Quantum Fourier transform.

ANIMESH DATTA (warwick.ac.uk/qinfo)

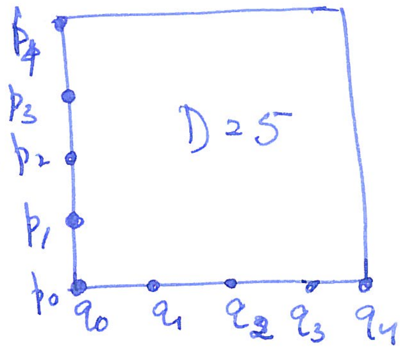
D-dimensional Hilbert space:

Orthonormal 'position' basis $|q_j\rangle = |e_j\rangle = |j\rangle \quad j = 0, \ldots, D-1$

$q_j = j/D$

Conjugate 'momentum' basis $|p_k\rangle, k = 0, \ldots, D-1$

$p_k \equiv k/D$



$2\pi\hbar D = 1 = \left(\text{area of phasespace}\right)$

$$|p_k\rangle = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} |q_j\rangle e^{2\pi i jk/D}$$

$$\langle q_j | p_k \rangle = \frac{1}{\sqrt{D}} e^{\frac{i}{\hbar} q_j p_k} = \frac{1}{\sqrt{D}} e^{2\pi i jk/D} \leftarrow \text{discrete FT coeffs.}$$

$$\langle q_j | p_k \rangle = \langle q_k | p_j \rangle = \langle q_{-j} | p_k \rangle^* = \langle p_k | q_{-j} \rangle$$

Matrix elements of a unitary matrix:

$$\frac{1}{D} \sum_{l=0}^{D-1} e^{2\pi i l(j-k)/D} = \frac{1}{D} \frac{1 - e^{2\pi i (j-k)}}{1 - e^{2\pi i (j-k)/D}} = \delta_{jk}.$$

The quantum Fourier transform is then defined as:

$$F|q_j\rangle = |p_j\rangle$$

$$\langle q_j | F | q_k \rangle = \langle q_j | p_k \rangle = e^{2\pi i jk/D} / \sqrt{D} \quad : \text{QFT matrix element}$$

$$y_k = \langle p_k | \psi \rangle = \langle q_k | F^\dagger | \psi \rangle = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} e^{2\pi i jk/D} \underbrace{\langle q_j | \psi \rangle}_{x_j}$$

$$\therefore \boxed{y_k = \frac{1}{\sqrt{D}} \sum_{j=0}^{D-1} e^{-2\pi i jk/D} x_j} \quad \underline{\text{DFT.}}$$

$$F|p_j\rangle = \sum_k \underbrace{F|q_k\rangle}_{p_k}\underbrace{\langle q_k|p_j\rangle}_{\langle p_k|q_{-j}\rangle} = \sum_k |p_k\rangle\langle p_k|q_{-j}\rangle = |q_{-j}\rangle$$

$$\Rightarrow \quad F^2|q_j\rangle = |q_{-j}\rangle \Rightarrow F^4 = 1 \quad \therefore F \text{ has eigenvalue } \pm 1, \pm i$$

What does all this have to do with qubits?

n qubits, $D = 2^n$. $|q_j\rangle = |j\rangle = |j_1\rangle \otimes \cdots |j_n\rangle$

Notation: $j = j_1 \cdots j_n = \sum_{l=1}^{n} j_l 2^{n-l}$

Binary rep. of j ($j_l = 0,1$).

