Fundamentals Of Computer Science Course Notes

Felipe Balbi

October 19, 2019

Week 1

Learning Objectives:

- Understand logical arguments and apply basic concepts of formal proof.

1.101 Introduction to propositional logic

Propositional Logic is a system that deals with propositions or statements.

Below there's an example of where we can apply propositional logic to derive conclusions.

Liars And Knights

Imagine there is an island with two types of people. Liars who always tell lies and knights who always tell the truth. One an excursion, you visit the island and encounter two people, person A and person B. Person A says "at least one of us is a liar", while person B says nothing. What conclusion can you draw?

With a little logical thinking, we can conclude that person A is a knight and person B is a liar.

1.103 Building blocks of logic

What is a proposition?

A **proposition** is a statement that can be either **true** or **false**. It must be one or the other, never both nor neither.

Examples of proposition:

- 2 is a prime number (T)
- 5 is an even number (F)

Not a proposition:

- x is a prime number

  In this case, it can be made into a proposition by assigning a value to x.
- Are you going to school?

  Because this is a question, we can't assign a truth value to the sentence.
- Do your homework now

  Being an order, it has no truth value.

Syntaxes of the propositional logic

Propositions are denoted by capitable letters: P, Q, R, and so on.

- P = carrots are orange
- Q = I went to a party yesterday

General statements are denoted by lowercase letters: p, q, r, and so on. They carry on a logical argument, are used in proofs, called propositional variables.

Connectives: change or combine propositions

Connectives transform **atomic** propositions into **compound** propositions.

Logical NOT ($\neg$) $\neg p$ is true if and only if p is false. Also called **negation**.

Logical OR ($\vee$) $p \vee q$ is true if and only if at least one of p or q is true or if both p and q are true. Also called **disjunction**.

Logical AND ($\wedge$) $p \wedge q$ is true if and only if both p and q are true and false otherwise. Also called **conjunction**.

Logical if then ($\implies$) $p \rightarrow q$ is true if and only if either p is false or q is true. Also called **implication** or **conditional**. p is called the **premise** and q is the **conclusion**.

Logical if and only if ($\iff$) $p \iff q$ is true if and only if both p and q are true or both are false. Also called **bi-conditional**.

Exclusive or ($\oplus$) $p \oplus q$ is true if and only if p or q is true but not both.

Translation from Logical Proposition to English

Let:

$$\begin{aligned} P &= \text{I study 20 hours a week} \\ Q &= \text{I attend all the lectures} \\ R &= \text{I will pass the exam} \\ S &= \text{I will be happy} \end{aligned} \tag{1}$$

Translate the following statement to English:

- $(P \vee Q) \implies (R \wedge S)$

  If **I study 20 hours a week** or **I attend all the lectures** then **I will pass the exam** and **I will be happy**.

Translation from English to Logical Proposition

Given the statement:

If UK does not exit the EU then skilled nurses will not leave the NHS and research grants will remain intact.

Translating to logical proposition we get:

$$\begin{aligned} P &= \text{UK exits the EU} \\ Q &= \text{Skilled nurses will leave the NHS} \\ R &= \text{Research grants will remain intact} \\ \neg P &\implies \neg Q \wedge R \end{aligned} \tag{2}$$

Note that we removed any **connectives** from our propositions as that's a good practice. This makes the logical statement easier to follow.

1.105 Truth Table: examples

A truth table is a set of all outcomes of propositions and connectives. The number of rows in a truth table, depends on the number of given propositions. If we have n propositions, our truth table will have 2^n rows.

