Finite Difference Taylor Series

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Abstract. We prove the equivalence of a Taylor series definition of the finite difference operation to its standard. The implication of approximation to the derivative is more explicit, and as a result, direct claims made of approximation-related properties of the finite difference operation can be proven with greater ease.

1 Equality Theorem and Proof

Theorem 1:
Given some $x \in \mathbb{R}$ and $h \in \mathbb{R}$, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous polynomial function on the interval $[x, x+h], h \in \mathbb{R}$. The following relationship then holds:

$$\Delta_h [f] (x) = \sum_{k=1}^{\deg f} \frac{h^k}{k!} f^{(k)} (x), \forall x \in \mathbb{R} \tag{1.1}$$

Proof. By induction on the degree of the input function f, we'll show Theorem 1.1 to be true.

The base case is when f is a constant function (ie: has a degree of 0). Theorem 1.1 is trivial for this case as the difference between terms of a constant is 0, and likewise is the derivative of a constant. To support the induction, we'll first assume that Theorem 1.1 is true for some $n \in \mathbb{N}$. Let's first redefine $f(x)$ to have a degree of $m = n + 1$. Using the Taylor series expansion of a general polynomial function and the definition of the finite Forward Difference Operation (f.d.o), we can express the f.d.o of $f(x)$ as the following:

$$\Delta_h [f] (x) = \sum_{k=0}^{m} \left(C_k (x+h)^k \right) - \sum_{k=0}^{m} \left(C_k x^k \right) \tag{1.2}$$

Using the binomial theorem, we have this expansion of $f(x+h)$:

$$f(x+h) = \sum_{k=0}^{m} \left(C_k \sum_{j=0}^{k} \binom{k}{j} x^{k-j} h^j \right) \tag{1.3}$$

To simplify the f.d.o expression, we'll need to extract $f(x)$ from the summation. This is done simply by incrementing j by 1 in $f(x+h)$ and writing in the first term explicitly:

$$f(x+h) = \sum_{k=0}^{m} \left(C_k \left[x^k + \sum_{j=1}^{k} \binom{k}{j} x^{k-j} h^j \right] \right) \tag{1.4}$$

From here we split the summation into two parts:

$$f(x+h) = \sum_{k=0}^{m} \left(C_k x^k \right) + \sum_{k=0}^{m} \left(C_k \sum_{j=1}^{k} \binom{k}{j} x^{k-j} h^j \right) \tag{1.5}$$

This allows us to cancel out $f(x)$ in the f.d.o giving us the following equality:

$$\Delta_h [f](x) = \sum_{k=0}^{m} \left(C_k \sum_{j=1}^{k} \binom{k}{j} x^{k-j} h^j \right) \tag{1.6}$$

Noticing now that the inner summation is empty for $k=0$, we can increment k by 1:

$$\Delta_h [f](x) = \sum_{k=1}^{m} \left(C_k \sum_{j=1}^{k} \binom{k}{j} x^{k-j} h^j \right) \tag{1.7}$$

Once again using the binomial theorem we expand the inner summation and move the coefficient into the summation. Further, we can move $\frac{1}{j!}$ into the h^j term:

$$\Delta_h [f](x) = \sum_{k=1}^{m} \left(\sum_{j=1}^{k} \left(\frac{h^j}{j!} \right) C_k \left(\frac{k!}{(k-j)!} \right) x^{k-j} \right) \tag{1.8}$$

We'll now need to use the concept of expressing a polynomial function as a sum of its ordered parts to reconcile the inner summation with a derivative over the entire function. To do this, we'll first need to define the ordered parts of a polynomial function. Then we'll need to define the n'th derivative of a polynomial function. We'll use these definitions to simplify the inner summation.

Definition 1. Let f_n be the notation for the n'th degree part of a polynomial function $f : \mathbb{R} \to \mathbb{R}$ of degree m. Simply put:

$$f_n(x) = C_n x^n \tag{1.9}$$

The sum of these parts is equivalent to the original function:

$$f(x) = \sum_{k=0}^{m} f_k(x) \tag{1.10}$$

Therefore the derivative of these parts is equivalent to the derivative of the original function:

$$f^{(n)}(x) = \sum_{k=0}^{m} f_k^{(n)}(x) \tag{1.11}$$

Definition 2. The nth derivative of some function $f : \mathbb{R} \to \mathbb{R}$ of degree m can be expressed as the following:

$$f^{(n)}(x) = \sum_{k=n}^{m} \frac{k!}{(k-n)!} C_k x^{k-n} \tag{1.12}$$

Returning to the proof, we notice that the inner summation is equivalent to the j'th derivative of the k'th degree part of f, so we can use Definition 1 to simplify the expression:

$$\Delta_h[f](x) = \sum_{k=1}^{m} \left(\sum_{j=1}^{k} \left(\frac{h^j}{j!} \right) f_k^{(j)}(x) \right) \tag{1.13}$$

Finally, we can use Definition 1 (1.11) to conclude that the inner summation is equivalent to the k'th derivative of f multiplied by $\frac{h^k}{k!}$:

$$\Delta_h[f](x) = \sum_{k=1}^{m} \left(\frac{h^k}{k!} f^{(k)}(x) \right) \tag{1.14}$$

Equation 1.14 is equivalent to our inductive hypothesis, so we can conclude that Theorem 1.1 is true for all $n \in \mathbb{N}$. $\square$