Qubit representation of F:

$$|p_j\rangle = \underset{\substack{\uparrow \\ \text{F-T on } n \text{ qubits}}}{F_n}|q_j\rangle = F_n|j\rangle = \frac{1}{2^{n/2}}\sum_{k=0}^{2^n-1}|k\rangle e^{2\pi i jk/D}$$

$$= \frac{1}{2^{n/2}}\sum_{k_1\ldots k_n=0}^{1}|k_1\rangle\otimes|k_2\rangle\cdots\otimes|k_n\rangle \underbrace{e^{2\pi i\left(\sum_{l=1}^{n}k_l 2^{n-l}\right)\frac{j}{D}}}_{\prod_{l=1}^{n} e^{2\pi i j k_l/2^l}}$$

$$= \frac{1}{2^{n/2}}\bigotimes_{l=1}^{n}\sum_{k_l=0}^{1}|k_l\rangle \underbrace{e^{2\pi i j k_l/2^l}}_{= e^{2\pi i (0.j_l\ldots j_n)k_l}}$$

$$\therefore \quad F_n|j\rangle = \frac{1}{2^{n/2}}\bigotimes_{l=1}^{n}\left(|0\rangle + e^{2\pi i j/2^l}|1\rangle\right)$$

$$\boxed{F_n|j\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i j/2}|1\rangle\right)\otimes\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i j/2^2}|1\rangle\right)\otimes\cdots\otimes\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i j/2^n}|1\rangle\right)}$$

This shows that F_n can be implemented by separate (controlled) operations on each qubit, giving an $O(n^2)$ algorithm.

Circuit for F_n:

③

$R_0 = 1\!\!1$
$R_1 = Z$
$R_2 = S$
$R_3 \doteq T$
$\vdots$
$R_{k+1}^2 \doteq R_k$

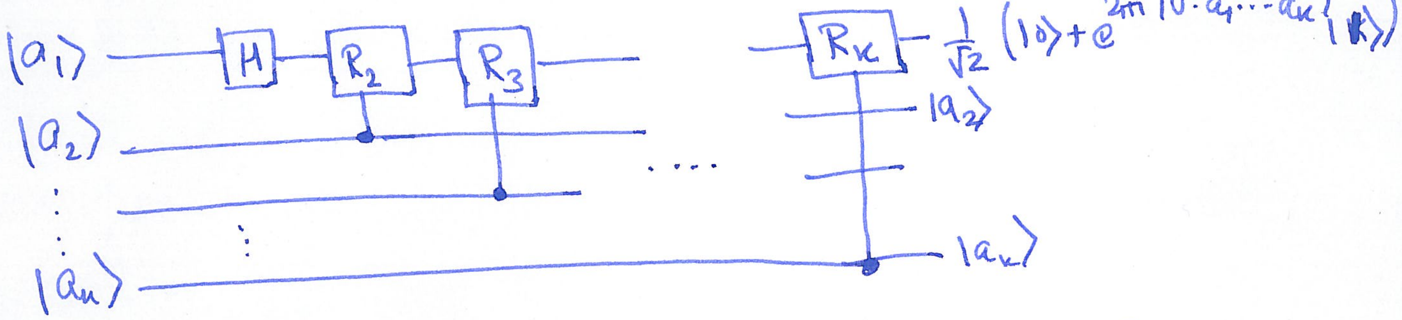
Let $R_k \equiv e^{2\pi i/2^{k+1}} \cdot e^{-\frac{iZ2\pi}{2^{k+1}}} = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix}$.

$$H|a\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^a|1\rangle\right)$$

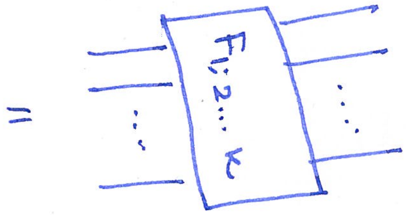
$$= \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\pi a}|1\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\,(0.a)}|1\rangle\right)$$

$$R_k^a\,(\cancel{\phi}\alpha|0\rangle + \beta|1\rangle) = \alpha|0\rangle + \beta e^{2\pi i a/2^k}|1\rangle = \alpha|0\rangle + \beta e^{2\pi i\,(0.\underbrace{0\cdots0}_{k-1\ 0\text{'s}}a)}|1\rangle$$

