Fundamentals Of Computer Science Course Notes

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Week 1

Learning Objectives:

- Understand logical arguments and apply basic concepts of formal proof.

1.101 Introduction to propositional logic

Propositional Logic is a system that deals with propositions or statements.

Below there's an example of where we can apply propositional logic to derive conclusions.

Liars And Knights

Imagine there is an island with two types of people. Liars who always tell lies and knights who always tell the truth. One an excursion, you visit the island and encounter two people, person A and person B. Person A says "at least one of us is a liar", while person B says nothing. What conclusion can you draw?

With a little logical thinking, we can conclude that person A is a knight and person B is a liar.

1.103 Building blocks of logic

What is a proposition?

A **proposition** is a statement that can be either **true** or **false**. It must be one or the other, never both nor neither.

Examples of proposition:

- 2 is a prime number (T)
- 5 is an even number (F)

Not a proposition:

- x is a prime number

  In this case, it can be made into a proposition by assigning a value to x.
- Are you going to school?

  Because this is a question, we can't assign a truth value to the sentence.
- Do your homework now

  Being an order, it has no truth value.

Syntaxes of the propositional logic

Propositions are denoted by capitable letters: P, Q, R, and so on.

- P = carrots are orange
- Q = I went to a party yesterday

General statements are denoted by lowercase letters: p, q, r, and so on. They carry on a logical argument, are used in proofs, called propositional variables.

Connectives: change or combine propositions

Connectives transform **atomic** propositions into **compound** propositions.

Logical NOT ($\neg$) $\neg p$ is true if and only if p is false. Also called **negation**.

Logical OR ($\vee$) $p \vee q$ is true if and only if at least one of p or q is true or if both p and q are true. Also called **disjunction**.

Logical AND ($\wedge$) $p \wedge q$ is true if and only if both p and q are true and false otherwise. Also called **conjunction**.

Logical if then ($\implies$) $p \rightarrow q$ is true if and only if either p is false or q is true. Also called **implication** or **conditional**. p is called the **premise** and q is the **conclusion**.

Logical if and only if ($\iff$) $p \iff q$ is true if and only if both p and q are true or both are false. Also called **bi-conditional**.

Exclusive or ($\oplus$) $p \oplus q$ is true if and only if p or q is true but not both.

Translation from Logical Proposition to English

Let:

$$\begin{aligned} P &= \text{I study 20 hours a week} \\ Q &= \text{I attend all the lectures} \\ R &= \text{I will pass the exam} \\ S &= \text{I will be happy} \end{aligned} \tag{1}$$

Translate the following statement to English:

- $(P \lor Q) \implies (R \land S)$

  If **I study 20 hours a week** or **I attend all the lectures** then **I will pass the exam** and **I will be happy**.

Translation from English to Logical Proposition

Given the statement:

If UK does not exit the EU then skilled nurses will not leave the NHS and research grants will remain intact.

Translating to logical proposition we get:

$$\begin{aligned} P &= \text{UK exits the EU} \\ Q &= \text{Skilled nurses will leave the NHS} \\ R &= \text{Research grants will remain intact} \\ \neg P &\implies \neg Q \land R \end{aligned} \tag{2}$$

Note that we removed any **connectives** from our propositions as that's a good practice. This makes the logical statement easier to follow.

1.105 Truth Table: examples

A truth table is a set of all outcomes of propositions and connectives. The number of rows in a truth table, depends on the number of given propositions. If we have n propositions, our truth table will have 2^n rows.

