Set Theory

Part 3

What is a Set?

Organizing Information

What is a Set?

- A *set* is an unordered collection of "objects"
- The collection objects are also called "members" or "elements"
- One of the most fundamental structures in mathematics

Set Notation

- We typically denote a set name using capital letter
- Members are separated with commas and encapsulated within curly brackets

Famous Sets (in Mathematics)

Letter	Name	Members
Z	Integers	…, -2, -1, 0, 1, 2, 3, …
N	Natural Numbers	1, 2, 3, 4, …
Q	Rational Numbers	a / b where both a and b are integers and b is not 0

Famous Sets (in Mathematics)

Letter	Name	Members
R	Real Numbers	All non-imaginary numbers. e.g. 1, 2.5, 3.1415….
U	Universal Set	All values of potential interest (U depends on context)

Set Notation: Membership

- Set notation uses a special symbol to denote if an object is a member of a set
- Below, the set V contains vegetables

```
potato ∈ V
```


Set Notation: Membership

- This is read as "potato is an element of V"
- …or "potato is a member of V"

```
potato ∈ V
```


Set Notation: Not a Member

- There is another special symbol that denotes an object is not a member of a set
- In the example below, the set F contains fluffy animals

```
lizard ∉ F
```


Defining Sets

How to specify items

Defining Sets

- Sets can be defined a number of different ways
- Each competing notation has advantages & disadvantages – depending on what you are defining

Set Notation: Explicit

- We can *explicitly* define this by listing each element
- For example, we can define a set *S* for members of the Three Stooges

```
S = {moe, larry, curly, shemp}
```

Set Notation: Pattern

- We can also specify a set by using a *pattern*.
- In the example below we are define a set of integers between 0 and 9.

```
A = {0, 1, 2, ... 9}
```

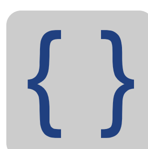

Empty Set

- An *empty set* contains no elements
- Can be represented with two curly-brackets (nothing in between)
- There is also a special symbol for empty sets

```
A = { }
A = ∅
```


Set Builder Notation

- A set can also be defined using *set builder notation*
- Consists of a variable name, a pipe symbol, and an true/false expression



By Characteristic

- The most basic form consists of a variable and an true/false statement
- In this example, everything that satisfies "x is an even integer" will be the set

```
{x | x is a even integer}
```


By Characteristic Examples

Expression	Result
{ x | x is an integer }	{ …, -1, 0, 1, 2, 3, … }
{ x | x is an even integer }	{ …, -2, 0, 2, 4, 6, … }
{ x | x is odd natural number}	{ 1, 3, 5, 7, 9, … }

Characteristic with Restriction

- Definitions can also be restricted by another set
- There are two different notations that *mean the same thing*

```
{x ∈ S | true/false expression on x}

{x | x ∈ S and true/false expression on x}
```

Characteristic Example

- Remember, Z is the set of integers in math
- It reads: *"All x where x is in Z and x is even"*

```
A = {x | x ∈ Z ∧ x is even}
```


By Characteristic Examples

Expression	Result
$\{ x \in Z \mid 0 < x < 5 \}$	{1, 2, 3, 4}
$\{ x \mid x \in N \wedge x < 7 \}$	{1, 2, 3, 4, 5, 6}

Characteristic with Structure

- The left-hand-side (before the pipe) doesn't have to just a variable name
- It can also be any mathematical expression

```
{f(x) | true/false expression using x}

{y | y = f(x) ∧ true/false using x}
```


Let's Try One…

- The second part of the notation must always be a true/false expression
- So, how do we create a set that contains:

{2, 4, 6, 8, 10, …}

Let's Try One…

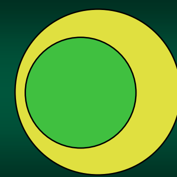
```
First approach:
A = {x | x ∈ N ∧ x is even}

Second approach:
A = {2x | x ∈ N}
```


How Does It Evaluate?

- Basically, when you look at something like: $\{ 2x \mid x \in N \}$, you should do the following
- Steps:
  1. Identify which variables make the right-hand-side true
  2. Plug them into the left-hand-side. These are the values in the set.

