CROSSWIND KITE POWER CURVE

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1. Fundamentals

Traditional wind turbines use the power coefficient, C_P, as a critical performance metric:

$$C_P = \frac{P}{\frac{1}{2}\rho v_w^3 A_{\text{swept}}} \tag{1}$$

This is the value that is optimized in the celebrated Betz limit, which says that the maximum achievable power coefficient is $C_P \approx 0.593$. Notably the power coefficient is referenced to the *swept area* of the wind turbine blades, which is reasonable given the limited swept area available to a traditional wind turbine.

For kite based systems, where system cost is correlated with kite size rather than the vast swept area, it makes sense to reference the power coefficient to *kite area*. This value is referred to as ζ:

$$\zeta = \frac{P}{\frac{1}{2}\rho v_w^3 A_k} \tag{2}$$

However, optimizing a system for the maximum C_P or ζ is not the same as optimizing for cost! In fact, optimizing a system for maximum ζ actually increases the maximum tension, a substantial cost driver, over a Betz limited system by a factor of two. To describe how efficiently a system converts tension into power, it is useful to introduce another dimensionless variable, the tension efficiency, η_T:

$$\eta_T = \frac{P}{T v_w} \tag{3}$$

According to actuator disk theory, for a Betz limited system, the wind velocity at the disk, where it may be assumed the force is applied, is $\frac{2}{3}v_w$. Thus, the tension efficiency for a Betz limited system is $\eta_T = \frac{2}{3}$.

To calculate the corresponding tension efficiency for a kite power system that attempts to maximize ζ, i.e. a Loyd limited system, it is first necessary to relate ζ to physical properties of the kite. Ignoring power efficiency losses and the typically low induction factors of kite propellers, the power generated by the kite is approximately, $P = D_{\text{prop}} v_k$, where D_{prop} is the

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drag on all the propellers and v_k is the airspeed of the kite. Substituting this into Eq. 2, it is found that

$$\zeta = C_{D_{\text{prop}}} \left(\frac{v_k}{v_w} \right)^3 \tag{4}$$

The kite-to-wind speed ratio, which figures prominently in Eq. 4, may be related to the lift and drag coefficients of the kite through simple force balance. Specifically, the airspeed of a kite traveling perpendicular to the wind is given by:

$$v_k = \sqrt{1 + \left(\frac{C_L}{C_D} \right)^2} v_w \approx \frac{C_L}{C_D} v_w \tag{5}$$

Here the drag coefficient, C_D, may be divided into two components: system drag, $C_{D_{\text{sys}}}$, and propeller drag, $C_{D_{\text{prop}}}$. The system drag coefficient includes parasitic drag, induced drag, and tether drag, and to a large degree is fixed during flight. The propeller drag coefficient, however, may be tuned over a wide range.

To optimize propeller drag for various metrics, e.g. ζ or η_T, it is useful to express the tunable propeller drag coefficient as a fraction, k, of the fixed system drag coefficient: $C_{D_{\text{prop}}} = kC_{D_{\text{sys}}}$. Now, the kite-to-wind speed ratio, λ, from Eq. 5 is

$$\lambda = \frac{v_k}{v_w} = \frac{C_L}{C_{D_{\text{sys}}}} \frac{1}{1+k} \tag{6}$$

Thus, ζ from Eq. 4 is

$$\zeta = \frac{C_L^3}{C_{D_{\text{sys}}}^2} \frac{k}{(1+k)^3} \tag{7}$$

And finally, using the approximation for tension, $T = \frac{1}{2}\rho v_k^2 A_k C_L$, the tension efficiency, η_T, from Eq. 3, is

$$\eta_T = \frac{k}{1+k} \tag{8}$$

There are many interesting things to note from the three dimensionless performance metrics described above (λ, ζ, and η_T):

- The kite speed is proportional to the system's lift-to-drag ratio, as expected, and this is tunable with the propeller drag.
- The C_L^3 in the ζ metric really encourages high lift coefficients. This effect is even more dramatic considering the high drag penalty imposed by the tether, which makes the increase in induced drag with C_L^2 less important when compared to other systems that optimize for the endurance metric.
- Maximizing ζ requires setting the propeller drag to half the system drag, $k = \frac{1}{2}$. From Eq. 6, this is equivalent to saying that the kite should fly at $\frac{2}{3}$ of its zero-propeller-drag speed.

