

Mathematics

(Chapter – 8) (Binomial Theorem) (Class – XI)

Exercise 8.1

Question 1:

Expand the expression $(1 - 2x)^5$

Answer 1:

By using Binomial Theorem, the expression $(1 - 2x)^5$ can be expanded as

$$\begin{aligned} & (1 - 2x)^5 \\ &= {}^5C_0(1)^5 - {}^5C_1(1)^4(2x) + {}^5C_2(1)^3(2x)^2 - {}^5C_3(1)^2(2x)^3 + {}^5C_4(1)^1(2x)^4 - {}^5C_5(2x)^5 \\ &= 1 - 5(2x) + 10(4x^2) - 10(8x^3) + 5(16x^4) - (32x^5) \\ &= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5 \end{aligned}$$

Question 2:

Expand the expression $\left(\frac{2}{x} - \frac{x}{2}\right)^5$

Answer 2:

By using Binomial Theorem, the expression $\left(\frac{2}{x} - \frac{x}{2}\right)^5$ can be expanded as

$$\begin{aligned} \left(\frac{2}{x} - \frac{x}{2}\right)^5 &= {}^5C_0\left(\frac{2}{x}\right)^5 - {}^5C_1\left(\frac{2}{x}\right)^4\left(\frac{x}{2}\right) + {}^5C_2\left(\frac{2}{x}\right)^3\left(\frac{x}{2}\right)^2 \\ &\quad - {}^5C_3\left(\frac{2}{x}\right)^2\left(\frac{x}{2}\right)^3 + {}^5C_4\left(\frac{2}{x}\right)\left(\frac{x}{2}\right)^4 - {}^5C_5\left(\frac{x}{2}\right)^5 \\ &= \frac{32}{x^5} - 5\left(\frac{16}{x^4}\right)\left(\frac{x}{2}\right) + 10\left(\frac{8}{x^3}\right)\left(\frac{x^2}{4}\right) - 10\left(\frac{4}{x^2}\right)\left(\frac{x^3}{8}\right) + 5\left(\frac{2}{x}\right)\left(\frac{x^4}{16}\right) - \frac{x^5}{32} \\ &= \frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5}{8}x^3 - \frac{x^5}{32} \end{aligned}$$

Question 3:

Expand the expression $(2x - 3)^6$

Answer 3:

By using Binomial Theorem, the expression $(2x - 3)^6$ can be expanded as

$$\begin{aligned}
(2x-3)^6 &= {}^6C_0(2x)^6 - {}^6C_1(2x)^5(3) + {}^6C_2(2x)^4(3)^2 - {}^6C_3(2x)^3(3)^3 \\
&\quad + {}^6C_4(2x)^2(3)^4 - {}^6C_5(2x)(3)^5 + {}^6C_6(3)^6 \\
&= 64x^6 - 6(32x^5)(3) + 15(16x^4)(9) - 20(8x^3)(27) \\
&\quad + 15(4x^2)(81) - 6(2x)(243) + 729 \\
&= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729
\end{aligned}$$

Question 4:

Expand the expression $\left(\frac{x}{3} + \frac{1}{x}\right)^5$

Answer 4:

By using Binomial Theorem, the expression $\left(\frac{x}{3} + \frac{1}{x}\right)^5$ can be expanded as

$$\begin{aligned}
\left(\frac{x}{3} + \frac{1}{x}\right)^5 &= {}^5C_0\left(\frac{x}{3}\right)^5 + {}^5C_1\left(\frac{x}{3}\right)^4\left(\frac{1}{x}\right) + {}^5C_2\left(\frac{x}{3}\right)^3\left(\frac{1}{x}\right)^2 \\
&\quad + {}^5C_3\left(\frac{x}{3}\right)^2\left(\frac{1}{x}\right)^3 + {}^5C_4\left(\frac{x}{3}\right)\left(\frac{1}{x}\right)^4 + {}^5C_5\left(\frac{1}{x}\right)^5 \\
&= \frac{x^5}{243} + 5\left(\frac{x^4}{81}\right)\left(\frac{1}{x}\right) + 10\left(\frac{x^3}{27}\right)\left(\frac{1}{x^2}\right) + 10\left(\frac{x^2}{9}\right)\left(\frac{1}{x^3}\right) + 5\left(\frac{x}{3}\right)\left(\frac{1}{x^4}\right) + \frac{1}{x^5} \\
&= \frac{x^5}{243} + \frac{5x^3}{81} + \frac{10x}{27} + \frac{10}{9x} + \frac{5}{3x^3} + \frac{1}{x^5}
\end{aligned}$$

Question 5:

Expand $\left(x + \frac{1}{x}\right)^6$

Answer 5:

By using Binomial Theorem, the expression $\left(x + \frac{1}{x}\right)^6$ can be expanded as

$$\begin{aligned}
\left(x + \frac{1}{x}\right)^6 &= {}^6C_0(x)^6 + {}^6C_1(x)^5\left(\frac{1}{x}\right) + {}^6C_2(x)^4\left(\frac{1}{x}\right)^2 \\
&\quad + {}^6C_3(x)^3\left(\frac{1}{x}\right)^3 + {}^6C_4(x)^2\left(\frac{1}{x}\right)^4 + {}^6C_5(x)\left(\frac{1}{x}\right)^5 + {}^6C_6\left(\frac{1}{x}\right)^6 \\
&= x^6 + 6(x)^5\left(\frac{1}{x}\right) + 15(x)^4\left(\frac{1}{x^2}\right) + 20(x)^3\left(\frac{1}{x^3}\right) + 15(x)^2\left(\frac{1}{x^4}\right) + 6(x)\left(\frac{1}{x^5}\right) + \frac{1}{x^6} \\
&= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}
\end{aligned}$$

Question 6:

Using Binomial Theorem, evaluate $(96)^3$

Answer 6:

96 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, binomial theorem can be applied.

It can be written that, $96 = 100 - 4$

$$\begin{aligned}
\therefore (96)^3 &= (100 - 4)^3 \\
&= {}^3C_0(100)^3 - {}^3C_1(100)^2(4) + {}^3C_2(100)(4)^2 - {}^3C_3(4)^3 \\
&= (100)^3 - 3(100)^2(4) + 3(100)(4)^2 - (4)^3 \\
&= 1000000 - 120000 + 4800 - 64 \\
&= 884736
\end{aligned}$$

Question 7:

Using Binomial Theorem, evaluate $(102)^5$

Answer 7:

102 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, Binomial Theorem can be applied.