Truth tables for each connective

What follows is a list of truth tables for each connective

Negation ($\neg$)

p	$\neg p$
1	0
0	1

Conjunction ($\wedge$)

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

Disjunction ($\vee$)

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

Implication ($\implies$)

p	q	$p \implies q$
1	1	1
1	0	0
0	1	1
0	0	1

Bi-conditional ($\iff$)

p	q	$p \iff q$
1	1	1
1	0	0
0	1	0
0	0	1

Exclusive Or ($\oplus$)

p	q	$p \oplus q$
1	1	0
1	0	1
0	1	1
0	0	0

Operator Precedence

When formulae are written without parenthesis, we must rely on rules of operator precedence. Logic operator precende rules are as follows:

$\neg \wedge \vee \implies \iff$

Example If we have the logical statement $p \implies p \wedge \neg q \vee s$, we can parse it following the steps below:

$$\begin{aligned} p &\implies p \wedge \neg q \vee s \\ p &\implies p \wedge (\neg q) \vee s \\ p &\implies (p \wedge (\neg q)) \vee s \\ p &\implies ((p \wedge (\neg q)) \vee s) \\ (p &\implies ((p \wedge (\neg q)) \vee s)) \end{aligned} \tag{3}$$

Constructing Truth Tables for Complex Formulae

Example 1

p	q	$p \wedge q$	$(p \wedge q) \implies p$
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	1

Example 2

p	q	$q \implies p$	$p \wedge (q \implies p)$
1	1	1	1
1	0	1	1
0	1	0	0
0	0	1	0

Comparing both examples

p	q	$(p \wedge q) \implies p$	$p \wedge (q \implies p)$
1	1	1	1
1	0	1	1
0	1	1	0
0	0	1	0

1.202 Tautology and consistency

Tautology

A formula that is **always** true regardless of the truth value of the proposition.

p	$\neg p$	$p \vee \neg p$
0	1	1
1	0	1

Consistent

A formula that is true **at least** for one scenario. All connectives are consistent.

The formula $p \wedge \neg p$ is **inconsistent** because it can never be true. Inconsistent formulae are also called **contradictions**.

1.204 Tautology and consistency: examples

Example 1: $p \vee (q \wedge \neg r)$

p	q	r	$\neg r$	$q \wedge \neg r$	$p \vee (q \wedge \neg r)$
0	0	0	1	0	0
0	0	1	0	0	0
0	1	0	1	1	1
0	1	1	0	0	0
1	0	0	1	0	1
1	0	1	0	0	1
1	1	0	1	1	1
1	1	1	0	0	1

This is a **Consistent** formula

Example 2: $(p \implies q) \implies (\neg q \vee r)$

p	q	r	$p \implies q$	$\neg q$	$(\neg \text{ q } \vee \text{ \$})$	$(p \implies q) \implies (\neg q \vee r)$
0	0	0	1	1	1	1
0	0	1	1	1	1	1
0	1	0	1	0	0	0
0	1	1	1	0	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	0	0	0
1	1	1	1	0	1	1

This is a **Consistent** formula

Example 3: $(p \implies q) \iff (\neg p \vee q)$

p	q	$p \implies q$	$\neg p$	$(\neg p \vee q)$	$(p \implies q) \iff (\neg p \vee q)$
0	0	1	1	1	1
0	1	1	1	1	1
1	0	0	0	0	1
1	1	1	0	1	1

This formula is a **Tautology**

Week 2

Learning Objectives:

- Understand logical arguments and apply basic concepts of formal proof.

1.301 Equivalences

Formulae A and B are equivalent if they have identical truth tables. Equivalence is denoted by the symbol $\equiv$

In other words, $A \equiv B$ means that A and B have the same truth values, regardless of how variables are assigned.

One thing to note is that $\equiv$ is **NOT** a connective.

De Morgan's Laws

$$\begin{aligned} \neg(p \wedge q) &\equiv \neg p \vee \neg q \\ \neg(p \vee q) &\equiv \neg p \wedge \neg q \end{aligned} \tag{4}$$

Truth Tables

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	0	1	1
1	1	0	0	1	0	0

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
0	0	1	1	0	1	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	1	0	0

$(p \implies q) \equiv (\neg p \vee q) \equiv \neg(p \wedge \neg q)$

p	q	$\neg p$	$\neg q$	$p \implies q$	$\neg p \vee q$	$\neg(p \wedge \neg q)$
0	0	1	1	1	1	1
0	1	1	0	1	1	1
1	0	0	1	0	0	0
1	1	0	0	1	1	1

