Efficient Order-Preserving Redistribution of Troves

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1 Introduction

The Liquity protocol [1] issues a USD-pegged stablecoin LUSD that is redeemable at face value against ETH: any owner of LUSD can exchange their stablecoins for equivalent value in the underlying collateral at any time. The ETH paid to the redeemer is taken from the borrowers' collateralized debt positions ("troves") in ascending order of collateral ratio. In other words, the system uses the redeemed LUSD to repay the debt on the riskiest trove with the currently lowest individual collateral ratio (ICR), and transfers a corresponding amount of ETH from the trove to the redeemer. If the redeemed LUSD is larger than the debt on the riskiest trove, the system proceeds with the second riskiest trove, and so on.

To allow for efficient redemptions despite Ethereum's gas constraints, the system keeps troves ordered by ICR, so that it can iterate over the linked list starting from the bottom. It is not feasible to sort the list after every operation.

Troves with an insufficient ICR are subject to liquidation. Given that Liquity's fallback liquidation mechanism [1] redistributes the collateral and debt of a liquidated trove between all remaining active troves, it must be ensured that redistributions do not break trove ordering.

Redistribution in proportion to the collateral size of active troves maintains ordering, as can be easily shown (see 9 Appendix). However, in practice, it is clear that redistributing in a "push" based manner - iterating over all troves and updating their collateral and debt - does not scale, and has computational complexity of $O(n)$, where n is the number of troves.

Previous work by Batog et al [2] derived a scalable $O(1)$ method to assign proportional rewards to a large number of recipients, as long as the basis for their rewards ("stakes") do not change over time. The method is "pull" based: instead of adjusting all recipient positions upon every reward event, the update is deferred to the moment at which an owner changes their position. In the described approach rewards are stored separately from the initial stakes, and do not compound. That is to say, past accumulated rewards are not included in future reward calculations.

It turns out that this approach thus cannot be applied to Liquity as is. As the system undergoes reward events, a given trove's ratio of initial collateral to its total collateral shrinks. Rewards are based on a smaller and smaller share of the total collateral. This is fine, as long as all active troves have experienced all reward events - in this case, ordering is maintained since all troves are affected by the same change.

However, a problem arises when a new trove is created after active troves have received reward shares. Such "fresh" troves (with no accumulated rewards) would thus gain an advantage in redistributions over older troves whose stakes may be smaller than their actual collateral due to the liquidations that have taken place in the meantime. In other words, a "fresh" trove that has experienced fewer rewards than the earlier troves would receive a disproportionate share of subsequent rewards relative to its collateral. Though, a trove's collateral ratio must always be based on its entire collateral, which does include accumulated rewards.

This discrepancy means that the reward distribution scheme described in [2] can break the ordering of troves by collateral ratio. To remedy this, we modify the original approach by introducing a "corrected stake", to ensure fresh troves do not receive disproportionate rewards. We then show that this corrected stake preserves trove ordering.

2 Troves and Liquidations

Troves are collateralized debt positions held by individual borrowers, defined by their amount of debt d (in LUSD) and collateral c (in ETH).

Let γ_i denote the individual collateral ratio (ICR) of a trove i, i.e. the ratio of its collateral to its debt:

$$\gamma_i = \frac{c_i}{d_i} \tag{1}$$

If γ_i falls below a minimum threshold δ, the trove is subject to liquidation. As long as there are sufficient LUSD tokens in the Stability Pool, the system can use them to repay the liquidated debt.

If the Stability Pool is empty, the trove's collateral and debt are redistributed to all active troves in the system as part of the liquidation process (a trove is called *active* as long as it has not been liquidated, or closed by their owner or by a redemption). Every existing borrower with an active trove thus receives a share of both the collateral and the debt of the liquidated trove.

Collateral and debt shares from liquidations are proportional to the *entire* collateral of the recipient troves, i.e. taking into account accumulated "rewards" from prior liquidations. Thus, in the event of a redistribution of a trove k, an active trove i receives the following collateral and debt shares x_i, y_i:

$$(x_i, y_i) = \left(\frac{c_i}{C} \cdot c_k, \frac{c_i}{C} \cdot d_k\right) \tag{2}$$

where $C = \sum c_j$ is the total collateral in the system.

The rewards compound across multiple redistribution events.

3 Trove Ordering

To ensure that the troves can be reliably accessed using a linked list, it is crucial to maintain the ordering of the troves (by ICR) throughout all redistribution events E_k.

The ordering is maintained if $\forall E_k, i, j : (\gamma_i > \gamma_j \implies \gamma'_i > \gamma'_j)$, where γ'_i and γ'_j is the ICR of troves i and j after E_k.

It is easy to prove that a redistribution in proportion to the collateral of the recipient trove as described by equation (2) troves maintains ordering. See 9 Appendix.

4 Efficient Reward Distribution

4.1 Scalable Reward Distribution with Fixed Stakes

A naive "push" based implementation of the redistribution ("strawman approach"), where rewards don't compound, would iterate over all participants and compute the debt and collateral shares for each recipient i separately whenever the system distributes such rewards. Thus, at a reward event t, every recipient would receive the following reward share:

$$r_{i,t} = s_i \cdot \frac{R_t}{S_t} \tag{3}$$

where R_t is the reward distributed at t, $r_{i,t}$ the reward share of recipient i, s_i the stake of recipient i, and S_t the sum of the stakes of all recipients, i.e. $\sum s_j$.

