Limits and Derivatives

Question1

Let for a differentiable function $f:(0,\infty)\to R, f(x)-f(y)\ge \log_e\left(\frac{x}{y}\right)+x-y, \forall x,y\in(0,\infty)$. Then $\sum_{n=1}^{20} f'\left(\frac{1}{n^2}\right)$ is equal to___

[27-Jan-2024 Shift 1]

Answer: 2890

Solution:

$$f(x)-f(y)\ge \ln x-\ln y+x-y$$

$$\frac{f(x)-f(y)}{x-y}\ge\frac{\ln x-\ln y}{x-y}+1$$

Let $x>y$

$$\lim_{y\to x} f'(x^-)\ge\frac{1}{x}+1 \quad \ldots(1)$$

Let $x<y$

$$\lim_{y\to x} f'(x^+)\le\frac{1}{x}+1 \quad \ldots(2)$$

$$f^1(x^-)=f^1(x^+)$$

$$f^1(x)=\frac{1}{x}+1$$

$$f'\left(\frac{1}{x^2}\right)=x^2+1$$

$$\sum_{x=1}^{20}(x^2+1)=\sum_{x-1}^{20}x^2+20$$

$$=\frac{20\times21\times41}{6}+20$$

$$=2890$$

Question2

Let $f(x)=x^3+x^2f'(1)+xf''(2)+f'''(3), x\in R$. Then $f'(10)$ is equal to__

[27-Jan-2024 Shift 1]

Answer: 202

Solution:

$$f(x) = x^3 + x^2 \cdot f'(1) + x \cdot f''(2) + f'''(3)$$

$$f'(x) = 3x^2 + 2xf'(1) + f''(2)$$

$$f''(x) = 6x + 2f'(1)$$

$$f'''(x) = 6$$

$$f'(1) = -5, f''(2) = 2, f'''(3) = 6$$

$$f(x) = x^3 + x^2 \cdot (-5) + x \cdot (2) + 6$$

$$f'(x) = 3x^2 - 10x + 2$$

$$f'(10) = 300 - 100 + 2 = 202$$

Question3

Suppose

$$f(x) = \frac{(2^x + 2^{-x})\tan x\sqrt{\tan^{-1}(x^2 - x + 1)}}{(7x^2 + 3x + 1)^3}$$

Then the value of $f'(0)$ is equal to

[29-Jan-2024 Shift 1]

Options:

A.

π

B.

0

C.

√π

D.

π/2

Answer: C

Solution:

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{(2^h + 2^{-h})\tan h\sqrt{\tan^{-1}(h^2 - h + 1)} - 0}{(7h^2 + 3h + 1)^3 h}$$

$$= \sqrt{\pi}$$

Question4

Let $y=\log_e\left(\frac{1-x^2}{1+x^2}\right), -1<x<1$. Then at $x=\frac{1}{2}$, the value of $225(y'-y'')$ is equal to

[29-Jan-2024 Shift 2]

Options:

A.

732

B.

746

C.

742

D.

736

Answer: D

Solution:

$$y=\log_e\left(\frac{1-x^2}{1+x^2}\right)$$

$$\frac{dy}{dx}=y'=\frac{-4x}{1-x^4}$$

Again,

$$\frac{d^2y}{dx^2}=y''=\frac{-4(1+3x^4)}{(1-x^4)^2}$$

Again

$$y'-y''=\frac{-4x}{1-x^4}+\frac{4(1+3x^4)}{(1-x^4)^2}$$

at $x=\frac{1}{2}$,

$$y'-y''=\frac{736}{225}$$

Thus $225(y'-y'')=225\times\frac{736}{225}=736$

Question5

Let g(x) be a linear function and f(x) = $\begin{cases} g(x) & , x\le 0 \\ \left(\frac{1+x}{2+x}\right)^{\frac{1}{x}} & , x>0 \end{cases}$ **is continuous at x = 0. If f′ (1) = f (−1), then the value of g(3) is**

[31-Jan-2024 Shift 1]

Options:

A.

$\frac{1}{3}\log_e\left(\frac{4}{9e^{1/3}}\right)$

B.

$\frac{1}{3}\log_e\left(\frac{4}{9}\right)+1$

C.

$\log_e\left(\frac{4}{9}\right)-1$

D.

$\log_e\left(\frac{4}{9e^{1/3}}\right)$

Answer: D

Solution:

Let $g(x) = ax + b$

Now function $f(x)$ in continuous at $x = 0$

$\therefore \lim\limits_{x \to 0^-} f(x) = f(0)$

$\Rightarrow \lim\limits_{x \to 0}\left(\frac{1+x}{2+x}\right)^{\frac{1}{x}} = b$

$\Rightarrow 0 = b$

$\therefore g(x) = ax$

Now, for $x > 0$

$f'(x) = \frac{1}{x}\cdot\left(\frac{1+x}{2+x}\right)^{\frac{1}{x}-1}\cdot\frac{1}{(2+x)^2}$

$+\left(\frac{1+x}{2+x}\right)^{\frac{1}{x}}\cdot\ln\left(\frac{1+x}{2+x}\right)\cdot\left(-\frac{1}{x^2}\right)$

$\therefore f'(1) = \frac{1}{9}-\frac{2}{3}\cdot\ln\left(\frac{2}{3}\right)$

And $f(-1) = g(-1) = -a$

$\therefore a = \frac{2}{3}\ln\left(\frac{2}{3}\right)-\frac{1}{9}$

$\therefore g(3) = 2\ln\left(\frac{2}{3}\right)-\frac{1}{3}$

$= \ln\left(\frac{4}{9\cdot e^{1/3}}\right)$

Question6

If $a=\lim_{x\to 0}\frac{\sqrt{1+\sqrt{1+x^4}}-\sqrt{2}}{x^4}$ and $b=\lim_{x\to 0}\frac{\sin^2 x}{\sqrt{2}-\sqrt{1+\cos x}}$, then the value of ab^3 is :

[27-Jan-2024 Shift 1]

Options:

A.

36

B.

32

C.

25

D.

30

Answer: B

Solution:

$$a=\lim_{x\to 0}\frac{\sqrt{1+\sqrt{1+x^4}}-\sqrt{2}}{x^4}$$

$$=\lim_{x\to 0}\frac{\sqrt{1+x^4}-1}{x^4\left(\sqrt{1+\sqrt{1+x^4}}+\sqrt{2}\right)}$$

$$=\lim_{x\to 0}\frac{x^4}{x^4\left(\sqrt{1+\sqrt{1+x^4}}+\sqrt{2}\right)\left(\sqrt{1+x^4}+1\right)}$$

Applying limit $a=\frac{1}{4\sqrt{2}}$

$$b=\lim_{x\to 0}\frac{\sin^2 x}{\sqrt{2}-\sqrt{1+\cos x}}$$

$$=\lim_{x\to 0}\frac{(1-\cos^2 x)(\sqrt{2}+\sqrt{1+\cos x})}{2-(1+\cos x)}$$

$$b=\lim_{x\to 0}(1+\cos x)(\sqrt{2}+\sqrt{1+\cos x})$$

Applying limits $b=2(\sqrt{2}+\sqrt{2})=4\sqrt{2}$

Now, $ab^3=\frac{1}{4\sqrt{2}}\times(4\sqrt{2})^3=32$

Question7

If $\lim_{x\to 0}\frac{3+\alpha\sin x+\beta\cos x+\log_e(1-x)}{3\tan^2 x}=\frac{1}{3}$, then $2\alpha-\beta$ is equal to :

[27-Jan-2024 Shift 2]

Options:

A.

2

B.

7

C.

5

D.

1

Answer: C

Solution:

$$\lim_{x \to 0} \frac{3 + \alpha sin\, x + \beta \cos x + \log_e(1 - x)}{3\tan^2 x} = \frac{1}{3}$$

$$\Rightarrow \lim_{x \to 0} \frac{3 + \alpha\left[x - \frac{x^3}{3!} + \ldots\right] + \beta\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!}\ldots\right] + \left(-x - \frac{x^2}{2} - \frac{x^3}{3}\ldots\right)}{3\tan^2 x} = \frac{1}{3}$$

$$\Rightarrow \lim_{x \to 0} \frac{(3+\beta) + (\alpha - 1)x + \left(-\frac{1}{2} - \frac{\beta}{2}\right)x^2 + \ldots}{3x^2} \times \frac{x^2}{\tan^2 x} = \frac{1}{3}$$

$$\Rightarrow \beta + 3 = 0,\ \alpha - 1 = 0 \text{ and } \frac{-\frac{1}{2} - \frac{\beta}{2}}{3} = \frac{1}{3}$$

$$\Rightarrow \beta = -3,\ \alpha = 1$$

$$\Rightarrow 2\alpha - \beta = 2 + 3 = 5$$

Question8

$$\lim_{x \to \frac{\pi}{2}} \left(\frac{1}{\left(x - \frac{\pi}{2}\right)^2} \int_{x^3}^{\left(\frac{\pi}{2}\right)^3} \cos\left(\frac{1}{t^3}\right) dt \right)$$ is equal to

[29-Jan-2024 Shift 1]

Options:

A.

$3\pi/8$

B.

$3\pi^2/4$

C.

$3\pi^2/8$

D.

$3\pi/4$

Answer: C

Solution:

Using L'hopital rule

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{0 - \cos x \times 3x^2}{2\left(x - \frac{\pi}{2}\right)}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\sin\left(x - \frac{\pi}{2}\right)}{2\left(x - \frac{\pi}{2}\right)} \times \frac{3\pi^2}{4}$$

$$= \frac{3\pi^2}{8}$$

Question9

Let the slope of the line 45x + 5y + 3 = 0 be $27r_1 + \frac{9r_2}{2}$ for some $r_1, r_2 \in R$. Then $\lim_{x \to 3}\left(\int_3^x \frac{8t^2}{\frac{3r_2x}{2} - r_2x^2 - r_1x^3 - 3x}\, dt\right)$ is equal

[29-Jan-2024 Shift 2]

Answer: 12

Solution:

According to the question,

$$27r_1 + \frac{9r_2}{2} = -9$$

$$\lim_{x \to 3} \frac{\int_3^x 8t^2\, dt}{\frac{3r_2x}{2} - r_2x^2 - r_1x^3 - 3x}$$

$$= \lim_{x \to 3} \frac{8x^2}{\frac{3r_2^2}{2} - 2r_2x - 3r_1x^2 - 3} \text{ (using LH' Rule)}$$

$$= \frac{72}{\frac{3r_2}{2} - 6r_2 - 27r_1 - 3}$$

$$= \frac{72}{-\frac{9r_2}{2} - 27r_1 - 3}$$

$$= \frac{72}{9-3} = 12$$

Question10

Let f : $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow$ R be a differentiable function such that f(0) = 1/2, If the $\lim_{x \to 0} \frac{x \int_0^x f(t)\,dt}{e^{x^2} - 1} = \alpha,$ then $8\alpha^2$ is equal to :

[30-Jan-2024 Shift 1]

Options:

A.

16

B.

2

C.

1

D.

4

Answer: B

Solution:

$$\lim_{x \to 0} \frac{x \int_0^x f(t)\,dt}{\left(\frac{e^{x^2} - 1}{x^2}\right) \times x^2}$$

$$\lim_{x \to 0} \frac{\int_0^x f(t)\,dt}{x} \quad \left(\lim_{x \to 0} \frac{e^{x^2} - 1}{x^2} = 1\right)$$

$$= \lim_{x \to 0} \frac{f(x)}{1} \text{ (using L Hospital)}$$

$$f(0) = \frac{1}{2}$$

$$\alpha = \frac{1}{2}$$

$$8\alpha^2 = 2$$

Question11

Let a be the sum of all coefficients in the expansion of $(1-2x+2x^2)^{2023}(3-4x^2+2x^3)^{2024}$ and $b = \lim_{x \to 0} \left(\frac{\int_0^x \frac{\log(1+t)}{t^{2024}+1} dt}{x^2} \right)$. **If the equations $cx^2 + dx + e = 0$ and $2bx^2 + ax + 4 = 0$ have a common root, where c, d, e ∈ R, then d:c : e equals**

[31-Jan-2024 Shift 1]

Options:

A.

2:1:4

B.

4:1:4

C.

1:2:4

D.

1:1:4

Answer: D

Solution:

Put $x = 1$

$\therefore a = 1$

$$b = \lim_{x \to 0} \frac{\int_0^x \frac{\ln(1+t)}{1+t^{2024}} dt}{x^2}$$

Using L' HOPITAL Rule

$$b = \lim_{x \to 0} \frac{\ln(1+x)}{(1+x^{2024})} \times \frac{1}{2x} = \frac{1}{2}$$

Now, $cx^2 + dx + e = 0$, $x^2 + x + 4 = 0$

$(D < 0)$

$$\therefore \frac{c}{1} = \frac{d}{1} = \frac{e}{4}$$

Question12

$$\lim_{x \to 0} \frac{e^{2|\sin x|} - 2|\sin x| - 1}{x^2}$$

[31-Jan-2024 Shift 1]

Options:

A.

is equal to -1

B.

does not exist

C.

is equal to 1

D.

is equal to 2

Answer: D

Solution:

$$\lim_{x \to 0} \frac{e^{2|\sin x|} - 2|\sin x| - 1}{x^2}$$

$$\lim_{x \to 0} \frac{e^{2|\sin x|} - 2|\sin x| - 1}{|\sin x|^2} \times \frac{\sin^2 x}{x^2}$$

Let $|\sin x| = t$

$$\lim_{t \to 0} \frac{e^{2t} - 2t - 1}{t^2} \times \lim_{x \to 0} \frac{\sin^2 x}{x^2}$$

$$= \lim_{t \to 0} \frac{2e^{2t} - 2}{2t} \times 1 = 2 \times 1 = 2$$

Question13

Let f : ⟶ R ⟶ (0, ∞) be strictly increasing function such that $\lim_{x \to \infty} \frac{f(7x)}{f(x)} = 1$. Then, the value of $\lim_{x \to \infty} \left[\frac{f(5x)}{f(x)} - 1 \right]$ **is equal to**

[31-Jan-2024 Shift 2]

Options:

A.

4

B.

0

C.

7/5

D.

1

Answer: B

Solution:

$f\ : R \longrightarrow (0, \infty)$

$\lim_{x \to \infty} \frac{f(7x)}{f(x)} = 1$

$\because f$ is increasing

$\therefore f(x) < f(5x) < f(7x)$

$\because \frac{f(x)}{f(x)} < \frac{f(5x)}{f(x)} < \frac{f(7x)}{f(x)}$

$1 < \lim_{x \to \infty} \frac{f(5x)}{f(x)} < 1$

$\therefore \left[\frac{f(5x)}{f(x)} - 1 \right]$

$\Rightarrow 1 - 1 = 0$

Question14

If $\lim_{x \to 0} \frac{ax^2 e^x - b\log_e(1+x) + cxe^{-x}}{x^2 sin\, x} = 1$, then $16(a^2 + b^2 + c^2)$ is equal to______

[31-Jan-2024 Shift 2]

Answer: 81

Solution:

$$\lim_{x \to 0} \frac{ax^2\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+.....\right)-b\left(x-\frac{x^2}{2}+\frac{x^3}{3}-........\right)+cx\left(1-x+\frac{x^2}{x!}-\frac{x^3}{3!}+......\right)}{x^3 \cdot \frac{\sin x}{x}}$$

$$= \lim_{x \to \infty} \frac{(c-b)x+\left(\frac{b}{2}-c+a\right)x^2+\left(a-\frac{b}{3}+\frac{c}{2}\right)x^3+.......}{x^3} = 1$$

$c-b=0, \quad \frac{b}{2}-c+a=0$

$a-\frac{b}{3}+\frac{c}{2}=1 \quad a=\frac{3}{4} \quad b=c=\frac{3}{2}$

$a^2+b^2+c^2=\frac{9}{16}+\frac{9}{4}+\frac{9}{4}$

$16(a^2+b^2+c^2)=81$

Question15

Let $\{x\}$ denote the fractional part of x and $f(x)=\frac{\cos^{-1}(1-\{x\}^2)\sin^{-1}(1-\{x\})}{\{x\}-\{x\}^3}, x \neq 0$. If L and R respectively denotes the left hand limit and the right hand limit of $f(x)$ at $x=0$, then $\frac{32}{\pi^2}(L^2+R^2)$ is equal to___ *[1-Feb-2024 Shift 1]*

Answer: 18

Solution:

Finding right hand limit

$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} f(h)$

$$= \lim_{h \to 0} \frac{\cos^{-1}(1-h^2)\sin^{-1}(1-h)}{h(1-h^2)}$$

$$= \lim_{h \to 0} \frac{\cos^{-1}(1-h^2)}{h}\left(\frac{\sin^{-1}1}{1}\right)$$

Let $\cos^{-1}(1-h^2) = \theta \Rightarrow \cos\theta = 1-h^2$

$$= \frac{\pi}{2}\lim_{\theta \to 0} \frac{\theta}{\sqrt{1-\cos\theta}}$$

$$= \frac{\pi}{2}\lim_{\theta \to 0} \frac{1}{\sqrt{\frac{1-\cos\theta}{\theta^2}}}$$

$$= \frac{\pi}{2}\frac{1}{\sqrt{1/2}}$$

$$R = \frac{\pi}{\sqrt{2}}$$

Now finding left hand limit

$$L = \lim_{x \to 0^-} f(x)$$

$$= \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} \frac{\cos^{-1}(1-\{-h\}^2)\sin^{-1}(1-\{-h\})}{\{-h\}-\{-h\}^3}$$

$$= \lim_{h \to 0} \frac{\cos^{-1}(1-(-h+1)^2)\sin^{-1}(1-(-h+1))}{(-h+1)-(-h+1)^3}$$

$$= \lim_{h \to 0} \frac{\cos^{-1}(-h^2+2h)\sin^{-1}h}{(1-h)(1-(1-h)^2)}$$

$$= \lim_{h \to 0}\left(\frac{\pi}{2}\right)\frac{\sin^{-1}h}{(1-(1-h)^2)}$$

$$= \frac{\pi}{2}\lim_{h \to 0}\left(\frac{\sin^{-1}h}{-h^2+2h}\right)$$

$$= \frac{\pi}{2}\lim_{h \to 0}\left(\frac{\sin^{-1}h}{h}\right)\left(\frac{1}{-h+2}\right)$$

$$L = \frac{\pi}{4}$$

$$\frac{32}{\pi^2}(L^2+R^2) = \frac{32}{\pi^2}\left(\frac{\pi^2}{2}+\frac{\pi^2}{16}\right)$$

$$= 16+2$$

$$= 18$$

Question16

Let $f(x) = \begin{cases} x-1 & x \text{ is even,} \\ 2x & x \text{ is odd,} \end{cases}$. $a \in N, f(f(f(a))) = 21$, then $\lim_{x \to a^-} \left\{ \frac{|x|^3}{a} - \left[\frac{x}{a} \right] \right\}$, where $[t]$ denotes the greatest integer less than or equal to t, is equal to :

[1-Feb-2024 Shift 2]

Options:

A.

121

B.

144

C.

169

D.

225

Answer: B

Solution:

$$f(x) = \begin{cases} x-1; & x = \text{ even} \\ 2x; & x = \text{ odd} \end{cases} .$$

$f(f(f(\text{a}))) = 21$

C - 1 : If $\text{a} =$ even

$f(\text{a}) = \text{a} - 1 =$ odd

$f(\text{f(a)}) = 2(\text{a} - 1) =$ even

$f(f(f(a))) = 2a - 3 = 21 \Rightarrow \text{a} = 12$

C - 2 : If $a =$ odd

$f(\text{a}) = 2\text{a} =$ even

$f(f(\text{a})) = 2\text{a} \quad 1 =$ odd

$f(f(f(\text{a}))) = 4\text{a} - 2 = 21$ (Not possible)

Hence $\text{a} = 12$

Now

$$\lim_{x \to 12^-} \left(\frac{|x|^3}{2} - \left[\frac{x}{12} \right] \right)$$

$$= \lim_{x \to 12^-} \frac{|x|^3}{12} - \lim_{x \to 12^-} \left[\frac{x}{12} \right]$$

$= 144 - 0 = 144.$

Question17

If $y=\frac{(\sqrt{x}+1)(x^2-\sqrt{x})}{x\sqrt{x}+x+\sqrt{x}}+\frac{1}{15}(3\cos^2 x-5)\cos^3 x$ then $96y'\left(\frac{\pi}{6}\right)$ is equal to :

[1-Feb-2024 Shift 2]

Answer: 105

Solution:

$$y=\frac{(\sqrt{x}+1)(x^2-\sqrt{x})}{x\sqrt{x}+x+\sqrt{x}}+\frac{1}{15}(3\cos^2 x-5)\cos^3 x$$

$$y=\frac{(\sqrt{x}+1)(\sqrt{x})((\sqrt{x})^3-1)}{(\sqrt{x})((\sqrt{x})^2+(\sqrt{x})+1)}+\frac{1}{5}\cos^5 x-\frac{1}{3}\cos^3 x$$

$$y=(\sqrt{x}+1)(\sqrt{x}-1)+\frac{1}{5}\cos^5 x-\frac{1}{3}\cos^3 x$$

$$y'=1-\cos^4 x\cdot(\sin x)+\cos^2 x(\sin x)$$

$$y'\left(\frac{\pi}{6}\right)=1-\frac{9}{16}\times\frac{1}{2}+\frac{3}{4}\times\frac{1}{2}$$

$$=\frac{32-9+12}{32}=\frac{35}{32}$$

$$=96y'\left(\frac{\pi}{6}\right)=105$$

Question18

$\lim\limits_{t\to 0}\left(1^{\frac{1}{\sin^2 t}}+2^{\frac{1}{\sin^2 t}}+\ldots.+n^{\frac{1}{\sin^2 t}}\right)^{\sin^2 t}$ **is equal to**

[24-Jan-2023 Shift 1]

Options:

A. n^2+n

B. n

C. $\frac{n(n+1)}{2}$

D. n^2

Answer: B

Solution:

Solution:

$\lim\limits_{t\to 0}\left(1^{\operatorname{cosec}^2 t}+2^{\operatorname{cosec}^2 t}+\ldots\ldots.+n^{\operatorname{cosec}^2 t}\right)^{\sin^2 t}$

$$= \lim_{t \to 0} n \left(\left(\frac{1}{n} \right)^{\operatorname{cosec}^2 t} + \left(\frac{2}{n} \right)^{\operatorname{cosec}^2 t} + \ldots\ldots + 1 \right)^{\sin^2 t}$$
$$= n$$