More Examples

Expression	Result
$\{ 2x + 1 \mid x \in Z \}$	$\{\ldots, -3, -1, 1, 3, 5, \ldots\}$
$\{ x \in Z \mid \text{sqrt}(x) \in Z \}$	$\{0, 1, 4, 9, 16, 25, \ldots\}$

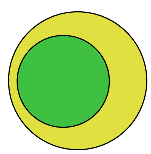


Subsets

Set Your Sets to Fun

Subsets

- Commonly, sets are compared to one another using set relationship operators
- Basically, set are defined on elements which they may have in common



Subsets

- Set A is considered a subset of set B if all the members of A are also members of B
- The subset operator is similar looking to the member operator

$$\{ 1, 4 \} \subseteq \{ 1, 3, 4, 5 \}$$

Subsets

- A set A is not a subset of B if A contains an element not found in B

$$\{ 3, 5 \} \nsubseteq \{ 3, 7 \}$$

Null is Always a Subset

- A null set contains no elements
- Hence, the null set is always a subset

$$\varnothing \subseteq \{ 2, 3 \}$$
$$\{\} \subseteq \{ 2, 3 \}$$

Proper Subsets

- Set A is a *proper subset* of B if A <u>is</u> a subset of B, but <u>not</u> equal to B
- Note: the notation lacks the underline – it is consistent with other operators like < and ≤

$\{\ 3,\ 5\ \} \subset \{\ 3,\ 5,\ 7\ \}$

$\{\ 1,\ 2\ \} \not\subset \{\ 1,\ 2\ \}$

Equality

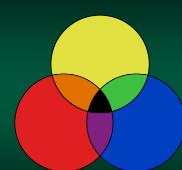
- Sets A and B are considered equal if-and-only-if…

for every $x \in A$ it is also true $x \in B$

for every $x \in B$ it is also true $x \in A$

Equality

- So, are { 1, 2, 3 } and { 2, 1, 3 } equal?
- How about { 1, 1, 2, 3, 3 } and { 3, 2, 1 }
- Answer is *yes!*
  - order does not matter in a set
  - multiple occurrences does not change if an element is a member

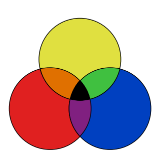


Venn Diagrams

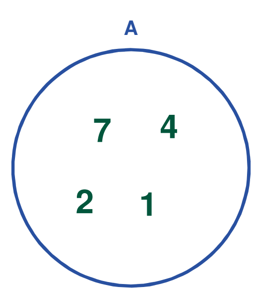
Graphically Representing Sets

Venn Diagrams

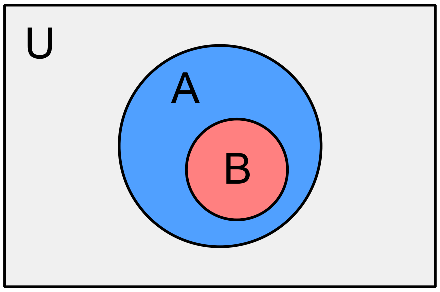
- Sets can also be abstractly representing graphically using Venn Diagrams
- Each set is represented by circle
- Overlaps between each set can show logical relations with set members



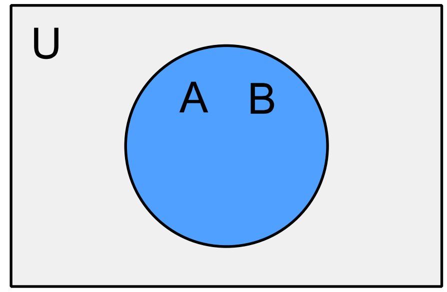
Example: A = {2, 7, 1, 4}



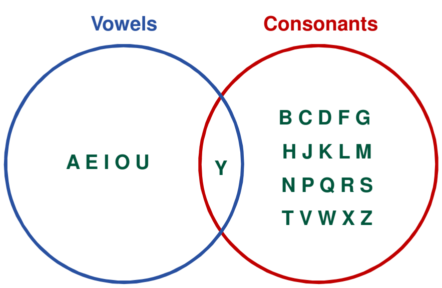
Subset Venn Diagram

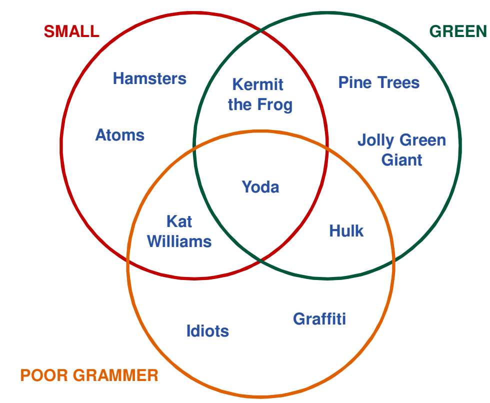


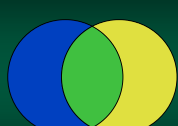
Equality



Example: Letters in English





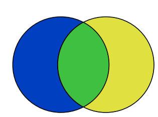


Set Operators

Defining Sets Using Sets

Operations on Sets

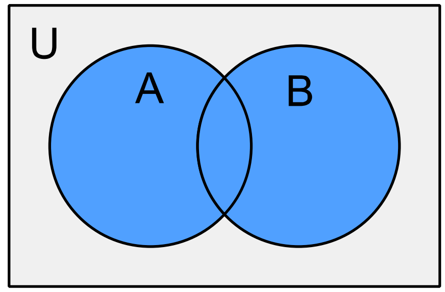
- New sets can be made from old sets using set operators.
- Just like new numbers can be created from old numbers: 1 + 2 = 3
- So, for the rest of this section, let U be the universe, and let A and B be sets



