The Bohr-Sommerfeld Model of the Radium Atom: A Mathematical Exploration

Accompanying Material for Manim Animation

February 26, 2025

1 Introduction

The Bohr-Sommerfeld model represents a critical transition point in our understanding of atomic structure, bridging classical physics and early quantum theory. While modern quantum mechanics has superseded this model, it remains a valuable historical framework that provides intuitive insights into atomic behavior.

In this mini-lecture, we explore the mathematical foundations of the Bohr-Sommerfeld model as applied to the radium atom (Ra, $Z = 88$), corresponding to the visual representation in our animation.

2 Historical Context

In 1913, Niels Bohr proposed his model of the hydrogen atom where electrons orbit the nucleus in circular paths with quantized angular momentum. Arnold Sommerfeld later extended this model in 1916 to include elliptical orbits and relativistic effects, creating what we now call the Bohr-Sommerfeld model.

This model was eventually replaced by the quantum mechanical description developed by Schrödinger, Heisenberg, and others in the mid-1920s, but its visualizations remain culturally significant in representing atomic structure.

3 Parametric Representation of Elliptical Orbits

The foundation of the Bohr-Sommerfeld model lies in the parametric description of elliptical electron orbits. Each orbit can be represented by the equations:

$$x(t) = a\cos(t) \tag{1}$$

$$y(t) = b\sin(t) \tag{2}$$

Where:

- a is the semi-major axis
- b is the semi-minor axis
- t is the parameter ranging from 0 to 2π

For a circular orbit, $a = b$, but most electron orbits in the Bohr-Sommerfeld model are elliptical, with $a \neq b$.

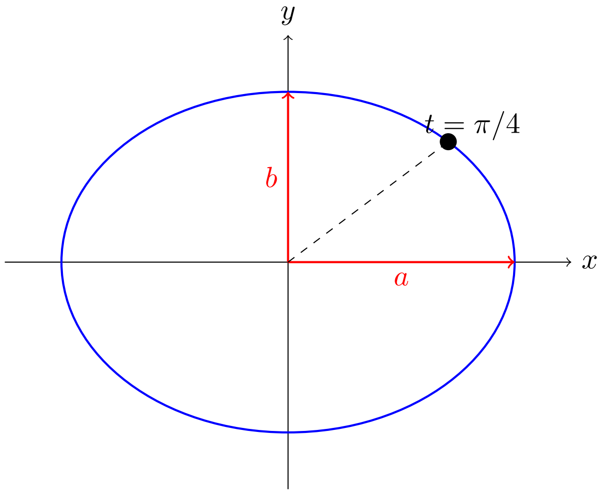

Figure 1: Parametric representation of an elliptical orbit showing the semi-major axis a, semi-minor axis b, and a point at parameter $t = \pi/4$

4 Quantum Conditions and Angular Momentum

The key quantum feature of the Bohr-Sommerfeld model is the quantization condition applied to the angular momentum:

$$\oint p_\phi \, d\phi = nh \tag{3}$$

This closed-loop integral expresses that the action (the canonical angular momentum p_ϕ integrated over a complete orbit) must equal an integer multiple of Planck's constant h. This quantization creates discrete allowed orbits rather than the continuous spectrum permitted by classical mechanics.

The physical interpretation is profound: the electron's angular momentum is quantized in units of $\hbar = h/2\pi$, which restricts electrons to specific orbital paths.

5 Orbital Families and Rotational Symmetry

The complex structure of multi-electron atoms like radium requires multiple orbital families. These families follow rotational symmetry patterns described by:

$$\theta_k = \frac{2\pi k}{n}, \quad \text{for } k = 0, 1, 2, \ldots, n-1 \tag{4}$$

Where:

- k ranges from 0 to $n - 1$
- n is the number of orbits in a family

This formula generates equally spaced rotational angles around the nucleus, creating the symmetric patterns visible in our visualization.

