9.2 Equivalence, and partial and total order relations

Notebook: Discrete Mathematics [CM1020]

Created: 2019-10-07 2:31 PM **Updated:** 2020-01-14 2:01 PM

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Cornell Notes	**Topic:** 9.2 Equivalence, and partial and total order relations	Course: BSc Computer Science Class: Discrete Mathematics-Lecture Date: January 14, 2020

Essential Question:

What is the difference between a equivalence class & relation? Also how is order demonstrated in relations?

Questions/Cues:

- What is an equivalence relation?
- What is an equivalence class?
- What is a partial order in relations?
- What is a total order in relations?

Notes

Definition of equivalence relation

Let **R** be a relation of elements on a set S. **R** is an **equivalence relation**

if and only if

R is **reflexive**, **symmetric** and **transitive**.

Example 1

- Let **R** be **relation** of elements in **Z**:
  $\mathbf{R} = \{ (a, b) \in Z^2 \mid a \bmod 2 = b \bmod 2 \}$

- We have already proved that this relation is:

  - **reflexive** as $a\ R\ a,\ \forall\ a \in Z$
  - **symmetric** as if $a\ R\ b$ then $b\ R\ a,\ \forall\ a, b \in Z$
  - **transitive** as if $a\ R\ b$ and $b\ R\ c$ then $a\ R\ c$, $\forall\ a, b, c \in Z$

- R is an **equivalence relation**.

Example 2

- Let **R** be a **relation** of elements in **Z**:
  $\mathbf{R} = \{ (a, b) \in Z^2 \mid a \leq b \}$

- We have already proved that this relation is:

  - **reflexive** as $a\ R\ a$ for all a in Z
  - **transitive** as if $a\ R\ b$ and $b\ R\ c$ then $a\ R\ c$, $\forall\ a, b, c \in Z$
  - **not Symmetric** as $2 \leq 3$ but $3 \nleq 2,\ \forall\ a, b \in Z$

- R is not an **equivalence relation**.

Definition of equivalence class

Let **R** be an **equivalence relation** on a set S. Then, the **equivalence class** of **a** $\in$ S is:

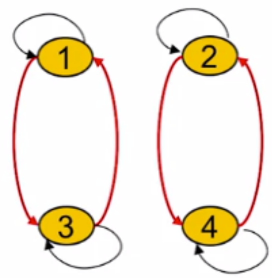
the **subset** of S containing all the **elements related** to **a** through '**R**'.

$$[a] = \{x : x \in S \text{ and } x\ R\ a\}$$

Example 1

- Let S = {1, 2, 3, 4} and **R** be a relation on elements in S:
  R = { (a, b) $\in$ S^2 | a mod 2 = b mod 2 }

- **R** is an **equivalence relation** with 2 equivalence **classes**:
  - [1] = [3] = {1, 3}
  - [2] = [4] = {2, 4}

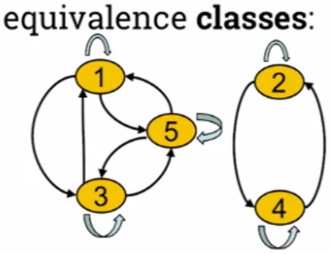


Example 2

- Let Z = {1, 2, 3, 4, 5} and **R** be relation of elements in Z:
  R = { (a, b) $\in$ Z^2 | a – b is an even number }

R = { (1,1), (2,2), (3,3), (4,4), (5,5), (2,4), (4,2), (1,3), (3,1), (1,5), (5,1), (3,5), (5,3) }

- **R** is an **equivalence relation** with 2 equivalence **classes**:
  - [1] = [3] = [5] = {1, 3, 5}
  - [2] = [4] = {2, 4}



Definition of partial order

Let **R** be a **relation** on elements in a set S. **R** is a partial order

if and only if

R is **reflexive**, **anti-symmetric** and **transitive**.

Example 1

- Let **R** be a **relation** of elements in **Z**:
  R = { (a, b)$\in$Z^2 | a$\leq$b }

- It can easily be proved that **R** is:
  - reflexive as **a ≤ a**, $\forall$ a $\in$ Z
  - transitive as if **a ≤ b** and **b ≤ c** then **a ≤ c**, $\forall$ a, b $\in$ Z
  - anti-symmetric as if **a ≤ b** and **b ≤ a** then **a = b**, $\forall$ a, b $\in$ Z

  - R is a **partial order**.

Example 2

- Let **R** be a **relation** of elements in **Z⁺**:
  R = { (a, b) $\in$ Z$^+$ | a divides b }

- It can easily be proved that **R** is:
  - reflexive as **a divides a**, $\forall$ a $\in$ Z$^+$
  - transitive as if **a divides b** and **b divides c** then **a divides c**, $\forall$ a, b, c $\in$ Z$^+$
  - anti-symmetric as **if a divides b** and **b divides a** then **a = b**, $\forall$ a, b $\in$ Z$^+$

- R is a **partial order**.

Definition of Total Order

Let R be a relation on elements in a set S. R is a

total order

if and only if

R is a partial order & $\{\forall a,b \in S \mid aRb \text{ or } bRa\}$

This means that R has to be a partial order & every two elements of the set S can be comparable with respect to the relation R

Example 1

- Let **R** be a **relation** of elements in **Z**:
  R = $\{ (a, b) \in Z^2 | a \leq b \}$

- It has been previously shown that **R** is a partial order

- Also, $\forall$ a, b $\in$ Z, a $\leq$ b or b $\leq$ a is true

- R is a **total order**.

Example 2

- Let **R** be a **relation** on elements in **Z⁺**:
  R = $\{ (a, b) \in Z^+ | \text{a divides b} \}$

- It has been proved that **R** is a **partial** order

- Z^+ contains elements that are **incomparable**, such as 5 and 7

- R **is not totally** ordered.

Summary

In this week, we learned what an equivalence relation & class are. Finally we explored the partial & total ordering of a relation.