DeFi’s Concentrated Liquidity From Scratch

Lecture 3 of 5

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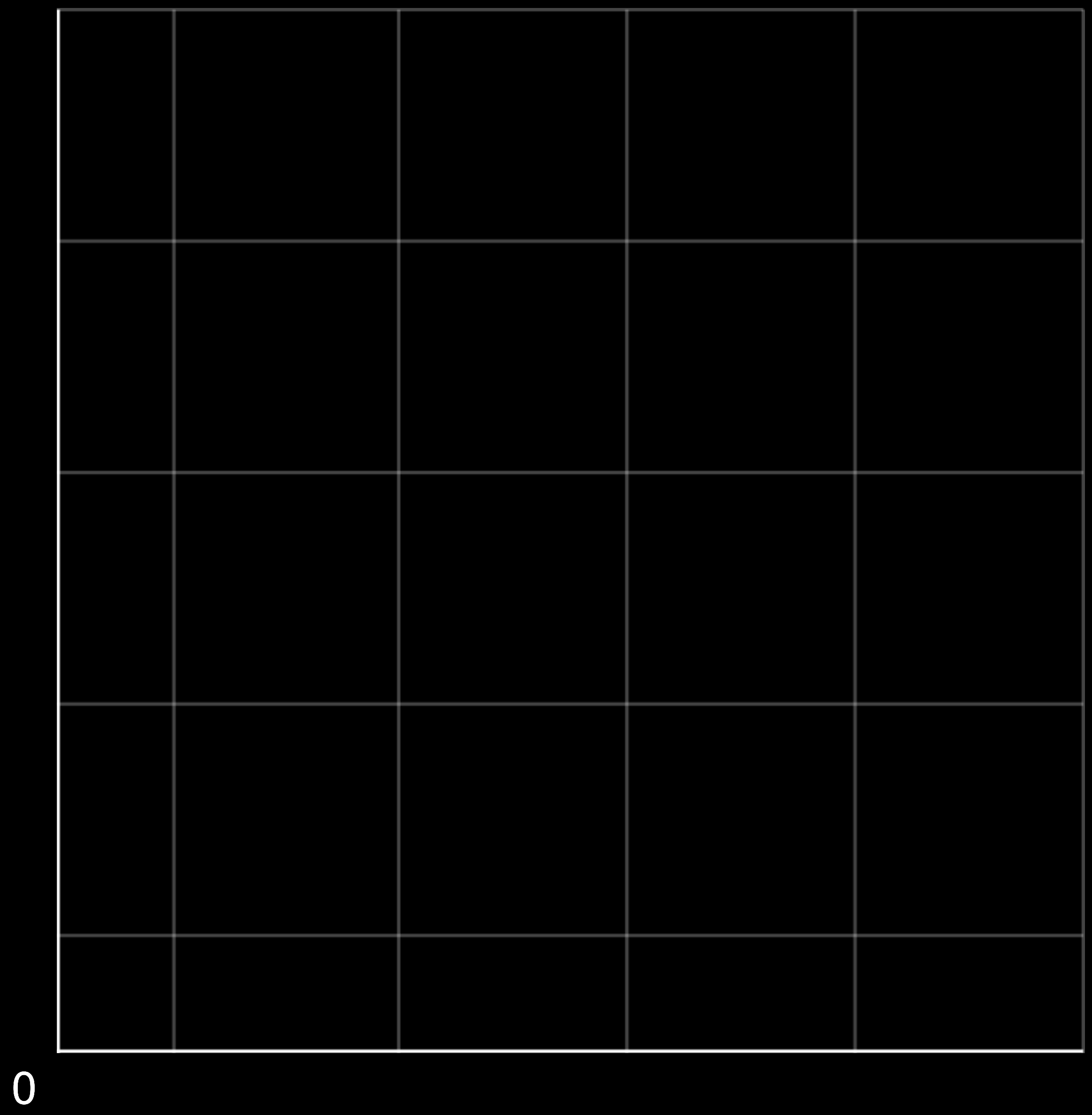
Project Lead, Bancor

<General Discussion>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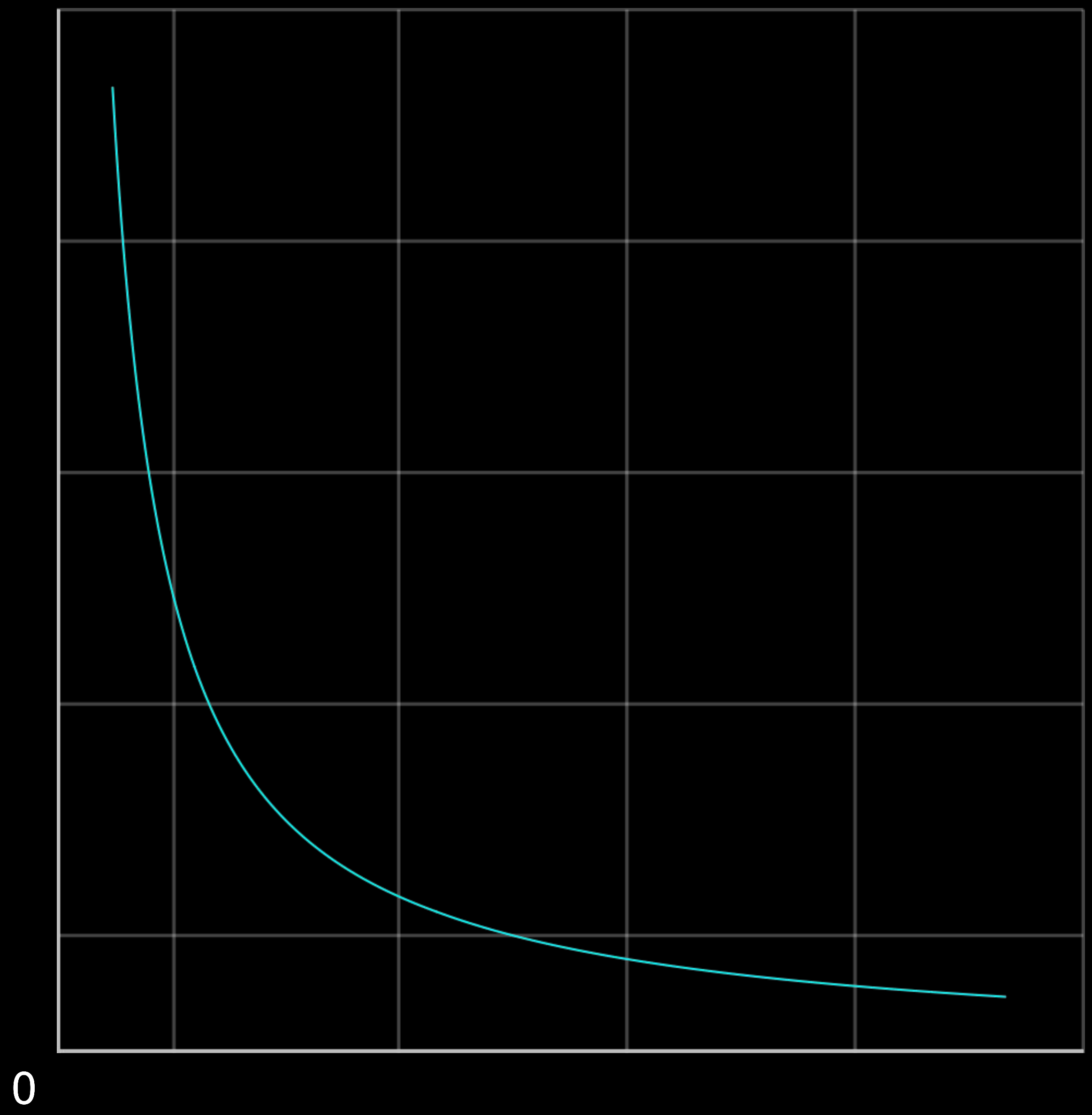
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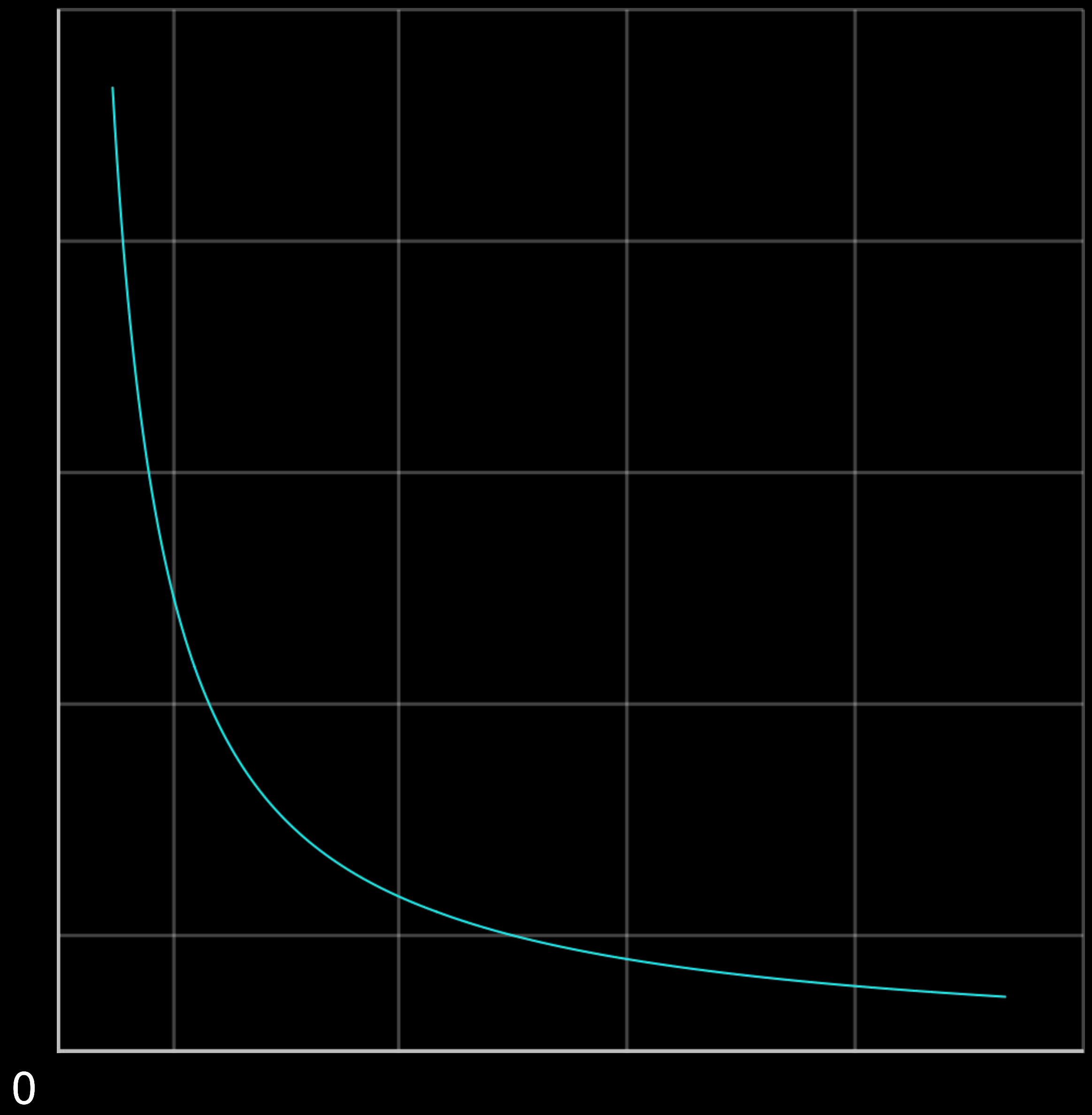
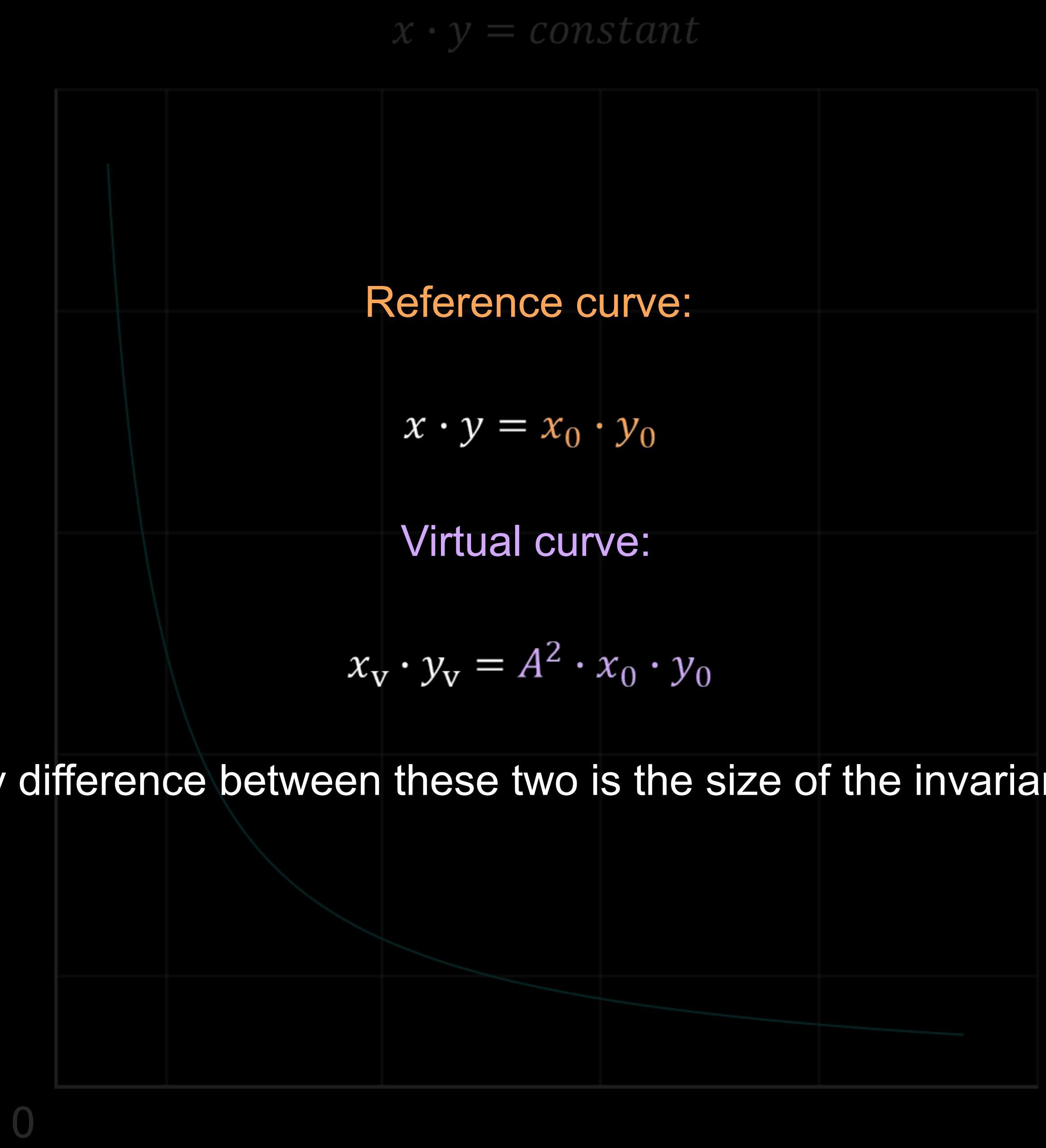
$$x \cdot y = constant$$