6 Mathematical Model of Radium's Orbital Structure

For our radium atom model, we define several orbital families:

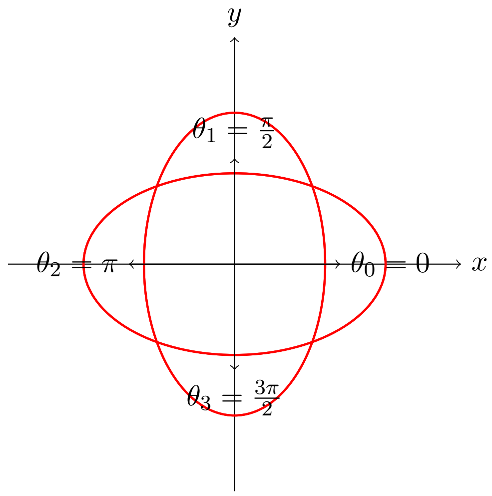

Figure 2: Rotational symmetry pattern for $n = 4$ orbital family (primary red orbits)

6.1 Primary Outer Orbits (Red)

The large red ellipses have parameters:

$$a_1 = 5.2 \times 10^{-12} \text{ m} \tag{5}$$
$$b_1 = 3.6 \times 10^{-12} \text{ m} \tag{6}$$

With 4 orbits rotated at angles:

$$\theta = \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\} \tag{7}$$

6.2 Secondary Orbits (Black/White)

The secondary ellipses have parameters:

$$a_2 = 4.8 \times 10^{-12} \text{ m} \tag{8}$$
$$b_2 = 3.2 \times 10^{-12} \text{ m} \tag{9}$$

With 6 orbits rotated at angles:

$$\theta = \left\{0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}\right\} \tag{10}$$

6.3 Inner Orbital Families

For the inner nested orbits, we use a scaling factor $\lambda = 0.7$:

$$a_i = \lambda^{i-2} \cdot a_2 \tag{11}$$
$$b_i = \lambda^{i-2} \cdot b_2 \tag{12}$$

With i ranging from 3 to 7, representing inner shells.

This recursive scaling creates the nested structure visible in the animation, with each successive shell being 70% the size of the previous one.

7 Transformation of Orbital Coordinates

The rotation of orbits is accomplished using the standard rotation matrix:

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{13}$$

Applied to the orbit coordinates:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = R_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a\cos(t) \\ b\sin(t) \end{pmatrix} \tag{14}$$

Expanding this equation gives us:

$$x' = a\cos(t)\cos\theta - b\sin(t)\sin\theta \tag{15}$$
$$y' = a\cos(t)\sin\theta + b\sin(t)\cos\theta \tag{16}$$

8 Nodal Points

The nodal points highlighted in our animation (labeled α, β, γ, δ) represent special configurations where:

$$\frac{dr}{dt} = 0 \tag{17}$$

These correspond to stationary points in the radial direction, occurring at:

$$t = \{0, \pi\} \text{ for major axis nodes } (\alpha, \beta) \tag{18}$$
$$t = \{\frac{\pi}{2}, \frac{3\pi}{2}\} \text{ for minor axis nodes } (\gamma, \delta) \tag{19}$$

These nodes were important in the original Bohr-Sommerfeld theory for understanding quantum behavior.

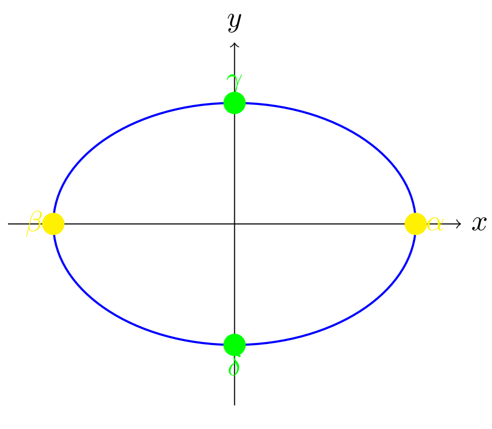

Figure 3: Nodal points on an elliptical orbit: major axis nodes (α, β) and minor axis nodes (γ, δ)

9 Electron Motion Dynamics

For our animation, electron motion along each orbit follows:

$$\mathbf{r}(t) = R_\theta \begin{pmatrix} a\cos(\omega t + \phi_0) \\ b\sin(\omega t + \phi_0) \end{pmatrix} \tag{20}$$

Where:

- ω is the angular frequency
- ϕ_0 is an initial phase offset

According to Kepler's third law, the angular frequency varies with the semi-major axis:

$$\omega \propto \frac{1}{a^{3/2}} \tag{21}$$

This produces the effect of electrons in inner orbits moving faster than those in outer orbits. In our animation, we implement this relationship to create physically realistic motion patterns.