Contrapositive: $(p \implies q) \equiv (\neg q \implies \neg p)$

p	q	$\neg p$	$\neg q$	$p \implies q$	$\neg q \implies \neg p$
0	0	1	1	1	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	0	0	1	1

1.304 First-order logic

Important Notions

- **Predicates** describe properties of objects

  A simple example could be `odd(3)`. Here we're applying the predicate `odd` to the object `3`. When arguments are applied to predicates, they become propositions and connectives for propositional logic can be employed in the usual manner:

$$Odd(3) \wedge Prime(3) = T \tag{5}$$

- **Quantifiers** allow reasoning on multiple objects

  The objects from a quantified statement are chosen from a *Domain*.

  - **Existential Quantifier** $\exists$

    We use it as follows: $\exists x$ some formula.

    When proving a formula based on the existential quantifier, it is enough to find **one** element which makes the formula true. In other words, existentially quantified statements are **false** unless there is a positive example.

  - **Universal Quantifier** $\forall$

    We use it as follows: $\forall x$ some formula.

    In order to satisfy the formula, we must prove that **every** x satisfies the formula.

    Note that a single counterexample is enough to disprove a universally quantified statement. In other words, universally quantified statements are **true** unless there is a false example.

Translations English - Logic

"All P's are Q's" translates into $\forall x(P(x) \implies Q(x))$

"No P's are Q's" translates into $\forall x(P(x) \implies \neg Q(x))$

"Some P's are Q's" translates into $\exists x(P(x) \wedge Q(x))$

"Some P's are not Q's" translates into $\exists x(P(x) \wedge \neg Q(x))$

Quantifiers to connectives

- Existential Quantifier

$\exists x P(x)$ and domain $D = \{x_1, x_2, \ldots, x_n\}$. This is equivalent to saying $P(x_1) \lor P(x_2) \lor \ldots \lor P(x_n)$

- Universal Quantifier

$\forall x P(x)$ and domain $D = \{x_1, x_2, \ldots, x_n\}$. This is equivalent to saying $P(x_1) \land P(x_2) \land \ldots \land P(x_n)$

Negation of Quantifiers

- Existential Quantifier

$$\begin{aligned} \neg \exists x P(x) &\equiv \neg(P(x_1) \lor P(x_2) \lor \ldots \lor P(x_n)) \\ &\equiv \neg P(x_1) \land \neg P(x_2) \land \ldots \land \neg P(x_n)) \\ &\equiv \forall x \neg P(x) \end{aligned} \tag{6}$$

- Universal Quantifier

$$\begin{aligned} \neg \forall x P(x) &\equiv \neg(P(x_1) \land P(x_2) \land \ldots \land P(x_n)) \\ &\equiv \neg P(x_1) \lor \neg P(x_2) \lor \ldots \lor \neg P(x_n)) \\ &\equiv \exists x \neg P(x) \end{aligned} \tag{7}$$

Example

$$\begin{aligned} \neg(\forall x(P(x) \implies Q(x)) &\equiv \exists x \neg(P(x) \implies Q(x)) \\ &\equiv \exists x \neg(\neg P(x) \lor Q(x)) \\ &\equiv \exists x(\neg\neg P(x) \land \neg Q(x)) \\ &\equiv \exists x(P(x) \land \neg Q(x)) \end{aligned} \tag{8}$$

Week 3

Learning Objectives:

- Correctly follow a sequence of justified steps to reach a conclusion statement.
- Prove a conclusion statement by first assuming it is false.
- Describe inductive steps.
- Understand logical arguments and apply basic concepts of formal proof.

2.01 What is a proof?

A proof is a sequence of logical statements that explains why a statement is true. Rosen's book defines a proof as follows:

> A proof is a valid argument that establishes the truth of a mathematical statement.

We need proofs to establish general truths about conjectures. For example a computer cannot confirm that **all** numbers have a certain property. Well, considering that numbers are infinite and computers work with finite amounts of memory, a computer will not be able to answer that question.

Given a theorem, often there are many ways in which we can prove it. There are commonly used proof techniques which we learn about.

