

Base Conversion

Decimal to base_n (quotient):

- Take the quotient of the decimal and divide by the base
- Note the remainder
- Repeat from step 1 until the quotient is zero
- Return the list of remainders, the first is the least significant digit of the new quotient. The last is the most significant.
- Eg: 273_{10} to base₂
 - $273 / 2 = 136 \text{ REM } 1$
 - $136 / 2 = 68 \text{ REM } 0$
 - $68 / 2 = 34 \text{ REM } 0$
 - $34 / 2 = 17 \text{ REM } 0$
 - $17 / 2 = 8 \text{ REM } 1$
 - $8 / 2 = 4 \text{ REM } 0$
 - $4 / 2 = 2 \text{ REM } 0$
 - $2 / 2 = 1 \text{ REM } 0$
 - $1 / 2 = 0 \text{ REM } 1$
 - $273_{10} = 100010001_2$

Decimal to base_n (fractional)

- Take the fractional of the decimal and multiply by the base
- Take the new quotient and note it separately
- Repeat from step 1 until the result is 0
- Return the list of quotients as the new fractional. The first is the most significant digit, the last is the least.
- Eg: 0.625_{10} to base₂
 - $0.625 * 2 = 1.25 \text{ QUOT } 1$
 - $0.25 * 2 = 0.5 \text{ QUOT } 0$
 - $0.5 * 2 = 1.0 \text{ QUOT } 1$
 - $0.0 * 2 = 0.0$
 - $0.625_{10} = 0.101_2$

Base_n to decimal (quotient)

- Multiply each digit by the base raised to the power of that digit's number.
- Add the results
- Eg: 101101_2
 - $1 * 2^0 = 1$
 - $0 * 2^1 = 0$
 - $1 * 2^2 = 4$
 - $1 * 2^3 = 8$
 - $0 * 2^4 = 0$
 - $1 * 2^5 = 32$
 - Sum = 45

Base_n to decimal (fractional)

- Multiply each digit by the base raised to the inverse power of that digit's number.

- Add the results
- Convert fraction to decimal
- Eg: 0.1101_2
 - $1 * 2^{-1} = 1/2$
 - $0 * 2^{-2} = 0$
 - $1 * 2^{-3} = 1/8$
 - $1 * 2^{-4} = 1/16$
 - Sum = $11/16$
 - $11/16 = 0.6875$

Base Arithmetic

As normal, but carry on the base number, not always 10.

Modular Arithmetic

Basics

Modular math = clock arithmetic

It is concerned with the remainders of division by a particular number

Remainders are always between 0 and $N - 1$. So working in (mod 5) we would expect remainders between 0 and 4.

N modulo K:

- If $N < K$ then $\text{Mod} = N$.
 - Eg.: $5 \pmod{7} \equiv 5 \pmod{7}$
- If $N > K$ then $\text{Mod} = N - (\text{quotient}(N/K) * K)$. In other words, find the remainder of N/K , it's not rocket science.
 - Eg.: $29 \pmod{7} \equiv 29 / 7 = 4 \text{ REM } 1 \equiv 1 \pmod{7}$
- Negation: $N \pmod{K}$ can also be expressed as $-(K - N) \pmod{K}$
 - Eg.: $255 \pmod{257} \equiv -2 \pmod{257}$

Special cases:

- (mod 10) Just take the least significant digit
- (mod 5) Least significant digit, if greater than five subtract five.
- (mod 9) Casting of 9s, probably a product of working in base₁₀. Add together all of the digits. Then add together the digits of the result. Continue until you have a single number. This is the remainder.
- (mod 3) Cast of 9s. Take result and divide by 3. Remainder is mod.
- (mod 11) Sum (even order digits) – Sum (odd order digits) = mod.

Arithmetic summary:

- $(X + Y) \pmod{K} \equiv ((X \pmod{K}) + (Y \pmod{K})) \pmod{K}$
- $(X - Y) \pmod{K} \equiv ((X \pmod{K}) - (Y \pmod{K})) \pmod{K}$
- $(X * Y) \pmod{K} \equiv ((X \pmod{K}) * (Y \pmod{K})) \pmod{K}$

Additive identity:

The additive identity of any (mod K) is 0, as when added to anything else it causes not change.

Additive inverse:

Any pair of mod results which add up to K are “additive inverses” as they result in mod 0.

Eg: $2 + 3 = 0 \pmod{5}$

This can also be stated as 3 being the -2 of $(\text{mod } 5)$, as when added to 2 it produces 0.

May be expressed as:

$$-23 \pmod{5}$$

Ans:

$$-23 \pmod{5} \equiv$$

$$-3 \equiv$$

$$2$$

Multiplication:

As in arithmetic summary, calculate $A \pmod{K}$ and $B \pmod{K}$, multiply the results then calculate $\text{Result} \pmod{K}$

Eg:

$$11 * 13 \pmod{9} \equiv$$

$$2 * 4 \pmod{9} \equiv$$

$$8$$

Multiplicative identity:

The multiplicative identity of any $(\text{mod } K)$ is 1, as when multiplied by anything else it doesn't change it.

Multiplicative inverse:

With linear numbers the inverse is the number which, when multiplied by the original, produces 1.

Eg. The inverse of 2 (or $2/1$ or 2^1) is $1/2$ (or 2^{-1})

The inverse of modular numbers is the same: The numbers which when multiplied will produce 1. This is arrived at by multiplying them normally then finding the modulo of the product.

