Logic & Arguments

Part 1

Logic Statements

Make Mr. Spock Proud

Logic Statements

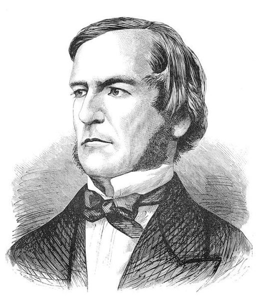
- Logic is used to construct <u>all</u> proofs and computer systems
- A *statement* is any declarative sentence that results in either true or false

Examples of Statements

- *There are exactly 35 people in this room*
- *Sacramento State is located next to a river*
- *10 + 2 = 11*
- *We had great choices for the 2016 Election*

Boolean Logic

- Discovered by George Boole
- First published in *The Mathematical Analysis of Logic (1847)*
- Revolutionized logic & proofs and is part of framework of modern of computer science



Boolean Operators

- Statements can be combined in *compound statements* using Boolean operators
- After statements are combined, they are still statements
- For example: "p and q"
  - given that p and q are both statements
  - then "p and q" is also a statement

Let's Review Boolean Operators

Operator	Name
`p and q`	True only if both p and q are true
`p or q`	True if either p or q true
`not p`	True if p false
`p xor q`	True if p and q are different
`p implies q`	True unless p is true and q is false

Logic Notation of Operators

Code-like	Logic
`p and q`	$p \land q$
`p or q`	$p \lor q$
`not p`	$\neg p$
`p xor q`	$p \oplus q$
`p implies q`	$p \rightarrow q$

Truth Table

p	q	$\neg p$	$p \lor q$	$p \land q$	$p \rightarrow q$
T	T	F	T	T	T
T	F	F	T	F	F
F	T	T	T	F	T
F	F	T	F	F	T

AND Example

True True

$1 < 2 \land 10 < 40$

Result: True

AND Example 2

True False

$1 < 2 \land 12 < 10$

Result: False

OR Example

True True

$1 < 2 \lor 10 < 40$

Result: True

OR Example 2

True — False

`5 > 3 ∨ 44 < 8`

Result: True

OR Example 3

False — False

`10 < 2 ∨ 12 < 10`

Result: False

NOT Example

True

`¬(1 < 2)`

Result: False

NOT Example

False

`¬(1 = 2)`

Result: True

Examples

`1 < 3 ∧ 10 < 40`	True
`1 = 3 ∧ 10 < 40`	False
`¬ (12 ≠ 12)`	True
`1 > 3 ∨ 30 < 20`	False

Implication

What is the implication of a little arrow?

Implication

- The only Boolean operator that causes confusion is implication
- However, its usage is **vital** to understand – since it is used your programs (even if you might not see it)

Implication

- For "p implies q"…
- p is called the *antecedent* (or hypothesis or assumption)
- q is called the *consequent* (or conclusion)

Implication

- "p implies q" is contradicted (false) **only** when…

  p is true and **q is false**

- In all other cases, it is true

Implication

- Consider the expression below
- The word "then" is alternative way of saying "implies"
- So, Is it True? False?

```
if x > 2 then x² > 4
```


Implication

- There are different values of x that will make the antecedent and consequent both true and false
- …but nothing that makes the expression false
- It is always true - no matter what the value of x

```
if x > 2 then x² > 4
```


Implies Example

True

True

$1 < 2 \rightarrow 10 < 40$

Result: True

Implies Example 2

True | False

`5 > 3 → 44 < 8`

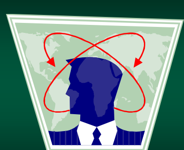
Result: False

Implies Example 3

False | False

`10 < 2 → 12 < 10`

Result: True

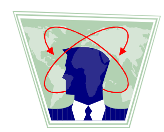


Analyzing Implication

Understanding is Power

Analyzing Implication

- Implication is both simple and complex
- It is used in all aspects of logical proof and the basis of all programs
- Understanding is complexity is essential to understanding logic (and discrete math)

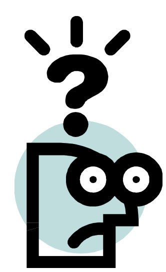


Implication Examples

- *If the Moon is made of cheese then the Moon is a tasty snack.*
- *If the flag has a bear then it is the Flag of California.*
- *If it is a fish then it is lives in water.*
- *If the university is Sacramento State then the mascot is a hornet.*

