Operational notes

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The following colors are **not** part of the final product, but serve as highlights in the editing/review process:

- text that needs attention from the Subject Matter Experts: Mirco, Anna,& Jan
- terms that have not yet been defined in the book
- text that needs advice from the communications/marketing team: Aaron & Shane
- text that needs to be completed or otherwise edited (by Sylvia)

NB: This PDF only includes the Elliptic Curves chapter

Todo list

MoonMath manual

TechnoBob and the Least Scruples crew

March 23, 2022

Contents

Chapter 5

Elliptic Curves

TODO: Elliptic Curve asymmetric cryptography examples. Private key, generator, public key.

Generally speaking, elliptic curves are "curves" defined in geometric planes like the Euclidean or the projective plane over some given field. One of the key features of elliptic curves over finite fields from the point of view of cryptography is that their set of points has a group law such that the resulting group is finite and cyclic, and it is believed that the discrete logarithm problem on these groups is hard.

A special class of elliptic curves are so-called **pairing-friendly curves**, which have a notation of a group pairing as defined in XXX. This pairing has cryptographically advantageous properties. Those curve are useful in the development of SNARKs, since they allow to compute so-called R1CS-satisfiability "in the exponent" MIRCO: (THIS HAS TO BE REWRITTEN WITH WAY MORE DETAIL)

add reference

In this chapter, we introduce epileptic curves as they are used in pairing-based approaches to the construction of SNARKs. The elliptic curves we consider are all defined over prime fields or prime field extensions and the reader should be familiar with the contend of the previous section on those fields.

In its most generality elliptic curves are defined as a smooth projective curve of genus 1 defined over some field $\mathbb{F}$ with a distinguished $\mathbb{F}$-rational point, but this definition is not very useful for the introductory character of this book. We will therefore look at 3 more practical definitions in the following sections, by introducing Weierstraß, Montgomery and Edwards curves. All of them are widely used in cryptography, and understanding them is crucial to being able to follow the rest of this book.

maybe remove this sentence?

5.1 Elliptic Curve Arithmetics

5.1.1 Short Weierstraß Curves

In this section, we introduce **short Weierstraß** curves, which are the most general types of curves over finite fields of characteristic greater than 3.

We start with their representation in affine space. This representation has the advantage that affine points correspond to pairs of numbers, which makes it more accessible for beginners. However, it has the disadvantage that a special "point at infinity" that is not a point on the curve, is necessary to describe the group structure. We introduce the elliptic curve group law and describe elliptic curve scalar multiplication, which is an instantiation of the exponential map from general cyclic groups.

affine space

Then we look at the projective representation of short Weierstraß curves. This has the advantage that no special symbol is necessary to represent the point at infinity but comes with the drawback that projective points are classes of numbers, which might be a bit unusual for a beginner.

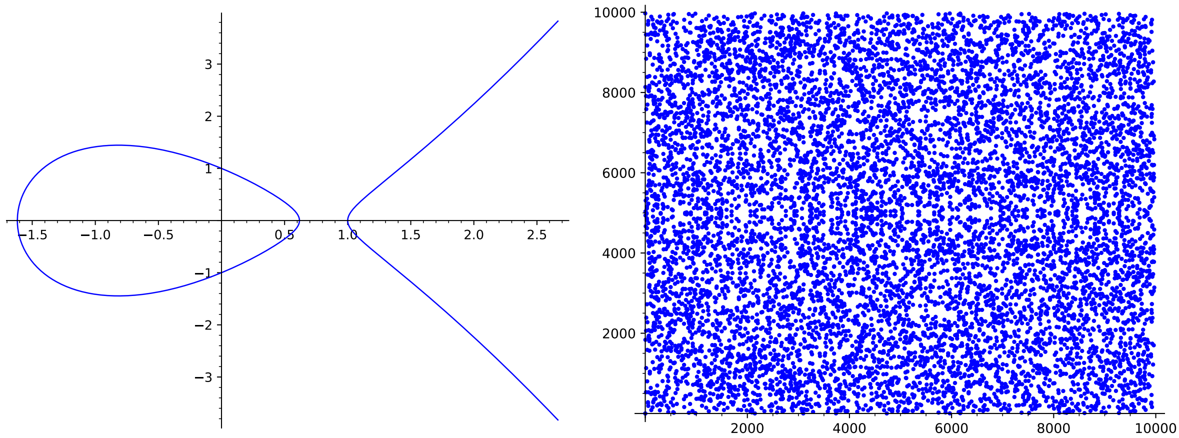
We finish this section with an explicit equivalence that transforms affine representations into projective ones and vice versa.

Affine short Weierstraß form Probably the least abstract and most straight-forward way to introduce elliptic curves for non-mathematicians and beginners is the so-called affine representation of a short Weierstraß curve. To see what this is, let $\mathbb{F}$ be a finite field of order q and $a,b \in \mathbb{F}$ two field elements such that $4a^3 + 27b^2 \bmod q \neq 0$. Then a **short Weierstraß elliptic curve** $E(\mathbb{F})$ over $\mathbb{F}$ in its affine representation is the set of all pairs of field elements $(x,y) \in \mathbb{F}\times\mathbb{F}$, that satisfy the short Weierstraß cubic equation $y^2 = x^3 + a\cdot x + b$, together with a distinguished symbol $\mathcal{O}$, called the **point at infinity**:

$$E(\mathbb{F}) = \{(x,y) \in \mathbb{F}\times\mathbb{F} \mid y^2 = x^3 + a\cdot x + b\} \bigcup \{\mathcal{O}\} \tag{5.1}$$

Notation and Symbols 7. In the literature, the set $E(\mathbb{F})$, which includes the symbol $\mathcal{O}$, is often called the set of **rational points** of the elliptic curve, in which case the curve itself is usually written as $E/\mathbb{F}$. However, in what follows, we will frequently identify an elliptic curve with its set of rational points and therefore use the notation $E(\mathbb{F})$ instead. This is possible in our case, since we only the group structure of the curve in consideration is relevant for us.

The term "curve" is used here because, in the ordinary 2 dimensional plane $\mathbb{R}^2$, the set of all points (x,y) that satisfy $y^2 = x^3 + a\cdot x + b$ looks like a curve. We should note however, that visualizing elliptic curves over finite fields as "curves" has its limitations, and we will therefore not stress the geometric picture too much, but focus on the computational properties instead. To understand the visual difference, consider the following two elliptic curves:



Both elliptic curves are defined by the same short Weierstraß equation $y^2 = x^3 - 2x + 1$, but the first curve is defined in the real affine plane $\mathbb{R}^2$, that is, the pair (x,y) contains real numbers, while the second one is defined in the affine plane $\mathbb{F}_{9973}^2$, which means that both x and y are from the prime field $\mathbb{F}_{9973}$. Every blue dot represents a pair (x,y) that is a solution to $y^2 = x^3 - 2x + 1$. As we can see, the second curve hardly looks like a geometric structure one would naturally call a curve. This shows that our geometric intuitions from $\mathbb{R}^2$ are obfuscated in curves over finite fields.

The identity $6\cdot(4a^3 + 27b^2) \bmod q \neq 0$ ensures that the curve is non-singular, which basically means that the curve has no cusps or self-intersections.

cusps

self-intersections

Throughout this book, the reader is advised to do as many computations in a pen-and-paper fashion as possible, as this is helps getting a deeper understanding of the details. However, when dealing with elliptic curves, computations can quickly become cumbersome and tedious, and one might get lost in the details. Fortunately, Sage is very helpful in dealing with elliptic curves. This book to introduces the reader to the great elliptic curve capabilities of Sage. One we to define elliptic curves and work is them goes like this:

rephrase

```
sage: F5 = GF(5) # define the base field
sage: a = F5(2) # parameter a
sage: b = F5(4) # parameter b
sage: # check non-sigularity
sage: F5(6)*(F5(4)*a^3+F5(27)*b^2) != F5(0)
True
sage: # short Weierstrass curve
sage: E = EllipticCurve(F5,[a,b]) # y^2 == x^3 + ax +b
sage: P = E(0,2) # 2^2 == 0^3 + 2*0 + 4
sage: P.xy() # affine coordinates
(0, 2)
sage: INF = E(0) # point at infinity
sage: try:  # point at infinity has no affine coordinates
....:     INF.xy()
....: except ZeroDivisionError:
....:     pass
sage: P = E.plot() # create a plotted version
```

The following three examples give a more practical understanding of what an elliptic curve is and how we can compute it. The reader is advised to read them carefully, and ideally, to also carry out the computation themselfs. We will repeatedly build on these examples in this chapter, and use the second example throughout this book.

Example 65. To provide the reader with an example of a small elliptic curve where all computation can be done with pen and paper, consider the prime field $\mathbb{F}_5$ from example 59 (page 60). quite familiar to readers who had worked through the examples and exercises in the previous chapter.

check reference

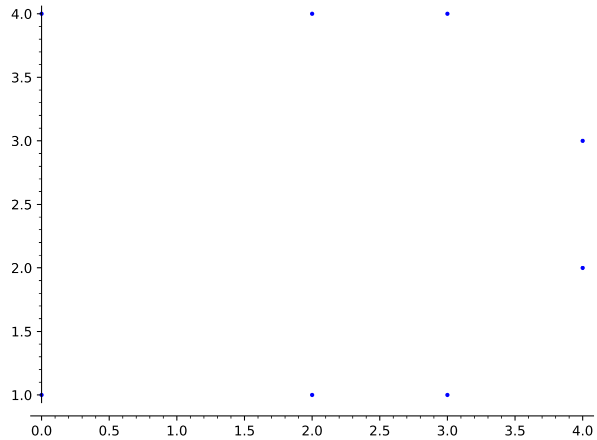
To define an elliptic curve over $\mathbb{F}_5$, we have to choose to numbers a and b from that field. Assuming we choose $a = 1$ and $b = 1$ then $4a^3 + 27b^2 \equiv 1 \pmod 5$ from which follows that the corresponding elliptic curve $E_1(\mathbb{F}_5)$ is given by the set of all pairs (x,y) from $\mathbb{F}_5$ that satisfy the equation $y^2 = x^3 + x + 1$, together with the special symbol $\mathcal{O}$, which represents the "point at infinity".

To get a better understand of that curve, observer that if we choose arbitrarily the pair $(x,y) = (1,1)$, we see that $1^2 \neq 1^3 + 1 + 1$ and hence $(1,1)$ is not an element of the curve $E_1(\mathbb{F}_5)$. On the other hand choosing for example $(x,y) = (2,1)$ gives $1^2 = 2^3 + 2 + 1$ and hence the pair $(2,1)$ is an element of $E_1(\mathbb{F}_5)$ (Remember that all computations are done in modulo 5 arithmetics).

Now since the set $\mathbb{F}_5 \times \mathbb{F}_5$ of all pairs (x,y) from $\mathbb{F}_5$ contains only $5 \cdot 5 = 25$ pairs, we can compute the curve, by just inserting every possible pair (x,y) into the short Weierstraß equation $y^2 = x^3 + x + 1$. If the equation holds, the pair is a curve point, if not that means that the point is not on the curve. Combining the result of this computation with the point at infinity gives the curve as follows:

$$E_1(\mathbb{F}_5) = \{\mathcal{O}, (0,1), (2,1), (3,1), (4,2), (4,3), (0,4), (2,4), (3,4)\}$$

This means that our elliptic curve is a set of 9 elements, 8 of which are pairs of numbers and one special symbol $\mathcal{O}$. Visualizing $E1$ gives the following plot:



In the development of SNARKs, it is sometimes necessary to do elliptic curve cryptography “in a circuit", which basically means that the elliptic curves need to be implemented in a certain SNARK-friendly way. We will look at what this means in chapter **??**. To be able to do this efficiently, it is desirable to have curves with special properties. The following example is a pen-and-paper version of such a curve, called **Baby-jubjub**, which parallels the definition of a cryptographically secure ~~curve extensively used in real-world SNARKs. The interested reader~~ is advised to read this example carefully, as we will use it and build on it in various places throughout the book. I feel like a lot of people won’t get the Lewis Carroll reference unless we make it more explicit

check reference

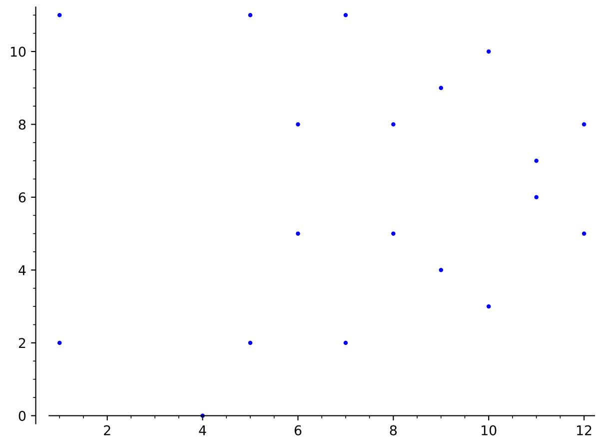
jubjub

Example 66 (Pen-JubJub). Consider the prime field $\mathbb{F}_{13}$ from exercise XXX. If we choose $a = 8$ and $b = 8$ then $4a^3 + 27b^2 \equiv 6 \pmod{13}$ and the corresponding elliptic curve is given by all pairs (x, y) from $\mathbb{F}_13$ such that $y^2 = x^3 + 8x + 8$ holds. We write *PJJ_13* for this curve and call it the *Pen-JubJub* curve.

Now, since the set $\mathbb{F}_{13} \times \mathbb{F}_{13}$ of all pairs (x, y) from $\mathbb{F}_{13}$ contains only $13 \cdot 13 = 169$ pairs, we can compute the curve, by just inserting every possible pair (x, y) into the short Weierstraß equation $y^2 = x^3 + 8x + 8$. We get

$$\begin{aligned} PJJ\_13 = \{&\mathcal{O}, (1,2), (1,11), (4,0), (5,2), (5,11), (6,5), (6,8), (7,2), (7,11), \\ &(8,5), (8,8), (9,4), (9,9), (10,3), (10,10), (11,6), (11,7), (12,5), (12,8)\} \end{aligned}$$

As we can see the curve consist of 20 points. 19 points from the affine plane and the point at infinity. To get a visual impression of the *PJJ_13* curve, we might plot all of its points (except the point at infinity) in the $\mathbb{F}_{13} \times \mathbb{F}_{13}$ affine plane. We get:



As we will see in what follows this curve is kind of special as it is possible to represent it in two alternative forms, called the Montgomery and the twisted Edwards form (See XXX and XXX).

add reference

add reference

Now that we have seen two pen-and-paper friendly elliptic curves, let us look at a curve that is used in actual cryptography. Cryptographically secure elliptic curve are not qualitatively different from the curves we looked at so far. The only difference is that the prime number modulus of the prime field is much larger. Typical examples use prime numbers, which have binary representations in the size of more than double the size of the desired security level. So if for example a security of 128 bit is desired, a prime modules of binary size ≥ 256 is chosen. The following example provides such a curve.

Example 67 (Bitcoin's Secp256k1 curve). To give an example of a real-world, cryptographically secure curve, let us look at curve Secp256k1, which is famous for being used in the public key cryptography of Bitcoin. The prime field $\mathbb{F}_p$ of Secp256k1 if defined by the prime number

$$p = 115792089237316195423570985008687907853269984665640564039457584007908834671663$$

which has a binary representation that need 256 bits. This implies that the $\mathbb{F}_p$ approximately contains 2^{256} many elements. So the underlying field is large. To get an image of how large the base field is, consider that the number 2^{256} is approximately in the same order of magnitude as the estimated number of atoms in the observable universe.

Curve Secp256k1 is then defined by the parameters $a, b \in \mathbb{F}_p$ with $a = 0$ and $b = 7$. Since $4 \cdot 0^3 + 27 \cdot 7^2 \bmod p = 1323$, those parameters indeed define an elliptic curve given by

$$Secp256k1 = \{(x,y) \in \mathbb{F}_p \times \mathbb{F}_p \mid y^2 = x^3 + 7\}$$

Clearly Secp256k1 is a curve, to large to do computations by hand, since it can be shown that *Secp256k1* contains

$$r = 115792089237316195423570985008687907852837564279074904382605163141518161494337$$

many elements, were r is a prime number that also has a binary representation of 256 bits. Cryptographically secure elliptic curves are therefore not useful in pen-and-paper computations. Fortunately Sage handles large curve efficiently:

```
sage: p = 11579208923731619542357098500868790785326998466564056403945758400790883467166 3
sage: # Hexadecimal representation
sage: p.str(16)
```

```
fffffffffffffffffffffffffffffffffffffffffffffffffffffffefffffc2f
sage: p.is_prime()
True
sage: p.nbits()
256
sage: Fp = GF(p)
sage: Secp256k1 = EllipticCurve(Fp,[0,7])
sage: r = Secp256k1.order() # number of elements
sage: r.str(16)
fffffffffffffffffffffffffffffffebaaedce6af48a03bbfd25e8cd0364141
sage: r.is_prime()
True
sage: r.nbits()
256
```

Exercise 35. Look up the definition of curve BLS12-381, implement it in Sage and compute its order.

