1st International Conference 'Economic Scientific Research - Theoretical, Empirical and Practical Approaches', ESPERA 2013

Implied volatility in black-scholes model with GARCH volatility

Muhammad Sheraz[a,]*, Vasile Preda[a,b]

[a]*Faculty of Mathematics and Computer Science, University of Bucharest, Bucharest, Romania*
[b]*National Institute of Economic Researcs, Calea 13 Septebrie nr.13, Bucharest, 050711, Romaniat, Romania*

Abstract

The famous Black-Scholes option pricing model is a mathematical description of financial market and derivative investment instruments. In this model volatility is a constant function, where trading option is indeed risky due to random components such as volatility. The notion of non-constant volatility was introduced in GARCH processes. Recently a Black-Scholes model with GARCH volatility has been introduced (Gong et al., 2010).In this article we derive an implied volatility formula for BS-Model with GARCH volatility. In this approach implied volatility patterns are due to market frictions and help us to support the evidence of fat-tailed return distributions against the disputed premise of lognormal returns in Black-Scholes model (Black and Scholes, 1973) .

Keywords: Option Pricing; Black-Scholes Model; GARCH Processes; Volatility;

1. Introduction

Fischer Black and Myron Scholes published an option valuation formula in their 1973's article()(Black and Scholes, 1973) that today is known as Black-Scholes model. The model has some restrictions for example, a risk free interest rate r (constant) and a

* Corresponding author. Tel.: +4-074870-5811.
E-mail address: muhammad_sheraz84@hotmail.com.

constant volatility σ (do not seem to be realistic).Trading option is risky due to possibly high random components such as volatility.

The concept of non-constant volatility has been introduced by GARCH processes .The study of stock price models under these processes is a new horizon in derivative investment instruments. Duan was the first to provide a solid theoretical foundation option pricing in this framework (Duan, 1992).Recently a new extension of model ((Black and Scholes, 1973 with GARCH volatility has been introduced (Gong et al., 2010).The volatility measures, the variation of price of financial instrument over time and implied volatility can be derived from the market price of a traded derivative. In 1986 the concept of implied volatilities was used for financial market research(Latane and Rendleman, 1976). Taylor series approximations have been frequently followed in pricing options. In Risk management particularly first and second order Taylor approximations are crucial. Black-Scholes formula has been considered for Taylor approximation for different purposes (Butler and Schachter, 1986, Latane and Rendleman, 1976).

In this article we consider the new model (Gong et al., 2010).In Section 2 we provide fundamental theory and tools. Section 3 is consisting of implied volatility formula for BS- call option of the model (Gong et al., 2010) and we compare the formula with original model (Black and Scholes, 1973). Finally in section 4 we present some concluding remarks.

2. Preliminaries

Let (Ω, F_t, P) be the probability space then price of an asset S_t at time t is a Geometric Brownian Motion (GBM).

$$dS_t = rS_t dt + \sigma S_t dW_t \qquad (1)$$

Here W_t is a standard Brownian motion and σ is volatility. We know that according to Black and Scholes model (Black and Scholes, 1973), A European call option can be written as:

$$C_{BS} = S\phi(d_1) - Ke^{-r\tau}\phi(d_2)$$

$$d_1 = \frac{\log(\frac{S}{K}) - (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, \ d_2 = d_1 - \sigma\sqrt{\tau} \qquad (2)$$

Where $\phi(.)$ is a cumulative distribution function for standardized normal random variable and $\tau = T - t$, S is the price of the asset , K is a strike price, r is interest rate and T is time to expiry .

Definition (Black and Scholes, 1973**)** If S its stock price, r is interest rate (risk free), then C is European call option that, gives its holder the right, but not the obligation to buy the one unit of underlying asset for a predetermined price K at the maturity date T .

Similarly a Put option P , gives its holder the right, but not the obligation to sell the specified amount of underlying asset for a predetermined price K at the maturity date T .