Question19

The set of all values of a for which $\lim\limits_{x \to a}([x-5]-[2x+2]) = 0$, where $[\propto]$ denotes the greater integer less than or equal to $\propto$ is equal to [24-Jan-2023 Shift 2]

Options:

A. $(-7.5, -6.5)$

B. $(-7.5, -6.5]$

C. $[-7.5, -6.5]$

D. $[-7.5, -6.5)$

Answer: A

Solution:

Solution:

$\lim\limits_{x \to a}([x-5]-[2x+2]) = 0$

$\lim\limits_{x \to a}([x]-5-[2x]-2) = 0$

$\lim\limits_{x \to a}([x]-[2x]) = 7$

$[a]-[2a] = 7$

$a \in I$, $a = -7$

$a \notin I$, $a = I + f$

Now, $[a]-[2a] = 7$

$-I-[2f] = 7$

Case-I: $f \in \left(0, \frac{1}{2}\right)$

$2f \in (0, 1)$

$-I = 7$

$I = -7 \Rightarrow a \in (-7, -6.5)$

Case-II: $f \in \left(\frac{1}{2}, 1\right)$

$2f \in (1, 2)$

$-I-1 = 7$

$I = -8 \Rightarrow a \in (-7.5, -7)$

Hence, $a \in (-7.5, -6.5)$

Question20

Let $x = 2$ be a root of the equation $x^2 + px + q = 0$ and $f(x) = \begin{cases} \dfrac{1-\cos(x^2-4px+q^2+8q+16)}{(x-2p)^4} & x \neq 2p \\ 0 & x = 2p \end{cases}$. Then $\lim\limits_{x \to 22^+}[f(x)]$ where [.] denotes greatest integer function, is

[29-Jan-2023 Shift 1]

Options:

A. 2

B. 1

C. 0

D. −1

Answer: C

Solution:

Solution:

$$\lim_{x \to 2p^+} \left(\frac{1 - \cos(x^2 - 4px + q^2 + 8q + 16)}{(x^2 - 4px + q^2 + 8q + 16)^2} \right) \left(\frac{(x^2 - 4px + q^2 + 8q + 16)^2}{(x - 2p)^2} \right)$$

$$\lim_{h \to 0} \frac{1}{2} \left(\frac{(2p + h)^2 - 4p(2p + h) + q^2 + 82 + 16}{h^2} \right)^2 = \frac{1}{2}$$

Using L'Hospital's

$$\lim_{x \to 2p^+} [f(x)] = 0$$

Question21

Let f, g and h be the real valued functions defined on ℝ as

$$f(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ 1 & x = 0 \end{cases}, g(x) = \begin{cases} \frac{\sin(x+1)}{(x+1)} & x \neq -1 \\ 1 & x = -1 \end{cases}. \text{ and } h(x) = 2[x] - f(x),$$

where [x] is the greatest integer ≤x. Then the value of $\lim_{x \to 1} g(h(x-1))$ is

[30-Jan-2023 Shift 2]

Options:

A. 1

B. sin (1)

C. −1

D. 0

Answer: A

Solution:

Solution:

$$\text{LHL} = \lim_{k \to 0} g(h(-k)), k > 0$$

$$= \lim_{k \to 0} g(-2 + 1) \because f(x) = -1 \; \forall x < 0$$

$$= g(-1) = 1$$

$$\text{RHL} = \lim_{k \to 0} g(h(k)), k > 0$$

$= \lim_{k \to 0} g(-1)$, $\because f(x) = 1,\ \forall x > 0$
$= 1$

Question22

$\lim_{x \to \infty} \frac{(\sqrt{3x+1} + \sqrt{3x-1})^6 + (\sqrt{3x+1} - \sqrt{3x-1})^6}{\left(x + \sqrt{x^2-1}\right)^6 + \left(x - \sqrt{x^2-1}\right)^6} \mathbf{x^3}$

[31-Jan-2023 Shift 2]

Options:

A. is equal to 9

B. is equal to 27

C. does not exist

D. is equal to $\frac{27}{2}$

Answer: B

Solution:

Solution:

$\lim_{x \to \infty} \frac{(\sqrt{3x+1} + \sqrt{3x-1})^6 + (\sqrt{3x+1} - \sqrt{3x-1})^6}{\left(x + \sqrt{x^2-1}\right)^6 + \left(x - \sqrt{x^2-1}\right)^6} x^3$

$$\lim_{x \to \infty} x^3 \times \left\{ \frac{x^3 \left\{ \left(\sqrt{3 + \frac{1}{x}} + \sqrt{3 - \frac{1}{x}} \right)^6 + \left(\sqrt{3 + \frac{1}{x}} - \sqrt{3 - \frac{1}{x}} \right)^6 \right\}}{x^6 \left\{ \left(1 + \sqrt{1 - \frac{1}{x^2}} \right)^6 + \left(1 - \sqrt{1 - \frac{1}{x^2}} \right)^6 \right\}} \right\}$$

$= \frac{(2\sqrt{3})^6 + 0}{2^6 + 0} = 3^3 = (27)$

Question23

If $f(x) = x^3 - x^2 f'(1) + x f''(2) - f'''(3)$, $x \in R$, then
[24-Jan-2023 Shift 2]

Options:

A. $3f(1) + f(2) = f(3)$

B. $f(3) - f(2) = f(1)$

C. $2f(0) - f(1) + f(3) = f(2)$

D. $f(1) + f(2) + f(3) = f(0)$

Answer: C

Solution:

Solution:
$f(x) = x^3 - x^2 f'(1) + x f''(2) - f'''(3), x \in R$
Let $f'(1) = a, f''(2) = b, f'''(3) = c$
$f(x) = x^3 - x^2 + bx - c$
$f'(x) = 3x^2 - 2ax + b$
$f''(x) = 6x - 2a$
$f'''(x) = 6$
$c = 6, a = 3, b = 6$
$f(x) = x^3 - 3x^2 + 6x - 6$
$f(1) = -2, f(2) = 2, f(3) = 12, f(0) = -6$
$2f(0) - f(1) + f(3) = 2 = f(2)$

Question24

Let $y(x) = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)(1 + x^{16})$. Then $y' - y''$ at $x = -1$ is equal to [25-Jan-2023 Shift 1]

Options:

A. 976

B. 464

C. 496

D. 944

Answer: C

Solution:

Solution:
$y = \dfrac{1 - x^{32}}{1 - x} \Rightarrow y - xy = 1 - x^{32}$
$y' - xy' - y = -32x^{31}$
$y'' - xy'' - y' - y' = -(32)(31)x^{30}$
at $x = -1 \Rightarrow y' - y'' = 496$

Question25

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function that satisfies the relation $f(x + y) = f(x) + f(y) - 1, \forall x, y \in \mathbb{R}$. If $f'(0) = 2$, then $|f(-2)|$ is equal to _____ [29-Jan-2023 Shift 1]

Answer: 3

Solution:

Solution:

$f(x+y) = f(x) + f(y) - 1$

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$f'(x) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = f'(0) = 2$

$f'(x) = 2 \Rightarrow dy = 2\,dx$

$y = 2x + C$

$x = 0, y = 1, c = 1$

$y = 2x + 1$

$|f(-2)| = |-4 + 1| = |-3| = 3$

Question26

Let f and g be twice differentiable functions on R such that
$f''(x) = g''(x) + 6x$
$f'(1) = 4g'(1) - 3 = 9$
$f(2) = 3g(2) = 12$
Then which of the following is NOT true?
[29-Jan-2023 Shift 2]

Options:

A. $g(-2) - f(-2) = 20$

B. If $-1 < x < 2$, then $|f(x) - g(x)| < 8$

C. $f'(x) - g'(x)\ | < 6 \Rightarrow -1 < x < 1\ |$

D. There exists $x_0 \in \left(1, \frac{3}{2}\right)$ such that $f(x_0) = g(x_0)$

Answer: B

Solution:

Solution:

$f''(x) = g''(x) + 6x \ldots (1)$

$f'(1) = 4g'(1) - 3 = 9 \ldots (2)$

$f(2) = 3g(2) = 12 \ldots (3)$

By integrating (1)

$f'(x) = g'(x) + 6\frac{x^2}{2} + C$

At $x = 1$

$f'(1) = g'(1) + 3 + C$

$\Rightarrow 9 = 4 + 3 + C \Rightarrow C = 3$

$\therefore f'(x) = g'(x) + 3x^2 + 3$

Again by integrating,

$f(x) = g(x) + \frac{3x^3}{3} + 3x + D$

At $x = 2$

$f(2) = g(2) + 8 + 3(2) + D$

$\Rightarrow 12 = 4 + 8 + 6 + D \Rightarrow D = -6$

So, $f(x) = g(x) + x^3 + 3x - 6$

$\Rightarrow f(x) - g(x) = x^3 + 3x - 6$

At $x = -2$

$\Rightarrow g(-2) - f(-2) = 20$ (Option (1) is true)

Now, for $-1 < x, 2$

$h(x) = f(x) - g(x) = x^3 + 3x - 6$

$\Rightarrow h'(x) = 3x^2 + 3$
$\Rightarrow h(x) \uparrow$
So, $h(-1) < h(x) < h(2)$
$\Rightarrow -10 < h(x) < 8$
$\Rightarrow |h(x)| < 10$ (option (2) is NOT true)
Now, $h'(x) = f'(x) - g'(x) = 3x^2 + 3$
If $|h'(x)| < 6 \Rightarrow |3x^2 + 3| < 6$
$\Rightarrow 3x^2 + 3 < 6$
$\Rightarrow x^2 < 1$
$\Rightarrow -1 < x < 1$ (option (3) is True)
If $x \in (-1, 1)$ $|f'(x) - g'(x)| < 6$
option (3) is true and now to solve
$f(x) - g(x) = 0$
$\Rightarrow x^3 + 3x - 6 = 0$
$h(x) = x^3 + 3x - 6$
here, $h(1) = -ve$ and $h\left(\frac{3}{2}\right) = +ve$
So there exists $x_0 \in \left(1, \frac{3}{2}\right)$ such that $f(x_0) = g(x_0)$
(option (4) is true)

Question27

Let

$$\mathbf{y = f(x) = \sin^3\left(\frac{\pi}{3}\left(\cos\left(\frac{\pi}{3\sqrt{2}}(-4x^3 + 5x^2 + 1)^{\frac{3}{2}}\right)\right)\right)}$$

Then, at x = 1,
[31-Jan-2023 Shift 1]

Options:

A. $2y' + \sqrt{3}\pi^2 y = 0$

B. $2y' + 3\pi^2 y = 0$

C. $\sqrt{2}y' - 3\pi^2 y = 0$

D. $y' + 3\pi^2 y = 0$

Answer: B

Solution:

$y = \sin^3(\pi/3 \cos g(x))$
$g(x) = \frac{\pi}{3\sqrt{2}}(-4x^3 + 5x^2 + 1)^{3/2}$
$g(1) = 2\pi/3$
$y' = 3\sin^2\left(\frac{\pi}{3}\cos g(x)\right) \times \cos\left(\frac{\pi}{3}\cos g(x)\right)$
$y'(1) = 3\sin^2\left(-\frac{\pi}{6}\right) \cdot \cos\left(\frac{\pi}{6}\right) \cdot \frac{\pi}{3}\left(-\sin\frac{2\pi}{3}\right) g'(1)$
$g'(x) = \frac{\pi}{3\sqrt{2}}(-4x^3 + 5x^2 + 1)^{1/2}(-12x^2 + 10x)$
$g'(1) = \frac{\pi}{2\sqrt{2}}(\sqrt{2})(-2) = -\pi$
$y'(1) = \frac{\text{not}\,6}{4} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\pi}{\text{not}}\left(\frac{-\sqrt{3}}{2}\right)(-\pi) = \frac{3\pi^2}{16}$
$y'(1) = \sin^3(\pi/3 \cos 2\pi/3) = -\frac{1}{8}$
$2y'(1) + 3\pi^2 y(1) = 0$

Question28

If $\int_{0}^{1}(x^{21}+x^{14}+x^{7})(2x^{14}+3x^{7}+6)^{1/7}dx = \frac{1}{l}(11)^{m/n}$ where 1, m, n ∈ ℕ, m and n are coprime then l + m + n is equal to ________.

[1-Feb-2023 Shift 1]

Answer: 63

Solution:

$\int(x^{20}+x^{13}+x^{6})(2x^{21}+3x^{14}+6x^{7})^{1/7}dx$

$2x^{21}+3x^{14}+6x^{7}=t$

$42(x^{20}+x^{13}+x^{6})dx = dt$

$$\frac{1}{42}\int_{0}^{11}t^{\frac{1}{7}}dt = \left(\frac{t^{\frac{8}{7}}}{\frac{8}{7}}\times\frac{1}{42}\right)_{0}^{11}$$

$$= \frac{1}{48}\left(t^{\frac{8}{7}}\right)_{0}^{11} = \frac{1}{48}(11)^{8/7}$$

$l = 48, m = 8, n = 7$

$l + m + n = 63$

Question29

$\lim\limits_{n\to\infty}\left\{\left(2^{\frac{1}{2}}-2^{\frac{1}{3}}\right)\left(2^{\frac{1}{2}}-2^{\frac{1}{5}}\right)\ldots\ldots\left(2^{\frac{1}{2}}-2^{\frac{1}{2n+1}}\right)\right\}$ **is equal to**

[6-Apr-2023 shift 2]

Options:

A. $\frac{1}{\sqrt{2}}$

B. $\sqrt{2}$

C. 1

D. 0

Answer: D

Solution:

$P = \lim_{n \to \infty} \left(2^{\frac{1}{2}} - 2^{\frac{1}{3}}\right)\left(2^{\frac{1}{2}} - 2^{\frac{1}{5}}\right) \ldots\ldots. \left(2^{\frac{1}{2}} - 2^{\frac{1}{2n+1}}\right)$

Let

$2^{\frac{1}{2}} - 2^{\frac{1}{3}} \rightarrow$ Smallest

$2^{\frac{1}{2}} - 2^{\frac{1}{2n+1}} \rightarrow$ Largest

Sandwich th.

$\left(2^{\frac{1}{2}} - 2^{\frac{1}{3}}\right)^n \leq P \leq \left(2^{\frac{1}{2}} - 2^{\frac{1}{22n+1}}\right)^n$

$\left(\begin{matrix} \text{lie b / w} \\ 0 \text{ and } 1 \end{matrix}\right)^n$

$\lim_{n \to \infty} \left(2^{\frac{1}{2}} - 2^{\frac{1}{3}}\right)^n = 0$

$\lim_{n \to \infty} \left(2^{\frac{1}{2}} - 2^{\frac{1}{2n+1}}\right)^n = 0$

$\therefore P = 0$

Question30

$\lim_{x \to 0} \left(\left(\frac{(1 - \cos^2(3x)}{\cos^3(4x)} \right) \left(\frac{\sin^3(4x)}{(\log_e(2x+1))^5} \right) \right)$ **is equal to ______**

[8-Apr-2023 shift 1]

Options:

A. 24

B. 9

C. 18

D. 15

Answer: C

Solution:

Solution:

$\lim_{x \to 0} \left[\frac{1 - \cos^2 3x}{9x^2} \right] \frac{9x^2}{\cos^3 4x} \cdot \frac{\left(\frac{\sin 4x}{4x}\right)^3 \times 64x^3}{\left[\frac{\ln(1+2x)}{2x}\right]^5 \times 32x^5}$

$\lim_{x \to 0} 2\left(\frac{1}{2} \times \frac{9}{1} \times \frac{1 \times 64}{1 \times 32}\right) = 18$

Question31

If $\alpha > \beta > 0$ are the roots of the equation $ax^2 + bx + 1 = 0$, and

$\lim_{x \to \frac{1}{\alpha}} \left(\frac{1 - \cos(x^2 + bx + a)}{2(1 - ax)^2} \right)^{\frac{1}{2}} = \frac{1}{k}\left(\frac{1}{\beta} - \frac{1}{\alpha}\right)$**, then k is equal to**

[8-Apr-2023 shift 2]

Options:

A. β

B. 2α

C. 2β

D. α

Answer: B

Solution:

Solution:

$\therefore ax^2 + bx + 1 = a(x-\alpha)(x-\beta) \therefore \alpha\beta = \frac{1}{a}$

$\therefore x^2 + bx + a = a(1-\alpha x)(1-\beta x)$

$$\therefore \lim_{x \to \frac{1}{\alpha}} \left\{ \frac{1-\cos(x^2+bx+a)}{2(1-\alpha x)^2} \right\}^{\frac{1}{2}} = \lim_{x \to \frac{1}{2}} \left\{ \frac{1-\cos a(1-\alpha x)(1-\beta x)}{2\{a(1-\alpha x)(1-\beta x)\}^2} \cdot a^2(1-\beta x)^2 \right\}^{\frac{1}{2}}$$

$$= \left[\frac{1}{2} \cdot \frac{1}{2} a^2 \left(1 - \frac{\beta}{\alpha}\right)^2 \right]^{\frac{1}{2}}$$

$$= \frac{1}{2} \frac{1}{\alpha\beta} \left(1 - \frac{\beta}{\alpha}\right) = \frac{1}{2} \left(\frac{1}{\alpha\beta} - \frac{1}{\alpha^2} \right)$$

$$= \frac{1}{2\alpha} \left(\frac{1}{\beta} - \frac{1}{\alpha} \right) = \frac{1}{k} \left(\frac{1}{\beta} - \frac{1}{\alpha} \right)$$

$\therefore k = 2\alpha$ Ans.

Question32

Among

(S1): $\lim_{n \to \infty} \frac{1}{n^2}(2+4+6+\dots+2n) = 1$

(S2): $\lim_{n \to \infty} \frac{1}{n^{16}}(1^{15}+2^{15}+3^{15}+\dots+n^{15}) = \frac{1}{16}$

[13-Apr-2023 shift 1]

Options:

A. Only (S1) is true

B. Both (S1) and (S2) are true

C. Both (S1) and (S2) are false

D. Only (S2) is true

Answer: B

Solution:

Solution:

$S_1 : \lim_{n \to \infty} \frac{n(n+1)}{n^2} = 1 \Rightarrow$ True

$S_2 : \lim_{n \to \infty} \frac{1}{n^{16}}(\Sigma r^{15}) = \lim_{n \to \infty} \frac{1}{n} \Sigma \left(\frac{r}{n} \right)^{15}$

$= \int_0^1 x^{15} dx = \frac{1}{16} \Rightarrow$ True

Question33

If $\lim_{x \to 0} \frac{e^{ax} - \cos(bx) - \frac{cxe^{-cx}}{2}}{1 - \cos(2x)} = 17$, then $5a^2 + b^2$ is equal to
[13-Apr-2023 shift 2]

Options:

A. 76

B. 72

C. 64

D. 68

Answer: D

Solution:

Solution:

$$\lim_{x \to 0} \frac{e^{ax} - \cos bx - \frac{cxe^{-cx}}{2}}{1 - \cos 2x} = 17$$

On expansion

$$\lim_{x \to 0} \frac{\left(1 + ax + \frac{(ax)^2}{2!} + \ldots\right) - \left(1 - \frac{(bx)^2}{2!} + \ldots\right) - \frac{cx}{2}\left(1 - cx + \frac{(cx)^2}{2!}\right)}{\left(\frac{1 - \cos 2x}{(2x)^2}\right) \times (2x)^2} = 17$$

$$\lim_{x \to 0} \frac{x\left(a - \frac{c}{2}\right) + x^2\left(\frac{a^2}{2} + \frac{b^2}{2} + \frac{c^2}{2}\right)}{\frac{1}{2}(4x^2)} = 17$$

For limit to be exist

$a - \frac{c}{2} = 0 \Rightarrow c = 2a$

$\Rightarrow \frac{\frac{a^2}{2} + \frac{b^2}{2} + \frac{c^2}{2}}{2} = 17$

$\Rightarrow \frac{a^2}{2} + \frac{b^2}{2} + \frac{4a^2}{2} = 34$

$\Rightarrow 5a^2 + b^2 = 68$

Question34

If $2x^y + 3y^x = 20$, then $\frac{dy}{dx}$ at (2, 2) is equal to:
[6-Apr-2023 shift 1]

Options:

A. $-\left(\frac{3+\log_e 8}{2+\log_e 4}\right)$

B. $-\left(\frac{2+\log_e 8}{3+\log_e 4}\right)$

C. $-\left(\frac{3+\log_e 4}{2+\log_e 8}\right)$

D. $-\left(\frac{3+\log_e 16}{4+\log_e 8}\right)$

Answer: B

Solution:

Solution:

$2x^y + 3y^x = 20$

$v_1^{v_2}\left(v_2 \frac{1}{v_1} + \ln v_1 \cdot v_2{}^1\right)$

$2x^y\left(y \cdot \frac{1}{x} + \ln x \frac{dy}{dx}\right) + 3y^x\left(x \frac{1}{y} \cdot \frac{dy}{dx} + \ln y \cdot 1\right) = 0$

Put $(2, 2)$

$2.4\left(1 + \ln 2 \frac{dy}{dx}\right) + 3 \cdot 4\left(1 \cdot \frac{dy}{dx} + \ln 2\right) = 0$

$\frac{dy}{dx}[8\ln 2 + 12] + 8 + 12\ln 2 = 0$

$\frac{dy}{dx} = -\left[\frac{2+3\ln 2}{3+2\ln 2}\right] = -\left[\frac{2+\ln 8}{3+\ln 4}\right]$

Question35

Let $f(x) = \frac{\sin x + \cos x - \sqrt{2}}{\sin x - \cos x}$, $x \in [0 . \pi] - \left\{\frac{\pi}{4}\right\}$. Then $f\left(\frac{7\pi}{12}\right) f''\left(\frac{7\pi}{12}\right)$ is equal to [8-Apr-2023 shift 1]

Options:

A. $\frac{-2}{3}$

B. $\frac{2}{9}$

C. $\frac{-1}{3\sqrt{3}}$

D. $\frac{2}{3\sqrt{3}}$

Answer: B

Solution:

Solution:

$f(x) = -\tan\left(\frac{x}{2} - \frac{\pi}{8}\right)$

$f'(x) = -\frac{1}{2}\sec^2\left(\frac{x}{2} - \frac{\pi}{8}\right)$

$f''(x) = -\sec^2\left(\frac{x}{2} - \frac{\pi}{8}\right) \cdot \tan\left(\frac{x}{2} \frac{-\pi}{8}\right) \cdot \frac{1}{2}$

$f\left(\frac{7\pi}{12}\right) = -\tan\left(\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

$f''\left(\frac{7\pi}{12}\right) = -\frac{1}{2}\sec^2\frac{\pi}{6} \cdot \tan\frac{\pi}{6} = \frac{-1}{2} \cdot \frac{4}{3} \times \frac{1}{\sqrt{3}} = \frac{-2}{3\sqrt{3}}$

$f\left(\frac{7\pi}{12}\right) \cdot f''\left(\frac{7\pi}{12}\right) = \frac{2}{9}$

Question36

Let k and m be positive real numbers such that the function

$f(x) = \begin{cases} 3x^2 + k\sqrt{x+1} & 0 < x < 1 \\ mx^2 + k^2 & x \geq 1 \end{cases}$ is differentiable for all $x > 0$. Then $\frac{8f'(8)}{f'\left(\frac{1}{8}\right)}$ is equal to ________.