Union

- A union of two sets combines all members of each set into a new one
- So, the result is two merged sets
- $A \cup B = \{ x \mid x \in A \vee x \in B \}$
- The symbol ∪ looks like U
  - which is also used for the "universe set"
  - be careful not the confuse the two

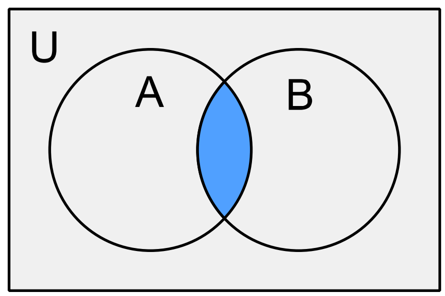
A ∪ B



Intersection

- The intersection of two sets contains only those elements that are found in both sets
- So, the result is where the two sets overlap
- $A \cap B = \{ x \mid x \in A \wedge x \in B \}$

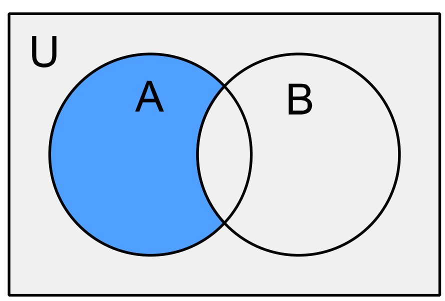
A ∩ B



Difference

- *Difference* (aka exclusion) removes all items found in set from another
- Typically, it is written as A – B even though it is not the same as subtraction
- $A - B = \{ x \mid x \in A \wedge x \notin B \}$

A - B



Difference – So Many Notations

- The notation for difference varies greatly
- Below are two different variations on the same notation

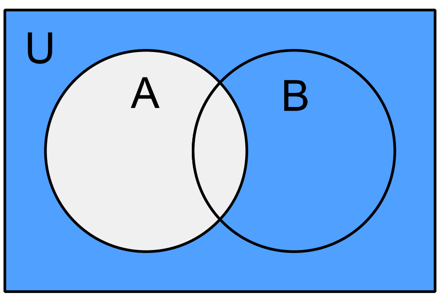
```
A – B
A \ B
```


Complement

- The *complement* of a set A, is all elements in the Universe, not in A
- Remember: what elements are in the Universe depends on the sets

$A' = \{ x \mid x \notin A \}$

A'



Different Notations Used

- Single postfix apostrophe
- An "over bar" (which is underlining on top)
- Superscript "c" for complement

$A' = \overline{A} = A^c$

Complement Example

- If set A is a subset of a set B, then the complement of A is all elements not in A but still in B
- Look at the following:

$A \subset B$
$A = \{1, 2, 3\}$
$B = \{1, 2, \dots 10\}$

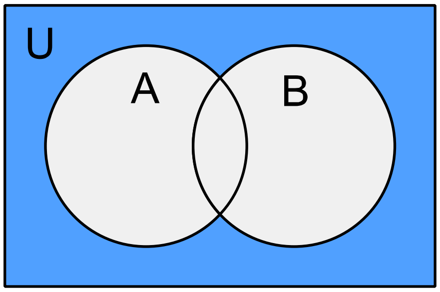
Complement Example

- Set A is a subset of a set B
- Therefore its "universe" is defined as the set of B

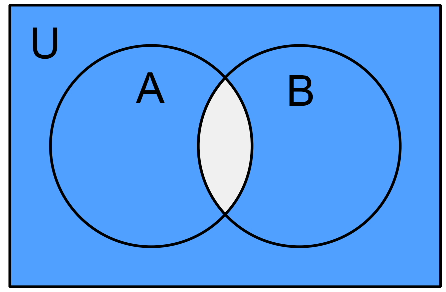
Therefore…

$A' = \{4, 5, 6, 7, 8, 9, 10\}$

(A ∪ B) '