Note that s_i does not depend on t, and is thus fixed throughout multiple reward events. The sum of s_i may vary over time because the set of them may change, stakes entering or leaving the set, but each amount s_i is fixed while it's in the set.

Based on this prerequisite, [2] suggests a scalable $O(1)$ method of distributing such rewards by deferring the reward computation. The total reward share of i from all reward events that occur between time t_1 and t_2 can be written as a sum of its reward shares with the (fixed) stake s_i being factored out:

$$\sum_{t=t_1+1}^{t_2} r_{i,t} = s_i \cdot \sum_{t=t_1+1}^{t_2} \frac{R_t}{S_t} \tag{4}$$

Let Q_t denote the sum of all rewards per total staked amount up to instant t:

$$Q_t = \sum_{k=0}^{t} \frac{R_t}{S_t} \tag{5}$$

Assuming stake s_i is deposited at moment t_1 and then withdrawn at moment $t_2 > t_1$, we can use the sum Q_t to compute the total reward share for participant i since

$$r_i = s_i \cdot \sum_{t=t_1+1}^{t_2} \frac{R_t}{S_t} \tag{6}$$

can be written as

$$r_i = s_i \cdot (Q_{t_2} - Q_{t_1}) \tag{7}$$

As Q_t is monotonic, we can simply track the current (latest) value of Q as a running sum, and snapshot it whenever we expect it to be required for a later computation, i.e. whenever a participant changes their stake.

To compute the total (accumulated) reward for participant i, we can then use following formula:

$$r_i = s_i \cdot (Q - Q_{t_1}) \tag{8}$$

4.2 Corrected Stake Approach for Variable Stakes

In Liquity, collateral and debt shares from liquidations are proportional to the entire collateral of the recipient troves. The redistribution must cope with the fact that some fraction of a trove's entire collateral is the accumulated reward from prior liquidations, and this fraction varies across troves.

The "strawman" approach in 4.1, with rewards proportional to the initial stake, neglects this: it over-rewards fresh troves, and under-rewards older troves.

We introduce a corrected stake s_i to restore proportional reward distribution:

$$s_i = \begin{cases} c_i & for\ C_\emptyset = 0 \\ c_i \cdot \frac{S_\emptyset}{C_\emptyset} & for\ C_\emptyset > 0 \end{cases} \tag{9}$$

The corrected stake s_i is chosen such that it earns rewards from liquidations equivalent to a trove that would have accumulated c_i total collateral by the time the fresh trove i was created.

$S_\emptyset$ and $C_\emptyset$ are the respective snapshots of the total stakes and total collateral in the system, taken immediately after the last liquidation event. Note that with this terminology, the total collateral includes the total stakes, and therefore $C_\emptyset \geq S_\emptyset$

By extending the original formula from [2], we can thus express the collateral and debt share received by a trove by:

$$r_i = \begin{cases} c_i \cdot (Q - Q_{t_1}) & for\ C_\emptyset = 0 \\ c_i \cdot \frac{S_\emptyset}{C_\emptyset} \cdot (Q - Q_{t_1}) & for\ C_\emptyset > 0 \end{cases} \tag{10}$$

Note that r_i can stand for the collateral or the debt share of a recipient trove since both of them can be represented as "rewards" through R_t.

The intuition behind the choice of corrected stake is that the corrected stake effectively models the fresh trove's collateral c_i as a total collateral, which includes 'virtual' accumulated rewards. The corrected stake earns rewards for the trove as if the trove had been in the system from the beginning - thus maintaining proportional reward growth.

5 System Order Evolution

As mentioned above, we aim to prove that the corrected stake approach leaves the ordering of the troves by collateral ratio unchanged throughout all liquidations, regardless of any borrowers that change their troves' collateral between liquidation events.

Given that Liquity's fallback liquidation mechanism [1] redistributes the collateral and debt of a liquidated trove between all remaining active troves, it must be ensured that redistributions do not break trove ordering.

To break up the proof, we introduce the notion of *system order*.

Definition 5.1 (System Order)**.** *We define the order n of a system of troves as the number of trove creations that are either the first one or immediately preceded by a liquidation. We usually denote a system of order n by Γ_n.*

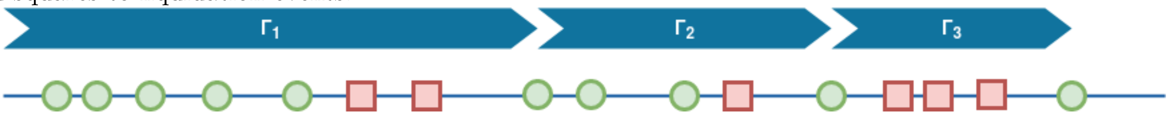
Thus, when the first trove is created, the system order equals 1. The first trove creation event after a series of liquidations triggers a system evolution, and the system order increases by 1.

Definition 5.2 (System order evolution)**.** *Given a system of troves Γ_n of order n, whose last event has been a liquidation, we define a "system order evolution", f, as a sequence of events in the following order:*

- *1 or more new troves are created*
- *0 or more troves are subsequently liquidated*

This sequence of events produces a new system $\Gamma_{n+1} = f(\Gamma_n)$ *of order* $n+1$.