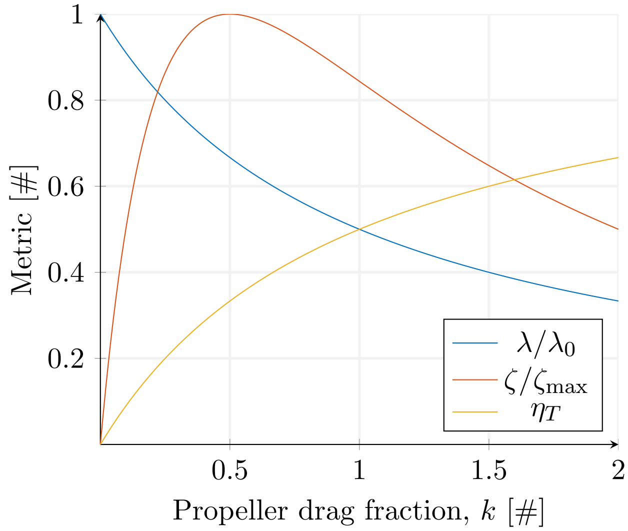

FIGURE 1. Kite performance metrics as a function of the propeller drag fraction.

- The tension efficiency is independent of any system parameter save the propeller drag fraction.
- Finally, optimizing for ζ results in a tension efficiency of $\eta_T = \frac{1}{3}$. Thus, maximizing ζ results in half the tension efficiency of a Betz limited system. Of course, it is not necessary to fly a system at maximum ζ; and if k is increased to 2, the system will achieve the same tension efficiency as a Betz limited device, but ζ will drop by a factor of 2.

2. Ideal power curve

Figure 2 shows the general form of a crosswind kite power curve. Below the cut-in wind speed, $v_{w,\mathrm{cut-in}}$, the system generates no power, or negative power if it is necessary to keep the kite flying. Then, there is a small range of low wind speeds where the system is limited by the requirement to maintain some minimum airspeed, $v_{k,\min}$. Next, the curve is Loyd-limited, or only limited by the maximum achievable ζ. Here the kite is trying to extract as much power as possible from the wind regardless of tension. At some wind speed, $v_{w,T}$, however, it becomes important to limit the tension. Here, the propeller drag fraction is increased from the Loyd optimal of $k = \frac{1}{2}$ to the maximum value available for the system or until the system's power limit is reached. This curve clearly lies below the Loyd-limited curve. The final section above $v_{w,P}$ is the power limited section. Power may be dumped either by further increasing k and slowing the wing down or by shifting the flight path off from downwind.

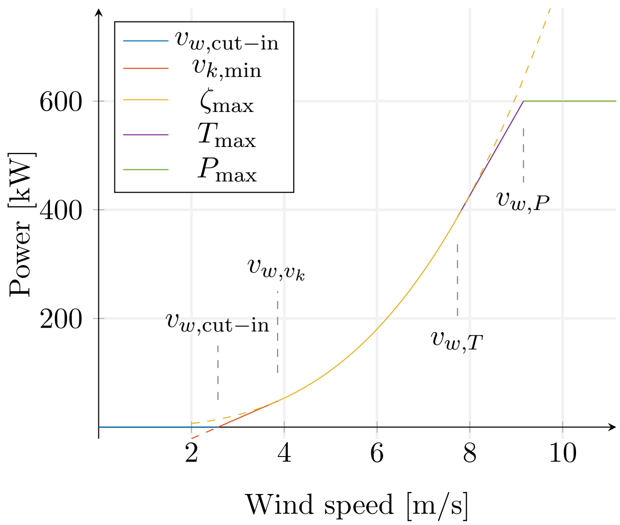

FIGURE 2. Representative ideal power curve showing the airspeed-limited, Loyd-limited, tension-limited, and power-limited segments. This particular power curve assumes: $A_k = 32.9\ \text{m}^2$, $v_{k,\min} = 30$ m/s, $\zeta_{\max} = 42.3$, $T_{\max} = 150$ kN, $P_0 = 96$ kW, and $P_{\max} = 600$ kW.