Eg. The inverse of 2 $(\text{mod } 5)$ is 3. As $2 * 3 = 6$ and $6 \pmod{5}$ is 1
This can also be written as $2^{-1} \pmod{5} \equiv 3$

Naive calculation of $X^{-1} \pmod{K}$:

- Multiply X by every N from 0 to K-1
- Find the $(\text{mod } K)$ of each of those numbers
- $X^{-1} \pmod{K} =$ whichever one results in 1

Calculation of $X^{-1} \pmod{K}$ when K is prime:

- Fermat's little theorem: $X^{K-1} \equiv 1 \pmod{K}$
- If we multiply both sides by X^{-1} we find that:
- $X^{-1} \equiv X^{K-2} \pmod{K}$
- So, we raise X to the power of K-2
- Find the result
- Then find the modulo $(\text{mod } K)$
- Which leaves us the inverse

Calculation of $X^{-1} \pmod{K}$ when K is not prime:

- Count how many numbers from 1 \rightarrow $K-1$ that are coprime with K
- Coprime: Has no factors in common other than 1
- Call this number N
- $X^{-1} \equiv X^{N-1} \pmod{K}$
- Eg:
 - $9^{-1} \pmod{22}$
 - factors of 22 are: 2, 11
 - [1- \rightarrow 21]=
[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21]
 - Remove numbers coprime with 22 (ie. Divisible by 2 or 11)
 - [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21]
 - $n = \text{len}[1,3,5,7,9,13,15,17,19,21] = 10$
 - $9^{-1} \equiv 9^9 \equiv 5 \pmod{22}$
- Check your results: $(X * X^{-1})\%K$ should equal 1
 - $(9*5)\%22 = 45\%22 = 1$

Exponentiation:

- Simplify the base to the lowest congruent integer.
 - Eg. $277^8 \equiv 2^8 \pmod{5}$
- Gradually raise this integer to the required power. This can be achieved by working out basic powers and multiplying them together to "add" the exponent.
 - Eg. $2^8 = 2^5 * 2^3 = 32 * 8 = 256$
- We can also multiply the exponent using powers
 - Eg. $2^{64} = (2^8)^8$
- Eg in $\pmod{5}$:
 - $2^3 \equiv 8 \equiv 3 \pmod{5}$
 - $2^5 \equiv 32 \equiv 2 \pmod{5}$
 - $2^8 \equiv 2^5 * 2^3 \equiv 3 * 2 \equiv 6 \equiv 1 \pmod{5}$
- NB: $X^{K-1} \pmod{K} = 1$ PROVIDED X/K is not a whole number
- This means that if K is prime, X will not evenly divide by K
- So all prime numbers are easier to work with in modular arithmetic, as we only ever have to calculate $K-1$ powers to calculate any $X^Y \pmod{K}$

Encryption using Modular Arithmetic:

RSA encryption works using a public/private key system (you already know this)

- For a given \pmod{K} , take any base (the plaintext character) and raise it to the power E .
- Resolve this power (find it's modulo) and transmit it, it is now encrypted.
- At the other end, raise it to the power D .
- When this new power is resolved, the resulting modulo is the original plaintext character.
- The public key is E combined with K
- The private key is D combined with K

Finding key pairs in mod K (where K is prime):

- E is the multiplicative inverse of D in (mod K-1)
- So $E * D \equiv 1 \pmod{K-1}$
- So by fermat's little theorem:
- $D \equiv E^{K-3} \pmod{K-1}$

Sequences

Sequences:

- Sequences are any list of numbers
- Some sequences have a logical progression, some do not
- If a sequence is infinite this is noted by placing a ... at the end
 - Eg.: 1,2,3,4,5,6...
- If there is a gap in the middle, we can also use ...
 - Eg.: 1,2,3...4,5,6
- Sequences can also be defined with formulae
 - Eg.: $X_n = n$

Arithmetic Sequences:

- An arithmetic sequence exists where each term is found by adding a constant number to the previous term.
 - Eg.: 1,5,9,13,17 ($n+4$)
- One expression is the "recurrence relation": $a_{n+1} = a_n + d$
 - This uses the current term to define the next one
 - d is the "common difference"
 - n is the current iteration
 - In order to use this definition the first term must be provided separately
- One can also write the "general term": $A_n = A_1 + d*(n-1)$
- In order to obtain term n using the recurrence relation, then we must calculate all the terms before n . In order to obtain n with the general term we simply slot our number into the formula.

Geometric Sequences:

- A geometric sequence exists where each term is found by multiplying the previous term by a constant number.
 - Eg.: 1,3,9,27,81,243
- The "recurrence relation" of this sequence is: $A_n = R*A_{n-1}$
 - R = The common ratio, or the factor used to create the new term
- The "general term" of this sequence is: $A_n = A_1 * R^{n-1}$
- As a rule: $A_n/A_{n+1} = R$
- This can be used on multiple terms to determine if the sequence is truly geometric
- Checking if a X is a term in a geometric sequence
 - Write out the general term
 - Resolve so that the formula is a single power
 - If X is a power of the base then yes. If not then no
- Negative ratios will result in alternating positive and negative terms
- Fractional ratios: Result in decreasing and convergent sequences

Series

- A series is the sum of the terms of a sequence
- The notation is:

$$\sum_{n=0}^{63} 2^n$$

- Sum of terms from A_0

to A_{63}

- Where each term is 2^n
- Triangular numbers: An arithmetic series where $D = 1$ produces triangular numbers
 - Eg.: $1+2+3+4+5 = 15$
 - They are called this as that number of objects can be arranged in a regular triangle, like bowling balls:
 - Eg.: 10
 - 0
 - 00
 - 000
 - 0000
- If a general term consists of the sum of two components, it can be treated as the summation of two separate summations.
- Eg.:
 - $(4)\sum(n=1) (n + 2^n) = (4)\sum(n=1)(n) + (4)\sum(n=1)(2^n)$
- The same is not true of two components multiplied together.
- If a series is multiplied by a number, or has another operation applied to it, it is the same as applying that operation to the general term.
- Eg.:
 - $5*(4)\sum(n=1)(n) = (4)\sum(n=1)(5n)$

Summing arithmetic sequences:

- Add the first and last terms
- Multiply this number by half the number of terms
- Eg.:
 - $1+4+7+10+13+16+19+22+25+28 =$
 - $(1+28)*5 =$
 - 145
- If the number of terms is odd, this formula still works, you just multiply by the half.
- Eg.:
 - $1+4+7+10+13 =$
 - $(1+13)*2.5 =$
 - 35

Summing geometric sequences:

- Where S is the sum of the terms of the sequence, N is the number of terms and R is the common ratio.
- $S = A_1 * ((1-R^n)/(1-R))$
- Eg.
 - Sequence = 1,2,4,8,16
 - $A_n = 2^{n-1}$
 - $R = 2$
 - $N = 5$
 - $S = (2^5-1)/(2-1)$
 - $= (32-1)/(1)$
 - $= 31/1$
 - $= 31$
- This can be proven:
 - $S = 1+2+4+8+16$
 - $2S = 2+4+8+16+32$
 - If we subtract S from $2S$ we get S , or:
 - $S = -1 \quad +32$
 - $S = 32-1$
 - $S = 31$
 - So we multiplied S by the common ratio (R), then subtracted S by the common ratio $- 1$ ($S*(R-1)$)
 - The result was $A_{n+1} - A_1*(R-1)$

Other summation formulae:

- Sum of numbers from 1-> N
 - $N(N+1)/2$
 - I.e. $1+2+3+4+5+6...+N$
 - Based on the formula for the sum of arithmetic series
- Sum of squares from 1-> N
 - $N(N+1)(2N+1)/6$
 - I.e. $1^2+2^2+3^2+4^2+5^2+6^2...+N^2$
- Sum of cubes from 1-> N
 - $N^2(N+1)^2/4$
 - I.e. $1^3+2^3+3^3+4^3+5^3+6^3...+N^3$

Convergence, Divergence & Limits

- Limits are a number which is approached but not reached by a sequence.
- Eg.:
 - $1, 1/2, 1/3, 1/4, 1/5 \dots \rightarrow 0$
 - May be noted as:
 -

$$\frac{1}{n} \xrightarrow{n \rightarrow +\infty} 0$$

$$\lim_{n \rightarrow +\infty} \frac{1}{n} = 0$$

- In order to find a limit, one can separate the series into parts.
 - Eg, $\sum (1/n + 5) = \sum (1/n) + \sum(5)$
 - $\sum (1/n) \rightarrow 0$ Therefore the limit is 5
- All arithmetic series are divergent and approach $+\infty$ or $-\infty$
- Geometric series are divergent if $R > 1$, and approach either $+\infty$ or $-\infty$
- If $R < -1$, then they are divergent with no limit
- If $R < 1$ & $R > -1$ they are convergent on 0
- If you're stuck, you can always sum a finite series to see where it goes.
- Harmonic series =
 - $1/1, 1/2, 1/3, 1/4 \dots 1/\infty$
 - ^ Not convergent
- If we add two convergent sequences, the result is also convergent and the sum of that series is equal to the sum of the original series added together.
 - Take two convergent sequences S_1 and S_2
 - $S_1 + S_2$ is also convergent
 - $\sum(S_1 + S_2) = \sum(S_1) + \sum(S_2)$
- If we multiply a convergent sequence by a constant, the result is convergent. Also, the limit of the new sequence is the same as the limit of the old sequence multiplied by the constant.
 - Where A_n is a convergent sequence and C is a constant:
 - $\text{Lim}(A_n) * C = \text{Lim}(A_n * C)$
 - And both are convergent
- When deciding the convergence of a sequence, one approach can be to consider similar sequences

- If $\sum A_n < \sum B_n < \sum C_n$ and both A_n and C_n are convergent, then B_n will be convergent also.
- As a bonus: $\lim(a_n) < \lim(b_n) < \lim(C_n)$
- Hypothesis: If you divide the terms of a series by N , the sum is $= \sum/N$

Functions and Graphing

2D Cartesian coordinate system:

I don't think I need to cover this

Function notation and definitions:

- Two common styles of function notation:
 - $f(x):x^2$
 - $y = x^2$
- Independent variable:
 - Normally plotted on the x-axis
 - So called because we can input any value into it
- Dependant variable:
 - Normally plotted on y-axis
 - So called because it's value is dependant on the value of the independant variable
- Domain: The series of values inputted as the independant variable
- Range: The series of values solved for the dependant variable

Interval notation:

- Expresses a set of values, eg: $0,1,2,3,4,5 = [0,5]$
- Square brackets "[" include the value they enclose
- Round brackets "(" exclude the value they enclose
- So $[0,5)$ means the set $\{0,1,2,3,4\}$
- If there is a gap in a set it may be expressed as two sets using venn notation
- Eg all positive and negative numbers except zero:
 - $(-\infty,0) \cup (0,\infty)$ using interval notation. The "U" represents the "union" of the two sets.
 - This can also be written as $\mathbb{R} \setminus \{0\}$
- Another example, all positive numbers and zero:
 - $[0,\infty)$
 - \mathbb{R}_0^+

Asymptote: Graphical equivalent of a limit. A line the graph approaches but will never touch.

Intercept: The point (or points) at which a graph cuts an axis.

Vertical intercept: Where it cuts the y axis

Horizontal intercept: Where it cuts the x axis

Vertex: A graph's point of greatest extent along a particular axis.

Types of graph:

- Line: $y = mx + c$
 - m is the slope
 - c is the vertical intercept

- Quadratic:
 - An arch type shape (parabola)
 - Caused by three terms summed together:
 - $y=ax^2+bx+c$
 - a is the coefficient of x^2 and affects the width of the arch
 - If a is positive, the parabola will be "u" shaped, if it is negative the arch will be drawn "n" shaped.
 - b is the coefficient of x and affects the position of the arch
 - c is the independent term (as it is not affected by the value of x) and defines the vertical intercept. Altering this value will move the parabola up or down the y axis.
 - We can also solve for x :
 - $x = (-b \pm \sqrt{b^2 - 4ac}) / 2a$
 - Vertex may be found by finding two values for x at the same y , then finding their center. Then solve for y with this x value.
 - Alternately, if the function can be restated as
 - $y=(x+a)^2 + b$
 - Then the vertex will be (a,b)
- Cubic:
 - Draws a compound curve which travels from one quarter to the opposite quarter (eg. Top-left to bottom-right)
 - As it approaches the origin it transitions through an S shaped bend
 - Caused by three terms summed together:
 - $y=ax^3+bx+c$
 - a is the coefficient of x^3 and affects the slope of the curves. Higher = narrower, lower = broader
 - If it is positive, it produces an ascending curve (bottom-left to top-right)
 - If it is negative it produces a descending curve (top-left to bottom-right)
 - b is the coefficient of x and affects the eccentricity of the s bend
 - A higher value will produce a more pronounced bend if the sign is opposite " a "
 - A higher value will produce a straighter bend if the sign is the same as " a "
 - c (the independent term) is the vertical intercept
 - A more negative value moves the curve down the y axis
 - A more positive value moves it up
- Higher order polynomials
 - Involving powers higher than x^3
 - If a term can be reduced to zero, then the graph will cross the x -axis at that point
 - Eg: $y = (x-4)(x+2)(x-2)(x+5)$
 - The first term will resolve as zero when x is 4, therefore there is a point at $(4,0)$
- Reciprocal function
 - X is the denominator of a fraction, 1 is the numerator

- Eg.: $y = 1/(x-5)$
- Forms a rectangular hyperbola
- Always has a value for X which will result in division by zero and cannot be computed. In this case $x = 5$
- This is a vertical asymptote
- So the domain is real numbers except 5:
 - $\mathbb{R} \setminus \{5\}$
 - $(-\infty, 5) \cup (5, \infty)$
- Rational function
 - X is the numerator, X^2 minus a constant is the denominator
 - Eg.: $x/(x^2-3)$
 - Has two vertical asymptotes (in this case, $\sqrt{3}$ and negative $\sqrt{3}$)
 - Also has a horizontal asymptote
- Piecewise function
 - Function broken down into separate conditional statements
 - Eg: $\text{if}(y \leq 5)\{y=x\}, \text{if}(y > 5)\{y=2x-2\}$
 - The sub function can be any other type of function

Transformations:

- Translation
 - Horizontal translation: Add/subtract value to/from every X before any other operation is performed
 - Eg: $y = x^2 + 3x$
 - Shift right by three units:
 - $y = (x - 3)^2 + 3(x - 3)$
 - Vertical translation: Add/subtract value after all other operations are performed.
 - Eg: $y = x^2 + 3x$
 - Shift up by two units:
 - $y = x^2 + 3x + 2$
- Rotation
- Scaling
 - Vertical scaling: Multiply all terms by the scalar after every other operation
 - Eg: $y = x^3 + x^2 + 5$
 - Scaled on the y axis by a factor of 2 becomes:
 - $y = 2x^3 + 2x^2 + 20$
 - or
 - $y = 2(x^3 + x^2 + 10)$
 - Horizontal scaling: Multiply every x by the inverse of the scalar prior to any other operation
 - Eg: $y = x^3 + x^2 + 5$
 - Scaled on the x axis by a factor of 2 becomes:
 - $y = (x*1/2)^3 + (x*1/2)^2 + 5$
- Reflection
 - Change the sign
 - Eg.: (5,2)
 - Reflected on the X axis becomes (5,-2)
 - Reflected on the Y axis becomes (-5,2)
 - Eg.: $y = x^2 + 2x + 3$
 - Y axis reflections: Invert the sign of the final result

- $y = -(x^2 + 2x + 3)$
- X axis reflections: Invert the sign of each X
- $y = (-x)^2 - 2(-x) + 3$

Kinematics:

- Displacement: Net distance moved from beginning to end. On a round trip displacement is zero
- Speed: Distance moved per unit of time
- Velocity:
- Acceleration: Rate of change of velocity. "Delta V"
- SUVAT Equations:
 - S: Position in time
 - U: Initial velocity
 - V: Final velocity
 - A: Acceleration
 - T: Time elapsed
 - $s = ut + \frac{1}{2}at^2$
 - $s = \frac{1}{2}(u+v)t$
 - $v^2 = u^2 + 2as$
 - $v = u + at$

Trigonometry

Angles:

- Acute: < 90 , > 0
- Right angle: $= 90$
- Obtuse: > 90 , < 180
- Reflex: > 180 , < 360

Triangles:

- Types:
 - Equilateral: Three equal sides
 - Isosceles: Two equal sides
 - Scalene: No equal sides
 - Right angle: One angle $= 90$ deg
- Rules:
 - All angles add up to 180 deg
 - Triangular inequality: The length of one side is less than the length of the other two.
- Notation:
 - ABC: Angles
 - abc: Sides opposite those angles
 - Eg.: If $C = 90$ deg then c = hypotenuse

Radians:

- 1 rad = the angle corresponding to a chord with length of r
- 360 degrees $= 2\pi$ rad
- $360/(\text{Angle in degrees}) = 2\pi \text{ rad}/(\text{Angle in radians})$
- Conversions:
 - Angle in Radians $= (\text{Degrees} * 2\pi)/360$
 - Angle in Radians $= \text{Degrees} * (\pi/180)$
 - Angle in Degrees $= (\text{Radians} * 360)/2\pi$
 - Angle in Degrees $= \text{Rads} * (180/\pi)$
- Rule of thumb: Calculate rad angles to 3 decimal places where you'd calculate degrees to 1 decimal place.