Many Ways to Say "Implies"

- *A implies B*
- *A → B*
- *B if A*
- *If A then B*
- *A given B*



"If I pan for gold then I'll get rich"

- Let's look at this statement closer
- It can be rewritten *"Pan of Gold→ Get Rich"* or, very tersely, *"P → R"*
- There four different combinations of the truth table

True → True

- If P is true, and R is true…
- *"We panned for gold and got rich"*
- Statement is true
  - we asserted that if P is true then R is true
  - since both are true, the statement is affirmed
  - true → true = true

False → True

- What if P is false and R is true
- *"We didn't pan for gold and got rich"*
- Statement is true
  - the fact we got rich (without panning for gold), doesn't mean that our statement is false
  - it has not contradicted the statement
  - false → true = true

False → False

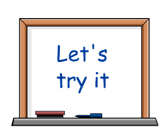
- What if P is false and R is false?
- *"We didn't pan for gold and didn't get rich"*
- Statement is true
  - the fact that both are false, still does not contraction our original statement
  - it stated "IF we pan for gold THEN we get rich"
  - false → false = true

True → False

- Finally what if P is true and R is false?
- *We panned for gold, but didn't get rich*
- Statement is false
  - we asserted if P is true then R must be true
  - however, since this contradicts the assertion, the result of the implication is false
  - true → false = false

Deconstructing Implication

- The implication logic can be broken down into the forms that are easier to remember
- This is actually quite important when we cover a few logical tricks later one
- So, let's look at the truth table for other logical operators



Analyzing Implication

- So, can implication be written using just logical "and", "or", or "not"?
- Yes, we can!

$p \rightarrow q$

Truth Table – Another way

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

True whenever p is false

True whenever q is true

Analyzing Implication

- So, we can definite p → q as "not p or q"
- Like before, let's prove in our truth table

$p \rightarrow q \equiv \neg p \lor q$

Truth Table – Or Logic

p	q	¬ p	¬p ∨ q	p → q
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Truth Table – One way to look at it

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

Only false for:

$p \land \neg q$

We can negate this.

Analyzing Implication

- So, we can definite p → q as "not (p and not q)"
- It doesn't look quite right, let's test it out

$p \rightarrow q \equiv \neg (p \land \neg q)$

Truth Table – And Logic

p	q	¬q	p ∧ ¬q	¬(p ∧ ¬q)	p → q
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	F	T	T
F	F	T	F	T	T

Analyzing Implication

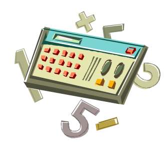
$$p \rightarrow q \equiv \neg(p \land \neg q)$$
$$\equiv \neg p \lor q$$

Logical Operator Precedence

It's just like algebra!

Logical Operator Precedence

- In *propositional logic*, compound statements can be combined with other statements using logical operators
- So, they can be chained together to form complex logic
- However, just like in algebra, there are possible issues



Algebra: Order of Operations

- Some mathematical operators have a high "precedence" than others
- They are computed first

```
3 + 6 / 3 * 2
```


Algebra: Order of Operations

- Knowing the correct order is vital
- For example, what is the result of the expression below?

```
3 + 6 / 3 * 2
```

Algebra: Order of Operations

- It is 7
- Divide and multiply are equal (and then done left to right), addition is done last

```
3 + 6 / 3 * 2  =  7
```


Many try, many fail

- Many students, using "PEMDAS" , think multiply is done before divide (M is before D)
- … or they just go left to right with no regard to precedence

```
3 + 6 / 3 * 2  =  4   WRONG!  F-!
```


Standard Precedence Levels

1	¬	Highest Level
2	∧	
3	∨ ⊕	
4	→	Lowest Level

Tautology & Contradiction

When the logic is quite elementary

Tautology & Contradictions

- Some statements are always true or false regardless of the variables used
- If the statement is always true, it is called *tautology*
- If the statement is always false, it is called a *contradiction*

Example Tautologies

- The following are examples of tautologies
- The result will always be **true**

```
p ∨ ¬p
p → p
p = p
```

Example Contradictions

- The following are examples of contradictions
- The result will always be **false**

```
p ∧ ¬p
p ⊕ p
```


Example

- So, what is the truth table for the example below?
- Let's create a truth table

```
(¬ p ∧ q) ∧ (p ∨ ¬q)
```


Logic Equivalence

Same meaning, different form

Logical Equivalence

- *Logical equivalence* is when two different statements are the same
- The truth tables for both statements are identical

