8.2 Rooted Trees & Binary Search Trees

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Cornell Notes	**Topic:** 8.2 Rooted Trees & Binary Search Trees	Course: BSc Computer Science Class: Discrete Mathematics-Lecture Date: January 03, 2020

Essential Question:

What are rooted trees & binary search trees?

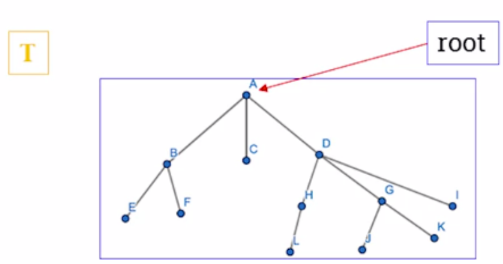
Questions/Cues:

- What is a rooted tree?
- How is a directed tree represented as a rooted tree?
- What is some terminology associated with rooted trees?
- What are the depth & height in a rooted tree?
- What are some special rooted trees?
- When is m-ary tree considered to be regular?
- What are some properties of m-ary rooted trees?
- What is isomorphism in trees & some properties related to this?
- What is isomorphism in rooted trees?
- What is a binary search tree?
- What is an application of binary search trees?
- What is the height of a binary search tree?
- What is the binary search algorithm?

Notes

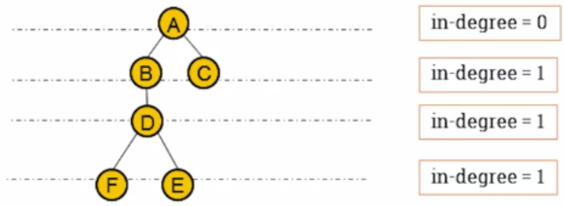
Definition of rooted trees

A rooted tree is a **directed tree** having one **distinguished** vertex r, called a root, such that for every vertex v there is a **directed path** from r to v.



Theorem

A directed tree is represented as a rooted tree **if and only if one vertex** has in-degree **0** whereas **all** other **vertices** have **in-degree 1**.



Terminology of rooted trees

A is the **root** of the tree

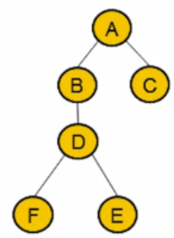
B is called the **parent of D**

E and **F** are the **children** of **D**

B and **A** are **ancestors** of **E** and **F** (**E** and **F** are **siblings**)

B and **D** are called **internal** nodes

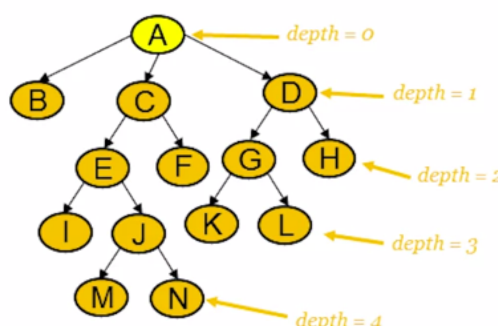
C, **E** and **F** are called **external nodes**.



Depth and height in a tree

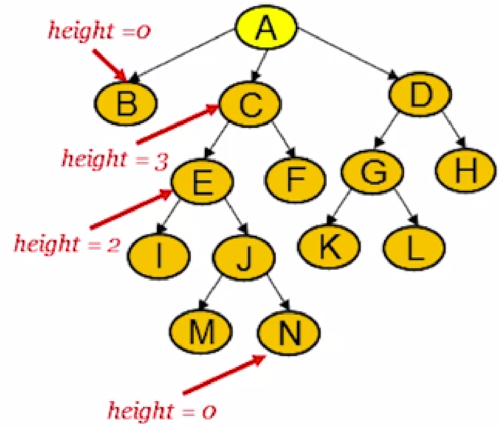
The **depth** or **path length** of a node in a tree is the number of edges from the root to that node.

The **height** of a node in a tree is the longest path from that node to a leaf.



The ***depth or the height*** of a tree is the maximum path length across all its nodes.

The depth (height) of this tree is **4**.

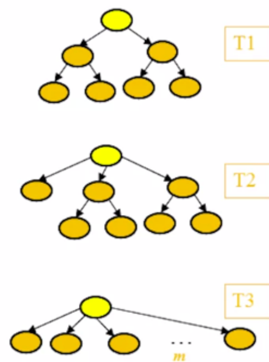


Special trees

Binary Trees
A binary tree is a rooted tree in which every vertex has 2 or fewer children.