Truth tables for each connective

What follows is a list of truth tables for each connective

Negation ($\neg$)

p	$\neg p$
1	0
0	1

Conjunction ($\wedge$)

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

Disjunction ($\vee$)

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

Implication ($\implies$)

p	q	$p \implies q$
1	1	1
1	0	0
0	1	1
0	0	1

Bi-conditional ($\iff$)

p	q	$p \iff q$
1	1	1
1	0	0
0	1	0
0	0	1

Exclusive Or ($\oplus$)

p	q	$p \oplus q$
1	1	0
1	0	1
0	1	1
0	0	0

Operator Precedence

When formulae are written without parenthesis, we must rely on rules of operator precedence. Logic operator precende rules are as follows:

$\neg \wedge \vee \implies \iff$

Example If we have the logical statement $p \implies p \wedge \neg q \vee s$, we can parse it following the steps below:

$$\begin{aligned} p &\implies p \wedge \neg q \vee s \\ p &\implies p \wedge (\neg q) \vee s \\ p &\implies (p \wedge (\neg q)) \vee s \\ p &\implies ((p \wedge (\neg q)) \vee s) \\ (p &\implies ((p \wedge (\neg q)) \vee s)) \end{aligned} \tag{3}$$

Constructing Truth Tables for Complex Formulae

Example 1

p	q	$p \wedge q$	$(p \wedge q) \implies p$
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	1

Example 2

p	q	$q \implies p$	$p \wedge (q \implies p)$
1	1	1	1
1	0	1	1
0	1	0	0
0	0	1	0

Comparing both examples

p	q	$(p \wedge q) \implies p$	$p \wedge (q \implies p)$
1	1	1	1
1	0	1	1
0	1	1	0
0	0	1	0

1.202 Tautology and consistency

Tautology

A formula that is **always** true regardless of the truth value of the proposition.

p	$\neg p$	$p \vee \neg p$
0	1	1
1	0	1

Consistent

A formula that is true **at least** for one scenario. All connectives are consistent.

The formula $p \wedge \neg p$ is **inconsistent** because it can never be true. Inconsistent formulae are also called **contradictions**.

1.204 Tautology and consistency: examples

Example 1: $p \vee (q \wedge \neg r)$

p	q	r	$\neg r$	$q \wedge \neg r$	$p \vee (q \wedge \neg r)$
0	0	0	1	0	0
0	0	1	0	0	0
0	1	0	1	1	1
0	1	1	0	0	0
1	0	0	1	0	1
1	0	1	0	0	1
1	1	0	1	1	1
1	1	1	0	0	1

This is a **Consistent** formula

Example 2: $(p \implies q) \implies (\neg q \vee r)$

p	q	r	$p \implies q$	$\neg q$	(¬ q ∨ $)	$(p \implies q) \implies (\neg q \vee r)$
0	0	0	1	1	1	1
0	0	1	1	1	1	1
0	1	0	1	0	0	0
0	1	1	1	0	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	0	0	0
1	1	1	1	0	1	1

This is a **Consistent** formula

Example 3: $(p \implies q) \iff (\neg p \vee q)$

p	q	$p \implies q$	$\neg p$	$(\neg p \vee q)$	$(p \implies q) \iff (\neg p \vee q)$
0	0	1	1	1	1
0	1	1	1	1	1
1	0	0	0	0	1
1	1	1	0	1	1

This formula is a **Tautology**

Week 2

Learning Objectives:

- Understand logical arguments and apply basic concepts of formal proof.

1.301 Equivalences

Formulae A and B are equivalent if they have identical truth tables. Equivalence is denoted by the symbol $\equiv$

In other words, $A \equiv B$ means that A and B have the same truth values, regardless of how variables are assigned.

One thing to note is that $\equiv$ is **NOT** a connective.

De Morgan's Laws

$$\begin{aligned} \neg(p \wedge q) &\equiv \neg p \vee \neg q \\ \neg(p \vee q) &\equiv \neg p \wedge \neg q \end{aligned} \tag{4}$$

Truth Tables

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
0	0	1	1	0	1	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	1	0	0

$(p \implies q) \equiv (\neg p \vee q) \equiv \neg(p \wedge \neg q)$

p	q	$\neg p$	$\neg q$	$p \implies q$	$\neg p \vee q$	$\neg(p \wedge \neg q)$
0	0	1	1	1	1	1
0	1	1	0	1	1	1
1	0	0	1	0	0	0
1	1	0	0	1	1	1

