ALGORITHMS HOMEWORK 1

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Homework Option. I would like to choose the homework heavy option.

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Suppose we are given a sorted array of n distinct elements that has been circularly shifted by k elements. We want to find an algorithm to find the maximum that runs in $O(log(n))$ time.

Lemma 1.1. *Consider two elements in our array, let us call them $A[i]$ and $A[j]$ where $i < j$. The elements between and including them are sorted if and only if $A[i] < A[j]$. In other words: given any two elements in a circularly sorted array, if they are in the correct order, then the elements between them are in correct order.*

Proof. To prove the forward implication we will actually show the contrapositive. Suppose the array slice $A[i..j]$ is not sorted, then $\exists k \ni A[k] > A[k+1]$ where $i \leq k \leq j$. We know then that since the array is circularly sorted that $\forall x \in A[i..k]$ and $\forall y \in A[(k+1)..j]$ we have $x > y$. Clearly $A[i] \in A[i..k]$ and $A[j]y \in A[(k+1)..j]$, therefore $A[i] > A[j]$

To show the converse, we simply note that for any sorted array, the following holds:

$$\forall i, j \ni i < j \Rightarrow A[i] < A[j]$$

Corollary 1.2. *Given a circularly sorted array and indices $i, j \in 0 \leq i < j \leq n$, the maximum is between i and j if and only if $A[i] < A[j]$.*

Proof. Since a circularly sorted array with distinct elements must have a unique maximum

Algorithm 1.3. (cirmax)
If the array consists of a single element, return that element.
Let $h = \lfloor n/2 \rfloor$; compare the $A[0]$ and $A[h]$ elements.

Case (Less than): return the value of cirmax($A[h..n]$)

Case (Greater than): return the value of cirmax($A[0..(h-1)]$)

Proof. If an array consists of a single element, clearly that element must be the maximum. If the first element $A[0]$ is less than the middle element $A[h]$ then by 1.2 the maximum occurs in the array slice $A[h..n]$. However if $A[0]$ is greater than $A[h]$, then, again by 1.2 the maximum occurs in $A[0..(h-1)]$ and we therefore only need consider that slice.

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