It can be written that, $102 = 100 + 2$

$$\begin{aligned}
\therefore (102)^5 &= (100+2)^5 \\
&= {}^5C_0(100)^5 + {}^5C_1(100)^4(2) + {}^5C_2(100)^3(2)^2 + {}^5C_3(100)^2(2)^3 \\
&\quad + {}^5C_4(100)(2)^4 + {}^5C_5(2)^5 \\
&= (100)^5 + 5(100)^4(2) + 10(100)^3(2)^2 + 10(100)^2(2)^3 + 5(100)(2)^4 + (2)^5 \\
&= 10000000000 + 1000000000 + 40000000 + 800000 + 8000 + 32 \\
&= 11040808032
\end{aligned}$$

Question 8:

Using Binomial Theorem, evaluate $(101)^4$

Answer 8:

101 can be expressed as the sum or difference of two numbers whose powers are easier to calculate and then, Binomial Theorem can be applied.

It can be written that, $101 = 100 + 1$

$$\begin{aligned}
\therefore (101)^4 &= (100+1)^4 \\
&= {}^4C_0(100)^4 + {}^4C_1(100)^3(1) + {}^4C_2(100)^2(1)^2 + {}^4C_3(100)(1)^3 + {}^4C_4(1)^4 \\
&= (100)^4 + 4(100)^3 + 6(100)^2 + 4(100) + (1)^4 \\
&= 100000000 + 4000000 + 60000 + 400 + 1 \\
&= 104060401
\end{aligned}$$

Question 9:

Using Binomial Theorem, evaluate $(99)^5$

Answer 9:

99 can be written as the sum or difference of two numbers whose powers are easier to calculate and then, Binomial Theorem can be applied.

It can be written that, $99 = 100 - 1$

$$\begin{aligned}
\therefore (99)^5 &= (100 - 1)^5 \\
&= {}^5C_0(100)^5 - {}^5C_1(100)^4(1) + {}^5C_2(100)^3(1)^2 - {}^5C_3(100)^2(1)^3 \\
&\quad + {}^5C_4(100)(1)^4 - {}^5C_5(1)^5 \\
&= (100)^5 - 5(100)^4 + 10(100)^3 - 10(100)^2 + 5(100) - 1 \\
&= 10000000000 - 500000000 + 10000000 - 100000 + 500 - 1 \\
&= 10010000500 - 500100001 \\
&= 9509900499
\end{aligned}$$

Question 10:

Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

Answer 10:

By splitting 1.1 and then applying Binomial Theorem, the first few terms of $(1.1)^{10000}$ can be obtained as

$$\begin{aligned}
(1.1)^{10000} &= (1 + 0.1)^{10000} \\
&= {}^{10000}C_0 + {}^{10000}C_1(1.1) + \text{Other positive terms} \\
&= 1 + 10000 \times 1.1 + \text{Other positive terms} \\
&= 1 + 11000 + \text{Other positive terms} \\
&> 1000
\end{aligned}$$

Hence, $(1.1)^{10000} > 1000$

Question 11:

Find $(a + b)^4 - (a - b)^4$. Hence, evaluate. $\left(\sqrt{3} + \sqrt{2}\right)^4 - \left(\sqrt{3} - \sqrt{2}\right)^4$

Answer 11:

Using Binomial Theorem, the expressions, $(a + b)^4$ and $(a - b)^4$, can be expanded as

$$\begin{aligned}
 (a+b)^4 &= {}^4C_0 a^4 + {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 + {}^4C_3 a b^3 + {}^4C_4 b^4 \\
 (a-b)^4 &= {}^4C_0 a^4 - {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 - {}^4C_3 a b^3 + {}^4C_4 b^4 \\
 \therefore (a+b)^4 - (a-b)^4 &= {}^4C_0 a^4 + {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 + {}^4C_3 a b^3 + {}^4C_4 b^4 \\
 &\quad - [{}^4C_0 a^4 - {}^4C_1 a^3 b + {}^4C_2 a^2 b^2 - {}^4C_3 a b^3 + {}^4C_4 b^4] \\
 &= 2({}^4C_1 a^3 b + {}^4C_3 a b^3) = 2(4a^3 b + 4ab^3) \\
 &= 8ab(a^2 + b^2)
 \end{aligned}$$

By putting $a = \sqrt{3}$ and $b = \sqrt{2}$, we obtain

$$\begin{aligned}
 (\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4 &= 8(\sqrt{3})(\sqrt{2}) \left[(\sqrt{3})^2 + (\sqrt{2})^2 \right] \\
 &= 8(\sqrt{6}) \{3 + 2\} = 40\sqrt{6}
 \end{aligned}$$

Question 12:

Find $(x+1)^6 + (x-1)^6$. Hence or otherwise evaluate. $(\sqrt{2}+1)^6 + (\sqrt{2}-1)^6$

Answer 12:

Using Binomial Theorem, the expressions, $(x+1)^6$ and $(x-1)^6$, can be expanded as

$$\begin{aligned}
 (x+1)^6 &= {}^6C_0 x^6 + {}^6C_1 x^5 + {}^6C_2 x^4 + {}^6C_3 x^3 + {}^6C_4 x^2 + {}^6C_5 x + {}^6C_6 \\
 (x-1)^6 &= {}^6C_0 x^6 - {}^6C_1 x^5 + {}^6C_2 x^4 - {}^6C_3 x^3 + {}^6C_4 x^2 - {}^6C_5 x + {}^6C_6 \\
 \therefore (x+1)^6 + (x-1)^6 &= 2[{}^6C_0 x^6 + {}^6C_2 x^4 + {}^6C_4 x^2 + {}^6C_6] \\
 &= 2[x^6 + 15x^4 + 15x^2 + 1]
 \end{aligned}$$

By putting $x = \sqrt{2}$ we obtain

$$\begin{aligned}
 (\sqrt{2}+1)^6 + (\sqrt{2}-1)^6 &= 2[(\sqrt{2})^6 + 15(\sqrt{2})^4 + 15(\sqrt{2})^2 + 1] \\
 &= 2(8 + 15 \times 4 + 15 \times 2 + 1) \\
 &= 2(8 + 60 + 30 + 1) \\
 &= 2(99) = 198
 \end{aligned}$$

Question 13:

Show that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.