Contrapositive: $(p \implies q) \equiv (\neg q \implies \neg p)$

p	q	$\neg p$	$\neg q$	$p \implies q$	$\neg q \implies \neg p$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	0	0	1	1

1.304 First-order logic

Important Notions

- **Predicates** describe properties of objects

  A simple example could be `odd(3)`. Here we're applying the predicate `odd` to the object `3`. When arguments are applied to predicates, they become propositions and connectives for propositional logic can be employed in the usual manner:

$$Odd(3) \wedge Prime(3) = T \tag{5}$$

- **Quantifiers** allow reasoning on multiple objects

  The objects from a quantified statement are chosen from a *Domain.*

  - **Existential Quantifier** $\exists$

    We use it as follows: $\exists x$ some formula.

    When proving a formula based on the existential quantifier, it is enough to find **one** element which makes the formula true. In other words, existentially quantified statements are **false** unless there is a positive example.

  - **Universal Quantifier** $\forall$

    We use it as follows: $\forall x$ some formula.

    In order to satisfy the formula, we must prove that **every** x satisfies the formula.

    Note that a single counterexample is enough to disprove a universally quantified statement. In other words, universally quantified statements are **true** unless there is a false example.

Translations English - Logic

"All P's are Q's" translates into $\forall x(P(x) \implies Q(x))$

"No P's are Q's" translates into $\forall x(P(x) \implies \neg Q(x))$

"Some P's are Q's" translates into $\exists x(P(x) \wedge Q(x))$

"Some P's are not Q's" translates into $\exists x(P(x) \wedge \neg Q(x))$

Quantifiers to connectives

- Existential Quantifier

$\exists x P(x)$ and domain $D = \{x_1, x_2, \ldots, x_n\}$. This is equivalent to saying $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$

- Universal Quantifier

$\forall x P(x)$ and domain $D = \{x_1, x_2, \ldots, x_n\}$. This is equivalent to saying $P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$

Negation of Quantifiers

- Existential Quantifier

$$\begin{aligned}\neg\exists x P(x) &\equiv \neg(P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)) \\ &\equiv \neg P(x_1) \wedge \neg P(x_2) \wedge \ldots \wedge \neg P(x_n)) \\ &\equiv \forall x \neg P(x)\end{aligned} \tag{6}$$

- Universal Quantifier

$$\begin{aligned}\neg\forall x P(x) &\equiv \neg(P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)) \\ &\equiv \neg P(x_1) \vee \neg P(x_2) \vee \ldots \vee \neg P(x_n)) \\ &\equiv \exists x \neg P(x)\end{aligned} \tag{7}$$

Example

$$\begin{aligned}\neg(\forall x(P(x) \implies Q(x)) &\equiv \exists x \neg(P(x) \implies Q(x)) \\ &\equiv \exists x \neg(\neg P(x) \vee Q(x)) \\ &\equiv \exists x(\neg\neg P(x) \wedge \neg Q(x)) \\ &\equiv \exists x(P(x) \wedge \neg Q(x))\end{aligned} \tag{8}$$

Week 3

Learning Objectives:

- Correctly follow a sequence of justified steps to reach a conclusion statement.
- Prove a conclusion statement by first assuming it is false.
- Describe inductive steps.
- Understand logical arguments and apply basic concepts of formal proof.

2.01 What is a proof?

A proof is a sequence of logical statements that explains why a statement is true. Rosen's book defines a proof as follows:

> A proof is a valid argument that establishes the truth of a mathematical statement.

We need proofs to establish general truths about conjectures. For example a computer cannot confirm that **all** numbers have a certain property. Well, considering that numbers are infinite and computers work with finite amounts of memory, a computer will not be able to answer that question.

Given a theorem, often there are many ways in which we can prove it. There are commonly used proof techniques which we learn about.

2.101 Direct proof

A direct proof exploits definitions and other mathematical theorems. It arrives at the desired statement by employing valid logical steps.