[8-Apr-2023 shift 2]

Answer: 309

Solution:

Solution:

function is differentiable $\forall x < 0$

so $f(1^-) = f(1) \ldots (1)$

$3 + \sqrt{2}k = m + k^2$

and $f_+^1(1^-) = f_-^1(1^+)$

$2m = 6 + \frac{k}{2\sqrt{2}}$

$m = 3 + \frac{k}{4\sqrt{2}} \ldots (2)$

$k^2 + 3 + \frac{k}{4\sqrt{2}} = 3 + \sqrt{2}k$

$k = \frac{7}{4\sqrt{2}}, 0$

$m = 3 + \frac{7}{32}$

$m = \frac{103}{32}$

$= \frac{8 \times 2 \times 8 \times \frac{103}{32}}{\frac{16}{12}}$

$= 103 \times 3 = 309$

Question37

$\lim_{x \to \frac{\pi}{2}} \left(\tan^2 x \left((2\sin^2 x + 3\sin x + 4)^{\frac{1}{2}} - (\sin^2 x + 6\sin x + 2)^{\frac{1}{2}} \right) \right)$ **is equal to**

[25-Jun-2022-Shift-2]

Options:

A. $\frac{1}{12}$

B. $-\frac{1}{18}$

C. $-\frac{1}{12}$

D. $\frac{1}{6}$

Answer: A

Solution:

Solution:

$$\lim_{x \to \frac{\pi}{2}} \tan^2 x \left\{ \sqrt{2\sin^2 x + 3\sin x + 4} - \sqrt{\sin^2 x + 6\sin x + 2} \right\}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\tan^2 x(\sin^2 x - 3\sin x + 2)}{\sqrt{2\sin^2 x + 3\sin x + 4} + \sqrt{\sin^2 x + 6\sin x + 2}}$$

$$= \frac{1}{6} \lim_{x \to \frac{\pi}{2}} \frac{(1 - \sin x)(2 - \sin x)}{\cos^2 x} \cdot \sin^2 x$$

$$= \frac{1}{6} \lim_{x \to \frac{\pi}{2}} \frac{(2 - \sin x)\sin^2 x}{1 + \sin x}$$

$$= \frac{1}{12}$$

Question38

$\lim_{x \to \frac{1}{\sqrt{2}}} \frac{\sin(\cos^{-1} x) - x}{1 - \tan(\cos^{-1} x)}$ **is equal to :**

[26-Jun-2022-Shift-1]

Options:

A. $\sqrt{2}$

B. $-\sqrt{2}$

C. $\frac{1}{\sqrt{2}}$

D. $-\frac{1}{\sqrt{2}}$

Answer: D

Solution:

$$\lim_{x \to \frac{1}{\sqrt{2}}} \frac{\sin(\cos^{-1}x) - x}{1 - \tan(\cos^{-1}x)}$$

Let $\cos^{-1}x = t$

$\Rightarrow x = \cos t$

When $x \to \frac{1}{\sqrt{2}}$, then $t \to \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \to \frac{\pi}{4}$

$$\therefore \lim_{t \to \frac{\pi}{4}} \frac{\sin t - \cos t}{1 - \tan(t)}$$

$$= \lim_{t \to \frac{\pi}{4}} \frac{\sin t - \cos t}{1 - \frac{\sin t}{\cos t}}$$

$$= \lim_{t \to \frac{\pi}{4}} \frac{(\sin t - \cos t)(\cos t)}{(\cos t - \sin t)}$$

$$= \lim_{t \to \frac{\pi}{4}} -\cos t$$

$$= -\lim_{t \to \frac{\pi}{4}} \cos t$$

$$= -\frac{1}{\sqrt{2}}$$

Question39

$\lim_{x \to 0} \frac{\cos(\sin x) - \cos x}{x^4}$ **is equal to :**

[26-Jun-2022-Shift-2]

Options:

A. $\frac{1}{3}$

B. $\frac{1}{4}$

C. $\frac{1}{6}$

D. $\frac{1}{12}$

Answer: C

Solution:

$$\lim_{x \to 0} \frac{\cos(\sin x) - \cos x}{x^4} = \lim_{x \to 0} \frac{2\sin(x + \sin x) \cdot \sin\left(\frac{x - \sin x}{2}\right)}{x^4}$$

$$= \lim_{x \to 0} 2 \cdot \left(\frac{\left(\frac{x + \sin x}{2}\right)\left(\frac{x - \sin x}{2}\right)}{x^4} \right)$$

$$= \lim_{x \to 0} \frac{1}{2} \cdot \left(\frac{\left(x + x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots\right)\left(x - x + \frac{x^3}{3!} \ldots\right)}{x^4} \right)$$

$$= \lim_{x \to 0} \frac{1}{2} \cdot \left(2 - \frac{x^2}{3!} + \frac{x^4}{5!} \ldots\right)\left(\frac{1}{3!} - \frac{x^2}{5!} - 1\right)$$

$$= \frac{1}{6}$$

Question40

Let a be an integer such that $\lim_{x \to 7} \frac{18 - [1 - x]}{[x - 3a]}$ exists, where [t] is greatest integer ≤t. Then a is equal to :
[27-Jun-2022-Shift-1]

Options:

A. −6

B. −2

C. 2

D. 6

Answer: A

Solution:

$\lim\limits_{x \to 7} \frac{18 - [1 - x]}{[x - 3a]}$ exist &a $\in$ I .

$= \lim\limits_{x \to 7} \frac{17 - [-x]}{[x] - 3a}$ exist

$\text{RH L} = \lim\limits_{x \to 7^{+}} \frac{17 - [-x]}{[x] - 3a} = \frac{25}{7 - 3a}\left[a \neq \frac{7}{3}\right]$

$\text{LH L} = \lim\limits_{x \to 7^{-}} \frac{17 - [-x]}{[x] - 3a} = \frac{24}{6 - 3a}[a \neq 2]$

For limit to exist

LH L = RH L

$\frac{25}{7 - 3a} = \frac{24}{6 - 3a}$

$\Rightarrow \frac{25}{7 - 3a} = \frac{8}{2 - a}$

$\therefore a = -6$

Question41

Let [t] denote the greatest integer ≤t and {t} denote the fractional part of t. The integral value of α for which the left hand limit of the function $f(x) = [1 + x] + \frac{\alpha^{2[x] + \{x\}} + [x] - 1}{2[x] + \{x\}}$ at x = 0 is equal to $\alpha - \frac{4}{3}$, is____

[27-Jun-2022-Shift-2]

Answer: 3

Solution:

Solution:

$f(x) = [1 + x] + \frac{a^{2[\,]+\{x\}} + [x] - 1}{2[x] + \{x\}}$

$\lim\limits_{x \to 0^{-}} f(x) = \alpha - \frac{4}{3}$

$\Rightarrow \lim\limits_{x \to 0^{-}} 1 + [x] + \frac{\alpha^{x + [x]} + [x] - 1}{x + [x]} = \alpha - \frac{4}{3}$

$\Rightarrow \lim\limits_{h \to 0^{-}} 1 - 1 + \frac{\alpha^{-h - 1} - 1 - 1}{-h - 1} = \alpha - \frac{4}{3}$

$\therefore \frac{\alpha^{-1} - 2}{-1} = \alpha - \frac{4}{3}$

$\Rightarrow 3\alpha^{2} - 10\alpha + 3 = 0$

$\therefore \alpha = 3$ or $\frac{1}{3}$

$\because \alpha$ in integer, hence $\alpha = 3$

Question42

The value of

$$\lim_{n \to \infty} 6 \tan\left\{\sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{r^{2} + 3r + 3}\right)\right\}$$ **is equal to :**

[28-Jun-2022-Shift-2]

Options:

A. 1

B. 2

C. 3

D. 6

Answer: C

Solution:

Solution:

$$\lim_{n\to\infty} 6\tan\left\{\sum_{r=1^{n}\tan}^{-1}\left(\frac{1}{r^2+3r+3}\right)\right\}$$

$$=\lim_{n\to\infty} 6\tan\left\{\sum_{r=1^{n}\tan}^{-1}\left(\frac{(r+2)-(r+1)}{1+(r+2)(r+1)}\right)\right\}$$

$$=\lim_{n\to\infty} 6\tan\left\{\sum_{r=1^{n}(\tan^{-1}(r+2)-\tan}^{-1}(r+1)\right)\right\}$$

$$=\lim_{n\to\infty} 6\tan\{\tan^{-1}(n+2)-\tan^{-1}2\}$$

$$=6\tan\left\{\frac{\pi}{2}-\cot^{-1}\left(\frac{1}{2}\right)\right\}$$

$$=6\tan\left(\tan^{-1}\left(\frac{1}{2}\right)\right)$$

$$=3$$

Question43

If $\lim\limits_{x\to 1}\frac{\sin(3x^2-4x+1)-x^2+1}{2x^3-7x^2+ax+b}=-2$, then the value of $(a-b)$ is equal to____ [28-Jun-2022-Shift-2]

Answer: 11

Solution:

$$\lim_{x\to 1}\frac{\left(\frac{\sin(3x^2-4x+1)}{3x^2-4x+1}\right)(3x^2-4x+1)-x^2+1}{2x^3-7x^2+ax+b}=-2$$

$$\Rightarrow\lim_{x\to 1}\frac{3x^2-4x+1-x^2+1}{2x^3-7x^2+ax+b}=-2$$

$$\Rightarrow\lim_{x\to 1}\frac{2(x-1)^2}{2x^3-7x^2+ax+b}=-2$$

So $f(x)=2x^3-7x^2+ax+b=0$ has $x=1$ as repeated root, therefore $f(1)=0$ and $f'(1)=0$ gives
$a+b+5$ and $a=8$
So, $a-b=11$

Question44

The value of $\lim_{x \to 1} \frac{(x^2 - 1)\sin^2(\pi x)}{x^4 - 2x^3 + 2x - 1}$ is equal to:
[29-Jun-2022-Shift-2]

Options:

A. $\frac{\pi^2}{6}$

B. $\frac{\pi^2}{3}$

C. $\frac{\pi^2}{2}$

D. π^2

Answer: D

Solution:

Solution:

$S = 1 + \frac{5}{6} + \frac{12}{6^2} + \frac{22}{6^3} + \ldots\ldots(1)$

$\frac{1}{6}S = \frac{1}{6} + \frac{5}{6^2} + \frac{12}{6^3} + \ldots\ldots(2)$

$S - \frac{1}{6}S = 1 + \frac{4}{6} + \frac{7}{6^2} + \frac{10}{6^3} + \ldots\ldots\ldots$

$\Rightarrow \frac{5S}{6} = 1 + \frac{4}{6} + \frac{7}{6^2} + \frac{10}{6^3} + \ldots\ldots\ (3)$

Now, multiplying both sides by $\frac{1}{6}$, we get

$\Rightarrow \frac{5S}{36} = \frac{1}{6} + \frac{4}{6^2} + \frac{7}{6^3} + \frac{10}{6^4} + .$

Subtract equation (4) from equation (3), we get

$\frac{25}{36}S = 1 + \frac{3}{6} + \frac{3}{6^2} + \frac{3}{6^3} + \ldots\ldots$

$\Rightarrow \frac{25S}{36} = 1 + \frac{\frac{3}{5}}{1 - \frac{1}{6}}$

$= 1 + \frac{3}{6} \times \frac{6}{5}$

$= 1 + \frac{3}{5} = \frac{8}{5}$

$\Rightarrow S = \frac{8}{5} \times \frac{36}{25} = \frac{288}{125}$

Question45

If $y = \tan^{-1}(\sec x^3 - \tan x^3)$, $\frac{\pi}{2} < x^3 < \frac{3\pi}{2}$, then
[24-Jun-2022-Shift-2]

Options:

A. $xy'' + 2y' = 0$

B. $x^2y'' - 6y + \frac{3\pi}{2} = 0$

C. $x^2y'' - 6y + 3\pi = 0$

D. $xy'' - 4y' = 0$

Answer: B

Solution:

Solution:

Let $x^3 = \theta \Rightarrow \frac{\theta}{2} \in \left(\frac{\pi}{4}, \frac{3\pi}{4} \right)$

$\therefore y = \tan^{-1}(\sec\theta - \tan\theta)$

$= \tan^{-1}\left(\frac{1 - \sin\theta}{\cos\theta} \right)$

$\therefore y = \frac{\pi}{4} - \frac{\theta}{2}$

$y = \frac{\pi}{4} - \frac{x^3}{2}$

$\therefore y' = \frac{-3x^2}{2}$

$y'' = -3x$

$\therefore x^2y'' - 6y + \frac{3\pi}{2} = 0$

Question46

Let f : R → R be defined as $f(x) = x^3 + x - 5$. If g(x) is a function such that $f(g(x)) = x$, $\forall' x' \in R$, then $g'(63)$ is equal to
[25-Jun-2022-Shift-1]

Options:

A. $\frac{1}{49}$

B. $\frac{3}{49}$

C. $\frac{43}{49}$

D. $\frac{91}{49}$

Answer: A

Solution:

Solution:

$f(x) = 3x^2 + 1$

$f'(x)$ is bijective function

and $f(g(x)) = x \Rightarrow g(x)$ is inverse of $f(x)$ $g(f(x)) = x$

$g'(f(x)) \cdot f'(x) = 1$

$g'(f(x)) = \dfrac{1}{3x^2 + 1}$

Put $x = 4$ we get

$g'(63) = \dfrac{1}{49}$

Question47

Let f : R → R satisfy $f(x + y) = 2^x f(y) + 4^y f(x)$, ∀x, y ∈ R. If f(2) = 3, then $14. \frac{f'(4)}{f'(2)}$ is equal to___
[26-Jun-2022-Shift-2]

Answer: 248

Solution:

$\because f(x+y) = 2^x f(y) + 4^y f(x) \ldots\ldots (1)$

Now, $f(y+x) 2^y f(x) + 4^x f(y) \ldots\ldots (2)$

$\therefore 2^x f(y) + 4^y f(x) = 2^y f(x) + 4^x f(y)$

$(4^y - 2^y) f(x) = (4^x - 2^x) f(y)$

$$\frac{f(x)}{4^x - 2^x} = \frac{f(y)}{4^y - 2^y} = k$$

$$\therefore f(x) = k(4^x - 2^x)$$

$$\because f(2) = 3 \text{ then } k = \frac{1}{4}$$

$$\therefore f(x) = \frac{4^x - 2^x}{4}$$

$$\therefore f'(x) = \frac{4^x \ln 4 - 2^x \ln 2}{4}$$

$$f'(x) = \frac{(2.4^x - 2^x)\ln 2}{4}$$

$$\therefore \frac{f'(4)}{f'(2)} = \frac{2.256 - 16}{2.16 - 4}$$

$$\therefore 14\frac{f'(4)}{f'(2)} = 248$$

Question48

If $\cos^{-1}\left(\frac{y}{2}\right) = \log_e\left(\frac{x}{5}\right)^5, \left|y\right| < 2$, then :
[27-Jun-2022-Shift-1]

Options:

A. $x^2y'' + xy' - 25y = 0$

B. $x^2y'' - xy' - 25y = 0$

C. $x^2y'' - xy' + 25y = 0$

D. $x^2y'' + xy' + 25y = 0$

Answer: D

Solution:

Solution:

$\cos^{-1}\left(\frac{y}{2}\right) = \log_e\left(\frac{x}{5}\right)^5 \quad \left|y\right| < 2$

Differentiating on both side

$-\frac{1}{\sqrt{1-\left(\frac{y}{2}\right)^2}} \times \frac{y'}{2} = \frac{5}{\frac{x}{5}} \times \frac{1}{5}$

$\frac{-xy'}{2} = 5\sqrt{1-\left(\frac{y}{2}\right)^2}$

Square on both side

$\frac{x^2y'^2}{4} = 25\left(\frac{4-y^2}{4}\right)$

Diff on both side

$2xy'^2 + 2y'y''x^2 = -25 \times 2yy'$

$xy' + y''x^2 + 25y = 0$

Question49

Let $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, $a \in R$. Then the sum of the squares of all the values of a, for which $2f'(10) - f'(5) + 100 = 0$, is
[27-Jun-2022-Shift-2]

Options:

A. 117

B. 106

C. 125

D. 136

Answer: C

Solution:

Solution:

$$f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}, a \in R$$

$f(x) = a(a^2 + ax) + 1(a^2x + ax^2)$

$= a(x + a)^2$

$f'(x) = 2a(x + a)$

Now, $2f'(10) - f'(5) + 100 = 0$

$\Rightarrow 2 \cdot 2a(10 + a) - 2a(5 + a) + 100 = 0$

$\Rightarrow 2a(a + 15) + 100 = 0$

$\Rightarrow a^2 + 15a + 50 = 0$

$\Rightarrow a = -10, -5$

$\therefore$ Sum of squares of values of a = 125

Question50

If $y(x) = (x^x)^x$, $x > 0$, then $\frac{d^2x}{dy^2} + 20$ at $x = 1$ is equal to____ [27-Jun-2022-Shift-2]

Answer: 16

Solution:

Solution:

$\because y(x) = (x^x)^x$

$\therefore y = x^{x^2}$

$\therefore \frac{dy}{dx} = x^2 \cdot x^{x^2 - 1} + x^{x^2} \ln x \cdot 2x$

$\therefore \frac{dx}{dy} = \frac{1}{x^{x^2+1}(1 + 2\ln x)} \cdots (i)$

Now, $\frac{d^2x}{dx} = \frac{d}{dx}\left(\left(x^{x^2+1}(1 + 2\ln x)\right)^{-1}\right) \cdot \frac{dx}{dy}$

$= \frac{-x\left(x^{x^2+1}(1 + 2\ln x)\right)^{-2} \cdot x^{x^2}(1 + 2\ln x)(x^2 + 2x^2 \ln x + 3)}{x^{x^2}(1 + 2\ln x)}$

$= \frac{-x^{x^2}(1 + 2\ln x)(x^3 + 3 + 2x^2 \ln x)}{\left(x^{x^2}(1 + 2\ln x)\right)^3}$

$\frac{d^2x}{dy^2(atx = 1)} = -4$

$\therefore \frac{d^2x}{dy^2(at x = 1)} + 20 = 16$

Question51

If $\lim_{n \to \infty} \left(\sqrt{n^2 - n - 1} + n\alpha + \beta \right) = 0$, then $8(\alpha + \beta)$ is equal to: [25-Jul-2022-Shift-1]

Options:

A. 4

B. -8

C. -4

D. 8

Answer: C

Solution:

Solution:

$\lim_{n \to \alpha} \left(\sqrt{n^2 - n - 1} + n\alpha + \beta \right) = 0$

[This limit will be zero when $\alpha < 0$ as when $\alpha > 0$ then overall limit will be ∞.]

$\Rightarrow \lim_{n \to \alpha} \frac{\left(\sqrt{n^2 - n - 1} + n\alpha + \beta \right)\left(\sqrt{n^2 - n - 1} - (n\alpha + \beta) \right)}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)} = 0$

$\Rightarrow \lim_{n \to \alpha} \frac{(n^2 - n - 1) - (n\alpha + \beta)^2}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)} = 0$

$\Rightarrow \lim_{n \to \alpha} \frac{n^2 - n - 1 - n^2\alpha^2 - 2n\alpha\beta - \beta^2}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)} = 0$

$\Rightarrow \lim_{n \to \alpha} \frac{n^2(1 - \alpha^2) - n(1 + 2\alpha\beta) - (1 + \beta^2)}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)}$

Here power of " n " in the numerator is 2 and power of " n " in the denominator is 1.

To get the value of limit equal to zero power of " n " should be equal in both numerator and denominator, otherwise value of limit will be infinite (∞).

$\therefore$ Coefficient of n^2 should be 0 in this case.

$\therefore 1 - \alpha^2 = 0$

$\Rightarrow \alpha = \pm 1$

But α should be < 0

$\therefore \alpha = +1$ not possible

$\therefore \alpha = -1$

$\Rightarrow \lim_{n \to \alpha} \frac{0 - n(1 + 2\alpha\beta) - (1 + \beta)}{n\left[\sqrt{1 - \frac{1}{n} - \frac{1}{n^2}} - \alpha - \frac{\beta}{n} \right]} = 0$

Divide numerator and denominator by n then we get,

$\Rightarrow \lim_{n \to \alpha} \frac{-(1 + 2\alpha\beta) - \frac{(1 + \beta)}{n}}{\sqrt{1 - \frac{1}{n} - \frac{1}{n^2}} - \alpha - \frac{\beta}{n}} = 0$

$\Rightarrow \frac{-(1 + 2\alpha\beta) - 0}{\sqrt{1 - 0 - 0} - \alpha - 0} = 0$

$\Rightarrow \frac{-(1 + 2\alpha\beta)}{1 - \alpha} = 0$

$\Rightarrow -(1 + 2\alpha\beta) = 0$

$\Rightarrow 1 + 2\alpha\beta = 0$

$\Rightarrow 2\alpha\beta = -1$

$\Rightarrow \beta = -\frac{1}{2\alpha} = -\frac{1}{2(-1)} = \frac{1}{2}$

$\therefore 8(\alpha + \beta)$

$= 8\left(-1 + \frac{1}{2}\right)$

$= 8 \times -\frac{1}{2}$

$= -4$

Question52

If $\lim_{n \to \infty}\left(\sqrt{n^2 - n - 1} + n\alpha + \beta\right) = 0$, then $8(\alpha + \beta)$ is equal to: [25-Jul-2022-Shift-1]

Options:

A. 4

B. −8

C. −4

D. 8

Answer: C

Solution:

Solution:

$\lim_{n \to \alpha}\left(\sqrt{n^2 - n - 1} + n\alpha + \beta\right) = 0$

[This limit will be zero when $\alpha < 0$ as when $\alpha > 0$ then overall limit will be ∞.]

