

# Lecture 4

## Fuzzy expert systems: Fuzzy logic

- Introduction, or what is fuzzy thinking?
- Fuzzy sets
- Linguistic variables and hedges
- Operations of fuzzy sets
- Fuzzy rules
- Summary

# Introduction, or what is fuzzy thinking?

- Experts rely on **common sense** when they solve problems.
- **How can we represent expert knowledge that uses vague and ambiguous terms in a computer?**
- Fuzzy logic is not logic that is fuzzy, but logic that is used to describe fuzziness. Fuzzy logic is the theory of fuzzy sets, sets that calibrate vagueness.
- Fuzzy logic is based on the idea that all things admit of degrees. Temperature, height, speed, distance, beauty – all come on a sliding scale. The motor is running *really hot*. Tom is a *very tall* guy.

- Boolean logic uses sharp distinctions. It forces us to draw lines between members of a class and non-members. For instance, we may say, Tom is tall because his height is 181 cm. If we drew a line at 180 cm, we would find that David, who is 179 cm, is small. Is David really a small man or we have just drawn an arbitrary line in the sand?
- Fuzzy logic reflects how people think. It attempts to model our sense of words, our decision making and our common sense. As a result, it is leading to new, more human, intelligent systems.

- Fuzzy, or multi-valued logic was introduced in the 1930s by **Jan Lukasiewicz**, a Polish philosopher. While classical logic operates with only two values 1 (true) and 0 (false), Lukasiewicz introduced logic that extended the range of truth values to all real numbers in the interval between 0 and 1. He used a number in this interval to represent the *possibility* that a given statement was true or false. For example, the possibility that a man 181 cm tall is really tall might be set to a value of 0.86. It is *likely* that the man is tall. This work led to an inexact reasoning technique often called **possibility theory**.

■ Later, in 1937, **Max Black** published a paper called “Vagueness: an exercise in logical analysis”. In this paper, he argued that a continuum implies degrees. Imagine, he said, a line of countless “chairs”. At one end is a Chippendale. Next to it is a near-Chippendale, in fact indistinguishable from the first item. Succeeding “chairs” are less and less chair-like, until the line ends with a log. When does a *chair* become a *log*? Max Black stated that if a continuum is discrete, a number can be allocated to each element. He accepted **vagueness as a matter of probability**.

- In 1965 **Lotfi Zadeh**, published his famous paper “Fuzzy sets”. Zadeh extended the work on possibility theory into a formal system of mathematical logic, and introduced a new concept for applying natural language terms. This new logic for representing and manipulating fuzzy terms was called **fuzzy logic**, and Zadeh became the Master of *fuzzy logic*.

## ■ *Why fuzzy?*

As Zadeh said, the term is concrete, immediate and descriptive; we all know what it means. However, many people in the West were repelled by the word *fuzzy*, because it is usually used in a negative sense.

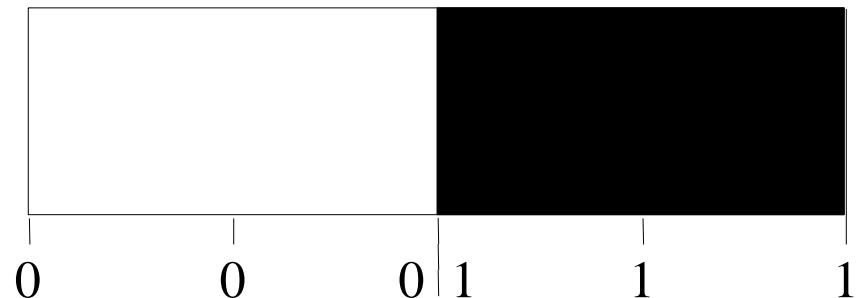
## ■ *Why logic?*

Fuzziness rests on fuzzy set theory, and fuzzy logic is just a small part of that theory.

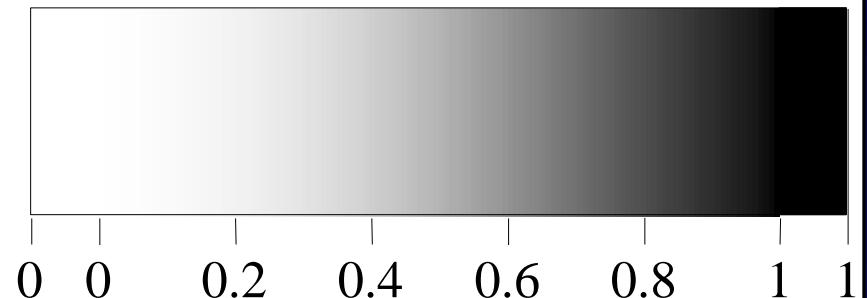
**Fuzzy logic is a set of mathematical principles for knowledge representation based on degrees of membership.**

Unlike two-valued Boolean logic, fuzzy logic is **multi-valued**. It deals with **degrees of membership** and **degrees of truth**. Fuzzy logic uses the continuum of logical values between 0 (completely false) and 1 (completely true). Instead of just black and white, it employs the spectrum of colours, accepting that things can be partly true and partly false at the same time.

# Range of logical values in Boolean and fuzzy logic



(a) Boolean Logic.



(b) Multi-valued Logic.

