4.2 Applications

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Cornell Notes	**Topic:** 4.2 Applications	Course: BSc Computer Science Class: Discrete Mathematics-Lecture Date: November 16, 2019

Essential Question:

What are the rules of inference and/or the rules of inference with quantifiers?

Questions/Cues:

- What are DeMorgan's laws for quantifiers?
- What is an argument?
- What are rules of inference?
- What are the steps to building a valid argument?
- What is fallacy and/or formal fallacy?
- What are the rules of inference with quantifiers?
- What are the steps to expressing complex statements?

Notes

- DeMorgan's laws for Quantifiers = stem from the need to consider negation of a quantified expression

  Example
  - S: “All the university’s computers are connected to the network.”
  - P: “There is at least one computer in the university operating on Linux.”

  Intuitively
  - The negation of S can be verified **if there is at least one** computer not connected to the network
  - The negation of P can be verified if **all university computers** are not operating on Linux

  - De Morgan’s laws formalise these intuitions.

  - Let P be a predicate over the variable x, it follows that:

De Morgan's laws for negating quantifiers:

$$\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$$

$$\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$$

Let S: "Every student of Computer Science has taken a course in Neural Networks."

- S can be expressed as: $\forall x\ P(x)$
- U = {students in CS}
- P(x): "x has taken a course in Neural Networks."

The **Negation** of S:

- "It is not the case that every student of Computer Science has taken a course in Neural Networks."
  $\neg(\forall x\ P(x)) \equiv \exists x\ \neg P(x)$
- This implies that: "There **is** at least one student who has **not** taken a course in Neural Networks."

Let R denote: "There is a student in Computer Science who didn't take a course in Machine Learning."

- R can be expressed as: $\exists x\ Q(x)$
- U = {students in CS}
- Q(x): "x didn't take a course in Machine Learning"

The **Negation** of S:

- "It is not the case that there is a student in Computer Science who didn't take a course in Machine Learning."
  $\neg(\exists x\ Q(x)) \equiv \forall x\ \neg Q(x)$
- This implies that: "Every student in Computer Science has taken a Machine Learning course."

In the case of nested quantifiers, we apply De Morgan's laws successively from left to right.

Example
Let P(x,y,z) denote a propositional function of variables: x, y and z.

$$\equiv \exists x \neg \exists y \ \forall z \ P(x, y, z)$$
$$\equiv \exists x \ \forall y \neg \forall z \ P(x, y, z)$$
$$\equiv \exists x \ \forall x \ \exists z \neg P(x, y, z)$$

$\neg \forall x \ \exists y \ \forall z \ P(x, y, z)$ is built by moving the negation to the right through all the quantifiers and replacing each $\forall$ with $\exists$, and vice versa

- Argument = in propositional logic is sequence of propositions
  - final proposition is called conclusion, other propositions in argument called premises (or hypotheses)
  - Argument is valid if the truth of its premises implies truth of the conclusion

Let's consider this **argument**:

- "If you have access to the internet, you can order a book on Machine Learning."
- "You have access to the internet."

∴ Therefore: "You can order a book on Machine Learning."

This argument is **valid** because whenever all its premises are true, the conclusion must also be true.

Let's consider this **argument** :

- "If you have access to Internet, you can order a book on Machine Learning."
- "You can order a book on Machine Learning."

∴ Therefore: "You have access to Internet."

This argument is **not valid** because we can imagine situations where the premises are true and the conclusion is false.

- Rules of inference = seen as building block in constructing incrementally complex valid arguments
    - we can use truth table to determine whether argument is T or F, but is a long process especially with multiple variables
    - ROI, a simplier way of proving validity of argument
    - Every ROI can be proved using a tautology

Modus ponens

- **Tautology:** $(p \land (p \to q)) \to q$
- **The rule of inference:**

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore q \end{array}$$

Example

- **p**: "It is snowing." **q**: "I will study Discrete Mathematics."
- "If it is snowing, I will study Discrete Mathematics."
- "It is snowing."
- Therefore: "I will study discrete mathematics."

    - In Modus ponens, if p implies q & premise is true, then conclusion is also true

Modus tollens

- **Tautology:** $(\neg q \land (p \to q)) \to \neg p$
- **The rule of inference:**

$$\begin{array}{l} \neg q \\ p \to q \\ \hline \therefore \neg p \end{array}$$

Example

- **p**: "It is snowing." **q**: "I will study discrete mathematics."
- "If it is snowing, I will study Discrete Mathematics."
- "I will not study Discrete Mathematics."
- Therefore: "It is not snowing."

    - In modus tollens, if premise not q is true and if conditional statement p implies q is true, conclusion not p is also true

Conjunction

- **Tautology:** $((p) \wedge (q)) \rightarrow (p \wedge q)$
- **The rule of inference:**

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

Example

- **p**: "I will study Programming." **q**: "I will study Discrete Mathematics."
- "I will study Programming."
- "I will study Discrete Mathematics."