Radicals/Surds:

- A surd is an irrational square root
 - Eg $\sqrt{2}$
- A root $*$ a root = the contents multiplies
 - Eg.: $\sqrt{2} * \sqrt{3} = \sqrt{6}$
- This can be used to simplify radicals of the same base
 - Eg.: $\sqrt{12} = \sqrt{4*3} = \sqrt{4}*\sqrt{3} = 2*\sqrt{3}$
- $X\sqrt{Y} * A\sqrt{B} = XA\sqrt{YB}$
- When factorising try for prime numbers
- A root / a root = the contents divided
 - Eg.: $\sqrt{30}/\sqrt{10} = \sqrt{30/10} = \sqrt{3}$

Pythagoras' Theorem:

- $a^2 + b^2 = c^2$

- Where abc is a right angled triangle
- And c is the hypotenuse
- Can be used to find a missing side
- Also to test if a triangle is right angled

Trigonometric Ratios:

- Unit triangle
 - Right angled triangle
 - Hypotenuse = 1
- To find the length of sides:
 - Sine = Opposite/Hypotenuse
 - Cosine = Adjacent/Hypotenuse
 - Tangent = Opposite/Adjacent
- To find the value of angles:
 - Arc-sine = Theta, given a certain O/H
 - Arc-cosine = Theta, given a certain A/H
 - Arc-tan = Theta, given a certain O/A
 - Also may be noted as: \sin^{-1} \cos^{-1} and \tan^{-1}

Trigonometric Rules:

- Sine rule:
 - $a/\sin(A) = b/\sin(B) = c/\sin(C)$
 - We can use this to solve triangles if:
 - We have two lengths and an angle opposite one of them
 - We have two angles and a length opposite one of them
- Cosine rule:
 - Length forms:
 - $a^2 = b^2 + c^2 - 2bc \cos(A)$
 - $b^2 = a^2 + c^2 - 2ac \cos(B)$
 - $c^2 = a^2 + b^2 - 2ab \cos(C)$
 - Angle forms:
 - $\cos(A) = (b^2 + c^2 - a^2)/2bc$
 - $\cos(B) = (a^2 + c^2 - b^2)/2ac$
 - $\cos(C) = (a^2 + b^2 - c^2)/2ab$
 - We can use this to solve triangles if:
 - We have two lengths and one of the opposite angles
 - We have two lengths and the angle between them
 - We have all three lengths
- Other rules:
 - $\sin^2(A) + \cos^2(A) = 1$
 - $\cos(A) = \sin(A + 90^\circ)$
 - $\sin(A)/\cos(A) = \tan(A)$

Trigonometric Functions

Angles, Quadrants and Coordinates:

- Quadrants run anticlockwise
 - 1st quadrant: Between 3:00 and 12:00
 - 2nd quadrant: Between 12:00 and 9:00
 - 3rd quadrant: Between 9:00 and 6:00
 - 4th quadrant: Between 6:00 and 3:00
- Measure angles going anticlockwise starting with the positive subaxis of X as the starting point.
 - Eg. (0,5) would have an angle of 90deg ($1/2\pi$ rad)

Unit circle:

- Circle with origin (0,0) and radius 1
- Used to define trigonometric ratios
- Trigonometric ratios only cover angles between 0 and 90 degrees. In order to use larger angles:
 - 90-180: $\text{Theta} = 180 - \text{angle}$
 - 180-270: $\text{Theta} = \text{angle} - 180$
 - 270-360: $\text{Theta} = 360 - \text{angle}$
- The sign of cos and sin will also need to be adjusted to account for projections into -x and -y space.
 - Quadrant 1: $+x(\cos) +y(\sin) +y/x(\tan)$
 - Quadrant 2: $-x(\cos) +y(\sin) -y/x(\tan)$
 - Quadrant 3: $-x(\cos) -y(\sin) +y/x(\tan)$
 - Quadrant 4: $+x(\cos) -y(\sin) -y/x(\tan)$
- P is a point in the first quadrant
 - Theta is the angle between the positive X semi-axis and the radius pointing to P
 - The compliment of P (with angle = $180-\text{theta}$) in the second quadrant may be noted as P^I
 - Similarly the compliment of P in the third and fourth quadrant may be noted as P^{II} and P^{III} respectively
- Converting from Sine to Cosine
 - To convert $\text{Sin}(x)$ to $\text{Cos}(x)$, translate left by $\pi/2$
 - So $\text{Cos}(x) = \text{Sin}(x + \pi/2)$
 - Equally, to convert $\text{Cos}(x)$ to $\text{Sin}(x)$, translate right by $\pi/2$
 - So $\text{Sin}(x) = \text{Cos}(x - \pi/2)$

Common Angles:

- 0 deg:
 - Sin: 0
 - Cos: 1
 - Tan: 0
- 30 deg:
 - Sin: 0.5
 - Cos: $\sqrt{3}/2$
 - Tan: $1/\sqrt{3}$
- 45 deg:

- Sin: $1/\sqrt{2}$
- Cos: $1/\sqrt{2}$
- Tan: 1
- 60 deg:
 - Sin: $\sqrt{3}/2$
 - Cos: 0.5 $\pi/3$
 - Tan: $\sqrt{3}$
- 90 deg:
 - Sin: 1
 - Cos: 0
 - Tan: NaN
- Convert angle to first quadrant
 - Eg: $320 = 40$
- $X = \cos(A) * R$
- $Y = \sin(A) * R$
 - Adjust sign appropriately
- 180 deg:
 - Sin: 0
 - Cos: -1
 - Tan: 0
- 270 deg:
 - Sin: -1
 - Cos: 0
 - Tan: NaN

Graphs of trigonometric functions:

- Sine is (unsurprisingly) a sine wave with a period of 2π and an origin at $(0,0)$
- Cosine is an offset sine wave with the same period and an origin at $-\pi/2$
- Tan is an s shaped curve which ranges from $-\infty$ to ∞ with an asymptote whenever sine = 0. It's origin is $(0,0)$ and it's period is π .
- Solving graphically:
 - If $y = \sin(x)$
 - And $\sin(x) = 1/2$
 - This means we are solving the function for $y=1/2$
 - Depending on the domain, this will produce a number of points.