# Fuzzy sets

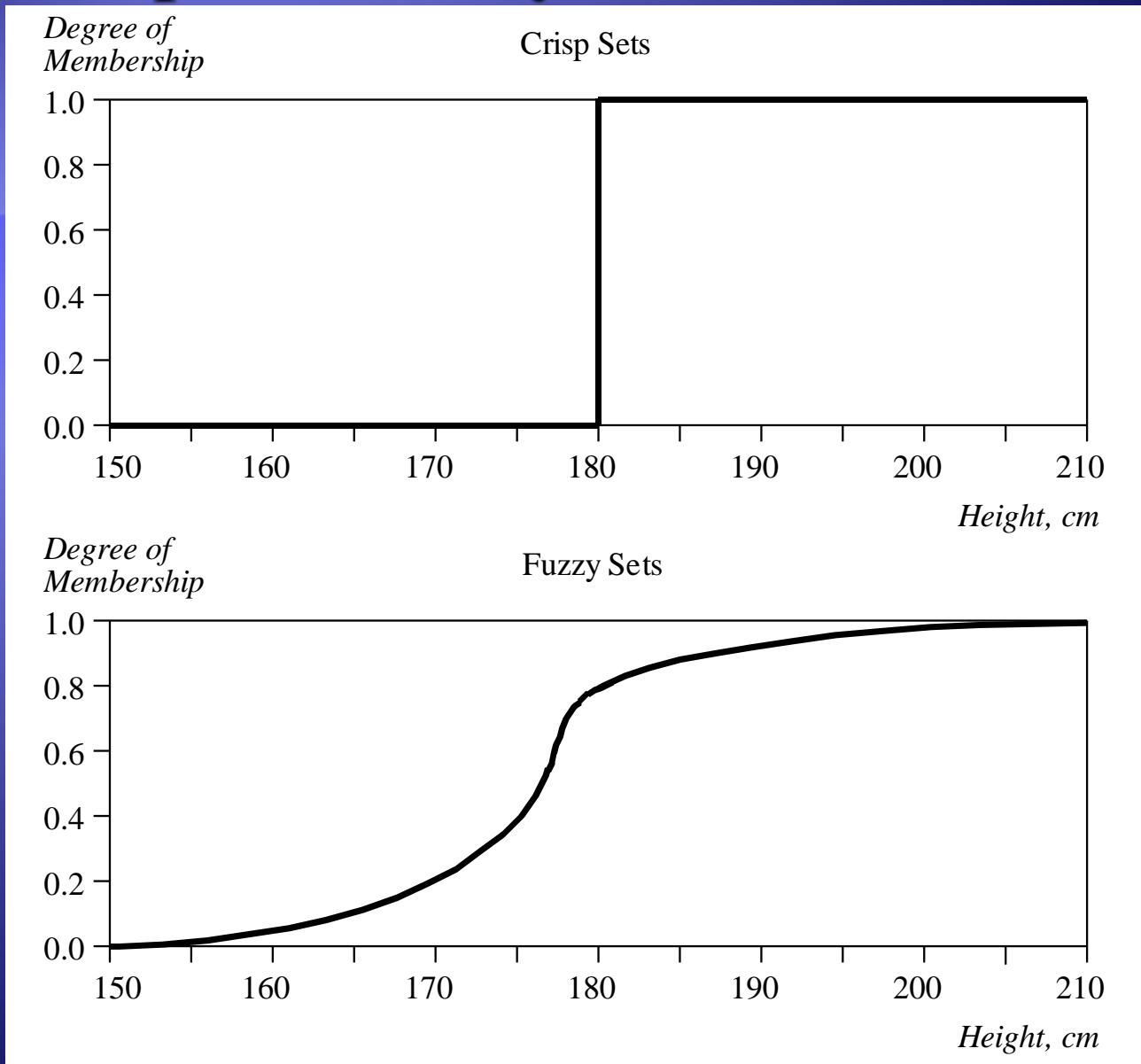
- The concept of a **set** is fundamental to mathematics.
- However, our own language is also the supreme expression of sets. For example, *car* indicates the *set of cars*. When we say *a car*, we mean one out of the set of cars.

- Crisp set theory is governed by a logic that uses one of only two values: true or false. This logic cannot represent vague concepts, and therefore fails to give the answers on the paradoxes.
- The basic idea of the fuzzy set theory is that an element belongs to a fuzzy set with a certain degree of membership. Thus, a proposition is not either true or false, but may be partly true (or partly false) to any degree. This degree is usually taken as a real number in the interval  $[0,1]$ .

- The classical example in fuzzy sets is *tall men*. The elements of the fuzzy set “tall men” are all men, but their degrees of membership depend on their height.

Name	Height, cm	Degree of Membership	
		<i>Crisp</i>	<i>Fuzzy</i>
Chris	208	1	1.00
Mark	205	1	1.00
John	198	1	0.98
Tom	181	1	0.82
David	179	0	0.78
Mike	172	0	0.24
Bob	167	0	0.15
Steven	158	0	0.06
Bill	155	0	0.01
Peter	152	0	0.00

# Crisp and fuzzy sets of “*tall men*”



- The  $x$ -axis represents the **universe of discourse** – the range of all possible values applicable to a chosen variable. In our case, the variable is the man height. According to this representation, the universe of men's heights consists of all tall men.
- The  $y$ -axis represents the **membership value of the fuzzy set**. In our case, the fuzzy set of “*tall men*” maps height values into corresponding membership values.

## A fuzzy set is a set with fuzzy boundaries.

- Let  $X$  be the universe of discourse and its elements be denoted as  $x$ . In the classical set theory, **crisp set  $A$  of  $X$  is defined as function  $f_A(x)$  called the characteristic function of  $A$**

$f_A(x): X \rightarrow \{0, 1\}$ , where

$$f_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

This set maps universe  $X$  to a set of two elements. For any element  $x$  of universe  $X$ , characteristic function  $f_A(x)$  is equal to 1 if  $x$  is an element of set  $A$ , and is equal to 0 if  $x$  is not an element of  $A$ .