Answer 13:

In order to show that $9^{n+1} - 8n - 9$ is divisible by 64, it has to be proved that, $9^{n+1} - 8n - 9 = 64k$, where k is some natural number

By Binomial Theorem,

$$(1+a)^m = {}^m C_0 + {}^m C_1 a + {}^m C_2 a^2 + \dots + {}^m C_m a^m$$

For $a = 8$ and $m = n + 1$, we obtain

$$\begin{aligned} (1+8)^{n+1} &= {}^{n+1} C_0 + {}^{n+1} C_1 (8) + {}^{n+1} C_2 (8)^2 + \dots + {}^{n+1} C_{n+1} (8)^{n+1} \\ \Rightarrow 9^{n+1} &= 1 + (n+1)(8) + 8^2 \left[{}^{n+1} C_2 + {}^{n+1} C_3 \times 8 + \dots + {}^{n+1} C_{n+1} (8)^{n-1} \right] \\ \Rightarrow 9^{n+1} &= 9 + 8n + 64 \left[{}^{n+1} C_2 + {}^{n+1} C_3 \times 8 + \dots + {}^{n+1} C_{n+1} (8)^{n-1} \right] \\ \Rightarrow 9^{n+1} - 8n - 9 &= 64k, \text{ where } k = {}^{n+1} C_2 + {}^{n+1} C_3 \times 8 + \dots + {}^{n+1} C_{n+1} (8)^{n-1} \text{ is a natural number} \end{aligned}$$

Thus, $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.

Question 14:

Prove that. $\sum_{r=0}^n 3^r {}^n C_r = 4^n$

Answer 14:

By Binomial Theorem,

$$\sum_{r=0}^n {}^n C_r a^{n-r} b^r = (a+b)^n$$

By putting $b = 3$ and $a = 1$ in the above equation, we obtain

$$\sum_{r=0}^n {}^n C_r (1)^{n-r} (3)^r = (1+3)^n$$

$$\Rightarrow \sum_{r=0}^n 3^r {}^n C_r = 4^n$$

Hence, proved.

Mathematics

(Chapter – 8) (Binomial Theorem) (Class – XI)

Exercise 8.2

Question 1:

Find the coefficient of x^5 in $(x + 3)^8$

Answer 1:

It is known that $(r + 1)^{\text{th}}$ term, (T_{r+1}) , in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^n C_r a^{n-r} b^r$$

Assuming that x^5 occurs in the $(r + 1)^{\text{th}}$ term of the expansion $(x + 3)^8$, we obtain

$$T_{r+1} = {}^8 C_r (x)^{8-r} (3)^r$$

Comparing the indices of x in x^5 and in T_{r+1} ,

we obtain $r = 3$

$$\text{Thus, the coefficient of } x^5 \text{ is } {}^8 C_3 (3)^3 = \frac{8!}{3!5!} \times 3^3 = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3 \cdot 2 \cdot 5!} \cdot 3^3 = 1512$$

Question 2:

Find the coefficient of a^5b^7 in $(a - 2b)^{12}$

Answer 2:

It is known that $(r + 1)^{\text{th}}$ term, (T_{r+1}) , in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^n C_r a^{n-r} b^r$$

Assuming that a^5b^7 occurs in the $(r + 1)^{\text{th}}$ term of the expansion $(a - 2b)^{12}$, we obtain

$$T_{r+1} = {}^{12} C_r (a)^{12-r} (-2b)^r = {}^{12} C_r (-2)^r (a)^{12-r} (b)^r$$

Comparing the indices of a and b in $a^5 b^7$ and in T_{r+1} ,

we obtain $r = 7$

Thus, the coefficient of a^5b^7 is

$${}^{12} C_7 (-2)^7 = -\frac{12!}{7!5!} \cdot 2^7 = -\frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 7!} \cdot 2^7 = -(792)(128) = -101376$$

Question 3:

Write the general term in the expansion of $(x^2 - y)^6$

Answer 3:

It is known that the general term T_{r+1} {which is the $(r + 1)^{\text{th}}$ term} in the binomial expansion of $(a + b)^n$ is given by $T_{r+1} = {}^n C_r a^{n-r} b^r$.

Thus, the general term in the expansion of $(x^2 - y^6)$ is

$$T_{r+1} = {}^6 C_r (x^2)^{6-r} (-y)^r = (-1)^r {}^6 C_r x^{12-2r} \cdot y^r$$

Question 4:

Write the general term in the expansion of $(x^2 - yx)^{12}$, $x \neq 0$

Answer 4:

It is known that the general term T_{r+1} {which is the $(r + 1)^{\text{th}}$ term} in the binomial expansion of $(a + b)^n$ is given by $T_{r+1} = {}^n C_r a^{n-r} b^r$.

Thus, the general term in the expansion of $(x^2 - yx)^{12}$ is

$$T_{r+1} = {}^{12} C_r (x^2)^{12-r} (-yx)^r = (-1)^r {}^{12} C_r x^{24-2r} \cdot y^r \cdot x^r = (-1)^r {}^{12} C_r x^{24-r} \cdot y^r$$

Question 5:

Find the 4th term in the expansion of $(x - 2y)^{12}$.

Answer 5:

It is known that $(r + 1)^{\text{th}}$ term, (T_{r+1}) , in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^n C_r a^{n-r} b^r$$

Thus, the 4th term in the expansion of $(x - 2y)^{12}$ is

$$T_4 = T_{3+1} = {}^{12} C_3 (x)^{12-3} (-2y)^3 = (-1)^3 \cdot \frac{12!}{3!9!} \cdot x^9 \cdot (2)^3 \cdot y^3 = -\frac{12 \cdot 11 \cdot 10}{3 \cdot 2} \cdot (2)^3 x^9 y^3 = -1760 x^9 y^3$$

Question 6:

Find the 13th term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$, $x \neq 0$

Answer 6:

It is known that $(r + 1)^{\text{th}}$ term, (T_{r+1}) , in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^n C_r a^{n-r} b^r$$

Thus, 13th term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$ is

$$\begin{aligned}
 T_{13} &= T_{12+1} = {}^{18}C_{12} (9x)^{18-12} \left(-\frac{1}{3\sqrt{x}}\right)^{12} \\
 &= (-1)^{12} \frac{18!}{12!6!} (9)^6 (x)^6 \left(\frac{1}{3}\right)^{12} \left(\frac{1}{\sqrt{x}}\right)^{12} \\
 &= \frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12!}{12! \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} \cdot x^6 \cdot \left(\frac{1}{x^6}\right) \cdot 3^{12} \left(\frac{1}{3^{12}}\right) \\
 &= 18564
 \end{aligned}$$

Question 7:

Find the middle terms in the expansions of $\left(3 - \frac{x^3}{6}\right)^7$

Answer 7:

It is known that in the expansion of $(a + b)^n$, if n is odd, then there are two middle terms,

namely $\left(\frac{n+1}{2}\right)^{\text{th}}$ term and $\left(\frac{n+1}{2}+1\right)^{\text{th}}$ term.