Inverses of trigonometric functions:

- Inverses may be graphed as the reflection of a section of their complement across the diagonal line $y=x$.
- When they are inverted, the domain and the range also invert
- ArcSin:
 - Domain: $[-1,1]$
 - Range: $[-\pi/2, \pi/2]$
- ArcCos:
 - Domain: $[-1,1]$
 - Range: $[0, \pi]$
- ArcTan:
 - Domain: \mathbb{R}

- Range: $(-1,1)$

Translations of trigonometric functions:

- The same as regular functions...
- Translation:
 - Vertical: add/subtract value after all other operations
 - Horizontal: add/subtract value to Xs before all other operations. Sign is inverse, so negative goes right, positive goes left.
- Scaling:
 - Vertical: Multiply by scalar after all other operations
 - Horizontal: Multiply all Xs by inverse of scalar before all other operations. Eg. If the scalar is 2, multiply by $1/2$.
- Reflection:
 - Vertical: Change the sign of the solution after all other operations
 - Horizontal: Change the sign of all Xs before any other operation

Solving Trigonometric Equations:

- Be aware that equations may have more than one valid solution.
- Such equations are often restricted to a specific range, eg. $-180 > x > 180$
- x must be solved for every potential value within this range:
 - Eg: $\sin(x - 30) = \sin(80)$
 - $x = 110$
 - Is also equivalent to:
 - $\sin(x - 30) = \sin(100)$
 - $x = 130$
- If the ratio is negative, restate it with a positive angle in the negative quadrant.
 - Eg. $-\tan(30) = \tan(150)$
- When finding equivalents of $\tan(x)$, \tan has a period of π or 180. So simply find the first value and add/subtract 180 from it.
- When considering possible values, search for the equivalent angles that share the same sign.
 - For example, $\sin(100)$ is the equivalent of $\sin(80)$ as 80 is in the first quadrant where the sign of sines are positive, and 100 is in the second quadrant where they are also positive.
 - For negative values we take all of the solutions and append 360k
 - $x = 110 + 360k$
 - $x = 130 + 360k$
 - Next we substitute values for k and see if they are congruent with our range.
 - In this case $k=1$ will produce values that exceed our range. $k = 0$ will work, $k = -1$ will work, $k=-2$ will exceed our range.
 - This gives us:

- $k = 0$
- $x = 110$
- $x = 130$
- $k = -1$
- $x = 110 - 360$
- $x = 130 - 360$
- We then solve for x
- $x = 110$
- $x = 130$
- $x = -230$
- $x = -250$
- When noting results, use set notation:
 - $X \text{ element}\{-250, -230, 110, 130\}$
- Or longer hand notation:
 - $x=-250$ or $x=-230$ or $x=110$ or $x=130$

Polar coordinates:

- Cartesian coordinates: Your distance from the x and y axes (*x deflection, y deflection*)
- Polar coordinates: Your distance from the origin and the angle from the positive X semi-axis (*radius, angle*)
- Eg : Cartesian(1,1) = Polar(sqrt(2), 45)

Converting from Cartesian to Polar:

- Eg, (2,1) to (r,a)
- The distance from the origin can be found using pythagoras' theorem
 - $r = \sqrt{2^2 + 1^2} = \sqrt{5} = 2.23$
 - $r = 2.23$
- The angle from the pos x semi-axis may be found via tan and cotan
 - $\tan(a) = 1/2$
 - $a = \text{Cotan}(1/2) = 26.56$
 - $a = 26.56$
- Cartesian (2,1) = Polar (2.23,26.56)
- Take into account the signs of the coordinates and convert to the relevant quadrant

Converting from Polar to Cartesian:

- Convert angle to first quadrant
 - Eg: $320 = 40$
- $X = \cos(A) * R$
- $Y = \sin(A) * R$
- Adjust sign appropriately

Exponential and Logarithmic Functions

Exponentiation:

- Whole integers: 4^3
 - Equivalent to $4 * 4 * 4$
 - Raising to a power of 0 = 1
 - Eg, $3^0 = 1$
 - Raising to a power of 1 = itself
 - Eg $6^1 = 6$
- Negative powers: 4^{-3}
 - Invert the base
 - So: $(1/4)^3$
 - Equivalent to $1/4 * 1/4 * 1/4$
- Fractional powers: $4^{2/3}$
 - Becomes: $\sqrt[3]{4^2}$
 - Or: $(\sqrt[3]{4})^2$
 - Denominator of the fraction is the root
 - Numerator of the fraction becomes the new power
 - Base remains the base
- Irrational powers $4^{\sqrt{3}}$
 - $4^{1.73}$
 - $100\sqrt[2]{2^{173}}$
 - Approximate to n decimal places

Exponential Arithmetic:

- When multiplying two powers with the same base:
 - Add the powers
 - Eg: $2^3 * 2^4 = 2^7$
- When dividing two powers with the same base:
 - Subtract the powers
 - Eg: $2^4 / 2^2 = 2^2$
- When multiplying two powers with the same exponent:
 - Multiply the bases
 - Eg: $2^2 * 5^2 = 10^2$
- When dividing two powers with the same exponent:
 - Divide the bases
 - Eg: $4^5 / 2^5 = 2^5$
- To find the inverse of an exponential function:
 - Invert the power and apply it to the result
 - Eg: $4^2 = 8$
 - So: $8^{1/2} = 4$
- Change the base of a power
 - Eg: $8^2 = 2^x$
 - Rephrase 8 as a power of 2
 - So: $8^2 = (2^3)^2 = 2^6$

Euler's Number:

- E = the limit of $(1+x/n)^n$

- Originally used when analysing compound interest
- Called the “natural base” as it is the exponential function equal to it’s own derivative
- I think this means that the limit of it’s own exponential function is itself

Solving exponential equations:

- Use exponential arithmetic techniques to find a common base
- Then use regular arithmetic with the exponents

Transforming exponential graphs:

- X axis reflection: Change sign of entire equation
 - Eg: $y = e^x$
 - Becomes: $y = -(e^x)$
- Y axis reflection: Change sign of exponent
 - Eg: $y = e^x$
 - Becomes: $y = e^{-x}$
- Horizontal translation: Add/subtract from x
 - Move left: $y = e^{x+1}$
 - Move right: $y = e^{x-1}$
- Vertical translation: Add/subtract from result of equation
 - Move up: $y = (e^x) + 1$
 - Move down: $y = (e^x) - 1$
- Horizontal scaling: Multiply/divide x by inverse of scalar
 - $y = e^{2x}$
- Vertical scaling: Multiply/divide result of equation by inverse of scalar
 - $y = 2(e^x)$