2.101 Direct proof

A direct proof exploits definitions and other mathematical theorems. It arrives at the desired statement by employing valid logical steps.

A direct proof is:

- Easy because there is no particular technique is used
- Not easy because the starting point is not obvious
- Know your definitions
- Allowed to use any theorem, axiom, logic, etc

Example 1:

Theorem 1. *If n and m are even numbers, then $n+m$ is also even*

Proof. What does even mean?

If an integer is even, it is twice another integer.

$$
\begin{aligned}
n &= 2k \\
m &= 2l
\end{aligned}
$$

k and l are integers

$$
\begin{aligned}
n+m &= 2k+2l \\
&= 2(k+l)
\end{aligned}
$$

Let $k+l=t$,

$$n+m=2t$$

$\therefore n+m$ is even. □

Example 2:

Theorem 2. *$\forall n \in \mathbb{N}, n^2 + n$ is even.*

Proof. If n is even, then $n = 2k$.

$$n^2 + n = (2k)^2 + 2k \,\text{Even}$$

If n is odd, then $n = 2k + 1$.

$$\begin{aligned} n^2 + n &= (2k+1)^2 + 2k + 1 \\ &= (2k)^2 + 2 \cdot 2k + 1^2 + 2k + 1^2 \\ &= 4k^2 + 6k + 2 \,\text{Even} \end{aligned}$$

□

2.102 Direct proof examples

Example 1:

Theorem 3. *If $a < b < 0$, then $a^2 > b^2$*

Proof. Assume $a < b$ and $a < 0$. Multiplying both sides of the inequality by a gives:

$$\begin{aligned} a \cdot a &< b \cdot a \\ a^2 &> b \cdot a \end{aligned}$$

Assume $a < b$ and $b < 0$. Multiplying both sides of the inequality by b gives:

$$\begin{aligned} a \cdot b &< b \cdot b \\ a \cdot b &> b^2 \end{aligned}$$

By the commutative property of multiplication we know that $a \cdot b = b \cdot a$, therefore:

$$\begin{aligned} a^2 > a \cdot b &> b^2 \\ \therefore a^2 &> b^2 \end{aligned}$$

□

Example 2:

Theorem 4. $\forall x \in \mathbb{N}, 2x^3 + x$ *is a multiple of* 3.

Proof. Factorizing $2x^3 + x$ gives $x(2x^2 + 1)$.

If x is a multiple of 3, the proof is complete.

If $x = 3k + 1$, then:

$$\begin{aligned} x(2x^2 + 1) &= (3k + 1)[2(3k + 1)^2 + 1] \\ &= (3k + 1)[2(9k^2 + 6k + 1) + 1] \\ &= (3k + 1)(18k^2 + 12k + 3] \\ &= 3(3k + 1)((6k^2 + 4k + 1) \end{aligned}$$

If $x = 3k + 2$, then:

$$\begin{aligned} x(2x^2 + 1) &= (3k + 2)[2(3k + 2)^2 + 1] \\ &= (3k + 2)[2(9k^2 + 12k + 4) + 1] \\ &= (3k + 2)(18k^2 + 24k + 9) \\ &= 3(3k + 2)(6k^2 + 8k + 3) \end{aligned}$$

□

2.202 Proof by contradiction

Proof by contradiction is also referred to as *indirect proof.* It follows a simple structure to prove that statement **A** is *true.*

We start by assuming **A** to be *false*, we follow just like a direct proof by employing mathematical definitions, theorems, axioms and logical steps until we arrive at a statement which **contradicts** our original assumption. This would show our original assumption to be incorrect. Therefore, if our assumption is **not** false, then it can only be true.

Example 1:

Theorem 5. *The square-root of two,* $\sqrt(2)$ *is irrational*

Proof. Assume $\sqrt(2)$ is rational. This means it can be written as a fraction, in lowest terms, of the form $\frac{p}{q}$ for $p, q \in \mathbb{N}$, $q \neq 0$.

