Finite Difference Taylor Series

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Abstract. We rediscover Taylor's definition of the finite difference operation and then prove the equivalence of this definition to the operation's standard definition. The implication of approximation to the derivative is more explicit, and as a result, direct claims made of approximation-related properties of the finite difference operation can be proven with greater ease.

1 Equality Theorem and Proof

Theorem 1:
Given some $x \in \mathbb{R}$ and $h \in \mathbb{R}$, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous polynomial function on the interval $[x, x+h], h \in \mathbb{R}$. The following relationship then holds:

$$\Delta_h [f] (x) = \sum_{k=1}^{\deg f} \frac{h^k}{k!} f^{(k)} (x), \forall x \in \mathbb{R} \tag{1.1}$$

Proof. By induction on the degree of the input function f, we'll show Theorem 1.1 to be true.

The base case is when f is a constant function (ie: has a degree of 0). Theorem 1.1 is trivial for this case as the difference between terms of a constant is 0, and likewise is the derivative of a constant. To support the induction, we'll first assume that Theorem 1.1 is true for some $n \in \mathbb{N}$. Let's first redefine $f(x)$ to have a degree of $m = n + 1$. Using the Taylor series expansion of a general polynomial function and the definition of the finite Forward Difference Operation (f.d.o), we can express the f.d.o of $f(x)$ as the following:

$$\Delta_h [f] (x) = \sum_{k=0}^{m} \left(C_k (x+h)^k \right) - \sum_{k=0}^{m} \left(C_k x^k \right) \tag{1.2}$$

Using the binomial theorem, we can expand $f(x+h)$ as the following:

$$f\left(x+h\right)=\sum_{k=0}^{m}\left(C_k\sum_{j=0}^{k}\binom{k}{j}x^{k-j}h^j\right) \tag{1.3}$$

To simplify the f.d.o expression, we'll need to extract $f\left(x\right)$ from the summation. This is simply done by incrementing j by 1 and writing in the first term explicitly:

$$f\left(x+h\right)=\sum_{k=0}^{m}\left(C_k\left[x^k+\sum_{j=1}^{k}\binom{k}{j}x^{k-j}h^j\right]\right) \tag{1.4}$$

From here we split the summation into two parts:

$$f\left(x+h\right)=\sum_{k=0}^{m}\left(C_k x^k\right)+\sum_{k=0}^{m}\left(C_k\sum_{j=1}^{k}\binom{k}{j}x^{k-j}h^j\right) \tag{1.5}$$

This allows us to cancel out $f\left(x\right)$ in the f.d.o giving us the following expression:

$$\Delta_h\left[f\right]\left(x\right)=\sum_{k=0}^{m}\left(C_k\sum_{j=1}^{k}\binom{k}{j}x^{k-j}h^j\right) \tag{1.6}$$

Noticing now that the inner summation is empty for $k=0$, we can increment k by 1:

$$\Delta_h\left[f\right]\left(x\right)=\sum_{k=1}^{m}\left(C_k\sum_{j=1}^{k}\binom{k}{j}x^{k-j}h^j\right) \tag{1.7}$$

We'll now need to use the concept of expressing a polynomial function as a sum of its ordered parts to reconcile the inner summation with a derivative over the entire function.

Definition 1.1. Let f_n be the notation for the n'th degree part of a polynomial function $f:\mathbb{R}\to\mathbb{R}$ of degree m. Simply put:

$$f_n\left(x\right)=C_n x^n \tag{1.1.1}$$

The sum of these parts is equivalent to the original function:

$$f\left(x\right)=\sum_{k=0}^{m}f_k\left(x\right) \tag{1.1.2}$$

Therefore the derivative of these parts is equivalent to the derivative of the original function:

$$f^{(n)}(x) = \sum_{k=0}^{m} f_k^{(n)}(x) \tag{1.1.3}$$

Definition 1.2. The nth derivative of some function $f : \mathbb{R} \to \mathbb{R}$ of degree m can be expressed as the following:

$$f_m^{(n)}(x) = \sum_{k=n}^{m} \frac{k!}{(k-n)!} C_k x^{k-n} \tag{1.2.1}$$

Once again using the binomial theorem we expand the inner summation and move the coefficient into the summation. Further, we can move $\frac{1}{j!}$ into the h^j term:

$$\Delta_h[f](x) = \sum_{k=1}^{m} \left(\sum_{j=1}^{k} C_k \left(\frac{h^j}{j!} \right) \left(\frac{k!}{(k-j)!} \right) x^{k-j} \right) \tag{1.2}$$

Noticing that the inner summation is equivalent to the j'th derivative of the k'th degree part of f, we can use Definition 1.2 to simplify the expression:

$$\Delta_h[f](x) = \sum_{k=1}^{m} \left(\sum_{j=1}^{k} f_k^{(j)}(x) \left(\frac{h^j}{j!} \right) \right) \tag{1.3}$$

Finally, we can use Definition 1.3 (1.1.3) to conclude that ...