Therefore, the middle terms in the expansion $\left(3 - \frac{x^3}{6}\right)^7$ are $\left(\frac{7+1}{2}\right)^{\text{th}} = 4^{\text{th}}$ term and $\left(\frac{7+1}{2}+1\right)^{\text{th}} = 5^{\text{th}}$ term

$$\begin{aligned}
 T_4 &= T_{3+1} = {}^7C_3 (3)^{7-3} \left(-\frac{x^3}{6}\right)^3 = (-1)^3 \frac{7!}{3!4!} \cdot 3^4 \cdot \frac{x^9}{6^3} \\
 &= -\frac{7 \cdot 6 \cdot 5 \cdot 4!}{3 \cdot 2 \cdot 4!} \cdot 3^4 \cdot \frac{1}{2^3 \cdot 3^3} \cdot x^9 = -\frac{105}{8} x^9 \\
 T_5 &= T_{4+1} = {}^7C_4 (3)^{7-4} \left(-\frac{x^3}{6}\right)^4 = (-1)^4 \frac{7!}{4!3!} (3)^3 \cdot \frac{x^{12}}{6^4} \\
 &= \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4! \cdot 3 \cdot 2} \cdot \frac{3^3}{2^4 \cdot 3^4} \cdot x^{12} = \frac{35}{48} x^{12}
 \end{aligned}$$

Thus, the middle terms in the expansion of $\left(3 - \frac{x^3}{6}\right)^7$ are $-\frac{105}{8}x^9$ and $\frac{35}{48}x^{12}$

Question 8:

Find the middle terms in the expansions of $\left(\frac{x}{3} + 9y\right)^{10}$

Answer 8:

It is known that in the expansion $(a + b)^n$, if n is even, then the middle term is $\left(\frac{n}{2} + 1\right)^{\text{th}}$ term.

Therefore, the middle term in the expansion of $\left(\frac{x}{3} + 9y\right)^{10}$ is $\left(\frac{10}{2} + 1\right)^{\text{th}} = 6^{\text{th}}$

$$\begin{aligned} T_6 = T_{5+1} &= {}^{10}C_5 \left(\frac{x}{3}\right)^{10-5} (9y)^5 = \frac{10!}{5!5!} \cdot \frac{x^5}{3^5} \cdot 9^5 \cdot y^5 \\ &= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 5!} \cdot \frac{1}{3^5} \cdot 3^{10} \cdot x^5 y^5 & [9^5 = (3^2)^5 = 3^{10}] \\ &= 252 \times 3^5 \cdot x^5 \cdot y^5 = 61236x^5y^5 \end{aligned}$$

Thus, the middle term in the expansion of $\left(\frac{x}{3} + 9y\right)^{10}$ is $61236x^5y^5$.

Question 9:

In the expansion of $(1 + a)^{m+n}$, prove that coefficients of a^m and a^n are equal.

Answer 9:

It is known that $(r + 1)^{\text{th}}$ term, (T_{r+1}) , in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r$$

Assuming that a^m occurs in the $(r + 1)^{\text{th}}$ term of the expansion $(1 + a)^{m+n}$, we obtain

$$T_{r+1} = {}^{m+n}C_r (1)^{m+n-r} (a)^r = {}^{m+n}C_r a^r$$

Comparing the indices of a in a^m and in T_{r+1} ,

we obtain

$$r = m$$

Therefore, the coefficient of a^m is

$${}^{m+n}C_m = \frac{(m+n)!}{m!(m+n-m)!} = \frac{(m+n)!}{m!n!} \quad \dots(1)$$

Assuming that a^n occurs in the $(k+1)^{\text{th}}$ term of the expansion $(1+a)^{m+n}$,

we obtain

$$T_{k+1} = {}^{m+n}C_k (1)^{m+n-k} (a)^k = {}^{m+n}C_k (a)^k$$

Comparing the indices of a in a^n and in T_{k+1} ,

we obtain

$$k = n$$

Therefore, the coefficient of a^n is

$${}^{m+n}C_n = \frac{(m+n)!}{n!(m+n-n)!} = \frac{(m+n)!}{n!m!} \quad \dots(2)$$

Thus, from (1) and (2), it can be observed that the coefficients of a^m and a^n in the expansion of $(1+a)^{m+n}$ are equal.

Question 10:

The coefficients of the $(r-1)^{\text{th}}$, r^{th} and $(r+1)^{\text{th}}$ terms in the expansion of $(x+1)^n$ are in the ratio 1:3:5. Find n and r .

Answer 10:

It is known that $(k+1)^{\text{th}}$ term, (T_{k+1}) , in the binomial expansion of $(a+b)^n$ is given by

$$T_{k+1} = {}^nC_k a^{n-k} b^k$$

Therefore, $(r-1)^{\text{th}}$ term in the expansion of $(x+1)^n$ is

$$T_{r-1} = {}^nC_{r-2} (x)^{n-(r-2)} (1)^{(r-2)} = {}^nC_{r-2} x^{n-r+2}$$

$(r+1)$ term in the expansion of $(x+1)^n$ is

$$T_{r+1} = {}^nC_r (x)^{n-r} (1)^r = {}^nC_r x^{n-r}$$

r^{th} term in the expansion of $(x + 1)^n$ is

$$T_r = {}^n C_{r-1} (x)^{n-(r-1)} (1)^{(r-1)} = {}^n C_{r-1} x^{n-r+1}$$

Therefore, the coefficients of the $(r - 1)^{\text{th}}$, r^{th} , and $(r + 1)^{\text{th}}$ terms in the expansion of $(x + 1)^n$ are ${}^n C_{r-2}$, ${}^n C_{r-1}$, and ${}^n C_r$ respectively. Since these coefficients are in the ratio 1:3:5, we obtain

