# Gale-Shapley Algorithm: Mathematical Deconstruction

Visual Mathematics Lecture Series

## Conceptual Framework

This analysis deconstructs the Gale-Shapley stable matching algorithm through six fundamental mathematical concepts, revealing deep connections across discrete mathematics and theoretical computer science.

## 1 Bipartite Graphs (Matching Foundations)

- **Visual Component**: Disjoint vertex sets $C$ (colleges) and $A$ (applicants) represented as colored circles
- **Formal Definition**:

$$\begin{aligned}
&C = \{c_1, c_2, \ldots, c_m\} \\
&A = \{a_1, a_2, \ldots, a_n\} \\
&E \subseteq C \times A
\end{aligned}$$

- **Mathematical Significance**: Models matching markets as 2-coloring in graph theory where edges represent permissible pairings. Connects to:
  - Hall's Marriage Theorem (existence conditions)
  - Maximum flow problems (optimization perspective)
  - Ramsey theory (structural constraints)

## 2 Preference Orderings (Combinatorial Structure)

- **Visual Component**: Permutation formulas emerging from representative nodes
- **Preference Formalization**:

$$\begin{aligned}
\text{Applicant } a_k &: c_{\sigma(1)} \succ c_{\sigma(2)} \succ \cdots \succ c_{\sigma(m)} \\
\text{College } c_i &: a_{\tau(1)} \succ a_{\tau(2)} \succ \cdots \succ a_{\tau(n)}
\end{aligned}$$

  where $\sigma, \tau$ are permutations maintaining total order
- **Mathematical Connections**:
  - Permutation groups (symmetric group $S_n$ actions)
  - Linear extensions in poset theory
  - Social choice theory (aggregation paradoxes)

## 3 Combinatorial Grid (Matching Space)

- **Visual Component**: 5x5 grid with alternating crossed cells
- **Mathematical Representation**:

$$\mathcal{M} = \{M \subseteq C \times A \mid (c, a) \in M \Rightarrow \text{compatibility constraints}\}$$

- **Interpretation**: Crossed cells represent forbidden pairings (modulo 2 condition). Relates to:
  - Latin squares (constrained arrangements)
  - Matrix permanents (counting perfect matchings)
  - Design theory (blocking sets)

# 4 Stability Condition (Optimality Criterion)

- **Core Formula**:

$$\nexists (a, c) \in A \times C : (a \prec_c a' \wedge c \prec_a c') \quad \text{(No blocking pairs)}$$

- **Mathematical Implications**:
  - Nash equilibrium in game theory
  - Linear programming duality
  - Fixed point theorems (matching as operators)

# 5 Algorithm Mechanics (Deferred Acceptance)

- **Procedural Steps**:
  1. While $\exists$ unfilled college $c$:
  2. Propose to highest-ranked unmatched $a$
  3. If $a$ prefers $c$ over current match:
     - Accept $c$, reject previous if necessary
- **Mathematical Properties**:
  - Monotonic convergence (Lyapunov functions)
  - Lattice structure of stable matches
  - Complexity class PPAD (algorithmic foundations)

# 6 Quota Constraints (Capacity Limits)

- **Capacity Formula**:

$$\forall c \in C, \ |M(c)| \leq q_c \quad \text{(Injective matching constraint)}$$

- **Mathematical Connections**:
  - Transportation problems (Hitchcock model)
  - Matroid theory (independent sets)
  - Multi-objective optimization (Pareto frontiers)

# Synthesis & Connections

The visualization sequence reveals an intricate web of mathematical relationships:

- Bipartite graphs provide the **relational framework**
- Preference permutations establish **individual utilities**
- Combinatorial grids model **solution spaces**
- Stability conditions ensure **game-theoretic equilibrium**
- Algorithm steps implement **constructive proofs**
- Quota constraints introduce **multiplicity bounds**

This integration demonstrates how matching algorithms synthesize concepts from graph theory, combinatorics, optimization, and economic theory into practical solutions for resource allocation problems.