Fundamentals Of Computer Science Course Notes

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Week 1

1.101 Introduction to propositional logic

Propositional Logic is a system that deals with propositions or statements.

Below there's an example of where we can apply propositional logic to derive conclusions.

Liars And Knights

Imagine there is an island with two types of people. Liars who always tell lies and knights who always tell the truth. One an excursion, you visit the island and encounter two people, person A and person B. Person A says "at least one of us is a liar", while person B says nothing. What conclusion can you draw?

With a little logical thinking, we can conclude that person A is a knight and person B is a liar.

1.103 Building blocks of logic

What is a proposition?

A **proposition** is a statement that can be either **true** or **false**. It must be one or the other, never both nor neither.

Examples of proposition:

- 2 is a prime number (T)
- 5 is an even number (F)

Not a proposition:

- x is a prime number

  In this case, it can be made into a proposition by assigning a value to x.

- Are you going to school?

  Because this is a question, we can't assign a truth value to the sentence.

- Do your homework now

  Being an order, it has no truth value.

Syntaxes of the propositional logic

Propositions are denoted by capitable letters: P, Q, R, and so on.

- P = carrots are orange
- Q = I went to a party yesterday

General statements are denoted by lowercase letters: p, q, r, and so on. They carry on a logical argument, are used in proofs, called propositional variables.

Connectives: change or combine propositions

Connectives transform **atomic** propositions into **compound** propositions.

Logical NOT ($\neg$) $\neg p$ is true if and only if p is false. Also called **negation**.

Logical OR ($\vee$) $p \vee q$ is true if and only if at least one of p or q is true or if both p and q are true. Also called **disjunction**.

Logical AND ($\wedge$) $p \wedge q$ is true if and only if both p and q are true and false otherwise. Also called **conjunction**.

Logical if then ($\implies$) $p \to q$ is true if and only if either p is false or q is true. Also called **implication** or **conditional**. p is called the **premise** and q is the **conclusion**.

Logical if and only if ($\iff$) $p \iff q$ is true if and only if both p and q are true or both are false. Also called **bi-conditional**.

Exclusive or ($\oplus$) $p \oplus q$ is true if and only if p or q is true but not both.

Translation from Logical Proposition to English

Let:

$$
\begin{aligned}
P &= \text{I study 20 hours a week} \\
Q &= \text{I attend all the lectures} \\
R &= \text{I will pass the exam} \\
S &= \text{I will be happy}
\end{aligned} \tag{1}
$$

Translate the following statement to English:

- $(P \lor Q) \implies (R \land S)$

  If **I study 20 hours a week** or **I attend all the lectures** then **I will pass the exam** and **I will be happy**.

Translation from English to Logical Proposition

Given the statement:

If UK does not exit the EU then skilled nurses will not leave the NHS and research grants will remain intact.

Translating to logical proposition we get:

$$
\begin{aligned}
P &= \text{UK exits the EU} \\
Q &= \text{Skilled nurses will leave the NHS} \\
R &= \text{Research grants will remain intact} \\
\neg P &\implies \neg Q \land R
\end{aligned} \tag{2}
$$

Note that we removed any **connectives** from our propositions as that's a good practice. This makes the logical statement easier to follow.

1.105 Truth Table: examples

A truth table is a set of all outcomes of propositions and connectives. The number of rows in a truth table, depends on the number of given propositions. If we have n propositions, our truth table will have 2^n rows.

Truth tables for each connective

What follows is a list of truth tables for each connective

Negation ($\neg$)

p	$\neg p$
1	0
0	1

Conjunction ($\wedge$)

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

Disjunction ($\vee$)

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

Implication ($\implies$)

p	q	$p \implies q$
1	1	1
1	0	0
0	1	1
0	0	1

Bi-conditional ($\iff$)

p	q	$p \iff q$
1	1	1
1	0	0
0	1	0
0	0	1

Exclusive Or ($\oplus$)

p	q	$p \oplus q$
1	1	0
1	0	1
0	1	1
0	0	0

Operator Precedence

When formulae are written without parenthesis, we must rely on rules of operator precedence. Logic operator precende rules are as follows:

$\neg \wedge \vee \implies \iff$

Example If we have the logical statement $p \implies p \wedge \neg q \vee s$, we can parse it following the steps below:

$$\begin{aligned}
p &\implies p \wedge \neg q \vee s \\
p &\implies p \wedge (\neg q) \vee s \\
p &\implies (p \wedge (\neg q)) \vee s \\
p &\implies ((p \wedge (\neg q)) \vee s) \\
(p &\implies ((p \wedge (\neg q)) \vee s))
\end{aligned} \tag{3}$$

Constructing Truth Tables for Complex Formulae

Example 1

p	q	$p \wedge q$	$(p \wedge q) \implies p$
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	1

Example 2

p	q	$q \implies p$	$p \wedge (q \implies p)$
1	1	1	1
1	0	1	1
0	1	0	0
0	0	1	0

Comparing both examples

p	q	$(p \wedge q) \implies p$	$p \wedge (q \implies p)$
1	1	1	1
1	0	1	1
0	1	1	0
0	0	1	0

1.202 Tautology and consistency

Tautology

A formula that is **always** true regardless of the truth value of the proposition.

p	$\neg p$	$p \vee \neg p$
0	1	1
1	0	1

Consistent

A formula that is true **at least** for one scenario. All connectives are consistent.