Affine compressed representation As we have seen in example XXX, cryptographically secure elliptic curves are defined over large prime fields, where elements of those fields typically need more than 255 bits storage on a computer. Since elliptic curve points consists of pairs of those field element, they need double that amount of storage.

add reference

To reduce the amount of space needed to represent a curve point note however, that up to a sign the y-coordinate of a curve point can be computed from the x-coordinate, by simply inserting x into the Weierstraß equation and then computing the roots of the result. This gives two results and it follows that we can represent a curve point in **compressed form** by simply storing the x-coordinate together with a single sign bit only, the latter of which deterministically decides which of the two roots to choose. In case that the y-coordinate is zero, both sign bit give the same result.

For example one convention could be to always choose the root closer to 0, when the sign bit is 0 and the root closer to the order of $\mathbb{F}$ when the sign bit is 1.

Example 68 (Pen-jubjub). To understand the concept of compressed curve points a bit better consider the *PJJ_13* curve from example XXX again. Since this curve is defined over the prime field $\mathbb{F}_{13}$ and numbers between 0 and 13 need approximately 4 bits to be represented, each *PJJ_13* -point need 8-bits of storage in uncompressed form, while it would need only 5 bits in compressed form. To see how this works, recall that in uncompressed form we have

add reference

$$
\begin{aligned}
PJJ\_13 = \{&\mathcal{O},(1,2),(1,11),(4,0),(5,2),(5,11),(6,5),(6,8),(7,2),(7,11),\\
&(8,5),(8,8),(9,4),(9,9),(10,3),(10,10),(11,6),(11,7),(12,5),(12,8)\}
\end{aligned}
$$

Using the technique of point compression, we can replace the y-coordinate in each (x,y) pair by a sign bit, indicating, whether or not y is closer to 0 or to 13. So y values in the range $[0,\ldots,6]$ having sign bit 0 and y-values in the range $[7,\ldots,12]$ having sign bit 1. Applying this to the points in *PJJ_13* gives the compressed representation:

$$
\begin{aligned}
PJJ\_13 = \{&\mathcal{O},(1,0),(1,1),(4,0),(5,0),(5,1),(6,0),(6,1),(7,0),(7,1),\\
&(8,0),(8,1),(9,0),(9,1),(10,0),(10,1),(11,0),(11,1),(12,0),(12,1)\}
\end{aligned}
$$

Note that the numbers $7,\ldots,12$ are the negatives (additive inverses) of the numbers $1,\ldots,6$ in modular 13 arithmetics and that $-0=0$. Calling the compression bit a “sign bit” therefore makes sense.

To recover the uncompressed point of say $(5,1)$, we insert the x-coordinate 5 into the Weierstraß equation and get $y^2 = 5^3+8\cdot 5+8 = 4$. As expected 4 is a quadratic residue in $\mathbb{F}_{13}$ with roots $\sqrt{4}=\{2,11\}$. Now since the sign bit of the point is 1, we have to choose the root closer to the modulus 13 which is 11. The uncompressed point is therefore $(5,11)$.

Looking at the previous examples, compression rate looks not very impressive. The following example therefore looks at the Secp256k1 curve to show that compression is actually useful.

Example 69. Consider the Secp256k1 curve from example XXX again. The following code involves Sage to generate a random affine curve point, we then apply our compression method

add reference

```
sage: P = Secp256k1.random_point().xy()
sage: P
(573274555909292870027549532819570308193155586251244694583622856308870288524 36, 24242609999426606897142811967939071817174686615886596221090801834998454950146)
sage: # uncompressed affine point size
sage: ZZ(P[0]).nbits()+ZZ(P[1]).nbits()
509
sage: # compute the compression
sage: if P[1] > Fp(-1)/Fp(2):
....:     PARITY = 1
....: else:
....:     PARITY = 0
sage: PCOMPRESSED = [P[0],PARITY]
sage: PCOMPRESSED
[57327455590929287002754953281957030819315558625124469458362285630887028852436, 0]
sage: # compressed affine point size
sage: ZZ(PCOMPRESSED[0]).nbits()+ZZ(PCOMPRESSED[1]).nbits()
255
```

Affine group law One of the key properties of an elliptic curve is that it is possible to define a group law on the set of its rational points, such that the point at infinity serves as the neutral element and inverses are reflections on the x-axis.

The origin of this law can be understood in a geometric picture and is known as the **chord-and-tangent rule**. In the affine representation of a short Weierstraß curve, the rule can be described in the following way:

- (Point addition) Let $P,Q \in E(\mathbb{F})\backslash\{\mathcal{O}\}$ with $P \neq Q$ be two distinct points on an elliptic curve, that are both not the point at infinity. Then the sum of P and Q is defined as follows: Consider the line l which intersects the curve in P and Q. If l intersects the elliptic curve at a third point R', define the sum $R = P \oplus Q$ of P and Q as the reflection of R' at the x-axis. If it does not intersect the curve at a third point define the sum to be the point at infinity $\mathcal{O}$. It can be shown, that no such chord-line will intersect the curve in more than three points, so addition is not ambiguous.

- (Point doubling) Let $P \in E(\mathbb{F})\backslash\{\mathcal{O}\}$ be a point on an elliptic curve, that is not the point at infinity. Then the sum of P with itself (the doubling) is defined as follows: Consider the line which is tangent to the elliptic curve at P, if this line intersects the elliptic curve at a second point R'. The sum $2P = P + P$ is then the reflection of R' at the x-axis. If it does not intersect the curve at a third point define the sum to be the point at infinity $\mathcal{O}$. It can be shown, It can be shown, that no such tangent-line will intersect the curve in more than two points, so addition is not ambiguous.

- (Point at infinity) We define the point at infinity $\mathcal{O}$ as the neutral element of addition, that is we define $P + \mathcal{O} = P$ for all points $P \in E(\mathbb{F})$.

It can be shown that the points of an elliptic curve form a commutative group with respect to the tangent and chord rule, such that $\mathcal{O}$ acts the neutral element and the inverse of any element $P \in E(\mathbb{F})$ is the reflection of P on the x-axis.

To translate the geometric description into algebraic equations, first observe that for any two given curve points $(x_1, y_1), (x_2, y_2) \in E(\mathbb{F})$, it can be shown that the identity $x_1 = x_2$ implies $y_2 = \pm y_1$, which shows that the following rules are a complete description of the affine addition law.

- (Neutral element) Point at infinity $\mathcal{O}$ is the neutral element.

- (Additive inverse) The additive inverse of $\mathcal{O}$ is $\mathcal{O}$ and for any other curve point $(x, y) \in E(\mathbb{F})\backslash\{\mathcal{O}\}$, the additive inverse is given by $(x, -y)$.

- (Addition rule) For any two curve points $P, Q \in E(\mathbb{F})$ addition is defined by one of the following three cases:

  1. (Adding the neutral element) If $Q = \mathcal{O}$, then the sum is defined as $P \oplus Q = P$.
  2. (Adding inverse elements) If $P = (x, y)$ and $Q = (x, -y)$, the sum is defined as $P \oplus Q = \mathcal{O}$.
  3. (Adding non self-inverse equal points) If $P = (x, y)$ and $Q = (x, y)$ with $y \neq 0$, the sum $2P = (x', y')$ is defined by

  $$x' = \left(\tfrac{3x^2+a}{2y}\right)^2 - 2x \quad , \quad y' = \left(\tfrac{3x^2+a}{2y}\right)^2 (x - x') - y$$

  4. (Adding non inverse different points) If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ such that $x_1 \neq x_2$, the sum $R = P + Q$ with $R = (x_3, y_3)$ is defined by

  $$x_3 = \left(\tfrac{y_2-y_1}{x_2-x_1}\right)^2 - x_1 - x_2 \quad , \quad y_3 = \left(\tfrac{y_2-y_1}{x_2-x_1}\right)(x_1 - x_3) - y_1$$

Note that short Weierstraß curve points P with $P = (x, 0)$ are inverse to themselfs, which implies $2P = \mathcal{O}$ in this case.

Notation and Symbols 8. Let $\mathbb{F}$ be a field and $E(\mathbb{F})$ be an elliptic curve over $\mathbb{F}$. We write $\oplus$ for the group law on $E(\mathbb{F})$ and $(E(\mathbb{F}), \oplus)$ for the group of rational points.

As we can see, it is very efficient to compute inverses on elliptic curves. However, computing the addition of elliptic curve points in the affine representation needs to consider many cases and involves extensive finite field divisions. As we will see in the next paragraph this can be simplified in projective coordinates.

To get some practical impression of how the group law on an elliptic curve is computed, let's look at some actual cases:

Example 70. Consider the elliptic curve $E_1(\mathbb{F}_5)$ from example XXX again. As we have seen, the set of rational points contains 9 elements and is given by

add reference

$$E_1(\mathbb{F}_5) = \{\mathcal{O}, (0,1), (2,1), (3,1), (4,2), (4,3), (0,4), (2,4), (3,4)\}$$

We know that this set defines a group, so we can add any two elements from $E_1(\mathbb{F}_5)$ to get a third element.

To give an example consider the elements $(0,1)$ and $(4,2)$. Neither of these elements is the neutral element $\mathcal{O}$ and since the x-coordinate of $(0,1)$ is different from the x-coordinate of $(4,2)$, we know that we have to use the chord rule, that is rule number 4 from XXX to compute the sum $(0,1) \oplus (4,2)$. We get

add reference

$$\begin{aligned}
x_3 &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 - x_1 - x_2 && \text{\# insert points} \\
&= \left(\frac{2-1}{4-0}\right)^2 - 0 - 4 && \text{\# simplify in } \mathbb{F}_5 \\
&= \left(\frac{1}{4}\right)^2 + 1 = 4^2 + 1 = 1 + 1 = 2
\end{aligned}$$

$$\begin{aligned}
y_3 &= \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x_1 - x_3) - y_1 && \text{\# insert points} \\
&= \left(\frac{2-1}{4-1}\right)(0-2) - 1 && \text{\# simplify in } \mathbb{F}_5 \\
&= \left(\frac{1}{4}\right) \cdot 3 + 4 = 4 \cdot 3 + 4 = 2 + 4 = 1
\end{aligned}$$

So in our elliptic curve $E_1(\mathbb{F}_5)$ we get $(0,1) \oplus (4,2) = (2,1)$ and indeed the pair $(2,1)$ is an element of $E_1(\mathbb{F}_5)$ as expected. On the other hand we have $(0,1) \oplus (0,4) = \mathcal{O}$, since both points have equal x-coordinates and inverse y-coordinates rendering them as inverse to each other. Adding the point $(4,2)$ to itself, we have to use the tangent rule, that is rule 3 from XXX. We get

add reference

$$\begin{aligned}
x' &= \left(\frac{3x^2 + a}{2y}\right)^2 - 2x && \text{\# insert points} \\
&= \left(\frac{3 \cdot 4^2 + 1}{2 \cdot 2}\right)^2 - 2 \cdot 4 && \text{\# simplify in } \mathbb{F}_5 \\
&= \left(\frac{3 \cdot 1 + 1}{4}\right)^2 + 3 \cdot 4 = \left(\frac{4}{4}\right)^2 + 2 = 1 + 2 = 3
\end{aligned}$$

$$\begin{aligned}
y' &= \left(\frac{3x^2 + a}{2y}\right)^2 (x - x') - y && \text{\# insert points} \\
&= \left(\frac{3 \cdot 4^2 + 1}{2 \cdot 2}\right)^2 (4-3) - 2 && \text{\# simplify in } \mathbb{F}_5 \\
&= 1 \cdot 1 + 3 = 4
\end{aligned}$$

So in our elliptic curve $E_1(\mathbb{F}_5)$ we get the doubling $2\cdot(4,2)$, that is $(4,2)\oplus(4,2)=(3,4)$ and indeed the pair $(3,4)$ is an element of $E_1(\mathbb{F}_5)$ as expected. The group $E_1(\mathbb{F}_5)$ has no self inverse points other then the neutral element $\mathcal{O}$, since no point has 0 as its y-coordinate. We can invoke Sage to double check the computations.

```
sage: F5 = GF(5)
sage: E1 = EllipticCurve(F5,[1,1])
sage: INF = E1(0) # point at infinity
sage: P1 = E1(0,1)
sage: P2 = E1(4,2)
sage: P3 = E1(0,4)
sage: R1 = E1(2,1)
sage: R2 = E1(3,4)
sage: R1 == P1+P2
True
sage: INF == P1+P3
True
sage: R2 == P2+P2
True
sage: R2 == 2*P2
True
sage: P3 == P3 + INF
True
```

Example 71 (Pen-jubjub). Consider the *PJJ_13*-curve from example XXX again and recall that its group of rational points is given by

add reference

$$PJJ\_13 = \{\mathcal{O},(1,2),(1,11),(4,0),(5,2),(5,11),(6,5),(6,8),(7,2),(7,11),\\ (8,5),(8,8),(9,4),(9,9),(10,3),(10,10),(11,6),(11,7),(12,5),(12,8)\}$$

In contrast to the group from the previous example, this group contains a self inverse point, which is different from the neutral element, given by $(4,0)$. To see what this means, observe that we can not add $(4,0)$ to itself using the tangent rule 3 from XXX, as the y-coordinate is zero. Instead we have to use rule 2, since $0=-0$. We therefore get $(4,0)\oplus(4,0)=\mathcal{O}$ in *PJJ_13*. The point $(4,0)$ is therefore inverse to itself, as adding it to itself gives the neutral element.

add reference

```
sage: F13 = GF(13)
sage: MJJ = EllipticCurve(F13,[8,8])
sage: P = MJJ(4,0)
sage: INF = MJJ(0) # Point at infinity
sage: INF == P+P
True
sage: INF == 2*P
True
```

Example 72. Consider the Secp256k1 curve from example XXX again. The following code involves Sage to generate a random affine curve point, we then apply our compression method

add reference

```
sage: P = Secp256k1.random_point()
```

```
sage: Q = Secp256k1.random_point()
sage: INF = Secp256k1(0)
sage: R1 = -P
sage: R2 = P + Q
sage: R3 = Secp256k1.order()*P
sage: P.xy()
(2437965124411773648884901383952245798298026200193112014924104
    592054125560358 2, 3815531853806256266340856886118837407064 3
    30105793105769280234966336891502774 7)
sage: Q.xy()
(6273267811834346524071370277009541823203325405903695727983144
    7554159754801518, 81206263702504109131546480004400274036228
    73257204518608057781722309607462714 2)
sage: (ZZ(R1[0]).str(16), ZZ(R1[1]).str(16))
('35e664c3768462813f30192e327e60c61508d279931cdbc639f3cb11c5b3
    157e', 'aba4dae1f8c83f0ac955259cd78622327b9f107d82937463dd8
    cded0c012750c')
sage: R2.xy()
(8315162076242884051827668971975027473477042355284820491860209
    945466147353499, 12808304373647884707293426644826593284347 8
    4573359628687283920496788161593119 0)
sage: R3 == INF
True
sage: P[1]+R1[1] == Fp(0) # -(x,y) = (x,-y)
True
```

Exercise 36. Consider the *PJJ_13*-curve from example XXX. add reference

1. Compute the inverse of $(10,10)$, $\mathcal{O}$, $(4,0)$ and $(1,2)$.
2. Compute the expression $3*(1,11)-(9,9)$.
3. Solve the equation $x+2(9,4)=(5,2)$ for some $x \in PJJ\_13$
4. Solve the equation $x\cdot(7,11)=(8,5)$ for $x \in \mathbb{Z}$

Scalar multiplication As we have seen in the previous section, elliptic curves $E(\mathbb{F})$ have the structure of a commutative group associated to them. It can moreover be shown, that this group is finite and cyclic, whenever the field is finite.