It can be represented graphically in the following way, where circles correspond to trove creations and squares to liquidation events:



Note that the order of a system remains constant throughout the event sequence of a system order evolution after the first creation event, and therefore the definition is consistent.

Let Γ_1 define a system of troves with past liquidations, in which all active troves have received reward shares from all past liquidations. Γ_1 is a first-order system, and contains only stakes of first-order troves. Each first-order stake s_i is equal to its collateral c_i, and $S = \sum s_i = \sum c_i$.

Let Γ_2 define an evolution of Γ_1, i.e. $\Gamma_2 = f(\Gamma_1)$. Γ_2 is a system with past liquidations, with $S = \sum s_i + \sum s_j$, where s_j is the stake of newly added troves j. Γ_2 is a second-order system, containing **both** *first-order* stakes $s_i = c_i$ which have experienced all liquidations, **and** *second-order* stakes s_j which have only experienced the liquidations after their creation.

In general, Γ_n is a system with n sequential pairs, each consisting of a trove creation period and a liquidation period. Troves created in a given trove creation period t have experienced only those liquidations that occurred in liquidation period t or greater.

Proposition 5.1. *At first-order, stake equals initial collateral.*

Proof. For first-order systems, all troves were added before any liquidation events occurred. The snapshot $C_\emptyset$ is equal to 0. Therefore:

$$s_i = c_i \tag{11}$$

for all s_i, c_i in an Γ_1 system. □

Definition 5.3 (System equivalence)**.** *Given two system of troves* Γ *and* Γ'*, we say that they are equivalent,* $\Gamma \equiv \Gamma'$*, if there exists a bijection* $\eta : \Gamma \to \Gamma'$ *between the troves in the system such that* $\forall i \in \Gamma$*, if* $j = \eta(i)$*,* $c_i = c_j$ *and* $d_i = d_j$*, where* c_i *and* c_j *are collaterals,* d_i *and* d_j *are debts, and both include accumulated rewards.*

Note that the systems may not be identical, in particular there could be equivalent troves belonging to different orders and with different stakes.

Proposition 5.2. *The relationship of "system equivalence" defined above is indeed an equivalence relation in the mathematical sense, i.e., it's reflexive, symmetric and transitive.*

Proof. Follows immediately from the definition from a bijection. □

Definition 5.4 (Equivalent system order evolution)**.** *Given two equivalent system of troves* $\Gamma \equiv \Gamma'$ *and a transformation for the first one* f*, we define an equivalent system order evolution* f' *over* Γ' *as the following sequence of events:*

- *The exact same new troves of* f *are created*
- *For each liquidation* $i \in \Gamma$ *of* f*, the trove* $j = \eta(i) \in \Gamma'$ *is liquidated.*

Proposition 5.3. *Given two equivalent systems of troves* $\Gamma \equiv \Gamma'$*, and a system order evolution* f *for the first one, and the equivalent evolution defined above:*

$$f(\Gamma) \equiv f'(\Gamma') \tag{12}$$

Graphically:

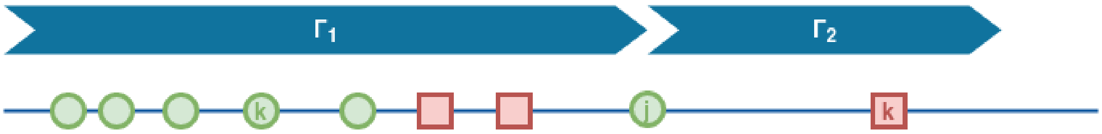
$$\begin{tikzcd} \Gamma \arrow[r, "\equiv"] \arrow[d, "f"'] & \Gamma' \arrow[d, "f'"] \\ f(\Gamma) \arrow[r, "\equiv"] & f'(\Gamma') \end{tikzcd}$$

Proof. For the first step of the evolution, the bijection is kept naturally by tying the new identical troves together.
The second step, liquidations, will remove troves from each system, and in a way that for each $i \in \Gamma$ removed, its image $\eta(i) \in \Gamma'$ is removed too. So the bijection will be maintained too. $\square$

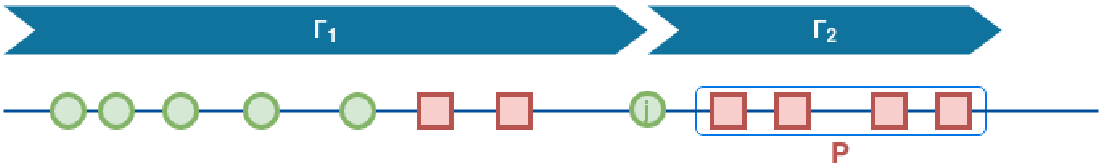
6 Outline of the Proof

We first prove that ICR equality is maintained with rewards proportional to corrected stakes - starting with the simplest case, and progressively generalizing, and finally we prove that order is maintained too.