$\Rightarrow \lim_{n \to \alpha} \frac{\left(\sqrt{n^2 - n - 1} + n\alpha + \beta\right)\left(\sqrt{n^2 - n - 1} - (n\alpha + \beta)\right)}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)} = 0$

$\Rightarrow \lim_{n \to \alpha} \frac{(n^2 - n - 1) - (n\alpha + \beta)^2}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)} = 0$

$\Rightarrow \lim_{n \to \alpha} \frac{n^2 - n - 1 - n^2\alpha^2 - 2n\alpha\beta - \beta^2}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)} = 0$

$\Rightarrow \lim_{n \to \alpha} \frac{n^2(1 - \alpha^2) - n(1 + 2\alpha\beta) - (1 + \beta^2)}{\sqrt{n^2 - n - 1} - (n\alpha + \beta)}$

Here power of " n " in the numerator is 2 and power of " n " in the denominator is 1.
To get the value of limit equal to zero power of " n " should be equal in both numerator and denominator, otherwise value of limit will be infinite (∞).

$\therefore$ Coefficient of n^2 should be 0 in this case.

$\therefore 1 - \alpha^2 = 0$

$\Rightarrow \alpha = \pm 1$

But α should be <0

$\therefore \alpha = +1$ not possible

$\therefore \alpha = -1$

$\Rightarrow \lim_{n \to \alpha} \frac{0 - n(1 + 2\alpha\beta) - (1 + \beta)}{n\left[\sqrt{1 - \frac{1}{n} - \frac{1}{n^2}} - \alpha - \frac{\beta}{n}\right]} = 0$

Divide numerator and denominator by n then we get,

$\Rightarrow \lim_{n \to \alpha} \frac{-(1 + 2\alpha\beta) - \frac{(1 + \beta)}{n}}{\sqrt{1 - \frac{1}{n} - \frac{1}{n^2}} - \alpha - \frac{\beta}{n}} = 0$

$\Rightarrow \frac{-(1 + 2\alpha\beta) - 0}{\sqrt{1 - 0 - 0} - \alpha - 0} = 0$

$\Rightarrow \frac{-(1+2\alpha\beta)}{1-\alpha} = 0$

$\Rightarrow -(1+2\alpha\beta) = 0$
$\Rightarrow 1+2\alpha\beta = 0$
$\Rightarrow 2\alpha\beta = -1$
$\Rightarrow \beta = -\frac{1}{2\alpha} = -\frac{1}{2(-1)} = \frac{1}{2}$
$\therefore 8(\alpha+\beta)$
$= 8\left(-1+\frac{1}{2}\right)$
$= 8 \times -\frac{1}{2}$
$= -4$

Question53

If $\lim_{n \to \infty} \frac{(n+1)^{k-1}}{n^{k+1}}[(nk+1)+(nk+2)+...+(nk+n)]$, then the integral value of k is $= 33 \cdot \lim_{n \to \infty} \frac{1}{n^{k+1}} \cdot [1^k + 2^k + 3^k + ... + n^k]$ equal to
[25-Jul-2022-Shift-1]

Answer: 5

Solution:

Solution:
$\lim_{n \to \infty}\left(\frac{n+1}{n}\right)^{k-1}\frac{1}{n}\sum_{r=1}^{n}\left(k+\frac{r}{n}\right) = 33\lim_{n \to \infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{r}{n}\right)^k$
$\Rightarrow \int_0^1 (k+x)dx = 33\int_0^1 x^k dx$
$\Rightarrow \frac{2k+1}{2} = \frac{33}{k+1}$
$\Rightarrow k = 5$

Question54

$\lim_{x \to \frac{\pi}{4}} \frac{8\sqrt{2}-(\cos x+\sin x)^7}{\sqrt{2}-\sqrt{2}\sin 2x}$ **is equal to**
[25-Jul-2022-Shift-2]

Options:

A. 14

B. 7

C. $14\sqrt{2}$

D. $7\sqrt{2}$

Answer: A

Solution:

Solution:

$$\lim_{x \to \frac{\pi}{4}} \frac{8\sqrt{2} - (\cos x + \sin x)^7}{\sqrt{2} - \sqrt{2}\sin 2x} \left(\frac{0}{0}. \text{ form } \right)$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{-7(\cos x + \sin x)^6(-\sin x + \cos x)}{-2\sqrt{2}\cos 2x} \text{ using L} - \text{H Rule}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{56(\cos x - \sin x)}{2\sqrt{2}\cos 2x} \left(\frac{0}{0} \right)$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{-56(\sin x + \cos x)}{-4\sqrt{2}\sin 2x} \text{ using L - H Rule}$$

$$= 7\sqrt{2} \cdot \sqrt{2} = 14$$

Question55

If the function

$$\mathbf{f(x) =} \begin{cases} \dfrac{\log_e(1 - x + x^2) + \log_e(1 + x + x^2)}{\sec x - \cos x} & , \ x \in \left(\dfrac{-\pi}{2}, \ \dfrac{\pi}{2} \right) - \{0\} \\ k & , \ x = 0. \end{cases}$$

is continuous at x = 0 , then k is equal to: [26-Jul-2022-Shift-1]

Options:

A. 1

B. −1

C. e

D. 0

Answer: A

Solution:

$$f(x) = \begin{cases} \dfrac{\log_e(1-x+x^2)+\log_e(1+x+x^2)}{\sec x - \cos x} & x \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right) - \{0\} \\ k & x = 0. \end{cases}$$

for continuity at $x = 0$

$$\lim_{x \to 0} f(x) = k$$

$$\therefore k = \lim_{x \to 0} \frac{\log_e(x^4+x^2+1)}{\sec x - \cos x}\left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\cos x \log_e(x^4+x^2+1)}{\sin^2 x}$$

$$= \lim_{x \to 0} \frac{\log_e(x^4+x^2+1)}{x^2}$$

$$= \lim_{x \to 0} \frac{\ln(1+x^2+x^4)}{x^2+x^4} \cdot \frac{x^2+x^4}{x^2}$$

$$= 1$$

Question56

Let $\beta = \lim_{x \to 0} \frac{\alpha x - (e^{3x} - 1)}{\alpha x (e^{3x} - 1)}$ for some $\alpha \in \mathbb{R}$. Then the value of $\alpha + \beta$ is : [26-Jul-2022-Shift-2]

Options:

A. $\frac{14}{5}$

B. $\frac{3}{2}$

C. $\frac{5}{2}$

D. $\frac{7}{2}$

Answer: C

Solution:

Solution:

$\beta = \lim_{x \to 0} \frac{\alpha x - (e^{3x} - 1)}{\alpha x(e^{3x} - 1)}, \alpha \in R$

$= \lim_{x \to 0} \frac{\frac{\alpha}{3} - \left(\frac{e^{3x} - 1}{3x}\right)}{\alpha x\left(\frac{e^{3x-1}}{3x}\right)}$

So, α = 3 (to make independent form)

$\beta = \lim_{x \to 0} \frac{1 - \left(\frac{e^{3x} - 1}{3x}\right)}{3x} = \frac{1 - \frac{\left(3x + \frac{9x^2}{2} + \ldots..\right)}{3x}}{3x}$

$= \frac{-\left(\frac{9}{2}x^2 + \frac{(3x)^3}{31} + \ldots..\right)}{9x^2} = \frac{-1}{2}$

$\therefore \alpha + \beta = 3 - \frac{1}{2} = \frac{5}{2}$

Question57

$\lim_{x \to 0} \left(\frac{(x + 2\cos x)^3 + 2(x + 2\cos x)^2 + 3\sin(x + 2\cos x)}{(x + 2)^3 + 2(x + 2)^2 + 3\sin(x + 2)} \right)^{\frac{100}{x}}$ is equal to _______.
[28-Jul-2022-Shift-1]

Answer: 1

Solution:

Let $x + 2\cos x = a$
$x + 2 = b$
as $x \to 0$, $a \to 2$ and $b \to 2$

$\lim_{x \to 0} \left(\frac{a^3 + 2a^2 + 3\ \text{sina}}{b^3 + 2b^2 + 3\ \text{sinb}} \right)^{\frac{100}{x}}$

$= e^{\lim_{x \to 0} \cdot \frac{100}{x} \cdot \frac{(a^3 - b^3) + 2(a^2 - b^2) + 3(\ \text{sina} - \ \text{sinb})}{b^3 + 2b^2 + 3\ \text{sinb}}}$

$\because \lim_{x \to 0} \frac{a - b}{x} = \lim_{x \to 0} \frac{2(\cos x - 1)}{x} = 0$

$= e^0$

$= 1$

Question58

If $\lim_{x \to 0} \frac{e^x + \beta e^{-x} + \gamma \sin x}{x\sin^2 x} = \frac{2}{3}$, where α, β, γ ∈ R, then which of the following is NOT correct?
[29-Jul-2022-Shift-1]

Options:

A. $\alpha^2+\beta^2+\gamma^2=6$

B. $\alpha\beta+\beta\gamma+\gamma\alpha+1=0$

C. $\alpha\beta^2+\beta\gamma^2+\gamma\alpha^2+3=0$

D. $\alpha^2-\beta^2+\gamma^2=4$

Answer: C

Solution:

Solution:

$\lim\limits_{x\to 0}\frac{\alpha e^x+\beta e^{-x}+\gamma\sin x}{x\sin^2 x}=\frac{2}{3}$

$\Rightarrow\alpha+\beta=0$ (to make indeterminant form) ...... (i)

Now,

$\lim\limits_{x\to 0}\frac{\alpha e^x-\beta e^{-x}+\gamma\cos x}{3x^2}=\frac{2}{3}$ (Using L-H Rule)

$\Rightarrow\alpha-\beta+\gamma=0$ (to make indeterminant form) ...... (ii)

Now,

$\lim\limits_{x\to 0}\frac{\alpha e^x+\beta e^{-x}-\gamma\sin x}{6x}=\frac{2}{3}$ (Using L-H Rule)

$\Rightarrow\frac{\alpha-\beta+\gamma}{6}=\frac{2}{3}$

$\Rightarrow\alpha-\beta+\gamma=4$...... (iii)

$\Rightarrow\gamma=-2$

and (i) + (ii)

$2\alpha=-\gamma$

$\Rightarrow\alpha=1$ and $\beta=-1$

and $\alpha\beta^2+\beta\gamma^2+\gamma\alpha^2+3=1-4-2+3=-2$

Question59

The value of $\log_e 2\,\frac{d}{dx}(\log_{\cos x}\operatorname{cosec} x)$ at $x=\frac{\pi}{4}$ is [26-Jul-2022-Shift-2]

Options:

A. $-2\sqrt{2}$

B. $2\sqrt{2}$

C. -4

D. 4

Answer: D

Solution:

Solution:

Let $f(x)=\log_{\cos x}\operatorname{cosec} x$

$=\frac{\log\operatorname{cosec} x}{\log\cos x}$

$$\Rightarrow f'(x) = \frac{\log\cos x \cdot \sin x \cdot \left(-\operatorname{cosec} x \cot x - \log\operatorname{cosec} x \cdot \frac{1}{\cos x} \cdot -\sin x\right)}{(\log\cos x)^2}$$

at $x = \frac{\pi}{4}$

$$f'\left(\frac{\pi}{4}\right) = \frac{-\log\left(\frac{1}{\sqrt{2}}\right) + \log\sqrt{2}}{\left(\log\frac{1}{\sqrt{2}}\right)^2} = \frac{2}{\log\sqrt{2}}$$

$\therefore \log_e 2 f'(x)$ at $x = \frac{\pi}{4} = 4$

Question60

Let $x(t) = 2\sqrt{2}\cos t\sqrt{\sin 2t}$ and

$y(t) = 2\sqrt{2}\ \sin t\sqrt{\sin 2t}$, $t \in \left(0, \frac{\pi}{2}\right)$

Then $\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$ at $t = \frac{\pi}{4}$ is equal to:

[28-Jul-2022-Shift-2]

Options:

A. $\frac{-2\sqrt{2}}{3}$

B. $\frac{2}{3}$

C. $\frac{1}{3}$

D. $\frac{-2}{3}$

Answer: D

Solution:

Solution:

$x = 2\sqrt{2}\cos t\sqrt{\sin 2t}$, $y = 2\sqrt{2}\ \sin t\sqrt{\sin 2t}$

$\therefore \frac{dx}{dt} = \frac{2\sqrt{2}\cos 3t}{\sqrt{\sin 2t}}$, $\frac{dy}{dt} = \frac{2\sqrt{2}\ \sin 3t}{\sqrt{\sin 2t}}$

$\therefore \frac{dy}{dx} = \tan 3t$, $\left(\text{at } t = \frac{\pi}{4}, \frac{dy}{dx} = -1\right)$

and $\frac{d^2y}{dx^2} = 3\sec^2 3t \cdot \frac{dt}{dx} = \frac{3\sec^2 3t \cdot \sqrt{\sin 2t}}{2\sqrt{2}\cos 3t}$

$\left(. \text{ At } .t = \frac{\pi}{4}, \frac{d^2y}{dx^2} = -3\right)$

$$\therefore \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = \frac{2}{-3} = \frac{-2}{3}$$

Question61

For the curve C : $(x^2+y^2-3)+(x^2-y^2-1)^5=0$, the value of $3y' - y^3y''$, at the point (α, α), $\alpha > 0$, on C, is equal to ________.
[27-Jul-2022-Shift-2]

Answer: 16

Solution:

Solution:
$\because C : (x^2+y^2-3)+(x^2-y^2-1)^5=0$ for point (α, α)
$\alpha^2+\alpha^2-3+(\alpha^2-\alpha^2-1)^5=0$
$\therefore \alpha=\sqrt{2}$
On differentiating $(x^2+y^2-3)+(x^2-y^2-1)^5=0$ we get
$x+yy'+5(x^2-y^2-1)^4(x-yy')=0$
When $x=y=\sqrt{2}$ then $y'=\frac{3}{2}$
Again on differentiating eq. (i) we get :
$1+(y')^2+yy''+20(x^2-y^2-1)(2x-2yy')(x-y'y)+5(x^2-y^2-1)^4(1-y'^2-yy'')=0$
For $x=y=\sqrt{2}$ and $y'=\frac{3}{2}$ we get $y''=-\frac{23}{4\sqrt{2}}$
$\therefore 3y'-y^3y''=3.\frac{3}{2}-(\sqrt{2})^3\cdot\left(-\frac{23}{4\sqrt{2}}\right)=16$

Question62

If $\lim\limits_{x\to 0}\frac{ax-(e^{4x}-1)}{ax(e^{4x}-1)}$ exists and is equal to b, then the value of $a-2b$ is ....
[2021, 25 Feb. Shift-11]

Answer: 5

Solution:

$\lim\limits_{x\to 0}\frac{ax-(e^{4x}-1)}{ax(e^{4x}-1)}=L(\text{ say })\left[\frac{0}{0}\text{ form }\right]$
Apply L - Hospital rule,
$L=\lim\limits_{x\to 0}\frac{a-4e^{4x}}{a(e^{4x}-1)+ax(4e^{4x})}$
[Limit exist everywhere except $a=4$]
Again, apply L-Hospital rule,
$L=\lim\limits_{x\to 0}\frac{-16e^{4x}}{a(4e^{4x})+a(4e^{4x})+ax(16e^{4x})}$
$=\frac{-16}{4a+4a}=\frac{-2}{a}$
$=\frac{-2}{4}=\frac{-1}{2}\quad(\text{use } a=4)$
Given, $L=b$
$\Rightarrow \frac{-2}{a}=\frac{-1}{2}=b$

Then, $a-2b=4-2\left(\frac{-1}{2}\right)=4+1=5$

Question63

$\lim_{n\to\infty} 1+\left(\frac{1+\frac{1}{2}+\ldots\ldots+\frac{1}{n}}{n^2}\right)^n$ **is equal to**

[2021, 25 Feb. Shift-1]

Options:

A. $\frac{1}{2}$

B. 0

C. $\frac{1}{e}$

D. 1

Answer: D

Solution:

Solution:

Given, limit form is 1^{∞}.

$L=e^{\lim_{n\to\infty}\left(\frac{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}}{n}\right)}$

$S=1+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)+\ldots$

Clearly,

$S<1+\left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)+\ldots+$

$\left(\left(\frac{1}{2^n}+\ldots+\frac{1}{2^n}\right)_{2^n \text{ times}}\right.$

$S<1+1+1+1+\ldots+1$

$S<n+1$

$\therefore\ L=e^{n\to\infty}\left(\frac{n+1}{2^{n+1}-1}\right)\Rightarrow L=e^0$

$\therefore\ L=1$

Question64

$\lim_{x\to 0}\frac{\int_0 (\sin\sqrt{t})dt}{x^3}$ **is equal to**

[2021, 24 Feb. Shift-1]

Options:

A. $\frac{2}{3}$

B. $\frac{3}{2}$

C. $\frac{1}{15}$

D. 0

Answer: A

Solution:

Solution:

Given, $\lim\limits_{x \to 0} \frac{\int_0^{x^2} (\sin \sqrt{t}) d\,t}{x^3}$

$\because$ It is of the form $\frac{0}{0}$.

By differentiating numerator and denominator,

$\lim\limits_{x \to 0} \frac{\sin \sqrt{x^2} \cdot 2x}{3x^2} = \lim\limits_{x \to 0} \frac{\sin x \cdot 2x}{3x^2}$

$= \frac{2}{3} \lim\limits_{x \to 0} \frac{\sin x}{x} = \frac{2}{3}(1) = \frac{2}{3}$

Question65

$\lim\limits_{n \to \infty} \tan \left\{ \sum\limits_{r=1}^{n} \tan^{-1}\left(\frac{1}{1+r+r^2} \right) \right\}$ is equal to

[2021, 24 Feb. Shift-I]

Answer: 1

Solution:

Solution:

Given, $\lim\limits_{n \to \infty} \tan \left\{ \sum\limits_{r=1}^{n} \tan^{-1}\left(\frac{1}{1+r+r^2} \right) \right\}$

$= \tan \left(\lim\limits_{n \to \infty} \sum\limits_{r=1}^{n} [\tan^{-1}(r+1) - \tan^{-1} r] \right)$

$= \tan \left(\lim\limits_{n \to \infty} \left(\tan^{-1}(n+1) - \frac{\pi}{4} \right) \right)$

$= \tan \left(\frac{\pi}{2} - \frac{\pi}{4} \right) = \tan \frac{\pi}{4} = 1$

Hence, the required value is 1.

Question66

The value of $\lim\limits_{h \to 0} 2 \left\{ \dfrac{\sqrt{3}\sin\left(\frac{\pi}{6}+h\right) - \cos\left(\frac{\pi}{6}+h\right)}{\sqrt{3}h(\sqrt{3}\cos h - \sin h)} \right\}$

[2021, 26 Feb. Shift-1]

Options:

A. $\frac{4}{3}$

B. $\frac{2}{\sqrt{3}}$

C. $\frac{3}{4}$

D. $\frac{2}{3}$

Answer: A

Solution:

Solution:

$$\lim_{h \to 0} 2 \left\{ \frac{\sqrt{3}\sin\left(\frac{\pi}{6}+h\right) - \cos\left(\frac{\pi}{6}+h\right)]}{\sqrt{3}h(\sqrt{3}\cosh - \sin h)} \right\}$$

$$= \lim_{h \to 0} 2 \left\{ \frac{2\left(\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{6}+h\right) - \frac{1}{2}\cos\left(\frac{\pi}{6}+h\right)\right)]}{2 \times \sqrt{3}h\left(\frac{\sqrt{3}}{2}\cosh - \frac{1}{2}\sin h\right)} \right\}$$

$$= \lim_{h \to 0} 2 \left\{ \frac{\cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}+h\right)}{\sqrt{3}h\left(\cos\frac{\pi}{6}\cosh - \sin\frac{\pi}{6}\sin h\right)} \right]$$

$$\left\{ \frac{\cos\left(\frac{\pi}{6}+h\right)}{6} \right\}$$

$$= \lim_{h \to 0} 2 \left\{ \frac{\sin\left(\frac{\pi}{6}+h-\frac{\pi}{6}\right)}{\sqrt{3}h\cos\left(h+\frac{\pi}{6}\right)} \right\}$$

$$= \lim_{h \to 0} \frac{2}{\sqrt{3}} \left\{ \frac{\sinh}{h\cos(h+\pi/6)} \right\}$$

$$= \frac{2}{\sqrt{3}} \cdot \lim_{h \to 0} \frac{\sinh}{h} \cdot \lim_{h \to 0} \frac{1}{\cos(h+\pi/6)}$$

$$= \frac{2}{\sqrt{3}} \cdot (1) \cdot \frac{1}{\cos(\pi/6)}$$

$$= \frac{2}{\sqrt{3}} \cdot 1 \cdot \frac{2}{\sqrt{3}} = \frac{4}{3}$$

Question67

The value of $\lim\limits_{n \to \infty} \dfrac{[r]+[2r]+\ldots\ldots+[nr]}{n^2}$**, where r is non-zero real number and [r] denotes the greatest integer less than or equal to r, is equal to**
[2021, 17 March Shift-11]

Options:

A. $\frac{r}{2}$

B. r

C. 2r

D. 0

Answer: A

Solution:

Solution:
As, we know that,
$r \le [r] < r + 1$
$2r \le [2r] < 2r + 1$
$3r \le [3r] < 3r + 1$
$\vdots \quad \vdots \quad \vdots$
$nr \le [nr] < nr + 1$
Adding $(r + 2r + 3r + 4r + \ldots + nr) \le [r] + [2r] + [3r]$
$+ [4r] + \ldots [nr] < (r + 1) + (2r + 1) + (3r + 1)$
$+ (4r + 1) + \ldots + (nr + 1)$
$\Rightarrow r(1 + 2 + 3 + 4 + \ldots + n) \le [r] + [2r]$
$+ [3r] + \ldots + [nr]$
$< (r + 2r + 3r + \ldots + nr) + (1 + 1 + 1 + \ldots + 1)_{n-\ \text{times}}$
$\Rightarrow r \cdot \frac{n(n+1)}{2} \le [r] + [2r] + [3r] + \ldots + [nr]$
$< \frac{r \cdot (n(n+1))}{2} + n$
$\Rightarrow \frac{r \cdot \left(\frac{n(n+1)}{2}\right)}{n^2}$
$\le \frac{[r] + [2r] + [3r] + \ldots + [nr]}{n^2}$

Now, $\lim_{n \to \infty} \frac{n(n+1) \cdot r}{2 \cdot n^2} = \lim_{n \to \infty} \frac{n \cdot n\left(1 + \frac{1}{n}\right) \cdot r}{2n^2}$

and $\lim_{n \to \infty} \frac{\frac{(1+0) \cdot r}{2} = \frac{r}{2}}{2}$

$= \lim_{n \to \infty} \frac{n\left(n + \frac{n(n+1)}{2}\right) + n}{n^2}$

$= \lim_{n \to \infty} \frac{n^2}{2n^2\left\{\left(1 + \frac{1}{n}\right) \cdot r + 2n + \frac{2}{n}\right\}} \cdot \ldots \Big(\text{ i)}$

$= \frac{(1+0) \cdot r + 0}{2} = \frac{r}{2} \ldots \text{(ii)}$

From Eqs. (i) and (ii), by Sandwich theorem, we conclude that, $\lim_{n \to \infty} \frac{[r] + [2r] + [3r] + \ldots + [nr]}{n^2} = \frac{r}{2}$

Sandwich Theorem
$\Rightarrow$ Let $g(x) \le f(x) \le h(x)$
and $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = 1$
$\therefore \lim_{x \to a} f(x) = 1$

Question68

If $\lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2$, then a + b + c is equal to ....
[2021, 16 March Shift-1]

Answer: 4

Solution:

Solution:

We have, $\lim_{x \to 0} \frac{ae^x - b\cos x + ce^{-x}}{x\sin x} = 2$

$\Rightarrow a\left(1 + x + \frac{x^2}{2!}\cdots\right) - b\left(1 - \frac{x^2}{2} + \frac{x^4}{4!}\cdots\right)$

$\lim_{x \to 0} \frac{+c\left(1 - x + \frac{x^2}{2!}\cdots\right)}{x\left(x - \frac{x^3}{3!} + \ldots\right)}$

$(a - b + c) + (a - c)x$

$\Rightarrow \lim_{x \to 0} \frac{+\left(\frac{a}{2} + \frac{b}{2} + \frac{c}{2}\right)x^2 + \ldots}{x^2 - \frac{x^4}{6} + \ldots}$

Here, in numerator, all the coefficients of x^k, where $k < 2$ has to be zero, then only limit will exist.