To understand the elliptic curve scalar multiplication, recall from XXX that every finite cyclic group of order q has a generator g and an associated exponential map $g^{(\cdot)}:\mathbb{Z}_q\to\mathbb{G}$, where g^n is the n-fold product of g with itself. add reference

Now, elliptic curve scalar multiplication is then nothing but the exponential map, written in additive notation. To be more precise let $\mathbb{F}$ be a finite field, $E(\mathbb{F})$ an elliptic curve of order r and P a generator of $E(\mathbb{F})$. Then the **elliptic curve scalar multiplication** with base P is given by

$$[\cdot]P:\mathbb{Z}_r\to E(\mathbb{F}); m\mapsto[m]P$$

where $[0]P=\mathcal{O}$ and $[m]P=P+P+\ldots+P$ is the m-fold sum of P with itself. Elliptic curve scalar multiplication is therefore nothing but an instantiation of the general exponential map, when using additive instead of multiplicative notation. This map is a homomorph of groups, which means that $[n+m]P = [n]P \oplus [m]P$.

As with all finite, cyclic groups the inverse of the exponential map exist and is usually called the **elliptic curve discrete logarithm map**. However, elliptic curve are believed to be XXX-groups, which means that we don't know of any efficient way to actually compute this map.

add reference

Scalar multiplication and its inverse, the elliptic curve discrete logarithm, define the elliptic curve discrete logarithm **problem**, which consists of finding solutions $m \in \mathbb{Z}_r$, such that

$$P = [m]Q \tag{5.2}$$

holds. Any solution m is usually called a **discrete logarithm** relation between P and Q. If Q is a generator of the curve, then there is a discrete logarithm relation between Q and any other point, since Q generates the group by repeatedly adding Q to itself. So for generator Q and point P, we know some discrete logarithm relation exist. However, since elliptic curves are believed to be XXX-groups, finding actual relations m is computationally hard, with runtimes approximately in the size of the order of the group. In practice, we often need the assumption that a discrete logarithm relation exists, but that at the same time no-one knows this relation.

add reference

One useful property of the exponential map in regard to the examples in this book, is that it can be used to greatly simplify pen-and-paper computations. As we have seen in example XXX, computing the elliptic curve addition law takes quit a bit of effort, when done without a computer. However, when g is a generator of small pen-and-paper elliptic curve group of order r, we can use the exponential map to write the group as

add reference

$$\mathbb{G} = \{[1]g \to [2]g \to [3]g \to \cdots \to [r-1]g \to \mathcal{O}\} \tag{5.3}$$

using cofactor clearing, which implies that $[r]g = \mathcal{O}$. "Logarithmic ordering" like this greatly simplifies complicated elliptic curve addition to the much simpler case of modular r addition. So in order to add two curve points P and Q, we only have to look up their discrete log relations with the generator, say $P = [n]g$ and $Q = [m]g$ and compute the sum as $P \oplus Q = [n+m]g$. This is, of course, only possible for small groups which we can organize as in XXX.

add reference

In the following example we will look at some implications of the fact that elliptic curves are finite cyclic groups. We will apply the fundamental theorem of finite cyclic groups and look how it reflects on the curves in consideration.

Example 73. Consider the elliptic curve group $E_1(\mathbb{F}_5)$ from example XXX. Since it is a finite cyclic group of order 9 and the prime factorization of 9 is $3 \cdot 3$, we can use the fundamental theorem of finite cyclic groups to reason about all its subgroups. In fact since the only prime factor of 9 is 3, we know that $E_1(\mathbb{F}_5)$ has the following subgroups:

add reference

- $\mathbb{G}_1 = E_1(\mathbb{F}_5)$ is a subgroup of order 9. By definition any group is a subgroup of itself.
- $\mathbb{G}_2 = \{(2,1),(2,4),\mathcal{O}\}$ is a subgroup of order 3. This is the subgroup associated to the prime factor 3.
- $\mathbb{G}_3 = \{\mathcal{O}\}$ is a subgroup of order 1. This is the trivial subgroup.

Moreover since $E_1(\mathbb{F}_5)$ and all its subgroups are cyclic, we know from XXX, that they must have generators. For example the curve point $(2,1)$ is a generator of the order 3-subgroup $\mathbb{G}_2$, since every element of $\mathbb{G}_2$ can be generated, by repeatedly adding $(2,1)$ to itself:

add reference

$$\begin{aligned}
[1](2,1) &= (2,1)\\
[2](2,1) &= (2,4)\\
[3](2,1) &= \mathcal{O}
\end{aligned}$$

Since $(2,1)$ is a generator we know from XXX, that it gives rise to an exponential map from the finite field $\mathbb{F}_3$ onto $\mathbb{G}_2$ defined by scalar multiplication

add reference

$$[\cdot](2,1) : \mathbb{F}_3 \to \mathbb{G}_2 \;:\; x \mapsto [x](2,1)$$

To give an example of a generator that generates the entire group $E_1(\mathbb{F}_5)$ consider the point $(0,1)$. Applying the tangent rule repeatedly we compute with some effort:

$$\begin{array}{rclcrcl}
[0](0,1) & = & \mathcal{O} & \quad & [1](0,1) & = & (0,1) \\
[2](0,1) & = & (4,2) & & [3](0,1) & = & (2,1) \\
[4](0,1) & = & (3,4) & & [5](0,1) & = & (3,1) \\
[6](0,1) & = & (2,4) & & [7](0,1) & = & (4,3) \\
[8](0,1) & = & (0,4) & & [9](0,1) & = & \mathcal{O}
\end{array}$$

Again, since $(2,1)$ is a generator we know from XXX, that it gives rise to an exponential map. However, since the group order is not a prime number, the exponential maps, does not map a from any field but from the residue class ring $\mathbb{Z}_9$ only:

add reference

$$[\cdot](0,1) : \mathbb{Z}_9 \to \mathbb{G}_1 \;:\; x \mapsto [x](0,1)$$

Using the generator $(0,1)$ and its associated exponential map, we can write $E(\mathbb{F}_1)$ i logarithmic order with respect to $(0,1)$ as explained in XXX. We get

add reference

$$E_1(\mathbb{F}_5) = \{(0,1) \to (4,2) \to (2,1) \to (3,4) \to (3,1) \to (2,4) \to (4,3) \to (0,4) \to \mathcal{O}\}$$

indicating that the first element is a generator and the n-th element is the scalar product of n and the generator. To see how logarithmic orders like this simplify the computations in small elliptic curve groups, consider example XXX again. In that example we use the chord and tangent rule to compute $(0,1) \oplus (4,2)$. Now in the logarithmic order of $E_1(\mathbb{F})$ we can compute that sum much easier, since we can directly see that $(0,1) = [1](0,1)$ and $(4,2) = [2](0,1)$. We can then deduce $(0,1) \oplus (4,2) = (2,1)$ immediately, since $[1](0,1) \oplus [2](0,1) = [3](0,1) = (2,1)$.

add reference

To give another example, we can immediately see that $(3,4) \oplus (4,3) = (4,2)$, without doing any expensive elliptic curve addition, since we know $(3,4) = [4](0,1)$ as well as $(4,3) = [7](0,1)$ from the logarithmic representation of $E_1(\mathbb{F}_5)$ and since $4+7=2$ in $\mathbb{Z}_9$, the result must be $[2](0,1) = (4,2)$.

Finally we can use $E_1(\mathbb{F}_5)$ as an example to understand the concept of cofactor clearing from XXX. Since the order of $E_1(\mathbb{F}_5)$ is 9 we only have a single factor, which happen to be the cofactor as well. Cofactor clearing then implies that we can map any element from $E_1(\mathbb{F}_5)$ onto its prime factor group $\mathbb{G}_2$ by scalar multiplication with 3. For example taking the element $(3,4)$ which is not in $\mathbb{G}_2$ and multiplying it with 3, we get $[3](3,4) = (2,1)$, which is an element of $\mathbb{G}_2$ as expected.

add reference

In the following example we will look at the subgroups of our pen-jubjub curve, define generators and compute the logarithmic order for pen-and-paper computations. Then we have another look at the principle of cofactor clearing.

Example 74. Consider the pen-jubjub curve *PJJ_13* from example XXX again. Since the order of *PJJ_13* is 20 and the prime factorization of 20 is $2^2 \cdot 5$, we know that the *PJJ_13* contains a "large" prime order subgroup of size 5 and a small prime oder subgroup of size 2.

add reference

To compute those groups, we can apply the technique of cofactor clearing in a try and repeat loop. We start the loop by arbitrarily choose an element $P \in PJJ\_13$. Then we multiply that element with the cofactor of the group, we want to compute. If the result is $\mathcal{O}$, we try a different element and repeat the process until the result is different from the point at infinity.

To compute a generator for the small prime order subgroup $(PJJ\_13)_2$, first observe that the cofactor is 10, since $20 = 2 \cdot 10$. We then arbitrarily choose the curve point $(5,11) \in PJJ\_13$ and compute $[10](5,11) = \mathcal{O}$. Since the result is the point at infinity, we have to try another curve point, say $(9,4)$. We get $[10](9,4) = (4,0)$ and we can deduce that $(4,0)$ is a generator of $(PJJ\_13)_2$. Logarithmic order of then gives

$$(PJJ\_13)_2 = \{(4,0) \to \mathcal{O}\}$$

as expected, since we know from example XXX that $(4,0)$ is self inverse, with $(4,0) \oplus (4,0) = \mathcal{O}$. Double checking the computations using Sage:

add reference

```
sage: F13 = GF(13)
sage: PJJ = EllipticCurve(F13,[8,8])
sage: P = PJJ(5,11)
sage: INF = PJJ(0)
sage: 10*P == INF
True
sage: Q = PJJ(9,4)
sage: R = PJJ(4,0)
sage: 10*Q == R
True
```

We can apply the same reasoning to the "large" prime order subgroup $(PJJ\_13)_5$, which contains 5 elements. To compute a generator for this group, first observe that the associated cofactor is 4, since $20 = 5 \cdot 4$. We choose the curve point $(9,4) \in PJJ\_13$ again and compute $[4](9,4) = (7,11)$ and we can deduce that $(7,11)$ is a generator of $(PJJ\_13)_5$. Using the generator $(7,11)$, we compute the exponential map $[\cdot](7,11) : \mathbb{F}_5 \to PJJ\_13$ and get

$$\begin{aligned}
[0](7,11) &= \mathcal{O}\\
[1](7,11) &= (7,11)\\
[2](7,11) &= (8,5)\\
[3](7,11) &= (8,8)\\
[4](7,11) &= (7,2)
\end{aligned}$$

We can use this computation to write the large order prime group $(PJJ\_13)_5$ of the pen-jubjub curve in logarithmic order, which we will use quite frequently in what follows. We get:

$$(PJJ\_13)_5 = \{(7,11) \to (8,5) \to (8,8) \to (7,2) \to \mathcal{O}\}$$

From this, we can immediately see that for example $(8,8) \oplus (7,2) = (8,5)$, since $3 + 4 = 2$ in $\mathbb{F}_5$.

From the previous two examples, the reader might get the impression, that elliptic curve computation can be largely replaced by modular arithmetics. This however, is not true in general, but only an arefact of small groups where it is possible to write the entire group in a logarithmic order. The following example gives some understanding, why this is not possible in cryptographically secure groups

Example 75. SEKTP BICOIN. DISCRET LOG HARDNESS PROHIBITS ADDITION IN THE FIELD...

Projective short Weierstraß form As we have seen in the previous section, describing elliptic curves as pairs of points that satisfy a certain equation is relatively straight-forward. However, in order to define a group structure on the set of points, we had to add a special point at infinity to act as the neutral element.

Recalling from the definition of projective planes XXX we know, that points at infinity are handled as ordinary points in projective geometry. It make therefore sense to look at the definition of a short Weierstraß curve in projective geometry.

add reference

To see what a short Weierstraß curve in projective coordinates is, let $\mathbb{F}$ be a finite field of order q and characteristic > 3, $a, b \in \mathbb{F}$ two field elements such that $4a^3 + 27b^2 \bmod q \neq 0$ and $\mathbb{F}\mathrm{P}^2$ the projective plane over $\mathbb{F}$. Then a **short Weierstraß elliptic curve** over $\mathbb{F}$ in its projective representation is the set

$$E(\mathbb{F}\mathrm{P}^2) = \{[X : Y : Z] \in \mathbb{F}\mathrm{P}^2 \mid Y^2 \cdot Z = X^3 + a \cdot X \cdot Z^2 + b \cdot Z^3\} \tag{5.4}$$

of all points $[X : Y : Z] \in \mathbb{F}\mathrm{P}^2$ from the projective plane, that satisfy the **homogenous** cubic equation $Y^2 \cdot Z = X^3 + a \cdot X \cdot Z^2 + b \cdot Z^3$.

To understand how the point at infinity is unified in this definition, recall from XXX that, in projective geometry points at infinity are given by homogeneous coordinates $[X : Y : 0]$. Inserting representatives $(x_1, y_1, 0) \in [X : Y : 0]$ from those classes into the defining homogenous cubic equations gives

add reference

$$\begin{aligned} y_1^2 \cdot 0 &= x_1^3 + a \cdot x_1 \cdot 0^2 + b \cdot 0^3 && \Leftrightarrow \\ 0 &= x_1^3 \end{aligned}$$

which shows that the only point at infinity that is also a point on a projective short Weierstraß curve is the class

$$[0, 1, 0] = \{(0, y, 0) \mid y \in \mathbb{F}\}$$

This point is the projective representation of $\mathcal{O}$. The projective representation of a short Weierstraß curve therefore has the advantage to not need a special symbol to represent the point at infinity $\mathcal{O}$ from the affine definition.

Example 76. To get an intuition of how an elliptic curve in projective geometry looks, consider curve $E_1(\mathbb{F}_5)$ from example (XXX). We know that in its affine representation, the set of rational points is given by

add reference

$$E_1(\mathbb{F}_5) = \{\mathcal{O}, (0,1), (2,1), (3,1), (4,2), (4,3), (0,4), (2,4), (3,4)\}$$

which is defined as the set of all pairs $(x, y) \in \mathbb{F}_5 \times \mathbb{F}_5$, such that the affine short Weierstraß equation $y^2 = x^3 + ax + b$ with $a = 1$ and $b = 1$ is satisfied.

To finde the projective representation of a short Weierstraß curve with the same parameters $a = 1$ and $b = 1$, we have to compute the set of projective points $[X : Y : Z]$ from the projective plane $\mathbb{F}_5\mathrm{P}^2$, that satisfy the homogenous cubic equation

$$y_1^2 z_1 = x_1^3 + 1 \cdot x_1 z_1^2 + 1 \cdot z_1^3$$

for any representative $(x_1, y_1, z_1) \in [X : Y : Z]$. We know from XXX, that the projective plane $\mathbb{F}_5\mathrm{P}^2$ contains $5^2 + 5 + 1 = 31$ elements, so we can take the effort and insert all elements into equation XXX and see if both sides match.

add reference

add reference

For example, consider the projective point $[0 : 4 : 1]$. We know from XXX, that this point in

add reference

the projective plane represents the line

$$[0:4:1] = \{(0,0,0),(0,4,1),(0,3,2),(0,2,3),(0,1,4)\}$$

in the three dimensional space $\mathbb{F}^3$. To check whether or not $[0:4:1]$ satisfies XXX, we can insert any representative, that is we can insert any element from XXX. Each element satisfies the equation if and only if any other satisfies the equation. So we insert $(0,4,1)$ and get

add reference

add reference

$$1^2 \cdot 1 = 0^3 + 1 \cdot 0 \cdot 1^2 + 1 \cdot 1^3$$

which tells us that the affine point $[0:4:1]$ is indeed a solution. And as we can see, would just as well insert any other representative. For example inserting $(0,3,2)$ also satisfies XXX, since

add reference

$$3^2 \cdot 2 = 0^3 + 1 \cdot 0 \cdot 2^2 + 1 \cdot 2^3$$

To find the projective representation of E_1, we first observe that the projective line at infinity $[1:0:0]$ is not a curve point on any projective short Weierstraß curve since it can not satisfy XXX for any parameter a and b. So we can exclude it from our consideration.

add reference

Moreover a point at infinity $[X:Y:0]$ can only satisfy equation XXX for any a and b, if $X=0$, which implies that the only point at infinity relavant for short Weierstraß elliptic curves is $[0:1:0]$, since $[0:k:0] = [0:1:0]$ for all k from the finite field. So we can exclude all points at infinity except the point $[0:1:0]$.

add reference

So all points that remain are the affine points $[X:Y:1]$. Inserting all of them into XXX we get the set of all projective curve points as

add reference

$$\begin{aligned} E_1(\mathbb{F}_5\mathrm{P}^2) = \{&[0:1:0],[0:1:1],[2:1:1],[3:1:1],\\ &[4:2:1],[4:3:1],[0:4:1],[2:4:1],[3:4:1]\}\end{aligned}$$

If we compare this with the affine representation we see that there is a 1:1 correspondence between the points in the affine representation XXX and the affine points in projective geometry and that the point $[0:1:0]$ represents the additional point $\mathcal{O}$ in the projective representation.

add reference

Exercise 37. Compute the projective representation of the pen-jubjub curve and the logarithmic order of its large prime order subgroup with respect to the generator $(7,11)$.