Thus, the basic building block is

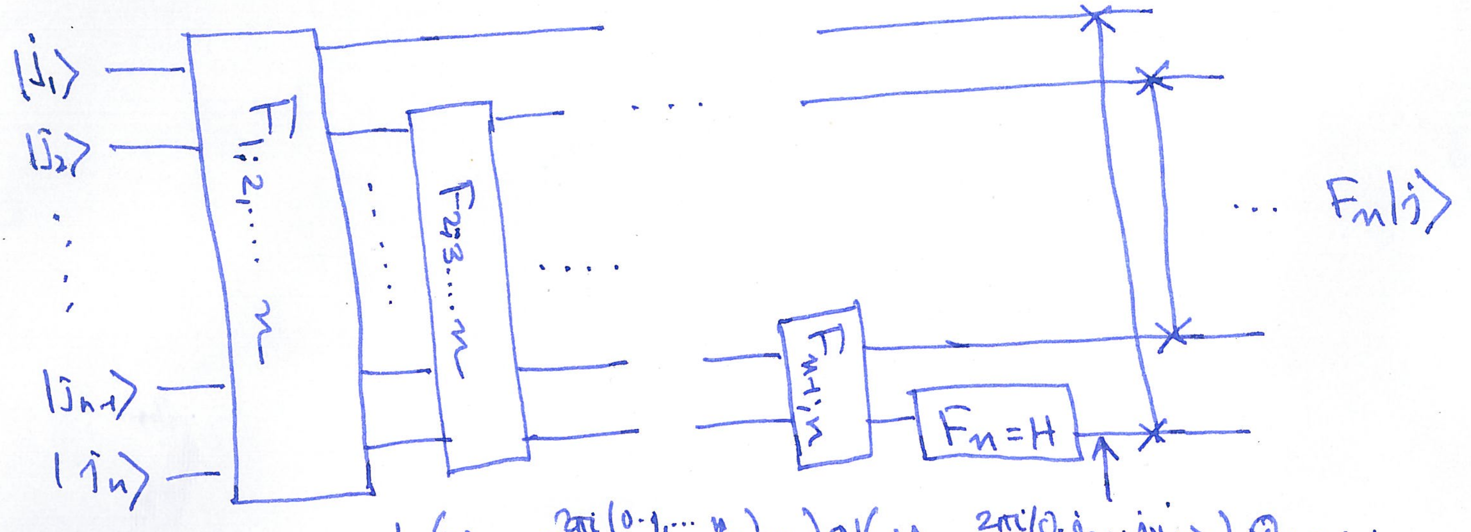


Resources: $O(k)$.



$$\underset{\text{target}}{F_{1;}}\underbrace{2,\ldots k}_{\text{control qubits}}$$

The full circuit :



$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i(0.j_1\cdots j_n)}|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i(0.j_2\cdots j_n}|1\rangle\right) \otimes \cdots$$

$$\cdots \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i(0.j_n)}|1\rangle\right)$$

The total resource count for the full circuit is ~~[illegible]~~ $O(n^2)$.