A direct proof is:

- Easy because there is no particular technique is used
- Not easy because the starting point is not obvious
- Know your definitions
- Allowed to use any theorem, axiom, logic, etc

Example 1:

Theorem 1. *If n and m are even numbers, then $n + m$ is also even*

Proof. What does even mean?

If an integer is even, it is twice another integer.

$$\begin{aligned} n &= 2k \\ m &= 2l \end{aligned}$$

k and l are integers

$$\begin{aligned} n + m &= 2k + 2l \\ &= 2(k + l) \end{aligned}$$

Let $k + l = t$,

$$n + m = 2t$$

$\therefore n + m$ is even. □

Example 2:

Theorem 2. *$\forall n \in \mathbb{N}, n^2 + n$ is even.*

Proof. If n is even, then $n = 2k$.

$$n^2 + n = (2k)^2 + 2k \,\text{Even}$$

If n is odd, then $n = 2k + 1$.

$$\begin{aligned} n^2 + n &= (2k+1)^2 + 2k + 1 \\ &= (2k)^2 + 2 \cdot 2k + 1^2 + 2k + 1^2 \\ &= 4k^2 + 6k + 2 \,\text{Even} \end{aligned}$$

□

2.102 Direct proof examples

Example 1:

Theorem 3. *If $a < b < 0$, then $a^2 > b^2$*

Proof. Assume $a < b$ and $a < 0$. Multiplying both sides of the inequality by a gives:

$$\begin{aligned} a \cdot a &< b \cdot a \\ a^2 &> b \cdot a \end{aligned}$$

Assume $a < b$ and $b < 0$. Multiplying both sides of the inequality by b gives:

$$\begin{aligned} a \cdot b &< b \cdot b \\ a \cdot b &> b^2 \end{aligned}$$

By the commutative property of multiplication we know that $a \cdot b = b \cdot a$, therefore:

$$\begin{aligned} a^2 > a \cdot b &> b^2 \\ \therefore a^2 &> b^2 \end{aligned}$$

□

Example 2:

Theorem 4. *$\forall x \in \mathbb{N}, 2x^3 + x$ is a multiple of 3.*

Proof. Factorizing $2x^3 + x$ gives $x(2x^2 + 1)$.

If x is a multiple of 3, the proof is complete.

If $x = 3k + 1$, then:

$$\begin{aligned} x(2x^2 + 1) &= (3k + 1)[2(3k + 1)^2 + 1] \\ &= (3k + 1)[2(9k^2 + 6k + 1) + 1] \\ &= (3k + 1)(18k^2 + 12k + 3] \\ &= 3(3k + 1)((6k^2 + 4k + 1) \end{aligned}$$

If $x = 3k + 2$, then:

$$\begin{aligned} x(2x^2 + 1) &= (3k + 2)[2(3k + 2)^2 + 1] \\ &= (3k + 2)[2(9k^2 + 12k + 4) + 1] \\ &= (3k + 2)(18k^2 + 24k + 9) \\ &= 3(3k + 2)(6k^2 + 8k + 3) \end{aligned}$$

□

2.202 Proof by contradiction

Proof by contradiction is also referred to as *indirect proof.* It follows a simple structure to prove that statement **A** is *true.*

We start by assuming **A** to be *false*, we follow just like a direct proof by employing mathematical definitions, theorems, axioms and logical steps until we arrive at a statement which **contradicts** our original assumption. This would show our original assumption to be incorrect. Therefore, if our assumption is **not** false, then it can only be true.

Example 1:

Theorem 5. *The square-root of two, $\sqrt(2)$ is irrational*

Proof. Assume $\sqrt(2)$ is rational. This means it can be written as a fraction, in lowest terms, of the form $\frac{p}{q}$ for $p, q \in \mathbb{N}$, $q \neq 0$.