Projective Group law As we have seen in XXX, one of the key properties of an elliptic curve is that it comes with a definition of a group law on the set of its rational points, described geometrically by the chord and tangent rule. This rule was kind of intuitive, with the exception of the distinguished point at infinity, which appeared whenever the chord or the tangent did not have a third intersection point with the curve.

add reference

One of the key features of projective coordinates is now, that in projective space it is guaranteed that any chord will always intersect the curve in three points and any tangent will intersect in two points including the tangent point. So the geometric picture simplifies as we don't need to consider external symbols and associated cases.

Again, it can be shown that the points of an elliptic curve in projective space form a commutative group with respect to the tangent and chord rule, such that the projective point $[0:1:0]$ is the neutral element and the additive inverse of a point $[X:Y:Z]$ is given by $[X:-Y:Z]$. The addition law is then usually described by the following algorithm, that minimizes the number of needed additions and multiplications in the base field.

Exercise 38. Compare that affine addition law for short Weierstraß curves with the projective addition rule. Which branch in the projective rule corresponds to which case in the affine law?

Algorithm 6 Projective Weierstraß Addition Law

Require: $[X_1 : Y_1 : Z_1], [X_2 : Y_2 : Z_2] \in E(\mathbb{FP}^2)$
  procedure ADD-RULE($[X_1 : Y_1 : Z_1], [X_2 : Y_2 : Z_2]$)
    if $[X_1 : Y_1 : Z_1] == [0 : 1 : 0]$ **then**
      $[X_3 : Y_3 : Z_3] \leftarrow [X_2 : Y_2 : Z_2]$
    else if $[X_2 : Y_2 : Z_2] == [0 : 1 : 0]$ **then**
      $[X_3 : Y_3 : Z_3] \leftarrow [X_1 : Y_1 : Z_1]$
    else
      $U_1 \leftarrow Y_2 \cdot Z_1$
      $U_2 \leftarrow Y_1 \cdot Z_2$
      $V_1 \leftarrow X_2 \cdot Z_1$
      $V_2 \leftarrow X_1 \cdot Z_2$
      if $V_1 == V_2$ **then**
        if $U_1 \neq U_2$ **then** $[X_3 : Y_3 : Z_3] \leftarrow [0 : 1 : 0]$
        else
          if $Y_1 == 0$ **then** $[X_3 : Y_3 : Z_3] \leftarrow [0 : 1 : 0]$
          else
            $W \leftarrow a \cdot Z_1^2 + 3 \cdot X_1^2$
            $S \leftarrow Y_1 \cdot Z_1$
            $B \leftarrow X_1 \cdot Y_1 \cdot S$
            $H \leftarrow W^2 - 8 \cdot B$
            $X' \leftarrow 2 \cdot H \cdot S$
            $Y' \leftarrow W \cdot (4 \cdot B - H) - 8 \cdot Y_1^2 \cdot S^2$
            $Z' \leftarrow 8 \cdot S^3$
            $[X_3 : Y_3 : Z_3] \leftarrow [X' : Y' : Z']$
          end if
        end if
      else
        $U = U_1 - U_2$
        $V = V_1 - V_2$
        $W = Z_1 \cdot Z_2$
        $A = U^2 \cdot W - V^3 - 2 \cdot V^2 \cdot V_2$
        $X' = V \cdot A$
        $Y' = U \cdot (V^2 \cdot V_2 - A) - V^3 \cdot U_2$
        $Z' = V^3 \cdot W$
        $[X_3 : Y_3 : Z_3] \leftarrow [X' : Y' : Z']$
      end if
    end if
    return $[X_3 : Y_3 : Z_3]$
  end procedure
Ensure: $[X_3 : Y_3 : Z_3] == [X_1 : Y_1 : Z_1] \oplus [X_2 : Y_2 : Z_2]$

Coordinate Transformations As we have seen in example XXX, there was a close relation between the affine and the projective representation of a short Weierstraß curve. This was no accident. In fact from a mathematical point of view projective and affine short Weierstraß curves describe the same thing as there is a one-to-one correspondence (an isomorphism) between both representations for any given parameters a and b.

add reference

To specify the isomorphism, let $E(\mathbb{F})$ and $E(\mathbb{F}\mathrm{P}^2)$ be an affine and a projective short Weierstraß curve defined for the same parameters a and b. Then the map

$$\Phi : E(\mathbb{F}) \to E(\mathbb{F}\mathrm{P}^2) \; : \; \begin{array}{lcl} (x,y) & \mapsto & [x:y:1] \\ \mathcal{O} & \mapsto & [0:1:0] \end{array} \tag{5.5}$$

maps points from a the affine representation to points from the projective representation of a short Weierstraß curve, that is if the pair of points (x,y) satisfies the affine equation $y^2 = x^3 + ax + b$, then all homogeneous coordinates $(x_1,y_1,z_1) \in [x:y:1]$ satisfy the projective equation $y_1^2 \cdot z_1 = x_1^3 + ay_1 \cdot z_1^2 + b \cdot z_1^3$. The inverse is given by the map

$$\Phi^{-1} : E(\mathbb{F}\mathbb{P}^2) \to E(\mathbb{F}) \; : \; [X:Y:Z] \mapsto \begin{cases} (\frac{X}{Z}, \frac{Y}{Z}) & \text{if } Z \neq 0 \\ \mathcal{O} & \text{if } Z = 0 \end{cases} \tag{5.6}$$

Note the only projective point $[X:Y:Z]$ with $Z \neq 0$ that satisfies XXX is given by the class $[0:1:0]$.

add reference

One key feature of Φ and its inverse is, that it respects the group structure, which means that $\Phi((x_1,y_1) \oplus (x_2,y_2))$ is equal to $\Phi(x_1,y_1) \oplus \Phi(x_2,y_2)$. The same holds true for the inverse map Φ^{-1}.

Maps with these properties are called **group isomorphisms** and from a mathematical point of view the existence of Φ implies, that both definition are equivalent and implementations can choose freely between both representations.

5.1.2 Montgomery Curves

History and use of them (optimized scalar multiplication)

Affine Montgomery Form To see what a Montgomery curve in affine coordinates is, let $\mathbb{F}$ be a finite field of characteristic > 2 and $A, B \in \mathbb{F}$ two field elements such that $B \neq 0$ and $A^2 \neq 4$. Then a **Montgomery elliptic curve** $M(\mathbb{F})$ over $\mathbb{F}$ in its affine representation is the set

$$M(\mathbb{F}) = \{(x,y) \in \mathbb{F} \times \mathbb{F} \mid B \cdot y^2 = x^3 + A \cdot x^2 + x\} \bigcup \{\mathcal{O}\} \tag{5.7}$$

of all pairs of field elements $(x,y) \in \mathbb{F} \times \mathbb{F}$, that satisfy the Montgomery cubic equation $B \cdot y^2 = x^3 + A \cdot x^2 + x$, together with a distinguished symbol $\mathcal{O}$, called the **point at infinity**.

Despite the fact that Montgomery curves look different then short Weierstraß curve, they are in fact just a special way to describe certain short Weierstraß curves. In fact every curve in affine Montgomery form can be transformed into an elliptic curve in Weierstraß form. To see that assume that a curve in Montgomery form $By^2 = x^3 + Ax^2 + x$ is given. The associated Weierstraß form is then

$$y^2 = x^3 + \frac{3 - A^2}{3B^2} \cdot x + \frac{2A^3 - 9A}{27B^3}$$

On the other hand, an elliptic curve $E(\mathbb{F})$ over base field $\mathbb{F}$ in Weierstraß form $y^2 = x^3 + ax + b$ can be converted to Montgomery form if and only if the following conditions hold:

- The number of points on $E(F)$ is divisible by 4
- The polynomial $z^3+az+b \in \mathbb{F}[z]$ has at least one root $z_0 \in \mathbb{F}$
- $3z_0^2+a$ is a quadratic residue in $\mathbb{F}$.

When these conditions are satisfied, then for $s = (\sqrt{3z_0^2+a})^{-1}$ the equivalent Montgomery curve is defined by the equation

$$sy^2 = x^3 + (3z_0 s)x^2 + x$$

If those properties are meet it is therefore possible to transform certain Weierstraß curve into Montgomery form. In the following example we will look at our pen-jubjub curve again and show that it is actually a Montgomery curve.

Example 77. Consider the prime field $\mathbb{F}_{13}$ and the pen-jubjub curve *PJJ_13* from example XXX. To see that it is a Montgomery curve, we have to check the properties from XXX:

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Since the order of *PJJ_13* is 20, which is divisible by 4, the first requirement is meet. Next, since $a = 8$ and $b = 8$, we have check if the polynomial $P(z) = z^3+8z+8$ has a root in $\mathbb{F}_{13}$. We simply evaluate P at all numbers $z \in \mathbb{F}_{13}$ a find that $P(4) = 0$, so a root is given by $z_0 = 4$. In a last step we have to check, that $3 \cdot z_0^2 + a$ has a root in $\mathbb{F}_{13}$. We compute

$$\begin{aligned} 3z_0^2 + a &= 3\cdot 4^2 + 8 \\ &= 3\cdot 3 + 8 \\ &= 9+8 \\ &= 4 \end{aligned}$$

To see if 4 is a quadratic residue, we can use Euler's criterion XXX to compute the Legendre symbol of 4. We get:

add reference

$$\left(\frac{4}{13}\right) = 4^{\frac{13-1}{2}} = 4^6 = 1$$

so 4 indeed has a root in $\mathbb{F}_{13}$. In fact computing a root of 4 in $\mathbb{F}_{13}$ is easy, since the integer root 2 of 4 is also one of its roots in $\mathbb{F}_{13}$. The other root is given by $13-4=9$.

Now since all requirements are meet, we have shown that *PJJ_13* is indeed a Montgomery curve and we can use XXX to compute its associated Montgomery form. We compute

add reference

$$\begin{aligned} s &= \left(\sqrt{3\cdot z_0^2+8}\right)^{-1} \\ &= 2^{-1} && \text{\# Fermat's little theorem} \\ &= 2^{13-2} && \text{\# } 2048 \bmod 13 = 7 \\ &= 7 \end{aligned}$$

The defining equation for the Montgomery form of our pen-jubjub curve is then given by the following equation

$$\begin{aligned} sy^2 &= x^3 + (3z_0 s)x^2 + x && \Rightarrow \\ 7\cdot y^2 &= x^3 + (3\cdot 4\cdot 7)x^2 + x && \Leftrightarrow \\ 7\cdot y^2 &= x^3 + 6x^2 + x \end{aligned}$$

So we get the defining parameters as $B=7$ and $A=6$ and we can write the pen-jubjub curve in its affine Montgomery representation as

$$PJJ\_13 = \{(x,y)\in\mathbb{F}_{13}\times\mathbb{F}_{13} \mid 7\cdot y^2 = x^3+6x^2+x\}\bigcup\{\mathcal{O}\}$$

Now that we have the abstract definition of our pen-jubjub curve in Montgomery form, we can compute the set of points, by inserting all pairs $(x,y)\in\mathbb{F}_{13}\times\mathbb{F}_{13}$ similar to how we computed the curve points in its Weierstraß representation. We get

$$\begin{aligned}PJJ\_13 = \{&\mathcal{O},(0,0),(1,4),(1,9),(2,4),(2,9),(3,5),(3,8),(4,4),(4,9),\\ &(5,1),(5,12),(7,1),(7,12),(8,1),(8,12),(9,2),(9,11),(10,3),(10,10)\}\end{aligned}$$

```
sage: F13 = GF(13)
sage: L_MPJJ = []
....: for x in F13:
....:     for y in F13:
....:         if F13(7)*y^2 == x^3 + F13(6)*x^2 +x:
....:             L_MPJJ.append((x,y))
sage: MPJJ = Set(L_MPJJ)
sage: # does not compute the point at infinity
```

Affine Montgomery coordinate transformation Comparing the Montgomery representation of the previous example with the Weierstraß representation of the same curve, we see that there is a 1:1 correspondence between the curve points in both examples. This is no accident. In fact if $M_{A,B}$ is a Montgomery curve and $E_{a,b}$ a Weierstraß curve with $a=\frac{3-A^2}{3B^2}$ and $b=\frac{2A^2-9A}{27B^3}$ then the function

$$\Phi : M_{A,B}\to E_{a,b} \;:\; (x,y)\mapsto\left(\frac{3x+A}{3B},\frac{y}{B}\right) \tag{5.8}$$

maps all points in Montgomery representation onto the points in Weierstraß representation. This map is a 1:1 correspondence (am isomorphism) and its inverse map is given by

$$\Phi^{-1} : E_{a,b}\to M_{A,B} \;:\; (x,y)\mapsto(s\cdot(x-z_0),s\cdot y) \tag{5.9}$$

where z_0 is a root of the polynomial $z^3+az+b\in\mathbb{F}[z]$ and $s=(\sqrt{3z_0^2+a})^{-1}$. Using this map, it is therefore possible for implementations of Montgomery curves to freely transit between the Weierstraß and the Montgomery representation. Note however, that according to XXX not every Weierstraß curve is a Montgomery curve, as all of the properties from XXX have to be satisfied. The map Φ^{-1} therefore does not always exists.

Example 78. Consider our pen-jubjub curve again. In example XXX we derive its Weierstraß representation and in example XXX we derive its Montgomery representation.

To see how the coordinate transformation Φ works in this example, let's map points from the Montogomery representation onto points from the Weierstraß representation. Inserting for example the point $(0,0)$ from the Montgomery representation XXX into Φ gives

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$$
\begin{aligned}
\Phi(0,0) &= \left(\frac{3\cdot 0 + A}{3B}, \frac{0}{B}\right)\\
&= \left(\frac{3\cdot 0 + 6}{3\cdot 7}, \frac{0}{7}\right)\\
&= \left(\frac{6}{8}, 0\right)\\
&= (4,0)
\end{aligned}
$$

So the Montgomery point $(0,0)$ maps to the self inverse point $(4,0)$ of the Weierstraß representation. On the other hand we can use our computations of $s = 7$ and $z_0 = 4$ from XXX, to compute the inverse map Φ^{-1}, which maps point on the Weiertraß representation to points on the Mongomery form. Inserting for example $(4,0)$ we get

add reference

$$
\begin{aligned}
\Phi^{-1}(4,0) &= (s\cdot(4 - z_0), s\cdot 0)\\
&= (7\cdot(4-4), 0)\\
&= (0,0)
\end{aligned}
$$

So as expected, the inverse maps maps the Weierstraß point back to where it came from on the Montgomery form. We can invoke Sage to proof that our computation of Φ is correct:

```
sage: # Compute PHI of Montgomery form:
sage: L_PHI_MPJJ = []
sage: for (x,y) in L_MPJJ: # LMJJ as defined previously
....:     v = (F13(3)*x + F13(6))/(F13(3)*F13(7))
....:     w = y/F13(7)
....:     L_PHI_MPJJ.append((v,w))
sage: PHI_MPJJ = Set(L_PHI_MPJJ)
sage: # Computation Weierstrass form
sage: C_WPJJ = EllipticCurve(F13,[8,8])
sage: L_WPJJ = [P.xy() for P in C_WPJJ.points() if P.order() >
    1]
sage: WPJJ = Set(L_WPJJ)
sage: # check PHI(Montgomery) == Weierstrass
sage: WPJJ == PHI_MPJJ
True
sage: # check the inverse map PHI^(-1)
sage: L_PHIINV_WPJJ = []
sage: for (v,w) in L_WPJJ:
....:     x = F13(7)*(v-F13(4))
....:     y = F13(7)*w
....:     L_PHIINV_WPJJ.append((x,y))
sage: PHIINV_WPJJ = Set(L_PHIINV_WPJJ)
sage: MPJJ == PHIINV_WPJJ
True
```

Montgomery group law So we see that Montgomery curves a special cases of short Weierstraß curves. As such they have a group structure defined on the set of their points, which can also be derived from a chord and tangent rule. In accordance with short Weierstraß curves, it can be shown that the identity $x_1 = x_2$ implies $y_2 = \pm y_1$, which shows that the following rules are a complete description of the affine addition law.