Phase estimation

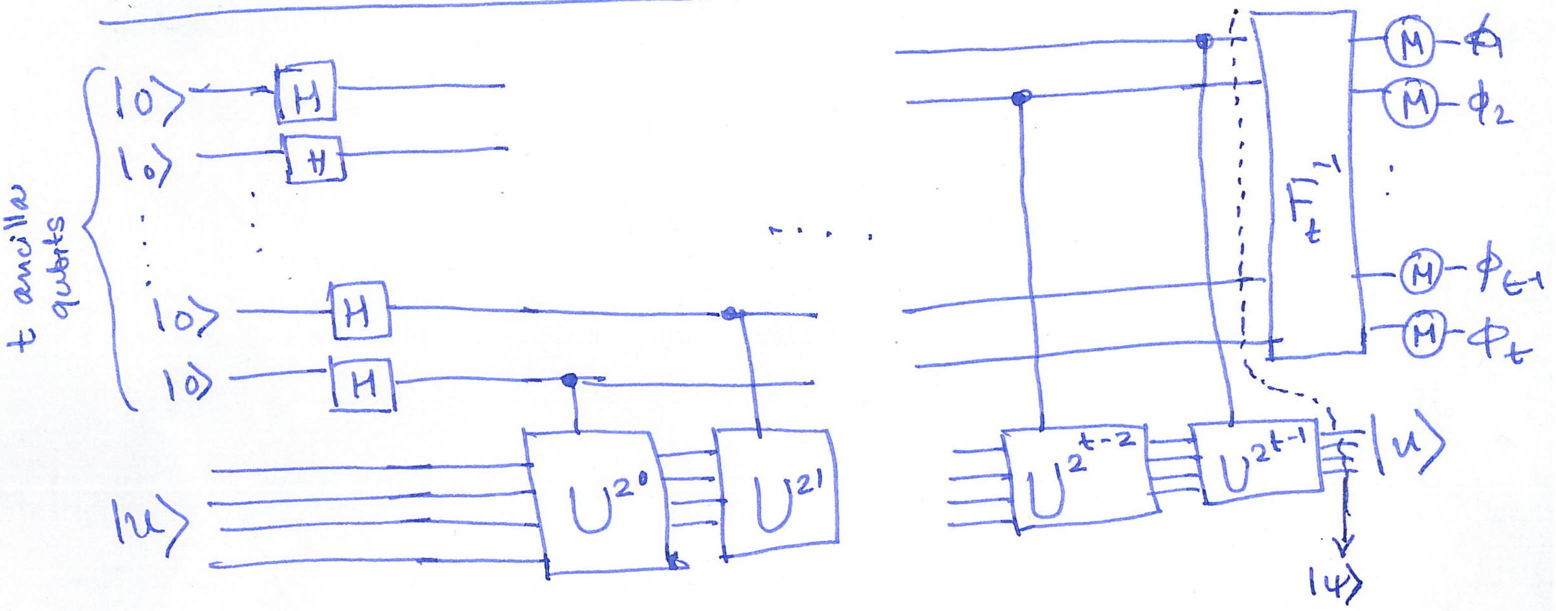
$$U|u\rangle = e^{2\pi i \phi}|u\rangle$$

ϕ → unknown phase as a fraction of 2π

$|u\rangle$ ↳ eigenvector of U that can be reliably prepared.

① Suppose $\phi = 0.\phi_1 \cdots \phi_t = \phi_1 \cdots \phi_t / 2^t$.

Phase estimation circuit:



$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i (2^{t-1}\phi)}|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i (2^{t-2}\phi)}|1\rangle\right) \otimes \cdots \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i (2^{0}\phi)}|1\rangle\right) \otimes |u\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i (0.\phi_t)}|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i (0.\phi_{t-1}\phi_t)}|1\rangle\right) \otimes \cdots \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i (0.\phi_1 \cdots \phi_t)}|1\rangle\right) \otimes |u\rangle$$

The controlled unitaries prepare a momentum state

$$|p_{2^t\phi}\rangle = F_t|\phi\rangle = \frac{1}{\sqrt{2}} \sum_{k=0}^{2^t-1} e^{2\pi i k\phi}|k\rangle$$ (periodic in position with period $\Delta q = 1/\phi = 2\pi\hbar/p_\phi$)

To determine ϕ, we need to determine the period $2\pi\hbar/p_\phi$, which can be obtained by a measurement in the momentum basis. As we may not know how to do that, we perform a FT which puts the phase information ~~to~~ into the standard basis.

② What happens when $\phi = \phi_1\phi_2\ldots$ has more than t digits?

Now let $|\phi\rangle$ denote the state that is input into the inverse FT:

$$|\phi\rangle = \frac{1}{2^{t/2}} \bigotimes_{l=1}^{t} \underbrace{\left(|0\rangle + e^{2\pi i \underbrace{(2^t\phi)}_{j}/2^l}|1\rangle\right)}_{\sum_{k_l=0}^{1} |k_l\rangle e^{2\pi i k_l (2^t\phi)/2^l}}$$