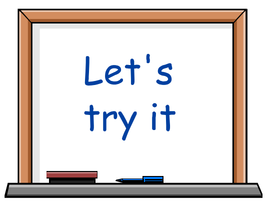


(A ∩ B) '



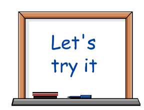
Let's Draw Some…

- Let's draw some Venn Diagrams using a several sets
- Using set operators we can highlight any area we want



Let's Draw Some

```
U = {a, b, c, d, e, f}

A = {a, b, c}
B = {b, c, d, e}
```



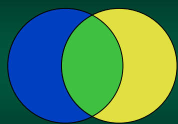
Let's Draw Some

```
Two sets:

In A but not in B: {a}
In B but not in A: {d, e}
In both A and B: {b, c}
In neither A nor B: {f}
```


Reasoning with Venn Diagrams

Say 100 people receive a questionnaire with two questions (1) Do you program Java?, (2) Do you program C# ?

If 65 said 'yes' to Java and 40 say 'yes' to both. How many program just program C#?



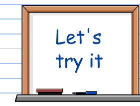
Set Algebra

Just a preview…

Let's Look at These…

Let's see if the following are the same:

$A \cup (B \cap C)$



$(A \cup B) \cap (A \cup C)$

Set Algebra

- We will cover more of this later, but set algebra shares the same principles as basic math
- You can visually treat the union as a "*" and the intersection as a "+"
- You can then factor out sets

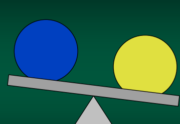
Another Look

$(A \cup B) \cap (A \cup C)$

$\rightarrow (A * B) + (A * C)$

$= A * (B + C)$

$A \cup (B \cap C)$

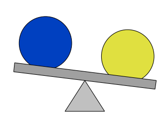


Cardinality

Counting Sets

Cardinality of a Set

- The *cardinality* of a set is the number of *distinct* elements
- This information is used in counting – the classification of the set's contents



Different Notations Used

- There are two different notations used
- The most common is the | pipe delimiters
- Alternatively, the "n" function is used

$$|A| \equiv n(A)$$

Examples

```
A = {1, 3, 5, 7}

|A| = 4
```


Examples

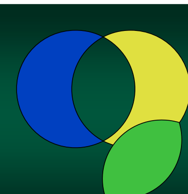
```
B = {1, 2, 3, 3, 3, 4}

|B| = 4
```


Counting

- If the set contains a finite number of elements, it is said to be *countable* – i.e. the cardinality is knowable
- If the set is infinitely large…
  - it is said to be *uncountable*
  - unless, the elements can be uniquely identified, then it is *countably infinite*

Countable Examples

Set	Result
$\{ x \mid x \in N \text{ and } x \leq 100\}$	Countable
$\{ 2x \mid x \in N \}$	Countably Infinite
$\{ x \mid x \in R \text{ and } 0 < x < 1\}$	Uncountable

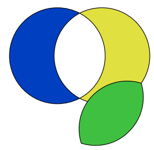


Inclusion-Exclusion

Counting by Subtracting!

Inclusion-Exclusion

- Sets can overlap – and can contain the same elements
- So, when counting items in sets, you must be careful not to count an item twice
- *Inclusion-exclusion* principle, can get the correct count



Disjoint Set Cardinality

- If sets A and B are disjoint then they have no elements in common
- Cardinality of the union is the sum of the cardinality of both A and B

$$|A \cup B| = |A| + |B|$$

Set Exclusion

- If sets A and B overlap they have elements in common
- The cardinality of the union is the sum of A and B excluding the intersection

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Set Exclusion

- Why?
- $|A| + |B|$ counts the intersection twice!
- So, we need to remove the duplicate count

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Set Exclusion

- Also note: this is the same equation for disjoint sets
- If disjoint, the intersection is $\varnothing$

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Let's look at this again…

Say 100 people receive a questionnaire with two questions (1) Do you program Java?, (2) Do you program C# ?

If 65 said 'yes' to Java and 40 say 'yes' to both. How many program just program C#?