- (Neutral element) Point at infinity $\mathcal{O}$ is the neutral element.
- (Additive inverse) The additive inverse of $\mathcal{O}$ is $\mathcal{O}$ and for any other curve point $(x,y) \in M(\mathbb{F}_q)\backslash\{\mathcal{O}\}$, the additive inverse is given by $(x,-y)$.
- (Addition rule) For any two curve points $P,Q \in M(\mathbb{F}_q)$ addition is defined by one of the following cases:
  1. (Adding the neutral element) If $Q = \mathcal{O}$, then the sum is defined as $P+Q = P$.
  2. (Adding inverse elements) If $P = (x,y)$ and $Q = (x,-y)$, the sum is defined as $P+Q = \mathcal{O}$.
  3. (Adding non self-inverse equal points) If $P = (x,y)$ and $Q = (x,y)$ with $y \neq 0$, the sum $2P = (x',y')$ is defined by

$$x' = (\tfrac{3x_1^2+2Ax_1+1}{2By_1})^2 \cdot B - (x_1+x_2) - A \quad , \quad y' = \tfrac{3x_1^2+2Ax_1+1}{2By_1}(x_1 - x') - y_1$$

  4. (Adding non inverse differen points) If $P = (x_1,y_1)$ and $Q = (x_2,y_2)$ such that $x_1 \neq x_2$, the sum $R = P+Q$ with $R = (x_3,y_3)$ is defined by

$$x' = (\tfrac{y_2-y_1}{x_2-x_1})^2 B - (x_1+x_2) - A \quad , \quad y' = \tfrac{y_2-y_1}{x_2-x_1}(x_1 - x') - y_1$$

5.1.3 Twisted Edwards Curves

As we have seen in XXX both Weierstraß and Montgomery curves have somewhat complicated addition and doubling laws as many cases have to be distinguished. Those cases translate to branches in computer programs.

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In the context of SNARK development two computational models for bounded computations, called **circuits** and **rank-1 constraint systems**, are used and program-branches are undesirably costly, when implemented in those models. It is therefore advantageous to look for curves with an addition/doubling rule, that requires no branches and as few field operations as possible.

Twisted Edwards curves are particular useful here as a subclass of these curves has a compact and easy to implement addition law that works for all point, including the point at infinity. Implementing that rule therefore needs no branching.

Twisted Edwards Form To see what an affine **twisted Edwards curve** looks like, let $\mathbb{F}$ be a finite field of characteristic > 2 and $a,d \in \mathbb{F}\backslash\{0\}$ two non zero field elements with $a \neq d$. Then a **twisted Edwards elliptic curve** in its affine representation is the set

$$E(\mathbb{F}) = \{(x,y) \in \mathbb{F} \times \mathbb{F} \mid a \cdot x^2 + y^2 = 1 + d \cdot x^2 y^2\} \tag{5.10}$$

of all pairs (x,y) from $\mathbb{F} \times \mathbb{F}$, that satisfy the twisted Edwards equation $a \cdot x^2 + y^2 = 1 + d \cdot x^2 y^2$. A twisted Edwards curve is called an Edwards curve (non twisted), if the parameter a is equal to 1 and is called a **SNARK-friendly** twisted Edwards curve if the parameter a is a quadratic residue and the parameter d is a quadratic non residue.

As we can see from the definition, affine twisted Edwards curve look somewhat different from Weierstraß curves as their affine representation does not need a special symbol to represent the point at infinity. In fact we we will see that the pair $(0,1)$ is always a point on any twisted Edwards curve and that it takes the role of the point at infinity.

Despite the different looks however, twisted Edwards curves are equivalent to Montgomery curves in the sense that for every twisted Edwards curve there is a Montgomery curve and a way to map the points of one curve in a 1:1 correspondence onto the other and vice versa. To see that assume that a curve in twisted Edwards form $a\cdot x^2+y^2=1+d\cdot x^2y^2$ is given. The associated Montgomery curve is then defined by the Montgomery equation

$$\frac{4}{a-d}y^2 = x^3+\frac{2(a+d)}{a-d}\cdot x^2+x \tag{5.11}$$

On the other hand a Montgomery curve $By^2=x^3+Ax^2+x$ with $B\neq 0$ and $A^2\neq 4$ can gives rise to a twisted Edwards curve defined by the equation

$$(\frac{A+2}{B})x^2+y^2 = 1+(\frac{A-2}{B})x^2y^2 \tag{5.12}$$

Recalling from XXX that Montgomery curves are just a special class of Weierstraß, we now know that twisted Edwards curve are special Weierstraß curves too. So the more general way to describe elliptic curves are Weierstraß curves.

add reference

Example 79. Consider the pen jubjub curve from example XXX again. We know from XXX that it is a Montgomery curve and since Montgomery curves are equivalent to twisted Edwards curve, we want to write that curve in twisted Edwards form. We use XXX and compute the parameters a and d as

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$$\begin{aligned}
a &= \frac{A+2}{B} && \text{\# insert A=6 and B=7}\\
&= \frac{8}{7}=3 && \# \ 7^{-1}=2
\end{aligned}$$

$$\begin{aligned}
d &= \frac{A-2}{B}\\
&= \frac{4}{7}=8
\end{aligned}$$

So we get the defining parameters as $a=3$ and $d=8$. Since our goal is to use this curve later on in implementations of pen-and-paper SNARKs, let's show that tiny-jubjub is moreover a **SNARK-friendly** twisted Edwards curve. To see that, we have to show that a is a quadratic residue and d is a quadratic non residue. We therefore compute the Legendre symbols of a and d using the Euler criterium. We get

$$\begin{aligned}
\left(\frac{3}{13}\right) &= 3^{\frac{13-1}{2}}\\
&= 3^6 = 1
\end{aligned}$$

$$\begin{aligned}
\left(\frac{8}{13}\right) &= 8^{\frac{13-1}{2}}\\
&= 8^6 = 12 = -1
\end{aligned}$$

which proofs that tiny-jubjub is SNARK-friendly. We can write the tiny-jubjub curve in its affine twisted Edwards representation as

$$TJJ\_13 = \{(x,y) \in \mathbb{F}_{13} \times \mathbb{F}_{13} \mid 3 \cdot x^2 + y^2 = 1 + 8 \cdot x^2 \cdot y^2\}$$

Now that we have the abstract definition of our pen-jubjub curve in twisted Edwards form, we can compute the set of points, by inserting all pairs $(x,y) \in \mathbb{F}_{13} \times \mathbb{F}_{13}$ similar to how we computed the curve points in its Weierstraß or Edwards representation. We get

$$\begin{aligned} PJJ\_13 = \{&(0,1),(0,12),(1,2),(1,11),(2,6),(2,7),(3,0),(5,5),(5,8),(6,4),\\ &(6,9),(7,4),(7,9),(8,5),(8,8),(10,0),(11,6),(11,7),(12,2),(12,11)\}\end{aligned}$$

```
sage: F13 = GF(13)
sage: L_EPJJ = []
....: for x in F13:
....:     for y in F13:
....:         if F13(3)*x^2 + y^2 == 1+ F13(8)*x^2*y^2:
....:             L_EPJJ.append((x,y))
sage: EPJJ = Set(L_EPJJ)
```

Twisted Edwards group law As we have seen, twisted Edwards curves are equivalent to Montgomery curves and as such also have a group law. However, in contrast to Montgomery and Weierstraß curves, the group law of SNARK-friendly twisted Edwards curves can be described by single computation, that works in all cases, no matter if we add the neutral element, inverse, or if have to double a point. To see how the group law looks like, first observe that the point $(0,1)$ is a solution to $a \cdot x^2 + y^2 = 1 + d \cdot x^2 \cdot y^2$ for any curve. The sum of any two points (x_1,y_1), (x_2,y_2) on an Edwards curve $E(\mathbb{F})$ is then given by

$$(x_1,y_1) \oplus (x_2,y_2) = \left(\frac{x_1y_2 + y_1x_2}{1 + dx_1x_2y_1y_2}, \frac{y_1y_2 - ax_1x_2}{1 - dx_1x_2y_1y_2}\right)$$

and it can be shown that the point $(0,1)$ serves as the neutral element and the inverse of a point (x_1,y_1) is given by $(-x_1,y1)$.

Example 80. Lets look at the tiny-jubjub curve in Edwards form from example XXX again. As we have seen, this curve is given by

add reference

$$\begin{aligned} PJJ\_13 = \{&(0,1),(0,12),(1,2),(1,11),(2,6),(2,7),(3,0),(5,5),(5,8),(6,4),\\ &(6,9),(7,4),(7,9),(8,5),(8,8),(10,0),(11,6),(11,7),(12,2),(12,11)\}\end{aligned}$$

To get an understanding of the twisted Edwards addition law, let's first add the neutral element $(0,1)$ to itself. We apply the group law XXX and get

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$$\begin{aligned}(0,1) \oplus (0,1) &= \left(\frac{0 \cdot 1 + 1 \cdot 0}{1 + 8 \cdot 0 \cdot 0 \cdot 1 \cdot 1}, \frac{1 \cdot 1 - 3 \cdot 0 \cdot 0}{1 - 8 \cdot 0 \cdot 0 \cdot 1 \cdot 1}\right)\\ &= (0,1)\end{aligned}$$

So as expected, adding the neutral element to itself gives the neutral element again. Now let's add the neutral element to some other curve point. We get

$$\begin{aligned}(0,1)\oplus(8,5) &= \left(\frac{0\cdot 5+1\cdot 8}{1+8\cdot 0\cdot 8\cdot 1\cdot 5}, \frac{1\cdot 5-3\cdot 0\cdot 8}{1-8\cdot 0\cdot 8\cdot 1\cdot 5}\right)\\ &= (8,5)\end{aligned}$$

Again as expected adding the neutral element to any element will give the element again. Given any curve point (x,y), we know that the inverse is given by $(-x,y)$. To see how the addition of a point to its inverse works out we therefore compute

$$\begin{aligned}(5,5)\oplus(8,5) &= \left(\frac{5\cdot 5+5\cdot 8}{1+8\cdot 5\cdot 8\cdot 5\cdot 5}, \frac{5\cdot 5-3\cdot 5\cdot 8}{1-8\cdot 5\cdot 8\cdot 5\cdot 5}\right)\\ &= \left(\frac{12+1}{1+5}, \frac{12-3}{1-5}\right)\\ &= \left(\frac{0}{6}, \frac{12+10}{1+8}\right)\\ &= \left(0, \frac{9}{9}\right)\\ &= (0,1)\end{aligned}$$

So adding a curve point to its inverse gives the neutral element, as expected. As we have seen from these examples the twisted Edwards addition law handles edge cases particularly nice and in a unified way.

5.2 Elliptic Curves Pairings

As we have seen in XXX some groups comes with the notation of a so-called pairing map, which is a non-degenerate bilinear map, from two groups into another group. add reference

In this section, we discuss **pairings on elliptic curves**, which form the basis of several zk-SNARKs and other zero knowledge proof schemes. The SNARKs derived from pairings have the advantage of constant-sized proof sizes, which is crucial to blockchains.

We start out by defining elliptic curve pairings and discussing a simple application which bears some resemblance to the more advanced SNARKs. We then introduce the pairings arising from elliptic curves and describe Miller's algorithm which makes these pairings practical rather than just theoretically interesting.

Elliptic curves have a few structures, like the Weil or the Tate map, that qualifies as pairing.

Embedding Degrees As we will see in what follows, every elliptic curve gives rise to a pairing map. However, as we will see in example XXX, not every such pairing is efficiently computable. So in order to distinguish curves with efficiently computable pairings from the rest, we need to start with an introduction to the so-called **embedding degree** of a curve. add reference

To understand this term, let $\mathbb{F}$ be a finite field, $E(\mathbb{F})$ an elliptic curve over $\mathbb{F}$, and n a prime number that divides the order of $E(\mathbb{F})$. The embedding degree of $E(\mathbb{F})$ with respect to n is then the smallest integer k such that n divides q^k-1.

Fermat's little theorem XXX implies, that every curve has at least **some** embedding degree k, since at least $k=n-1$ is always a solution to the congruency $q^k \equiv 1 \pmod{n}$ which implies that the remainder of the integer division of q^k-1 by n is 0. add reference

Example 81. To get a better intuition of the embedding degree, let's consider the elliptic curve $E_1(\mathbb{F}_5)$ from example XXX. We know from XXX that the order of $E_1(\mathbb{F}_5)$ is 9 and since the only prime factor of 9 is 3, we compute the embedding degree of $E_1(\mathbb{F}_5)$ with respect to 3.

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To find that embedding degree we have to find the smallest integer k, such that 3 divides $q^k - 1 = 5^k - 1$. We try and increment until we find a proper k.

$$\begin{aligned} k = 1&: 5^1 - 1 = 4 && \text{not divisible by } 3 \\ k = 2&: 5^2 - 1 = 24 && \text{divisible by } 3 \end{aligned}$$

So we know that the embedding degree of $E_1(\mathbb{F}_5)$ is 2 relative to the the prime factor 3.

Example 82. Lets consider the tiny jubjub curve *TJJ_13* from example XXX. We know from XXX that the order of *TJJ_13* is 20 and that the order therefore has two prime factors. A "large" prime factor 5 and a small prime factor 2.

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We start by computing the ebedding degree of *TJJ_13* with respect to the large prime factor 5. To find that embedding degree we have to find the smallest integer k, such that 5 divides $q^k - 1 = 13^k - 1$. We try and increment until we find a proper k.

$$\begin{aligned} k = 1&: 13^1 - 1 = 12 && \text{not divisible by } 5 \\ k = 2&: 13^2 - 1 = 168 && \text{not divisible by } 5 \\ k = 3&: 13^3 - 1 = 2196 && \text{not divisible by } 5 \\ k = 4&: 13^4 - 1 = 28560 && \text{divisible by } 5 \end{aligned}$$

So we know that the embedding degree of *TJJ_13* is 4 relative to the the prime factor 5.

In real-world applications, like on pairing friendly elliptic curves as for example BLS_12-381, usually only the embedding degree of the large prime factor are relevant, which in case of out tiny-jubjub curve, is represented by 5. It should however, be noted that every prime factor of a curves order has its own notation of embedding degree despite the fact that this is mostly irrelevant in applications.

To find the embedding degree of the small prime factor 2 we have to find the smallest integer k, such that 2 divides $q^k - 1 = 13^k - 1$. We try and increment until we find a proper k.

$$k = 1: 13^1 - 1 = 12 \qquad \text{divisible by } 2$$

So we know that the embedding degree of *TJJ_13* is 1 relative to the the prime factor 2. So as we have seen, different prime factors can have different embedding degrees in general.

```
sage: p = 13
sage: # large prime factor
sage: n = 5
sage: for k in range(1,5): # Fermat's little theorem
....:     if (p^k-1)%n == 0:
....:         break
sage: k
4
sage: # small prime factor
sage: n = 2
sage: for k in range(1,2): # Fermat's little theorem
....:     if (p^k-1)%n == 0:
....:         break
```

```
sage: k
1
```

Example 83. To give an example of a cryptographically secure real-world elliptic curve that does not have a small embedding degree let's look at curve secp256k1 again. We know from XXX that the order of this curve is a prime number, so we only have a single embedding degree.

add reference

To test potential embedding degrees k, say in the range $1 \ldots 1000$, we can invoke Sage and compute:

```
sage: p = 11579208923731619542357098500868790785326998466564056
    4039457584007908834671663
sage: n = 11579208923731619542357098500868790785283756427907490
    4382605163141518161494337
sage: for k in range(1,1000):
....:     if (p^k-1)%n == 0:
....:         break
sage: k
999
```

So we see that secp256k1 has at least no embedding degree $k < 1000$, which renders secp256k1 as a curve that has no small embedding degree. A property that is of importance later on.

Elliptic Curves over extension fields Suppose that p is a prime number and $\mathbb{F}_p$ its associated prime field. We know from XXX, that the fields $\mathbb{F}_{p^m}$ are extensions of $\mathbb{F}_p$ in the sense that $\mathbb{F}_p$ is a subfield of $\mathbb{F}_{p^m}$. This implies that we can extend the affine plane an elliptic curve is defined on, by changing the base field to any extension field. To be more precise let $E(\mathbb{F}) = \{(x,y) \in \mathbb{F} \times \mathbb{F} \mid y^2 = x^3 + a \cdot x + b\}$ be an affine short Weierstraß curve, with parameters a and b taken from $\mathbb{F}$. If $\mathbb{F}'$ is any extension field of $\mathbb{F}$, then we extend the domain of the curve by defining

add reference

$$E(\mathbb{F}') = \{(x,y) \in \mathbb{F}' \times \mathbb{F}' \mid y^2 = x^3 + a \cdot x + b\} \tag{5.13}$$

So while we did not change the defining parameters, we consider curve points from the affine plane over the extension field now. Since $\mathbb{F} \subset \mathbb{F}'$ it can be shown that the original elliptic curve $E(\mathbb{F})$ is a sub curve of the extension curve $E(\mathbb{F}')$.