(and reverse the steps leading to the qubit form of the FT)

$$= \frac{1}{2^{t/2}} \sum_{k_1,\ldots,k_n} |k_1\rangle\otimes\cdots\otimes|k_n\rangle e^{2\pi i\left(\sum_{l=1}^{t} k_l 2^{t-l}\phi\right)}$$

$$= \frac{1}{2^{t/2}}\sum_{k=0}^{2^t-1}|k\rangle e^{2\pi i k\phi}$$

$$|0\rangle^{\otimes t}|u\rangle \longrightarrow |\phi\rangle|u\rangle = \frac{1}{2^{t/2}}\sum_{k=0}^{2^t-1}|k\rangle U^k|u\rangle$$ ↳ the state before the $F^\dagger$

$$\langle q_j|F^\dagger|\phi\rangle = \frac{1}{2^t}\sum_{k=0}^{2^t-1} e^{2\pi i k(\phi - j/2^t)}$$

⑥

This implies $$|0\rangle^{\otimes t}|\psi\rangle \longrightarrow \frac{1}{2^{t/2}} \sum_{k=0}^{2^t-1} |k\rangle U^k|\psi\rangle .$$

To get to the result, look at the state after the Hadamard.
The controlled unitaries leads to

$$|j\rangle|\psi\rangle \rightarrow |j_1\rangle \otimes \cdots \otimes |j_t\rangle \underbrace{U^{2^{t-1} j_1} \cdots U^{2^1 j_{t-1}} U^{2^0 j_t}}_{U^{\sum_{k=1}^{t} 2^{t-k} j_k} = U^j} |\psi\rangle$$

$$\rightarrow |j\rangle U^j |\psi\rangle$$

So, with the Hadamards, the transformation is

$$|0\rangle^{\otimes t}|\psi\rangle \longrightarrow \frac{1}{2^{t/2}} \sum_{j=0}^{2^t-1} |j\rangle U^j |\psi\rangle .$$

So, if $|\psi\rangle$ is a superposition of multiple eigenstates of U, the output measurement will yield one of the eigenvalues with the probability from the superposition

$$\langle q_j | F^{+} | \phi\rangle = \frac{1}{2^t} \sum_{t=0}^{2^t-1} \left(e^{2\pi i(\phi - j/2t)}\right)^k$$

$$= \frac{1}{2^t} \frac{1 - e^{2\pi i(\phi 2^t - j)}}{1 - e^{2\pi i(\phi - j/2t)}}$$

$b = \lfloor \phi 2^t \rceil$ and $\phi 2^t = b + \delta_j$ $0 \leq \delta < 1$,

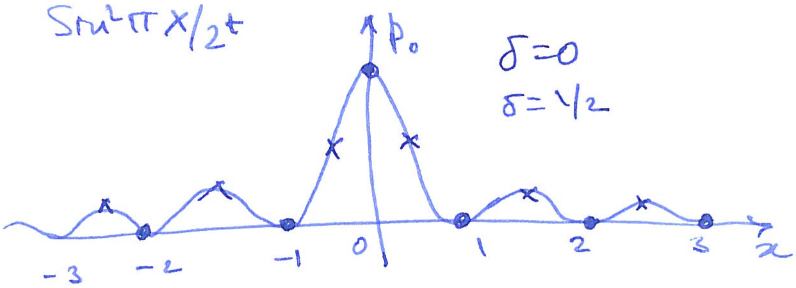
$$\phi = \underset{\substack{\uparrow \\ \text{t-bit approx. of } \phi}}{b 2^{-t}} + \delta 2^{-t} \longrightarrow \text{error in t-bit approx.}$$

$$\langle q_{b+l} | F^{+} | \phi\rangle = \frac{1}{2^t} \frac{1 - e^{2\pi i(\delta - l)}}{1 - e^{2\pi i(\delta - l)/2^t}}$$

$$= \frac{1}{2^t} \frac{e^{i\pi(\delta-l)/2} \sin\left[\pi(\delta-l)\right]}{e^{i\pi(\delta-l)/2^t} \sin\left[\pi(\delta-l)/2^t\right]}$$

$$\therefore \left|\langle q_{b+l}|F^{+}|\phi\rangle\right|^2 = \frac{1}{2^{2t}} \frac{\sin^2 \pi(\delta-l)}{\sin^2(\pi(l-\delta)/2^t)}$$

$$= \frac{1}{2^{2t}} \frac{\sin^2 \pi x}{\sin^2 \pi x/2^t} \quad ; x = l-\delta$$