Using the Formula…

- The union of Java & C# contains 100
- Java set contains 65
- The intersection contains 40

$$|J \cup C| = |J| + |C| - |J \cap C|$$

Using the Formula…

- The union of Java & C# contains 100
- Java set contains 65
- The intersection contains 40

$$100 = 65 + |C| - 40$$

Using the Formula…

- The union of Java & C# contains 100
- Java set contains 65
- The intersection contains 40

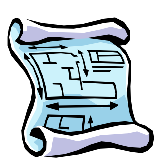
$$\begin{aligned} |C| &= 100 - 65 + 40 \\ &= 75 \end{aligned}$$

Tuples

Order is Important

Tuples & Sets

- To denote sets, we delimit the list of members with curly brackets
- For example, the prime numbers between 1 and 10 is {2, 3, 5, 7}
- Order does not matter, so {2, 3, 5, 7} = {7, 5, 3, 2}



Tuples

- However, in many cases the *order is important*
- These are called *n-tuples* where "n" is the number of elements
- 2-tuples are also called *ordered pairs*

Tuple Notation

- To denote a tuple – we delimit the elements by either parenthesis, angle brackets or square brackets
- Curly-brackets are never used to avoid obvious confusion

```
( 1, 2, 3 )
< 1, 2, 3 >
[ 1, 2, 3 ]
```


Tuple Examples

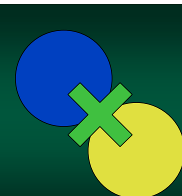
- Order is important, so any element out of position will cause inequality

```
( 1, 2, 3 ) ≠ ( 3, 2, 1 )
```


Tuple Examples

- Logic generally applies to algorithms since, in procedural programming, order is important
- The following is a tuple of events in California History

```
( Sutter's Fort Built, Bear Flag Revolt,
  Gold Rush, California Joins Union )
```

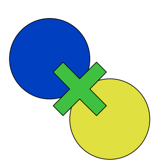



Cross Products

Databases use this…

Products

- Sets can be multiplied, which will result in a set of tuples
- Well, a set of ordered pairs, to be more specific
- $A \times B = \{ (x, y) \mid x \in A \text{ and } y \in B \}$



Example Product

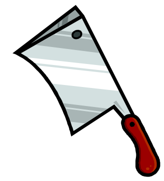
```
Given:
  A = {1, 2}
  B = {x, y}

A × B = { (1, x), (2, x),
          (1, y), (2, y) }
```

Partitions

Cutting a Set Into Pieces

Partitions

- A *partition* of a set A is a collection of non-empty disjoint sets whose union is A
- So, it is like the set A was "chopped", cleanly, into subsets



Requirements

- Each subset must be mutually exclusive
- … unless they are identical *(because duplicates don't count in sets)*

Partition Example

- The following is a valid partition of the set {1, 2, 3, … 9}

```
{ {1}, {2,3,5,7}, {4,6}, {8,9} }
```


Partition Example

- The following is a partition of N.

```
N = { {1}, {2,3}, {4,5,6}, ... }
```


Countable Examples

For the set {1, 2, 3, 4}…

Set	Partition?
`{ {1}, {2}, {3}, {4} }`	**Yes**
`{ {1,2}, {1,2}, {3,4} }`	**Yes.** {1,2} is duplicate
`{ {1,2,3}, {2,4} }`	**No.**

Power Series

All the combinations

Power Series

- A *power set* of a set S is a set of all the subsets of S
- This also, obviously, contains the null set
- The notation for the power set S is *P*(S)

Power Set Example

```
G = {a, b}

P(G) = { ø, {a}, {b}, {a,b} }
```


Power Set Example 2

```
H = {a, b, c}

P(H) = { ø, {a}, {b}, {c},
         {a,b}, {a,c}, {b,c}
         {a,b,c} }
```


Power Set Example 3

```
I = {a, b, c, d}

P(I) = { ø,
         {a}, {b}, {c}, {d}
         {a,b}, {a,c}, {a,d},
         {b,c}, {b,d}, {c,d},
         {a,b,c}, {a,b,d}, {b,c,d},
         {a,b,c,d} }
```


Cardinality of Power Sets

- What is the cardinality of power sets?
- Is it possible to determine | P(S) | if we know | S |
- This will be important later…

Let's Look at the Examples

```
G = {a, b}

P(G) = { ø, {a}, {b}, {a,b} }

|G|    = 2
|P(G)| = 4
```


Power Set Example 2

```
H = {a, b, c}

P(H) = { ø, {a}, {b}, {c},
         {a,b}, {a,c}, {b,c}
         {a,b,c} }

|G|    = 3
|P(G)| = 8
```


Power Set Example 3

```
I = {a, b, c, d}

P(I) = {ø, {a}, {b}, {c}, {d}
        {a,b}, {a,c}, {a,d},
        {b,c}, {b,d}, {c,d},
        {a,b,c}, {a,b,d}, {b,c,d},
        {a,b,c,d}}

|I|    = 4
|P(I)| = 16
```


Cardinality of Power Set

- The cardinality of a power set is 2^n where n is the cardinality of the original set
- This is used in statistics… covered later

$$|P(S)| = 2^{|S|}$$