Example 84. Consider the prime field $\mathbb{F}_5$ from example XXX and the elliptic curve $E_1(\mathbb{F}_5)$ from example XXX. Since we know from XXX that $\mathbb{F}_{5^2}$ is an extension field of $\mathbb{F}_5$, we can extend the definition of $E_1(\mathbb{F}_5)$ to define a curve over $\mathbb{F}_{5^2}$:

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$$E_1(\mathbb{F}_{5^2}) = \{(x,y) \in \mathbb{F} \times \mathbb{F} \mid y^2 = x^3 + x + 1\}$$

Since $\mathbb{F}_{5^2}$ contains 25 points, in order to compute the set $E_1(\mathbb{F}_{5^2})$, we have to try $25 \cdot 25 = 625$ pairs, which is probably a bit too much for the average motivated reader. Instead, we involve Sage to compute the curve for us. To do so choose the representation of $\mathbb{F}_{5^2}$ from XXX. We get:

add reference

```
sage: F5= GF(5)
sage: F5t.<t> = F5[]
sage: P = F5t(t^2+2)
sage: P.is_irreducible()
True
```

```
sage: F5_2.<t> = GF(5^2, name='t', modulus=P)
sage: E1F5_2 = EllipticCurve(F5_2,[1,1])
sage: E1F5_2.order()
27
```

So curve $E_1(\mathbb{F}_{5^2})$ consist of 27 points, in contrast to curve $E_1(\mathbb{F}_5)$, which consists of 9 points. Printing the points gives

$$
\begin{aligned}
E_1(\mathbb{F}_{5^2}) = \{&\mathcal{O},(0,4),(0,1),(3,4),(3,1),(4,3),(4,2),(2,4),(2,1),\\
&(4t+3,3t+4),(4t+3,2t+1),(3t+2,t),(3t+2,4t),\\
&(2t+2,t),(2t+2,4t),(2t+1,4t+4),(2t+1,t+1),\\
&(2t+3,3),(2t+3,2),(t+3,2t+4),(t+3,3t+1),\\
&(3t+1,t+4),(3t+1,4t+1),(3t+3,3),(3t+3,2),(1,4t)\}
\end{aligned}
$$

As we can see, curve $E_1(\mathbb{F}_5)$ sits inside curve $E(\mathbb{F}_{5^2})$, which is implied from $\mathbb{F}_5$ being a subfield of $\mathbb{F}_{5^2}$.

Full Torsion groups The fundamental theorem of finite cyclic groups XXX implies, that every prime factor n of a cyclic groups order defines a subgroup of the size of the prime factor. We called such a subgroup an n-torsion group. We have seen many of those subgroups in the examples XXX and XXX.

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Now when we consider elliptic curve extensions as defined in XXX, we could ask, what happens to the n-torsion groups in the extension. One might intuitively think that their extension just parallels the extension of the curve. For example when $E(\mathbb{F}_p)$ is a curve over prime field $\mathbb{F}_p$, with some n-torsion group $\mathbb{G}$ and when we extend the curve to $E(\mathbb{F}_{p^m})$, then there is a bigger n-torsion group, such that $\mathbb{G}$ is a subgroup. Naively this would make sense, as $E(\mathbb{F}_p)$ is a sub-curve of $E(\mathbb{F}_{p^m})$.

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However, the real situation is a bit more surprising than that. To see that, let $\mathbb{F}_p$ be a prime field and $E(\mathbb{F}_p)$ an elliptic curve of order r, with embedding degree k and n-torsion group $E(\mathbb{F}_p)[n]$ for same prime factor n of r. Then it can be shown that the n-torsion group $E(\mathbb{F}_{p^m})[n]$ of a curve extension is equal to $E(\mathbb{F}_p)[n]$, as long as the power m is less then the embedding degree k of $E(\mathbb{F}_p)$.

However, for the prime power p^m, for any $m \geq k$, $E(\mathbb{F}_{p^m})[n]$ is strictly larger then $E(\mathbb{F}_p)[n]$ and contains $E(\mathbb{F}_p)[n]$ as a subgroup. We call the n-torsion group $E(\mathbb{F}_{p^k})[n]$ of the extension of E over $\mathbb{F}_{p^k}$ the **full n-torsion group** of that elliptic curve. It can be shown that it contains n^2 many elements and consists of $n+1$ subgroups, one of which is $E(\mathbb{F}_p)[n]$.

So roughly speaking, when we consider towers of curve extensions $E(\mathbb{F}_{p^m})$, ordered by the prime power m, then the n-torsion group stays constant for every level m small then the embedding degree, while it suddenly blossoms into a larger group on level k, with $n+1$ subgroups and it then stays like that for any level m larger then k. In other words, once the extension field is big enough to find one more point of order n (that is not defined over the base field), then we actually find all of the points in the full torsion group.

Example 85. Consider curve $E_1(\mathbb{F}_5)$ again. We know that it contains a 3-torsion group and that the embedding degree of 3 is 2. From this we can deduce that we can find the full 3-torsion group $E_1[3]$ in the curve extension $E_1(\mathbb{F}_{5^2})$, the latter of which we computed in XXX.

add reference

Since that curve is small, in order to find the full 3-torsion, we can loop through all elements of $E_1(\mathbb{F}_{5^2})$ and check check the defining equation $[3]P = \mathcal{O}$. Invoking Sage, we compute

```
sage: INF = E1F5_2(0) # Point at infinity
sage: L_E1_3 = []
sage: for p in E1F5_2:
....:     if 3*p == INF:
....:         L_E1_3.append(p)
sage: E1_3 = Set(L_E1_3) # Full 3-torsion set
```

we get

$$E_1[3] = \{\mathcal{O}, (1,t), (1,4t), (2,1), (2,4), (2t+1,t+1), (2t+1,4t+4), (3t+1,t+4), (3t+1,4t+1)\}$$

Example 86. Consider the tiny jubjub curve from example XXX. we know from XXX that it contains a 5-torsion group and that the embedding degree of 5 is 4. This implies that we can find the full 5-torsion group $TJJ\_13[5]$ in the curve extension $TJJ\_13(\mathbb{F}_{13^4})$.

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To compute the full torsion, first observe that since $\mathbb{F}_{13^4}$ contains 28561 element, computing $TJJ\_13(\mathbb{F}_{13^4})$ means checking $28561^2 = 815730721$ elements. From each of these curve points P, we then have to check the equation $[5]P = \mathcal{O}$. Doing this for 815730721 is a bit to slow even on a computer.

Fortunately, Sage has a way to loop through points of given order efficiently. The following Sage code then gives a way to compute the full torsion group:

```
sage: # define the extension field
sage: F13= GF(13) # prime field
sage: F13t.<t> = F13[] # polynomials over t
sage: P = F13t(t^4+2) # irreducible polynomial of degree 4
sage: P.is_irreducible()
True
sage: F13_4.<t> = GF(13^4, name='t', modulus=P) # F_{13^4}
sage: TJJF13_4 = EllipticCurve(F13_4,[8,8]) # tiny jubjub
  extension
sage: # compute the full 5-torsion
sage: L_TJJF13_4_5 = []
sage: INF = TJJF13_4(0)
sage: for P in INF.division_points(5): # [5]P == INF
....:     L_TJJF13_4_5.append(P)
sage: len(L_TJJF13_4_5)
25
sage: TJJF13_4_5 = Set(L_TJJF13_4_5)
```

So, as expected, we get a group that contains $5^2 = 25$ elements. As it's rather tedious to write this group down and as we don't need it in what follows, we skip writing it. To see that the embedding degree 4 is actually the smallest prime power to find the full 5-torsion group, let's compute the 5-torsion group over of the tiny-jubjub curve the extension field $\mathbb{F}_{13^3}$. We get

```
sage: # define the extension field
sage: P = F13t(t^3+2) # irreducible polynomial of degree 3
sage: P.is_irreducible()
True
sage: F13_3.<t> = GF(13^3, name='t', modulus=P) # F_{13^3}
sage: TJJF13_3 = EllipticCurve(F13_3,[8,8]) # tiny jubjub
  extension
```

```
sage: # compute the 5-torsion
sage: L_TJJF13_3_5 = []
sage: INF = TJJF13_3(0)
sage: for P in INF.division_points(5): # [5]P == INF
....:     L_TJJF13_3_5.append(P)
sage: len(L_TJJF13_3_5)
5
sage: TJJF13_3_5 = Set(L_TJJF13_3_5) # full $5$-torsion
```

So as we can see the 5-torsion group of tiny-jubjub over $\mathbb{F}_{13^3}$ is equal to the 5-torsion group of tiny-jubjub over $\mathbb{F}_{13}$ itself.

Example 87. Let's look at curve Secp256k1. We know from XXX that the curve is of some prime order r and hence the only n-torsion group to consider is the curve itself. So the curve group is the r-torsion.

add reference

However, in order to find the full r-torsion of Secp256k1, we need to compute the embedding degree k and as we have seen in XXX it is at least not small. We know from Fermat's little theorem that a finite embedding degree must exist, though. It can be shown that it is given by

add reference

$$k = 19298681539552699237261830834781317547292737984581739710086052358636024 9056$$

which is a 256-bit number. So the embedding degree is huge, which implies that the fiel extension $\mathbb{F}_{p^k}$ is huge too. To understand how big $\mathbb{F}_{p^k}$ is, recall that an element of $\mathbb{F}_{p^m}$ can be represented as a string $[x_0, \ldots, x_m]$ of m elements, each containing a number from the prime field $\mathbb{F}_p$. Now in the case of Secp256k1, such a representation has k-many entries, each of 256 bits in size. So without any optimizations, representing such an element would need $k \cdot 256$ bits, which is too much to be represented in the observable universe.

Torsion-Subgroups As we have stated above, any full n-torsion group contains $n+1$ cyclic subgroups, two of which are of particular interest in pairing-based elliptic curve cryptography. To characterize these groups we need to consider the so-called **Frobenius** endomorphism

$$\pi : E(\mathbb{F}) \to E(\mathbb{F}) : \begin{array}{ccc} (x,y) & \mapsto & (x^p, y^p) \\ \mathcal{O} & \mapsto & \mathcal{O} \end{array} \tag{5.14}$$

of an elliptic curve $E(\mathbb{F})$ over some finite field $\mathbb{F}$ of characteristic p. It can be shown that π maps curve points to curve points. The first thing to note is that in case that $\mathbb{F}$ is a prime field, the Frobenius endomorphism acts trivially, since $(x^p, y^p) = (x, y)$ on prime fields, due to Fermat's little theorem XX. So the Frobenius map is more interesting over prime field extensions.

With the Frobenius map at hand, we can now characterize two important subgroups of the full n-torsion. The first subgroup is the n-torsion group that already exists in the curve over the base field. In pairing-based cryptography this group is usually written as $\mathbb{G}_1$, assuming that the prime factor 'n' in the definition is implicitly given. Since we know that the Frobenius map, acts trivially on curve over the prime field we can define $\mathbb{G}_1$ as:

$$\mathbb{G}_1[n] := \{(x,y) \in E[n] \mid \pi(x,y) = (x,y)\ \} \tag{5.15}$$

In more mathematical terms this definition means, that $\mathbb{G}_1$ is the **Eigenspace** of the Frobenius map with respect to the **Eigenvalue** 1.

Now it can be shown, that there is another subgroup of the full n-torsion group that can be characterized by the Frobenius map. In the context of so-called type 3 pairing-based cryptography this subgroup is usually called $\mathbb{G}_2$ and it defined as

$$\mathbb{G}_2[n] := \{(x,y) \in E[n] \mid \pi(x,y) = [p](x,y)\} \tag{5.16}$$

So in mathematical terms $\mathbb{G}_2$ is the **Eigenspace** of the Frobenius map with respect to the **Eigenvalue** p.

Notation and Symbols 9. If the prime factor n of the curves order is clear from the context, we sometimes simply write $\mathbb{G}_1$ and $\mathbb{G}_2$ to mean $\mathbb{G}_1[n]$ and $\mathbb{G}_2[n]$, respectively.

It should be noted, however, that sometimes other definitions of $\mathbb{G}_2$ appear in the literature, however, in the context of pairing-based cryptography, this is the most common one. It is particularly useful, as we can define hash functions that map into $\mathbb{G}_2$, which is not possible for all subgroups of the full n-torsion.

Example 88. Consider the curve $E_1(\mathbb{F}_5)$ from example XXX again. As we have seen this curve has embedding degree $k = 2$ and a full 3-torsion group is given by

$$E_1[3] = \{\mathcal{O},(2,1),(2,4),(1,t),(1,4t),(2t+1,t+1),(2t+1,4t+4),(3t+1,t+4),(3t+1,4t+1)\}$$

According to the general theory, $E_1[3]$ contains 4 subgroups and we chan characterize the subgroups $\mathbb{G}_1$ and $\mathbb{G}_2$ using the Frobenius endomorphism. Unfortunately at the time of this writing Sage did have a predefined Frobenius endomorphism for elliptic curves, so we have to use the Frobenius endomorphism of the underlying field as a temporary workaround. We compute

```
sage: L_G1 = []
sage: for P in E1_3:
....:     PiP = E1F5_2([a.frobenius() for a in P]) # pi(P)
....:     if P == PiP:
....:         L_G1.append(P)
sage: G1 = Set(L_G1)
```

So as expected the group $\mathbb{G}_1 = \{\mathcal{O},(2,4),(2,1)\}$ is identical to the 3-torsion group of the (unextended) curve over the prime field $E_1(\mathbb{F}_5)$. We can use almost the same algorithm to compute the group $\mathbb{G}_2$ and get

```
sage: L_G2 = []
sage: for P in E1_3:
....:     PiP = E1F5_2([a.frobenius() for a in P]) # pi(P)
....:     pP = 5*P # [5]P
....:     if pP == PiP:
....:         L_G2.append(P)
sage: G2 = Set(L_G2)
```

so we compute the the second subgroup of the full 3-torsion group of curve E_1 as the set $\mathbb{G}_2 = \{\mathcal{O},(1,t),(1,4t)\}$.