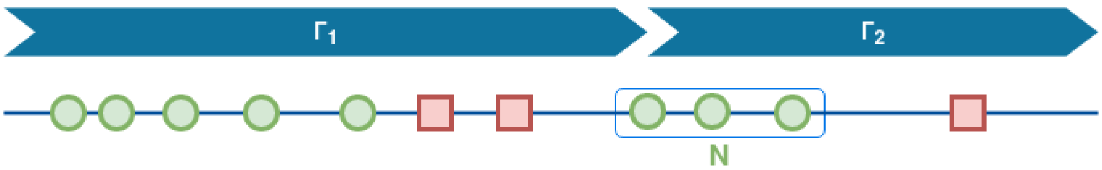
- Lemma 7.1. We consider a system transition $\Gamma_1 \to \Gamma_2$ triggered by the creation of a trove j, followed by a liquidation of a trove k. For any trove i with $\gamma_i = \gamma_j$ in Γ_1, the corrected stake approach maintains $\gamma_i = \gamma_j$ in Γ_2.



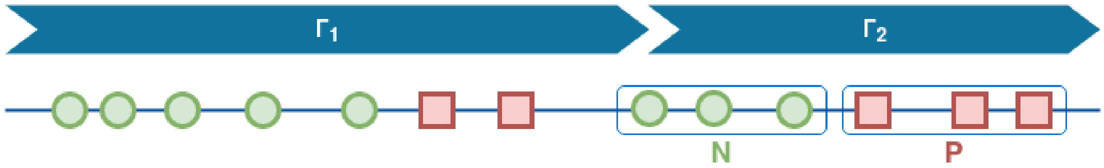
- Lemma 7.2. We consider a system transition $\Gamma_1 \to \Gamma_2$ triggered by the creation of a trove j, followed by a set of liquidations P, such that $|P| > 1$ and $j \notin P$. For any trove i with $\gamma_i = \gamma_j$ in Γ_1, the corrected stake approach maintains $\gamma_i = \gamma_j$ in Γ_2.



- Lemma 7.3. We consider a system transition $\Gamma_1 \to \Gamma_2$ triggered by the creation of a trove, followed by some more creations, up to a set N, such that $|N| > 1$, and then by a single liquidation. For any trove i in Γ_1 and any trove j in N with $\gamma_i = \gamma_j$, the corrected stake approach maintains $\gamma_i = \gamma_j$ in Γ_2.



- Theorem 7.4. We consider a system transition $\Gamma_1 \to \Gamma_2$ triggered by the creation of a trove, followed by some more creations, up to a set N, and then by a set of liquidations P, such that $|N|, |P| > 0$. For any trove i in Γ_1 and any trove $j \in N, j \notin P$ with $\gamma_i = \gamma_j$, the corrected stake approach maintains $\gamma_i = \gamma_j$ in Γ_2.
  In other words, in a second-order system with M previous liquidations, and N second-order troves added after the last liquidation, ICR equality between any first-order trove and corresponding second-order trove with the same ICR holds across P subsequent liquidation events.

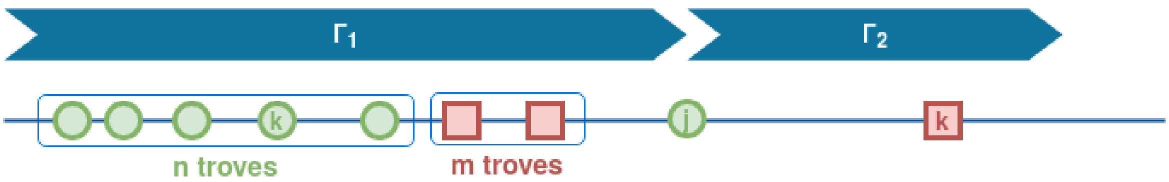


- Lemma 7.5. For every 2^{nd} order system Γ_2 there exists an equivalent first-order system Γ_1 that contains only first-order stakes which have experienced all liquidations.
- Proposition 7.6. For every n^{th} order system Γ_n there exists an equivalent first-order system Γ_1 that contains only first-order stakes which have experienced all liquidations.
- Corollary 7.6.1. In an n^{th} order system with M previous liquidations, and N second-order troves added after the last liquidation, ICR equality between an $(n-1)^{th}$ order trove and n^{th} order trove holds across P liquidation events.
- Theorem 8.1. For every system Γ_n containing troves i and j with $\gamma_i > \gamma_j$ that transitions to Γ_{n+1}, the corrected stake approach maintains $\gamma_i > \gamma_j$.

7 Corrected Stake Approach Preserves Equality

Lemma 7.1. *We consider a system transition $\Gamma_1 \to \Gamma_2$ triggered by the creation of a trove j, followed by a liquidation of a trove k. For any trove i with $\gamma_i = \gamma_j$ in Γ_1, the corrected stake approach maintains $\gamma_i = \gamma_j$ in Γ_2.*

Proof. A first-order system of $n+m$ troves undergoes m trove liquidations, before evolving to second-order.



Consider the m past liquidations from the point of view of an active first-order trove i. As per (9), the stake of trove i is $s_i = c_i$.

Let's define the reward from liquidating trove k received by trove i as the pair $R_i^k = (x_i^k, y_i^k)$ where x_i^k is the share of trove k's collateral received by trove i and y_i^k is the share trove k's debt absorbed by trove i.

Let's call x_i the total accumulated collateral from rewards earned by trove i. At the end of the first-order stage this is the sum of its collateral rewards from m past liquidations.