Two sources of error:

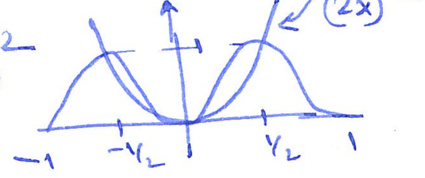
1. $\delta 2^{-t}$ is the error in determination of ϕ because of ϕ having more than t bits
2. $l 2^{-t}$ is the error in determination of ϕ because the measurement doesn't yield b.

$|l| \leq 2^{m-1}$ means an error $\leq m$ bits, giving ϕ to $N=t-m$ bits.

$$\therefore \underbrace{p(|l| > 2^{m-1})}_{\text{prob. of getting less than } N=t-m \text{ bits}} = \sum_{l=2^{t-1}+1}^{-2^{m-1}-1} \left|\langle q_{b+l}|F^{+}|\phi\rangle\right|^2 + \sum_{l=2^{m-1}+1}^{2^{t-1}} \left|\langle q_{b+l}|F^{+}|\phi\rangle\right|^2$$

$$l = -2^{t-1}+1, \ldots, 2^{t-1}$$

Note: $\sin^2 \pi x \leq 1$, $\sin^2 \pi x \geq (2x)^2$



$$\frac{l-\delta}{2^t} \leq \frac{1}{2^t}\left(2^{t-1}-\delta\right) = \frac{1}{2} - \frac{\delta}{2^t} \leq \frac{1}{2}$$

$$\frac{l-\delta}{2^t} \geq \frac{1}{2^t}\left(-2^{t-1}+1-\delta\right) = -\frac{1}{2} + \frac{1-\delta}{2^t} \geq -\frac{1}{2}.$$

$$\therefore \left|\frac{l-\delta}{2^t}\right| \leq \frac{1}{2}$$

and so, $\sin^2 \pi\left(\frac{l-\delta}{2^t}\right) \geq \left(\frac{2(l-\delta)}{2^t}\right)^2 = \frac{4(l-\delta)^2}{2^{2t}}.$

Also, $|\langle q_{b+l}|F^+|\phi\rangle|^2 \leq \frac{1}{4(l-\delta)^2}$,

<u>Putting it all together,</u> $p(|l|>2^{m-1}) \leq \frac{1}{4}\left(\sum_{l=-2^{t-1}+1}^{-2^{m-1}} \underbrace{\left(\frac{1}{l-\delta}\right)^2}_{\leq 1/l^2} + \sum_{l=2^{m-1}+1}^{2^{t}-1} \underbrace{\left(\frac{1}{l-\delta}\right)^2}_{\leq \frac{1}{(l-1)^2}}\right)$

$$\leq \frac{1}{4}\left(\sum_{l=-2^{t-1}+1}^{-2^{m-1}} 1/l^2 + \sum_{l=2^{m-1}}^{2^{t-1}-1} 1/l^2\right)$$

$$= \frac{1}{2}\sum_{l=2^{m-1}}^{2^{t-1}-1} 1/l^2 \leq \frac{1}{2}\int_{2^{m-1}-1}^{2^{t-1}-1} dl/l^2 \leq \frac{1}{2}\int_{2^{m-1}-1}^{\infty}\frac{dl}{l^2}$$

$$= \frac{1}{2}\frac{1}{2^{m-1}-1}.$$

$\therefore\ \varepsilon = \left(\begin{array}{l}\text{prob. of getting less than}\\ N=t-m \text{ bits of } \phi\end{array}\right) = 1 - \left(\begin{array}{l}\text{prob. of getting } N=t-m\\ \text{bits or more}\end{array}\right)$

$$\leq \frac{1}{2}\frac{1}{2^{m-1}-1} = \frac{1}{2}\frac{1}{2^{t-N-1}-1}.$$

$$\boxed{m \leq 1+\log\left(1+\frac{1}{2\varepsilon}\right)}.$$