) $\in$ S^2 | a mod 2 = b mod 2 }

$$M_R = \begin{matrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{matrix}$$

Definition of anti-symmetric

A relation **R** on a set S is said to be **anti-symmetric** if and only if $\forall$ a, b $\in$ S, if a **R** b and b **R** a then a **=** b.

This means that if the relation is anti-symmetric, then no two different elements are related both ways

Example

- Let **R** be a relation on elements in Z:

$$R = \{ (a,b) \in Z^2 \mid a \leq b \}$$

- Let a, b $\in$ Z, a **R** b and b **R** a
  - $a \leq b$ and $b \leq a$
  - $a = b$

- R is **anti-symmetric**.

Digraph of an anti-symmetric relation

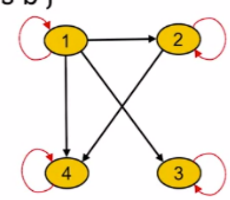
The **digraph** of an **anti-symmetric** relation on elements in a set S contains **no parallel** edges between any two difference vertices.

Digraph of an anti-symmetric relation

Example

Let S = {1, 2, 3, 4} and **R** be relation on elements in S

$$\mathbf{R} = \{ (a,b) \in S^2 \mid a \text{ divides } b \}$$



Matrix of an anti-symmetric relation

Let $\mathbf{M_r} = [\mathbf{m_{ij}}]$ be the matrix of an **anti-symmetric** relation.
If $i \neq j$ and $\mathbf{m_{ij}} \neq 0$ then $\mathbf{m_{ji}} = \mathbf{0}$.

Matrix of an anti-symmetric relation

Example

Let S = {1, 2, 3, 4} and let **R** be relation of elements in S
$\mathbf{R} = \{ (a,b) \in S^2 \mid a \text{ divides } b \}$

$$M_R = \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

Exercise

Let **R** be the relation defined by the Matrix M_R

$$M_R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Is R reflexive? Symmetric? Anti-symmetric?

- Clearly R is **not reflexive**: $m_{1,1}=0$.
- It is not **symmetric** because $m_{2,1}=1$, $m_{1,2}=0$.
- However, it is **anti-symmetric**.

Definition of transitivity

A relation **R** on set S is called **transitive** if and only if
$\forall$ a, b, c $\in$ S, if (**a R b** and **b R c**) then **a R c**.

$\mathbf{R = \{ (x,y) \in N^2 \mid x \leq y \}}$

- Yes, it is transitive as $\forall$ x, y, z $\in$ **N**
  if $x \leq y$ **and** $y \leq z$ then $x \leq z$.

R = { (2,3),(3,2),(2,2) }

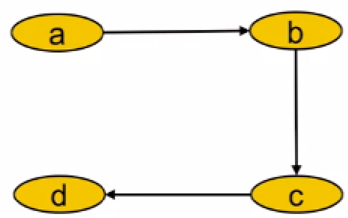
•No, it is not transitive because 3R2 and 2R3 but 3R3.

R= { (a,b) | a is an ancestor of b }

•Yes, it is transitive because if a is an ancestor of b and b is an ancestor of c, then a is an ancestor of c.

Example

Let **S = {a, b, c, d}** and let **R** be a relation on **S** represented by the following digraph:



Not transitive as: a R b and b R c, however a $\not{R}$ c

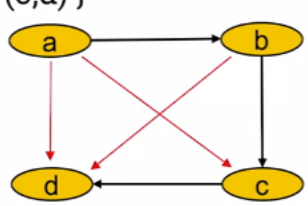
Or

b R c and c R d, however b $\not{R}$ d

Transitive closure of a relation

What is the minimum number of edges that need to be added to make this relation transitive?

R = { (a,b), (b,c), (c,d) }



$R_{enhanced}$ = { (a,b), (b,c), (a,c), (c, d), (b,d), (a,d) } (transitive closure of R)

Summary

In this week, we learned what a relation is, the properties of relations, how to combine relations, the representations of relations & transitive closure of a relation.