(For simplicity, let's assume that troves $n+1, ..., n+m$ are the liquidated troves, in that order, and $1, ..., n$ are those that remain)

For a remaining trove:

$$x_i = x_i^{n+1} + ... + x_i^{n+m} = \sum_{k=n+1}^{n+m} x_i^k = \sum_{j=1}^{m} x_i^{n+j} \tag{13}$$

For a liquidated trove the sum would be truncated at the previous liquidated trove:

$$x_i = x_i^{n+1} + ... + x_i^{i-1} = \sum_{k=n+1}^{i-1} x_i^k \tag{14}$$

With each liquidation, c_j collateral is removed from the system. Again, as per (9), stake equals collateral. Thus, the S numerator in each liquidation is reduced by c_j, where j denotes the index of the liquidated trove.

Let

$$C_{n+m} = \sum_{i=1}^{n+m} c_i \tag{15}$$

and

$$L_m = \sum_{j=1}^{m} c_{n+j} \tag{16}$$

We now sum all reward events, noting that the liquidated trove's collateral is removed from the S denominator at each reward. Getting back to (13), note that for each previous liquidated trove $n + j$, the share of collateral that corresponds to trove i is:

$$x_i^j = \frac{c_i}{C_{n+m} - L_j}(c_{n+j} + x_{n+j}) \tag{17}$$

The denominator $C_{n+m} - L_j$ corresponds to the fact that the amount of total initial stakes has been reduced by the amount of liquidated collateral from troves until j, L_j. The collateral to redistribute is $c_{n+j} + x_{n+j}$, which corresponds to the initial collateral of the liquidated trove, c_{n+j}, plus the collateral it has earned from previous liquidations, x_{n+j}.

Therefore:

$$x_i = c_i \left[\frac{c_{n+1} + x_{n+1}}{C_{n+m} - L_1} + \frac{c_{n+2} + x_{n+2}}{C_{n+m} - L_2} + \frac{c_{n+3} + x_{n+3}}{C_{n+m} - L_3} + ... + \frac{c_{n+m} + x_{n+m}}{C_{n+m} - L_m} \right] \tag{18}$$

i.e.

$$x_i = c_i \sum_{j=1}^{m} \frac{c_{n+j} + x_{n+j}}{\sum\limits_{i=1}^{n+m} c_i - \sum\limits_{p=1}^{j} c_{n+p}} \tag{19}$$

(Note, that for liquidation of a given trove j, the redistributed collateral is the sum of its collateral c_{n+j} plus its accumulated collateral reward x_{n+j} which has itself been earned from liquidations $[n+1, n+2, n+3, \dots n+j-1]$. Thus, liquidations have a "roll-up" effect - though, it is not important for our result. In fact, it can also be proved that $x_i = c_i \frac{L_m}{C_n}$)

We label the main sum expression H.

Rewriting trove i's accumulated reward:

$$x_i = Hc_i \tag{20}$$

Summing over all n active troves gives the total accumulated rewards for active troves in the system:

$$X_n = \sum_{i=1}^{n} Hc_i \tag{21}$$

$$X_n = H\ C_n \tag{22}$$

Note that after m liquidations, the system snapshots update from initially:

$$S_\emptyset = C_\emptyset = C_{n+m} \tag{23}$$

to:

$$S_\emptyset = C_n \tag{24}$$

$$C_\emptyset = C_n + X_n \tag{25}$$

(Note that it can also be proved that that $X_n = L_m$ and therefore $C_\emptyset = C_n + L_m = C_{n+m}$)

Now, a fresh trove is added, j, with collateral c_j (and the system becomes second-order). Let the ICR of trove j equal the ICR of an active first-order trove i.

$$\gamma_j = \gamma_i \tag{26}$$

$$\gamma_j = \frac{c_j}{d_j} \tag{27}$$

$$\gamma_i = \frac{c_i + x_i}{d_i + y_i} \tag{28}$$

Where c_j, d_j and c_i, d_i are the collateral and debt values of trove j and trove i respectively.

x_i, y_i are the respective accumulated collateral and debt rewards for trove i earned by its stake over its lifetime.

The ICR equality identity (26) yields the following relation:

$$c_j = \frac{d_j}{d_i + y_i}(c_i + x_i) \tag{29}$$

i.e.

$$c_j = \lambda(c_i + x_i) \tag{30}$$

where

$$\lambda = \frac{d_j}{d_i + y_i} \tag{31}$$

Trove j's stake s_j is given by the corrected stake rule (9), that is:

$$s_j = \frac{c_j \cdot S_\emptyset}{C_\emptyset} \tag{32}$$

Which by (24) and (25) gives:

$$s_j = \frac{c_j \cdot C_n}{C_n + X_n} \tag{33}$$

Now, trove k liquidates. Upon liquidation, the second-order trove j and the first-order trove i earn the following rewards:

$$\begin{aligned} r_{cj} &= as_j \\ r_{dj} &= bs_j \\ r_{ci} &= as_i \\ r_{di} &= bs_i \end{aligned} \tag{34}$$

where

$$a = \frac{c_k + x_k}{S} \tag{35}$$

$$b = \frac{d_k + y_k}{S} \tag{36}$$

And since s_i is a first-order stake:

$$s_i = c_i \tag{37}$$

To show ICR equivalence after the reward event, we must first obtain s_j as a linear function of c_i. Recall our definition of trove j's stake from (33):

$$s_j = \frac{c_j \cdot C_n}{C_n + X_n} \tag{38}$$

Now, substituting in the expression for i's collateral, (30), we obtain:

$$s_j = \frac{\lambda(c_i + x_i)C_n}{C_n + X_n} \tag{39}$$

Substituting in the expressions for accumulated reward x_j from (20), and total accumulated reward X_n from (22):

$$s_j = \frac{\lambda(c_i + Hc_i)C_n}{C_n + HC_n} \tag{40}$$

And factorizing:

$$s_j = \frac{\lambda c_i(C_n + HC_n)}{C_n + HC_n} \tag{41}$$

Canceling yields:

$$s_j = \lambda c_i \tag{42}$$

We now compare ICRs of trove j and trove i, after liquidation of trove k, respectively γ'_j and γ'_i.