It is possible to describe the power curve from Fig. 2 in terms of the fundamental limitations of the system: $v_{k,\min}$, $\zeta_{\max}$, $T_{\max}$, and either $k_{\max}$ or $P_{\max}$ depending on whether the power limit is set by the propellers or the electrical power system. In the following, the functional form of the power curve in each of the sections is derived.

The cut-in wind speed is the wind speed at which the kite maintains some minimum airspeed, $v_{k,\min}$, while using zero thrust, i.e. $k = 0$.

$$v_{w,\text{cut-in}} = \frac{C_{D_{\text{sys}}}}{C_L} v_{k,\min} \tag{9}$$

Below the cut-in wind speed, it generally makes sense to land the kite. However, if it is necessary that the kite keep flying, then k must be chosen to satisfy the minimum airspeed requirement. It turns out that the minimum airspeed requirement also constrains the power curve slightly above $v_{w,\text{cut-in}}$ up to the airspeed-limit wind speed threshold, $v_{w,v_k} = \frac{3}{2} v_{w,\text{cut-in}}$.

$$k = \frac{C_L}{C_{D_{\text{sys}}}} \frac{v_w}{v_{k,\min}} - 1, \quad v_w < v_{w,v_k} \tag{10}$$

Thus, the power curve in the airspeed-limited section is

$$P(v_w) = P_0 \left(\frac{v_w}{v_{w,\text{cut-in}}} - 1 \right), \quad v_w < v_{w,v_k} \tag{11}$$

where $P_0 = \frac{1}{2}\rho v_{k,\min}^3 A_k C_{D_{\text{sys}}}$ is the power necessary to keep the kite flying at its minimum airspeed without wind.

The Loyd-limited section is simple and follows the expected v_w^3 function. The propeller drag is held at the optimal $k = \frac{1}{2}$ value and thus the power curve is:

$$P(v_w) = \frac{1}{2}\rho v_w^3 A_k \zeta_{\max}, \quad v_{w,v_k} < v_w < v_{w,T} \tag{12}$$

where $v_{w,T}$ is the tension-limit wind speed threshold to be derived below.

The next section is the tension-limited section. Recall that here the power curve is shifted down by slowing the kite down by increasing the propeller drag. To determine the appropriate propeller drag fraction that meets the tension requirement, combine equations 2 and 3, substituting $T_{\max}$ for tension:

$$\frac{1}{2}\rho v_w^3 A_k \frac{C_L^3}{C_{D_{\text{sys}}}^2} \frac{k}{(1+k)^3} = T_{\max} v_w \frac{k}{1+k} \tag{13}$$

and solve for k:

$$k = \frac{3}{2}\sqrt{\frac{\frac{1}{2}\rho A_k \zeta_{\max}}{T_{\max}/3}} \cdot v_w - 1 \tag{14}$$

The tension limited section begins when $k > \frac{1}{2}$. Thus, the threshold wind speed, $v_{w,T}$, for the tension limited section, expressed in terms of limits of the system, is:

$$v_{w,T} = \sqrt{\frac{T_{\max}/3}{\frac{1}{2}\rho A_k \zeta_{\max}}} \tag{15}$$

From this it is also possible to derive an extremely simple form for the maximum power in the tension limited section as a function of wind speed:

$$P(v_w) = T_{\max}\left(v_w - \frac{2}{3}v_{w,T}\right), \quad v_{w,T} < v_w < v_{w,P} \tag{16}$$