$a - b + c = 0$

$a - c = 0$

$\Rightarrow a = c$

$\Rightarrow b = 2a$

After solving limit,

$\frac{a + b + c}{2} = 2$

So, $a + 2a + a = 4$

$\Rightarrow a = 1$

$\therefore a = 1, b = 2$ and $c = 1$

$a + b + c = 1 + 2 + 1 = 4$

Question69

If $\lim_{x \to 0} \frac{\sin^{-1}x - \tan^{-1}x}{3x^3}$ is equal to L, then the value of (6L + 1) is [2021, 18 March Shift-1]

Options:

A. $\frac{1}{6}$

B. $\frac{1}{2}$

C. 6

D. 2

Answer: D

Solution:

Solution:

Given, $L = \lim_{x \to 0} \frac{\sin^{-1}x - \tan^{-1}x}{3x^3}$

$\Rightarrow L$

$$\left(x+\frac{1^2\cdot x^3}{3!}+\frac{1^2\cdot 3^2\cdot x^5}{5!}+\ldots\right)$$

$$=\lim_{x\to 0}\frac{-\left(x-\frac{x^3}{3}+\frac{x^5}{5}\cdots\right)}{3x^3}\Bigg)$$

$\Rightarrow$(using expansion of $\sin^{-1}x$ and $\tan^{-1}x$)

$$=\lim_{x\to 0}\frac{\left(\frac{x^3}{3!}+\frac{9x^5}{5!}+\ldots\right)-\left(-\frac{x^3}{3}+\frac{x^5}{5}\cdots\right)}{3x^3}\Bigg)$$

$\Rightarrow L$

$$=\lim_{x\to 0}\frac{x^3\left[\left(\frac{1}{6}+\frac{9x^2}{120}+\ldots\right)+\left(\frac{1}{3}-\frac{x^2}{5}+\ldots\right)\right]}{3x^3}\Bigg)$$

$$\Rightarrow L=\frac{\frac{1}{6}+\frac{1}{3}}{3}=\frac{\frac{1+2}{6}}{3}=\frac{1}{6}$$

$$\therefore 6L+1=6\times\frac{1}{6}+1=2$$

Question70

The value of $\lim_{x\to 0^+}\frac{\cos^{-1}(x-[x]^2)\cdot\sin^{-1}(x-[x]^2)}{x-x^3}$ where [x] denotes the greatest integer ≤x is
[2021, 17 March Shift-1]

Options:

A. π

B. 0

C. $\frac{\pi}{4}$

D. $\frac{\pi}{2}$

Answer: D

Solution:

Solution:

$$\lim_{x\to 0^+}\frac{\cos^{-1}(x-[x]^2)\cdot\sin^{-1}(x-[x]^2)}{x-x^3}$$

$x\to 0+h$

$$=\lim_{h\to 0}\frac{\cos^{-1}(h-0)\cdot\sin^{-1}(h-0)}{h-h^3}$$

$$=\lim_{h\to 0}\frac{\cos^{-1}h\cdot\sin^{-1}h}{h(1-h)(1+h)}$$

$$=\lim_{h\to 0}\left(\frac{\sin^{-1}h}{h}\right)\left[\frac{\cos^{-1}h}{(1-h)(1+h)}\right]=1\cdot\frac{\pi}{2}$$

$$\text{RHL}=\frac{\pi}{2}$$

Question71

The value of the limit $\lim_{\theta \to 0} \frac{\tan(\pi\cos^2\theta)}{\sin(2\pi\sin^2\theta)}$ is equal t'o
[2021, 17 March Shift-II]

Options:

A. $-\frac{1}{2}$

B. $-\frac{1}{4}$

C. 0

D. $\frac{1}{4}$

Answer: A

Solution:

Solution:
Method (I)

$$\text{Let } L = \lim_{\theta \to 0} \left(\frac{\tan(\pi\cos^2\theta)}{\sin(2\pi\sin^2\theta)} \right)$$

$$= \lim_{\theta \to 0} \left(\frac{\tan[\pi(1-\sin^2\theta)]}{\sin(2\pi\sin^2\theta)} \right)$$

$$= \lim_{\theta \to 0} \left(\frac{\tan(\pi - \pi\sin^2\theta)}{\sin(2\pi\sin^2\theta)} \right)$$

$$= \lim_{\theta \to 0} \left(\frac{-\tan(\pi\sin^2\theta)}{\sin(2\pi\sin^2\theta)} \right)$$

$$= \lim_{\theta \to 0} \left[\frac{\frac{-\tan(\pi\sin^2\theta)}{(\pi\sin^2\theta)} \times (\pi\sin^2\theta)}{\frac{\sin(2\pi\sin^2\theta)}{(2\pi\sin^2\theta)} \times (2\pi\sin^2\theta)} \right] = \frac{-1}{2}$$

Method (II)

$$\text{Let } L = \lim_{\theta \to 0} \left[\frac{\tan(\pi\cos^2\theta)}{\sin(2\pi\sin^2\theta)} \right] \left(\text{F orm } \frac{0}{0} \right)$$

[Using L-Hospital Rule]

$$L = \lim_{\theta \to 0} \frac{\sec^2(\pi\cos^2\theta)(-2\pi\cos\theta \cdot \sin\theta)}{\cos(2\pi\sin^2\theta) \cdot (4\pi\sin\theta \cdot \cos\theta)}$$

$$= \frac{-1}{2} \times \frac{(-1)^2}{1} = \frac{-1}{2}$$

Question72

If $f(x) = \sin\left(\cos^{-1}\left(\frac{1-2^{2x}}{1+2^{2x}}\right)\right)$ and its first derivative with respect to x is $-\frac{b}{a}\log_e 2$ when x = 1, where a and b are integers, then the minimum value of $a^2 - b^2|$ is ....
[2021, 17 March Shift-1]

Answer: 481

Solution:

Solution:

$f(x) = \sin\left(\cos^{-1}\left(\frac{1-2^{2x}}{1+2^{2x}}\right)\right)$

Let 2^{2x} be ${}^{2}\tan^{2}\theta$.

$\therefore f(x) = \sin\left[\cos^{-1}\left(\frac{1-\tan^{2}\theta}{1+\tan^{2}\theta}\right)\right]$

$= \sin[\cos^{-1}(\cos 2\theta)] = \sin 2\theta$

$= \frac{2\tan\theta}{1+\tan^{2}\theta} = \frac{2\cdot 2^{x}}{1+2^{2x}}$

$f(x) = 2\cdot\left(\frac{2^{x}}{1+2^{2x}}\right)$

$f'(x) = 2\cdot\left[\frac{(1+2^{2x})(2^{x}\log 2) - 2^{x}(2^{2x}\log 2\cdot 2)}{(1+2^{2x})^{2}}\right]$

$f'(1) = 2\left(\frac{5\cdot 2\log 2 - 2\cdot 8\log 2}{5^{2}}\right)$

$= \left(-\frac{12}{25}\right)\log 2$

$= \frac{-b}{a}\log e\,2$

$\Rightarrow a = 25$ and $b = 12$

$\therefore |a^{2} - b^{2}|_{\min} = |25^{2} - 12^{2}| = 481$

Question73

Let $f : R \to R$ be a function such that $f(2) = 4$ and $f'(2) = 1$. Then, the value of $\lim_{x\to 2}\frac{x^{2}f(2) - 4f(x)}{x-2}$ is equal to [2021, 27 July Shift-1]

Options:

A. 4

B. 8

C. 16

D. 12

Answer: D

Solution:

Solution:

$f(2) = 4,\ f'(2) = 1$

Now, $\lim_{x\to 2}\frac{x^{2}f(2) - 4f(x)}{x-2}$

Applying L-Hospital Rule as $\frac{0}{0}$ form on putting $x = 2$

So, $\lim_{x\to 2}\frac{2xf(2) - 4f'(x)}{1}$

$= 2\cdot 2\cdot f(2) - 4f'(2)$

$= 4\cdot 4 - 4\cdot 1 = 12$

Question74

If $\lim_{x \to 0} \frac{\alpha x e^{x} - \beta \log_{e}(1 + x) + \gamma x^{2} e^{-x}}{x \sin^{2} x}$ = 10, α, β, γ ∈ R, then the value of α + β + γ is .... [2021, 20 July Shift-2]

Answer: 3

Solution:

Solution:

$\lim_{x \to 0} \frac{\alpha x e^{x} - \beta \log_{e}(1 + x) + \gamma x^{2} e^{-x}}{x \sin^{2} x} = 10$

Now, $\lim_{x \to 0} \frac{\alpha x e^{x} - \beta \log_{e}(1 + x) + \gamma x^{2} e^{-x}}{x \sin^{2} x}$

$\alpha x(1 + x + x^{2} / 2 + \ldots) - \beta$

$= \lim_{x \to 0} \frac{\left(x - \frac{x^{2}}{2} + \frac{x^{3}}{3} \ldots \right) - \gamma x^{2}(1 - x + x^{2} / 2 \ldots)}{x \sin^{2} x}$

$x(\alpha + \beta) + x^{2}(\alpha + \beta / 2 + \gamma) + x^{3}$

$= \lim_{x \to 0} \frac{\left(\frac{\alpha}{2} - \frac{\beta}{3} - \gamma \right)^{x} \ldots}{x \sin^{2} x}$

For limit to exist, the numerator must have degree greater than or equal to denominator.

Degree of denominator $= 3$

∴ For limit to exist, $\alpha - \beta = 0$ . . . (i)

and $\alpha + \frac{\beta}{2} + \gamma = 0$ . . . (ii)

Also, for terms greater than degree '3', gives 0 as $x \to 0$

∴ $\frac{\alpha}{2} - \frac{\beta}{3} - \gamma = 10$ . . . (iii)

From Eq. (i), $\beta = \alpha$

From Eq. (ii),

$\gamma = \left(\frac{\alpha}{2} + \alpha \right) = - \frac{3\alpha}{2}$

Putting these in Eq. (iii),

$\frac{\alpha}{2} - \frac{\alpha}{3} + \frac{3\alpha}{2} = 10 \Rightarrow \frac{3\alpha - 2\alpha + 9\alpha}{6} = 10$

$\Rightarrow \frac{10\alpha}{6} = 10 \Rightarrow \alpha = 6$

Now, $\alpha = \beta \Rightarrow \beta = 6$

Again, $\gamma = \frac{-3\alpha}{2} \Rightarrow \gamma = -9$

$\therefore \alpha + \beta + \gamma = 6 + 6 - 9 = 12 - 9 = 3$

Question75

The value of $\lim_{x \to 0} \left(\frac{x}{\sqrt[8]{1 - \sin x} - \sqrt[8]{1 + \sin x}} \right)$ is equal to [2021, 27 July Shift-II]

Options:

A. 0

B. 4

C. -4

D. -1

Answer: C

Solution:

Solution:

$$\lim_{x \to 0}\left(\frac{x}{\sqrt[8]{1-\sin x} - \sqrt[8]{1+\sin x}} \right)$$

Rationalise denominator three times, $\lim_{x \to 0}\left(\frac{x}{\sqrt[8]{1-\sin x} - \sqrt[8]{1+\sin x}} \right)$

$$\left(\frac{\sqrt[8]{1-\sin x} + \sqrt[8]{1+\sin x}}{\sqrt[8]{1-\sin x} + \sqrt[8]{1+\sin x}} \right)$$

$$\left(\frac{\sqrt[4]{1-\sin x} + \sqrt[4]{1+\sin x}}{\sqrt[4]{1-\sin x} + \sqrt[4]{1+\sin x}} \right)$$

$$\left(\frac{\sqrt{1-\sin x} + \sqrt{1+\sin x}}{\sqrt{1-\sin x} + \sqrt{1+\sin x}} \right)$$

$$= \lim_{x \to 0}\left[\frac{x}{(1-\sin x)-(1+\sin x)} \right]$$

$(\sqrt[8]{1-\sin x} + \sqrt[8]{1+\sin x})$

$(\sqrt[4]{1-\sin x} + \sqrt[4]{1+\sin x})(\sqrt{1-\sin x} + \sqrt{1+\sin x})$

$$= \lim_{x \to 0}\left[\frac{x}{-2\sin x} \right](\sqrt[8]{1-\sin x} + \sqrt[8]{1+\sin x})$$

$(\sqrt[4]{1-\sin x}\ \sqrt[4]{1+\sin x})(\sqrt{1-\sin x} + \sqrt{1+\sin x})$

$$= \left(-\frac{1}{2}\right)(2)(2)(2) \quad \left[\because \lim_{x \to 0}\frac{\sin x}{x} = 1\right]$$

$$= -4$$

Question76

If the value of $\lim_{x \to 0}(2 - \cos x\sqrt{\cos 2x})^{\left(\frac{x+2}{x^2}\right)}$ is equal to e^a, then a is equal to ....
[2021, 20 July Shift-1]

Answer: 3

Solution:

Solution:

$$\lim_{x \to 0}(2 - \cos x\sqrt{\cos 2x})^{\frac{x+2}{x^2}} = 1^{\infty}$$

$$\Rightarrow \lim_{x \to 0}(1 + 1 - \cos x\sqrt{\cos 2x})^{\frac{x+2}{x^2}}$$

$\Rightarrow e^{x \to 0}(1 - \cos x \sqrt{\cos 2x}) \left(\frac{x+2}{x^2} \right)$

$\Rightarrow \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} -$

$\Rightarrow \cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} -$

$= \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots \right) \left(1 - 2x^2 + \frac{2}{3}x^4 \ldots \right) \frac{1}{2}$

We have to extract till the coefficient of x^2 as denominator is x^2.

So, $\left(1 - \frac{x^2}{2} \right)(1 - 2x^2) \frac{1}{2} = \left(1 - \frac{x^2}{2} \right)(1 - x^2)$

$= \left(1 - \frac{x^2}{2} - x^2 + \frac{x^4}{2} \right) = \left(1 - \frac{3}{2}x^2 \right)$

So, $e^{\lim_{x \to 0}(1 - \cos x \sqrt{\cos 2x}) \left(\frac{x+2}{x^2} \right)}$

$= e^{\lim_{x \to 0} \left[1 - \left(\frac{3x^2}{2} \right) \right] \left(\frac{x+2}{x^2} \right)}$

$= e^{\lim_{x \to 0} \left(\frac{3x^2}{2} \right) \left(\frac{x+2}{x^2} \right)} = e^3$

$\therefore\ e^a = e^3 \Rightarrow a = 3$

Question77

If $\alpha_1\beta$ are the distinct roots of $x^2 + bx + c = 0$, then $\lim_{x \to \beta} \frac{e^{2(x^2+bx+c)} - 1 - 2(x^2+bx+c)}{(x-\beta)^2}$ is equal to [2021, 27 Aug. Shift-1]

Options:

A. $b^2 + 4c$

B. $2(b^2 + 4c)$

C. $2(b^2 - 4c)$

D. $b^2 - 4c$

Answer: C

Solution:

Solution:

$\because \alpha, \beta$ are distinct roots of $x^2 + bx + c = 0$

$\Rightarrow\ x^2 + bx + c = (x - \alpha)(x - \beta) = 0$

Now, $\lim_{x \to \beta} \frac{e^{2(x^2+bx+c)} - 1 - 2(x^2+bx+c)}{(x-\beta)^2}$

$= \lim_{x \to \beta} \frac{e^{2(x-\alpha)(x-\beta)} - 1 - 2(x-\alpha)(x-\beta)}{(x-\beta)^2}$

$= \lim_{h \to 0} \frac{e^{2(\beta-\alpha+h)h} - 1 - 2(\beta-\alpha+h)h}{h^2}$

$1 + 2(\beta - \alpha + h)h + \frac{|2(\beta-\alpha+h)h|^2}{2!}$

$= \lim_{h \to 0} \frac{+\ldots - 1 - 2h(\beta-\alpha+h)}{h^2}$

$= \lim_{h \to 0} \frac{2(\beta-\alpha+h)^2h^2 + \ldots}{h^2}$

$= \ 2(\beta - \alpha)^2 = 2(b^2 - 4c)$

Question78

If $\lim_{x \to \infty}\left(\sqrt{x^2 - x + 1} - ax\right) = b$, then the ordered pair (a, b) is [2021, 27 Aug. Shift-2]

Options:

A. $\left(1, \ \frac{1}{2}\right)$

B. $\left(1, -\frac{1}{2}\right)$

C. $\left(-1, \ \frac{1}{2}\right)$

D. $\left(-1, -\frac{1}{2}\right)$

Answer: B

Solution:

Solution:

Given, $\lim_{x \to \infty}\left(\sqrt{x^2 - x + 1} - ax\right) = b$

$\Rightarrow \lim_{x \to \infty}\left(\sqrt{x^2 - x + 1} - ax\right)$

$\frac{\left(\sqrt{x^2 - x + 1}\right) + ax}{\left(\sqrt{x^2 - x + 1} + ax\right)} = b$

$\Rightarrow \lim_{x \to \infty} \frac{x^2 - x + 1 - a^2x^2}{\sqrt{x^2 - x + 1} + ax} = b$

Limit exists only if $a^2 = 1$

$\therefore \ \lim_{x \to \infty} \frac{-x + 1}{\sqrt{x^2 - x + 1} + ax} = \pm$

$\Rightarrow \lim_{x \to \infty} \frac{-1 + \frac{1}{x}}{\sqrt{1 - \frac{1}{x} + \frac{1}{x^2}} + a} = b \Rightarrow \ \frac{-1}{1 + a} = b$

But $\ a \neq -1$

$a = 1$

$b = -\frac{1}{2} \ (a, b) = \left(1, -\frac{1}{2}\right)$

Question79

$\lim_{x \to 2}\left(\sum_{n=1}^{9} \frac{x}{n(n+1)x^2 + 2(2n+1)x + 4}\right)$ is equal to [2021, 26 Aug. Shift-II]

Options:

A. $\frac{9}{44}$

B. $\frac{5}{24}$

C. $\frac{1}{5}$

D. $\frac{7}{36}$

Answer: A

Solution:

Solution:

We have,

$$S = \lim_{x \to 2} \sum_{n=1}^{9} \frac{x}{n(n+1)x^2 + 2(2n+1)x + 4}$$

$$= \sum_{n=1}^{9} \frac{2}{4(n^2 + 3n + 2)}$$

$$= \frac{1}{2} \sum_{n=1}^{9} \frac{(n+2) - (n+1)}{(n+1)(n+2)}$$

$$= \frac{1}{2} \sum_{n=1}^{9} \left[\frac{1}{n+1} - \frac{1}{n+2} \right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \ldots + \left(\frac{1}{10} - \frac{1}{11} \right) \right]$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{11} \right) = \frac{1}{2} \times \left(\frac{11-2}{2 \times 11} \right) = \frac{9}{44}$$

Question80

$\lim_{x \to 0} \frac{\sin^2(\pi \cos^4 x)}{x^4}$ is equal to

[2021, 31 Aug. Shift-1]

Options:

A. π^2

B. $2\pi^2$

C. $4\pi^2$

D. 4π

Answer: C

Solution:

Solution:

$$= \lim_{x \to 0} \frac{\sin^2(\pi \cos^4 x)}{x^4}$$

$$= \lim_{x \to 0} \frac{\sin^2[\pi(1 - \cos^4 x)]}{[\pi(1 - \cos^4 x)]^2} \cdot \frac{\pi^2(1 - \cos^4 x)^2}{x^4}$$

$$= \lim_{x \to 0} \pi^2 \frac{\sin^4 x(1 + \cos^2 x)^2}{x^4}$$

$= \lim_{x \to 0} \pi^2(1 + \cos^2 x)^2 = 4\pi^2$

Question81

If $\alpha = \lim_{x \to \frac{\pi}{4}} \frac{\tan^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)}$ and $\beta = \lim_{x \to 0}(\cos x)^{\cot x}$ are the roots of the quation, $ax^2 + bx - 4 = 0$, then the ordered pair (a, b) is
[2021, 31 Aug. Shift-II]

Options:

A. $(1, -3)$

B. $(-1, 3)$

C. $(-1, -3)$

D. $(1, 3)$

Answer: D

Solution:

Solution:

$\alpha = \lim_{x \to \frac{\pi}{4}} \frac{\tan^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)}$

$= \lim_{x \to \frac{\pi}{4}} \frac{\tan x(\tan x + 1)(\tan x - 1)}{\cos\left(x + \frac{\pi}{4}\right)}$

$= \lim_{x \to \frac{\pi}{4}} \frac{\frac{\sin x}{\cos x} \cdot \left(\frac{\sin x - \cos x}{\cos x}\right)\left(\frac{\sin x + \cos x}{\cos x}\right)}{\frac{1}{\sqrt{2}}(\cos x - \sin x)}$

$= \lim_{x \to \frac{\pi}{4}} \frac{-\sqrt{2}\sin x(\sin x + \cos x)}{\cos 3x}$

$= \frac{-\sqrt{2} \times \frac{1}{\sqrt{2}} \times \sqrt{2}}{\frac{1}{2\sqrt{2}}} = -4$

and $\beta = \lim_{x \to 0}(\cos x)^{\cot x}$

$= e^{x \to 0} \frac{\cos x - 1}{\tan x}$

$= e^{x \to 0} - \frac{\sin x}{\sec x} = e^0 = 1$

Equation whose roots are x and β, is

$x^2 + 3x - 4 = 0$

$\therefore = 1 . b = 3$

Question82

If $y^{1/4} + y^{-1/4} = 2x$, and $(x^2 - 1)\frac{d^2y}{dx^2} + \alpha x\frac{dy}{dx} + \beta y = 0$ then $|\alpha - \beta|$ is equal to

[2021, 27 Aug. Shift-1]

Answer: 17

Solution:

Solution:

Given, $y^{\frac{1}{4}} + y^{-\frac{1}{4}} = 2x$

$\Rightarrow (y^{1/4} + y^{-1/4})^2 = (2x)^2$

$\Rightarrow (y^{1/4} + y^{-1/4})^2 = 4x^2$

Differentiating w.r.t. x, we get

$\frac{1}{4y}\left(y^{\frac{1}{4}} - y^{-\frac{1}{4}}\right)\frac{dy}{dx} = 2$

$\Rightarrow \left(y^{\frac{1}{4}} - y^{-\frac{1}{4}}\right)\frac{dy}{dx} = 8y$. . . (i)

Now, $y^{\frac{1}{4}} - y^{-\frac{1}{4}}$

$= \sqrt{\left(y^{\frac{1}{4}} + y^{-\frac{1}{4}}\right)^2 - 4}$

$\Rightarrow y^{\frac{1}{4}} - y^{-\frac{1}{4}} = 2\sqrt{x^2 - 1}$ . . . (ii)

$\Rightarrow \left(\sqrt{x^2 - 1}\right)\frac{dy}{dx} = 4y$

[using Eqs. (i) and (ii)] Squaring on both sides,

$(x^2 - 1)\left(\frac{dy}{dx}\right)^2 = 16y^2$

Again, differentiating w.r.t. x

$(x^2 - 1) \cdot 2\frac{dy}{dx} \cdot \frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^2 = 32y\frac{dy}{dx}$

On dividing by $\frac{2dy}{dx}$, we get

$(x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 16y$

or $(x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 16y = 0$

Comparing with

$(x^2 - 1)\frac{d^2y}{dx^2} + \alpha x\frac{dy}{dx} + \beta y = 0$

$\therefore\ \alpha = 1, \beta = -16$

$\therefore\ |\alpha - \beta| = |1 + 16| = 17$

Question83

. If y = y(x) is an implicit function of x such that $\log_e(x + y) = 4xy$, then $\frac{d^2y}{dx^2}$ at x = 0 is equal to

[2021, 26 Aug. Shift-1]

Answer: 40

Solution:

Solution:
We have, $\ln(x+y) = 4xy$
$\Rightarrow\ x + y = e^{4xy}$
$\Rightarrow\ 1 + \frac{dy}{dx} = \left(4x\frac{dy}{dx} + 4y\right)e^{4xy}$
If $x = 0$, then $y = 1$
At (0, 1), $\frac{dy}{dx} = 3$
$\frac{d^2y}{dx^2} = e^{4xy}\left(4x\frac{dy}{dx} + 4y\right)^2$
$+e^{4xy}\left(4x\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 4\frac{dy}{dx}\right)$
At $x = 0$,
$\frac{d^2y}{dx^2} = 16 + 24 = 40$

Question84

Let $f(x) = \cos\left(2\tan^{-1}\sin\left(\cot^{-1}\sqrt{\frac{1-x}{x}}\right)\right)$ $0 < x < 1$. Then, [2021, 26 Aug. Shift-1]

Options:

A. $(1-x)^2 f'(x) - 2(f(x))^2 = 0$

B. $(1+x)^2 f'(x) + 2(f(x))^2 = 0$

C. $(1-x)^2 f'(x) + 2(f(x))^2 = 0$

D. $(1+x)^2 f'(x) - 2f(x))^2 = 0$

Answer: C

Solution:

Solution:
$f(x) = \cos\left(2\tan^{-1}\sin\left(\cot^{-1}\sqrt{\frac{1-x}{x}}\right)\right)$
$\cot^{-1}\sqrt{\frac{1-x}{x}} = \sin^{-1}\sqrt{x}$
$\therefore\ f(x) = \cos(2\tan^{-1}\sin\sin^{-1}\sqrt{x})$
or $f(x) = \cos(2\tan^{-1}\sqrt{x})$
$= \cos\tan^{-1}\left(\frac{2\sqrt{x}}{1-x}\right)$
$f(x) = \cos\cos^{-1}\left(\frac{1-x}{1+x}\right)$
$f(x) = \frac{1-x}{1+x}$
$f'(x) = \frac{-(1+x)-(1-x)}{(1+x)^2} = \frac{-2}{(1+x)^2}$
$f'(x)(1-x)^2 = -2\left(\frac{1-x}{1+x}\right)^2$
$(1-x)^2 f'(x) + 2[f(x)]^2 = -2\left(\frac{1-x}{1+x}\right)^2$

$+2\left(\frac{1-x}{1+x}\right)^2 = 0$

Question85

Let f : R → R be a continuous function. Then, $\lim_{x \to \frac{\pi}{4}} \frac{\frac{\pi}{4}\int_2^{\sec^2 x} f(x)dx}{x^2 - \frac{\pi^2}{16}}$ **is equal to**

[2021, 01 Sep. Shift-II]

Options:

A. $f(2)$

B. $2f(2)$

C. $2f(\sqrt{2})$

D. $4f(2)$

Answer: B

Solution:

Solution:
Using L-Hopital's rule

$$\lim_{x \to \frac{\pi}{4}} \frac{\frac{\pi}{4} \cdot 2\sec x \cdot \sec x \cdot \tan x \cdot f(\sec^2 x) - 0}{2x}$$

[using Leibnitz theorem]

$$= \frac{\frac{\pi}{4} \cdot 2(\sqrt{2})^2 \cdot (1) f(2)}{2 \cdot \frac{\pi}{4}} = 2f(2)$$

Question86

$\lim_{x \to 2} \frac{3^x + 3^{3-x} - 12}{3^{-\frac{x}{2}} - 3^{1-x}}$ **is equal to _________.**

[NA Jan. 7, 2020 (I)]

Answer: 36

Solution:

Solution:
Let $3^x = t^2$

$$\lim_{t \to 3} \frac{t^2 + \frac{27}{t^2} - 12}{\frac{1}{t} - \frac{3}{t^2}} = \lim_{t \to 3} \frac{t^4 - 12t^2 + 27}{t - 3}$$

$$= \lim_{t \to 3} \frac{(t^2 - 3)(t + 3)(t - 3)}{t - 3}$$

$$= (3^2 - 3)(3 + 3) = 36$$

Question87

$\lim_{x \to 0} \left(\frac{3x^2 + 2}{7x^2 + 2} \right)^{1/x^2}$ is equal to:

[Jan. 8, 2020 (I)]

Options:

A. $\frac{1}{e}$

B. $\frac{1}{e^2}$

C. e^2

D. e

Answer: B

Solution:

Solution:

Let $R = \lim_{x \to 0} \left(\frac{3x^2 + 2}{7x^2 + 2} \right)^{\frac{1}{x^2}} = e^{\lim_{x \to 0} \left\{ \frac{3x^2 + 2}{7x^2 + 2} - 1 \right\}}$

$= e^{\lim_{x \to 0} \frac{1}{x^2} \left\{ \frac{-4x^2}{7x^2 + 2} \right\}} = e^{\frac{-4}{2}} = e^{-2} = \frac{1}{e^2}$

Question88

$\lim_{x \to 0} \int_0^x \frac{t \sin(10t)\, d\, t}{x}$ is equal to:

[Jan. 8, 2020 (II)]

Options:

A. 0

B. $\frac{1}{10}$

C. $-\frac{1}{5}$

D. $-\frac{1}{10}$

Answer: A

Solution:

Solution:
Using L' Hospital rule,
$\lim_{x \to 0} \frac{x \sin(10x)}{1} = 0$

Question89

If $\lim_{x \to 1} \frac{x + x^2 + x^3 + \ldots\ldots + x^n - n}{x - 1}$ = 820, ($n \in N$) then the value of n is equal to ________.
[NA Sep. 02, 2020 (I)]

Answer: 40

Solution:

Solution:
$\lim_{x \to 1} \frac{x + x^2 + x^3 + \ldots\ldots + x^n - n}{x - 1} = 820 \left(\frac{0}{0} \text{case}\right)$
$\lim_{x \to 1} \frac{1 + 2x + 3x^2 + \ldots\ldots + nx^{n-1}}{1} = 820$(Using L' Hospital rule)
$\Rightarrow 1 + 2 + 3 + \ldots\ldots + n = 820$
$\Rightarrow \frac{n(n+1)}{2} = 820 \Rightarrow n^2 + n - 1640 = 0$
$\Rightarrow n = 40, n \in N$

Question90

$\lim_{x \to 0} \left(\tan\left(\frac{\pi}{4} + x\right)\right)^{1/x}$ is equal to :
[Sep. 02, 2020 (II)]

Options:

A. e

B. 2

C. 1

D. e^2

Answer: D

Solution:

Solution:

$\lim_{x \to 0} \left(\frac{1 + \tan x}{1 - \tan x}\right)^{1/x}$

$\Rightarrow e^{\lim_{x \to 0} \frac{1}{x}\left[\tan\left(\frac{\pi}{4} + x\right) - 1\right]} \Rightarrow e^{\lim_{x \to 0} \frac{1}{x}\left(\frac{1 + \tan x}{1 - \tan x} - 1\right)}$

$\Rightarrow e^{\lim_{x \to 0} \left(\frac{2 \tan x}{1 - \tan x}\right)\frac{1}{x}} = e^{\lim_{x \to 0} \left(\frac{\tan x}{x}\right)\left(\frac{2}{1 - \tan x}\right)} = e^2$

Question91

If

$$\lim_{x \to 0} \left\{ \frac{1}{x^8} \left(1 - \cos\frac{x^2}{2} - \cos\frac{x^2}{4} + \cos\frac{x^2}{2}\cos\frac{x^2}{4} \right) \right\} = 2^{-k},$$

then the value of k is ________.
[NA Sep. 03, 2020 (I)]

Answer: 8

Solution:

Solution:

$\lim_{x \to 0} \frac{\left(1 - \cos\frac{x^2}{2}\right)}{x^4} \frac{\left(1 - \cos\frac{x^2}{4}\right)}{x^4} = 2^{-k}$

$\Rightarrow \lim_{x \to 0} \frac{2\sin^2\frac{x^2}{4}}{\frac{x^4}{16} \times 16} \times \frac{2\sin^2\frac{x^2}{8}}{\frac{x^4}{64} \times 64} = 2^{-k}$

$\Rightarrow \frac{4}{16 \times 64} = 2^{-8} = 2^{-k}\left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$

$\therefore k = 8$

Question92

Let [t] denote the greatest integer ≤t. If for some $\lambda \in R - \{0, 1\}$, $\lim_{x \to 0} \left|\frac{1 - x + |x|}{\lambda - x + [x]}\right| = L$, then L is equal to:
[Sep. 03, 2020 (I)]

Options:

A. 1

B. 2

C. $\frac{1}{2}$

D. 0

Answer: B

Solution:

Solution:

Given $\lim_{x \to 0} \left| \frac{1 - x + |x|}{\lambda - x + [x]} \right| = L$

Here, L.H.L. $= \lim_{h \to 0} \left| \frac{1 + h + h}{\lambda + h - 1} \right| = \left| \frac{1}{\lambda - 1} \right|$

R.H.L. $= \lim_{h \to 0} \left| \frac{1 - h + h}{\lambda + h + 0} \right| = \left| \frac{1}{\lambda} \right|$

Given that limit exists. Hence L.H.L. = R.H.L.

$\Rightarrow |\lambda - 1| = |\lambda|$

$\Rightarrow \lambda = \frac{1}{2}$ and $L = \left| \frac{1}{\lambda} \right| = 2$

Question93

$\lim_{x \to a} \frac{(a + 2x)^{\frac{1}{3}} - (3x)^{\frac{1}{3}}}{(3a + x)^{\frac{1}{3}} - (4x)^{\frac{1}{3}}}$ **$(a \neq 0)$ is equal to**

[Sep. 03, 2020 (II)]

Options:

A. $\left(\frac{2}{3}\right)^{\frac{4}{3}}$

B. $\left(\frac{2}{3}\right)\left(\frac{2}{9}\right)^{\frac{1}{3}}$

C. $\left(\frac{2}{9}\right)^{\frac{4}{3}}$

D. $\left(\frac{2}{9}\right)\left(\frac{2}{3}\right)^{\frac{1}{3}}$

Answer: B

Solution:

Solution:

$\lim_{x \to a} \frac{(a + 2x)^{\frac{1}{3}} - (3x)^{\frac{1}{3}}}{(3a + x)^{\frac{1}{3}} - (4x)^{\frac{1}{3}}} \quad \left[\frac{0}{0}\text{. case}\right]$

Apply L'Hospital rule

$\lim_{x \to a} \frac{\frac{1}{3}(a + 2x)^{-2/3} / 2 - \frac{1}{3} \cdot (3x)^{-2/3} \cdot 3}{\frac{1}{3}(3a + x)^{-2/3} \cdot - \frac{1}{3}(4x)^{-2/3} \cdot 4}$

$= \frac{\frac{1}{3}(3a)^{-2/3} \cdot (2 - 3)}{\frac{1}{3}(4a)^{-2/3} \cdot (1 - 4)} = \frac{3^{-2/3}}{4^{-2/3}} \cdot \frac{1}{3} = \frac{2^{4/3}}{9^{1/3}} \cdot \frac{1}{3} = \frac{2}{3} \cdot \left(\frac{2}{9}\right)^{1/3}$

Question94

$\lim_{n \to \infty} \tan \left\{ \sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{1+r+r^2} \right) \right\}$ is equal to__

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Answer: 1

Solution:

Solution:

$$\lim_{n \to \infty} \tan\left(\sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{1+r(r+1)} \right) \right)$$

$$= \lim_{n \to \infty} \tan\left(\sum_{r=1}^{n} \tan^{-1}\left(\frac{r+1-r}{1+r(r+1)} \right) \right)$$

$$= \tan\left(\lim_{n \to \infty} \sum_{r=1}^{n} [\tan^{-1}(r+1) - \tan^{-1}(r)] \right)$$

$$= \tan\left(\lim_{n \to \infty} \left(\tan^{-1}(n+1) - \frac{\pi}{4} \right) \right)$$

$$= \tan\left(\frac{\pi}{4} \right) = 1$$

Question95

Substitution & Rationalisation If α is the positive root of the equation, $p(x) = x^2 - x - 2 = 0$, then $\lim_{x \to \alpha^+} \frac{\sqrt{1-\cos(p(x))}}{x+\alpha-4}$ is equal to:

[Sep. 05, 2020 (I)]

Options:

A. $\frac{3}{2}$

B. $\frac{3}{\sqrt{2}}$

C. $\frac{1}{\sqrt{2}}$

D. $\frac{1}{2}$

Answer: B

Solution:

Solution:

$x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$

$\Rightarrow x = 2, -1 \Rightarrow \alpha = 2$

$$\therefore \lim_{x \to 2^+} \frac{\sqrt{1 - \cos(x^2 - x - 2)}}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{\sqrt{2}\left|\sin\left(\frac{x^2 - x - 2}{2}\right)\right|}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{\sqrt{2}\sin(x^2 - x - 2) . 2}{\left(\frac{x^2 - x - 2}{2}\right)} \times \frac{(x^2 - x - 2)}{2(x - 2)}$$

$$= \frac{1}{\sqrt{2}} \lim_{x \to 2^+} \left(\frac{\sin\left(\frac{x^2 - x - 2}{2}\right)}{\frac{x^2 - x - 2}{2}} \right) \times \lim_{x \to 2^+} \frac{(x - 2)(x + 1)}{(x - 2)}$$

$$= \frac{1}{\sqrt{2}} \times 1 \times 3 = \frac{3}{\sqrt{2}}$$

Question96

$$\lim_{x \to 0} \frac{x\left(e^{\left(\sqrt{1 + x^2 + x^4} - 1\right)/x} - 1\right)}{\sqrt{1 + x^2 + x^4} - 1}$$

[Sep. 05, 2020 (II)]

Options:

A. is equal to $\sqrt{e}$

B. is equal to

C. is equal to 0

D. does not exist

Answer: B

Solution:

Solution:

Let $L = \lim_{x \to 0} \frac{x\left(e^{\frac{\sqrt{1 + x^2 + x^4} - 1}{x}} - 1\right)}{\sqrt{1 + x^2 + x^4} - 1}$

$$= \lim_{x \to 0} \frac{e^{\frac{\sqrt{1 + x^2 + x^4} - 1}{x}} - 1}{\frac{\sqrt{1 + x^2 + x^4} - 1}{x}}$$

Put $\frac{\sqrt{1 + x^2 + x^4} - 1}{x} = t$ when $x \to 0 \Rightarrow t \to 0$

$\therefore L = \lim_{t \to 0} \frac{e^t - 1}{t} = 1$

Question97

$\lim_{x \to \frac{\pi}{4}} \frac{\cot^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)}$ **is:**

[Jan. 12, 2019 (I)]

Options:

A. 4

B. $4\sqrt{2}$

C. $8\sqrt{2}$

D. 8

Answer: D

Solution:

Solution:

$$\lim_{x \to \frac{\pi}{4}} \frac{\cot^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)} = \lim_{x \to \frac{\pi}{4}} \frac{\cot^3 x\left(1 - \frac{\tan x}{\cos^3 x}\right)}{\cos(x + \pi / 4)}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{(1 - \tan^4 x)}{\tan^3 x \cos(x + \pi / 4)}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{(1 + \tan^2 x)(1 - \tan x)(1 + \tan x)}{\tan^3 x\left(\frac{\cos x - \sin x}{\sqrt{2}}\right)}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{(1 + \tan^2 x)(1 + \tan x)(\cos x - \sin x)}{\frac{\sin^3 x}{\cos^2 x}\left(\frac{\cos x - \sin x}{\sqrt{2}}\right)}$$

$$= \frac{(2)(2)}{\frac{1}{(\sqrt{2})(\sqrt{2})}} = 8$$

Question98

$\lim_{x \to 1^-} \frac{\sqrt{\pi} - \sqrt{2\sin^{-1} x}}{\sqrt{1 - x}}$ **is equal to:**

[Jan. 12, 2019 (II)]

Options:

A. $\frac{1}{\sqrt{2\pi}}$

B. $\sqrt{\frac{2}{\pi}}$

C. $\sqrt{\frac{\pi}{2}}$

D. $\sqrt{\pi}$

Answer: B

Solution:

$$\lim_{x\to 1^-}\frac{\sqrt{\pi}-\sqrt{2\sin^{-1}x}}{\sqrt{1-x}} = \lim_{h\to 0} f(1-h)$$

$$= \lim_{h\to 0}\frac{\sqrt{\pi}-\sqrt{2\sin^{-1}(1-h)}}{\sqrt{1-(1-h)}}$$

$$= \lim_{h\to 0}\frac{\sqrt{\pi}-\sqrt{2\sin^{-1}(1-h)}}{\sqrt{h}}$$

$$= \lim_{h\to 0}\frac{-\frac{1}{2\sqrt{2\sin^{-1}(1-h)}}\times 2\times\frac{1}{\sqrt{1-(1-h)^2}}(-1)}{\frac{1}{2\sqrt{h}}}$$

$$= \lim_{h\to 0}\frac{\frac{1}{\sqrt{2\sin^{-1}(1-h)}}\frac{1}{\sqrt{h(2-h)}}}{\frac{1}{2\sqrt{h}}}$$

$$= 2\times\frac{1}{\sqrt{\pi}}\times\frac{1}{\sqrt{2}} = \sqrt{\frac{2}{\pi}}$$

Question99

Let [x] denote the greatest integer less than or equal to x. Then : $\lim_{x\to 0}\frac{\tan(\pi\sin^2 x)+(|x|-\sin(x[x]))^2}{x^2}$:

[Jan. 11, 2019 (I)]

Options:

A. does not exist

B. equals π

C. equals $\pi + 1$

D. equals 0

Answer: A

Solution:

Solution:

RHL is, $\lim_{x\to 0^+}\frac{\tan(\pi\sin^2 x)+(x-0)^2}{x^2}$

$$= \lim_{x\to 0^+}\left(\frac{\tan(\pi\sin^2 x)}{x^2}+1\right) = 1+\pi$$

And LHL is, $\lim_{x\to 0^-}\frac{\tan(\pi\sin^2 x)+(-x+\sin x)^2}{x^2}$

$$= \lim_{x\to 0^-}\frac{\tan(\pi\sin^2 x)+x^2+\sin^2 x-2x\sin x}{x^2}$$

$= \pi+1+1-2 = \pi$

Since, LH L $\neq$ RH L

Hence, limit does not exist.