$$\frac{{}^n C_{r-2}}{{}^n C_{r-1}} = \frac{1}{3} \quad \text{and} \quad \frac{{}^n C_{r-1}}{{}^n C_r} = \frac{3}{5}$$

$$\begin{aligned}\frac{{}^n C_{r-2}}{{}^n C_{r-1}} &= \frac{n!}{(r-2)!(n-r+2)!} \times \frac{(r-1)!(n-r+1)!}{n!} = \frac{(r-1)(r-2)!(n-r+1)!}{(r-2)!(n-r+2)(n-r+1)!} \\ &= \frac{r-1}{n-r+2}\end{aligned}$$

$$\therefore \frac{r-1}{n-r+2} = \frac{1}{3}$$

$$\Rightarrow 3r-3 = n-r+2$$

$$\Rightarrow n-4r+5=0 \quad \dots(1)$$

$$\begin{aligned}\frac{{}^n C_{r-1}}{{}^n C_r} &= \frac{n!}{(r-1)!(n-r+1)!} \times \frac{r!(n-r)!}{n!} = \frac{r(r-1)!(n-r)!}{(r-1)!(n-r+1)(n-r)!} \\ &= \frac{r}{n-r+1}\end{aligned}$$

$$\therefore \frac{r}{n-r+1} = \frac{3}{5}$$

$$\Rightarrow 5r = 3n - 3r + 3$$

$$\Rightarrow 3n - 8r + 3 = 0 \quad \dots(2)$$

Multiplying (1) by 3 and subtracting it from (2), we obtain

$$4r - 12 = 0$$

$$\Rightarrow r = 3$$

Putting the value of r in (1), we obtain n

$$-12 + 5 = 0$$

$$\Rightarrow n = 7$$

Thus, $n = 7$ and $r = 3$

Question 11:

Prove that the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

Answer 11:

It is known that $(r + 1)^{\text{th}}$ term, (T_{r+1}) , in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^n C_r a^{n-r} b^r$$

Assuming that x^n occurs in the $(r + 1)^{\text{th}}$ term of the expansion of $(1 + x)^{2n}$, we obtain

$$T_{r+1} = {}^{2n} C_r (1)^{2n-r} (x)^r = {}^{2n} C_r (x)^r$$

Comparing the indices of x in x^n and in T_{r+1} , we obtain $r = n$

Therefore, the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is

$${}^{2n} C_n = \frac{(2n)!}{n!(2n-n)!} = \frac{(2n)!}{n!n!} = \frac{(2n)!}{(n!)^2} \quad \dots(1)$$

Assuming that x^n occurs in the $(k + 1)^{\text{th}}$ term of the expansion $(1 + x)^{2n-1}$, we obtain

$$T_{k+1} = {}^{2n-1} C_k (1)^{2n-1-k} (x)^k = {}^{2n-1} C_k (x)^k$$

Comparing the indices of x in x^n and T_{k+1} , we obtain $k = n$

Therefore, the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$ is

$$\begin{aligned} {}^{2n-1} C_n &= \frac{(2n-1)!}{n!(2n-1-n)!} = \frac{(2n-1)!}{n!(n-1)!} \\ &= \frac{2n.(2n-1)!}{2n.n!(n-1)!} = \frac{(2n)!}{2.n!n!} = \frac{1}{2} \left[\frac{(2n)!}{(n!)^2} \right] \quad \dots(2) \end{aligned}$$

From (1) and (2), it is observed that

$$\begin{aligned} \frac{1}{2} ({}^{2n} C_n) &= {}^{2n-1} C_n \\ \Rightarrow {}^{2n} C_n &= 2 ({}^{2n-1} C_n) \end{aligned}$$

Therefore,

the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

Hence, proved.

Question 12:

Find a positive value of m for which the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

Answer 12:

It is known that $(r + 1)^{\text{th}}$ term, (T_{r+1}) , in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^n C_r a^{n-r} b^r$$

Assuming that x^2 occurs in the $(r + 1)^{\text{th}}$ term of the expansion $(1 + x)^m$, we obtain

$$T_{r+1} = {}^m C_r (1)^{m-r} (x)^r = {}^m C_r (x)^r$$

Comparing the indices of x in x^2 and in T_{r+1} , we obtain $r = 2$

Therefore, the coefficient of x^2 is ${}^m C_2$

It is given that the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

$$\therefore {}^m C_2 = 6$$

$$\Rightarrow \frac{m!}{2!(m-2)!} = 6$$

$$\Rightarrow \frac{m(m-1)(m-2)!}{2 \times (m-2)!} = 6$$

$$\Rightarrow m(m-1) = 12$$

$$\Rightarrow m^2 - m - 12 = 0$$

$$\Rightarrow m^2 - 4m + 3m - 12 = 0$$

$$\Rightarrow m(m-4) + 3(m-4) = 0$$

$$\Rightarrow (m-4)(m+3) = 0$$

$$\Rightarrow (m-4) = 0 \text{ or } (m+3) = 0$$

$$\Rightarrow m = 4 \text{ or } m = -3$$

Thus, the positive value of m , for which the coefficient of x^2 in the expansion $(1 + x)^m$ is

6, is 4.

Mathematics

(Chapter – 8) (Binomial Theorem)
(Class – XI)

Miscellaneous Exercise on Chapter 8

Question 1:

Find a , b and n in the expansion of $(a + b)^n$ if the first three terms of the expansion are 729, 7290 and 30375, respectively.

Answer 1:

It is known that $(r + 1)^{\text{th}}$ term, (T_{r+1}) , in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^n C_r a^{n-r} b^r$$

The first three terms of the expansion are given as 729, 7290, and 30375 respectively.