- In the fuzzy theory, fuzzy set  $A$  of universe  $X$  is defined by function  $\mu_A(x)$  called the *membership function* of set  $A$

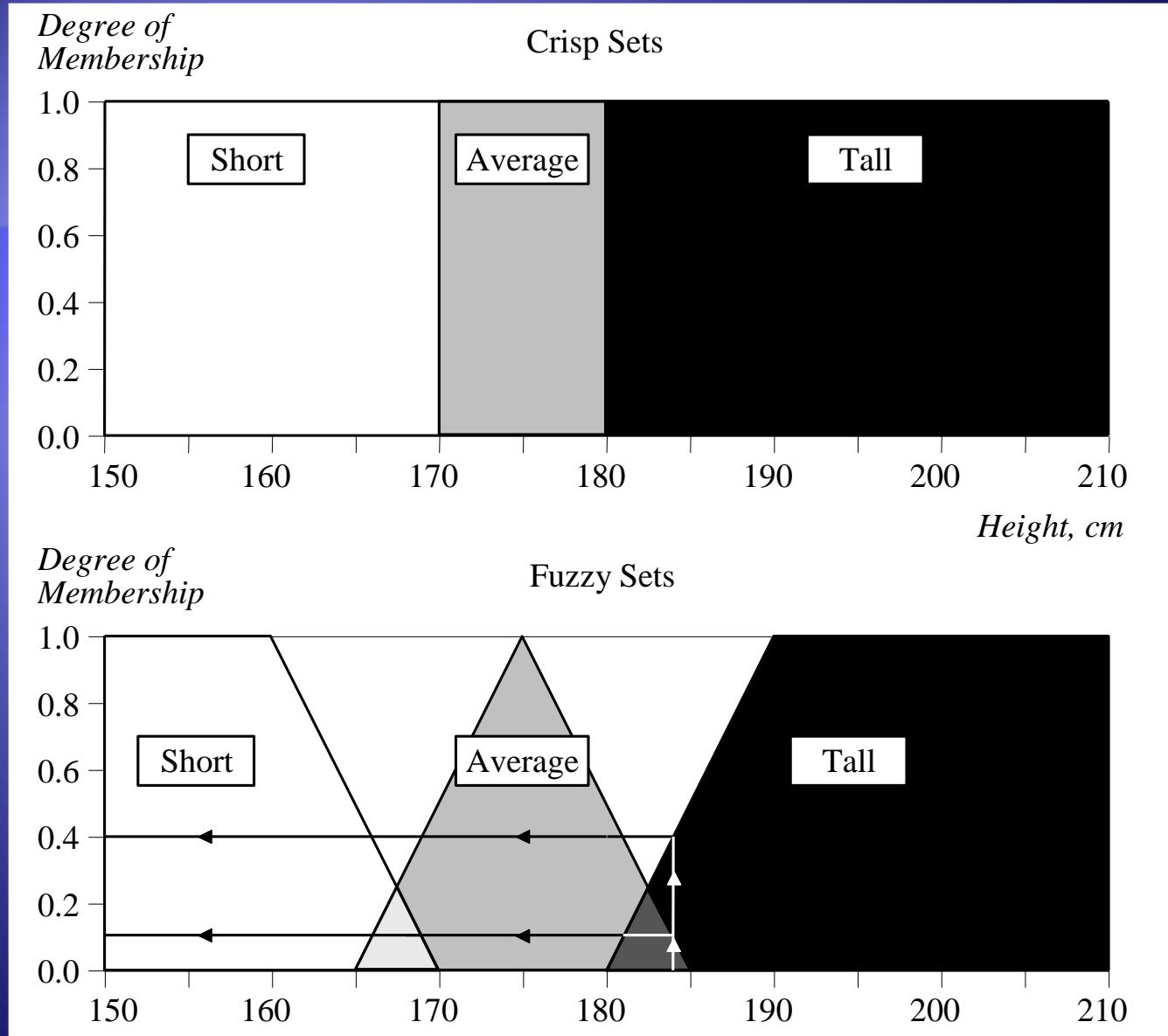
$\mu_A(x): X \rightarrow [0, 1]$ , where  $\mu_A(x) = 1$  if  $x$  is totally in  $A$ ;  
 $\mu_A(x) = 0$  if  $x$  is not in  $A$ ;  
 $0 < \mu_A(x) < 1$  if  $x$  is partly in  $A$ .

This set allows a continuum of possible choices. For any element  $x$  of universe  $X$ , membership function  $\mu_A(x)$  equals the degree to which  $x$  is an element of set  $A$ . This degree, a value between 0 and 1, represents the **degree of membership**, also called **membership value**, of element  $x$  in set  $A$ .