If $\frac{p}{q}$ is in lowest terms, it means the fraction cannot be further simplified. Therefore, we have:

$$\begin{aligned}
\sqrt(2) &= \frac{p}{q} \\
\left(\sqrt(2)\right)^2 &= \left(\frac{p}{q}\right)^2 \\
2 &= \frac{p^2}{q^2} \\
2q^2 &= p^2
\end{aligned}$$

From this, we can see that p must be even. Which means $p = 2k$. Therefore:

$$\begin{aligned}
2q^2 &= p^2 \\
2q^2 &= (2k)^2 \\
2q^2 &= 4k^2 \\
q^2 &= 2k^2
\end{aligned}$$

From this, we can see that q must also be even. Which means our fraction $\frac{p}{q}$ cannot be in lowest terms. Therefore, $\sqrt(2)$ cannot be a rational number, so it is irrational. □

Example 2:

Theorem 6. *There is an infinite number of prime numbers.*

Proof. Assume there are finitely many prime numbers. Let the set of prime numbers be $P = \{p_1, p_2, \ldots, p_n\}$. Let $N = (p_1 \cdot p_2 \cdot \ldots \cdot p_n) + 1$.

If we divide N by any of the prime numbers in our list of prime numbers, it will have a remainder of 1. Therefore, N is, itself, a prime number. □

2.203 Proof by contraposive

This technique exploits equivalent classes of logical statements. Let us remember that $a \implies b \equiv \neg b \implies \neg a$.

In some cases, when we need to prove $a \implies b$, it may be easier to prove its contrapositive $(\neg b \implies \neg a)$ is true.

Example 1:

Theorem 7. $\forall n \in \mathbb{N}, Odd(n^3 + 1) \implies Even(n)$

Proof. By means of the contrapositive $\forall n \in \mathbb{N}, Odd(n) \implies Even(n^3 + 1)$.

$$n = 2k + 1 \forall k \in \mathbb{N}$$

$$\begin{aligned}
n^3 + 1 &= (2k+1)^3 + 1 \\
&= (2k+1)(2k+1)(2k+1) + 1 \\
&= 8k^3 + 12k^2 + 6k + 2 \\
&= 2(4k^3 + 6k^2 + 3k + 1)
\end{aligned}$$

□

Example 2:

Theorem 8. *Suppose* $x, y \in \mathbb{R}$, $y^3 + yx^2 \leq x^3 + xy^2 \implies y \leq x$

Proof. By contrapositive $y > x \implies y^3 + yx^2 > x^3 + xy^2$.

Assuming $y > x$, we know that $y - x > 0$. Let us multiply both sides of the inequality by $x^2 + y^2$. Therefore:

$$\begin{aligned}
(y^2 + x^2)(y - x) &> 0(y^2 + x^2) \\
y^3 + yx^2 - xy^2 - x^3 &> 0 \\
y^3 + yx^2 &> x^3 + xy^2
\end{aligned}$$

□

Week 4

Learning Objectives:

- Correctly follow a sequence of justified steps to reach a conclusion statement.
- Prove a conclusion statement by first assuming it is false.
- Describe inductive steps.
- Understand logical arguments and apply basic concepts of formal proof.

2.301 Proof by induction

Mathematical induction is a useful proof method that has several steps that must be followed. We can consider mathematical induction as a row of standing dominoes and we want to prove that all dominoes fall.

In order to prove that all dominoes fall, we must first and foremost prove that the first domino falls. After that we prove that if one domino falls, the next one must also fall.

Mathematically, if $P(0)$ is true and $\forall k \in \mathbb{N} P(k) \implies P(k+1)$, then we can conclude that $\forall n \in \mathbb{N} P(n)$ is true.

Proof by induction has three important steps:

- The *Base Case* or *Basis*

  Here we prove that $P(0)$ is true. This allows us to prove that the theorem **starts** true.

- The *Inductive Step*

  Here we prove that $P(k) \implies P(k+1)$. Note that we never assume this to be true. We must always carefully prove it. Because we're trying to prove an implication, we assume $P(k)$ to be true and prove $P(k+1)$ to be true when $P(k)$ is true. The reason for this is that if $P(k)$ is false, then the implication is true anyway. The assumption that $P(k) is true is called the *Inductive Hypothesis*.

