Greedy approach to Egyptian fraction

Author : Oksana Dura

April 2021

1 Introduction to the Egyptian fraction

An Egyptian fraction is a finite sum of distinct unit fractions. Below is an example of the of this method:

$$\frac{5}{8} = \frac{1}{2} + \frac{1}{8} \tag{1}$$

As it can be seen from above in the expression each fraction has a numerator equal to 1 and a denominator that is a positive integer. This type of method does not allow the expression to have fractions with the same denominator, they have to be different. The value of an expression of this type has to be a rational number in the form $\frac{a}{b}$. Every positive rational number can be represented by an Egyptian fraction.

1.0.1 Problem

Implementation of the Egyptian fraction. The program takes input from the user positive rational fraction expressed in the format $\frac{a}{b}$ and outputs that fraction as the sum of distinct unit fractions. The task requires that I use greedy approach when solving the problem.

$$\begin{aligned} \frac{30}{31} = {} & \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \frac{1}{42} + \frac{1}{56} + \frac{1}{72} + \frac{1}{90} + \\ & \frac{1}{110} + \frac{1}{132} + \frac{1}{156} + \frac{1}{182} + \frac{1}{210} + \frac{1}{240} + \frac{1}{272} + \\ & \frac{1}{306} + \frac{1}{342} + \frac{1}{380} + \frac{1}{420} + \frac{1}{462} + \frac{1}{506} + \frac{1}{552} + \\ & \frac{1}{600} + \frac{1}{650} + \frac{1}{702} + \frac{1}{756} + \frac{1}{812} + \frac{1}{870} + \frac{1}{930} \end{aligned}$$

Figure 1: Egyptian fraction

2 Greedy approach

For the homework it was required to approach the Egyptian fraction using the greedy algorithm. The greedy algorithm consists on subtracting the largest possible uniform fraction from the given fraction. The condition is that the difference between the given fraction and the uniform fraction is non-negative. The following theorem is useful for the problem.

Theorem 1 *Subtracting the largest possible unit fraction that keeps a non negative difference from a proper irreducible fraction* $\frac{a}{b}$ *, that is, such that* $\frac{a}{b} - \frac{1}{k} \geq 0$, *the resulting fraction numerator is strictly less than the one of the initial fraction and the resulting fraction denominator is strictly greater than the one of the initial fraction.*

The theorem is helpful for understanding the problem and for giving an initial idea on how the output will look like. The denominators of the expected output will become bigger hence the fraction will become significantly smaller.

2.0.1 Limitations

- Numerator is greater then zero, because dividing zero with a positive integer gives zero hence cannot be represented as an Egyptian fraction.
- Division with zero is not possible.
- Denominator needs to be greater than 0.

2.0.2 Approaches for solving the problem:

- If the numerator divides the denominator with module equal to zero, then the fraction can be expressed as $\frac{1}{\frac{b}{a}}$.
- If the denominator divides the numerator with module equal to zero, then the output is an integer not a fraction.
- If numerator is greater than denominator then we get an integer and the module of dividing the numerator with the denominator becomes the new numerator and the denominator remains the same. The values loop again throw the function.
- The numerator is less then the denominator. In this case a new value is introduced. The new value n is the ceiling of the division between the denominator and the numerator. The fraction $\frac{1}{n}$ is the biggest fraction that when subtracting from the initial fraction is bigger/equal to zero. After having subtracted this fraction from the initial one, we are left with a new fraction $\frac{oldNumerator * n - oldDenominator}{oldDenominator * n}$. The new fraction recurs again in the function.