Therefore, we obtain

$$T_1 = {}^n C_0 a^{n-0} b^0 = a^n = 729 \quad \dots(1)$$

$$T_2 = {}^n C_1 a^{n-1} b^1 = n a^{n-1} b = 7290 \quad \dots(2)$$

$$T_3 = {}^n C_2 a^{n-2} b^2 = \frac{n(n-1)}{2} a^{n-2} b^2 = 30375 \quad \dots(3)$$

Dividing (2) by (1), we obtain

$$\begin{aligned} \frac{n a^{n-1} b}{a^n} &= \frac{7290}{729} \\ \Rightarrow \frac{nb}{a} &= 10 \end{aligned} \quad \dots(4)$$

Dividing (3) by (2), we obtain

$$\begin{aligned} \frac{n(n-1)a^{n-2}b^2}{2na^{n-1}b} &= \frac{30375}{7290} \\ \Rightarrow \frac{(n-1)b}{2a} &= \frac{30375}{7290} \\ \Rightarrow \frac{(n-1)b}{a} &= \frac{30375 \times 2}{7290} = \frac{25}{3} \\ \Rightarrow \frac{nb}{a} - \frac{b}{a} &= \frac{25}{3} \\ \Rightarrow 10 - \frac{b}{a} &= \frac{25}{3} \quad [\text{Using (4)}] \\ \Rightarrow \frac{b}{a} &= 10 - \frac{25}{3} = \frac{5}{3} \end{aligned} \quad \dots(5)$$

From (4) and (5), we obtain

$$n \cdot \frac{5}{3} = 10$$

$$\Rightarrow n = 6$$

Substituting $n = 6$ in equation (1), we obtain a^6

$$= 729$$

$$\Rightarrow a = \sqrt[6]{729} = 3$$

From (5), we obtain

$$\frac{b}{3} = \frac{5}{3} \Rightarrow b = 5$$

Thus, $a = 3$, $b = 5$, and $n = 6$.

Question 2:

Find a if the coefficients of x^2 and x^3 in the expansion of $(3 + ax)^9$ are equal.

Answer 2:

It is known that $(r + 1)^{\text{th}}$ term, (T_{r+1}) , in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = {}^n C_r a^{n-r} b^r$$

Assuming that x^2 occurs in the $(r + 1)^{\text{th}}$ term in the expansion of $(3 + ax)^9$, we obtain

$$T_{r+1} = {}^9 C_r (3)^{9-r} (ax)^r = {}^9 C_r (3)^{9-r} a^r x^r$$

Comparing the indices of x in x^2 and in T_{r+1} , we obtain

$$r = 2$$

Thus, the coefficient of x^2 is

$${}^9 C_2 (3)^{9-2} a^2 = \frac{9!}{2!7!} (3)^7 a^2 = 36 (3)^7 a^2$$

Assuming that x^3 occurs in the $(k + 1)^{\text{th}}$ term in the expansion of $(3 + ax)^9$, we obtain

$$T_{k+1} = {}^9 C_k (3)^{9-k} (ax)^k = {}^9 C_k (3)^{9-k} a^k x^k$$

Comparing the indices of x in x^3 and in T_{k+1} , we obtain $k = 3$

Thus, the coefficient of x^3 is

$${}^9 C_3 (3)^{9-3} a^3 = \frac{9!}{3!6!} (3)^6 a^3 = 84 (3)^6 a^3$$

It is given that the coefficients of x^2 and x^3 are the same.

$$84(3)^6 a^3 = 36(3)^7 a^2$$

$$\Rightarrow 84a = 36 \times 3$$

$$\Rightarrow a = \frac{36 \times 3}{84} = \frac{104}{84}$$

$$\Rightarrow a = \frac{9}{7}$$

Thus, the required value of a is $9/7$.

Question 3:

Find the coefficient of x^5 in the product $(1 + 2x)^6 (1 - x)^7$ using binomial theorem.

Answer 3:

Using Binomial Theorem, the expressions, $(1 + 2x)^6$ and $(1 - x)^7$, can be expanded as

$$\begin{aligned}(1+2x)^6 &= {}^6C_0 + {}^6C_1(2x) + {}^6C_2(2x)^2 + {}^6C_3(2x)^3 + {}^6C_4(2x)^4 \\ &\quad + {}^6C_5(2x)^5 + {}^6C_6(2x)^6 \\ &= 1 + 6(2x) + 15(2x)^2 + 20(2x)^3 + 15(2x)^4 + 6(2x)^5 + (2x)^6 \\ &= 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6\end{aligned}$$

$$\begin{aligned}(1-x)^7 &= {}^7C_0 - {}^7C_1(x) + {}^7C_2(x)^2 - {}^7C_3(x)^3 + {}^7C_4(x)^4 \\ &\quad - {}^7C_5(x)^5 + {}^7C_6(x)^6 - {}^7C_7(x)^7 \\ &= 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7 \\ \therefore (1+2x)^6 (1-x)^7 &= (1 + 12x + 60x^2 + 160x^3 + 240x^4 + 192x^5 + 64x^6)(1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7)\end{aligned}$$

The complete multiplication of the two brackets is not required to be carried out. Only those terms, which involve x^5 , are required.

The terms containing x^5 are

$$\begin{aligned}1(-21x^5) + (12x)(35x^4) + (60x^2)(-35x^3) + (160x^3)(21x^2) + (240x^4)(-7x) + (192x^5)(1) \\ = 171x^5\end{aligned}$$

Thus, the coefficient of x^5 in the given product is 171.

Question 4:

If a and b are distinct integers, prove that $a - b$ is a factor of $a^n - b^n$, whenever n is a positive integer. [Hint: write $a^n = (a - b + b)^n$ and expand]

Answer 4:

In order to prove that $(a - b)$ is a factor of $(a^n - b^n)$, it has to be proved that

$a^n - b^n = k(a - b)$, where k is some natural number

It can be written that, $a = a - b + b$

$$\begin{aligned} \therefore a^n &= (a - b + b)^n = [(a - b) + b]^n \\ &= {}^nC_0(a - b)^n + {}^nC_1(a - b)^{n-1}b + \dots + {}^nC_{n-1}(a - b)b^{n-1} + {}^nC_nb^n \\ &= (a - b)^n + {}^nC_1(a - b)^{n-1}b + \dots + {}^nC_{n-1}(a - b)b^{n-1} + b^n \\ \Rightarrow a^n - b^n &= (a - b)[(a - b)^{n-1} + {}^nC_1(a - b)^{n-2}b + \dots + {}^nC_{n-1}b^{n-1}] \\ \Rightarrow a^n - b^n &= k(a - b) \end{aligned}$$

where, $k = [(a - b)^{n-1} + {}^nC_1(a - b)^{n-2}b + \dots + {}^nC_{n-1}b^{n-1}]$ is a natural number

This shows that $(a - b)$ is a factor of $(a^n - b^n)$, where n is a positive integer.