Reference curve:

$$x \cdot y = x_0 \cdot y_0$$

Virtual curve:

$$x_{\mathrm{v}} \cdot y_{\mathrm{v}} = A^2 \cdot x_0 \cdot y_0$$

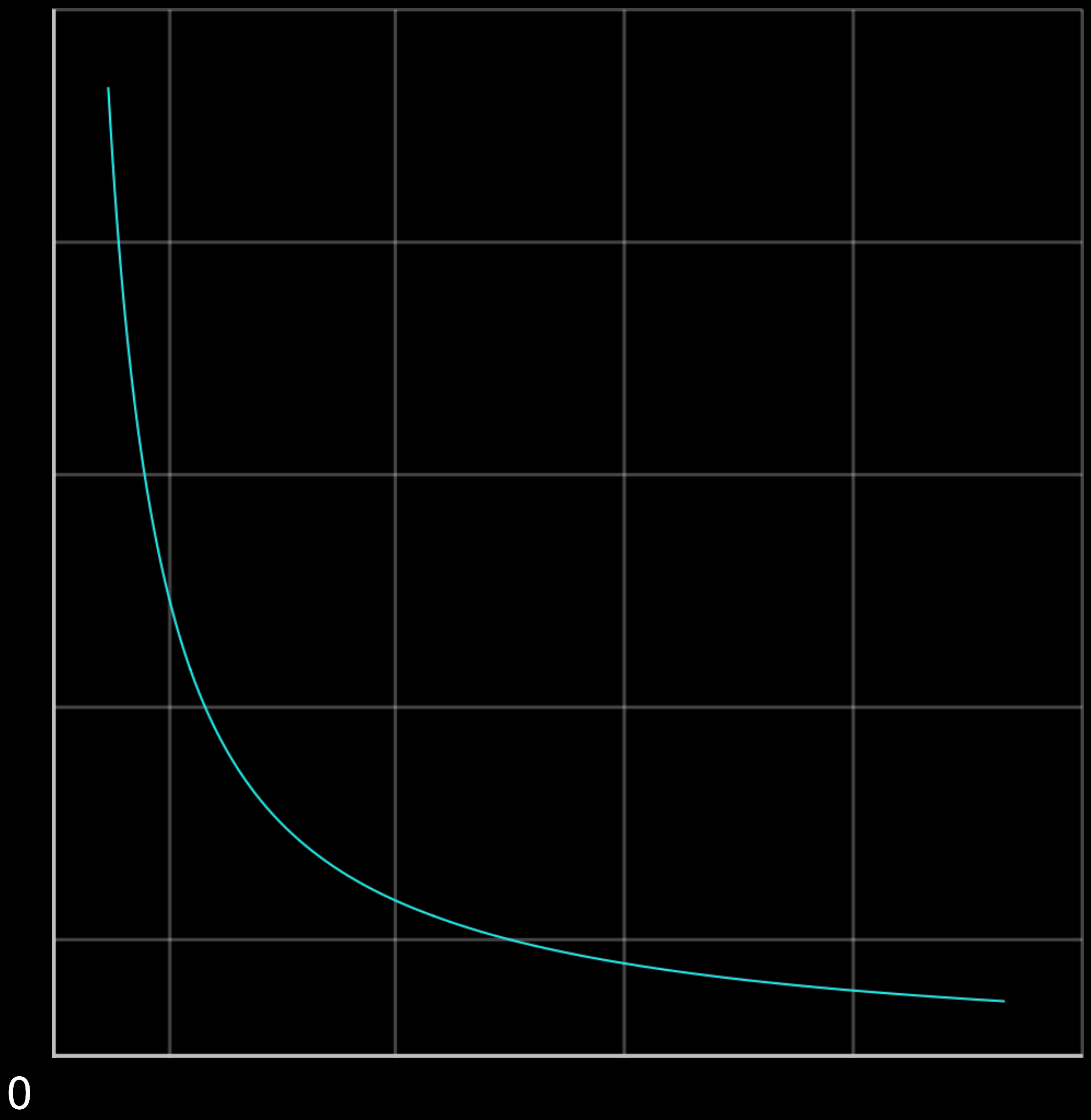
The only difference between these two is the size of the invariant part.