Communitive Law

- Both ∧ and ∨ are communitive
- This means the left-hand and right-hand operands can be switched (symmetric relation)

```
p ∧ q ≡ q ∧ p
p ∨ q ≡ q ∨ p
```


Idempotent Law

- When a statement is combined with itself, it is equivalent to just the statement (no duplicate)
- This applies to both ∧ and ∨

```
p ∧ p ≡ p
p ∨ p ≡ p
```

Involution Law

- One of the most basic equivalences in logic is the *double negation*
- It is fairly obvious, so not more needs to be said

```
¬ ¬ p ≡ p
```


Complement Law

- When a statement is used with its complement (itself negated), it will result in either true or false
- So, it is always as tautology or contradiction

```
p ∧ ¬ p ≡ false
p ∨ ¬ p ≡ true
```


Identity Law

- Some operands will have no effect on the truth table of a statement
- In this case, the statement can be simplified

```
p ∧ true   ≡  p
p ∨ false  ≡  p
```


Domination Law

- This might look similar to the identity law, but look very careful at which operator is being used
- These will result in either true or false.

```
p ∨ true   ≡  true
p ∧ false  ≡  false
```


Associative Law

- Some operators in math are *associative*
- For example: (a + b) + c = a + (b + c)
- Same applies to ∧ and ∨

```
p ∧ (q ∧ r)  ≡  (p ∧ q) ∧ r
```


Distributive Law

- Math has operators that are *distributive*
- For example: a * (b + c) = (a * b) + (a * c)
- Works for both ∧ and ∨

```
p ∧ (q ∨ r)  ≡  (p ∧ q) ∨ (p ∧ r)

p ∨ (q ∧ r)  ≡  (p ∨ q) ∧ (p ∨ r)
```

DeMorgan's Law

- *DeMorgan's Law* states important rule for logical equivalency
- These are used to convert And operators to Or and vice-versa

DeMorgan's Law

- So, it states you can change the operator from ∧ to ∨ or vice-versa
- If you negate both operands

$\neg (p \land q) \equiv \neg p \lor \neg q$

$\neg (p \lor q) \equiv \neg p \land \neg q$

Truth Table – Testing Not-Or

p	q	¬ p	¬ q	¬p ∧ ¬q	¬ (p ∨ q)
T	T	F	F	F	F
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

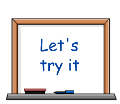
Truth Table – Testing Not-And

p	q	¬ p	¬ q	¬p ∨ ¬q	¬ (p ∧ q)
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

Example

- So, what is the truth table for the example below?
- Let's create a truth table

$(p \land q) \lor (\neg p \land q)$



Boolean Algebra

- However, truth tables become unwieldy as the number of variables increase.
- Logical algebra is another way to evaluate equivalence
- Equivalences can be used to generate one expression from another

Example Simplification

$(p \land q) \lor (\neg p \land q) =$

$(p \lor \neg p) \land q =$ After using Distributive Law

$\text{true} \land q =$ After using Complement Law

q After using Identify Law

Arguments

Proving a Point

Arguments

- A combination of true statements can be used to claim another as true
- An *argument* is a collection of statements (called *premises*), which, when all are true, imply a consequence

Example Argument

```
raining → wet outside
not wet outside
_____________________
?
```

Our conclusion goes here

Example Argument 2

```
raining → wet outside
not raining
_____________________
?
```

What conclusion goes here? Does it work?

Example Argument 3

```
x is even  or  x is odd
x is even
_______________________
?
```

Obvious! But why?

When an Argument is Valid

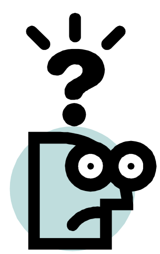
- When all the premises are true then the consequence must be true
- If all the premises are true, but the conclusion can be false, the argument is disproven

When an Argument is Valid

- However, if any premise is false, then the argument is not disproven – *it is still valid*
- We can often prove arguments by building truth tables

Argument Notation

- Arguments can be written out several ways
- The most common approach is to write each premise on a different line
- The consequence is written below the premises separated with horizontal line



Common Notation

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline p \rightarrow r \end{array}$$

Another Argument Notation

- Arguments can be written on a single line
- Premises are separated with commas
- The consequence follows the symbol $\vdash$ or $\therefore$

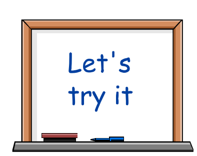
"Therefore"

$p \rightarrow q,\ q \rightarrow r\ \vdash\ p \rightarrow r$

Let's Try This

$$\begin{array}{l} p \rightarrow (q \rightarrow r) \\ q \\ \hline p \rightarrow r \end{array}$$

Valid!