When variance of the log of stock returns changes with time i.e. $\sigma = \theta_t$ then a new formula has recently been presented (Gong et al., 2010). The call option for the model can be written as:

$$C = SE_{\theta_t}[\phi(d_1)] - Ke^{-rT}E_{\theta_t}[\phi(d_2)]$$

$$d_1 = f(\theta_t^2) = \frac{\log(\frac{S}{K}) + rT + \frac{1}{2}\theta_t^2}{\theta_t}, \ d_2 = g(\theta_t^2) = d_1 - \theta_t \qquad (3)$$

Here, θ_t is a stationary GARCH process having mean μ_θ and variance σ_θ^2 .Option pricing based on GARCH models has been studied under the assumption that the innovations are standard normal (i.e. under normal GARCH).

Definition (Christian et al.,2010**)** A process of the following form is called GARCH (p,q) process

$$\theta_t^2 = \omega + \sum_{i=1}^p \alpha_i \varepsilon_{t-1}^2 + \sum_{j=1}^q \beta_j \sigma_{t-j}^2$$

$$\varepsilon_t = \theta_t z_t \ , z_t \ \square \ N(0, \sigma_t^2), \ \omega > 0, \alpha_i \geq 0, \beta_j \geq 0$$

(4)

Proposition 1 If we consider equation (3) then we have following results.

$$E_{\theta_t}[\phi(d_1)] = \phi\left[\frac{\log(\frac{S}{K}) + rT + \frac{1}{2}E(\theta_t^2)}{\sqrt{E(\theta_t^2)}}\right] \Rightarrow d_1 = \frac{\log(\frac{S}{K}) + rT + \frac{1}{2}E(\theta_t^2)}{\sqrt{E(\theta_t^2)}}$$

$$E_{\theta_t}[\phi(d_2)] = \phi\left[\frac{\log(\frac{S}{K}) + rT - \frac{1}{2}E(\theta_t^2)}{\sqrt{E(\theta_t^2)}}\right] \Rightarrow d_2 = \frac{\log(\frac{S}{K}) + rT - \frac{1}{2}E(\theta_t^2)}{\sqrt{E(\theta_t^2)}}$$

(5)

Then we have

$$d_1 - d_2 = \sqrt{E(\theta_t^2)}$$

Proof: The proof of the proposition is immediate by using, $d_1 - d_2$.

Definition [8] Let $C_t(K,T)$ be the market price of standard European Call option with strike price $K > 0$ and maturity date T at time $t \in [0,T)$ .The implied volatility $\sigma_t(K,T)$ is then defined as the value of the volatility parameter which compare the market price of the option with the price given by the formula.

$$C_t(K,T) = C_{BS}(t, S_t, K, T, r, \sigma_t(K,T))$$

(6)

3. Implied Volatility Formula in Black-Scholes Model with GARCH Volatility

The Black-Scholes structure relates the price of an option to the current time t , stock price S_t , σ volatility of the stock, the interest rate r ,the maturity date T and strike price K . As we know the model introduce that volatility is a constant function throughout the life of an option but empirical research conflicts with the assumption of the model .The implied volatility by the market price can be obtained by inverting the option pricing formula. In this section we use (Gong et al., 2010), and obtain implied volatility, in addition we use the same procedure for the original BS-model, and compare our resulted equations.

CASE-I: When $\sigma = \theta_t$ and θ_t is a GARCH process.

We know that formula for call option of new model with GARCH volatility process(Gong et al., 2010).

$$C = SE_{\theta_t}[\phi(d_1)] - Ke^{-rT}E_{\theta_t}[\phi(d_2)]$$

We know the following expansion

$$N(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left[x - \frac{x^3}{6} + \frac{x^5}{40} \cdots \right]$$ (7)

Using *(7)* for the call option of Black-Scholes with GARCH volatility, also we follow the relation $d_1 - d_2 = \sqrt{E(\theta_t^2)}$ and we suppose for simplicity $X = Ke^{-rT}$.