Logarithms:

- Inverse of exponentiation
 - Where $X = b^Y$
 - $Y = \text{Log}_b(X)$
- Logarithms in base e are “natural logarithms”
 - Eg. $e^x = 24$
 - Or: $\text{Log}_e(24) = x$
 - Or: $x = \text{Ln}24$
 - Ln means “natural logarithm”
- On a calculator:
 - $\text{Ln} = \text{Log}_e$
 - $\text{Log} = \text{log}_{10}$
- Cannot find the log of a negative number: These are imaginary
- Also cannot find the log of 0

Logarithmic Arithmetic:

- Adding logs of same base:
 - Multiply the logs
 - $\text{Log}_2(4) + \text{Log}_2(3) = \text{Log}_2(4 * 3) = \text{Log}_2(12)$
- Subtracting logs of same base:
 - Divide the logs
 - $\text{Log}_2(9) - \text{Log}_2(3) = \text{Log}_2(9/3) = \text{Log}_2(3)$
- Logarithm of a power:

- Exponent multiplied by Log of base
- $\text{Log}_2(9^3) = 3 * \text{Log}_2(9)$
- For inverse bases with the same log
 - Change the sign of the result
 - $\text{Log}_2 16 = 4$
 - $\text{Log}_{1/2} 16 = -4$
 - This is just another way of saying that negative powers produce fractions
- Changing base:
 - Changing from base b to d
 - $\log_b(x) = \log_d(x)/\log_d(b)$
 - So, $\log_3(5) = \log_{10}(5)/\log_{10}(3) = 1.46$

Logarithmic functions:

- Inverse of exponential functions
 - Reflection of an exponential curve across the 45 degree line $y = x$
 - Take the table of values for exponential functions, and flip the axes
 - So if a point is (1, 2) it will now be (2, 1)
- When the base is greater than zero
 - The result is a relatively flat curve, it's x value grows extremely quickly as y increases
 - It has a vertical asymptote of 0 (if not translated)
 - When $x = 1$, y is 0
 - When $x < 1$, y is negative
 - When $x > 1$, y is positive
- When the base is between one and zero:
 - The graph is reflected on the x axis
 - This is presumably the inverse graph of the whole version of that number
- For all bases:
 - Where $X = 1$, $Y = 0$
 - Where $X = (\text{the base})$, $Y = 1$

Logarithmic Equations:

- Graphical solution:
 - Literally just plot it on a graph and find the result
- Calculator:
 - In some cases it may be possible to simply solve the equation directly on your calculator
 - Most calculators only have ln and log(base 10) functions so we may need to change the base
- Algebraically:
 - Refer to arithmetic section
 - Not always necessary to provide a numeric answer. Rephrasing it as a power can be acceptable
 - Remember that we can't find the log of negative numbers

Transformations of Logarithmic Graphs:

- X axis reflection: Change sign of entire equation
 - Eg: $y = \text{Log}(x)$

- Becomes: $y = -\text{Log}(x)$
- Y axis reflection: Change sign of log
 - Eg: $\text{Log}(x)$
 - Becomes: $\text{Log}(x)$
- Horizontal translation: Add/subtract from x
 - Move left: $y = e^{x+1}$
 - Move right: $y = e^{x-1}$
- Vertical translation: Add/subtract from result of equation
 - Move up: $y = (e^x) + 1$
 - Move down: $y = (e^x) - 1$
- Horizontal scaling: Multiply/divide x by inverse of scalar
 - $y = e^{2x}$
- Vertical scaling: Multiply/divide result of equation by inverse of scalar
 - $y = 2(e^x)$

Inverting a Function:

- Eg: $y = 5e^{x-1}$
- First step, swap the variables
 - $x = 5e^{y-1}$
- Solve for y
 - $x/5 = e^{y-1}$
 - $\text{Ln}(x/5) = y - 1$
 - $y = \text{Ln}(x/5) + 1$

Calculus

Blah

Vectors

Vector basics:

- Vectors are a direction and magnitude
- As opposed to scalars which are just magnitude
- They have a tail (origin) and head (destination)
- Notation:
 - Vector = $\vec{v}(\frac{1}{2})$
 - Where 1 is the x displacement and 2 is the y displacement
 - Magnitude = $|\vec{v}| = \sqrt{1^2+2^2} = \sqrt{5}$
- $V(\text{displacement}) = C(\text{head point}) - A(\text{tail point})$

Arithmetic:

- In order to check if vectors are parallel, find out if one is a multiple of the other
- The complement of a vector:
 - The same magnitude in the opposite direction
 - Switch the sign on each coordinate
 - $\vec{v} = (1,2)$
 - $-\vec{v} = (-1,-2)$
- A point plus a vector is a point
 - The tail is point A, the vector extends from it, the head is point B
- A vector plus a vector is a vector
 - The head of Vector A is the tail of Vector B. Vector C extends from tail A to head B
- A vector minus a vector is a vector
 - Same as adding, but add the complement
- A point (head) minus a point (tail) is a vector
 - This is essentially reversing a point plus a vector. We start at the head and work backwards to find the tail. This reverse operation defines the vector.
- A vector multiplied by a value (scalar) is a vector
 - Just multiply each coordinate by the value
- A vector divided by a value (scalar) is a vector
 - Just divide each coordinate by the value
- Dot Product:
 - Notation: $\vec{v} \cdot \vec{w}$
 - Operation: $(v(x)*w(x)) + (v(y)*w(y))$
 - Where θ is the angle between \vec{v} and \vec{w}
 - The dot product is $\cos\theta * |\vec{v}| * |\vec{w}|$
 - This can be used to find:
 - \vec{v} projected on to $\vec{w} = \vec{v} \cdot \vec{w} / |\vec{w}|$
 - \vec{w} projected on to $\vec{v} = \vec{v} \cdot \vec{w} / |\vec{v}|$
 - $\theta = \arccos((\vec{v} \cdot \vec{w})/(|\vec{w}|*|\vec{v}|))$
- A unit vector has magnitude of 1
 - Notation:
 - Vector: \vec{v}
 - Unit Vector: \vec{w}