If $\frac{p}{q}$ is in lowest terms, it means the fraction cannot be further simplified. Therefore, we have:

$$\begin{aligned}
\sqrt(2) &= \frac{p}{q} \\
\left(\sqrt(2)\right)^2 &= \left(\frac{p}{q}\right)^2 \\
2 &= \frac{p^2}{q^2} \\
2q^2 &= p^2
\end{aligned}$$

From this, we can see that p must be even. Which means $p = 2k$. Therefore:

$$\begin{aligned}
2q^2 &= p^2 \\
2q^2 &= (2k)^2 \\
2q^2 &= 4k^2 \\
q^2 &= 2k^2
\end{aligned}$$

From this, we can see that q must also be even. Which means our fraction $\frac{p}{q}$ cannot be in lowest terms. Therefore, $\sqrt(2)$ cannot be a rational number, so it is irrational. □

Example 2:

Theorem 6. *There is an infinite number of prime numbers.*

Proof. Assume there are finitely many prime numbers. Let the set of prime numbers be $P = \{p_1, p_2, \ldots, p_n\}$. Let $N = (p_1 \cdot p_2 \cdot \ldots \cdot p_n) + 1$.

If we divide N by any of the prime numbers in our list of prime numbers, it will have a remainder of 1. Therefore, N is, itself, a prime number. □

2.203 Proof by contraposive

This technique exploits equivalent classes of logical statements. Let us remember that $a \implies b \equiv \neg b \implies \neg a$.

In some cases, when we need to prove $a \implies b$, it may be easier to prove its contrapositive $(\neg b \implies \neg a)$ is true.

Example 1:

Theorem 7. $\forall n \in \mathbb{N}, Odd(n^3 + 1) \implies Even(n)$

Proof. By means of the contrapositive $\forall n \in \mathbb{N}, Odd(n) \implies Even(n^3 + 1)$.

$$n = 2k + 1 \forall k \in \mathbb{N}$$

$$\begin{aligned} n^3 + 1 &= (2k+1)^3 + 1 \\ &= (2k+1)(2k+1)(2k+1) + 1 \\ &= 8k^3 + 12k^2 + 6k + 2 \\ &= 2(4k^3 + 6k^2 + 3k + 1) \end{aligned}$$

□

Example 2:

Theorem 8. *Suppose* $x, y \in \mathbb{R}$, $y^3 + yx^2 \leq x^3 + xy^2 \implies y \leq x$

Proof. By contrapositive $y > x \implies y^3 + yx^2 > x^3 + xy^2$.

Assuming $y > x$, we know that $y - x > 0$. Let us multiply both sides of the inequality by $x^2 + y^2$. Therefore:

$$\begin{aligned} (y^2 + x^2)(y - x) &> 0(y^2 + x^2) \\ y^3 + yx^2 - xy^2 - x^3 &> 0 \\ y^3 + yx^2 &> x^3 + xy^2 \end{aligned}$$

□

Week 4

Learning Objectives:

- Correctly follow a sequence of justified steps to reach a conclusion statement.
- Prove a conclusion statement by first assuming it is false.
- Describe inductive steps.
- Understand logical arguments and apply basic concepts of formal proof.

2.301 Proof by induction

Mathematical induction is a useful proof method that has several steps that must be followed. We can consider mathematical induction as a row of standing dominoes and we want to prove that all dominoes fall.

In order to prove that all dominoes fall, we must first and foremost prove that the first domino falls. After that we prove that if one domino falls, the next one must also fall.

Mathematically, if $P(0)$ is true and $\forall k \in \mathbb{N} P(k) \implies P(k+1)$, then we can conclude that $\forall n \in \mathbb{N} P(n)$ is true.

Proof by induction has three important steps:

- The *Base Case* or *Basis*

  Here we prove that $P(0)$ is true. This allows us to prove that the theorem **starts** true.

- The *Inductive Step*

  Here we prove that $P(k) \implies P(k+1)$. Note that we never assume this to be true. We must always carefully prove it. Because we're trying to prove an implication, we assume $P(k)$ to be true and prove $P(k+1)$ to be true when $P(k)$ is true. The reason for this is that if $P(k)$ is false, then the implication is true anyway. The assumption that $P(k) is true is called the *Inductive Hypothesis.*

- The *Conclusion by Induction*

  We finish the proof by writing $\therefore \forall n \in \mathbb{N} P(n)$ is true.