$$x \cdot y = constant$$

$$x \cdot y = 12$$

$$3 \cdot y = 12$$

$$y = \frac{12}{3} = 4$$



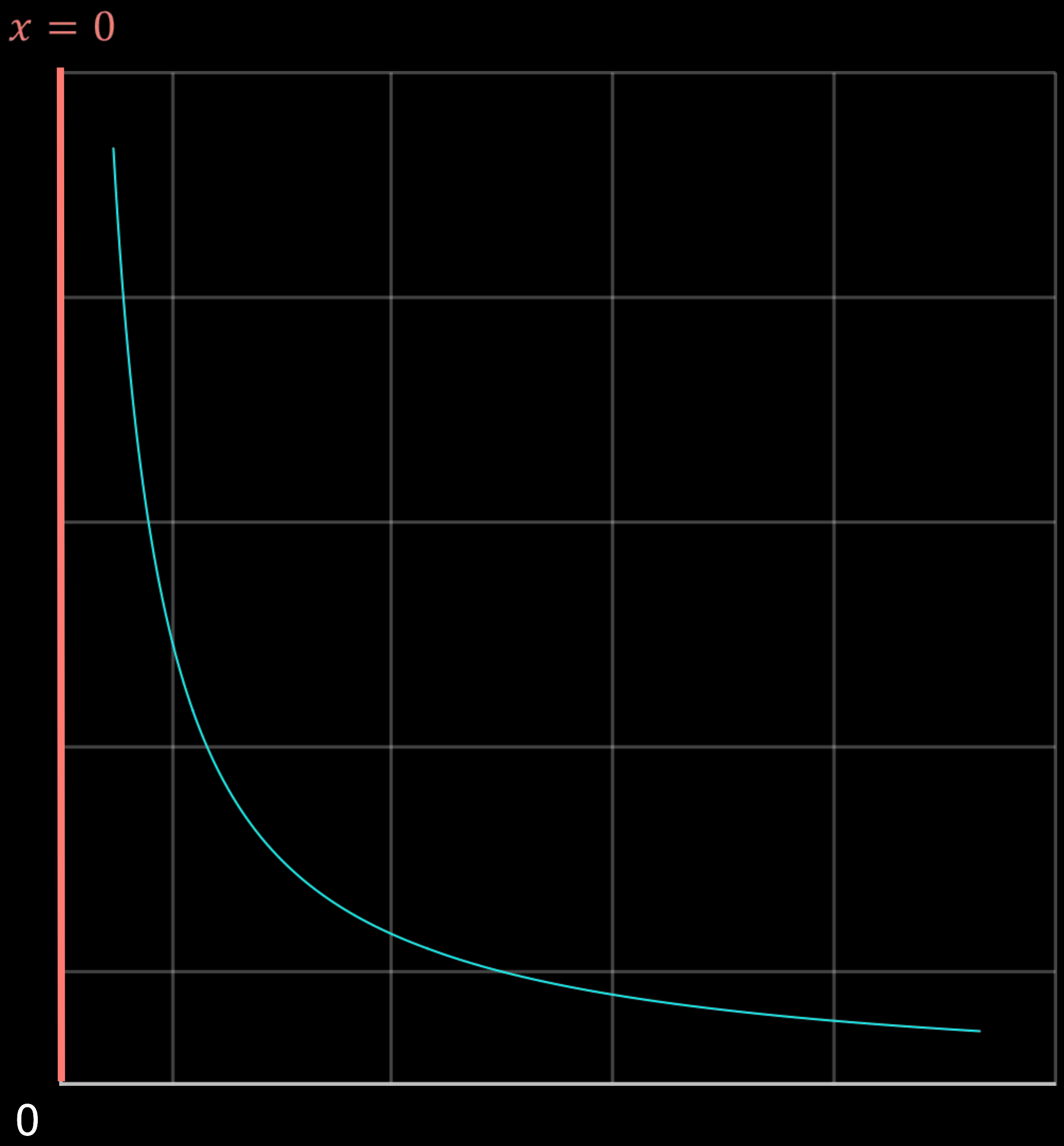
$$x \cdot y = constant$$

$$x \cdot y = 12$$

$$0 \cdot y = 12$$

$$y = \frac{12}{0} = undefined$$

$\therefore x = 0$ is an asymptote



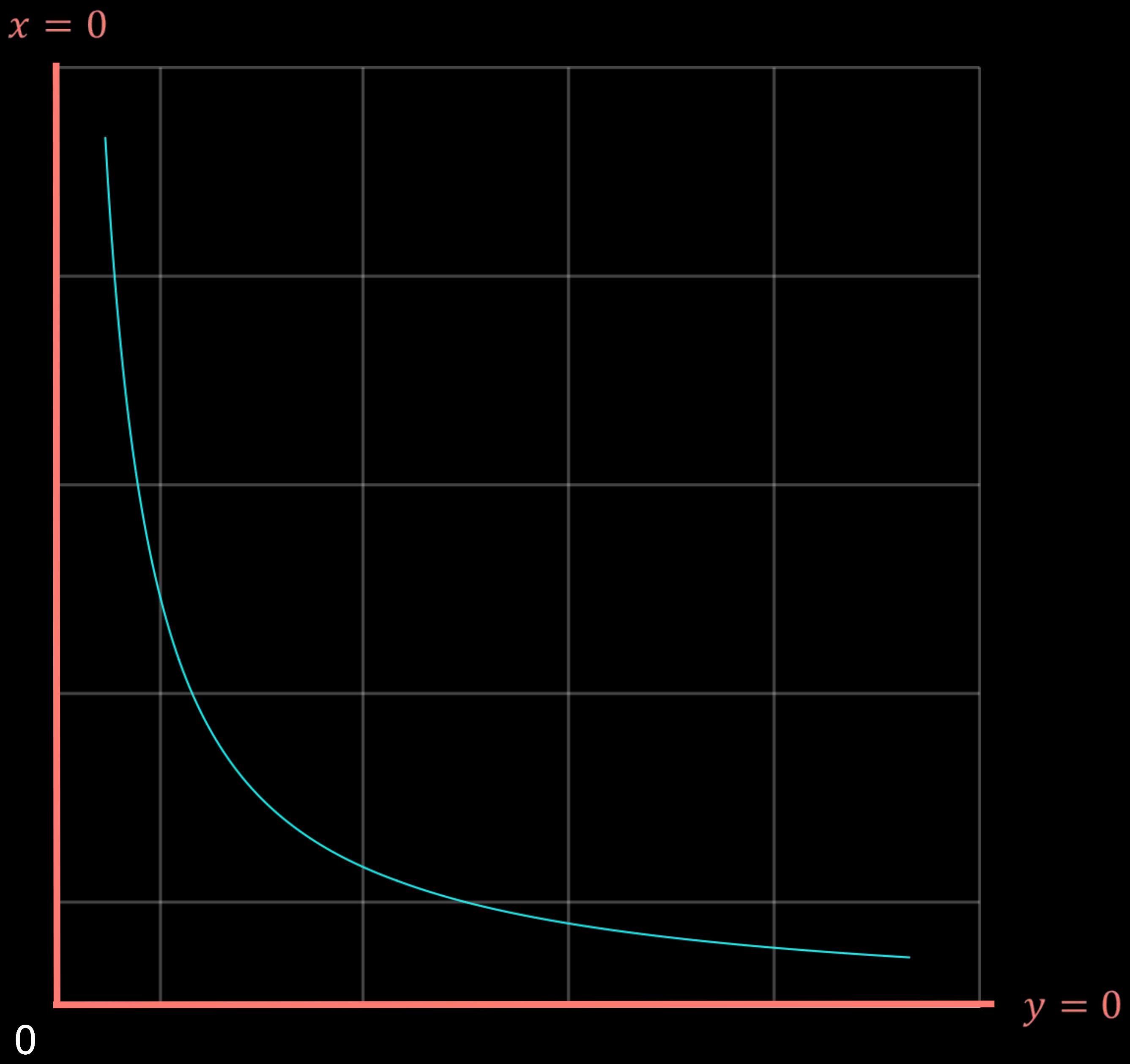
$x \cdot y = constant$

$x \cdot y = 12$

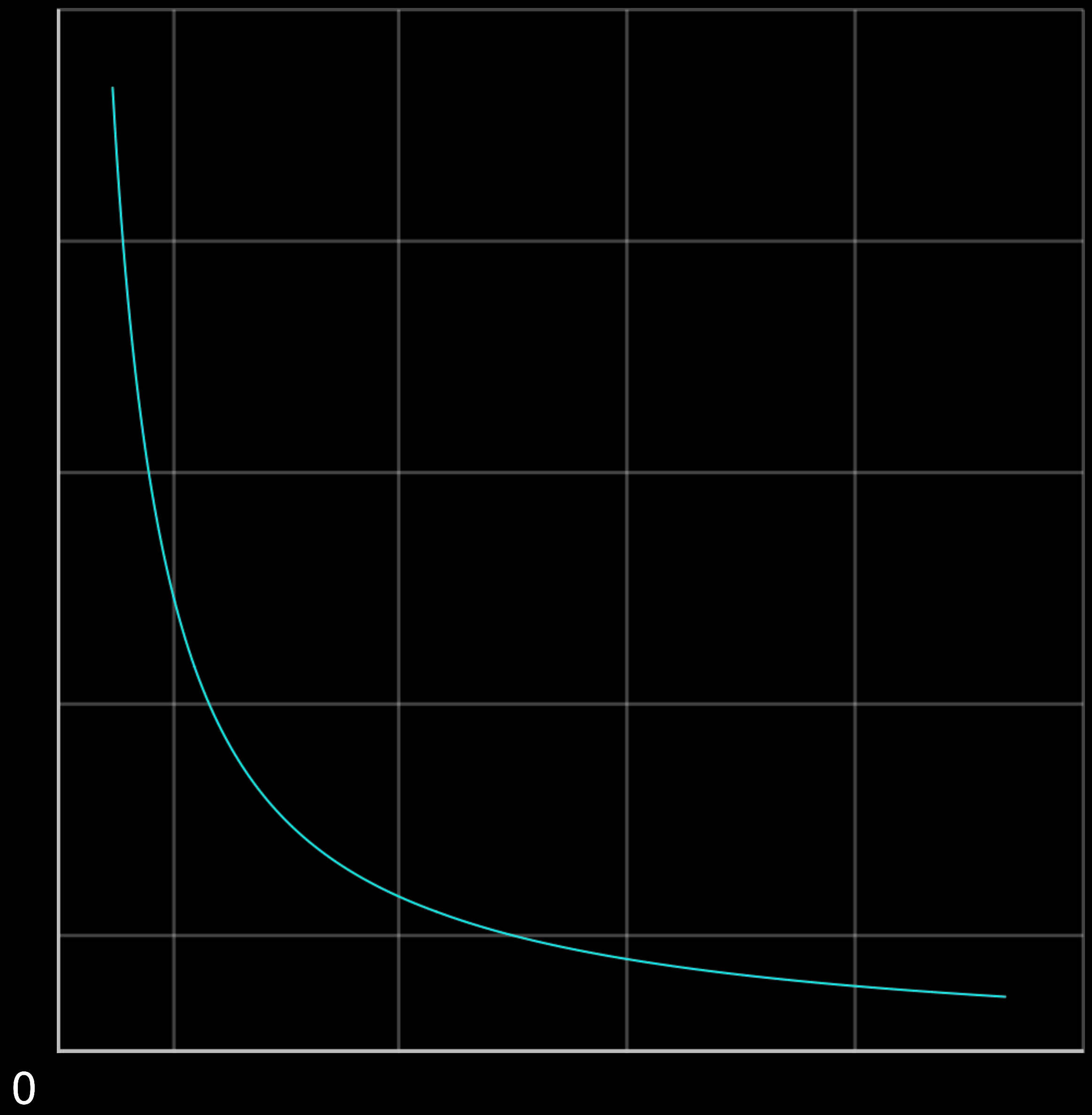
$x \cdot 0 = 12$

$x = \frac{12}{0} = undefined$

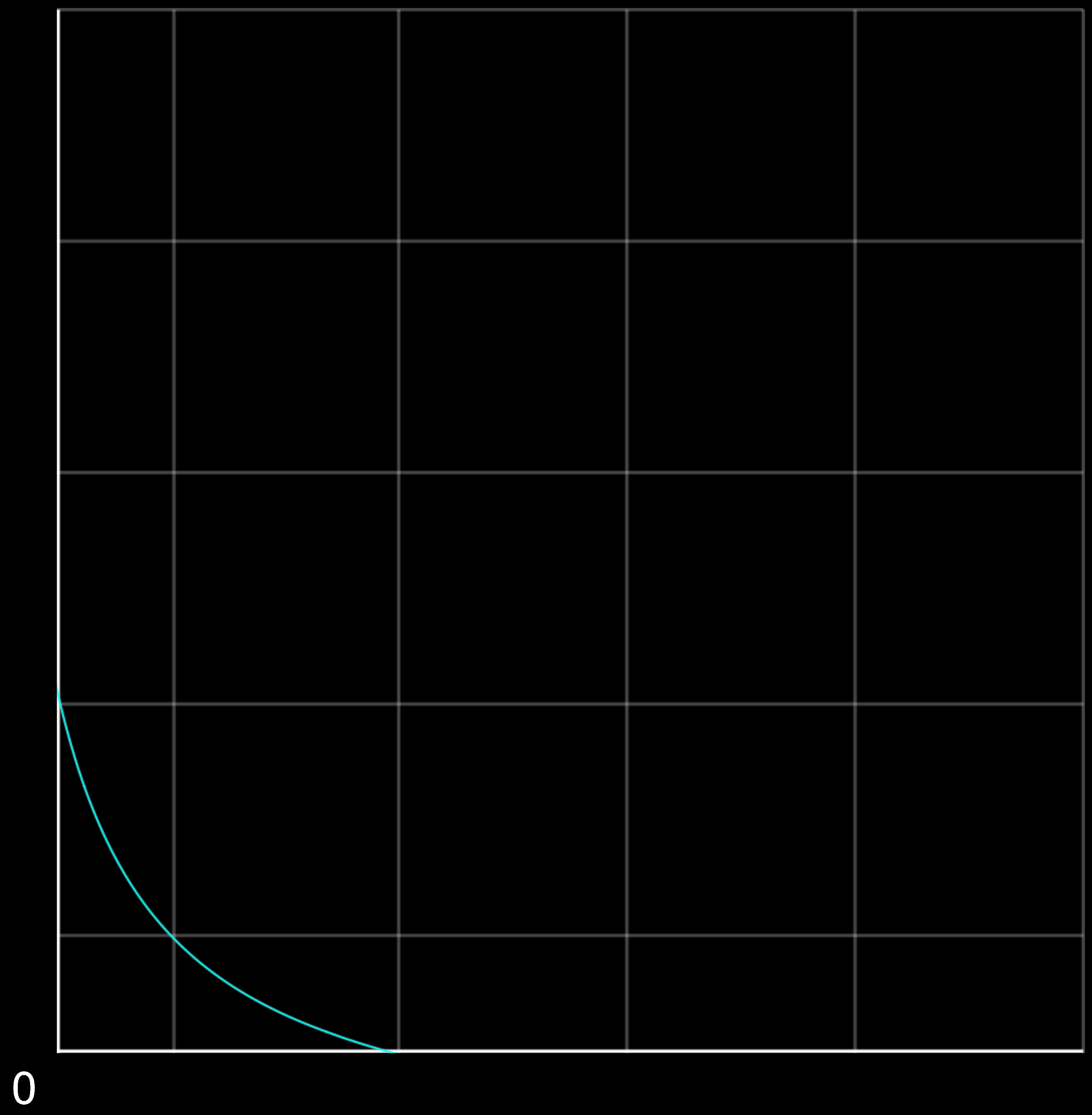
$\therefore y = 0$ is an asymptote



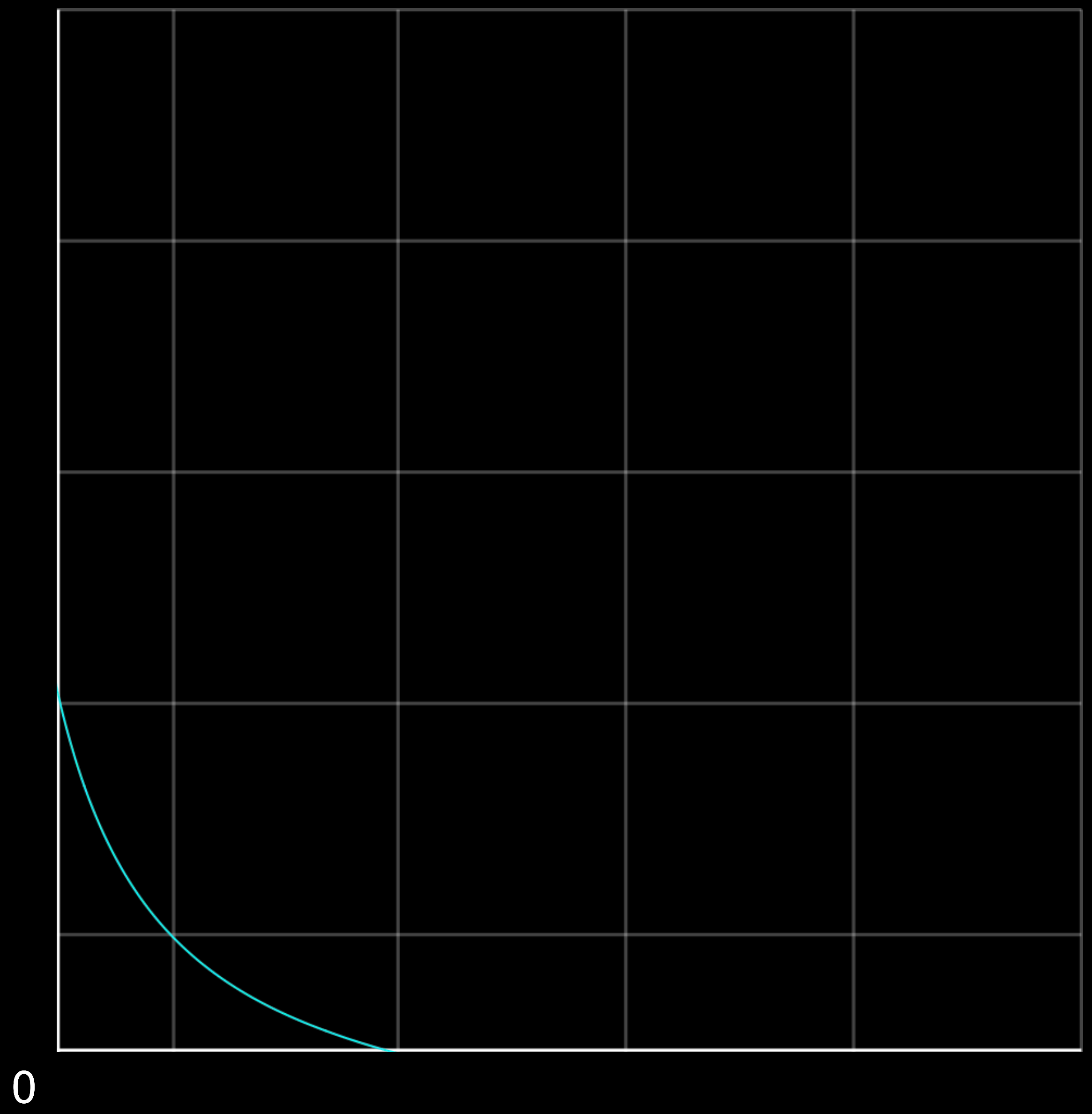
$$x \cdot y = constant$$



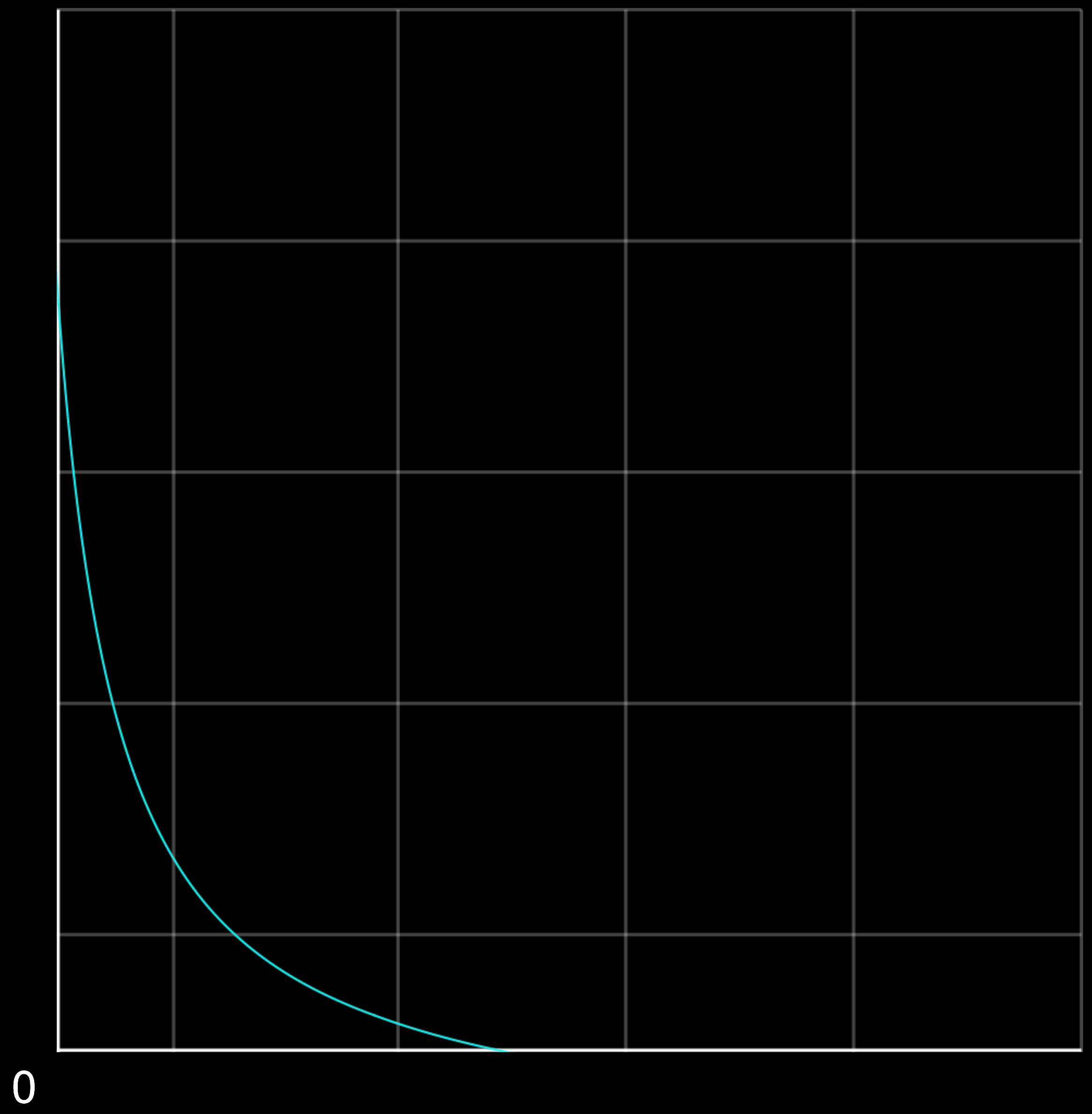
$\cancel{x \cdot y = constant}$



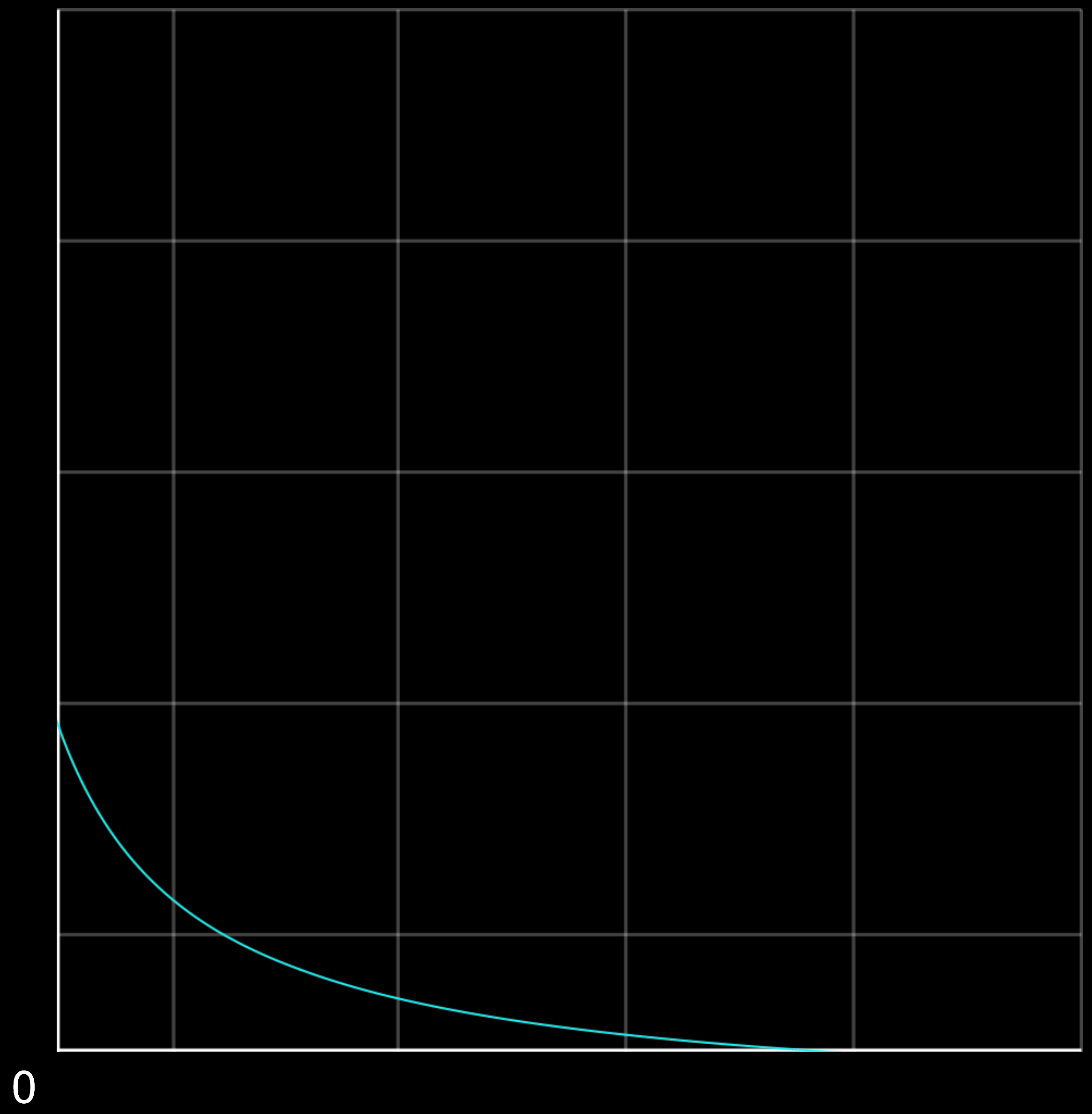