  - Therefore: "I will study Programming and Discrete Mathematics."

- In conjunction, if premise p is true and if premise q is true, the conclusion p and q is also true

Simplification

- **Tautology:** $(p \wedge q) \rightarrow p$
- **The rule of inference:**

$$\begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}$$

Example

- **p**: "I will study Programming." **q**: "I will study Discrete Mathematics."
- I will study Programming and Discrete Mathematics

  - Therefore: "I will study Discrete Mathematics."

- In simplification, if the premise p and q is true, the conclusion p is also true

Addition

- **Tautology:** $p \rightarrow (p \lor q)$
- **The rule of inference:**

$$\begin{array}{l} p \\ \hline \therefore p \lor q \end{array}$$

Example

- **p**: "I will visit Paris." **q**: "I will study Discrete Mathematics."
- "I will visit Paris."

- Therefore: "I will visit Paris or I will study Discrete Mathematics."

  - In addition, if the premise p is true, the conclusion p or q is also true.

Hypothetical syllogism

- **Tautology:** $((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r)$
- **The rule of inference :**

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Example

- **p**: "It is snowing." **q**: "I will study Discrete Mathematics."
- **r**: "I will pass the quizzes."
- "If it is snowing, I will study Discrete Mathematics."
- "If I study Discrete Mathematics, I will pass the quizzes."

- Therefore: "If it is snowing, I will pass the quizzes."

  - In hypothetical syllogism, if the premise p implies q is true and the premise q implies r is true, the conclusion p implies r is also true.

Disjunctive syllogism

- **Tautology:** $((p \vee q) \wedge \neg p) \rightarrow q$
- **The rule of inference:**

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Example

- **p**: "I will study Art." **q**: "I will study Discrete Mathematics."
- "I will study Art or I will study Discrete Mathematics."
- "I will not study Discrete Mathematics."

- Therefore: "I will study Art."

- In Disjunctive Syllogism, if the premise p or q is true and the premise not p is true, the conclusion q is also true

Resolution

- **Tautology:** $((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$
- **The rule of inference:**

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$$

Example

- **p**: "It is raining." **q**: "It is snowing." **r** : "It is cold."
- "It is raining or it is snowing."
- "It is not raining or it is cold."

- Therefore: "It is snowing or it is cold."

- In Resolution, if the premise p or q is true and the premise not p or r is true, the conclusion q or r is also true

To build a **valid argument** we need to follow the steps below:

- If initially written in English, transform the statement into an argument form by choosing a variable for each simple proposition
- Start with the hypothesis of the argument
- Build a sequence of steps in which each step follows from the previous step by applying:
  - **rules of inference**
  - **laws of logic**
- The final step of the argument is the conclusion.

Let's build a valid argument from the following premises:

$\neg p$: "It is not cold tonight."

$q \to p$: "We will go to the theatre only if it is cold."

$\neg q \to r$: "If we do not go to the theatre, we will watch a movie at home."

$r \to s$: "If we watch a movie at home, we will need to make popcorn."

Propositional variables:

p: "It is cold tonight." **r**: "We will watch a movie at home."
q: "We will go to the theatre." **s**: "We will need to make popcorn."

	Step	*Justification*
1	$q \to p$	*Hypothesis*
2	$\neg p$	*Hypothesis*
3	$\therefore \neg q$	*Modus tollens 1, 2*
4	$\neg q \to r$	*Hypothesis*
5	$\therefore r$	*Modus ponens 3, 4*
6	$r \to s$	*Hypothesis*
7	$\therefore s$	*Modus ponens 5, 6*

Conclusion: "We will need to make popcorn."

- Fallacy = use of incorrect argument when reasoning
  - Formal fallacies can be expressed in propositional logic and proved to be incorrect

- Some of the widely used **formal fallacies** are:
  - affirming the consequent
  - a conclusion that denies premises
  - contradictory premises
  - denying the antecedent
  - existential fallacy
  - exclusive premises.

Let's consider the following argument:

- If you have internet access, you can order this book.
- You can order this book.

- Therefore, you have Internet access.

- This argument can be formalised as: if $p \rightarrow q$ and q, then p
- Where **p**: "You have Internet access."
  q: "You can order this book."
- The proposition $((p \rightarrow q) \wedge q) \rightarrow p$ is not a tautology, because it is false when p is false and q is true
- This is an incorrect argument using the fallacy of affirming the consequent (or conclusion).

- Rules of inference with quantifiers either remove or reintroduce quantifiers within a statement.

Universal instantiation (UI)

The rule of inference:

$$\frac{\forall x\, P(x)}{\therefore P(c)}$$

Example

- All computer science students study discrete mathematics.

∴ Therefore, John, who is a computer science student, studies discrete mathematics.