$$\gamma'_j = \frac{c_j + r_{cj}}{d_j + r_{dj}} \tag{43}$$

$$\gamma'_i = \frac{c_i + x_i + r_{ci}}{d_i + y_i + r_{di}} \tag{44}$$

Using (34), the individual rewards as functions of stakes:

$$\gamma_j' = \frac{c_j + as_j}{d_j + bs_j} \tag{45}$$

$$\gamma_i' = \frac{c_i + x_i + as_i}{d_i + y_i + bs_i} \tag{46}$$

Now, substituting our definitions for s_i (37) and s_j (42):

$$\gamma_j' = \frac{c_j + a\lambda c_i}{d_j + b\lambda c_i} \tag{47}$$

$$\gamma_i' = \frac{c_i + x_i + ac_i}{d_i + y_i + bc_i} \tag{48}$$

Using identities (30) for c_j, and (31) for d_j:

$$\gamma_j' = \frac{\lambda(c_i + x_i + ac_i)}{\lambda(d_i + y_i + bc_i)} \tag{49}$$

$$\gamma_i' = \frac{c_i + x_i + ac_i}{d_i + y_i + bc_i} \tag{50}$$

Thus, canceling λ:

$$\gamma_j' = \gamma_i' \tag{51}$$

□

Lemma 7.2. *We consider a system transition $\Gamma_1 \to \Gamma_2$ triggered by the creation of a trove j, followed by a set of liquidations P, such that $|P| > 1$ and $j \notin P$. For any trove i with $\gamma_i = \gamma_j$ in Γ_1, the corrected stake approach maintains $\gamma_i = \gamma_j$ in Γ_2.*

Proof. If instead of a single liquidation event at a given system order, we have now a set of P liquidation events, of arbitrary size > 1, it is clear that ICR equality holds across all P events:

Since ICR equality holds across one liquidation event, it will hold across the next, and thus hold for all.

Liquidation events do not alter the stakes that earn shares of liquidated collateral and debt - and for a given stake, the individual trove reward term given in (34) depends only on reward sizes and stakes. □

Lemma 7.3. *We consider a system transition $\Gamma_1 \to \Gamma_2$ triggered by the creation of a trove, followed by some more creations, up to a set N, such that $|N| > 1$, and then by a single liquidation. For any trove i in Γ_1 and any trove j in N with $\gamma_i = \gamma_j$, the corrected stake approach maintains $\gamma_i = \gamma_j$ in Γ_2.*

Proof. With N second-order troves added between consecutive liquidation events, the stake s_j of any given second-order trove is given by (9):

$$s_j = \frac{c_j \cdot S_\emptyset}{C_\emptyset} \tag{52}$$

The snapshots of the system state after the last liquidation event ($S_\emptyset$, $C_\emptyset$) remain constant until the next liquidation. It is clear that all N second-order stakes s_j have been corrected by the same constant factor.

Thus, s_j in the N second-order troves case is equal to s_j in the single second-order trove case.

As such, the logic of the proof of Lemma 7.1 applies - and ICR equality between a second-order trove and first-order trove holds across a liquidation event, no matter how many fresh troves are added in between.
The fact that the liquidated trove is first or second-order doesn't have any impact on that proof either: it would only vary the amounts x_k and y_k in 35 and 36, which may be even zero if k is the last created trove, but we don't make any assumption on those amounts along the proof. $\square$

Theorem 7.4. *We consider a system transition $\Gamma_1 \to \Gamma_2$ triggered by the creation of a trove, followed by some more creations, up to a set N, and then by a set of liquidations P, such that $|N|, |P| > 0$. For any trove i in Γ_1 and any trove $j \in N, j \notin P$ with $\gamma_i = \gamma_j$, the corrected stake approach maintains $\gamma_i = \gamma_j$ in Γ_2.*
In other words, in a second-order system with M previous liquidations, and N second-order troves added after the last liquidation, ICR equality between a first-order trove and second-order trove holds across P subsequent liquidation events.

Proof. Proofs of Lemmas 7.2 and 7.3 can be applied independently to extend Lemma 7.1 into this general result. $\square$

Lemma 7.5. *For every 2^{nd} order system Γ_2 there exists an equivalent first-order system Γ_1 that contains only first-order stakes which have experienced all liquidations.*

Proof. We now show that a second-order system is equivalent to a first-order system.

Consider a hypothetical first-order trove i and an actual second-order trove j. Let both troves have identical ICR, and also let trove i's total collateral and debt equal trove j's initial collateral and initial debt respectively:

$$c_i + x_i = c_j \tag{53}$$

$$d_i + y_i = d_j \tag{54}$$

Clearly, the ratio $\lambda = \frac{d_j}{d_i + y_i} = 1$.