Example 89. Considering the tiny-jubjub curve *TJJ_13* from example XXX. In example XXX we computed its full 5 torsion, which is a group that has 6 subgroups. We compute $G1$ using Sage as:

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```
sage: L_TJJ_G1 = []
```

```
sage: for P in TJJF13_4_5:
....:     PiP = TJJF13_4([a.frobenius() for a in P]) # pi(P)
....:     if P == PiP:
....:         L_TJJ_G1.append(P)
sage: TJJ_G1 = Set(L_TJJ_G1)
```

We get $\mathbb{G}1 = \{\mathcal{O},(7,2),(8,8),(8,5),(7,11)\}$

```
sage: L_TJJ_G1 = []
sage: for P in TJJF13_4_5:
....:     PiP = TJJF13_4([a.frobenius() for a in P]) # pi(P)
....:     pP = 13*P # [5]P
....:     if pP == PiP:
....:         L_TJJ_G1.append(P)
sage: TJJ_G1 = Set(L_TJJ_G1)
```

$\mathbb{G}_2 = \{\mathcal{O},(9t^2+7,t^3+11t,),(9t^2+7,12t^3+2t),(4t^2+7,5t^3+10t),(4t^2+7,8t^3+3t)\}$

Example 90. Consider Bitcoin's curve Secp256k1 again. Since the group $\mathbb{G}_1$ is identical to the torsion group of the unextended curve and since Secp256k1 has prime order, we know, that, in this case, $\mathbb{G}_1$ is identical to Secp256k1. It is however, infeasible not just to compute $\mathbb{G}_2$ itself, but to even compute an average element of $\mathbb{G}_2$ as elements need too much storage to be representable in this universe.

The Weil Pairing In this part we consider a pairing function defined on the subgroups $\mathbb{G}_1[r]$ and $\mathbb{G}_2[r]$ of the full r-torsion $E[r]$ of a short Weierstraß elliptic curve. To be more precise let $E(\mathbb{F}_p)$ be an elliptic curve of embedding degree k, such that r is a prime factor of its order. Then the **Weil pairing** is a bilinear, non-degenerate map

$$e(\cdot,\cdot):\mathbb{G}_1[r]\times\mathbb{G}_2[r]\to\mathbb{F}_{p^k}\ ;\ (P,Q)\mapsto(-1)^r\cdot\frac{f_{r,P}(Q)}{f_{r,Q}(P)} \tag{5.17}$$

where the extension field elements $f_{r,P}(Q), f_{r,Q}(P)\in\mathbb{F}_{p^k}$ are computed by **Miller's algorithm**: Understanding in detail how the algorithm works requires the concept of **divisors**, which we don't really need in this book. The interested reader might look at [REFERENCES]

In real-world application of pairing friendly elliptic curves, the embedding degree is usually a small number like 2, 4, 6 or 12 and the number r is the largest prime factor of the curves order.

Example 91. Consider curve $E_1(\mathbb{F}_5)$ from example XXX. Since the only prime factor of the groups order is 3 we can not compute the Weil pairing on this group using our definition of Miller's algorithm. In fact since $\mathbb{G}_1$ is of order 3, executing the if statement on line XXX will lead to a division by zero error in the computation of the slope m.

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Example 92. Consider the tiny-jubjub curve $TJJ\_13(\mathbb{F}_{13})$ from example XXX again. We want to instantiate the general definition of the Weil pairing for this example. To do so, recall that we have see in example XXX, its embedding degree is 4 and that we have the following type-3 pairing groups:

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$$\mathbb{G}_1 = \{\mathcal{O},(7,2),(8,8),(8,5),(7,11)\}$$
$$\mathbb{G}_2 = \{\mathcal{O},(9t^2+7,t^3+11t),(9t^2+7,12t^3+2t),(4t^2+7,5t^3+10t),(4t^2+7,8t^33+3t)\}$$

where $\mathbb{G}_1$ and $\mathbb{G}_2$ are subgroups of the full 5-torsion found in the curve $TJJ\_13(\mathbb{F}_{13^4})$. The type-3 Weil pairing is a map $e(\cdot,\cdot):\mathbb{G}_1\times\mathbb{G}_2\to\mathbb{F}_{13^4}$. From the first if-statement in Miller's

Algorithm 7 Miller's algorithm for short Weierstraß curves $y^2 = x^3 + ax + b$

Require: $r > 3$, $P \in E[r]$, $Q \in E[r]$ and
$b_0, \ldots, b_t \in \{0,1\}$ with $r = b_0 \cdot 2^0 + b_1 \cdot 2^1 + \ldots + b_t \cdot 2^t$ and $b_t = 1$

procedure MILLER'S ALGORITHM(P, Q)
- **if** $P = \mathcal{O}$ or $Q = \mathcal{O}$ or $P = Q$ **then**
  - **return** $f_{r,P}(Q) \leftarrow (-1)^r$
- **end if**
- $(x_T, y_T) \leftarrow (x_P, y_P)$
- $f_1 \leftarrow 1$
- $f_2 \leftarrow 1$
- **for** $j \leftarrow t-1, \ldots, 0$ **do**
  - $m \leftarrow \frac{3 \cdot x_T^2 + a}{2 \cdot y_T}$
  - $f_1 \leftarrow f_1^2 \cdot (y_Q - y_T - m \cdot (x_Q - x_T))$
  - $f_2 \leftarrow f_2^2 \cdot (x_Q + 2x_T - m^2)$
  - $x_{2T} \leftarrow m^2 - 2x_T$
  - $y_{2T} \leftarrow -y_T - m \cdot (x_{2T} - x_T)$
  - $(x_T, y_T) \leftarrow (x_{2T}, y_{2T})$
  - **if** $b_j = 1$ **then**
    - $m \leftarrow \frac{y_T - y_P}{x_T - x_P}$
    - $f_1 \leftarrow f_1 \cdot (y_Q - y_T - m \cdot (x_Q - x_T))$
    - $f_2 \leftarrow f_2 \cdot (x_Q + (x_P + x_T) - m^2)$
    - $x_{T+P} \leftarrow m^2 - x_T - x_P$
    - $y_{T+P} \leftarrow -y_T - m \cdot (x_{T+P} - x_T)$
    - $(x_T, y_T) \leftarrow (x_{T+P}, y_{T+P})$
  - **end if**
- **end for**
- $f_1 \leftarrow f_1 \cdot (x_Q - x_T)$
- **return** $f_{r,P}(Q) \leftarrow \frac{f_1}{f_2}$

end procedure

algorithm, we can deduce that $e(\mathcal{O},Q)=1$ as well as $e(P,\mathcal{O})=1$ for all arguments $P\in\mathbb{G}_1$ and $Q\in\mathbb{G}_2$. So in order to compute a non-trivial Weil pairing we choose the arguments $P=(7,2)$ and $Q=(9t^2+7,12t^3+2t)$.

In order to compute the pairing $e((7,2),(9t^2+7,12t^3+2t))$ we have to compute the extension field elements $f_{5,P}(Q)$ and $f_{5,Q}(P)$ applying Miller's algorithm. Do do so first observe that we have $5=1\cdot 2^0+0\cdot 2^1+1\cdot 2^2$, so we get $t=2$ as well as $b_0=1$, $b_1=0$ and $b_2=1$. The loop therefore needs to be executed two times.

Computing $f_{5,P}(Q)$, we initiate $(x_T,y_T)=(7,2)$ as well as $f_1=1$ and $f_2=1$. Then

j	b_j	m	f_1	f_2	x_{2T}	y_{2T}	x_{T+P}	y_{T+P}
1	$\cdot$							

$$
\begin{aligned}
m &= \frac{3\cdot x_T^2+a}{2\cdot y_T}\\
&= \frac{3\cdot 2^2+1}{2\cdot 4}=\frac{3}{3}\\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
f_1 &= f_1^2\cdot(y_Q-y_T-m\cdot(x_Q-x_T))\\
&= 1^2\cdot(t-4-1\cdot(1-2))=t-4+1\\
&= t+2
\end{aligned}
$$

$$
\begin{aligned}
f_2 &= f_2^2\cdot(x_Q+2x_T-m^2)\\
&= 1^2\cdot(1+2\cdot 2-1^2)=(1+4-1)\\
&= 4
\end{aligned}
$$

$$
\begin{aligned}
x_{2T} &= m^2-2x_T\\
&= 1^2-2\cdot 2=-3\\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
y_{2T} &= -y_T-m\cdot(x_{2T}-x_T)\\
&= -4-1\cdot(2-2)=-4\\
&= 1
\end{aligned}
$$

So we update $(x_T,y_T)=(2,1)$ and since $b_0=1$ we have to execute the if statement on line XXX in the for loop. However, since we only loop a single time, we don't need to compute

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y_{T+P}, since we only need the updated x_T in the final step. We get:

$$\begin{aligned} m &= \frac{y_T - y_P}{x_T - x_P} \\ &= \frac{1-4}{2-x_P} \end{aligned}$$

$$f_1 = f_1 \cdot (y_Q - y_T - m \cdot (x_Q - x_T))$$

$$f_2 = f_2 \cdot (x_Q + (x_P + x_T) - m^2)$$

$$x_{T+P} = m^2 - x_T - x_P$$

5.3 Hashing to Curves

Elliptic curve cryptography frequently requires the ability to hash data onto elliptic curves. If the order of the curve is not a prime number hashing to prime number subgroups is also of importance. In the context of pairing-friendly curves it is also sometimes necessary to hash specifically onto the group $\mathbb{G}_1$ or $\mathbb{G}_2$.

As we have seen in XXX, many general methods are known to hash into groups in general and finite cyclic groups in particular. As elliptic groups are cyclic those method can be utilized in this case, too. However, in what follows we want to describe some methods special to elliptic curves, that are frequently used in applications.

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Try and increment hash functions One of the most straight-forward ways to hash a bitstring onto an elliptic curve point, in a secure way, is to use a cryptographic hash function together with one of the methods we described in XXX to hash to the modular arithmetics base field of the curve. Ideally the hash function generates an image that is at least one bit longer than the bit representation of the base field modulus.

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The image in the base field can then be interpreted as the x-coordinate of the curve point and the two possible y-coordinates are then derived from the curve equation, while one of the bits that exceeded the modulus determines which of the two y-coordinates to choose.

Such an approach would be easy to implement and deterministic and it will conserve the cryptographic properties of the original hash function. However, not all x-coordinates generated in such a way, will result in quadratic residues, when inserted into the defining equation. It follows that not all field elements give rise to actual curve points. In fact on a prime field, only half of the field elements are quadratic residues and hence assuming an even distribution of the hash values in the field, this method would fail to generate a curve point in about half of the attempts.

One way to account for this problem is the so-called **try and increment** method. Its basic assumption is, that hashing different values, the result will eventually lead to a valid curve point.

Therefore instead of simply hashing a string s to the field the concatenation of s with additional bytes is hashed to the field instead, that is a try and increment hash as described in XXX is used. If the first **try** of hashing to the field does not result in a valid curve point, the counter is

add reference

Algorithm 8 Hash-to-$E(\mathbb{F}_r)$

Require: $r \in \mathbb{Z}$ with $r.nbits() = k$ and $s \in \{0,1\}^*$
Require: Curve equation $y^2 = x^3 + ax + b$ over $\mathbb{F}_r$
  procedure TRY-AND-INCREMENT(r,k,s)
    $c \leftarrow 0$
    repeat
      $s' \leftarrow s||c\_bits()$
      $z \leftarrow H(s')_0 \cdot 2^0 + H(s')_1 \cdot 2^1 + \ldots + H(s')_k \cdot 2^k$
      $x \leftarrow z^3 + a \cdot z + b$
      $c \leftarrow c + 1$
    until $z < r$ and $x^{\frac{r-1}{2}} \bmod r = 1$
    if $H(s')_{k+1} == 0$ **then**
      $y \leftarrow \sqrt{x}$ #(root in $\mathbb{F}_r$)
    else
      $y \leftarrow r - \sqrt{x}$ #(root in $\mathbb{F}_r$)
    end if
    return (x,y)
  end procedure
Ensure: $(x,y) \in E(\mathbb{F}_r)$

incremented and the hashing is repeated again. This is done until a valid curve point is found eventually.

This method has the advantageous that is relatively easy to implement in code and that it preserves the cryptographic properties of the original hash function. However, it is not guaranteed to find a valid curve point, as there is a chance that all possible values in the chosen size of the counter will fail to generate a quadratic residue. Fortunately it is possible to make the probability for this arbitrarily small by choosing large enough counters and relying on the (approximate) uniformity of the hash-to-field function.

If the curve is not of prime order, the result will be a general curve point that might not be in the "large" prime order subgroup. A so-called **cofactor clearing** step is then necessary to project the curve point onto the subgroup. This is done by scalar multiplication with the cofactor of prime order with respect to the curves order.

Example 93. Consider the tiny jubjub curve from example XXX. We want to construct a try and increment hash function, that hashes a binary string s of arbitrary length onto the large prime order subgroup of size 5.

add reference

Since the curve as well as our targeted subgroup are defined over the field $\mathbb{F}_{13}$ and the binary representation of 13 is $13.bits() = 1101$, we apply SHA256 from Sage's crypto library on the concatenation $s||c$ for some binary counter string and use the first 4 bits of the image to try to hash into $\mathbb{F}_{13}$. In case we are able to hash to a value z, such that $z^3 + 8 \cdot z + 8$ is a quadratic residue in $\mathbb{F}_{13}$, we use the 5-th bit to decide which of the two possible roots of $z^3 + 8 \cdot z + 8$ we will choose as the y-coordinate. The result is then a curve point different from the point at infinity. To project it to a point of $\mathbb{G}_1$, we multiply it with the cofactor 4. If the result is still not the point at infinity, it is the result of the hash.

To make this concrete let $s =' 100110011110101101000001111'$ be our binary string that we want to hash onto $\mathbb{G}_1$. We use a 4-bit binary counter, starting at zero, i.e we choose $c = 0000$. Invoking Sage we define the try-hash function as

```
sage: import hashlib
```

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```
sage: def try_hash(s,c):
....:     s_1 = s+c
....:     hasher = hashlib.sha256(s_1.encode('utf-8'))
....:     digest = hasher.hexdigest()
....:     d = Integer(digest,base=16)
....:     sign = d.str(2)[-5:-4]
....:     d = d.str(2)[-4:]
....:     z = Integer(d,base=2)
....:     return (z,sign)
sage: try_hash('10011001111010110100000111','0000')
(15, '1')
```

As we can see, our first attempt to hash into $\mathbb{F}_{13}$ was not successful as 15 is not a number in $\mathbb{F}_{13}$, so we increment the binary counter by 1 and try again:

```
sage: try_hash('10011001111010110100000111','0001')
(3, '0')
```

And we find a hash into $\mathbb{F}_{13}$. However, this point is not guaranteed to define a curve point. To see that we insert $z = 3$ into the right side of the Weierstraß equation of the tiny.jubjub curve and compute $3^3 + 8 * 3 + 8 = 7$, but 7 is not a quadratic residue in $\mathbb{F}_{13}$ since $7^{\frac{13-1}{2}} = 7^6 = 12 = -1$. So 3 is a not a suitable point and we have to increment the counter two more times:

```
sage: try_hash('10011001111010110100000111','0010')
(3, '0')
sage: try_hash('10011001111010110100000111','0011')
(6, '1')
```

Since $6^3 + 8 \cdot 6 + 8 = 12$ and we have $\sqrt{12} \in \{5, 8\}$, we finally found the valid x coordinate $x = 6$ for the curve point hash. Now since the sign bit of this hash is 1, we choose the larger root $y = 8$ as the y-coordinate and get the hash

$$H('10011001111010110100000111') = (6, 8)$$

which is a valid curve point point on the tiny jubjub curve. Now in order to project it onto the "large" prime order subgroup we have to do cofactor clearing, that is we have to multiply the point with the cofactor 4. We get

$$[4](6, 8) = \mathcal{O}$$

so the hash value is still not right. We therefore have to increment the counter two times again, until we finally find a correct hash to $\mathbb{G}_1$

```
sage: try_hash('10011001111010110100000111','0100')
(0, '1')
sage: try_hash('10011001111010110100000111','0101')
(12, '0')
```

Since $12^3 + 8 \cdot 12 + 8 = 12$ and we have $\sqrt{12} \in \{5, 8\}$, we found another valid x coordinate $x = 12$ for the curve point hash. Now since the sign bit of this hash is 0, we choose the smaller root $y = 5$ as the y-coordinate and get the hash

$$H('10011001111010110100000111') = (12, 5)$$

which is a valid curve point point on the tiny jubjub curve and in order to project it onto the "large" prime order subgroup we have to do cofactor clearing, that is we have to multiply the point with the cofactor 4. We get

$$[4](12,5) = (8,5)$$

So hashing the binary string $'100110011110101101000001111'$ onto $\mathbb{G}_1$ gives the hash value $(8,5)$ as a result.

5.4 Constructing elliptic curves

Cryptographically secure elliptic curves like Secp256k1 from example XXX are known for quite some time. In the latest advancements of cryptography, it is however, often necessary to design and instantiate elliptic curves from scratch, that satisfy certain very specific properties.

add reference

For example, in the context of SNARK development it was necessary to design a curve that can be efficiently implemented incide of a so-called circuit, in order to enable primitives like elliptic curve signature schemes in a zero knowledge proof. Such a curve is give by the Baby-jubjub curve [XXX] and we have paralleled its definition by introducing the tiny-jubjub curve from example XX. As we have seen those curves are instances of so-called twisted Edwards curves and as such have easy to implement addition laws that work without branching. However, we introduced the tiny-jubjub curve out of thin air, as we just gave the curve parameters without explaining how we came up with them.

add reference

Another requirement in the context of many so-called pairing-based zero knowledge proofing systems is the existing of a suitable, pairing-friendly curve with a specified security level and a low embedding degree as defined in XXX. Famous examples are the BLS_12 or the NMT curves.

add reference

The major goal of this section is to explain the most important method to design elliptic curves with predefined properties from scratch, called the **complex multiplication method**. We will apply this method in section to synthesize a particular BLS_6 curve, the most insecure BLS_6 curve, which will serve as the main curve to build our pen-and-paper SNARKs on. As we will see, this curve has a "large" prime factor subgroup of order 13, which implies, that we can use our tiny-jubjub curve to implement certain elliptic curve cryptographic primitives in circuits over that BLS_6 curve.

Before we introduce the complex multiplication method, we have to explain a few properties of elliptic curves that are of key importance in understanding the complex multiplication method.

The Trace of Frobenius To understand the complex multiplication method of elliptic curve, we have to define the so-called **trace** of an elliptic curve first.