The formula $p \wedge \neg p$ is **inconsistent** because it can never be true. Inconsistent formulae are also called **contradictions**.

1.204 Tautology and consistency: examples

Example 1: $p \vee (q \wedge \neg r)$

p	q	r	$\neg r$	$q \wedge \neg r$	$p \vee (q \wedge \neg r)$
0	0	0	1	0	0
0	0	1	0	0	0
0	1	0	1	1	1
0	1	1	0	0	0
1	0	0	1	0	1
1	0	1	0	0	1
1	1	0	1	1	1
1	1	1	0	0	1

This is a **Consistent** formula

Example 2: $(p \implies q) \implies (\neg q \vee r)$

p	q	r	$p \implies q$	$\neg q$	(¬ q ∨ $)	$(p \implies q) \implies (\neg q \vee r)$
0	0	0	1	1	1	1
0	0	1	1	1	1	1
0	1	0	1	0	0	0
0	1	1	1	0	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	0	0	0
1	1	1	1	0	1	1

This is a **Consistent** formula

Example 3: $(p \implies q) \iff (\neg p \vee q)$

p	q	$p \implies q$	$\neg p$	$(¬ p ∨ q)	$(p \implies q) \iff (\neg p \vee q)$
0	0	1	1	1	1
0	1	1	1	1	1
1	0	0	0	0	1
1	1	1	0	1	1

This formula is a **Tautology**

Week 2

1.301 Equivalences

Formulae A and B are equivalent if they have identical truth tables. Equivalence is denoted by the symbol $\equiv$

In other words, $A \equiv B$ means that A and B have the same truth values, regardless of how variables are assigned.

One thing to note is that $\equiv$ is **NOT** a connective.

De Morgan's Laws

$$\begin{aligned} \neg(p \wedge q) &\equiv \neg p \vee \neg q \\ \neg(p \vee q) &\equiv \neg p \wedge \neg q \end{aligned} \tag{4}$$

Truth Tables

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
0	0	1	1	0	1	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	1	0	0

$(p \implies q) \equiv (\neg p \vee q) \equiv \neg(p \wedge \neg q)$

p	q	$\neg p$	$\neg q$	$p \implies q$	$\neg p \vee q$	$\neg(p \wedge \neg q)$
0	0	1	1	1	1	1
0	1	1	0	1	1	1
1	0	0	1	0	0	0
1	1	0	0	1	1	1

Contrapositive: $(p \implies q) \equiv (\neg q \implies \neg p)$

p	q	$\neg p$	$\neg q$	$p \implies q$	$\neg q \implies \neg p$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	0	0	1	1

1.304 First-order logic

Important Notions

- **Predicates** describe properties of objects

  A simple example could be `odd(3)`. Here we're applying the predicate `odd` to the object `3`. When arguments are applied to predicates, they become propositions and connectives for propositional logic can be employed in the usual manner:

  $$Odd(3) \land Prime(3) = T \tag{5}$$

- **Quantifiers** allow reasoning on multiple objects

  The objects from a quantified statement are chosen from a *Domain.*

  - **Existential Quantifier** $\exists$

    We use it as follows: $\exists x$ some formula.

    When proving a formula based on the existential quantifier, it is enough to find **one** element which makes the formula true. In other words, existentially quantified statements are **false** unless there is a positive example.

  - **Universal Quantifier** $\forall$

    We use it as follows: $\forall x$ some formula.

    In order to satisfy the formula, we must prove that **every** x satisfies the formula.

    Note that a single counterexample is enough to disprove a universally quantified statement. In other words, universally quantified statements are **true** unless there is a false example.

Translations English - Logic

"All P's are Q's" translates into $\forall x(P(x) \implies Q(x))$

"No P's are Q's" translates into $\forall x(P(x) \implies \neg Q(x))$

"Some P's are Q's" translates into $\exists x(P(x) \land Q(x))$

"Some P's are not Q's" translates into $\exists x(P(x) \land \neg Q(x))$

Quantifiers to connectives

- Existential Quantifier

$\exists x P(x)$ and domain $D = \{x_1, x_2, \ldots, x_n\}$. This is equivalent to saying $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$

- Universal Quantifier

$\forall x P(x)$ and domain $D = \{x_1, x_2, \ldots, x_n\}$. This is equivalent to saying $P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$

Negation of Quantifiers

- Existential Quantifier

$$\begin{aligned}\neg \exists x P(x) &\equiv \neg(P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)) \\ &\equiv \neg P(x_1) \wedge \neg P(x_2) \wedge \ldots \wedge \neg P(x_n)) \\ &\equiv \forall x \neg P(x)\end{aligned} \tag{6}$$

- Universal Quantifier

$$\begin{aligned}\neg \forall x P(x) &\equiv \neg(P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)) \\ &\equiv \neg P(x_1) \vee \neg P(x_2) \vee \ldots \vee \neg P(x_n)) \\ &\equiv \exists x \neg P(x)\end{aligned} \tag{7}$$

Example

$$\begin{aligned}\neg(\forall x(P(x) \implies Q(x)) &\equiv \exists x \neg(P(x) \implies Q(x)) \\ &\equiv \exists x \neg(\neg P(x) \vee Q(x)) \\ &\equiv \exists x(\neg\neg P(x) \wedge \neg Q(x)) \\ &\equiv \exists x(P(x) \wedge \neg Q(x))\end{aligned} \tag{8}$$