$$(x + H) \cdot (y + V) = constant$$



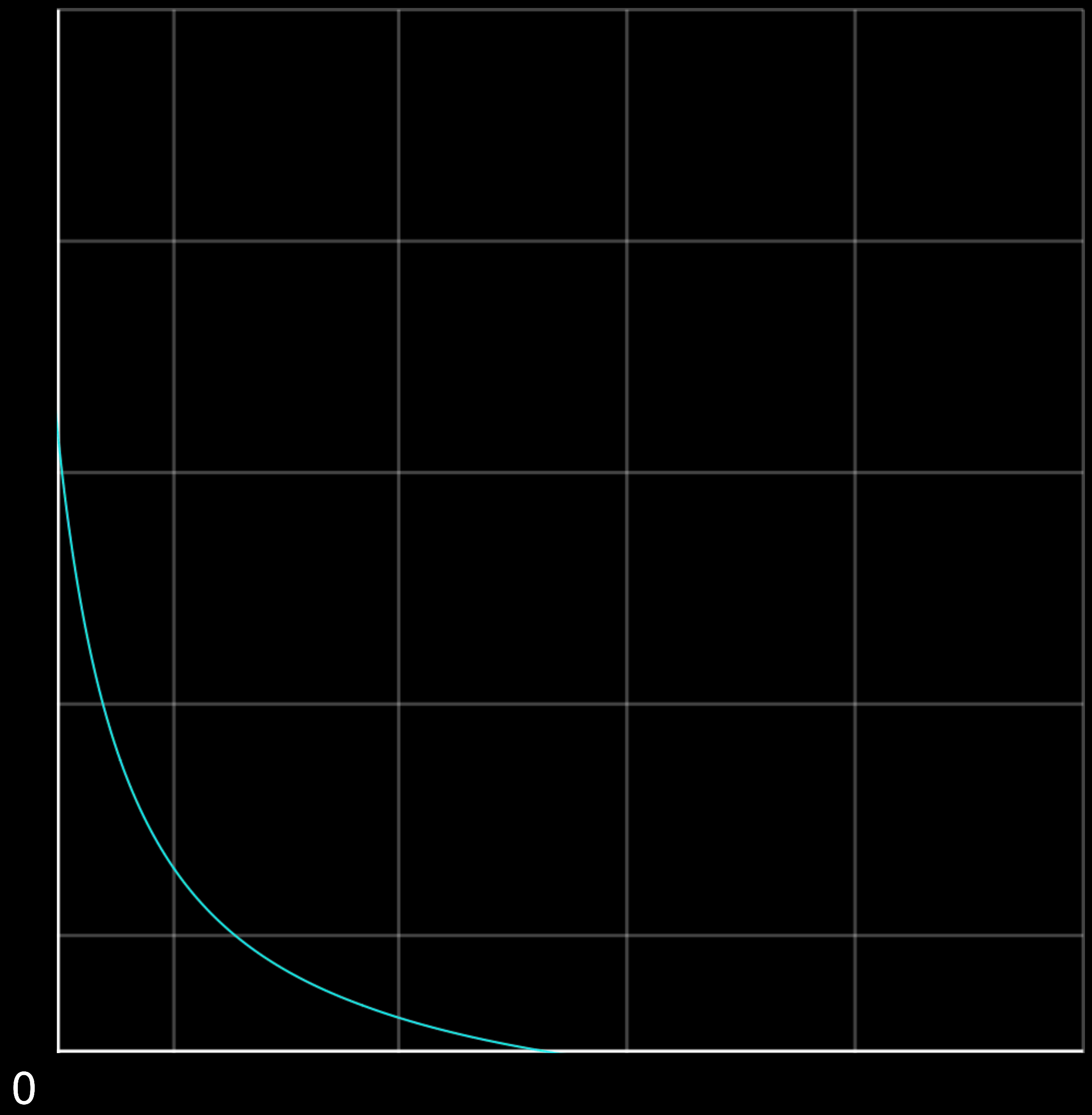
$$(x + H) \cdot (y + V) = constant$$



$$(x + H) \cdot (y + V) = constant$$



$$(x + H) \cdot (y + V) = constant$$



$$(x + H) \cdot (y + V) = constant$$

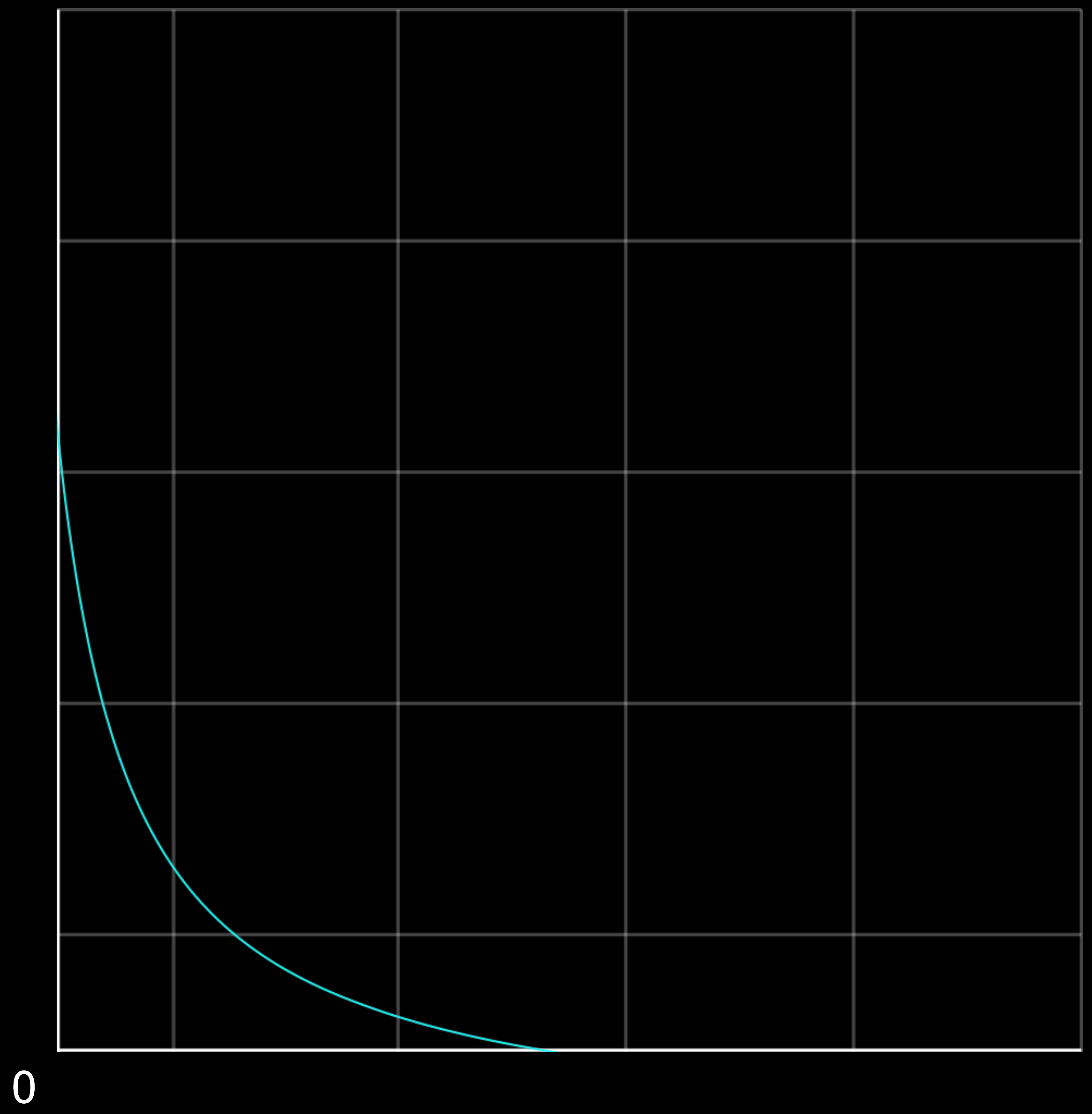
$$(0 + H) \cdot (y + V) = constant$$

$$y + V = \frac{constant}{H}$$

$$y = \frac{constant}{H} - V$$

$\therefore x = 0$ is *not* an asymptote

What if $x = -H$?



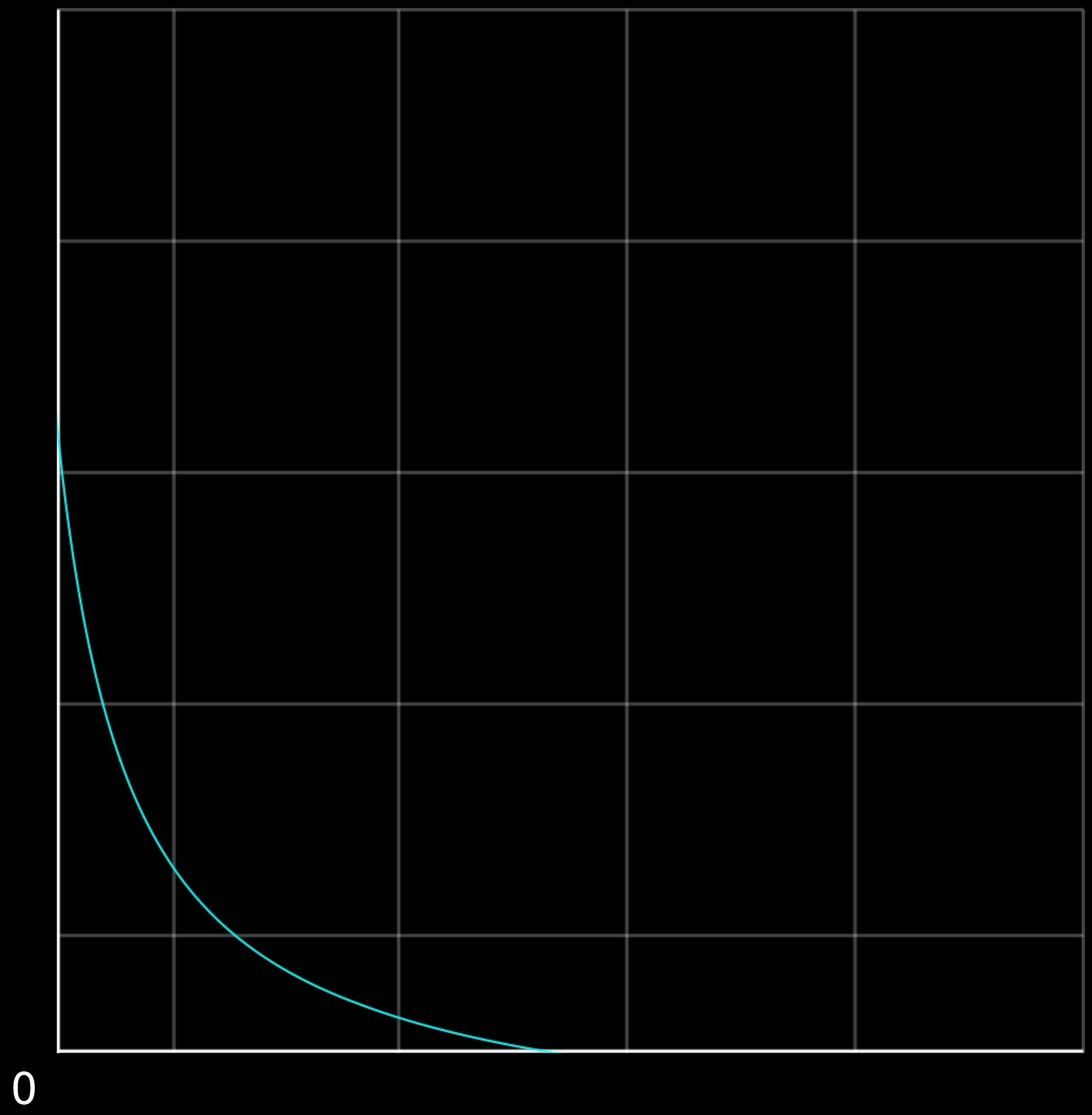
$$(x + H) \cdot (y + V) = constant$$

$$(-H + H) \cdot (y + V) = constant$$

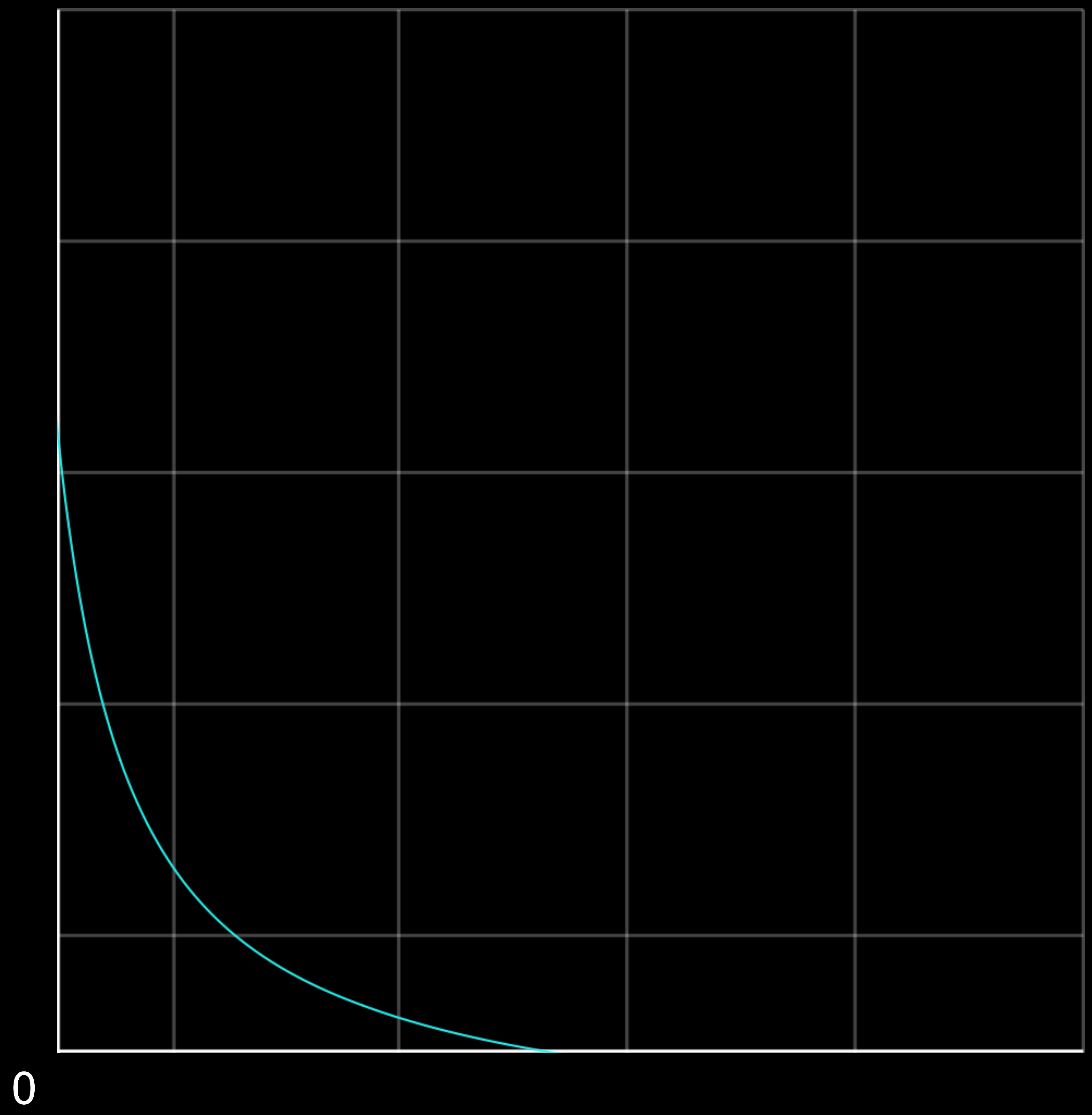
$$y + V = \frac{constant}{0} = undefined$$

$\therefore x = -H$ is an asymptote.

But *where* is that?