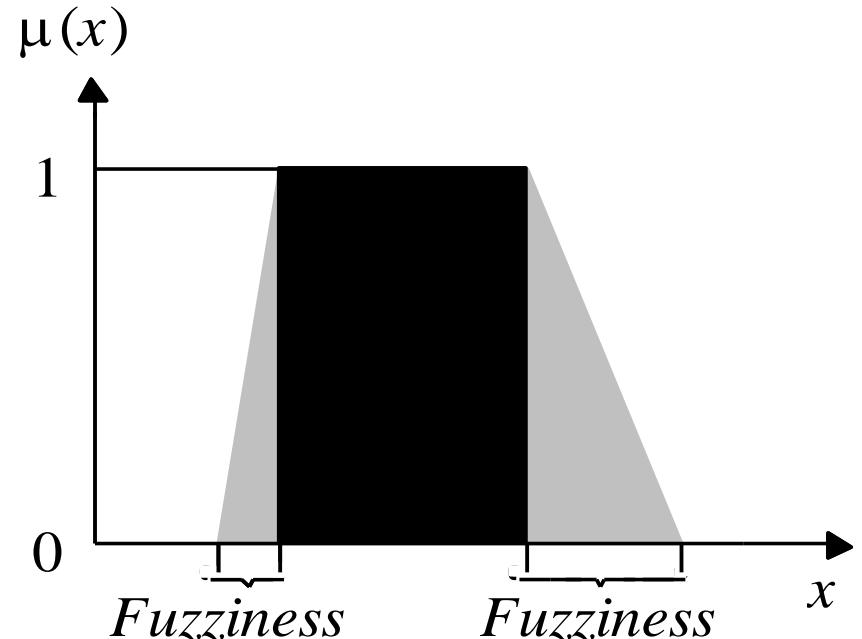
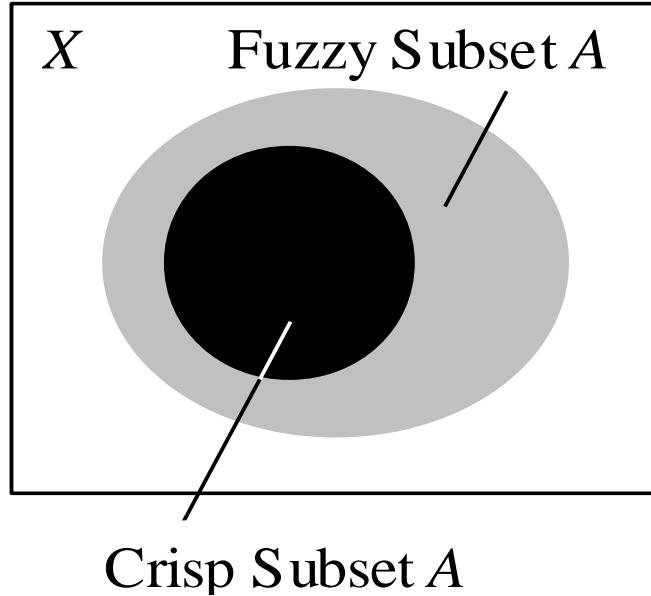
# How to represent a fuzzy set in a computer?

- First, we determine the membership functions. In our “*tall men*” example, we can obtain fuzzy sets of *tall*, *short* and *average* men.
- The universe of discourse – the men’s heights – consists of three sets: *short*, *average* and *tall men*. As you will see, a man who is 184 cm tall is a member of the *average men* set with a degree of membership of 0.1, and at the same time, he is also a member of the *tall men* set with a degree of 0.4.

# Crisp and fuzzy sets of short, average and tall men



# Representation of crisp and fuzzy subsets



Typical functions that can be used to represent a fuzzy set are sigmoid, gaussian and pi. However, these functions increase the time of computation. Therefore, in practice, most applications use **linear fit functions**.

# Linguistic variables and hedges

- At the root of fuzzy set theory lies the idea of linguistic variables.
- A **linguistic variable** is a **fuzzy variable**. For example, the statement “John is tall” implies that the linguistic variable *John* takes the linguistic value *tall*.

In fuzzy expert systems, linguistic variables are used in fuzzy rules. For example:

**IF           wind is strong**

**THEN sailing is good**

**IF           project\_duration is long**

**THEN completion\_risk is high**

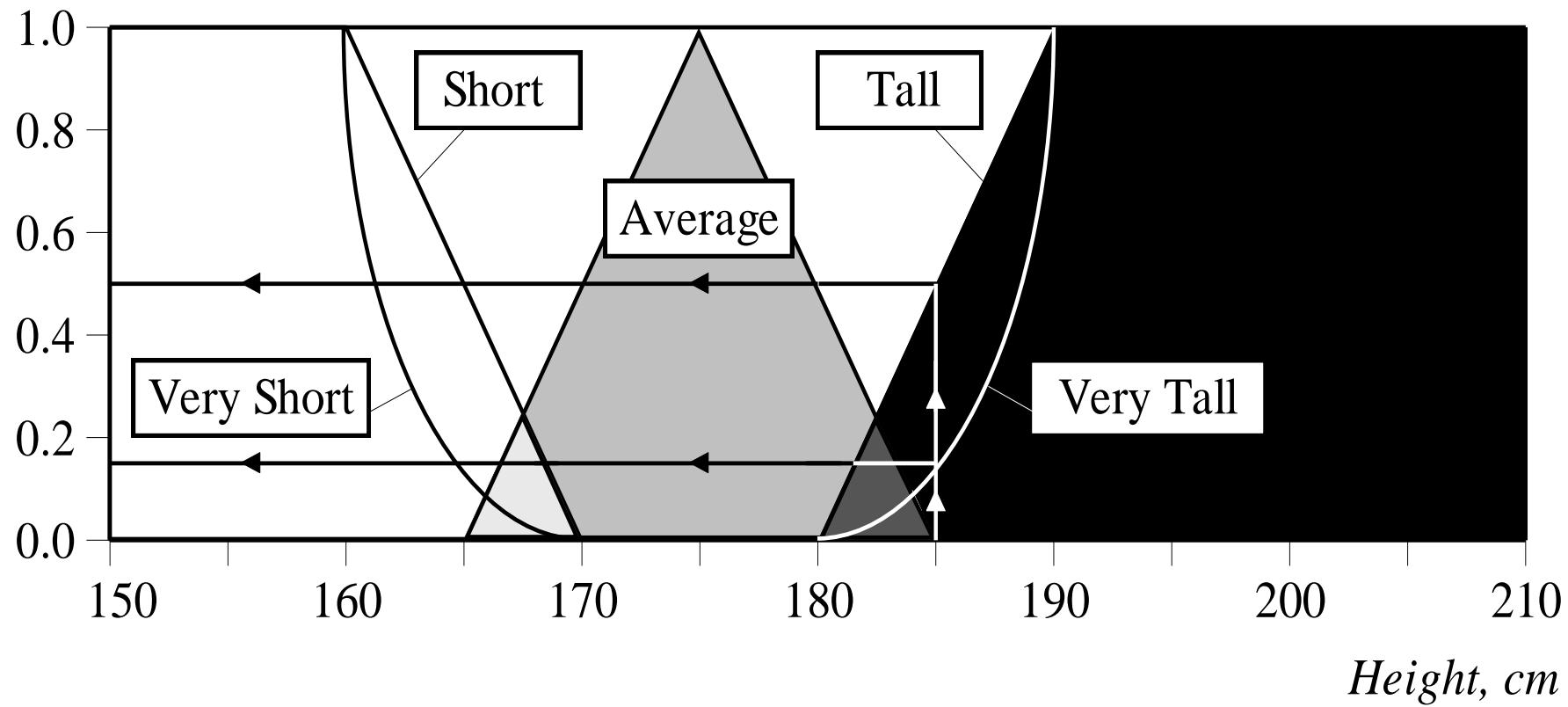
**IF           speed is slow**

**THEN stopping\_distance is short**

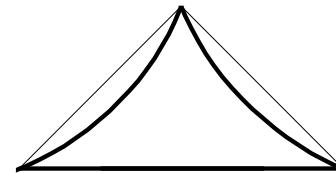
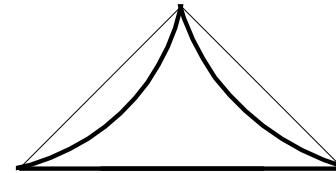
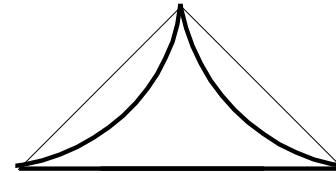
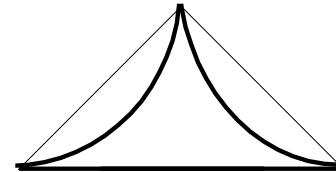
- The range of possible values of a linguistic variable represents the universe of discourse of that variable. For example, the universe of discourse of the linguistic variable *speed* might have the range between 0 and 220 km/h and may include such fuzzy subsets as *very slow*, *slow*, *medium*, *fast*, and *very fast*.
- A linguistic variable carries with it the concept of fuzzy set qualifiers, called *hedges*.
- **Hedges are terms that modify the shape of fuzzy sets. They include adverbs such as *very*, *somewhat*, *quite*, *more or less* and *slightly*.**

# Fuzzy sets with the hedge *very*

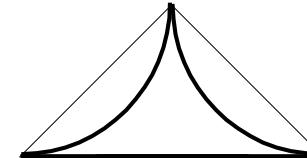
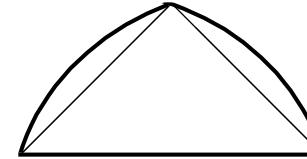
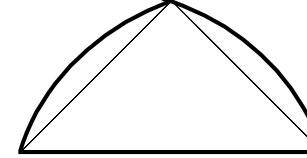
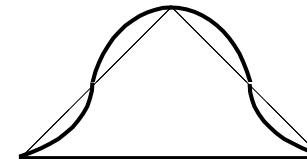
*Degree of Membership*



# Representation of hedges in fuzzy logic

<i>Hedge</i>	<i>Mathematical Expression</i>	<i>Graphical Representation</i>
A little	$[\mu_A(x)]^{1.3}$	
Slightly	$[\mu_A(x)]^{1.7}$	
Very	$[\mu_A(x)]^2$	
Extremely	$[\mu_A(x)]^3$	

# Representation of hedges in fuzzy logic (continued)

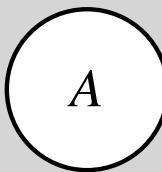
<i>Hedge</i>	<i>Mathematical Expression</i>	<i>Graphical Representation</i>
Very very	$[\mu_A(x)]^4$	
More or less	$\sqrt{\mu_A(x)}$	
Somewhat	$\sqrt{\mu_A(x)}$	
Indeed	$2 [\mu_A(x)]^2$ if $0 \leq \mu_A \leq 0.5$ $1 - 2 [1 - \mu_A(x)]^2$ if $0.5 < \mu_A \leq 1$	

# Operations of fuzzy sets

The classical set theory developed in the late 19th century by Georg Cantor describes how crisp sets can interact. These interactions are called **operations**.