10 Energy Levels

The energy of each orbit in the Bohr model is given by:

$$E_n = -\frac{Z^2 e^4 m}{8\varepsilon_0^2 h^2 n^2} = -\frac{13.6\,\mathrm{eV}\cdot Z^2}{n^2} \tag{22}$$

For radium with $Z = 88$, the ground state energy is:

$$E_1 = -13.6\,\mathrm{eV}\cdot 88^2 = -105,318.4\,\mathrm{eV} \tag{23}$$

Higher energy levels follow the n^{-2} relationship:

Energy Level	**Quantum Number**	**Energy (eV)**
Ground state	$n = 1$	$-105,318.4$
First excited	$n = 2$	$-26,329.6$
Second excited	$n = 3$	$-11,702.0$
Third excited	$n = 4$	$-6,582.4$

Table 1: Energy levels for the radium atom in the Bohr-Sommerfeld model

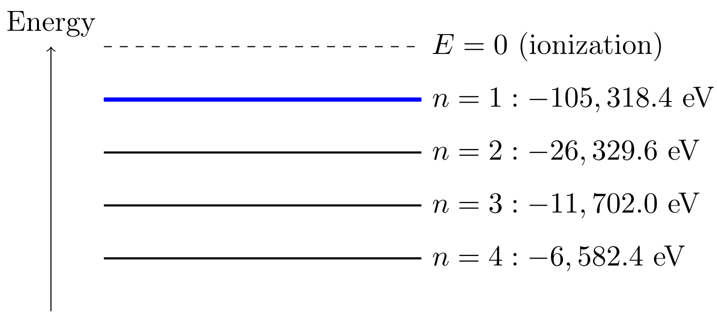

Figure 4: Energy level diagram for radium showing the first four quantum states

11 Limitations of the Bohr-Sommerfeld Model

Despite its elegance, the Bohr-Sommerfeld model had several limitations:

1. It successfully explained the hydrogen spectrum but was less accurate for multi-electron atoms
2. It couldn't explain spectral line intensities
3. It couldn't account for the full complexity of atomic spectra
4. It lacked a proper theoretical foundation, which quantum mechanics later provided

12 Connection to Modern Quantum Mechanics

The Bohr-Sommerfeld quantization rule:

$$\oint p_\phi \, d\phi = nh \tag{24}$$

Is now recognized as a special case of the more general WKB approximation in quantum mechanics. The modern quantum mechanical description replaced discrete orbits with probability distributions described by wave functions.

The principal quantum number n in the Bohr model corresponds to the same quantum number in modern quantum theory, but with profoundly different interpretations:

Feature	**Bohr-Sommerfeld Model**	**Modern Quantum Mechanics**
Electron location	Definite orbits	Probability distributions
Quantum numbers	Orbital parameters	State descriptors
Angular momentum	$n\hbar$	Complex eigenvalues
Visualization	Geometrical orbits	Abstract Hilbert space

Table 2: Comparison between Bohr-Sommerfeld and modern quantum descriptions

13 Conclusion

The Bohr-Sommerfeld model of the radium atom, as visualized in our animation, represents a fascinating historical approach to understanding atomic structure. Its mathematical formalism provided critical insights that helped guide the development of quantum mechanics.

While we now know that electrons don't actually follow discrete elliptical paths as depicted, these visualizations remain powerful conceptual tools for understanding the quantized nature of atomic systems and appreciating the historical development of quantum theory.

The animation we've developed using Manim brings these mathematical concepts to life, offering a visual exploration of a model that, while outdated, forms an essential chapter in the development of our understanding of the atomic world.

References

[1] Bohr, N. (1913). *On the Constitution of Atoms and Molecules.* Philosophical Magazine, 26(153), 1-25.

[2] Sommerfeld, A. (1916). *Zur Quantentheorie der Spektrallinien.* Annalen der Physik, 356(17), 1-94.

[3] Eisberg, R., & Resnick, R. (1985). *Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles.* John Wiley & Sons.