The start of the power limited section may either be set by the fundamental limitations of the electrical power system, $P_{\max}$, or by the maximum drag fraction, $k_{\max}$, achievable with the propellers. In a balanced system, these limits should be consistent with each other, but it is useful to derive the form of the power curve for each case. In the power system limited case, the threshold wind speed, $v_{w,P}$, for the power limited section is:

$$v_{w,P} = \frac{P_{\max}}{T_{\max}} + \frac{2}{3}v_{w,T} \tag{17}$$

In the propeller drag limited case, there is essentially a maximum tension efficiency, which can be used to convert the tension limit to a power limit. Thus,

$$v_{w,P} = \frac{2}{3}(1 + k_{\max})v_{w,T} \tag{18}$$

and the maximum power, P_{max}, is

$$P_{\mathrm{max}} = \frac{2(T_{\mathrm{max}}/3)^{3/2} k_{\mathrm{max}}}{\sqrt{\frac{1}{2}\rho A \zeta_{\mathrm{max}}}} \tag{19}$$

Putting this all together in one place, the power curve is defined by:

$$P(v_w) = \begin{cases} 0 & v_w \leq v_{w,\mathrm{cut-in}} \\ P_0 \left(\frac{v_w}{v_{w,\mathrm{cut-in}}} - 1 \right) & v_{w,\mathrm{cut-in}} < v_w \leq \frac{3}{2} v_{w,\mathrm{cut-in}} \\ \frac{1}{2}\rho v_w^3 A_k \zeta_{\mathrm{max}} & \frac{3}{2} v_{w,\mathrm{cut-in}} < v_w \leq v_{w,T} \\ T_{\mathrm{max}} \left(v_w - \frac{2}{3} v_{w,T} \right) & v_{w,T} < v_w \leq v_{w,P} \\ P_{\mathrm{max}} & v_{w,P} < v_w \leq v_{w,\mathrm{cut-out}} \\ 0 & v_w > v_{w,\mathrm{cut-out}} \end{cases} \tag{20}$$

3. Efficiency losses

3.1. Air-to-grid losses. The simplified analysis in Sec. 1 assumes that the power generated is equal to propeller drag times airspeed, $P = D_{\mathrm{prop}} v_k$; however this does not take into account the efficiency of the propellers in converting aerodynamic power into mechanical shaft power. Moreover, the shaft power from the propellers goes through multiple conversions, from mechanical to electrical and between AC and DC, and is transmitted down the tether. Each of the conversions and transmissions has an associated power loss and efficiency. The total efficiency between the aerodynamic forces on the kite and the electrical power at the grid, $\eta_{\mathrm{air-grid}}$, is given by:

$$\eta_{\mathrm{air-grid}} = \eta_{\mathrm{prop}} \cdot \eta_{\mathrm{motor}} \cdot \eta_{\mathrm{kite\ inv.}} \cdot \eta_{\mathrm{tether}} \cdot \eta_{\mathrm{ground\ inv.}} \tag{21}$$

The efficiency loss for the tether is simply the power loss from resistive heating in the tether $i^2 R_{\mathrm{tether}}$:

$$\eta_{\mathrm{tether}} = 1 - \frac{P_k R_{\mathrm{tether}}}{V_k^2} \tag{22}$$

Efficiency	Typical value	Source
η_{prop}	0.81	XROTOR, Rev. 2 props
η_{motor}	0.95	YASA engineering report
$\eta_{\mathrm{kite\ inv.}}$	0.96	Dyno measurements
η_{tether}	0.95	Eq. 22
$\eta_{\mathrm{ground\ inv.}}$	0.96	Satcon datasheet
$\eta_{\mathrm{air-grid}}$	0.67	Eq. 21

Table 1. Total efficiency stack-up.

3.2. **Off-downwind losses.** The simplified analysis in Sec. 1 also assumes that the flight path of the kite is normal to the direction of the wind. To modify this analysis to account for a flight path that is centered at some elevation and azimuth angle off downwind, simply replace the wind velocity, v_w, with the wind velocity perpendicular to the flight path, $v_{w,\perp}$.

$$v_{w,\perp} = v_w \cos\theta \tag{23}$$

Here θ is the angle between the normal vector of the flight path and the wind vector.

3.3. **Gravity losses.** Significant power is lost due to various effects from the velocity changes caused by gravity. The exact form of this loss depends on the propeller strategy used. There are two naive propeller strategies, neither optimal, which can be used to determine the worst case gravity loss.