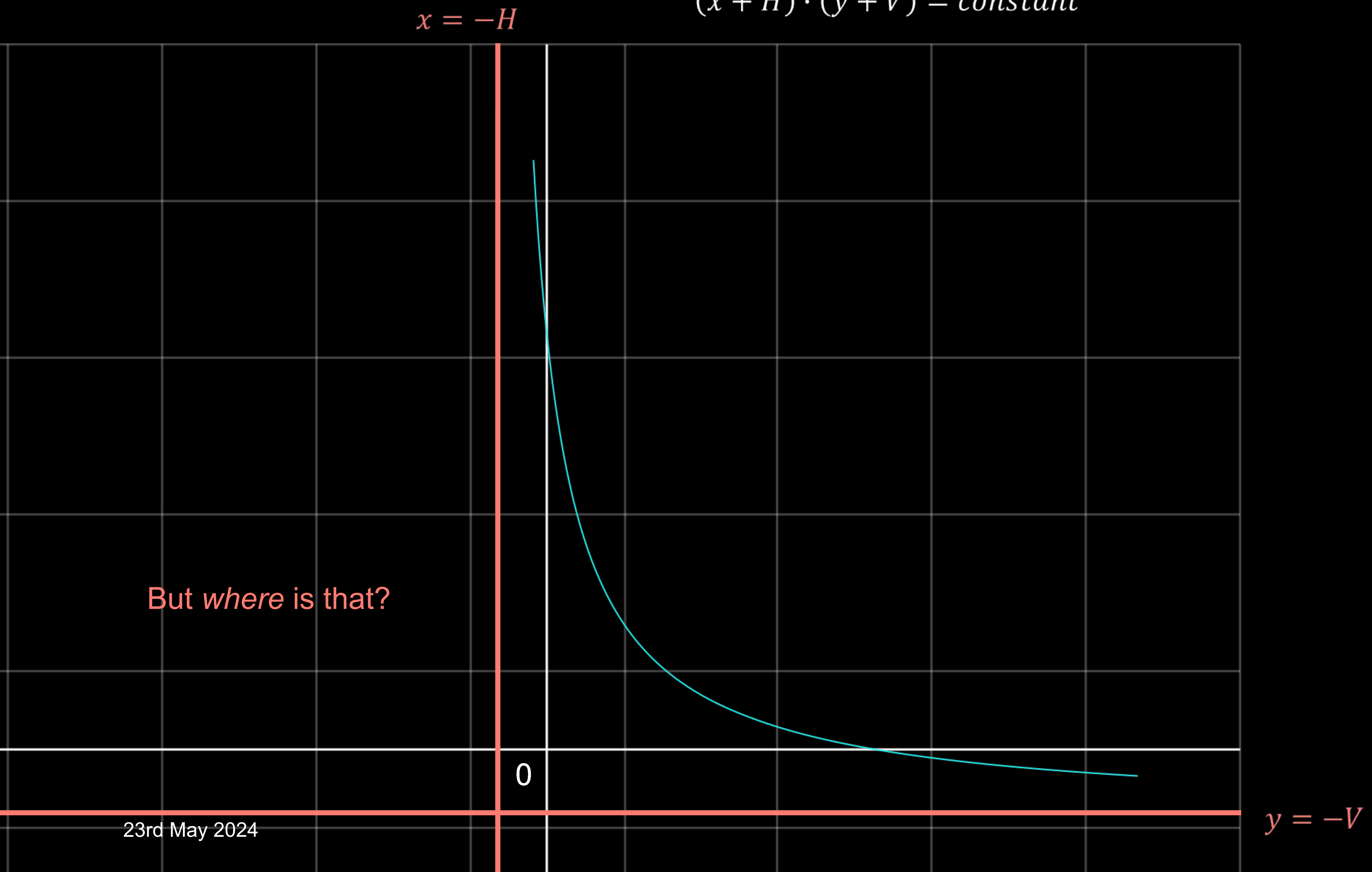


$$(x + H) \cdot (y + V) = constant$$



But *where* is that?

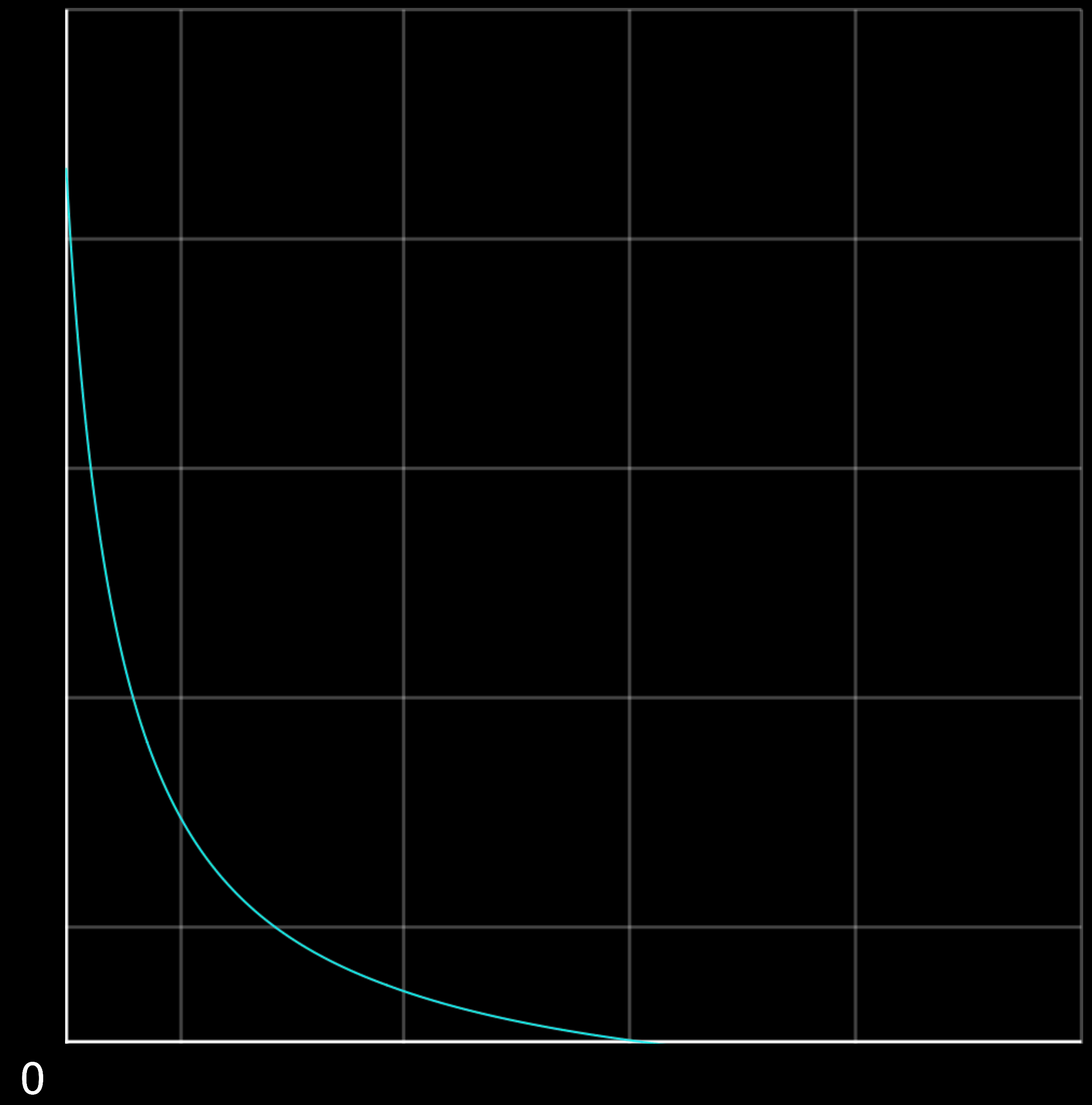
$$(x + H) \cdot (y + V) = constant$$



$x \cdot y = constant \longrightarrow (x + H) \cdot (y + V) = constant$

How the sausage is made.

$x \cdot y = constant \longrightarrow (x + H) \cdot (y + V) = constant$



<Desmos>

$x \cdot y = constant_1$ | $x \cdot y = x_0 \cdot y_0$

"Start with the canonical case…"

$$x \cdot y = constant_1$$

$$x \cdot y = x_0 \cdot y_0$$

“Start with the canonical case…”

$$x \cdot y = constant_2 \cdot constant_1$$

$$x_v \cdot y_v = A^2 \cdot x_0 \cdot y_0$$

“…then make the curve larger.”

$$x \cdot y = constant_1$$

$$x \cdot y = x_0 \cdot y_0$$

"*Start with the canonical case…*"

$$x \cdot y = constant_2 \cdot constant_1$$

$$x_v \cdot y_v = A^2 \cdot x_0 \cdot y_0$$

"*…then make the curve larger.*"

$$(x + H) \cdot (y + V) = constant_2 \cdot constant_1$$

$$\big(x + x_0 \cdot (A - 1)\big) \cdot \big(y + y_0 \cdot (A - 1)\big) = A^2 \cdot x_0 \cdot y_0$$

"*Then move it left and down so that its coordinates correspond 1:1 with token balances.*"

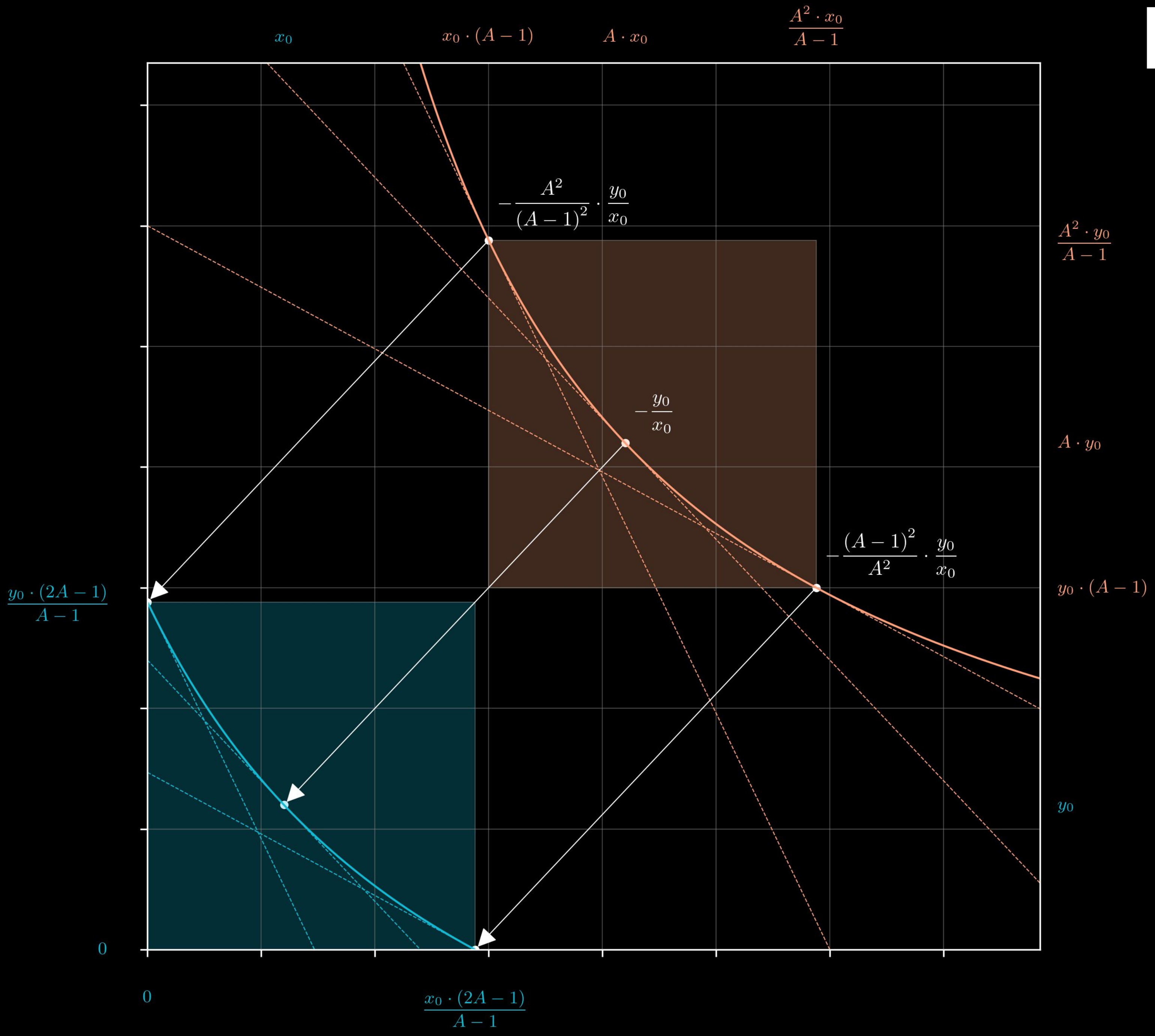
reference curve	virtual curve	real curve
$x \cdot y = constant_1$	$x \cdot y = constant_2 \cdot constant_1$	$(x + H) \cdot (y + V) = constant_2 \cdot constant_1$
$x \cdot y = x_0 \cdot y_0$	$x_{\mathrm{v}} \cdot y_{\mathrm{v}} = A^2 \cdot x_0 \cdot y_0$	$\left(x + x_0 \cdot (A - 1)\right) \cdot \left(y + y_0 \cdot (A - 1)\right) = A^2 \cdot x_0 \cdot y_0$

reference curve	virtual curve	real curve
$x \cdot y = constant_1$	$x \cdot y = constant_2 \cdot constant_1$	$(x + H) \cdot (y + V) = constant_2 \cdot constant_1$
$x \cdot y = x_0 \cdot y_0$	$x_\mathrm{v} \cdot y_\mathrm{v} = A^2 \cdot x_0 \cdot y_0$	$\big(x + \underbrace{x_0 \cdot (A - 1)}_{x_{\mathrm{asym}} = -x_0 \cdot (A - 1)}\big) \cdot \big(y + \underbrace{y_0 \cdot (A - 1)}_{y_{\mathrm{asym}} = -y_0 \cdot (A - 1)}\big) = A^2 \cdot x_0 \cdot y_0$

$$\frac{y_{\text{asym}} = -y_0 \cdot (A-1)}{x_{\text{asym}} = -x_0 \cdot (A-1)} = \frac{-y_0 \cdot (A-1)}{-x_0 \cdot (A-1)} = \frac{y_{\text{asym}}}{x_{\text{asym}}} = \frac{y_0}{x_0} = P_0 = \frac{y_{\text{int}}}{x_{\text{int}}} = \sqrt{P_{\text{high}} \cdot P_{\text{low}}}$$