- The *Conclusion by Induction*

  We finish the proof by writing $\therefore \forall n \in \mathbb{N} P(n)$ is true.

It's common practice to end a proof with the $\square$ symbol. Referred to as **QED** (from Latin *quod erat demonstrandum*).

2.303 Example of a correct proof

$\sum_{i=0}^{n-1} 2^i = 2^n - 1$

Theorem 9. *The sum of the first n powers of 2, is $2^n - 1$.*

Proof. Let $P(n) = 2^0 + 2^1 + \ldots + 2^{n-1} = 2^n - 1$, prove that $P(n)$ is valid for all n.

Basis: Prove that $P(1)$ is true.

$$\begin{aligned} P(1) &= 2^{1-1} \\ &= 2^0 \\ &= 1 \\ &= 2^1 - 1 \end{aligned}$$

Indutive Step: Prove that $P(k) \implies P(k+1)$ is true.

Assuming $P(k) = 2^0 + 2^1 + \ldots + 2^{k-1} = 2^k - 1$ to be true, prove that $P(k+1) = 2^0 + 2^1 + \ldots + 2^{k-1} + 2^k = 2^{k+1} - 1$ is also true.

$$P(k+1) = 2^0 + 2^1 + \ldots + 2^{k-1} + 2^k$$

By Inductive Hypothesis we know that $2^0 + 2^1 + \ldots + 2^{k-1} = 2^k - 1$, therefore:

$$
\begin{aligned}
P(k+1) &= 2^k - 1 + 2^k \\
&= 2^k + 2^k - 1 \\
&= 2^{k+1} - 1
\end{aligned}
$$

Conclusion: $P(k+1)$ is true, $\therefore \forall n \in \mathbb{N} 2^n - 1$ is true. $\square$

$\forall n n < 3^n$

Theorem 10. *$n < 3^n$, for all $n \in \mathbb{N}$*

Proof. Let $P(n) = n < 3^n$, prove by induction that $P(n)$ is true for all n.

Basis: Prove $P(0)$ is true.

$$
\begin{aligned}
P(0) &= 0 < 3^0 \\
&= 0 < 1
\end{aligned}
$$

Inductive Step: Prove $P(k) \implies P(k+1)$.

Assuming $P(k) = k < 3^k$ to be true, prove that $P(k+1) = (k+1) < 3^{k+1}$ is true.

$$
\begin{aligned}
k &< 3^k \\
k+1 &< 3^k + 1 \\
k+1 &< 3^k + 1 < 3^k + 3^k + 3^k \\
k+1 &< 3^k + 1 < 3 \cdot 3^k \\
k+1 &< 3^k + 1 < 3^{k+1} \\
k+1 &< 3^{k+1}
\end{aligned}
$$

Conclusion: $P(k+1)$ is true, $\therefore \forall n \in \mathbb{N} n < 3^n$ is true. $\square$

2.305 Example of an incorrect proof

We're going to see how easy it is to make a mistake in a proof if we don't follow the steps correctly.

$n + 1 < n \forall n \in \mathbb{N}$

Theorem 11. *$n + 1 < n$, for all $n \in \mathbb{N}$.*

Proof. INCORRECT!!!

Let $P(n) = n + 1 < n \forall n \in \mathbb{N}$.

Prove $P(k) \implies P(k+1)$. Assuming $P(k) istrue, so$k + 1 < k.$Show$P(k + 1)$istrue.Adding 1 tobothsidesoftheinequalityweget$k + 1 + 1 < k + 1.$Let$l = k + 1$weget$l + 1 < l, soP(k + 1)$isalsotrue.Therefore$P(n) is true. □

In this proof, we didn't prove the base case, so the proof in invalid.

2.401 Conclusion

We have learned about several powerful proof techniques. We explored Proof by Induction, which exploits the fact that natural numbers are like a chain.

We have also seen how contrapositive proofs are, sometimes, easier then proving the original statement. We have also witnessed how proofs can go wrong if we miss an important step.