# Cantor's sets

*Not A*



Complement

*B*

*A*

Containment

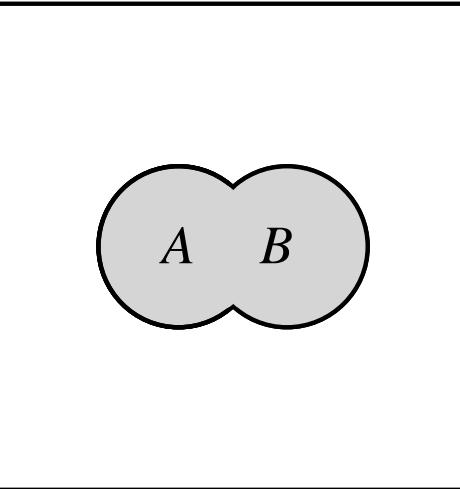
*A* *B*



Intersection

Union

*A* *B*



## ■ Complement

*Crisp Sets:* *Who does not belong to the set?*

*Fuzzy Sets:* *How much do elements not belong to the set?*

The complement of a set is an opposite of this set. For example, if we have the set of *tall men*, its complement is the set of *NOT tall men*. When we remove the tall men set from the universe of discourse, we obtain the complement. If  $A$  is the fuzzy set, its complement  $\neg A$  can be found as follows:

$$\mu_{\neg A}(x) = 1 - \mu_A(x)$$

## ■ Containment

*Crisp Sets*: Which sets belong to which other sets?

*Fuzzy Sets*: Which sets belong to other sets?

Similar to a Chinese box, a set can contain other sets. The smaller set is called the **subset**. For example, the set of *tall men* contains all tall men; *very tall men* is a subset of *tall men*. However, the *tall men* set is just a subset of the set of *men*. In crisp sets, all elements of a subset entirely belong to a larger set. In fuzzy sets, however, each element can belong less to the subset than to the larger set. Elements of the fuzzy subset have smaller memberships in it than in the larger set.

## ■ Intersection

Crisp Sets: *Which element belongs to both sets?*

Fuzzy Sets: *How much of the element is in both sets?*

In classical set theory, an intersection between two sets contains the elements shared by these sets. For example, the intersection of the set of *tall men* and the set of *fat men* is the area where these sets overlap. In fuzzy sets, an element may partly belong to both sets with different memberships. A fuzzy intersection is the lower membership in both sets of each element. The fuzzy intersection of two fuzzy sets  $A$  and  $B$  on universe of discourse  $X$ :

$$\mu_A \cap_B (x) = \min [\mu_A(x), \mu_B(x)] = \mu_A(x) \cap \mu_B(x),$$

where  $x \in X$

## ■ Union

Crisp Sets: Which element belongs to either set?

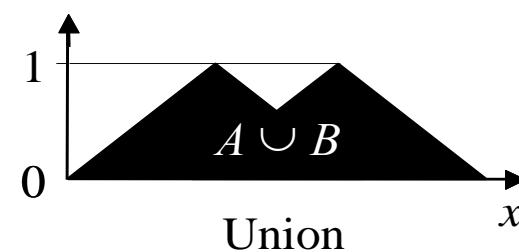
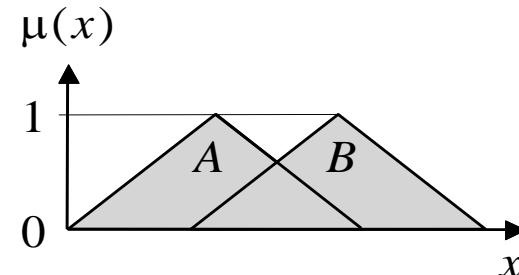
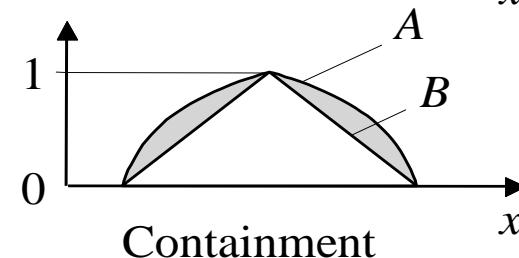
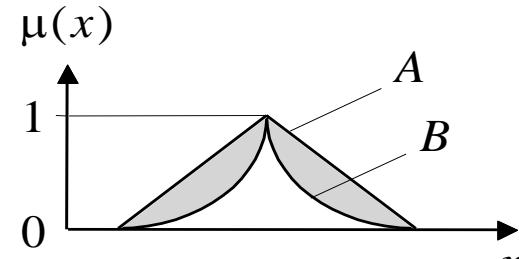
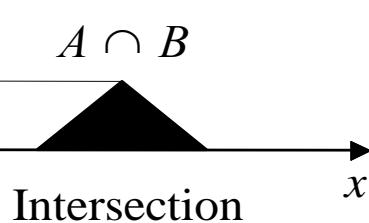
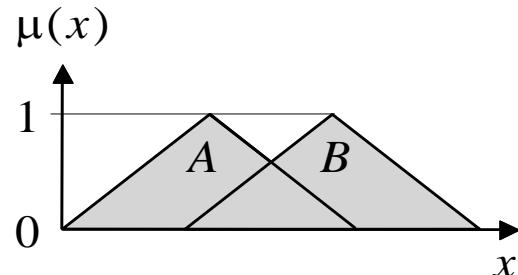
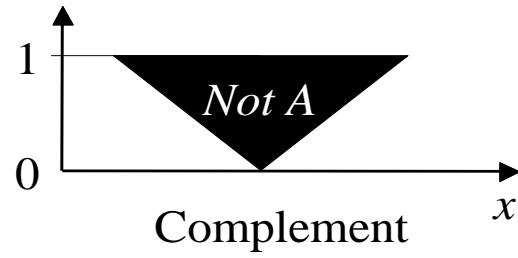
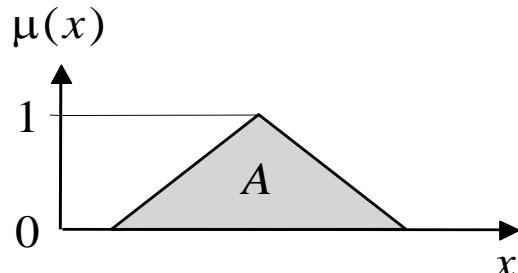
Fuzzy Sets: How much of the element is in either set?