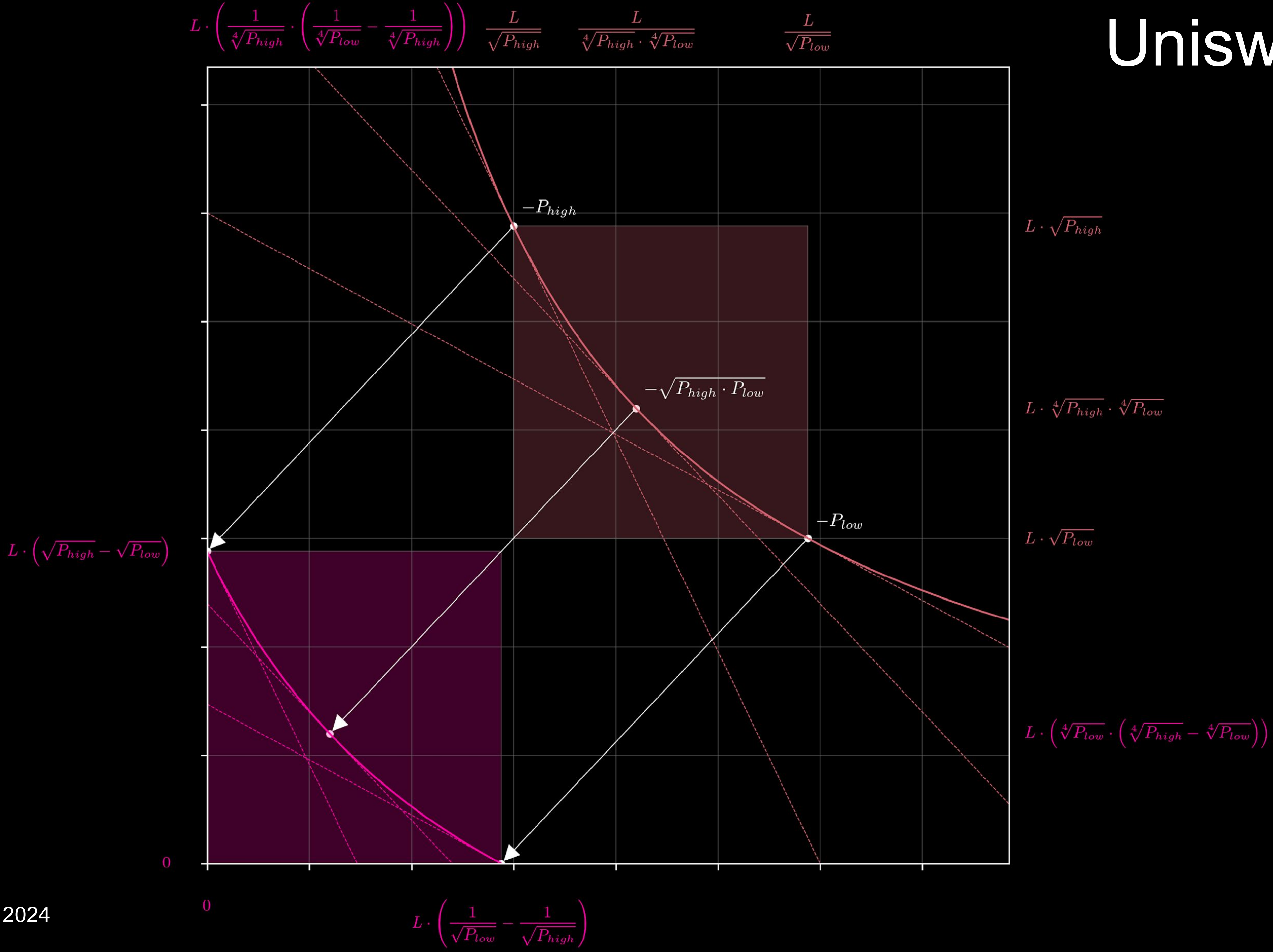
<\General Theory>

Bancor v2



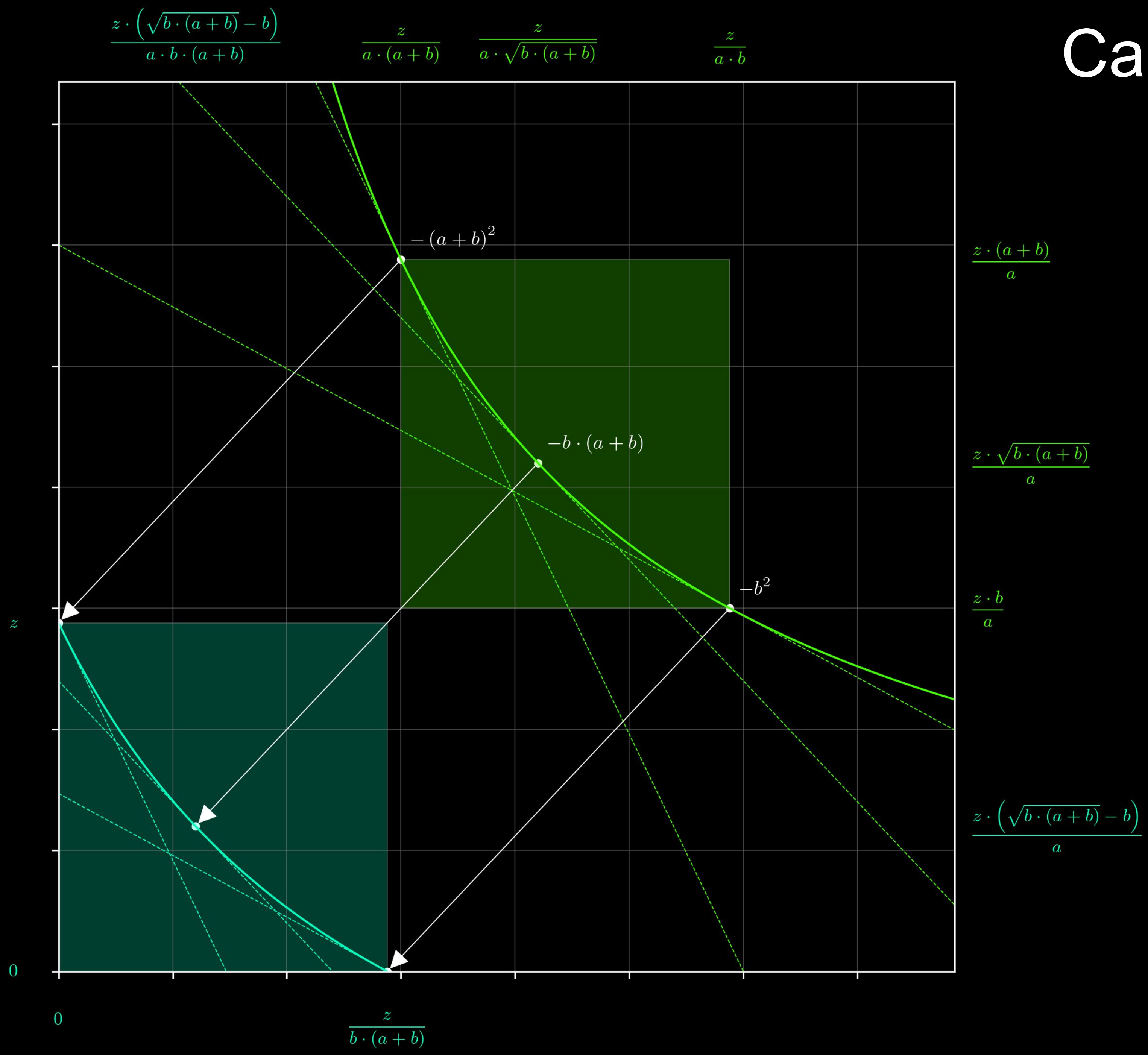
x_0
$x_0 \cdot (A-1)$
$A \cdot x_0$
$\frac{A^2 \cdot x_0}{A-1}$
$-\frac{A^2}{(A-1)^2} \cdot \frac{y_0}{x_0}$
$\frac{A^2 \cdot y_0}{A-1}$
$-\frac{y_0}{x_0}$
$A \cdot y_0$
$-\frac{(A-1)^2}{A^2} \cdot \frac{y_0}{x_0}$
$\frac{y_0 \cdot (2A-1)}{A-1}$
$y_0 \cdot (A-1)$
y_0
0
0
$\frac{x_0 \cdot (2A-1)}{A-1}$

Uniswap v3



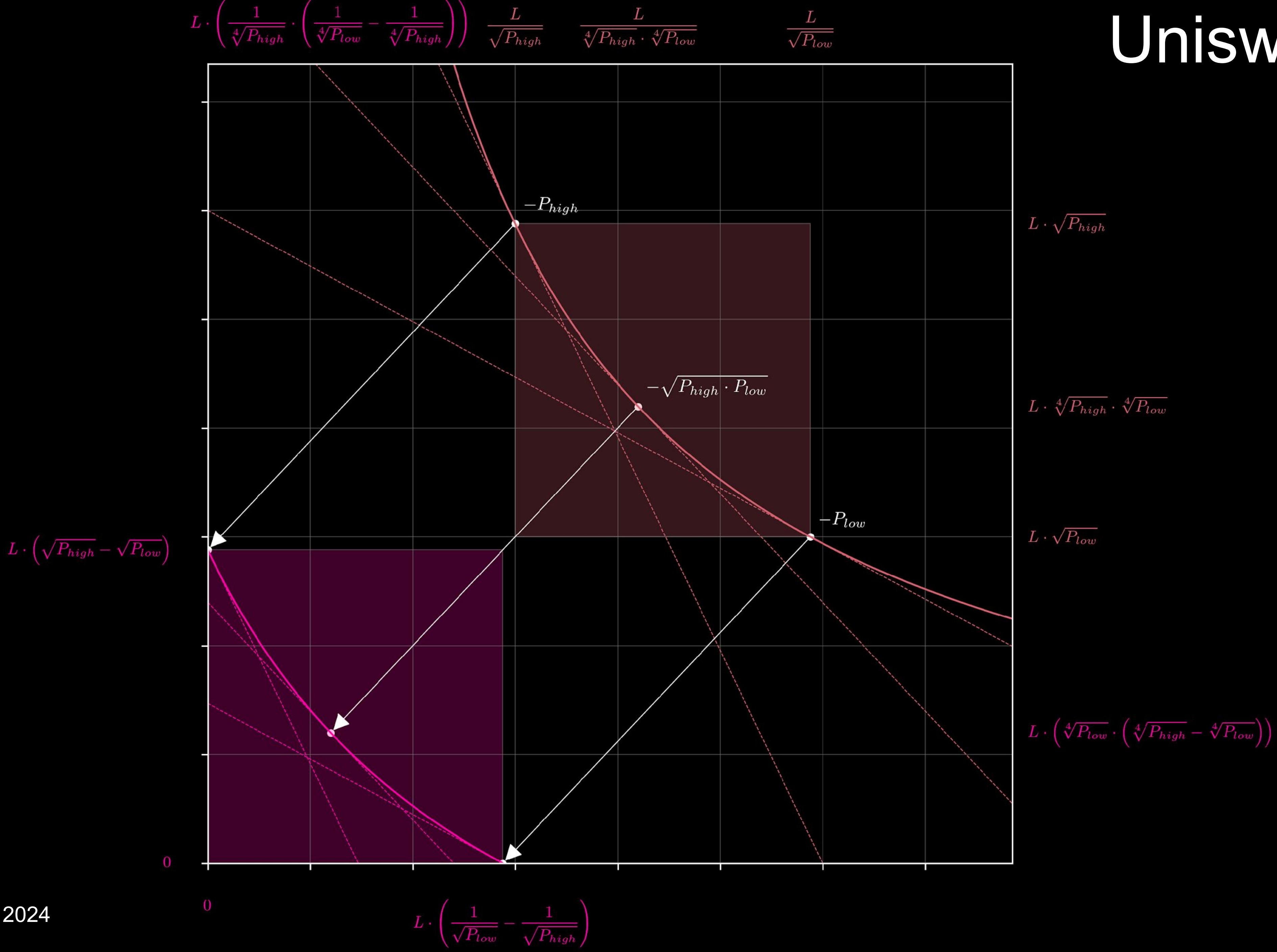
$L \cdot \left(\frac{1}{\sqrt[4]{P_{high}}} \cdot \left(\frac{1}{\sqrt[4]{P_{low}}} - \frac{1}{\sqrt[4]{P_{high}}} \right) \right)$
$\frac{L}{\sqrt{P_{high}}}$
$\frac{L}{\sqrt[4]{P_{high}} \cdot \sqrt[4]{P_{low}}}$
$\frac{L}{\sqrt{P_{low}}}$
$-P_{high}$
$-\sqrt{P_{high} \cdot P_{low}}$
$-P_{low}$
$L \cdot \sqrt{P_{high}}$
$L \cdot \sqrt[4]{P_{high}} \cdot \sqrt[4]{P_{low}}$
$L \cdot \sqrt{P_{low}}$
$L \cdot \left(\sqrt[4]{P_{low}} \cdot \left(\sqrt[4]{P_{high}} - \sqrt[4]{P_{low}} \right) \right)$
$L \cdot \left(\sqrt{P_{high}} - \sqrt{P_{low}} \right)$
0
0
$L \cdot \left(\frac{1}{\sqrt{P_{low}}} - \frac{1}{\sqrt{P_{high}}} \right)$

Carbon DeFi



$\frac{z\cdot\left(\sqrt{b\cdot(a+b)}-b\right)}{a\cdot b\cdot(a+b)}$
$\frac{z}{a\cdot(a+b)}$
$\frac{z}{a\cdot\sqrt{b\cdot(a+b)}}$
$\frac{z}{a\cdot b}$
$-(a+b)^2$
$-b\cdot(a+b)$
$-b^2$
$\frac{z\cdot(a+b)}{a}$
$\frac{z\cdot\sqrt{b\cdot(a+b)}}{a}$
$\frac{z\cdot b}{a}$
$\frac{z\cdot\left(\sqrt{b\cdot(a+b)}-b\right)}{a}$
z
0
0
$\frac{z}{b\cdot(a+b)}$

Uniswap v3


$L \cdot \left(\frac{1}{\sqrt[4]{P_{high}}} \cdot \left(\frac{1}{\sqrt[4]{P_{low}}} - \frac{1}{\sqrt[4]{P_{high}}}\right)\right)$
$\frac{L}{\sqrt{P_{high}}}$
$\frac{L}{\sqrt[4]{P_{high}} \cdot \sqrt[4]{P_{low}}}$
$\frac{L}{\sqrt{P_{low}}}$
$-P_{high}$
$-\sqrt{P_{high} \cdot P_{low}}$
$-P_{low}$
$L \cdot \sqrt{P_{high}}$
$L \cdot \sqrt[4]{P_{high}} \cdot \sqrt[4]{P_{low}}$
$L \cdot \sqrt{P_{low}}$
$L \cdot \left(\sqrt[4]{P_{low}} \cdot \left(\sqrt[4]{P_{high}} - \sqrt[4]{P_{low}}\right)\right)$
$L \cdot \left(\sqrt{P_{high}} - \sqrt{P_{low}}\right)$
0
0
$L \cdot \left(\frac{1}{\sqrt{P_{low}}} - \frac{1}{\sqrt{P_{high}}}\right)$

<Uniswap v3>

$$\left(x + x_0 \cdot (A - 1)\right) \cdot \left(y + y_0 \cdot (A - 1)\right) = A^2 \cdot x_0 \cdot y_0$$

$$P_{\text{high}} = \frac{A^2}{(A-1)^2} \cdot \frac{y_0}{x_0} \;\Rightarrow\; \sqrt{P_{\text{high}}} = \frac{A}{A-1} \cdot \frac{\sqrt{y_0}}{\sqrt{x_0}} \;\Rightarrow\; A - 1 = \frac{A}{\sqrt{P_{\text{high}}}} \cdot \frac{\sqrt{y_0}}{\sqrt{x_0}}$$

$$P_{\text{low}} = \frac{(A-1)^2}{A^2} \cdot \frac{y_0}{x_0} \;\Rightarrow\; \sqrt{P_{\text{low}}} = \frac{A-1}{A} \cdot \frac{\sqrt{y_0}}{\sqrt{x_0}} \;\Rightarrow\; \frac{1}{A-1} = \frac{1}{\sqrt{P_{\text{low}}} \cdot A} \cdot \frac{\sqrt{y_0}}{\sqrt{x_0}} \;\Rightarrow\; A - 1 = \sqrt{P_{\text{low}}} \cdot A \cdot \frac{\sqrt{x_0}}{\sqrt{y_0}}$$

$$A - 1 = \frac{A}{\sqrt{P_{\text{high}}}} \cdot \frac{\sqrt{y_0}}{\sqrt{x_0}}$$

$$A - 1 = \sqrt{P_{\text{low}}} \cdot A \cdot \frac{\sqrt{x_0}}{\sqrt{y_0}}$$

$$\big(x + x_0 \cdot (A - 1)\big) \cdot \big(y + y_0 \cdot (A - 1)\big) = A^2 \cdot x_0 \cdot y_0$$

$$\left(x + x_0 \cdot \frac{A}{\sqrt{P_{\text{high}}}} \cdot \frac{\sqrt{y_0}}{\sqrt{x_0}}\right) \cdot \left(y + y_0 \cdot \sqrt{P_{\text{low}}} \cdot A \cdot \frac{\sqrt{x_0}}{\sqrt{y_0}}\right) = A^2 \cdot x_0 \cdot y_0$$

$$\left(x + x_0 \cdot \frac{A}{\sqrt{P_{\mathrm{high}}}} \cdot \frac{\sqrt{y_0}}{\sqrt{x_0}}\right) \cdot \left(y + y_0 \cdot \sqrt{P_{\mathrm{low}}} \cdot A \cdot \frac{\sqrt{x_0}}{\sqrt{y_0}}\right) = A^2 \cdot x_0 \cdot y_0$$

$$\left(x + x_0 \cdot \frac{A}{\sqrt{P_{\text{high}}}} \cdot \frac{\sqrt{y_0}}{\sqrt{x_0}}\right) \cdot \left(y + y_0 \cdot \sqrt{P_{\text{low}}} \cdot A \cdot \frac{\sqrt{x_0}}{\sqrt{y_0}}\right) = A^2 \cdot x_0 \cdot y_0$$

$$\downarrow$$

$$\left(x + \frac{A \cdot \sqrt{x_0} \cdot \sqrt{y_0}}{\sqrt{P_{\text{high}}}}\right) \cdot \left(y + A \cdot \sqrt{x_0} \cdot \sqrt{y_0} \cdot \sqrt{P_{\text{low}}}\right) = A^2 \cdot x_0 \cdot y_0$$

$$A \cdot \sqrt{x_0} \cdot \sqrt{y_0} = L$$

$$\left(x + \frac{A \cdot \sqrt{x_0} \cdot \sqrt{y_0}}{\sqrt{P_{\text{high}}}}\right) \cdot \left(y + A \cdot \sqrt{x_0} \cdot \sqrt{y_0} \cdot \sqrt{P_{\text{low}}}\right) = A^2 \cdot x_0 \cdot y_0$$

$$A \cdot \sqrt{x_0} \cdot \sqrt{y_0} = L$$

$$\left(x + \frac{A \cdot \sqrt{x_0} \cdot \sqrt{y_0}}{\sqrt{P_{\text{high}}}}\right) \cdot \left(y + A \cdot \sqrt{x_0} \cdot \sqrt{y_0} \cdot \sqrt{P_{\text{low}}}\right) = A^2 \cdot x_0 \cdot y_0$$

$$\left(x + \frac{L}{\sqrt{P_{\text{high}}}}\right) \cdot \left(y + L \cdot \sqrt{P_{\text{low}}}\right) = L^2$$

Bancor v2

$$\left(x + x_0 \cdot (A - 1)\right) \cdot \left(y + y_0 \cdot (A - 1)\right) = A^2 \cdot x_0 \cdot y_0$$

Uniswap v3

$$\left(x + \frac{L}{\sqrt{P_{\mathrm{high}}}}\right) \cdot \left(y + L \cdot \sqrt{P_{\mathrm{low}}}\right) = L^2$$

Both are defined by three constants:

- A, x_0, y_0
- $L, \sqrt{P_{\mathrm{high}}}, \sqrt{P_{\mathrm{low}}}$

Both refer to the same object.