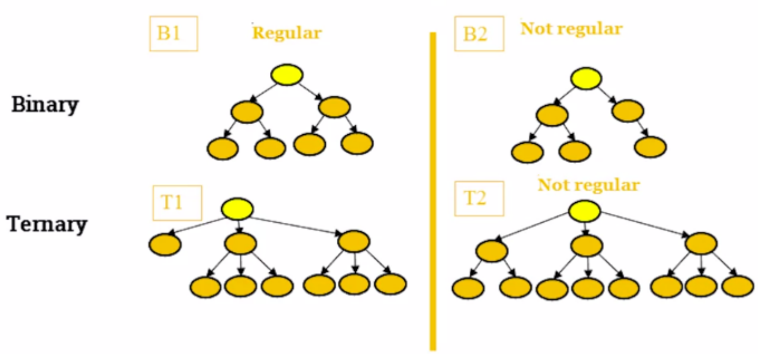
Ternary Trees
A ternary tree is a rooted tree in which every vertex has 3 or fewer children.

m-ary Trees
A m-ary tree is a rooted tree in which every vertex has m or fewer children.



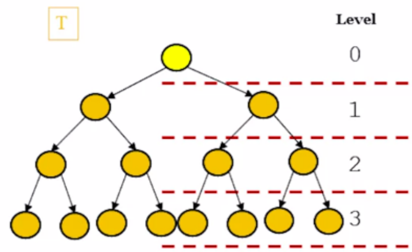
Regular rooted trees

An **m-ary** tree is **regular** if every one of its **internal** nodes **has exactly m** children.



Properties

An **m-ary tree** has at most m^h vertices at level **h**.



Level	Binary Tree	Ternary Tree	m-ary Tree
0	1	1	1
1	2^1	3^1	m^1
2	2^2	3^2	m^2
3	2^3	3^3	m^3
⋮	⋮	⋮	⋮
Max no. of nodes per level: h	2^h	3^h	m^h

Note** Also the maximum number of edges in an m-ary of h levels = $\frac{m^{h+1}-1}{m-1}$

Isomorphic trees

Two trees T_1 and T_2 are isomorphic if there is a **bijection**:

$$f: V(T_1) \rightarrow V(T_2)$$

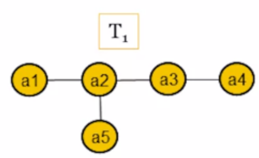
which **preserves adjacency** and **non-adjacency**.

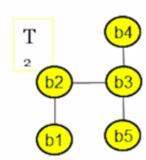
That is, if uv is in $E(T_1)$ and $f(u)f(v)$ is in $E(T_2)$.

Notation:

$T_1 \cong T_2$ means that T_1 and T_2 are isomorphic.

Example





$f: V(T_1) \rightarrow V(T_2)$

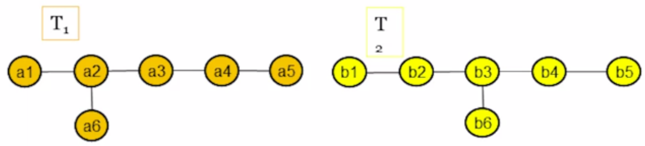
with

$f(a1) = b4$
$f(a2) = b3$
$f(a3) = b2$
$f(a4) = b1$
$f(a5) = b5$

Properties

Two trees with **different degree sequences** are **not isomorphic**.

Two trees with the **same degree** sequence **are *not necessarily*** **isomorphic**.



T_1 and T_2 have the **same degree sequence**: 3,2,2,1,1,1
T1 and T2 are **not isomorphic**.

Isomorphic rooted trees

Two isomorphic trees are **isomorphic as rooted trees** if and only if there is a **bijection** that maps the **root** of one tree to the root of the other.

Properties

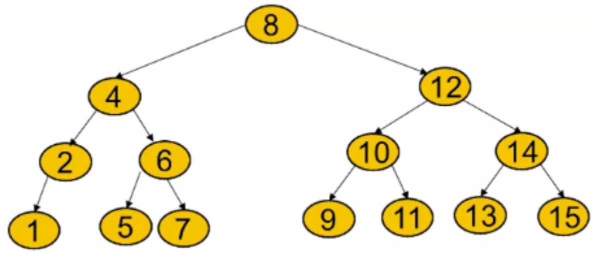
Isomorphic trees **may** or **may not** be **isomorphic** as **rooted trees**.