The union of two crisp sets consists of every element that falls into either set. For example, the union of *tall men* and *fat men* contains all men who are tall **OR** fat. In fuzzy sets, the union is the reverse of the intersection. That is, the union is the largest membership value of the element in either set. The fuzzy operation for forming the union of two fuzzy sets  $A$  and  $B$  on universe  $X$  can be given as:

$$\mu_A \cup_B (x) = \max [\mu_A(x), \mu_B(x)] = \mu_A(x) \cup \mu_B(x),$$

where  $x \in X$

# Operations of fuzzy sets



# Fuzzy rules

In 1973, **Lotfi Zadeh** published his second most influential paper. This paper outlined a new approach to analysis of complex systems, in which Zadeh suggested capturing human knowledge in fuzzy rules.

# What is a fuzzy rule?

A fuzzy rule can be defined as a conditional statement in the form:

**IF**  $x$  is  $A$   
**THEN**  $y$  is  $B$

where  $x$  and  $y$  are linguistic variables; and  $A$  and  $B$  are linguistic values determined by fuzzy sets on the universe of discourses  $X$  and  $Y$ , respectively.

# What is the difference between classical and fuzzy rules?

A classical IF-THEN rule uses binary logic, for example,

Rule: 1

IF speed is > 100

THEN stopping\_distance is long

Rule: 2

IF speed is < 40

THEN stopping\_distance is short

The variable *speed* can have any numerical value between 0 and 220 km/h, but the linguistic variable *stopping\_distance* can take either value *long* or *short*. In other words, classical rules are expressed in the black-and-white language of Boolean logic.

We can also represent the stopping distance rules in a fuzzy form:

Rule: 1

IF speed is fast

THEN stopping\_distance is long

Rule: 2

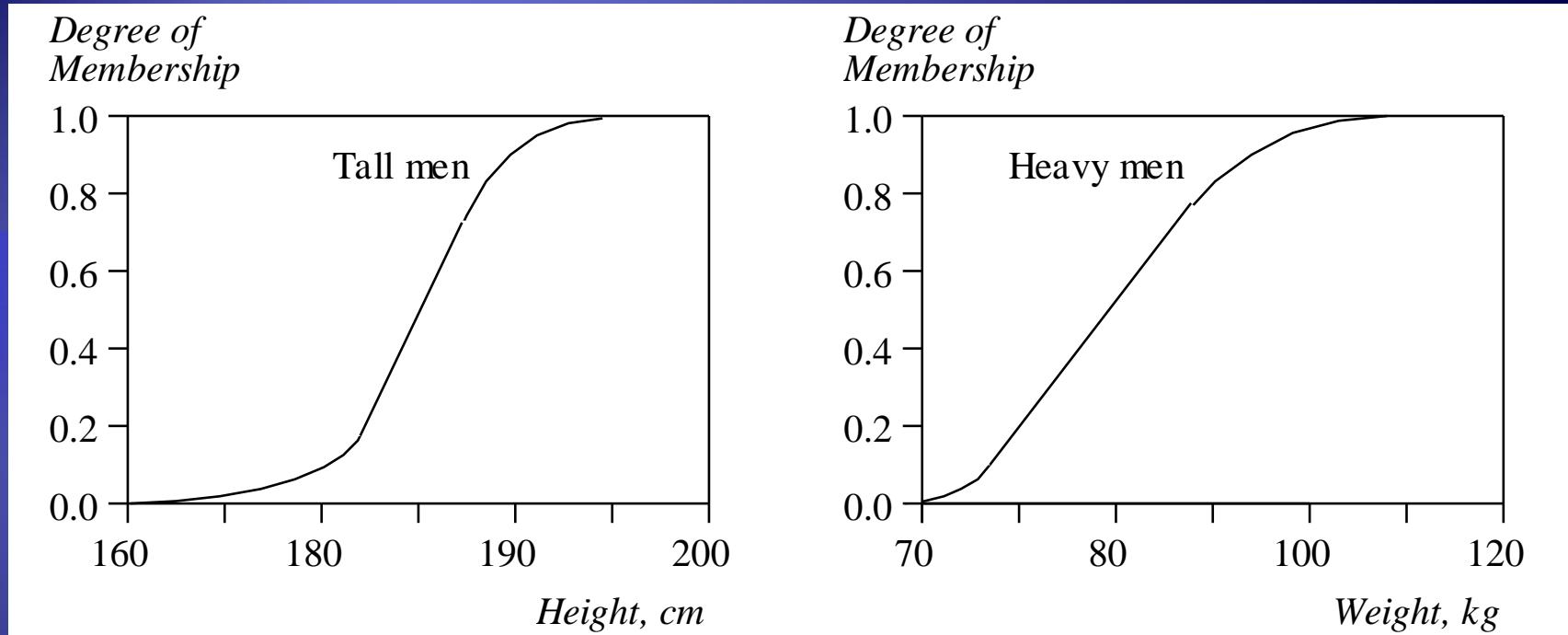
IF speed is slow

THEN stopping\_distance is short

In fuzzy rules, the linguistic variable *speed* also has the range (the universe of discourse) between 0 and 220 km/h, but this range includes fuzzy sets, such as *slow*, *medium* and *fast*. The universe of discourse of the linguistic variable *stopping\_distance* can be between 0 and 300 m and may include such fuzzy sets as *short*, *medium* and *long*.