Bancor v2

$$(x + x_0 \cdot (A-1)) \cdot (y + \ldots$$

Uniswap v3

$$\left(x + \frac{L}{\sqrt{P_{\text{high}}}}\right) \cdot \left(y + L \cdot \sqrt{P_{\text{low}}}\right) = L^2$$

Why reparametrize the invariant?

Both are defined by three constants:

- A, x_0, y_0
- $L, \sqrt{P_{\text{high}}}, \sqrt{P_{\text{low}}}$

Both refer to the same object.

Why reparametrize the invariant?

<Discussion>

Both are defined by three constants:

- A, x_0, y_0
- $L, \sqrt{P_{\text{high}}}, \sqrt{P_{\text{low}}}$

Both refer to the same object.

Why reparametrize the invariant?

System architecture and implementation.

Bancor v2

$$\big(x + x_0 \cdot (A-1)\big) \cdot \big(y + y_0 \cdot (A-1)\big) = A^2 \cdot x_0 \cdot y_0$$

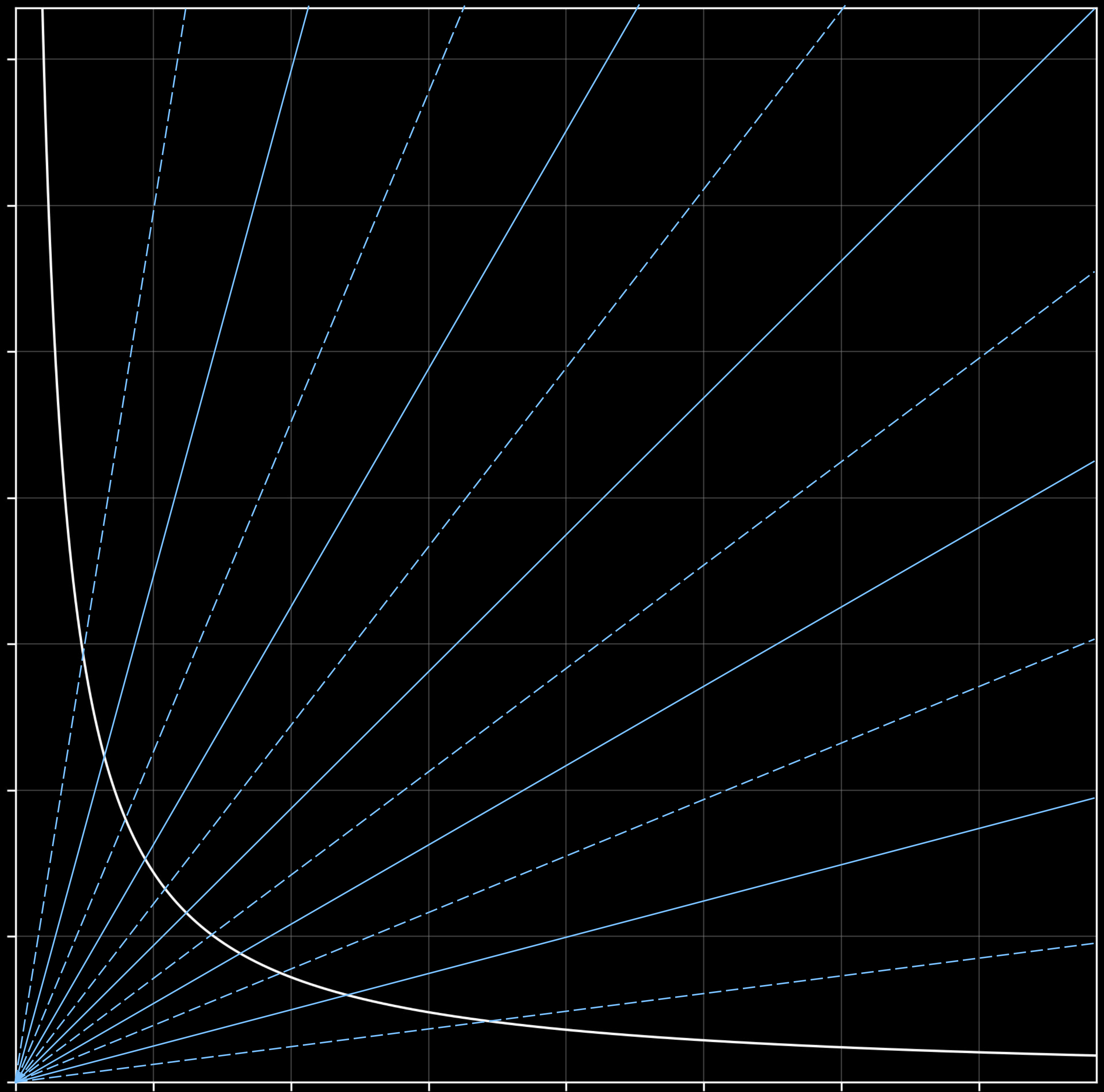
Uniswap v3

$$\left(x + \frac{L}{\sqrt{P_{\text{high}}}}\right) \cdot \left(y + L \cdot \sqrt{P_{\text{low}}}\right) = L^2$$

Both are defined by three constants:

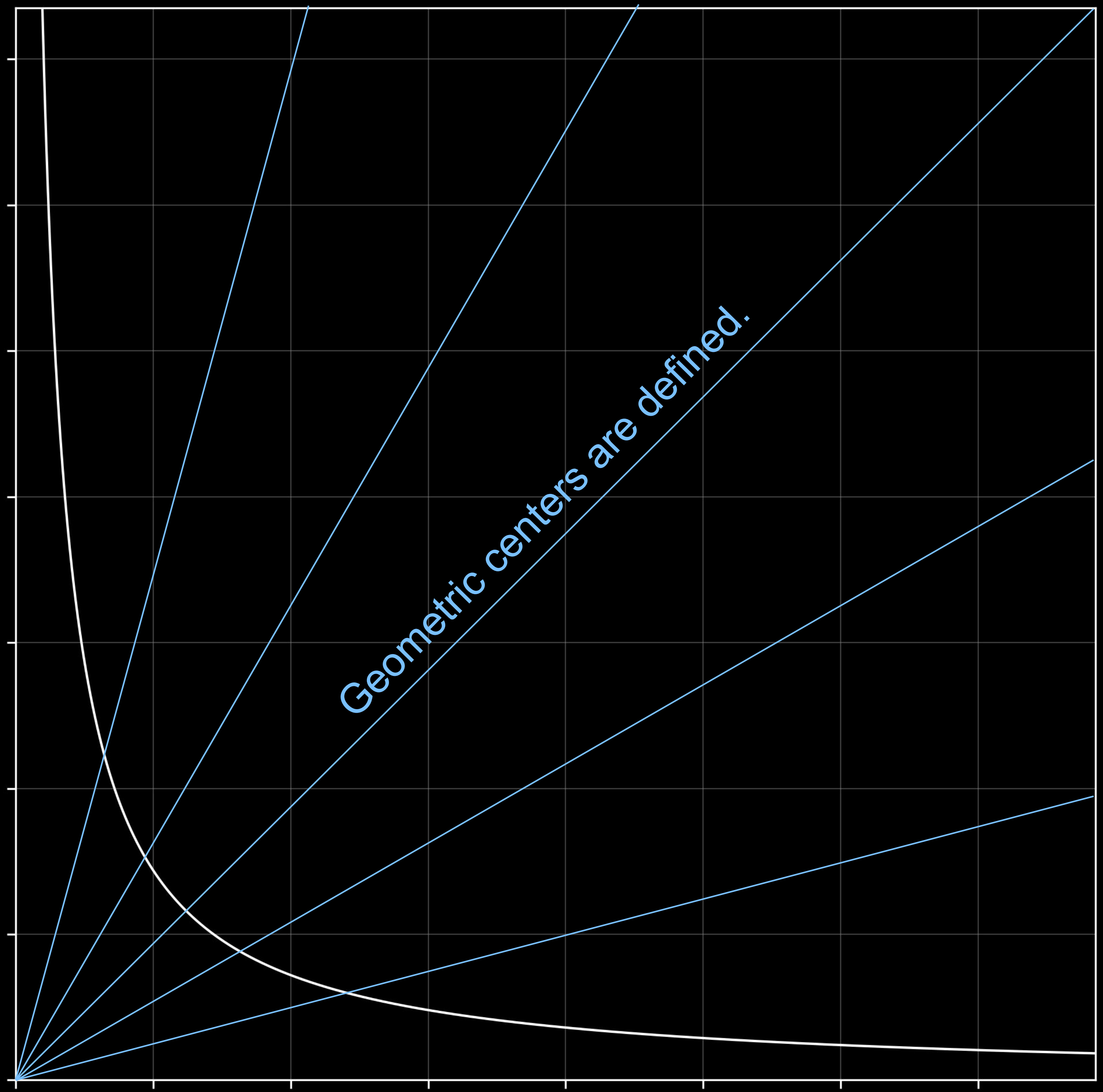
- A, x_0, y_0
- $L, \sqrt{P_{\text{high}}}, \sqrt{P_{\text{low}}}$

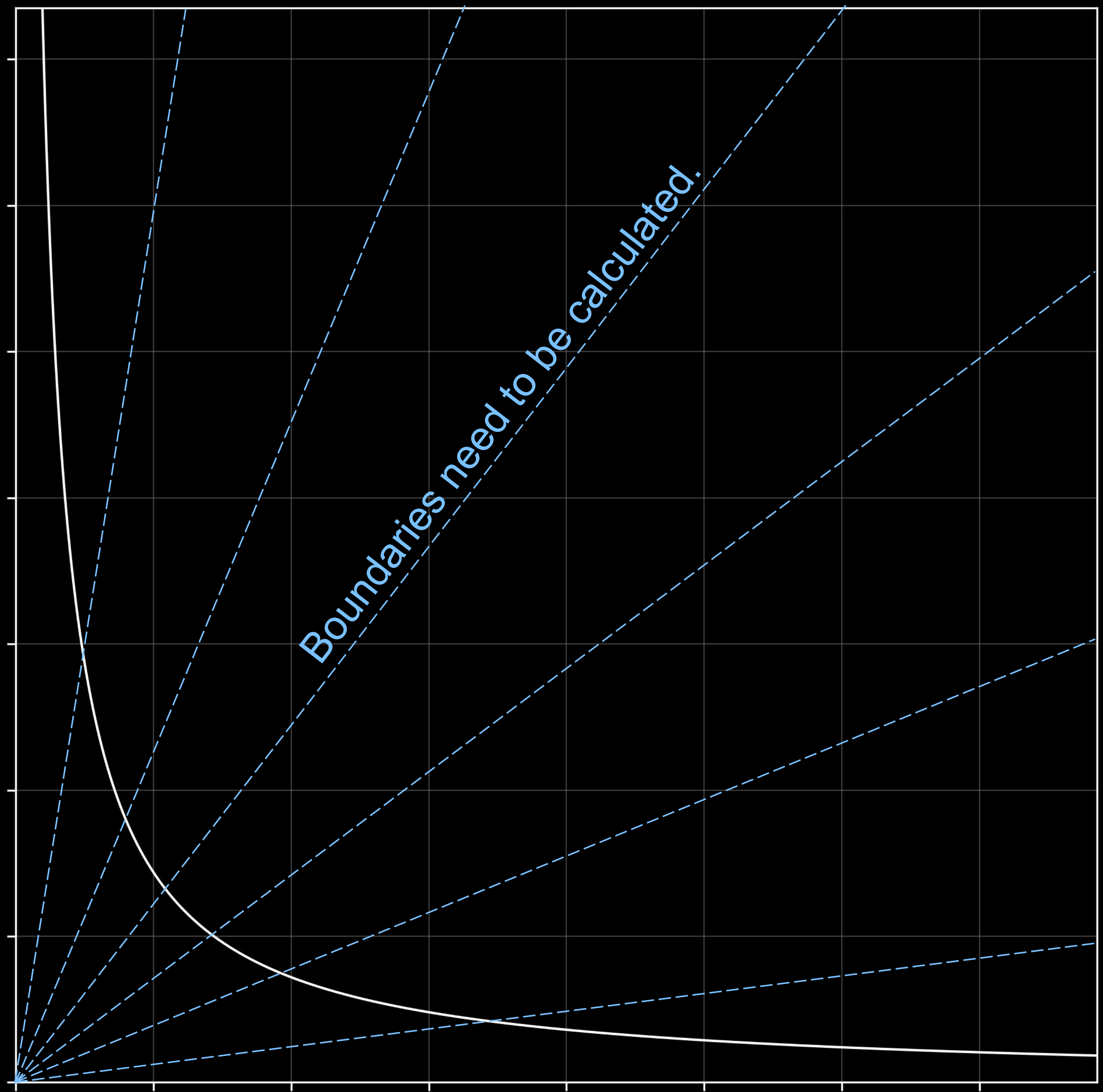
Both refer to the same object.