We substitute $\lambda = 1$ into the second-order system expression for s_i, from equation (42), to yield:

$$s_j = c_i \tag{55}$$

Thus, any second-order stake is equivalent to some hypothetical first-order stake $s_i = c_i$, which has accumulated collateral reward $x_i = (c_j - c_i)$ and debt reward $y_i = (d_j - d_i)$.

Therefore any second-order system is equivalent to a first-order system that contains only first-order stakes which have experienced all liquidations. We write:

$$\Gamma_2 \equiv \Gamma_1 \tag{56}$$

$\square$

Proposition 7.6. *For every n^{th} order system Γ_n there exists an equivalent first-order system Γ_1 that contains only first-order stakes which have experienced all liquidations.*

Proof. We prove first by induction that for any n^{th} order system Γ_n there exits an equivalent $(n-1)^{th}$ order system Γ_{n-1}.
We have already proved it for $n = 2$ in previous lemma (it doesn't make sense for $n = 1$).
Now we show that if it's true for n then it's true for $n + 1$, i.e.:

$$\forall \Gamma_n \ \exists \Gamma_{n-1} \ s.t. \ \Gamma_n \equiv \Gamma_{n-1} \Rightarrow \forall \Gamma_{n+1} \ \exists \Gamma_n \ s.t. \ \Gamma_{n+1} \equiv \Gamma_n \tag{57}$$

Let Γ_{n+1} an $(n+1)^{th}$ order system, and let Γ_n an n^{th} order system and f a system order evolution that originate Γ_{n+1}, i.e.:

$$\Gamma_{n+1} = f(\Gamma_n) \tag{58}$$

We are assuming that there is $(n-1)^{th}$ order system Γ_{n-1} such that $\Gamma_n \equiv \Gamma_{n-1}$. By proposition 5.3, we can define an evolution f' such that: $f(\Gamma_n) \equiv f'(\Gamma_{n-1})$. Therefore:

$$\Gamma_{n+1} = f(\Gamma_n) \equiv f'(\Gamma_{n-1}) \tag{59}$$

And by the definition of f', $\Gamma_n := f'(\Gamma_{n-1})$ is an n^{th} order system.

So, for every $n > 1$, if Γ_n is an n^{th} order system, there is an $(n-1)^{th}$ order system Γ_{n-1} such that $\Gamma_n \equiv \Gamma_{n-1}$, and for the transitive property of equivalence, we finally can descend to find a first-order system Γ_1 such that:

$$\Gamma_n \equiv \Gamma_1 \tag{60}$$

□

Having shown all n^{th} order systems are equivalent to a first-order system, we now extend our previous conclusion to n^{th} order systems:

Corollary 7.6.1. *In an n^{th} order system with M previous liquidations, and N second-order troves added after the last liquidation, ICR equality between an $(n-1)^{th}$ order trove and n^{th} order trove holds across P liquidation events.*

8 Corrected Stake Approach Preserves Order

Theorem 8.1. *For every system Γ_n containing troves i and j with $\gamma_i > \gamma_j$ that transitions to Γ_{n+1}, the corrected stake approach maintains $\gamma_i > \gamma_j$.*

Proof. Here we show that ICR ordering is preserved with corrected stakes across a liquidation event.

We make use of the first-order equivalence result, namely, that with corrected stakes:

$$\Gamma_n = \Gamma_1 \tag{61}$$

i.e:

Any n^{th} order system of troves is equivalent to a first-order system of troves. For a given fresh trove with stake s_i and collateral c_i, the stake s_i is equivalent to some hypothetical first-order stake c_j which has accumulated collateral reward $x_j = (c_i - c_j)$ and debt reward $y_j = (d_i - d_j)$.

Due to this equivalence between first and n^{th}-order systems, if ordering is preserved for first-order systems, it is preserved for n^{th} order systems.

Now consider a first-order system of troves, with stakes equal to their initial collateral.

Let trove i and trove j be troves with initial collateral c_i, c_j accumulated collateral and debt rewards x_i, y_i and x_j, y_j respectively:

$$\gamma_i = \frac{c_i + x_i}{d_i + x_i} \tag{62}$$

$$\gamma_j = \frac{c_j + x_j}{d_j + y_j} \tag{63}$$

Let their ICRs be such that:

$$\gamma_i > \gamma_j \tag{64}$$

Since, a first-order trove's collateral and debt rewards are always in direct proportion to its initial collateral, we can write the accumulated rewards as:

$$x_i = Ac_i \tag{65}$$

$$x_j = Ac_j \tag{66}$$

and

$$y_i = Bc_i \tag{67}$$

$$y_j = Bc_j \tag{68}$$

Where A is the sum of all 'collateral rewards per unit staked', and B is the sum of all 'debt rewards per unit staked'. This yields ICRs:

$$\gamma_i = c_i \frac{1 + A}{d_i + Bc_i} \tag{69}$$

$$\gamma_j = c_j \frac{1 + A}{d_j + Bc_j} \tag{70}$$

And the initial ICR inequality becomes:

$$c_i \frac{1 + A}{d_i + Bc_i} > c_j \frac{1 + A}{d_j + Bc_j} \tag{71}$$

Cross multiplying and canceling the common denominator yields:

$$c_i \left(1 + A\right) \left(d_j + Bc_j\right) > c_j \left(1 + A\right) \left(d_i + Bc_i\right) \tag{72}$$

Then expanding:

$$c_i \left(d_j + Bc_j\right) > c_j \left(d_i + Bc_i\right) \tag{73}$$

$$c_i d_j + Bc_i c_j > c_j d_i + Bc_i c_j \tag{74}$$

And canceling terms:

$$c_i d_j > c_j d_i \tag{75}$$

Finally yielding the result:

$$\frac{d_j}{c_j} > \frac{d_i}{c_i} \tag{76}$$

We will later use this to prove that the inequality of ICRs holds across a liquidation event.