$$C = S\left[\frac{1}{2} + \frac{1}{\sqrt{2\pi}}(d_1) \right] - X\left[\frac{1}{2} + \frac{1}{\sqrt{2\pi}}(d_2) \right]$$

$$C = S\left[\frac{1}{2} + \frac{1}{\sqrt{2\pi}}(d_1) \right] - X\left[\frac{1}{2} + \frac{1}{\sqrt{2\pi}}(d_1 - \sqrt{E(\theta_t^2)}) \right]$$

$$C = \frac{(S-X)}{2} + \frac{S-X}{\sqrt{2\pi}}(d_1) + \frac{X}{\sqrt{2\pi}}\sqrt{E(\theta_t^2)}$$

We want to get an equation in terms of θ_t and we can simplify the above equation by replacing d_1 with equivalent expression. In addition for simplicity we suppose, $u = \log(\frac{S}{K}) + rT$

$$C = \frac{\sqrt{2\pi}(S-X)\sqrt{E(\theta_t^2)} + 2(S-X)[u + \frac{E(\theta_t^2)}{2}] + 2XE(\theta_t^2)}{2\sqrt{2\pi}\sqrt{E(\theta_t^2)}}$$

$$2\sqrt{2\pi}C\sqrt{E(\theta_t^2)} = \sqrt{2\pi}(S-X)\sqrt{E(\theta_t^2)} + 2u(S-X) + (S-X)E(\theta_t^2) + 2XE(\theta_t^2)$$

$$(S+X)E(\theta_t^2) + \sqrt{2\pi}(S-X)\sqrt{E(\theta_t^2)} - 2\sqrt{2\pi}\sqrt{E(\theta_t^2)}C + 2u(S-X) = 0$$

$$(S+X)E(\theta_t^2) + \sqrt{2\pi}\left[(S-X) - 2C \right]\sqrt{E(\theta_t^2)} + 2u(S-X) = 0$$

Here we have, $\alpha = (S+X)$, $\beta = \sqrt{2\pi}\left[(S-X) - 2C \right]$ and $\gamma = 2u(S-X)$, then we may write the above equation as follows:

$$\alpha E(\theta_t^2) + \beta\sqrt{E(\theta_t^2)} + \gamma = 0$$ (8)

We suppose , $\sqrt{E(\theta_t^2)} = x$ then $E(\theta_t^2) = x^2$,then we may write equation *(8)* as follows:

$$\alpha x^2 + \beta x + \gamma = 0$$ (9)

The equation *(9)* is a simple quadratic equation and discriminate of the equation can be written as $\Delta = \beta^2 - 4\alpha\gamma$

and roots of the equation are : $x = \dfrac{-\beta \pm \sqrt{\Delta}}{2\alpha}$. In addition in the case of non-negative root of equation *(9)* the signs of coefficients α, β and γ are crucial. Furthermore if $\alpha > 0, \beta < 0, \gamma < 0$ then discriminate, $\Delta > 0$ always which implies there exist at least one positive real root of the equation *(9)*. However the signs of coefficients depend upon values of stock price, strike price.

Here we study a special case when value of stock price S is equal to strike price K i.e. $S = K$, then option is called at the money option (ATM). If we consider $r = 0$ then in equation *(9)* the coefficient $\gamma = 0$ and we obtained following equation:

$$\alpha x^2 + \beta x = 0 \qquad\qquad (10)$$

Here the roots of the equation *(10)* are: $x = 0, \ x = \dfrac{-\beta}{\alpha}$, where $\alpha = 2S$ and $\beta = -2\sqrt{2\pi} C$. In other words we can also say sum of the roots of equation *(9)* is equal to the at the money option.

Example 1 Consider the data as used in(Gong et al., 2010). i.e. $S = 425.73$, $C_{GARCH} = 25.33635043$, $r = 0$, then at the money option we have , $x = \dfrac{-\beta}{\alpha} = \dfrac{-\sqrt{2\pi} C_{GARCH}}{S} = 0.1491$

CASE-II: Using Black-Scholes assumption of volatility

In this case we use the formula for call option i.e. C_{BS} and we know that

$$C_{BS} = S\phi(d_1) - Ke^{-r\tau}\phi(d_2)$$

We know the following expansion

$$N(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\left[x - \frac{x^3}{6} + \frac{x^5}{40} \cdots \right]$$