Valid Arguments

Proving a Point

Valid Arguments

- *Rules of Inference* are valid arguments that are commonly used in proofs
- Most of these are obvious to you… it is natural logical thought

Rules of Inference

- *Modus Pones* (aka Law of Detachment)
- *Modus Tollens*
- *Disjunctive Syllogism*
- *Hypothetical Syllogism*

Modus Ponens

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline q \end{array}$$

Rules of Inference

Modus Ponens Example

If it is a fish, then it lives in water.

It is a fish.

Therefore, it lives in water!

Modus Tollens

$$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \neg p \end{array}$$

Rules of Inference

Modus Tollens Example

If it is a fish, then it lives in water.

It doesn't live in water.

Therefore, it is <u>not</u> a fish!

Disjunctive Syllogism

$p \lor q$

$\neg p$

---

q

Rules of Inference

Disjunctive Syllogism Example

It breathes water or air.

It doesn't breath water.

Therefore, it breathes air.

Hypothetical Syllogism

$p \rightarrow q$

$q \rightarrow r$

$p \rightarrow r$

Rules of Inference

Hypothetical Syllogism Example

If is a trout, then it is a fish

If it is a fish, then it lives in water.

Therefore, a trout lives in water!

Let's Apply The Logic

- Let's these rules on the argument below
- We can use either a truth table or logical deduction

```
If I study then I will an A
If I don't watch Game of Thrones then I will study
I didn't get an A
__________________________
I watched Game of Thrones
```

Simply Logic: letters

- First, let's simply the structure of the argument so we can see the logic
- We will assign each part a letter

```
s = studied
a = got an A
g = watched Game of Thrones
```


Simply Logic: New Form

```
1.  s → a
2.  ¬ g → s
3.  ¬ a
    ______
    g
```


Modus Tollens – study, no A

```
1.  s → a
2.  ¬ g → s
3.  ¬ a
    ______
    g
```

1 and 3: Modus Tollens

Modus Tollens – study, no A

```
1.  ¬ s
2.  ¬ g → s
3.  ¬ a
    ______
    g
```

1 and 3: Modus Tollens

"Did not study"

Modus Tollens – study, Pokemon

```
1.  ¬ s
2.  ¬ g → s
3.  ¬ a
    ______
    g
```

1 and 2: Modus Tollens

Modus Tollens – study, Pokemon

```
1.  ¬ s
2.  ¬ ¬ g
3.  ¬ a
    ______
    g
```

1 and 2: Modus Tollens

"Did not ***not*** play Pokemon"

Double Negation

```
1.  ¬ s
2.  ¬ ¬ g
3.  ¬ a
    ______
    g
```

Double negation

Double Negation

```
1.  ¬ s
2.  g
3.  ¬ a
    ______
    g
```

2:
Double Negation

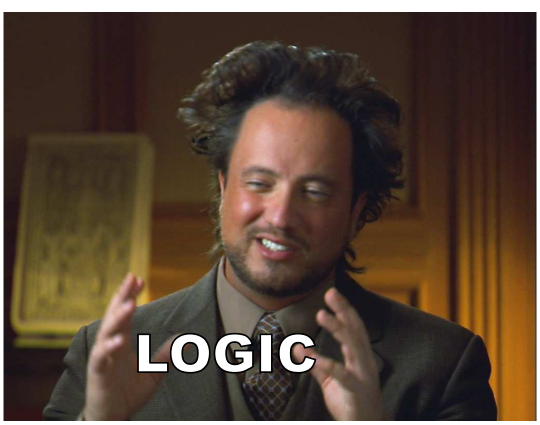
"Watched Game of Thrones"

Logical Fallacies

"This is most illogical" – Mr. Spock

Logical Fallacies

- There are a number of fallacious arguments that, while they might look logical, are wrong
- The following slides contain some of them
- For fun, apply them to current political discourse or *History Channel 2*



Fallacy of the Converse

- *Fallacy of the Converse* is based on assumption that if the conclusion is true then the hypothesis is true
- Also called:
  - *affirming the consequent*
  - *converse error*

Fallacy of the Converse Example

If it is a fish, then it lives in water.