not to scale

not to scale





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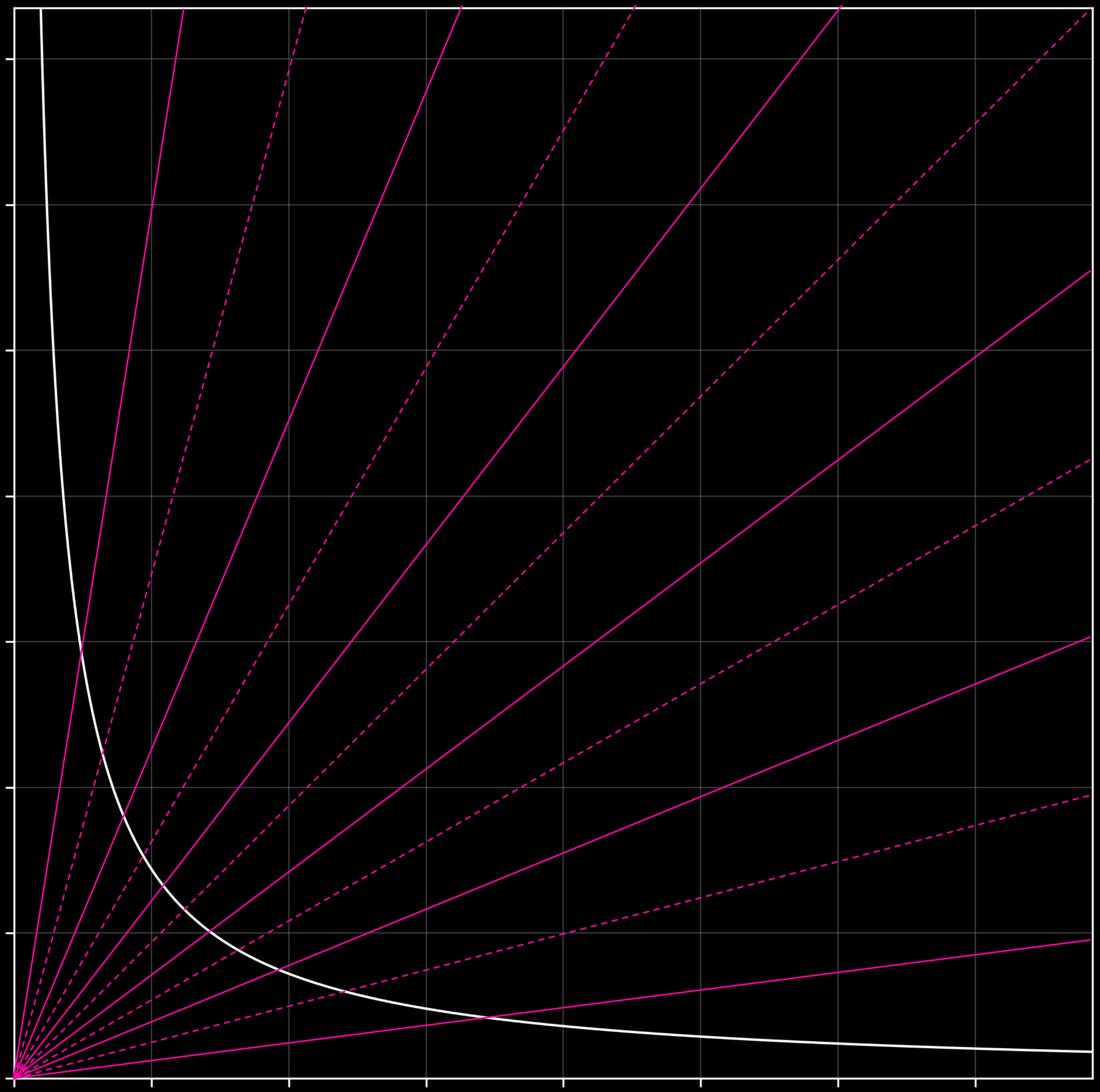
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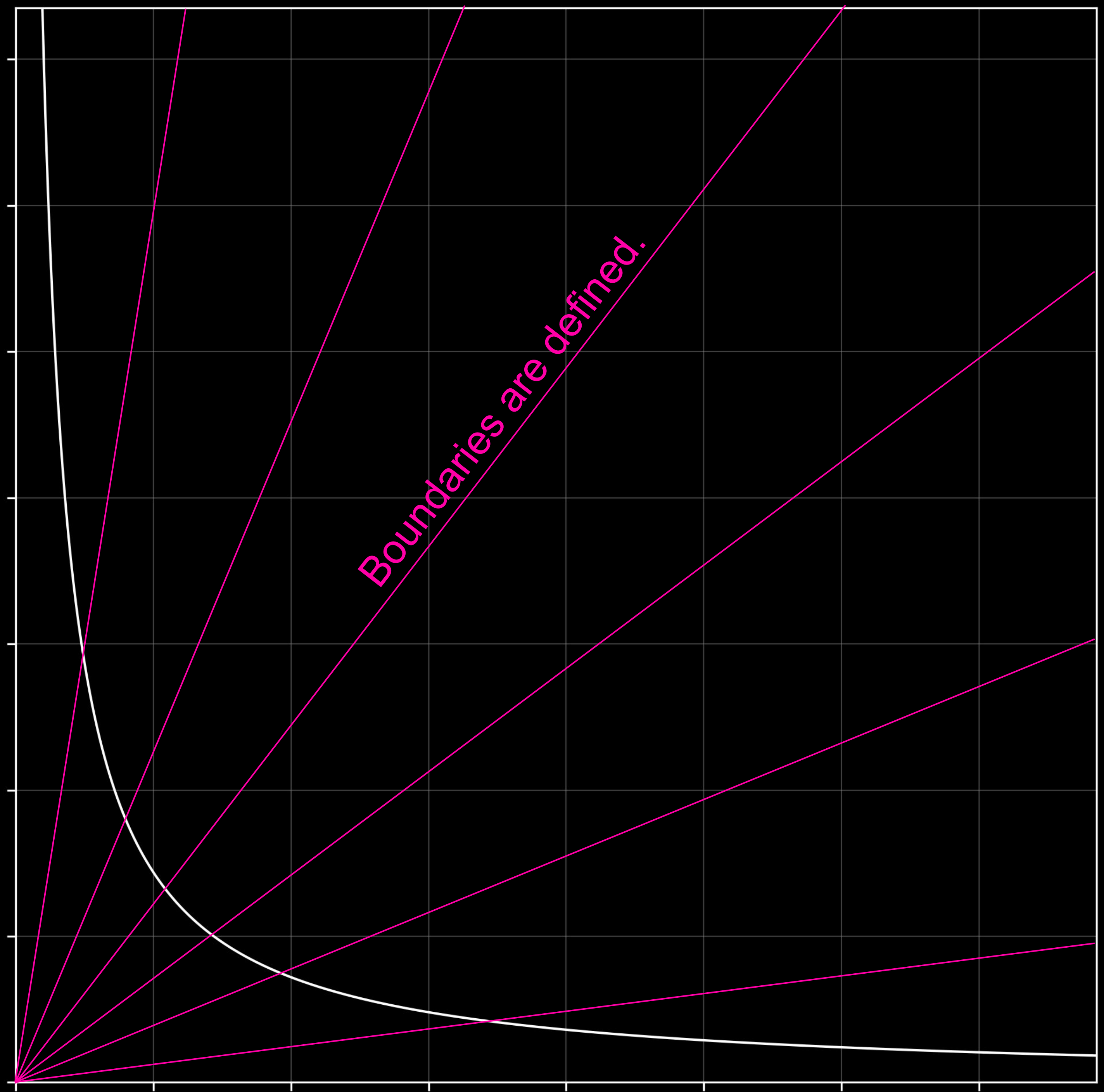

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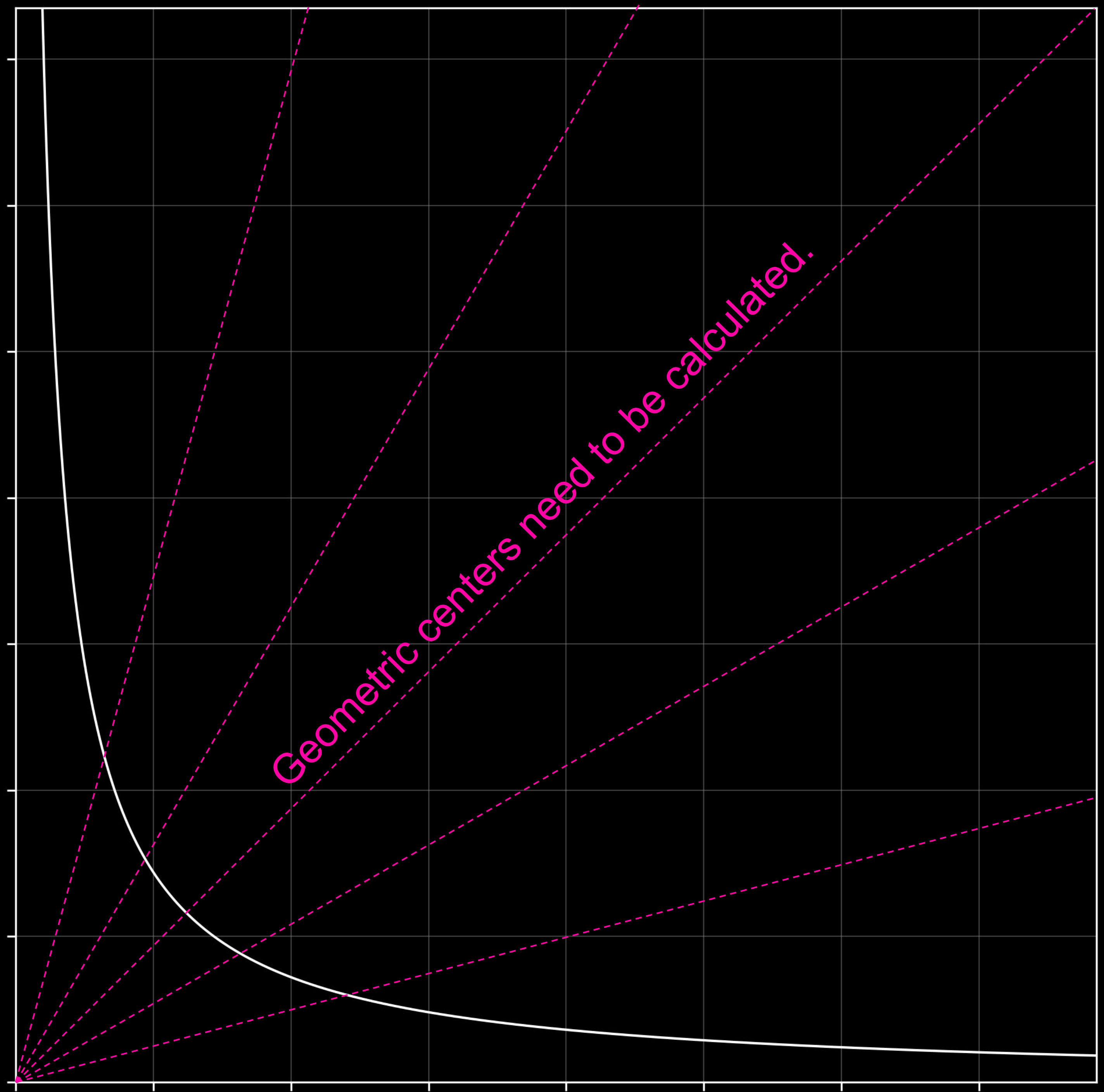
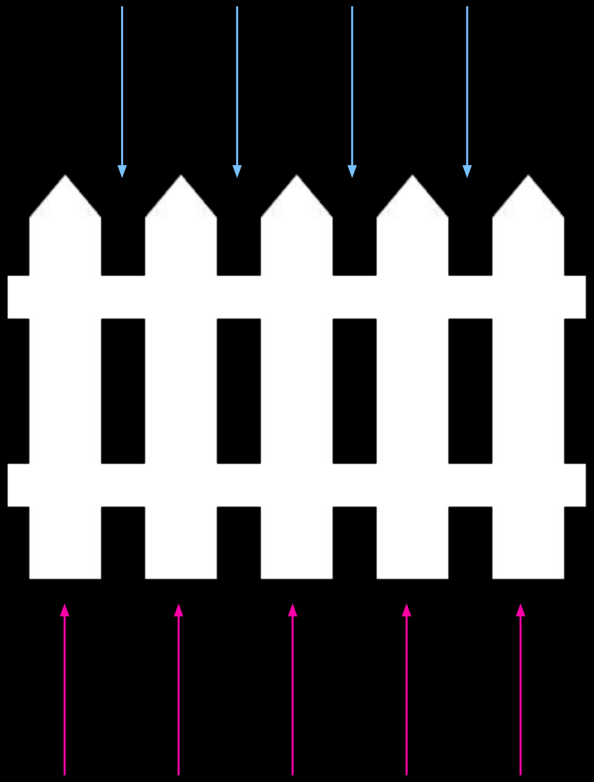


$$\big(x + x_0 \cdot (A-1)\big) \cdot \big(y + y_0 \cdot (A-1)\big) = A^2 \cdot x_0 \cdot y_0$$

Parameterizes the space between posts.



Parameterizes the post placement.

$$\left(x + \frac{L}{\sqrt{P_{\text{high}}}}\right) \cdot \left(y + L \cdot \sqrt{P_{\text{low}}}\right) = L^2$$

$$\left(x + x_0 \cdot (A-1)\right) \cdot \left(y + y_0 \cdot (A-1)\right) = A^2 \cdot x_0 \cdot y_0$$

Parameterizes the space between posts.

So what?

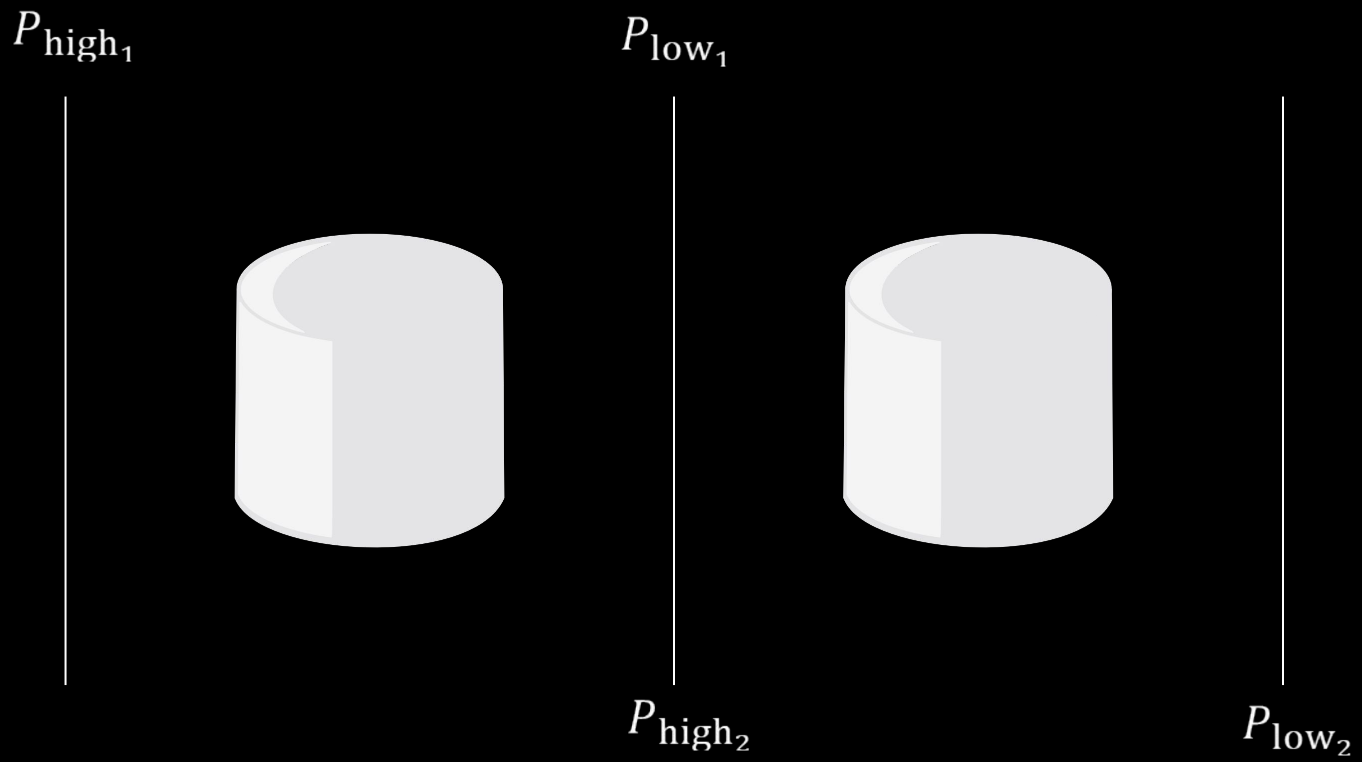
Parameterizes the post placement.

$$\left(x + \frac{L}{\sqrt{P_{\text{high}}}}\right) \cdot \left(y + L \cdot \sqrt{P_{\text{low}}}\right) = L^2$$

Homework, 23rd May

1. Create *two* systems of two, discrete, concentrated liquidity pools.
2. The P_{low} boundary of the first pool (P_{low_1}) must be equal to the P_{high} boundary of the second pool (P_{high_2}).
3. Therefore, the two pools are contiguous with each other with respect to this common price boundary.
4. In the first system of two contiguous pools, use only the curve parameters x_0, y_0 and A.
5. In the second system of two pools, use only the curve parameters $\sqrt{P_{\text{high}}}$, $\sqrt{P_{\text{low}}}$ and L.

Homework, 23rd May



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DeFi’s Concentrated Liquidity From Scratch

Lecture 3 of 5

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Token Engineering Academy

CARBON DEFI

Bancor