  - Universal Instantiation is used to conclude that P(c) is true where is c is a particular member of the domain, given the premise $\forall\, xP(x)$. This rule of inference removes the universal quantifier
  -

Universal generalization (UG)

The rule of inference:

$$\frac{P(c) \text{ for an arbitrary element of the domain}}{\forall x\, P(x)}$$

Example

- **DS** = {all data science students}
- Let **c** be an arbitrary element in **DS**.
- **c** studies machine learning.

∴ Therefore, ∀**x** ∈DS, **x** studies machine learning.

- Universal generalization is used to conclude that $\forall\ xP(x)$ is true by taking an arbitrary element C from the domain and showing that P(c) is true. This rule of inference introduces the universal quantifier

Existential instantiation (EI)

The rule of inference:

$$\frac{\exists x\, P(x)}{\therefore P(c) \text{ for some element of the domain}}$$

Example

- **DS** = {all data science students}
- There exists a student of data science who uses *Python Pandas Library*.

∴ There is a student, c, who is using *Python Pandas Library*.

- Existential instantiation is used to conclude that there is an element c in the domain for which P(c) is true. If we know that is x P(x) is true, we cannot selected an arbitrary value but rather acknowledge it exists, let name it "c" and use in argument. This rule of inference removes the existential quantifier

Existential generalization (EG)

The rule of inference:

$$\frac{P(c) \text{ for some element of the domain}}{\therefore\ \exists x\, P(x)}$$

Example

- **DS** = {all data science students}
- John, a student of data science, got an A in the machine learning course.

∴ Therefore, there exists someone in DS who got an A in machine learning.

- Existential generalization is used to conclude $\exists\, xP(x)$ is true when P(c) is true for some elements c of the domain. This rule of inference introduces the existential quantifier

Universal modus ponens

The rule of inference

$$\frac{\begin{array}{l}\forall x\, P(x) \rightarrow Q(x) \\ P(a) \text{ for some element of the domain}\end{array}}{Q(a)}$$

Example

- **DS** = {all computer science students}
- Every computer science student studying data science will study machine learning.
- John is studying data science.

∴ Therefore, John will study machine learning.

- Universal modus ponens is combination of universal instantiation and modus ponens. Universal modus ponens concludes that if for all x in the domain P(x) implies Q(x) and P(a) is true for some elements of the domain, we can conclude Q(a) is also true.

Universal modus tollens

The rule of inference:

$$\forall x\, P(x) \rightarrow Q(x)$$
$$\neg Q(a) \text{ for some element of the domain}$$
$$\overline{\neg P(a)}$$

Example

- **CS** = {all computer science students}
- Every computer science student studying data science will study machine learning.
- John is not studying machine learning.

∴ Therefore, John is not studying data science.

- Universal modus tollens is a combination of universal instantiation and modus tollens. Universal modus tollens is used to conclude that if for all x in the domain P(x) implies Q(x) and if Q(a) is false for some element of the domain, we can conclude that P(a) is also false.

Expressing complex statements

Given a statement in natural language, we can formalise it using the following steps as appropriate:

1. Determine the universe of discourse of variables.
2. Reformulate the statement by making "**for all**" and "**there exists" explicit**
3. Reformulate the statement by introducing **variables** and defining **predicates**
4. Reformulate the statement by introducing **quantifiers** and **logical operations**.

Express the statement S: "there exists a real number between any two not equal real numbers".

- The universe of discourse is: real numbers.
- Introduce variables and predicates:
  - "For all real numbers x and y, there exists z between x and y."
- Introduce quantifiers and logical operations:
  - $\forall x\ \forall y$ if $x < y$ then $\exists z$ where $x < z < y$

Express the statement S: “every student has taken a course in machine learning”.

The expression will depend on the choice of the universe of discourse

Case 1: U = {all students}

- Let M(x) be: “x has taken a course in machine learning.”
- S can be expressed as: $\forall x\, M(x)$

Case 2: U = {all people}

- Let S(x) be: “x is a student” and M(x) the same as in case 1
- S can be expressed as $\forall x\, (S(x) \rightarrow M(x))$

Note: $\forall x\, (S(x) \wedge M(x))$ is **not** correct.

Express the statement S: “some student has taken a course in machine learning”.

The expression will depend on the choice of the universe of discourse.

Case 1: U = {all students}

- Let M(x) be: “x has taken a course in machine learning.”
- S can be expressed as: $\exists x\, M(x)$

Case 2: U = {all people}

- Let S(x) be: “x is a student” and M(x) the same as in case 1.
- S can be expressed as $\exists x\, (S(x) \wedge M(x))$.

Note: $\exists x\, (S(x) \rightarrow M(x))$ is **not** correct in this case.

Summary

In this week, we learned about DeMorgan's laws for quantifiers, rules of inference, rules of inferences with quantifiers and fallacies and/or formal fallacies. Alongside we explored the steps to building a valid argument and the steps to expressing complex statements.