T1 and T2 are **isomorphic** as **graphs**
but **not isomorphic** as **rooted trees**

Definition

A binary search tree is a **binary tree** in which the vertices are **labelled** with items so that a **label of a vertex is greater than** the labels of all vertices in the **left subtree** of this vertex and **is less than** the labels of all vertices in the **right subtree** of this vertex.

Example



Applications

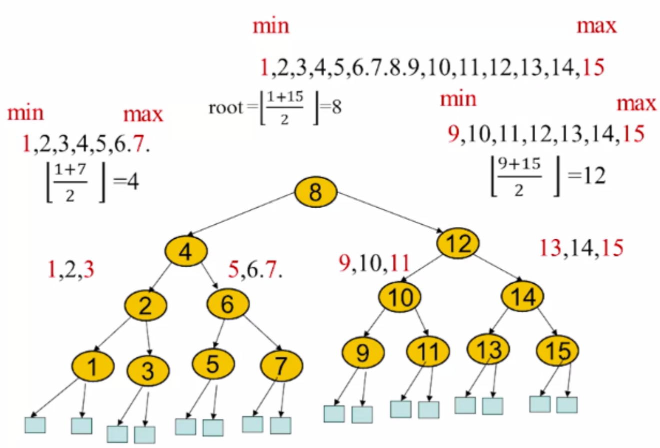
The use applies in the case where we want to **store a modifiable collection** in a **computer's memory** and be able to **search**, **insert** or **remove** elements from the collection in an efficient way.

Binary search trees can be used to solve these kind of **problems**.

Example

Build a binary search tree to store 15 records and find the height of the this trees.

Solution



Height of the tree

Method 1 $\qquad 2^{h-1} < 1+\text{N} \le 2^{h}$

$$\equiv$$

$$\text{h-1} < \log 2(1+\text{N}) \le \text{h}$$

$$\equiv$$

Method 2 $\qquad \text{h}= \lceil log_2\,(N+1) \rceil$

For example: if N=15 then h= 4

$$2^{4-1} < 1+15< \ \le 2^{4}$$

$$\text{h}= \lceil log_2\,(15+1) \rceil = \lceil log_2\,(16) \rceil = 4$$

Exercise

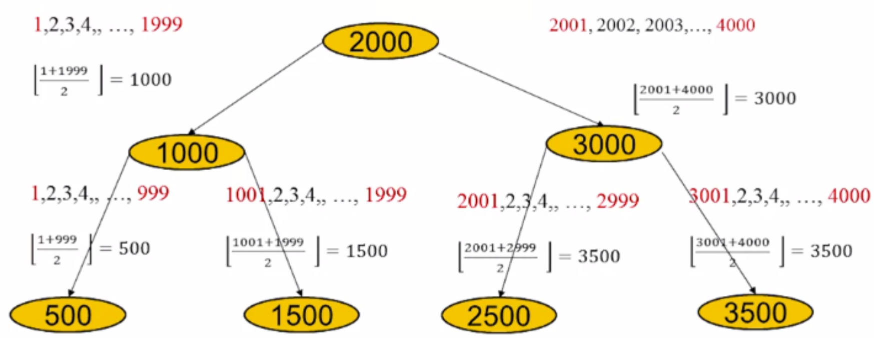
Find the first 3 level of a binary search tree to store 4000 records.

Find the height of this tree.

Solution

$$1,2,3,4,,\ \ldots,\ 4000$$

$$\text{root} = \left\lfloor \frac{1+4000}{2} \right\rfloor = 2000$$



Height of the tree

Method 1

$$2^{h-1} < 1 + N \leq 2^h$$

$$2^{12-1} < 1 + 4000 \leq 2^{12}$$

$$h = 12$$

Method 2

$$h = \lceil log_2 (N + 1) \rceil$$

$$h = \lceil log_2 (4000 + 1) \rceil = \lceil log_2 (4001) \rceil = 12$$

Binary search algorithm

The algorithm starts by comparing the searched element to the middle term of the list.

The list is then split into two smaller sub-lists of the same size, or where one of these smaller lists has one fewer term than the other.

The search continues by restricting the search to the appropriate sub-list based on the comparison of the searched element and term in the middle.

Example

Search for **21** in the **list** of :

3 4 5 7 8 10 10 12 14 15 17 18 20 21 22 24

Summary

In this week, we learned what rooted tree is, special rooted trees, properties & terminology associated with rooted trees, what m-ary trees are & what a binary search tree is.