- Fuzzy rules relate fuzzy sets.
- In a fuzzy system, all rules fire to some extent, or in other words they fire partially. If the antecedent is true to some degree of membership, then the consequent is also true to that same degree.

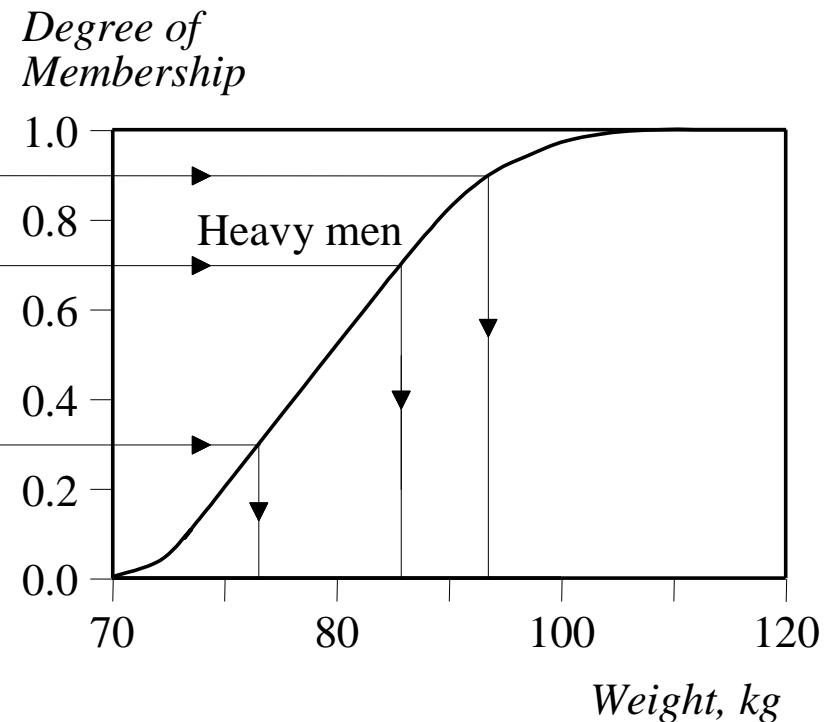
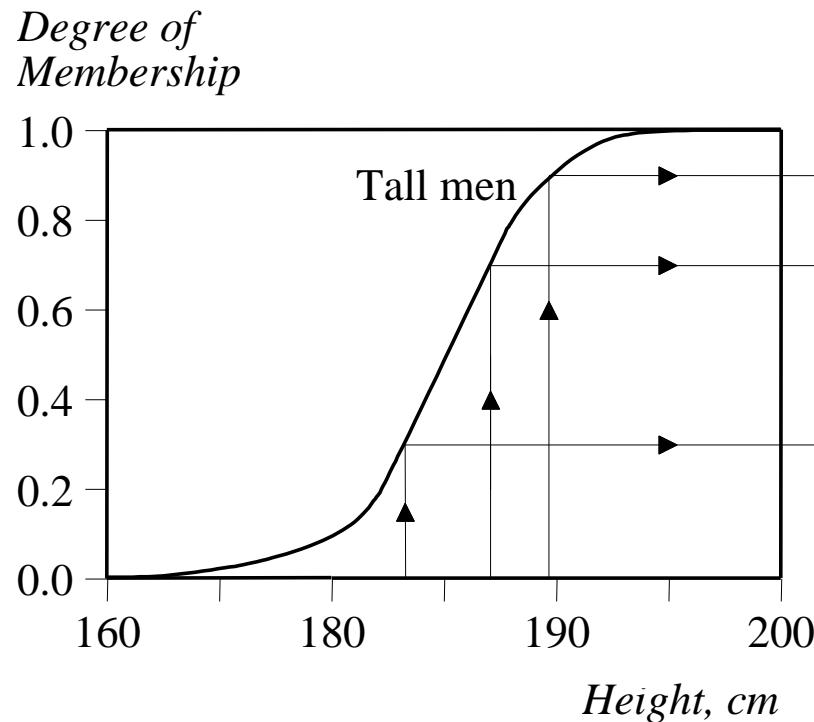
# Fuzzy sets of *tall* and *heavy* men



These fuzzy sets provide the basis for a weight estimation model. The model is based on a relationship between a man's height and his weight:

IF        **height is *tall***  
THEN    **weight is *heavy***

The value of the output or a truth membership grade of the rule consequent can be estimated directly from a corresponding truth membership grade in the antecedent. This form of fuzzy inference uses a method called **monotonic selection**.



A fuzzy rule can have multiple antecedents, for example:

IF project\_duration is long  
AND project\_staffing is large  
AND project\_funding is inadequate  
THEN risk is high

IF service is excellent  
OR food is delicious  
THEN tip is generous

The consequent of a fuzzy rule can also include multiple parts, for instance:

IF        temperature is hot  
THEN    hot\_water is reduced;  
          cold\_water is increased