Question100

$\lim_{x\to 0}\frac{x\cot(4x)}{\sin^2 x\cot^2(2x)}$ is equal to

[Jan. 11, 2019 (II)]

Options:

A. 0

B. 2

C. 4

D. 1

Answer: D

Solution:

Solution:

$$\lim_{x \to 0} \frac{x \cot 4x}{\sin^2 x \cdot \cot^2 2x} = \lim_{x \to 0} \frac{x \cdot \tan^2 2x}{\sin^2 x \cdot \tan 4x}$$

$$= \lim_{x \to 0} \left(\frac{x}{\sin x}\right)^2 \cdot \left(\frac{\tan 2x}{2x}\right)^2 \cdot \left(\frac{4x}{\tan 4x}\right) \cdot \frac{4}{2^2} = 1$$

Question101

For each t ∈ R, let [t] be the greatest integer less than or equal to t. Then,

$$\lim_{x \to 1+} \frac{(1 - |x| + \sin|1-x|) \sin\left(\frac{\pi}{2}[1-x]\right)}{|1-x|[1-x]}$$

[Jan. 10, 2019 (I)]

Options:

A. equals 1

B. equals 0

C. equals – 1

D. does not exist

Answer: B

Solution:

Solution:

$$\lim_{x \to 1^+} \frac{(1 - |x| + \sin(|1-x|)) \sin\left(\frac{\pi}{2}[1-x]\right)}{|1-x|[1-x]}$$

$$= \lim_{h \to 0} \frac{(1 - |1+h| + \sin(|1-1-h|)) \sin\left(\frac{\pi}{2}[1-1-h]\right)}{|1-1-h|[1-1-h]}$$

$$= \lim_{h \to 0} \frac{(1 - 1 - h + \sinh) \sin\left(\frac{\pi}{2}(-1)\right)}{h([0-h])}$$

$$= \lim_{h \to 0} \frac{(-h + \sin h)\sin\left(-\frac{\pi}{2}\right)}{h(-1)} = 0$$

Question102

$$\lim_{y \to 0} \frac{\sqrt{1 + \sqrt{1 + y^4}} - \sqrt{2}}{y^4}$$

[Jan. 9, 2019 (I)]

Options:

A. exists and equals $\frac{1}{4\sqrt{2}}$

B. exists and equals $\frac{1}{2\sqrt{2}(\sqrt{2}+1)}$

C. exists and equals $\frac{1}{2\sqrt{2}}$

D. does not exist

Answer: A

Solution:

Solution:

$$L = \lim_{y \to 0} \frac{\sqrt{1 + \sqrt{1 + y^4}} - \sqrt{2}}{y^4}$$

$$= \lim_{y \to 0} \frac{\left(\sqrt{1 + \sqrt{1 + y^4}} - \sqrt{2}\right)\left(\sqrt{1 + \sqrt{1 + y^4}} + \sqrt{2}\right)}{y^4\left(\sqrt{1 + \sqrt{1 + y^4}} + \sqrt{2}\right)}$$

$$= \lim_{y \to 0} \frac{1 + \sqrt{1 + y^4} - 2}{y^4\left(\sqrt{1 + \sqrt{1 + y^4}} + \sqrt{2}\right)}$$

$$= \lim_{y \to 0} \frac{\left(\sqrt{1 + y^4} - 1\right)\left(\sqrt{1 + y^4 + 1}\right)}{y^4\left(\sqrt{1 + \sqrt{1 + y^4}} + \sqrt{2}\right)\left(\sqrt{1 + y^4} + 1\right)}$$

$$= \lim_{y \to 0} \frac{1 + y^4 - 1}{y^4\left(\sqrt{1 + \sqrt{1 + y^4}} + \sqrt{2}\right)\left(\sqrt{1 + y^4} + 1\right)}$$

$$= \frac{1}{2\sqrt{2} \times 2} = \frac{1}{4\sqrt{2}}$$

Question103

For each $x \in R$, let $[x]$ be greatest integer less than or equal to x. Then $\lim_{x \to 0} \frac{x([x] + |x|)\sin[x]}{x}$ is equal to:

[Jan. 09, 2019 (II)]

Options:

A. – sin 1

B. 1

C. sin 1

D. 0

Answer: A

Solution:

Solution:

$$\lim_{x \to 0^-} \frac{x([x] + |x|) \cdot \sin[x]}{|x|}$$
$$= \lim_{x \to 0} \frac{(0-h)([0-h] + |0-h|) \cdot \sin[0-h]}{|0-h|}$$
$$= \lim_{h \to 0} \frac{(-h)(-1+h)\sin(-1)}{h}$$
$$= \lim_{h \to 0} (1-h)\sin(-1) = -\sin 1$$

Question104

Let $f(x) = 5 - |x - 2|$ and $g(x) = |x + 1|$, $x \in R$. If $f(x)$ attains maximum value at α and $g(x)$ attains minimum value at β then $\lim_{x \to -\alpha\beta} \frac{(x-1)(x^2 - 5x + 6)}{x^2 - 6x + 8}$ is equal to :
[April 12, 2019(II)]

Options:

A. 1 / 2

B. −3 / 2

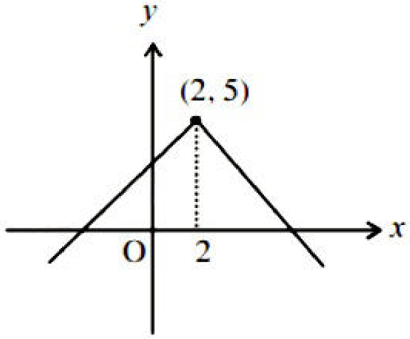
C. −1 / 2

D. 3 / 2

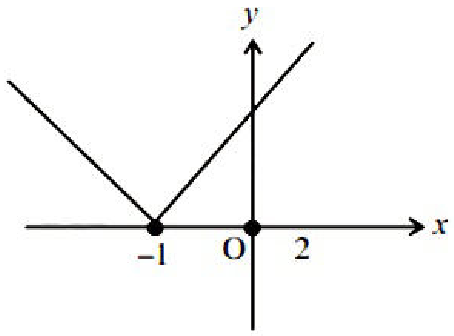
Answer: A

Solution:

Solution:
$f(x) = 5 - |x - 2|$
Graph of $y = f(x)$



By the graph $f(x)$ is maximum at $x = 2$
$\therefore \alpha = 2$ $g(x) = |x + 1|$
Graph of $y = g(x)$



By the graph g(x) is minimum at $x = -1$
$\therefore\ \beta = -1$
Now, $\lim\limits_{x \to 2} \frac{(x-1)(x-2)(x-3)}{(x-2)(x-4)}$
$= \lim\limits_{x \to 2} \frac{(x-1)(x-3)}{x-4} = \frac{1}{2}$

Question105

$\lim\limits_{x \to 0} \frac{x + 2\sin x}{\sqrt{x^2 + 2\sin x + 1} - \sqrt{\sin^2 x - x + 1}}$ **is**

[April 12, 2019(II)]

Options:

A. 6

B. 2

C. 3

D. 1

Answer: B

Solution:

Solution:
Given limit is,
$\lim\limits_{x \to 0} \frac{x + 2\sin x}{\sqrt{x^2 + 2\sin x + 1} - \sqrt{\sin^2 x - x + 1}}$
On rationalising,
$= \lim\limits_{x \to 0} \frac{(x + 2\sin x)\left[\sqrt{x^2 + 2\sin x + 1} + \sqrt{\sin^2 x - x + 1}\right]}{(x^2 - \sin^2 x) + (x + 2\sin x)}$
$= \lim\limits_{x \to 0} \frac{\left[1 + 2\left(\frac{\sin x}{x}\right)\right]\left[\sqrt{x^2 + 2\sin x + 1} + \sqrt{\sin^2 x - x + 1}\right]}{\left(x - \frac{\sin^2 x}{x}\right) + \left(1 + 2\left(\frac{\sin x}{x}\right)\right)}$
$= \frac{3 \times 2}{3} = 2 \quad \left[\because \lim\limits_{x \to 0} \frac{\sin}{x} = 1\right]$

Question106

If $\lim\limits_{x \to 1} \frac{x^4 - 1}{x - 1} = \lim\limits_{x \to k} \frac{x^3 - k^3}{x^2 - k^2}$**, then k is:**

[April 10, 2019 (I)]

Options:

A. $\frac{8}{3}$

B. $\frac{3}{8}$

C. $\frac{3}{2}$

D. $\frac{4}{3}$

Answer: A

Solution:

Solution:

Given, $\lim\limits_{x \to 1} x^4 - 1x - 1 = \lim\limits_{x \to K} \left(\frac{x^3 - k^3}{x^2 - k^2} \right)$

Taking L.H.S. $\lim\limits_{x \to 1} \frac{x^4 - 1}{x - 1} \quad \left(\frac{0}{0} \text{form} \right)$

$\underset{x \to 1}{\text{Lt}} \frac{4x^3}{1} = 4$ [Using L Hospital's Rule]

$\therefore \lim\limits_{x \to K} \frac{x^3 - k^3}{x^2 - k^2} = 4$

$\Rightarrow \lim\limits_{x \to K} \frac{3x^2}{2x} = 4$ [Using L Hospital's Rule]

$\Rightarrow \frac{3}{2} k = 4 \Rightarrow k = \frac{8}{3}$

Question107

If $\lim\limits_{x \to 1} \frac{x^2 - ax + b}{x - 1} = 5$, then a + b is equal to : [April 10, 2019 (II)]

Options:

A. –4

B. 5

C. –7

D. 1

Answer: C

Solution:

Solution:

$\lim\limits_{x \to 1} \frac{x^2 - ax + b}{x - 1} = 5$

$\because$ limit is finite $\therefore 1 - a + b = 0$

$\Rightarrow \lim\limits_{x \to 1} \frac{2x - a}{1} = 5 \left(\frac{0}{0} \text{form} \right)$ (By L Hospital's rule)

$\Rightarrow 2 - a = 5 \Rightarrow a = -3$ and $b = -4$

Then a + b = −3 − 4 = −7

Question108

$\lim_{x \to 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1 + \cos x}}$ **equals:**

[April 8, 2019 (I)]

Options:

A. $4\sqrt{2}$

B. $\sqrt{2}$

C. $2\sqrt{2}$

D. 4

Answer: A

Solution:

Solution:

$\lim_{x \to 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1 + \cos x}}$

$= \lim_{x \to 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{2\cos^2 \frac{x}{2}}} \left[\frac{0}{0} \right]$

$= \lim_{x \to 0} \frac{\sin^2 x}{\sqrt{2}\left[1 - \cos \frac{x}{2} \right]} = \lim_{x \to 0} \frac{\sin^2 x}{2\sqrt{2}\sin^2 \frac{x}{4}}$

$= \lim_{x \to 0} \frac{\left(\frac{\sin x}{x} \right)^2 . 16}{2\sqrt{2}\left(\frac{\sin \frac{x}{4}}{\frac{x}{4}} \right)^2} = \frac{16}{2\sqrt{2}} = 4\sqrt{2}$

Question109

If α and β are the roots of the equation $375x^2 - 25x - 2 = 0$,then $\lim_{n \to \infty} \sum_{r=1}^{n} \alpha^r + \lim_{n \to \infty} \sum_{r=1}^{n} \beta^r$ is equal to :

[April 12, 2019 (I)]

Options:

A. $\frac{21}{346}$

B. $\frac{29}{358}$

C. $\frac{1}{12}$

D. $\frac{7}{116}$

Answer: C

Solution:

Solution:
Given equation is, $375x^2 - 25x - 2 = 0$
Sum and product of the roots are,
$\alpha + \beta = \frac{25}{375}$ and $\alpha\beta = \frac{-2}{375}$

$\lim_{n \to \infty} \sum_{r=1}^{n} (\alpha^r + \beta^r)$
$= (\alpha + \alpha^2 + \alpha^3 + \ldots\ldots.\infty\ (\beta + \beta^2 + \beta^3 \ldots\ldots + \infty)$
$= \alpha 1 - \alpha + \beta 1 - \beta\ = \frac{\alpha + \beta - 2\alpha\beta}{1 - (\alpha + \beta) + \alpha\beta}$
$= \frac{\frac{25}{375} + \frac{4}{375}}{1 - \frac{25}{375} - \frac{2}{375}}\ = \frac{29}{375 - 25 - 2} = \frac{29}{348} = \frac{1}{12}$

Question110

Let $f : R \to R$ be a differentiable function satisfying $f'(3) + f'(2) = 0$. Then $\lim_{x \to 0} \left(\frac{1 + f(3 + x) - f(3)}{1 + f(2 - x) - f(2)} \right)^{\frac{1}{x}}$ is equal to :
[April 08, 2019 (II)]

Options:

A. 1

B. e^{-1}

C. e

D. e^2

Answer: A

Solution:

Solution:
$I\ = \lim_{x \to 0} \left(\frac{1 + f(3 + x) - f(3)}{1 + f(2 - x) - f(2)} \right)^{\frac{1}{x}}$ [1^∞ form]
$\Rightarrow I\ = e^{l}1$, where
$I_1 = \lim_{x \to 0} \left(\left(\frac{1 + f(3 + x) - f(3)}{1 + f(2 - x) - f(2)} - 1 \right) \right)\ \left(\frac{1}{x} \right)$
$= \lim_{x \to 0} \left(\frac{1}{x} \right) \left(\frac{f(3 + x) - f(3) - f(2 - x) + f(2)}{1 + f(2 - x) - f(2)} \right)\ \left(\frac{0}{0} \text{form} \right)$
By L. Hospital Rule,
$I_1 = \lim_{x \to 0} \left(\frac{f'(3 + x) + f'(2 - x)}{1} \right)\ \lim_{x \to 0} \left(\frac{1}{1 + f(2 - x) - f(2)} \right)$
$= f'(3) + f'(2) = 0$
$\Rightarrow I\ = e^{I_1} = e^0 = 1$

Question111

$\lim_{x \to 0} \frac{x \tan 2x - 2x \tan x}{(1 - \cos 2x)^2}$ **equals.**

[Online April 15, 2018]

Options:

A. 1

B. $-\frac{1}{2}$

C. $\frac{1}{4}$

D. $\frac{1}{2}$

Answer: D

Solution:

Solution:

Let, $L = \lim_{x \to 0} \frac{(x \tan 2x - 2x \tan x)}{(1 - \cos 2x)^2} = \lim_{x \to 0} K$ (say)

$$\Rightarrow K = \frac{x\left[\frac{2 \tan x}{1 - (\tan x)^2}\right] - 2x \tan x}{(1 - (1 - 2\sin^2 x))^2}$$

$$= \frac{2x \tan x - [2x \tan x - 2x\tan^3 x]}{4\sin^4 x \times (1 - \tan^2 x)}$$

$$= \frac{2x\tan^3 x}{4\sin^4 x \times (1 - \tan^2 x)} = \frac{2x\tan^3 x}{4\sin^4 x \times \left(\frac{\cos^2 x - \sin^2 x}{\cos^2 x}\right)}$$

$$= \frac{2x\frac{\sin^3 x}{\cos^3 x}}{4\sin^4 x \times \left(\frac{\cos^2 x - \sin^2 x}{\cos^2 x}\right)}$$

$$\Rightarrow K = \frac{x}{2 \sin x \times (\cos^2 x - \sin^2 x) \cos x}$$

$$\therefore L = \lim_{x \to 0} \frac{x}{2 \sin x} \times \lim_{x \to 0} \frac{1}{\cos x(\cos^2 x - \sin^2 x)}$$

$$= \lim_{x \to 0} \frac{x}{2 \sin x} \times \lim_{x \to 0} \frac{1}{\cos 0(\cos^2 0 - \sin^2 0)} = \frac{1}{2}$$

Question112

For each $t \in R$, let $[t]$ be the greatest integer less than or equal to t. Then

$$\lim_{x \to 0^+} \mathbf{x}\left(\left[\frac{1}{x}\right] + \left[\frac{2}{x}\right] + \ldots\ldots + \left[\frac{15}{x}\right]\right)$$

[2018]

Options:

A. is equal to 15.

B. is equal to 120.

C. does not exist (in R).

D. is equal to 0.

Answer: B

Solution:

Solution:

Since, $\lim_{x \to 0^+} x\left(\left[\frac{1}{x}\right] + \left[\frac{2}{x}\right] + \ldots\ldots + \left[\frac{15}{x}\right] \right)$

$= \lim_{x \to 0^+} x\left(\frac{1+2+3+\ldots..+15}{x} \right) - \left(\left\{\frac{1}{x}\right\} + \left\{\frac{2}{x}\right\} + \ldots\ldots. + \left\{\frac{15}{x}\right\} \right)$

$\because 0 \le \left\{\frac{r}{x}\right\} < 1 \Rightarrow 0 \le x\left\{\frac{r}{x}\right\} < x$

$\therefore \lim_{x \to 0^+} x\left(\frac{1+2+3+\ldots\ldots+15}{x} \right) = \frac{15 \times 16}{2} = 120$

Question113

$\lim_{x \to 0} \frac{(27+x)^{\frac{1}{3}} - 3}{9 - (27+x)^{\frac{2}{3}}}$ **equals.**

[Online April 16, 2018]

Options:

A. $-\frac{1}{3}$

B. $\frac{1}{6}$

C. $-\frac{1}{6}$

D. $\frac{1}{3}$

Answer: C

Solution:

Solution:

Let $L = \lim_{x \to 0} \frac{(27+x)^{\frac{1}{3}} - 3}{9 - (27+x)^{\frac{2}{3}}}$

Here 'L' is in the indeterminate form i.e., $\frac{0}{0}$

$\therefore$ using the L 'Hospital rule we get:

$L = \lim_{x \to 0} \frac{\frac{1}{3}(27+x)^{\frac{-2}{3}}}{-\frac{2}{3}(27+x)^{\frac{-1}{3}}} = \frac{\frac{1}{3} \times (27)^{\frac{-2}{3}}}{\frac{-2}{3} \times 27^{\frac{-1}{3}}} = -\frac{1}{6}$

Question114

Let $f(x)$ be a polynomial of degree 4 having extreme values at $x = 1$ and $x = 2$. If $\lim_{x \to 0} \left(\frac{f(x)}{x^2} + 1 \right) = 3$ then $f(-1)$ is equal to

[Online April 15, 2018]

Options:

A. $\frac{1}{2}$

B. $\frac{3}{2}$

C. $\frac{5}{2}$

D. $\frac{9}{2}$

Answer: D

Solution:

Solution:
$\because f(x)$ has extremum values at $x = 1$ and $x = 2$
$\because f'(1) = 0$ and $f'(2) = 0$
As, $f(x)$ is a polynomial of degree 4.
Suppose $f(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E$
$\because \lim_{x \to 0} \left(\frac{f(x)}{x^2} + 1 \right) = 3$
$\Rightarrow \lim_{x \to 0} \left(\frac{Ax^4 + Bx^3 + Cx^2 + Dx + E}{x^2} + 1 \right) = 3$
$\Rightarrow \lim_{x \to 0} \left(Ax^2 + Bx + C + \frac{D}{x} + \frac{E}{x^2} + 1 \right) = 3$
As limit has finite value, so $D = 0$ and $E = 0$
Now $A(0)^2 + B(0) + C + 0 + 0 + 1 = 3$
$\Rightarrow c + 1 = 3 \Rightarrow c = 2$
$f'(x) = 4Ax^3 + 3Bx^2 + 2Cx + D$
$f'(1) = 0 \Rightarrow 4A(1) + 3B(1) + 2C(1) + D = 0$
$\Rightarrow 4A + 3B = -4$ ......(i)
$f'(2) = 0 \Rightarrow 4A(8) + 3B(4) + 2C(2) + D = 0$
$\Rightarrow 8A + 3B = -2$ ....(i)
From equations (i) and (ii), we get
$A = \frac{1}{2}$ and $B = -2$
So, $f(x) = \frac{x^4}{2} - 2x^3 + 2x^2$
Therefore ,$f(-1) = \frac{(-1)^4}{2} - 2(-1)^3 + 2(-1)^2$
$= \frac{1}{2} + 2 + 2 = \frac{9}{2}$. Hence $f(-1) = \frac{9}{2}$

Question115

$\lim_{x \to \frac{\pi}{2}} \frac{\cot x - \cos x}{(\pi - 2x)^3}$ equals:

[2017]

Options:

A. $\frac{1}{4}$

B. $\frac{1}{24}$

C. $\frac{1}{16}$

D. $\frac{1}{8}$

Answer: C

Solution:

Solution:

$$\lim_{x \to \frac{\pi}{2}} \frac{\cot x(1-\sin x)}{-8\left(x-\frac{\pi}{2}\right)^3} = \lim_{x \to \frac{\pi}{2}} \frac{\cot x(1-\sin x)}{8\left(\frac{\pi}{2}-x\right)^3}$$

Put $\frac{\pi}{2} - x = t \Rightarrow$ as $x \to \frac{\pi}{2} \Rightarrow t \to 0$

$$= \lim_{t \to 0} \frac{\cot\left(\frac{\pi}{2}-t\right)\left(1-\sin\left(\frac{\pi}{2}-t\right)\right)}{8t^3}$$

$$= \lim_{t \to 0} \frac{\tan t(1-\cos t)}{8t^3} = \lim_{t \to 0} \frac{\tan t}{8t} \cdot \frac{1-\cos t}{t^2}$$

$$= \frac{1}{8} \cdot 1 \cdot \frac{1}{2} = \frac{1}{16}$$

Question116

$\lim_{x \to 3} \frac{\sqrt{3x}-3}{\sqrt{2x-4}-\sqrt{2}}$ is equal to:

[Online April 8, 2017]

Options:

A. $\sqrt{3}$

B. $\frac{1}{\sqrt{2}}$

C. $\frac{\sqrt{3}}{2}$

D. $\frac{1}{2\sqrt{2}}$

Answer: B

Solution:

Solution:

$$\text{Let} A = \lim_{x \to 3} \frac{\sqrt{3x}-3}{\sqrt{2x-4}-\sqrt{2}}$$

Rationalise

$\Rightarrow A = \lim_{x \to 3} \frac{(3x-9) \times (2x-4+\sqrt{2})}{\{(2x-4)-2\} \times (\sqrt{3x}+3)}$

$= \lim_{x \to 3} \frac{3(x-3)}{2(x-3)} \times \frac{\sqrt{2x-4}+\sqrt{2}}{(\sqrt{3x}+3)} = \frac{3}{2} \times \frac{2\sqrt{2}}{6} = \frac{1}{\sqrt{2}}$

Question117

Let p = $\lim_{x \to 0^+} (1+\tan^2\sqrt{x})^{\frac{1}{2x}}$ then $\log p$ is equal to

[2016]

Options:

A. $\frac{1}{2}$

B. $\frac{1}{4}$

C. 2

D. 1

Answer: A

Solution:

Solution:

$\ln p = \lim_{x \to 0^+} \frac{1}{2x} \ln(1+\tan^2\sqrt{x})$

$\lim_{x \to 0^+} \frac{1}{x} \ln(\sec\sqrt{x})$

Applying L hospital's rule :

$= \lim_{x \to 0^+} \frac{\sec\sqrt{x}\tan\sqrt{x}}{\sec\sqrt{x} \,.\, 2\sqrt{x}} = \lim_{x \to 0^+} \frac{\tan\sqrt{x}}{2\sqrt{x}} = \frac{1}{2}$

Question118

$\lim_{x \to 0} \frac{(1-\cos 2x)^2}{2x\tan x - x\tan 2x}$ **is**

[Online April 10, 2016]

Options:

A. 2

B. $-\frac{1}{2}$

C. –2

D. $\frac{1}{2}$

Answer: C

Solution:

Solution:

$\lim_{x \to 0} \text{frac}\,(1 - \cos 2x)^2\, 2x \tan x - x \tan 2x$

$$= \lim_{x \to 0} \frac{(2\sin^2 x)^2}{2x\left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots..\right) - x\left(2x + \frac{2^3x^3}{3} + 2\frac{2^5x^5}{15} + \ldots\ldots\right)}$$

$$= \lim_{x \to 0} \frac{4\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\right)^4}{x^4\left(\frac{2}{3} - \frac{8}{5}\right) + x^6\left(\frac{4}{15} - \frac{64}{15}\right)}$$

$$= \lim_{x \to 0} \frac{4\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots\ldots.\right)^4}{-2 + x^2\left(-\frac{60}{15}\right) + \ldots\ldots}$$

(dividing numerator & denominator by x^4)

= –2

Question119

If $\lim_{x \to \infty}\left(1 + \frac{a}{x} - \frac{4}{x^2}\right)^{2x} = e^3$, then ′a′ is equal to:

[Online April 9, 2016]

Options:

A. 2

B. $\frac{3}{2}$

C. $\frac{1}{2}$

D. $\frac{2}{3}$

Answer: B

Solution:

Solution:

$\lim_{x \to \infty}\left(1 + \frac{a}{x} - \frac{4}{x^2}\right)^{2x}$(1^∞ form)

$= e\left[\lim_{x \to \infty}\left(1 + \frac{a}{x} - \frac{4}{x^2} - 1\right)2x\right]$

$= e \lim_{x \to \infty}\left(2a - \frac{8}{x}\right) = e^{2a}$

$\therefore 2a = 3 \Rightarrow a = \frac{3}{2}$

Question120

$\lim_{x \to 0} \frac{(1 - \cos 2x)(3 + \cos x)}{x \tan 4x}$ **is equal to**

[2015]

Options:

A. 2

B. $\frac{1}{2}$

C. 4

D. 3

Answer: A

Solution:

Solution:
Multiply and divide by x in the given expression, we get

$$\lim_{x \to 0} \frac{(1 - \cos 2x)}{x^2} \frac{(3 + \cos x)}{1} \cdot \frac{x}{\tan 4x}$$

$$= \lim_{x \to 0} \frac{2\sin^2 x}{x^2} \cdot \frac{3 + \cos x}{1} \cdot \frac{x}{\tan 4x}$$

$$= 2 \lim_{x \to 0} \frac{\sin^2 x}{x^2} . \lim_{x \to 0} 3 + \cos x . \lim_{x \to 0} \frac{x}{\tan 4x}$$

$$= 2.4 \frac{1}{4} \lim_{x \to 0} \frac{4x}{\tan 4x} = 2.4 . \frac{1}{4} = 2$$

Question121

$\lim_{x \to 0} \frac{e^{x^2} - \cos x}{\sin^2 x}$ **is equal to:**

[Online April 10, 2015]

Options:

A. 2

B. 3

C. $\frac{3}{2}$

D. $\frac{5}{4}$

Answer: C

Solution:

Solution:

$$\lim_{x \to 0} \frac{2xe^{x^2} + \sin x}{2 \sin x \cos x}$$

$$\lim_{x \to 0} \left(\frac{x}{\sin x} e^{x^2} + \frac{1}{2} \right) \frac{1}{\cos x} = 1 + \frac{1}{2} = \frac{3}{2}$$

Question122

Let f (x) be a polynomial of degree four having extreme values at x = 1 and x = 2. If $\lim_{x \to 0} \left[1 + \frac{f(x)}{x^2}\right] = 3$, then f (2) is equal to :

[2015]

Options:

A. 0

B. 4

C. – 8

D. – 4

Answer: A

Solution:

Solution:

$\lim_{x \to 0} \left[1 + \frac{f(x)}{x^2}\right] = 3$

$\Rightarrow \lim_{x \to 0} \frac{f(x)}{x^2} = 2$

So, f (x) contain terms in x^2, x^3 and x^4.

Let $f(x) = a_1x^2 + a_2x^3 + a_3x^4$

Since $\lim_{x \to 0} \frac{f(x)}{x^2} = 2 \Rightarrow a_1 = 2$

Hence, $f(x) = 2x^2 + a_2x^3 + a_3x^4$

$f'(x) = 4x + 3a_2x^2 + 4a_3x^3$

As given : $f'(1) = 0$ adn $f'(2) = 0$

Hence, $4 + 3a_2 + 4a_3 = 0$...(i)

and $8 + 12a_2 + 32a_3 = 0$...(ii)

By 4x(i) − (ii), we get

$16 + 12a_2 + 16a_3 - (8 + 12a_2 + 32a_3) = 0$

$\Rightarrow 8 - 16a_3 = 0 \Rightarrow a_3 = 1 / 2$

and by eqn. (i), $4 + 3a_2 + 4 / 2 = 0 \Rightarrow a_2 = -2$

$\Rightarrow f(x) = 2x^2 - 2x^3 + \frac{1}{2}x^4$

$f(2) = 2 \times 4 - 2 \times 8 + \frac{1}{2} \times 16 = 0$

Question123

$\lim_{x \to 0} \frac{\sin(\pi \cos^2 x)}{x^2}$ is equal to:

[2014]

Options:

A. $-\pi$

B. π

C. $\frac{\pi}{2}$

D. 1

Answer: B

Solution:

Solution:

$\lim\limits_{x \to 0} \frac{\sin(\pi\cos^2 x)}{x^2}$

$= \lim\limits_{x \to 0} \frac{\sin[\pi(1 - \sin^2 x)]}{x^2}$

$= \lim\limits_{x \to 0} \sin \frac{(\pi - \pi\sin^2 x)}{x^2}$ $[\because \sin(\pi - \theta) = \sin\theta]$

$= \lim\limits_{x \to 0} \sin \frac{(\pi\sin^2 x)}{\pi\sin^2 x} \times \frac{\pi\sin^2 x}{x^2}$

$= \lim\limits_{x \to 0} 1 \times \pi \left(\frac{\sin x}{x}\right)^2 = \pi$

Question124

If

$\lim\limits_{x \to 2} \frac{\tan(x-2)\{x^2 + (k-2)x - 2k\}}{x^2 - 4x + 4}$ **= 5,**

then k is equal to:
[Online April 11, 2014]

Options:

A. 0

B. 1

C. 2

D. 3

Answer: D

Solution:

Solution:

$\lim\limits_{x \to 2} \frac{\tan(x-2)\{x^2 + (k-2)x - 2k\}}{x^2 - 4x + 4} = 5$

$\Rightarrow \lim\limits_{x \to 2} \frac{\tan(x-2)\{x^2 + kx - 2x - 2k\}}{(x-2)^2} = 5$

$\Rightarrow \lim\limits_{x \to 2} \frac{\tan(x-2)\{x(x-2) + k(x-2)\}}{(x-2) \times (x-2)} = 5$

$\Rightarrow \lim\limits_{x \to 2} \left(\frac{\tan(x-2)}{(x-2)}\right) \times \lim\limits_{x \to 2} \left(\frac{(k+x)(x-2)}{(x-2)}\right) = 5$

$\Rightarrow 1 \times \lim\limits_{x \to 2} (k+x) = 5 \left\{ \because \lim\limits_{h \to 0} \frac{\tan h}{h} = 1 \right\}$

or $k + 2 = 5$

$\Rightarrow k = 3$

Question125

$\lim_{x \to 0} \frac{(1-\cos 2x)(3+\cos x)}{x \tan 4x}$ is equal to
[2013]

Options:

A. $-\frac{1}{4}$

B. $\frac{1}{2}$

C. 1

D. 2

Answer: D

Solution:

Solution:
Multiply and divide by x in the given expression, we get
$\lim_{x \to 0} \frac{(1-\cos 2x)}{x^2} \frac{(3+\cos x)}{1} \cdot \frac{x}{\tan 4x} \left[\because 1-\cos 2x = 2\sin^2 \frac{x}{2} \right]$
$= \lim_{x \to 0} \frac{2\sin^2 x}{x^2} \cdot \frac{3+\cos x}{1} \cdot \frac{x}{\tan 4x}$
$= 2\lim_{x \to 0} \frac{\sin^2 x}{x^2} \cdot \lim_{x \to 0}(3+\cos x) \cdot \lim_{x \to 0} \frac{4x}{\tan 4x} \times \frac{1}{4}$
$= 2.4 \cdot \frac{1}{4} = 2$

Question126

Let $f(1) = -2$ and $f'(x) \geq 4.2$ for $1 \leq x \leq 6$. The possible value of $f(6)$ lies in the interval :
[April 25, 2013]

Options:

A. [15,19)

B. $(-\infty, 12)$

C. [12,15)

D. $[19, \infty)$

Answer: D

Solution:

Solution:
Given $f(1) = -2$ and $f'(x) \geq 4.2$ for $1 \leq x \leq 6$
Consider $f'(x) = \frac{f(x+h) - f(x)}{h}$
$\Rightarrow f(x+h) - f(x) = f'(x) \cdot h \geq (4.2)h$

So, $f(x+h) \ge f(x) + (4.2)h$
put $x = 1$ and $h = 5$, we get
$f(6) \ge f(1) + 5(4.2) \Rightarrow f(6) \ge 19$
Hence $f(6)$ lies in $[19, \infty)$

Question127

$\lim_{x \to 0} \frac{\sin(\pi \cos^2 x)}{x^2}$ equals
[Online May 26, 2012]

Options:

A. $-\pi$

B. 1

C. -1

D. π

Answer: D

Solution:

Solution:
Consider, $\lim_{x \to 0} \frac{\sin(\pi \cos^2 x)}{x^2}$
$= \lim_{x \to 0} \frac{\sin(\pi - \pi \sin^2 x)}{x^2}$ $[\because \sin(\pi - \theta) = \sin\theta]$
$= \lim_{x \to 0} \frac{\sin(\pi \sin^2 x)}{\pi \sin^2 x} \times \frac{(\pi \sin^2 x)}{x^2} = \pi$

Question128

$\lim_{x \to 0} \left(\frac{x - \sin x}{x}\right) \sin\left(\frac{1}{x}\right)$
[Online May 7, 2012]

Options:

A. equals 1

B. equals 0

C. does not exist

D. equals – 1

Answer: B

Solution:

Solution:

Consider $\lim_{x \to 0} \left(\frac{x - \sin x}{x} \right) \sin\left(\frac{1}{x} \right)$

$= \lim_{x \to 0} \left[\frac{x\left(1 - \frac{\sin x}{x}\right)}{x} \right] \times \lim_{x \to 0} \sin\left(\frac{1}{x} \right)$

$= \lim_{x \to 0} \left[1 - \frac{\sin x}{x} \right] \times \lim_{x \to 0} \sin\left(\frac{1}{x} \right)$

$= \left[1 - \lim_{x \to 0} \frac{\sin x}{x} \text{right} \right] \times \lim_{x \to 0} \sin\left(\frac{1}{x} \right)$

$= 0 \times \lim_{x \to 0} \sin\left(\frac{1}{x} \right) = 0$

Question129

If $f(x) = 3x^{10} - 7x^{8} + 5x^{6} - 21x^{3} + 3x^{2} - 7$, then $\lim_{\alpha \to 0} \frac{f(1 - \alpha) - f(1)}{\alpha^3 + 3\alpha}$ is

[Online May 19, 2012]

Options:

A. $-\frac{53}{3}$

B. $\frac{53}{3}$

C. $-\frac{55}{3}$

D. $\frac{55}{3}$

Answer: B

Solution:

Solution:
Let $f(x) = 3x^{10} - 7x^{8} + 5x^{6} - 21x^{3} + 3x^{2} - 7$
$f'(x) = 30x^{9} - 56x^{7} + 30x^{5} - 63x^{2} + 6x$
$f'(1) = 30 - 56 + 30 - 63 + 6$
$= 66 - 63 - 56 = -53$
Consider $\lim_{\alpha \to 0} \frac{f(1 - \alpha) - f(1)}{\alpha^3 + 3\alpha}$
$= \lim_{\alpha \to 0} \frac{f'(1 - \alpha)(-1) - 0}{3\alpha^2 + 3}$ (By using L'hospital rule)
$= \frac{f'(1 - 0)(-1)}{3(0)^2 + 3} = \frac{-f'(1)}{3} = \frac{53}{3}$

Question130

Let $f : R \to [0, \infty)$ be such that $\lim_{x \to 5} f(x)$ exists and $\lim_{x \to 5} \frac{(f(x))^2 - 9}{\sqrt{|x - 5|}} = 0$

Then $\lim_{x \to 5} f(x)$ equals :

[2011 RS]

Options:

A. 0

B. 1

C. 2

D. 3

Answer: D

Solution:

Solution:

Given that $\lim_{x \to 5} \frac{(f(x))^2 - 9}{\sqrt{|x-5|}} = 0$

$\Rightarrow \lim_{x \to 5} [(f(x))^2 - 9] = 0$

$\Rightarrow \left[\lim_{x \to 5} f(x)\right]^2 = 9 \Rightarrow \lim_{x \to 5} f(x) = 3$

Question131

$\lim_{x \to 2} \left(\frac{\sqrt{1 - \cos\{2(x-2)\}}}{x-2} \right)$

[2011]

Options:

A. equals $\sqrt{2}$

B. equals $-\sqrt{2}$

C. equals $\frac{1}{\sqrt{2}}$

D. does not exist

Answer: D

Solution:

Solution:

$\lim_{x \to 2} \frac{\sqrt{1 - \cos\{2(x-2)\}}}{x-2} \left[\because 1 - \cos\theta = 2\sin^2\frac{v}{2}\right]$

$= \lim_{x \to 2} \frac{\sqrt{2}\,|\sin(x-2)|}{x-2}$

$\underset{(at x = 2)}{L.H.L} = -\lim_{x \to 2} \frac{\sqrt{2}\sin(x-2)}{(x-2)} = -\sqrt{2}$

$\underset{(at\, x = 2)}{R.H.L} = \lim_{x \to 2} \frac{\sqrt{2}\sin(x-2)}{(x-2)} = \sqrt{2}$

Thus $\underset{(at x = 2)}{L.H.L} \neq \underset{(at\, x = 2)}{R.H.L}$

Hence, $\lim_{x \to 2} \frac{\sqrt{1 - \cos\{2(x-2)\}}}{x-2}$ does not exist.

Question132

Let f : R → R be a positive increasing function with $\lim_{x \to \infty} \frac{f(3x)}{f(x)} = 1$ then $\lim_{x \to \infty} \frac{f(2x)}{f(x)} =$
[2010]

Options:

A. $\frac{2}{3}$

B. $\frac{3}{2}$

C. 3

D. 1

Answer: D

Solution:

Solution:
Given that f(x) is a positive increasing function.
$\therefore 0 < f(x) < f(2x) < f(3x)$
Divided by $f(x)$
$\Rightarrow 0 < 1 < \frac{f(2x)}{f(x)} < \frac{f(3x)}{f(x)}$
$\Rightarrow \lim_{x \to \infty} 1 \le \lim_{x \to \infty} \frac{f(2x)}{f(x)} \le \lim_{x \to \infty} \frac{f(3x)}{f(x)}$
By Sandwich Theorem.
$\Rightarrow \lim_{x \to \infty} \frac{f(2x)}{f(x)} = 1$

Question133

Let α and β be the distinct roots of $ax^2 + bx + c = 0$,then $\lim_{x \to \alpha} \frac{1 - \cos(ax^2 + bx + c)}{(x - \alpha)^2}$ is equal to
[2005]

Options:

A. $\frac{a^2}{2}(\alpha - \beta)^2$

B. 0

C. $\frac{-a^2}{2}(\alpha - \beta)^2$

D. $\frac{1}{2}(\alpha - \beta)^2$

Answer: A

Solution:

Solution:
Given that
$ax^2 + bx + c = a(x-\alpha)(x-\beta)$
$\lim\limits_{x \to \alpha} \dfrac{1-\cos a(x-\alpha)(x-\beta)}{(x-\alpha)^2}$

$= \lim\limits_{x \to \alpha} \dfrac{2\sin^2\left(a\dfrac{(x-\alpha)(x-\beta)}{2}\right)}{(x-\alpha)^2}$

$= \lim\limits_{x \to \alpha} \dfrac{2}{(x-\alpha)^2} \times \dfrac{\sin^2\left(a\dfrac{(x-\alpha)(x-\beta)}{2}\right)}{\dfrac{a^2(x-\alpha)^2(x-\beta)^2}{4}} \times \dfrac{a^2(x-\alpha)^2(x-\beta)^2}{4}$

$= \dfrac{a^2}{2}(\alpha-\beta)^2.$

Question134

If $\lim\limits_{x \to \infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x} = e^2$, then the values of a and b, are [2004]

Options:

A. $a = 1$ and $b = 2$

B. $a = 1, b \in R$

C. $a \in R, b = 2$

D. $a \in R, b \in R$

Answer: B

Solution:

Solution:

We know that $\lim\limits_{x \to \infty}(1+x)^{\frac{1}{x}} = e$

Given that $\lim\limits_{x \to \infty}\left(1+\frac{a}{x}+\frac{b}{x^2}\right)^{2x} = e^2$

$\Rightarrow \lim\limits_{x \to \infty}\left[\left(1+\frac{a}{x}+\frac{b}{x^2}\right)\left(\dfrac{1}{\frac{a}{x}+\frac{b}{x^2}}\right)\right]^{2x\left(\frac{a}{x}+\frac{b}{x^2}\right)} = e^2$

$\Rightarrow e^{\lim\limits_{x \to \infty}\left[a+\frac{b}{x}\right]} = e^2 \Rightarrow e^{2a} = e^2$
$\Rightarrow a = 1$ and $b \in R$

Question135

$\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\left[1-\tan\left(\frac{x}{2}\right)\right][1-\sin x]}{\left[1+\tan\left(\frac{x}{2}\right)\right][\pi-2x]^3}$ is [2003]

Options:

A. ∞

B. $\frac{1}{8}$

C. 0

D. $\frac{1}{32}$

Answer: D

Solution:

Solution:

$$\lim_{x \to \frac{\pi}{2}} \frac{\tan\left(\frac{\pi}{4} - \frac{x}{2}\right) . (1 - \sin x)}{(\pi - 2x)^3}$$

Let $x = \frac{\pi}{2} + y;\ y \to 0$

$$= \lim_{y \to 0} \frac{\tan\left(-\frac{y}{2}\right) . (1 - \cos y)}{(-2y)^3}$$

$$= \lim_{y \to 0} \frac{-\tan\frac{y}{2}\, 2\sin^2\frac{y}{2}}{(-8) . \frac{y^3}{8} . 8} \left[\because 1 - \cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

$$= \lim_{y \to 0} \frac{1}{32} \frac{\tan\frac{y}{2}}{\left(\frac{y}{2}\right)} . \left[\frac{\sin y/2}{y/2}\right]^2 = \frac{1}{32} \left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = \lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1\right]$$

Question136

$\lim_{x \to 0} \frac{\log x^n - [x]}{[x]}$, n ∈ N , ([x] denotes greatest integer less than or equal to x) [2002]

Options:

A. has value -1

B. has value 0

C. has value 1

D. does not exist

Answer: D

Solution:

Solution:

Since, $\lim_{x \to 0^-} [x] = -1 \neq \lim_{x \to 0^+} [x] = 0$. So $\lim_{x \to 0} [x]$ does not exist, hence the required limit does not exist.

Question137

$$\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+2}\right)^x$$

[2002]

Options:

A. e^4

B. e^2

C. e^3

D. 1

Answer: A

Solution:

Solution:

$$\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+2}\right)^x = \lim_{x\to\infty}\left(1+\frac{4x+1}{x^2+x+2}\right)^x$$

$$= \lim_{x\to\infty}\left[\left(1+\frac{4x+1}{x^2+x+2}\right)^{\frac{x^2+x+2}{4x+1}}\right]^{\frac{(4x+1)x}{x^2+x+2}}$$

$$= e^{\lim\limits_{x\to\infty}\frac{4x^2+x}{x^2+x+2}}\left[\because \lim_{x\to\infty}(1+\lambda x)^{\frac{1}{x}} = e^{\lambda}\right]$$

$$= e^{\lim\limits_{x\to\infty}\frac{4+\frac{1}{x}}{1+\frac{1}{x}+\frac{2}{x^2}}} = e^4\left[\because \frac{1}{\infty} = 0\right]$$

Question138

$\lim\limits_{x\to0}\frac{\sqrt{1-\cos 2x}}{\sqrt{2}x}$ **is**

[2002]

Options:

A. 1

B. –1

C. zero

D. does not exist

Answer: D

Solution:

Solution:

$\lim_{x \to 0} \frac{\sqrt{1 - \cos 2x}}{\sqrt{2}x}$;

$\lim_{x \to 0} \frac{\sqrt{2\sin^2 x}}{\sqrt{2}x} \Rightarrow \lim_{x \to 0} \frac{|\sin x|}{x}$ $\left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$

The limit of above does not exist as

$LHS = -1 \neq RHL = 1$

Question139

Let $f(x) = 4$ and $f'(x) = 4$. Then $\lim_{x \to 2} \frac{xf(2) - 2f(x)}{x - 2}$ is given by [2002]

Options:

A. 2

B. -2

C. -4

D. 3

Answer: C

Solution:

Given that $f(2) = 4$ and $f(2) = 4$

We have, $\lim_{x \to 2} \frac{xf(2) - 2f(x)}{x - 2}$, $\left(\frac{0}{0}\right)$

Applying L-Hospital's rule, we get

$= \lim_{x \to 2} f(2) - 2f'(x) = f(2) - 2f'(2)$

$= 4 - 2 \times 4 = -4$