Using *(7)* for the call option of Black-Scholes with GARCH volatility, also we follow the relation $d_1 - d_2 = \sigma\sqrt{\tau}$ and we suppose for simplicity $X = Ke^{-r\tau}$.

$$C_{BS} = S\left[\frac{1}{2} + \frac{1}{\sqrt{2\pi}}(d_1) \right] - X\left[\frac{1}{2} + \frac{1}{\sqrt{2\pi}}(d_2) \right]$$

$$C_{BS} = S\left[\frac{1}{2} + \frac{1}{\sqrt{2\pi}}(d_1) \right] - X\left[\frac{1}{2} + \frac{1}{\sqrt{2\pi}}(d_1 - \sigma\sqrt{\tau}) \right]$$

$$C_{BS} = \frac{(S - X)}{2} + \frac{S - X}{\sqrt{2\pi}}(d_1) + \frac{X}{\sqrt{2\pi}}\sigma\sqrt{\tau}$$

We want to get an equation in terms of θ_t and we can simplify the above equation by replacing d_1 with

equivalent expression. In addition for simplicity we suppose, $v = \log(\frac{S}{K}) + r\tau$.

$$C_{BS} = \frac{\sqrt{2\pi}(S-X)\sigma\sqrt{\tau} + 2(S-X)[v + \frac{\sigma^2\tau}{2}] + 2X\sigma^2\tau}{2\sqrt{2\pi}\sigma\sqrt{\tau}}$$

$$(S+X)\sigma^2\tau + \sqrt{2\pi}\left[(S-X) - 2C_{BS}\right]\sigma\sqrt{\tau} + 2v(S-X) = 0$$

Here we have, $\alpha = (S+X)$, $\beta = \sqrt{2\pi}\left[(S-X) - 2C_{BS}\right]$ and $\gamma = 2v(S-X)$, then we may write the above equation as follows:

$$\alpha\sigma^2\tau + \beta\sigma\sqrt{\tau} + \gamma = 0 \qquad\qquad (11)$$

We suppose , $\sigma\sqrt{\tau} = y$ then $\sigma^2\tau = y^2$,then we may write equation (8) as follows:

$$\alpha y^2 + \beta y + \gamma = 0 \qquad\qquad (12)$$

The equation *(12)* is similar to equation *(9)* therefore we can use the same concept for roots of the equation *(12)* as discussed for *(9)*.

Conclusions

In this article, some extensions of Black and Scholes model with GARCH volatility have been derived. We have used Taylor approximation as discussed in (Gong et al., 2010). However the value of implied volatility might depend upon nature of coefficients of our resulted quadratic equation. In addition using our method in the Black-Scholes model with GARCH volatility, the implied volatility of the stock, which varies over can be determined .The underlying asset price process is continuous and distribution may turned out to be asymmetric.

References

Black, F.,Scholes, M. ,1973. The Pricing of Options and Corporate Liabilities, Journal of Political Economy 81, p.637-659.
Butler, J.S., Schachter, B. ,1986. Unbiased Estimation of the Black-Scholes Formula, Journal of Financial Economics 15 (3), p. 341-357.
Christian, F., Jean, Z.M., 2010. GARCH models, John Wiley and sons.
Corrado, C.J., Miller,T.W., 1996. A Note on a Simple Accurate Formula to Compute Implied Standard Deviations, Journal of Banking and Finance 20, 595-603.
Duan, J.C., 1992. The GARCH Option Pricing Model, Mathematical Finance 5, p. 3-32.
Gong, H.,Thavaneswaran, A., Sing,J.,2010. A Black-Scholes Model with GARCH Volatility, The Mathematical Scientist, 35, p. 37-42.
Latane, H.A., Rendleman R.J., 1976. Standard Deviation of Stock Price Ratios Implied by Option Premia , Journal of Finance 31, p.369-382.
Reinhold, H., 2004.Lecture Notes in Economics and Mathematical Systems: Stochastic Implied Volatility, Springer-Verlag,Berlin Hidelberg.