Question 5:

Evaluate. $(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6$

Answer 5:

Firstly, the expression $(a + b)^6 - (a - b)^6$ is simplified by using Binomial Theorem. This can be done as

$$\begin{aligned} (a + b)^6 &= {}^6C_0a^6 + {}^6C_1a^5b + {}^6C_2a^4b^2 + {}^6C_3a^3b^3 + {}^6C_4a^2b^4 + {}^6C_5a^1b^5 + {}^6C_6b^6 \\ &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6 \\ (a - b)^6 &= {}^6C_0a^6 - {}^6C_1a^5b + {}^6C_2a^4b^2 - {}^6C_3a^3b^3 + {}^6C_4a^2b^4 - {}^6C_5a^1b^5 + {}^6C_6b^6 \\ &= a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6 \\ \therefore (a + b)^6 - (a - b)^6 &= 2[6a^5b + 20a^3b^3 + 6ab^5] \end{aligned}$$

Putting $a = \sqrt{3}$ and $b = \sqrt{2}$, we obtain

$$\begin{aligned} (\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6 &= 2[6(\sqrt{3})^5(\sqrt{2}) + 20(\sqrt{3})^3(\sqrt{2})^3 + 6(\sqrt{3})(\sqrt{2})^5] \\ &= 2[54\sqrt{6} + 120\sqrt{6} + 24\sqrt{6}] \\ &= 2 \times 198\sqrt{6} \\ &= 396\sqrt{6} \end{aligned}$$

Question 6:

Find the value of $\left(a^2 + \sqrt{a^2 - 1}\right)^4 + \left(a^2 - \sqrt{a^2 - 1}\right)^4$

Answer 6:

Firstly, the expression $(x + y)^4 + (x - y)^4$ is simplified by using Binomial Theorem.

This can be done as

$$\begin{aligned}(x+y)^4 &= {}^4C_0x^4 + {}^4C_1x^3y + {}^4C_2x^2y^2 + {}^4C_3xy^3 + {}^4C_4y^4 \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

$$\begin{aligned}(x-y)^4 &= {}^4C_0x^4 - {}^4C_1x^3y + {}^4C_2x^2y^2 - {}^4C_3xy^3 + {}^4C_4y^4 \\ &= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4\end{aligned}$$

$$\therefore (x+y)^4 + (x-y)^4 = 2(x^4 + 6x^2y^2 + y^4)$$

Putting $x = a^2$ and $y = \sqrt{a^2 - 1}$, we obtain

$$\begin{aligned}\left(a^2 + \sqrt{a^2 - 1}\right)^4 + \left(a^2 - \sqrt{a^2 - 1}\right)^4 &= 2 \left[\left(a^2\right)^4 + 6\left(a^2\right)^2 \left(\sqrt{a^2 - 1}\right)^2 + \left(\sqrt{a^2 - 1}\right)^4 \right] \\ &= 2 \left[a^8 + 6a^4(a^2 - 1) + (a^2 - 1)^2 \right] \\ &= 2 \left[a^8 + 6a^6 - 6a^4 + a^4 - 2a^2 + 1 \right] \\ &= 2 \left[a^8 + 6a^6 - 5a^4 - 2a^2 + 1 \right] \\ &= 2a^8 + 12a^6 - 10a^4 - 4a^2 + 2\end{aligned}$$

Question 7:

Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

Answer 7:

$$0.99 = 1 - 0.01$$

$$\begin{aligned}\therefore (0.99)^5 &= (1 - 0.01)^5 \\ &= {}^5C_0(1)^5 - {}^5C_1(1)^4(0.01) + {}^5C_2(1)^3(0.01)^2 \quad (\text{Approximately}) \\ &= 1 - 5(0.01) + 10(0.01)^2 \\ &= 1 - 0.05 + 0.001 \\ &= 1.001 - 0.05 \\ &= 0.951\end{aligned}$$

Thus, the value of $(0.99)^5$ is approximately 0.951.

Question 8:

Find n , if the ratio of the fifth term from the beginning to the fifth term from the end in

the expansion of $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$ is $\sqrt{6} : 1$

Answer 8:

In the expansion, $(a+b)^n = {}^n C_0 a^n + {}^n C_1 a^{n-1} b + {}^n C_2 a^{n-2} b^2 + \dots + {}^n C_{n-1} a b^{n-1} + {}^n C_n b^n$

Fifth term from the beginning $= {}^n C_4 a^{n-4} b^4$

Fifth term from the end $= {}^n C_{n-4} a^4 b^{n-4}$

Therefore, it is evident that in the expansion of $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$ the fifth term from the

beginning is ${}^n C_4 \left(\sqrt[4]{2}\right)^{n-4} \left(\frac{1}{\sqrt[4]{3}}\right)^4$ and the fifth term from the end is ${}^n C_{n-4} \left(\sqrt[4]{2}\right)^4 \left(\frac{1}{\sqrt[4]{3}}\right)^{n-4}$

$${}^n C_4 \left(\sqrt[4]{2}\right)^{n-4} \left(\frac{1}{\sqrt[4]{3}}\right)^4 = {}^n C_4 \frac{\left(\sqrt[4]{2}\right)^n}{\left(\sqrt[4]{2}\right)^4} \cdot \frac{1}{3} = {}^n C_4 \frac{\left(\sqrt[4]{2}\right)^n}{2} \cdot \frac{1}{3} = \frac{n!}{6 \cdot 4! (n-4)!} \left(\sqrt[4]{2}\right)^n \quad \dots(1)$$

$${}^n C_{n-4} \left(\sqrt[4]{2}\right)^4 \left(\frac{1}{\sqrt[4]{3}}\right)^{n-4} = {}^n C_{n-4} \cdot 2 \cdot \frac{\left(\sqrt[4]{3}\right)^4}{\left(\sqrt[4]{3}\right)^n} = {}^n C_{n-4} \cdot 2 \cdot \frac{3}{\left(\sqrt[4]{3}\right)^n} = \frac{6n!}{(n-4)! 4!} \cdot \frac{1}{\left(\sqrt[4]{3}\right)^n} \quad \dots(2)$$

It is given that the ratio of the fifth term from the beginning to the fifth term from the end is $\sqrt{6} : 1$. Therefore, from (1) and (2), we obtain

$$\frac{n!}{6 \cdot 4! (n-4)!} \left(\sqrt[4]{2}\right)^n : \frac{6n!}{(n-4)! 4!} \cdot \frac{1}{\left(\sqrt[4]{3}\right)^n} = \sqrt{6} : 1$$

$$\Rightarrow \frac{\left(\sqrt[4]{2}\right)^n}{6} : \frac{6}{\left(\sqrt[4]{3}\right)^n} = \sqrt{6} : 1$$

$$\Rightarrow \frac{\left(\sqrt[4]{2}\right)^n}{6} \times \frac{\left(\sqrt[4]{3}\right)^n}{6} = \sqrt{6}$$

$$\Rightarrow \left(\sqrt[4]{6}\right)^n = 36\sqrt{6}$$

$$\Rightarrow 6^{\frac{n}{4}} = 6^{\frac{5}{2}}$$

$$\Rightarrow \frac{n}{4} = \frac{5}{2}$$

$$\Rightarrow n = 4 \times \frac{5}{2} = 10$$

Thus, the value of n is 10.