The first propeller strategy, which is the easiest to analyze, is the constant airspeed strategy, where a term is added to the nominal propeller drag to exactly cancel the force of gravity. The loss here occurs because the conversion of power from aerodynamic power to electrical grid power and back is not perfectly efficient. The exact form of the power loss with this strategy is

$$P_{\mathrm{grav}} = \frac{m g_{\|} v_k}{\pi} \left(\eta_{\mathrm{air-grid}} - 1/\eta_{\mathrm{air-grid}}\right) \tag{24}$$

The second propeller strategy is the constant drag fraction strategy. Here, the control system ignores the speed variations caused by gravity. One way to view this strategy is that the kite is storing the excess energy during the down stroke as the kinetic energy in the kite and then reusing this energy on the upstroke. Storing and reusing the energy in this manner is more efficient than converting the power to and from electrical power.

To analyze this strategy, assume that the velocity of the kite takes the approximate form

$$v_k \approx v_{k,0} + \Delta v \sin\psi \tag{25}$$

where ψ is the angle around the loop. The velocity modulation fraction, $a = \Delta v / v_{k,0}$, is approximately:

$$a = \frac{\Delta v}{v_{k,0}} \approx \frac{m g_{\|} R_{\mathrm{path}}}{m v_{k,0}^2 + \frac{1}{2}\rho v_{k,0}^2 A_k C_{D_{\mathrm{sys}}}(1+k) R_{\mathrm{path}}} \tag{26}$$

There are two main power loss mechanisms due to the velocity variation. The first mechanism is simply that the kite is no longer flying at the optimal airspeed. The effect of this can be found by integrating the extra drag force on the kite, $D_{\mathrm{extra}} = \frac{1}{2}\rho C_D A_k v_k \Delta v$, around the loop. The second mechanism is that, while the kite generates the same amount of energy per loop, the loops take longer for larger modulations. Combining these effects, the power loss due to gravity is:

$$P_{\text{grav}} = P(v_w)\left(1 - \sqrt{1-a^2} - \frac{a^2}{2}\sqrt{1-a^2}\right) \tag{27}$$

Note that this quantity is defined to be positive for a power less.

4. Realistic power curve

Equation 28 describes the modified power curve after applying the efficiency losses from the previous section:

$$P(v_w) = \begin{cases} 0 & v_w \leq v_{w,\text{cut-in}} \\ \eta_{\text{air-grid}} \cdot P_0 \left(\frac{v_w}{v_{w,\text{cut-in}}} - 1\right) - P_{\text{grav}} & v_{w,\text{cut-in},0} < v_w \leq \frac{3}{2} v_{w,\text{cut-in},0} \\ \eta_{\text{air-grid}} \cdot \frac{1}{2}\rho v_w^3 A_k \zeta_{\max} \cos^3\theta - P_{\text{grav}} & \frac{3}{2} v_{w,\text{cut-in},0} < v_w \leq v_{w,T} \\ \eta_{\text{air-grid}} \cdot T_{\max}\cos\theta\left(v_w - \frac{2}{3}v_{w,T}\right) - P_{\text{grav}} & v_{w,T} < v_w \leq v_{w,P} \\ P_{\max} & v_{w,P} < v_w \leq v_{w,\text{cut-out}} \\ 0 & v_w > v_{w,\text{cut-out}} \end{cases} \tag{28}$$

Note that because of the complicated form of the gravity power losses (see Eq. 24 and Eq. 27), they are included as a wind speed independent constant for now.

$$P_0 = \frac{1}{2}\rho v_{k,\min}^3 A_k C_{D_{\text{sys}}} \tag{29}$$

$$v_{w,\text{cut-in},0} = \frac{C_{D_{\text{sys}}}}{C_L}\frac{v_{k,\min}}{\cos\theta} \tag{30}$$

$$v_{w,\text{cut-in}} = \left(1 + \frac{P_{\text{grav}}}{\eta_{\text{air-grid}} P_0}\right)\frac{C_{D_{\text{sys}}}}{C_L}\frac{v_{k,\min}}{\cos\theta} \tag{31}$$

$$v_{w,T} = \frac{1}{\cos\theta}\sqrt{\frac{T_{\max}/3}{\frac{1}{2}\rho A_k \zeta_{\max}}} \tag{32}$$

$$v_{w,P} = \frac{P_{\max} + P_{\text{grav}}}{\eta_{\text{air-grid}} T_{\max}\cos\theta} + \frac{2}{3}v_{w,T} \tag{33}$$