It's common practice to end a proof with the $\square$ symbol. Referred to as **QED** (from Latin *quod erat demonstrandum*).

2.303 Example of a correct proof

$\sum_{i=0}^{n-1} 2^i = 2^n - 1$

Theorem 9. *The sum of the first n powers of 2, is* $2^n - 1$.

Proof. Let $P(n) = 2^0 + 2^1 + \ldots + 2^{n-1} = 2^n - 1$, prove that $P(n)$ is valid for all n.

Basis: Prove that $P(1)$ is true.

$$\begin{aligned} P(1) &= 2^{1-1} \\ &= 2^0 \\ &= 1 \\ &= 2^1 - 1 \end{aligned}$$

Indutive Step: Prove that $P(k) \implies P(k+1)$ is true.

Assuming $P(k) = 2^0 + 2^1 + \ldots + 2^{k-1} = 2^k - 1$ to be true, prove that $P(k+1) = 2^0 + 2^1 + \ldots + 2^{k-1} + 2^k = 2^{k+1} - 1$ is also true.

$$P(k+1) = 2^0 + 2^1 + \ldots + 2^{k-1} + 2^k$$

By Inductive Hypothesis we know that $2^0 + 2^1 + \ldots + 2^{k-1} = 2^k - 1$, therefore:

$$
\begin{aligned}
P(k+1) &= 2^k - 1 + 2^k \\
&= 2^k + 2^k - 1 \\
&= 2^{k+1} - 1
\end{aligned}
$$

Conclusion: $P(k+1)$ is true, $\therefore \forall n \in \mathbb{N} 2^n - 1$ is true. $\square$

$\forall n n < 3^n$

Theorem 10. *$n < 3^n$, for all $n \in \mathbb{N}$*

Proof. Let $P(n) = n < 3^n$, prove by induction that $P(n)$ is true for all n.

Basis: Prove $P(0)$ is true.

$$
\begin{aligned}
P(0) &= 0 < 3^0 \\
&= 0 < 1
\end{aligned}
$$

Inductive Step: Prove $P(k) \implies P(k+1)$.

Assuming $P(k) = k < 3^k$ to be true, prove that $P(k+1) = (k+1) < 3^{k+1}$ is true.

$$
\begin{aligned}
k &< 3^k \\
k+1 &< 3^k + 1 \\
k+1 &< 3^k + 1 < 3^k + 3^k + 3^k \\
k+1 &< 3^k + 1 < 3 \cdot 3^k \\
k+1 &< 3^k + 1 < 3^{k+1} \\
k+1 &< 3^{k+1}
\end{aligned}
$$

Conclusion: $P(k+1)$ is true, $\therefore \forall n \in \mathbb{N} n < 3^n$ is true. $\square$

2.305 Example of an incorrect proof

We're going to see how easy it is to make a mistake in a proof if we don't follow the steps correctly.

$n + 1 < n \forall n \in \mathbb{N}$

Theorem 11. *$n + 1 < n$, for all $n \in \mathbb{N}$.*

Proof. INCORRECT!!!

Let $P(n) = n + 1 < n \forall n \in \mathbb{N}$.

Prove $P(k) \implies P(k+1)$. Assuming $P(k) istrue, so$k + 1 < k.$Show$P(k + 1)$istrue. Adding 1 to both sides of the inequality we get$k + 1 + 1 < k + 1.$Let$l = k + 1$weget$l + 1 < l, soP(k + 1)$is also true. Therefore$P(n) is true. $\square$

In this proof, we didn't prove the base case, so the proof in invalid.

2.401 Conclusion

We have learned about several powerful proof techniques. We explored Proof by Induction, which exploits the fact that natural numbers are like a chain.

We have also seen how contrapositive proofs are, sometimes, easier then proving the original statement. We have also witnessed how proofs can go wrong if we miss an important step.