Question 9:

Expand using Binomial Theorem $\left(1 + \frac{x}{2} - \frac{2}{x}\right)^4, x \neq 0$

Answer 9:

Using Binomial Theorem, the given expression $\left(1 + \frac{x}{2} - \frac{2}{x}\right)^4$ can be expanded as

$$\begin{aligned}
 & \left[\left(1 + \frac{x}{2}\right) - \frac{2}{x} \right]^4 \\
 &= {}^4C_0 \left(1 + \frac{x}{2}\right)^4 - {}^4C_1 \left(1 + \frac{x}{2}\right)^3 \left(\frac{2}{x}\right) + {}^4C_2 \left(1 + \frac{x}{2}\right)^2 \left(\frac{2}{x}\right)^2 - {}^4C_3 \left(1 + \frac{x}{2}\right) \left(\frac{2}{x}\right)^3 + {}^4C_4 \left(\frac{2}{x}\right)^4 \\
 &= \left(1 + \frac{x}{2}\right)^4 - 4 \left(1 + \frac{x}{2}\right)^3 \left(\frac{2}{x}\right) + 6 \left(1 + x + \frac{x^2}{4}\right) \left(\frac{4}{x^2}\right) - 4 \left(1 + \frac{x}{2}\right) \left(\frac{8}{x^3}\right) + \frac{16}{x^4} \\
 &= \left(1 + \frac{x}{2}\right)^4 - \frac{8}{x} \left(1 + \frac{x}{2}\right)^3 + \frac{24}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} - \frac{16}{x^2} + \frac{16}{x^4} \\
 &= \left(1 + \frac{x}{2}\right)^4 - \frac{8}{x} \left(1 + \frac{x}{2}\right)^3 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4} \quad \dots(1)
 \end{aligned}$$

Again by using Binomial Theorem, we obtain

$$\begin{aligned}
 \left(1 + \frac{x}{2}\right)^4 &= {}^4C_0 (1)^4 + {}^4C_1 (1)^3 \left(\frac{x}{2}\right) + {}^4C_2 (1)^2 \left(\frac{x}{2}\right)^2 + {}^4C_3 (1)^1 \left(\frac{x}{2}\right)^3 + {}^4C_4 \left(\frac{x}{2}\right)^4 \\
 &= 1 + 4 \times \frac{x}{2} + 6 \times \frac{x^2}{4} + 4 \times \frac{x^3}{8} + \frac{x^4}{16} \\
 &= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} \quad \dots(2)
 \end{aligned}$$

$$\begin{aligned}
 \left(1 + \frac{x}{2}\right)^3 &= {}^3C_0 (1)^3 + {}^3C_1 (1)^2 \left(\frac{x}{2}\right) + {}^3C_2 (1) \left(\frac{x}{2}\right)^2 + {}^3C_3 \left(\frac{x}{2}\right)^3 \\
 &= 1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8} \quad \dots(3)
 \end{aligned}$$

From (1), (2), and (3), we obtain

$$\begin{aligned}
& \left[\left(1 + \frac{x}{2} \right) - \frac{2}{x} \right]^4 \\
&= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x} \left(1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8} \right) + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4} \\
&= 1 + 2x + \frac{3}{2}x^2 + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x} - 12 - 6x - x^2 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4} \\
&= \frac{16}{x} + \frac{8}{x^2} - \frac{32}{x^3} + \frac{16}{x^4} - 4x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - 5
\end{aligned}$$

Question 10:

Find the expansion of $(3x^2 - 2ax + 3a^2)^3$ using binomial theorem.

Answer 10:

Using Binomial Theorem, the given expression $(3x^2 - 2ax + 3a^2)^3$ can be expanded as

$$\begin{aligned}
& \left[(3x^2 - 2ax) + 3a^2 \right]^3 \\
&= {}^3C_0 (3x^2 - 2ax)^3 + {}^3C_1 (3x^2 - 2ax)^2 (3a^2) + {}^3C_2 (3x^2 - 2ax)(3a^2)^2 + {}^3C_3 (3a^2)^3 \\
&= (3x^2 - 2ax)^3 + 3(9x^4 - 12ax^3 + 4a^2x^2)(3a^2) + 3(3x^2 - 2ax)(9a^4) + 27a^6 \\
&= (3x^2 - 2ax)^3 + 81a^2x^4 - 108a^3x^3 + 36a^4x^2 + 81a^4x^2 - 54a^5x + 27a^6 \\
&= (3x^2 - 2ax)^3 + 81a^2x^4 - 108a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6 \quad ... (1)
\end{aligned}$$

Again by using Binomial Theorem, we obtain

$$\begin{aligned}
& (3x^2 - 2ax)^3 \\
&= {}^3C_0 (3x^2)^3 - {}^3C_1 (3x^2)^2 (2ax) + {}^3C_2 (3x^2)(2ax)^2 - {}^3C_3 (2ax)^3 \\
&= 27x^6 - 3(9x^4)(2ax) + 3(3x^2)(4a^2x^2) - 8a^3x^3 \\
&= 27x^6 - 54ax^5 + 36a^2x^4 - 8a^3x^3 \quad ... (2)
\end{aligned}$$

From (1) and (2), we obtain

$$\begin{aligned}
& (3x^2 - 2ax + 3a^2)^3 \\
&= 27x^6 - 54ax^5 + 36a^2x^4 - 8a^3x^3 + 81a^2x^4 - 108a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6 \\
&= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6
\end{aligned}$$