3 Tests:

Figure 2: The user tries to input values for the numerator less than 1

```
C:\Users\User\Desktop\EgyptianFraction\cmake-build-debug\EgyptianFraction.exe
Enter the numerator:-1
Enter the numerator:0
Enter the numerator:-1
Enter the numerator:-11222
Enter the numerator:
```

Figure 3: The user tries to input values for the denominator less than 1.

```
C:\Users\User\Desktop\EgyptianFraction\cmake-build-debug\EgyptianFraction.exe
Enter the numerator:1
Enter the denominator:0
Enter the denominator:0
Enter the denominator:0
Enter the denominator:-1
Enter the de
nominator:-1
Enter the denominator:
```

Figure 4: Case when the division of the denominator with the numerator has
has module zero.

```
C:\Users\User\Desktop\EgyptianFraction\cmake-build-debug\EgyptianFraction.exe
Enter the numerator:6
Enter the denominator:12
The fraction: 6/12 can be represented with the Egyptian method as:
 1/2
Process finished with exit code 0
```

Figure 5: Case when the division of the numerator with the denominator has has module zero.

```
C:\Users\User\Desktop\EgyptianFraction\cmake-build-debug\EgyptianFraction.exe
Enter the numerator:100
Enter the denominator:2
The fraction: 100/2 can be represented with the Egyptian method as:
 an integer 50
Process finished with exit code 0
```

Figure 6: Case the numerator is bigger than the denominator.

```
C:\Users\User\Desktop\EgyptianFraction\cmake-build-debug\EgyptianFraction.exe
Enter the numerator:25
Enter the denominator:4
The fraction: 25/4 can be represented with the Egyptian method as:
 6 + 1/4
Process finished with exit code 0
```

Figure 7: Case the denominator is bigger than the numerator.

```
C:\Users\User\Desktop\EgyptianFraction\cmake-build-debug\EgyptianFraction.exe
Enter the numerator:27
Enter the denominator:89
The fraction: 27/89 can be represented with the Egyptian method as:
 1/4 + 1/19 + 1/1353 + 1/9151692
Process finished with exit code 0
```

4 Conclusions

Greedy approach of the Egyptian fraction consist of finding the largest possible unit fraction such that $\frac{a}{b} - \frac{1}{k} \geq 0$.

Automatically we get a question is the greedy approach the best method to be used in solving the Egyptian problem? In order to reach to a conclusion we consider some examples.

Example 1 Consider the fraction $\frac{3}{7}$. Expressing it as an Egyptian fraction using the greedy approach we get:

$$\frac{3}{7} = \frac{1}{3} + \frac{1}{11} + \frac{1}{231} \tag{2}$$

But it can also be expressed as:

$$\frac{3}{7} = \frac{1}{6} + \frac{1}{7} + \frac{1}{14} + \frac{1}{21} \tag{3}$$

We noticed that the denominators in the second method are smaller than the denominators in the first method, but we have added an extra fraction.
The same fraction can be expressed as:

$$\frac{3}{7} = \frac{1}{4} + \frac{1}{7} + \frac{1}{28} \tag{4}$$

We notice that for the same number of components the denominators in the third example are relatively smaller than the ones in the first example. From this example it is easy to see that the greedy approach is not the best to be used to find the Egyptian fraction having relatively small denominators.

Now we need to consider if the greedy method is the best approach at finding the shortest expansion.
Consider the fraction $\frac{5}{21}$. Expressing it as an Egyptian fraction using the greedy approach we get:

$$\frac{5}{21} = \frac{1}{25} + \frac{1}{757} + \frac{1}{763309} + \frac{1}{873960180913} + \frac{1}{1025410058030422033} \tag{5}$$

But it can also be expressed as the following:

$$\frac{5}{21} = \frac{1}{33} + \frac{1}{121} + \frac{1}{363} \tag{6}$$

From the examples above it is easy to reach to the following conclusions:

- The greedy approach works for all proper functions.
- The greedy approach has limitations.
- The denominators of the fractions can grow really big.
- It is not always the shortest expansion.

References

"Algorithms for Egyptian Fractions." Wolfram Demonstrations Project, 2007,doi:10.3840/001716.

Wagon, Stan. "Egyptian Fractions." Mathematica® in Action, 1999, pp. 321–329., doi:10.1007/978-1-4612-1454-$0_1$6.