5. Mean power production

The utility of a power curve is that it enables the calculation of the mean power production, $\bar{P}$, at a site with a given wind distribution. The mean power production is found by integrating the power curve weighted by the probability distribution function of the wind speeds:

$$\bar{P} = \int_0^{v_{w,\text{cut-out}}} dv\, p(v) P(v) \tag{34}$$

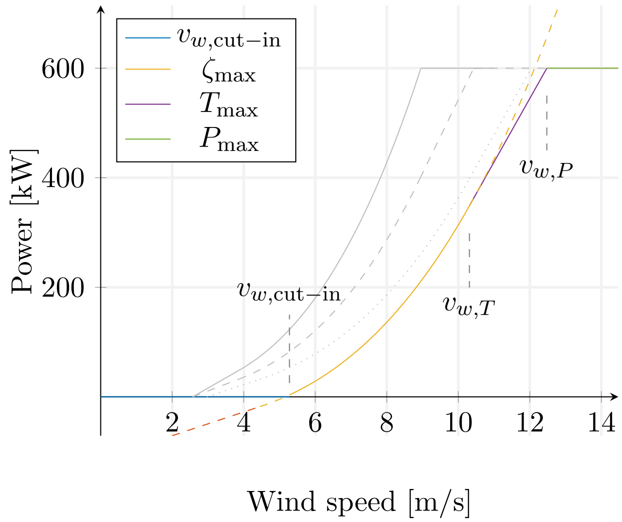

FIGURE 3. Realistic power curve (colored) versus the ideal power curve (solid gray), power curve with conversion losses (dashed gray), power curve with conversion and off-downwind losses (dotted gray). This particular power curve assumes: $A_k = 32.9$ m^2, $v_{k,\min} = 30$ m/s, $\zeta_{\max} = 42.3$, $T_{\max} = 200$ kN, $P_0 = 96$ kW, $P_{\max} = 600$ kW, $\theta = 30°$, $\eta_{\text{air-grid}} = 0.67$, and $P_{\text{grav}} = 50$ kW.

Assuming a Rayleigh distribution, Eq. 35, for the distribution of wind speeds, it is possible to derive a closed form solution for the mean power production of a kite system in terms of the site's mean wind speed, $\bar{v}$, and the limits of the kite system.

$$p(v) = \frac{\pi}{2}\frac{v}{\bar{v}^2}\exp\left(-\frac{\pi}{4}\frac{v^2}{\bar{v}^2}\right) \tag{35}$$

The mean power is composed of terms from the Loyd-limited, tension limited, and power limited sections of the power curve.

$$\bar{P} = \bar{P}_\zeta + \bar{P}_T + \bar{P}_P - \bar{P}_{\text{grav}} \tag{36}$$

Ignoring the gravity losses for now, the functional form of each of these components is:

$$\begin{aligned}\bar{P}_\zeta = \eta_{\text{air-grid}} \cdot \frac{1}{2}\rho\zeta_{\max}\cos^3\theta A\bar{v}^3 \\ \times \left[\frac{6}{\pi}\operatorname{erf}\left(\frac{\sqrt{\pi}}{2}\frac{v_{w,T}}{\bar{v}}\right) - \frac{v_{w,T}}{\bar{v}}\exp\left(-\frac{\pi}{4}\frac{v_{w,T}^2}{\bar{v}^2}\right)\left(\frac{6}{\pi} + \frac{v_{w,T}^2}{\bar{v}^2}\right)\right]\end{aligned} \tag{37}$$