It lives in water.

Therefore, it is a fish!

Fallacy of the Converse

$p \rightarrow q$
q
―――――
p

p can still be false

Fallacy of the Inverse

- *Fallacy of the Inverse* is based on assumption that if the hypothesis is false, then the conclusion is also false
- Also called:
  - denying the antecedent
  - inverse error

Fallacy of the Inverse Example

If it is a cat, then it is furry.

It is not a cat.

Therefore, it is not furry!

Fallacy of the Inverse

$p \rightarrow q$
$\neg p$
―――――
$\neg q$

q can be either true or false

Fallacy of Affirming a Disjunct

- *Fallacy of Affirming a Disjunct* is based on assumption that if there are two attributes and one is true, then the other must be false.
- Other names:
  - *fallacy of the alternative*
  - *false exclusionary disjunct*

Affirming a Disjunct Example

Suspect is either a politician or a lawyer.

Suspect is a politician.

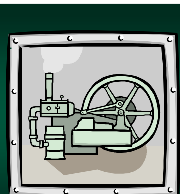
Therefore, the suspect isn't a lawyer.

Fallacy of Affirming a Disjunct

$p \lor q$
p

$\neg q$

Just because p is true, doesn't mean q has to be false.

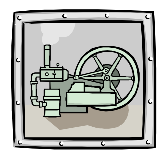


Boolean Algebra & Proofs

Proofs and Logic are one

Boolean Algebra

- Remember Boolean algebra laws: Associative, Commutative, etc…
- These can be used to expand an expression… and then simplify it in a different form



Example

if a or b is true and
a and b is false

then
a = not b

Example (with a rewrite)

We can rewrite it as an argument:

$a \lor b$ = true
$a \land b$ = false
---
a = $\neg$b

The Strategy

- We *could* use a Truth Table to prove this
- Let's use Boolean Algebra to prove if this is correct

```
a ∨ b = true
a ∧ b = false
____________
a = ¬b
```

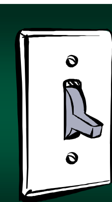

The Approach

1. Start with a
2. Try to get to ¬ b
3. Use the premises to replace values

```
a ∨ b = true
a ∧ b = false
____________
a = ¬b
```



```
a =  a
  =  a   ∧  true                 Identity
  =  a   ∧  (b ∨ ¬b)             Complement
  =  a ∧ b    ∨   a ∧ ¬b         Distributive
  =  false    ∨   a ∧ ¬b         Premise
  =  b ∧ ¬b   ∨   a ∧ ¬b         Complement
  =  ¬b  ∧  (b ∨ a)              Distributive
  =  ¬b  ∧   true                Premise
  =  ¬b                          Identity
```

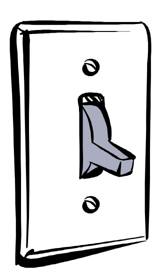



Duals

Don't "Flip" Out

Duels

- The dual of a theorem is created by:
  - flip all true and false
  - flip all and all ∨ and ∧
- One interesting thing about Boolean Algebra is that *whenever an argument is true, so is its dual.*



Example

```
Show p  =  p ∨ p

p  = p
   = p  ∨  false              Identity
   = p  ∨  (p ∧ ¬p)           Complement
   = (p ∨ p) ∧ (p ∨ ¬p)       Distributive
   = (p ∨ p) ∧  true          Complement
   =  p ∨ p                   Identity
```

Example

```
The dual of this law: p  =  p ∧ p

p  = p
   = p  ∧  true            Identity
   = p  ∧  (p ∨ ¬p)        Complement
   = (p ∧ p) ∨ (p ∧ ¬p)    Distributive
   = (p ∧ p) ∨  false      Complement
   =  p ∧ p                Identity
```


Example

```
The dual of this law: p  =  p ∧ p

p  = p
   = p  ∧  true            Identity
   = p  ∧  (p ∨ ¬p)        Complement
   = (p ∧ p) ∨ (p ∧ ¬p)    Distributive
   = (p ∧ p) ∨  false      Complement
   =  p ∧ p                Identity
```


Conclusion

- The second proof was simple because all the initial laws were true – as were their duals
- So, any proof of a law *P* using the laws of Boolean algebra can be rewritten using the duals, which would prove the dual of P