Week 5

Learning Objectives

- Explore finite or countable discrete structures in the context of computer science.
- Consider how different rules can be applied to appreciate the number of possible outcomes for an event.
- Explore relationships between sets and elements within or across sets.
- Consider how elements in a set can be counted.

3.01 Introduction

During this topic, we study key principles in counting. We study the Pigeon-hole principle and learn to apply to prove theorems.

3.101 Counting

How many outfits can we pick from a collection of 5 pairs of trousers and 7 shirts? Essentially this translates to:

$$
\begin{aligned}
&\binom{7}{1} \cdot \binom{5}{1} \\
&= \frac{7!}{1! \cdot (7-1)!} \cdot \frac{5!}{1! \cdot (5-1)!} \\
&= \frac{7 \cdot 6!}{6!} \cdot \frac{5 \cdot 4!}{4!} \\
&= 7 \cdot 5 \\
&= 35
\end{aligned}
$$

Product Rule

The product rule says that if a job can be split into two separate tasks, if there are n ways of doing task 1 and m ways of doing task 2 then the job can be done in $n \cdot m$ ways.

A generalization of this is to state that if there are k tasks and each task can be achieved in n_i ways, then there are $n_1 \cdot n_2 \cdot \ldots \cdot n_k$ ways of achieving the task.

Example 1 How many strings of length 5 can we make with uppercase English letters?

We have 26 uppercase letters in the English alphabet and there are no restrictions to repetition. For each letter we have 26 options, therefore we can make as many as $26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 26^5 = 11881376$.

Example 2 How many strings of length 5 can we make with 3 uppercase English letters and 2 digits?

This is going to be $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 = 26^3 \cdot 10^2 = 1757600$

Sum Rule

The sum rule states that if a job can be done in n ways **or** m ways, then it can be done in $n + m$ ways.

Example 1 A teacher is choosing a student to be her assistent from 5 different classes. The classes contain 28, 21, 24, 25, and 27 students. How many possible ways are there to pick an assistant?

There are $28 + 21 + 24 + 25 + 27 = 125$ ways of picking an assistant.

3.102 Complex counting

Continuing with counting, looking at more advanced techniques.

Example 1

For most accounts, you need to choose a password. In this example, the password must be five to seven characters long. Each drawn from uppercase letters or digits. The password must contain at least one upper case letter.

Let's split this work by length:

Passwords	Length 5	Length 6	Length 7
All passwords (1)	36^5	36^6	36^7
No Letters (2)	10^5	10^6	10^7
Valid Passwords (1 - 2)	60 366 176	2 175 782 336	78 354 164 096
		Total	80 590 312 608

Subtraction Rule

The subtraction rule applies when lists have items in common. This rule is also known as *Inclusion-Exclusion Principle.*

This rule states that if a choice can be made from two lists containing n and m items, then the number of ways to make a choice from these two lists is $n + m -$ items in common.

Example 1 How many integers less than 100 are divisible by either 2 or 3. In other words, we're talking about the cardinality of the union of the set of numbers divisible by 2 and less than 100 and the set of numbers divisible by 3 and less than 100.

However, this would be cumbersome to calculate. A simpler way is to first calculate how many numbers between 1 and 99 are divisible by 2 using $\lfloor \frac{99}{2} \rfloor = 49$. Similarly, we can calculate how many numbers between 1 and 99 are divisible by 3 using $\lfloor \frac{99}{3} \rfloor = 33$.

We must remember to decrement numbers that are divisible by both 2 and 3. Such numbers are divisible by 6, therefore $\lfloor \frac{99}{6} \rfloor = 16$.

So the answer to our original question is $49 + 33 - 16 = 66$.

3.201 The Pigeonhole Principle

The Pidgeonhole Principle states that if n items are put into m containers, with $n > m$, then at least one container must contain more than one item. The Pidgeonhole Principle is also know as *Dirichlet's drawer principle.*

The generalized pigeonhole principle states that:
Theorem 12. *If there are N objects to be placed in k boxes, then at least one box contains the $\lceil \frac{N}{k} \rceil$ objects.*

Proof. **By contradiction**

Assume **none** of the boxes contains more than $\lceil \frac{N}{k} \rceil - 1$ objects. Since we have k boxes, we can conclude that Number of Objects $\leq k(\lceil \frac{N}{k} \rceil - 1) < k(\frac{N}{k} + 1 - 1) = N$.