- $\vec{w} = \hat{v}$
- Circumflex indicated vector converted to unit vector
- To convert from vector \vec{v} to unit vector \vec{w} :
 - Find $|\vec{v}|$ (magnitude of \vec{v})
 - Divide \vec{v} by $|\vec{v}|$
 - Ie. Multiply \vec{v} by $1/|\vec{v}|$
 - This gives \vec{w} or \hat{v}

Matrices

Blah

Combinatorics and Probability

Definitions:

- Combination: Set of objects from a superset. Order is unimportant
- Permutation: Set of objects from a superset. Order is important

Permutations:

- Order matters
- Number of permutations may be calculated with factorials
 - Eg: A,B,C,D
 - Arranged as all possible permutations of four letters
 - $4! = 4*3*2*1 = 24$
- If the permutation set is smaller than the superset, then divide by the difference in number of places
 - Eg: A,B,C,D,E,F
 - Superset length is 6
 - Arranged as all possible permutations of 4 letters
 - Leaves a remainder of 2
 - $(4!)/(2!) = (4*3*2*1)/(2*1) = 24/2 = 12$
- If some of the elements of the superset are indistinguishable
 - We may wish to restrict the subsets to "distinguishable permutations"
 - This means that permutations containing indistinguishable elements in different orders will not be counted
 - Formula is:
 - $(n!)/(n_1!n_2!\dots n_k!)$
 - Where $n!$ Is the total number of permutations
 - n_x is the total number of that indistinguishable element
 - Eg: M,I,S,S,I,S,S,I,P,P,I
 - $n_1 (M) = 1$
 - $n_2 (I) = 4$
 - $n_3 (S) = 4$
 - $n_4 (P) = 2$
 - $k = 4$
 - $n = 11$
 - $n!/n_1!*n_2!*n_3!*n_4!$
 - $(11!)/(1!)(4!)(4!)(2!)$
 - 34650

Combinations:

- Order unimportant
 - Eg: C,O,M,P,U,T,E,R
 - C,O,M = M,C,O
- Find the number of permutations, then divide by the number of permutations of each subset
 - Eg: C,O,M,P,U,T,E,R
 - Find the number of possible combinations of 3 letters
 - First find the number of permutations:

- 8 letters
- $8!/5! = 336$
- Next divide by the number of permutations of each subset
 - $336/3! = 336/6 = 56$
- Also can use the notation nC_r
 - N = number of elements in the superset
 - R = number of elements in each subset
 - ${}^nC_r = n!/((n-r)!*r!)$

Probability:

- Notation: Probability of x occurring = $P(x)$
- Sample space: The set of all possible outcomes
- Event: The set of outcome(s) whose probability is being assessed
- Probability = # of outcomes in the event / # of outcomes in sample space
 - Eg, flipping a coin:
 - Sample space: TT, TH, HH, HT
 - Event: HH
 - Probability = $1/4$
- Maximum probability under normal conditions is 1. The only way to get a higher probability is to have an event space larger than the sample space.

Probability of event A OR event B:

- If there is no overlap in the event space (events are mutually exclusive), add them
 - $P(A \cup B) = P(A) + P(B)$
- If there is an overlap (not mutually exclusive), add them, and subtract the overlap
 - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- Eg: Probability of rolling at least one "2" when rolling two dice
 - Events 1: The first dice has a 2 {21,22,23,24,25,26}
 - Events 2: The second dice has a 2 {12,22,32,42,52,62}
 - Overlap: Both dice have a 2 {22}
 - Probability 1 = $6/36 = 1/6$
 - Probability 2 = $6/36 = 1/6$
 - Overlap = $1/36$
 - Total probability = $6/36 + 6/36 - 1/36 = 11/36$

Probability of event A AND event B

- If they are independent (the outcome of one does not affect the outcome of the other) then multiply their probabilities
 - $P(A \cap B) = P(A) * P(B)$
 - Eg, rolling a dice twice. What is the chance of getting a 2 and then a 3.
 - Probability of rolling a 2 = $1/6$
 - Probability of rolling a 3 = $1/6$
 - Probability of rolling 2 followed by 3 = $1/36$

Complement of an event:

- Sample space – event space = complement set
- Notation: Complement A = A' or A^c or \bar{A}
- Sara prefers: A'
- $P(A') = 1 - P(A)$

Conditional probability: Probability of B given that A has happened

- Our sample space is the event space of A
- $P(A|B) = P(A \cap B) / P(B)$

Statistics

Arithmetic mean:

- Tries to find an average number to represent the dataset
- Notation: Arithmetic mean of $x = \bar{x}$ (sample mean) or μ (actual mean)
- Formula: (Sum of data points)/(number of data points)

Variance:

- Tries to give you an idea of how close or far the dataset is to the arithmetic mean
- Long method:
 - Find the difference between each datapoint and the mean
 - Square the results
 - Find the mean of the results
- Notation: σ^2 (actual variance) s^2 (sample variance)
- Actual Formula: (Sum of datapoints - μ)²/(number of data points)
- Sample Formula: (Sum of datapoints - μ)²/(number of data points - 1)

Standard Deviation:

- Square root of variance
- More accurate measure of deviance from the mean
- Notation: σ (actual SD) s (sample SD)

Normal Distribution:

- Approximation of the natural curve most natural distributions of properties follow
- Inputs: μ , σ^2
- Formula: $1/(\sqrt{\sigma^2 2\pi}) * e^{-(x-\mu)^2/2\sigma^2}$

Bayes Theorem:

- Given that: $P(A|B) = P(A \cap B) / P(B)$
 - $P(A \cap B) = P(A|B) * P(B)$
 - $P(B|A) = (P(A|B) * P(B)) / P(A)$
- Terminology:
 - Posterior = $P(B|A)$
 - Prior = $P(A|B)$
 - Likelihood = $P(B)$
 - Evidence = $P(A)$