Now consider a liquidation event occurs. Upon a trove liquidation, r_c collateral and r_d debt are distributed to all active troves. Each active trove earns rewards proportional to its initial collateral, thus:

$$\gamma_i' = \frac{c_i\,(1+A) + ac_i}{d_i + Bc_i + bc_i} \tag{77}$$

$$\gamma_j' = \frac{c_j\,(1+A) + ac_j}{d_j + Bc_j + bc_j} \tag{78}$$

Where:

$$a = \frac{r_c}{S} \tag{79}$$

$$b = \frac{r_d}{S} \tag{80}$$

Collecting terms:

$$\gamma_i = \frac{c_i\,(1+a+A)}{d_i + (1+B)\,c_i} \tag{81}$$

$$\gamma_j = \frac{c_j\,(1+a+A)}{d_j + (1+B)\,c_j} \tag{82}$$

And taking reciprocals:

$$\frac{1}{\gamma_i'} = \frac{d_i + (1+B)\,c_i}{c_i\,(1+a+A)} \tag{83}$$

$$\frac{1}{\gamma_j'} = \frac{d_j + (1+B)\,c_j}{c_j\,(1+a+A)} \tag{84}$$

Rearranging, and separating the constant term:

$$\frac{1}{\gamma_i'} = \left[\frac{\frac{d_i}{c_i}}{1+a+A}\right] + \left[\frac{1+B}{1+a+A}\right] \tag{85}$$

$$\frac{1}{\gamma_j'} = \left[\frac{\frac{d_j}{c_j}}{1+a+A}\right] + \left[\frac{1+B}{1+a+A}\right] \tag{86}$$

Recall our earlier result (76): $\frac{d_i}{c_i} < \frac{d_j}{c_j}$. Thus:

$$\frac{1}{\gamma_i'} < \frac{1}{\gamma_j'} \tag{87}$$

Then taking reciprocals, finally yields:

$$\gamma_i' > \gamma_j' \tag{88}$$

Therefore, trove ordering holds across a liquidation event in first-order systems, and thus holds across a liquidation event in n^{th} order systems.

References

[1] R. Lauko, R. Pardoe. *Liquity: Decentralized Borrowing Protocol (Whitepaper), 2020.* https://docsend.com/view/bwiczmy

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9 Appendix

We prove that after a redistribution proportional to the collateral of active troves, the ordering of active troves by ICR is preserved.

Consider two active troves i and j with ICR γ_i, γ_j such that $\gamma_i > \gamma_j$.

Let c_i, d_i and c_j, d_j denote the collateral and debt of troves i and j respectively, and let C denote the total active collateral in the system.

Given the definition of ICR we have

$$\frac{c_i}{d_i} > \frac{c_j}{d_j} \tag{89}$$

In the event of a redistribution of a trove k, its collateral c_k and debt d_k is redistributed to all active troves, proportional to their shares of C.

Troves i and j thus receive the following pairs of collateral and debt rewards (x_i, y_i) and (x_j, y_j):

$$(x_i, y_i) = \left(\frac{c_i}{C} \cdot c_k, \frac{c_i}{C} \cdot d_k\right) \tag{90}$$

$$(x_j, y_j) = \left(\frac{c_j}{C} \cdot c_k, \frac{c_j}{C} \cdot d_k\right) \tag{91}$$

As a result, their ICRs will change to:

$$\gamma_i' = \frac{c_i + x_i}{d_i + y_i} \tag{92}$$

$$\gamma_j' = \frac{c_j + x_j}{d_j + y_j} \tag{93}$$

Let $a = \frac{c_k}{C}$ and $b = \frac{d_k}{C}$. Substituting the rewards, this leads to:

$$\gamma_i' = \frac{c_i + a \cdot c_i}{d_i + b \cdot c_i} \tag{94}$$

$$\gamma_j' = \frac{c_j + a \cdot c_j}{d_j + b \cdot c_j} \tag{95}$$

The equations (94) and (95) can be rewritten as:

$$\frac{1}{\gamma_i'} = \left(\frac{d_i}{c_i} + b\right) \cdot \frac{1}{1+a} \tag{96}$$

$$\frac{1}{\gamma_j'} = \left(\frac{d_j}{c_j} + b\right) \cdot \frac{1}{1+a} \tag{97}$$

Taking the reciprocal of inequality (89):

$$\frac{d_i}{c_i} < \frac{d_j}{c_j} \tag{98}$$

and applying it to equations (96) and (97), canceling out common constants, yields:

$$\frac{1}{\gamma_i'} < \frac{1}{\gamma_j'} \tag{99}$$

Finally, taking the reciprocal of (99), yields:

$$\gamma_i' > \gamma_j' \tag{100}$$

Thus, ordering of active troves by ICR is preserved across a reward event that is proportional to the collateral of the active troves. □