(38)

$$\begin{aligned}\bar{P}_T = \eta_{\text{air-grid}} \cdot T_{\max} \cos\theta \bar{v} \\ \times \Bigg[\operatorname{erf}\left(\frac{\sqrt{\pi}}{2}\frac{v_{w,P}}{\bar{v}}\right) - \operatorname{erf}\left(\frac{\sqrt{\pi}}{2}\frac{v_{w,T}}{\bar{v}}\right) \\ + \left(\frac{2}{3}\frac{v_{w,T}}{\bar{v}} - \frac{v_{w,P}}{\bar{v}}\right)\exp\left(-\frac{\pi}{4}\frac{v_{w,P}^2}{\bar{v}^2}\right) + \frac{1}{3}\frac{v_{w,T}}{\bar{v}}\exp\left(-\frac{\pi}{4}\frac{v_{w,T}^2}{\bar{v}^2}\right)\Bigg]\end{aligned}$$

(39) $$\bar{P}_P = P_{\max}\exp\left(-\frac{\pi}{4}\frac{v_{w,P}^2}{\bar{v}^2}\right)$$

(40) $$\bar{P}_{\text{grav}} = P_{\text{grav}}\left[\exp\left(-\frac{\pi}{4}\frac{v_{w,\text{cut-in}}^2}{\bar{v}^2}\right) - \exp\left(-\frac{\pi}{4}\frac{v_{w,P}^2}{\bar{v}^2}\right)\right]$$

6. Appendix

6.1. **Gravity power loss.** Here is a sketch of the derivation of the velocity modulation for the constant drag fraction strategy (see Eq. 26).

(41) $$m\dot{v}_k = mg_{\|}\cos\psi + \frac{1}{2}\rho v_a A(C_L v_w - C_D v_k)$$

(42) $$v_k(t) = v_{k,0} + \delta v_k(t)$$

Assuming δv_k is small:

(43) $$m\delta\dot{v}_k = mg_{\|}\cos\psi - \frac{1}{2}\rho v_{k,0} C_D A \delta v_k$$

(44) $$\Delta v_k = \int_{side}^{top} d\delta v_k$$

(45) $$= \int_0^{\pi/2} d\psi \frac{R_{\text{path}}}{v_{k,0}}\left(g_{\|}\cos\psi - \frac{1}{2}\rho\frac{v_{k,0}}{m}AC_D\delta v_k\right)$$

(46) $$\frac{\Delta v}{v_{k,0}} \approx \frac{mg_{\|}R_{\text{path}}}{mv_{k,0}^2 + \frac{1}{2}\rho v_{k,0}^2 A_k C_D R_{\text{path}}}$$

The time it takes to go around the loop is

(47) $$T_{\text{loop}} = \int_0^{2\pi} d\psi \frac{R}{v_{k,0}+\delta v} = \frac{R}{v_{k,0}}\int_0^{2\pi} d\psi \frac{1}{1 + a\sin\psi} = \frac{T_{\text{loop},0}}{\sqrt{1-a^2}}$$

where $a = \Delta v / v_{k,0}$.

The energy made in a loop is

$$E_{\mathrm{loop}} = \int_0^{2\pi} d\psi R \frac{1}{2}\rho v_k^2 A_k \eta_{\mathrm{air-grid}} k C_{D_{\mathrm{sys}}} \tag{48}$$

$$= \int_0^{2\pi} d\psi R \frac{1}{2}\rho v_{k,0}^2 (1 + a\sin\psi)^2 A_k \eta_{\mathrm{air-grid}} k C_{D_{\mathrm{sys}}} \tag{49}$$

$$= \frac{1}{2}\rho v_{k,0}^2 R A_k \eta_{\mathrm{air-grid}} k C_{D_{\mathrm{sys}}} \left(1 + \frac{a^2}{2}\right) 2\pi \tag{50}$$

The power loss due to gravity may be found by comparing the $a = 0$ power to the expected power.

$$P_{\mathrm{grav}} = \frac{E_{\mathrm{loop},0}}{T_{\mathrm{loop},0}} - \frac{E_{\mathrm{loop}}}{T_{\mathrm{loop}}} \tag{51}$$

$$= P_0 \left(1 - \sqrt{1 - a^2} - \frac{a^2}{2}\sqrt{1 - a^2}\right) \tag{52}$$