From that we conclude that Number of Objects $\leq N$, which contradicts our original statement that there are exactly N objects. $\square$

Example 1

How many cards from a standard deck of 52 cards must be selected to guarantee that 5 cards are from the same suit?

There are 4 suits in a standard deck of cards. If we pick 16 cards and spread them evenly among the suits, we will end up with 4 cards for each suit. At the moment we pick the 17^{th} card, it must go to one of the 4 suits, therefore giving 5 cards from the same suit.

We can verify this with $\lceil \frac{17}{4} \rceil = 5$.

3.202 The Pigeonhole Principle: examples

Example 1

In a group of 4 integers, show that there are at least two with the same remainder when divided by 3.

There are exactly three possible remainders when dividing numbers by three. Either the number is divisible by 3, giving a remainder of zero, or it is 1 above a multiple of three, or 2 above a multiple of three.

In any group of 4 integers, we will have at least two numbers with the same remainder when divided by three. This means that we have three boxes (the three possible remainders) and 4 objects (our 4 randomly selected integers). By the pidgeonhole principle, we know that at least one box will contain more than one object.

Example 2

A bag contains 7 blue balls and 4 red balls, how many must be selected to guarantee that three balls are of the same color.

In this case, there are 2 boxes (the colors) and 11 objects. In the worst case, we pick colors evenly. Assuming we have picked 4 balls (2 blue and 2 red), when we pick the 5^{th} ball, it must be either blue or red, therefore giving us our 3 balls of the same color.

In other words, we want to find the number which satisfies $\lceil \frac{x}{2} \rceil = 3$.

Example 3

Select 5 integers from the set $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$; show that at least two integers add up to 9.

The pairs making up 9 are $(1, 8)$, $(2, 7)$, $(3, 6)$, $(4, 5)$. If we label those pairs A, B, C, D, we can see that all numbers in the set belong to one of 4 boxes.

There are 4 boxes and 5 objects, therefore at least one box will have more than 1 object.

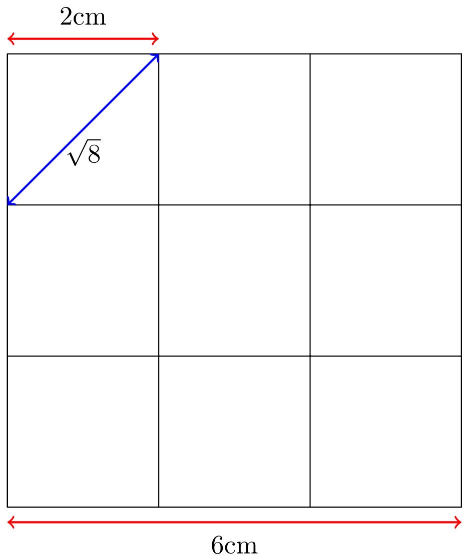
Example 4

There are n people in the room; every pair is either friends or not friends. Show that there are at least two people with the same number of friends.

We know that there are n in the room and the amount of friends people can have are limited to $1, 2, 3, \ldots, n-1$ or $0, 1, 2, \ldots, n-2$. In both of these cases we will have $n-1$ boxes and n people. Consequently, at least one box will have more than one person.

Example 5

Show that if there are 10 dots on a square of $6cm \times 6cm$, there are at least two dots within $\sqrt{8}$ cm.



We can see that a $6cm \times 6cm$ square divided into 9 equal smaller squares of $2cm \times 2cm$. The smaller squares have a hypotenuse of $hyp = \sqrt{2^2 + 2^2} = \sqrt{8}$ (indicated by the blue line).

We have 9 boxes that are $\sqrt{8}$ apart, but have 10 objects. Therefore at least two will be within $\sqrt{8}$ distance from each other.