We know from XXX that elliptic curves over finite fields are cyclic groups of finite order. An interesting question therefore is, if it is possible to estimate the number of elements that curve contains. Since an affine short Weierstraß curve consists of pairs (x,y) of elements from a finite field $\mathbb{F}_q$ plus the point at infinity and the field $\mathbb{F}_q$ contains q elements, the number of curve points can not be arbitrarily large, since it can contain at most q^2+1 many elements.

add reference

There is however, a more precise estimation, usually called the **Hasse bound**. To understand it, let $E(\mathbb{F}_q)$ be an affine short Weierstraß curve over a finite field $\mathbb{F}_w$ of order q and let $|E(\mathbb{F}_q)|$ be the order of the curve. Then there is an integer $t \in \mathbb{Z}$ called the **trace of Frobenius** of the curve, such that $|t| \leq 2\sqrt{q}$ and

$$|E(\mathbb{F})| = q+1-t \tag{5.18}$$

A positive trace therefore implies, that the curve contains less points than the underlying field and a negative trace means that the curve contains more point. However, the estimation $|t| \leq 2\sqrt{q}$ implies that the difference is not very large in either direction and the number of elements in an elliptic curve is always approximately in the same order of magnitude as the size of the curve's basefield.

Example 94. Consider the elliptic curve $E_1(\mathbb{F}_5)$ from example XXX. We know that it contains 9 curve points. Since the order of $\mathbb{F}_5$ is 5 we compute the trace of $E_1(\mathbb{F})$ to be $t = -3$, since the Hasse bound is given by

add reference

$$9 = 5 + 1 - (-3)$$

And indeed we have $|t| \leq 2\sqrt{q}$, since $\sqrt{5} > 2.23$ and $|-3| = 3 \leq 4.46 = 2 \cdot 2.23 < 2 \cdot \sqrt{5}$.

Example 95. To compute the trace of the tiny-jubjub curve, oberse from example XXX, that the order of *PJJ_13* is 20. Since the order of $\mathbb{F}_{13}$ is 13, we can therefore use the Hasse bound and compute the trace as $t = -6$, since

oberse

add reference

$$20 = 13 + 1 - (-6)$$

Again we have $|t| \leq 2\sqrt{q}$, since $\sqrt{13} > 3.60$ and $|-6| = 6 \leq 7.20 = 2 \cdot 3.60 < 2 \cdot \sqrt{13}$.

Example 96. To compute the trace of Secp256k1, recall from example XXX, that this curve is defined over a prime field with p elements and that the order of that group is given by r, with

add reference

$$p = 115792089237316195423570985008687907853269984665640564039457584007908834671663$$
$$r = 115792089237316195423570985008687907852837564279074904382605163141518161494337$$

Using the Hesse bound $r = p + 1 - t$, we therefore compute $t = p + 1 - r$, which gives the trace of curve Secp256k1 as

$$t = 432420386565659656852420866390673177327$$

So as we can see Secp256k1 contains less elements than its underlying field. However, the difference is tiny, since the order of Secp256k1 is in the same order of magnitude as the order of the underlying field. Compared to p and r, t is tiny.

```
sage: p = 115792089237316195423570985008687907853269984665640564039457584007908834671663
sage: r = 115792089237316195423570985008687907852837564279074904382605163141518161494337
sage: t = p + 1 -r
sage: t.nbits()
129
sage: abs(RR(t)) <= 2*sqrt(RR(p))
True
```

The j-invariant As we have seen in XXX two elliptic curve $E_1(\mathbb{F})$ defined by $y^2 = x^3 + ax + b$ and $E_2(\mathbb{F})$ defined by $y^2 + a'x + b'$ are strictly isomorphic, if and only if there is a quadratic residue $d \in \mathbb{F}$, such that $a' = ad^2$ and $b' = bd^3$.

add reference

There is however, a more general way to classify elliptic curves over finite fields $\mathbb{F}_q$, based on the so-called j-invariant of an elliptic curve:

$$j(E(\mathbb{F}_q)) = (1728 \bmod q)\frac{4 \cdot a^3}{4 \cdot a^3 + (27 \bmod q) \cdot b^2} \tag{5.19}$$

with $j(E(\mathbb{F}_q)) \in \mathbb{F}_q$. We will not go into the details of the j-invariant, but state only, that two elliptic curves $E_1(\mathbb{F})$ and $E_2(\mathbb{F}')$ are isomorphic over the algebraic closures of $\mathbb{F}$ and $\mathbb{F}'$, if and only if $\overline{\mathbb{F}} = \overline{\mathbb{F}'}$ and $j(E_1) = j(E_2)$.

So the j-invariant is an important tool to classify elliptic curves and it is needed in the complex multiplication method to decide on an actual curve instantiation, that implements abstractly chosen properties.

Example 97. Consider the elliptic curve $E_1(\mathbb{F}_5)$ from example XXX. We compute its j-invariant as

add reference

$$
\begin{aligned}
j(E_1(\mathbb{F}_5)) &= (1728 \bmod 5)\frac{4\cdot 1^3}{4\cdot 1^3 + (27 \bmod 5)\cdot 1^2}\\
&= 3\frac{4}{4+2}\\
&= 3\cdot 4 && = 2
\end{aligned}
$$

Example 98. Consider the elliptic curve *PJJ_13* from example XXX. We compute its j-invariant as

add reference

$$
\begin{aligned}
j(E_1(\mathbb{F}_5)) &= (1728 \bmod 13)\frac{4\cdot 8^3}{4\cdot 8^3 + (27 \bmod 13)\cdot 8^2}\\
&= 12\cdot\frac{4\cdot 5}{4\cdot 5 + 1\cdot 12}\\
&= 12\cdot\frac{7}{7+12}\\
&= 12\cdot 7\cdot 6^{-1}\\
&= 12\cdot 7\cdot 11\\
&01
\end{aligned}
$$

Example 99. Consider Secp256k1 from example XXX. We compute its j-invariant using Sage:

add reference

```
sage: p = 1157920892373161954235709850086879078532699846656405
    64039457584007908834671663
sage: F = GF(p)
sage: j = F(1728)*((F(4)*F(0)^3)/(F(4)*F(0)^3+F(27)*F(7)^2))
sage: j == F(0)
True
```

The Complex Multiplication Method As we have seen in the previous sections, elliptic curves have various defining properties, like their order and their prime factors, the embedding degree, or the cardinality of the base field. The so-called **complex multiplication** (CM) gives a practical method for constructing elliptic curves with pre-defined restrictions on the order and the base field.

The method usually starts by choosing a base field $\mathbb{F}_q$ of the curve $E(\mathbb{F}_q)$ we want to construct, such that $q = p^m$ for some prime number p and " $m \in \mathbb{N}$ with $m \geq 1$. We assume $p > 3$ to simplify things in what follows.

Next the trace of Frobenius $t \in \mathbb{Z}$ of the curve is chosen, such that p and t are coprime, i.e. such that $gcd(p,t) = 0$ holds true. The choice of t also defines the curves order r, since

$r = p+1-t$ by the Hasse bound XXX, so choosing t, will define the large order subgroup as well as all small cofactors. r has to be defined in such a way, that the elliptic curve meats the security requirements of the application it is designed for. add reference

Note that the choice of p and t also determines the embedding degree k of any prime order subgroup of the curve, since k is defined as the smallest number, such that the prime order n divides the number $q^k - 1$.

In order for the complex multiplication method to work, both q and t can not be arbitrary, but must be chosen in such a way that two additional integers $D \in \mathbb{Z}$ and $v \in \mathbb{Z}$ exist, such that $D < 0$ as well as $D \bmod 4 = 0$ or $D \bmod 4 = 1$ and the equation

$$4q = t^2 + |D|v^2 \tag{5.20}$$

holds. If those numbers exist, we call D the **CM-discriminant** and we know that we can construct a curve $E(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$, such that the order of the curve is $|E(\mathbb{F}_q)| = q+1-t$.

It is the content of the complex multiplication method to actually construct such a curve, that is finding the parameters a and b from $\mathbb{F}_q$ in the defining Weiertraß equation, such that the curve has the desired order r.

Finding solutions to equation XXX, can be achieved in different ways, which we will not look much into. In general it can be said, that there are well known constraints for elliptic curve families like the BLS (ECT) families, that provides families of solutions. In what follows we will look at one type curves the BLS-family, which gives an entire range of solutions. add reference

Assuming that proper parameters q, t, D and v are found, we have to compute the so-called **Hilbert class polynomial** $H_D \in \mathbb{Z}[x]$ of the CM-discriminant D, which is a polynomial with integer coefficients. To do so, we first have to compute the following set:

$$\begin{aligned} ICG(D) = \{(A,B,C) \mid A,B,C \in \mathbb{Z}, D = B^2 - 4AC, gcd(A,B,C) = 1,\\ |B| \leq A \leq \sqrt{\frac{|D|}{3}}, A \leq C, \text{ if } B < 0 \text{ then } |B| < A < C\} \end{aligned}$$

One way to compute this set, is to first compute the integer $A_{max} = Floor(\sqrt{\frac{|D|}{3}})$, then loop through all the integers A in the range $[0,\ldots,A_{max}]$ as well as through all the integers B in the range $[-A_{max},\ldots,A_{max}]$ and to see if there is an integer C, that satisfies $D = B^2 - 4AC$ and the rest of the requirements in XXX. add reference

To compute the Hilbert class polynomial, the so-called **j-function** (or j-invariant) is needed, which is a complex function defined on the upper half $\mathbb{H}$ of the complex plane $\mathbb{C}$, usually written as

$$j : \mathbb{H} \to \mathbb{C} \tag{5.21}$$

Roughly speaking what this means is that the j-functions takes complex numbers $(x+i\cdot y)$ with positive imaginary part $y > 0$ as inputs and returns a complex number $j(x+i\cdot y)$ as result.

For the sake of this book, it is not important to actually understand the j-function, and we can use Sage to compute it in a similar way as we would use Sage to computer any other well known function. It should be noted however, that the computation of the j-function in Sage is sometimes prone to precision errors. For example, the j-function has a root in $\frac{-1+i\sqrt{3}}{2}$, which Sage only approximates. Therefore using Sage to compute the j-function, we need to take precision loss into account and eventually round to the nearest integer.

```
sage: z = ComplexField(100)(0,1)
```
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```
sage: z # (0+1i)
1.0000000000000000000000000000*I
sage: elliptic_j(z)
1728.0000000000000000000000000
sage: # j-function only defined for positive imaginary
   arguments
sage: z = ComplexField(100)(1,-1)
sage: try:
....:     elliptic_j(z)
....: except PariError:
....:     pass
sage: # root at (-1+i sqrt(3))/2
sage: z = ComplexField(100)(-1,sqrt(3))/2
sage: elliptic_j(z)
-2.6445453750358706361219364880e-88
sage: elliptic_j(z).imag().round()
0
sage: elliptic_j(z).real().round()
0
```

With a way to compute the j-function and the precomputed set $ICG(D)$ at hand, we can now compute the Hilbert class polynomial as

$$H_D(x) = \Pi_{(A,B,C)\in ICG(D)}\left(x - j\left(\frac{-B+\sqrt{D}}{2A}\right)\right) \tag{5.22}$$

So what we do is we loop over all elements (A,B,C) from the set $ICG(D)$ and compute the j-function at the point $\frac{-B+\sqrt{D}}{2A}$, where D is the CM-discriminant that we decided in a previous step. The result then defines a factor of the Hilbert class polynomial and all factors are multiplied together.

It can be shown, that the Hilbert class polynomial is an integer polynomial, but actual computations need high precision arithmetics to avoid approximation errors, that usually occur in computer approximations of the j-function as shown above. So in case the calculated Hilbert class polynomial does not have integer coefficients, we need to round the result to the nearest integer. Given that the precision we used was high enough, the result will be correct.

In the next step we use the Hilbert class polynomial $H_D \in \mathbb{Z}[x]$ and project it to a polynomial $H_{D,q} \in \mathbb{F}_q[x]$ with coefficients in the base field $\mathbb{F}_q$ as chosen in the first step. We do this by simply computing the new coefficients as the old coefficients modulus p, that is if $H_D(x) = a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0$ we compute the q-modulus of each coefficient $\tilde{a}_j = a_j \bmod p$ which defines the **projected Hilbert class polynomial** as

$$H_{D,p}(x) = \tilde{a}_m x^m + \tilde{a}_{m-1}x^{m-1} + \ldots + \tilde{a}_1 x + \tilde{a}_0$$

We then search for roots of $H_{D,p}$, since every root j_0 of $H_{D,p}$ defines a family of elliptic curves over $\mathbb{F}_q$, which all have a j-invariant XXX equal to j_0. We can pick any root and all of them will lead to proper curves eventually. add reference

However some of the curves with the correct j-invariant might have an order different from the one we initially decided on. So we need a way to decide on a curve with the correct order.

To compute such a curve, we have to distinguish a few different cases, based on our choice of the root j_0 and of the CM-discriminant D. If $j_0 \neq 0$ or $j_0 \neq 1728 \bmod q$ we compute

$c_1 = \frac{j_0}{(1728 \bmod q) - j_0}$ and then we chose some arbitrary quadratic non-residue $c_2 \in \mathbb{F}_q$ and some arbitrary cubic non residue $c_3 \in \mathbb{F}_q$.

The following table is guaranteed to define a curve with the correct order $r = q + 1 - t$, for the trace of Frobenius t we initially decided on:

- Case $j_0 \neq 0$ and $j_0 \neq 1728 \bmod q$. A curve with the correct order is defined by one of the following equations

$$y^2 = x^3 + 3c_1 x + 2c_1 \quad \text{or} \quad y^2 = x^3 + 3c_1 c_2^2 x + 2c_1 c_2^3 \tag{5.23}$$

- Case $j_0 = 0$ and $D \neq -3$. A curve with the correct order is defined by one of the following equations

$$y^2 = x^3 + 1 \quad \text{or} \quad y^2 = x^3 + c_2^3 \tag{5.24}$$

- Case $j_0 = 0$ and $D = -3$. A curve with the correct order is defined by one of the following equations

$$\begin{aligned} y^2 = x^3 + 1 \quad &\text{or} \quad y^2 = x^3 + c_2^3 \quad \text{or} \\ y^2 = x^3 + c_3^2 \quad &\text{or} \quad y^2 = c_3^2 c_2^3 \quad \text{or} \\ y^2 = x^3 + c_3^{-2} \quad &\text{or} \quad y^2 = x^3 + c_3^{-2} c_2^3 \end{aligned}$$

- Case $j_0 = 1728 \bmod q$ and $D \neq -4$. A curve with the correct order is defined by one of the following equations

$$y^2 = x^3 + x \quad \text{or} \quad y^2 = x^3 + c_2^2 x \tag{5.25}$$

- Case $j_0 = 1728 \bmod q$ and $D = -4$. A curve with the correct order is defined by one of the following equations

$$\begin{aligned} y^2 = x^3 + x \quad &\text{or} \quad y^2 = x^3 + c_2 x \quad \text{or} \\ y^2 = x^3 + c_2^2 x \quad &\text{or} \quad y^2 = x^3 + c_2^3 x \end{aligned}$$

To decide the proper defining Weierstraß equation, we therefore have to compute the order of any of the potential curves above and then choose the one that fits out initial requirements. Since it can be shown that the Hilbert class polynomials for the CM-discriminants $D = -3$ and $D = -4$ are given by $H_{-3,q}(x) = x$ and $H_{-4,q} = x - (1728 \bmod q)$ (EXERCISE) the previous cases are exhaustive.

To summarize, using the complex multiplication method, it is possible to synthesize elliptic curve with predefined order over predefined base fields from scratch. However, the curves that are constructed this way are just some representatives of a larger class of curves, all of which have the same order. In applications it is therefore sometimes more advantageous to choose a different representative from that class. To do so recall from XXX, that any curve defined by the Weierstraß equation $y^2 = x^3 + axb$ is isomorphic to a curve of the form $y^2 = x^3 + ad^2x + bd^3$ for some quadratic residue $d \in \mathbb{F}_q$. add reference

So in order to find a nice representative (e.g. with small parameters a and b) in a last step, the designer might choose a quadratic residue d such that the transformed curve looks the way they wanted it.

Example 100. Consider curve $E_1(\mathbb{F}_5)$ from example XXX. We want to use the complex multiplication method to derive that curve from scratch. Since $E_1(\mathbb{F}_5)$ is a curve of order $r = 9$ over the prime field of order $q = 5$, we know from example XX that its trace of Frobenius is $t = -3$, which also implies that q and $|t|$ are coprime.

add reference

We then have to find parameters $D, v \in \mathbb{Z}$ with $D < 0$ and $D \bmod 4 = 0$ or $D \bmod 4 = 1$, such that $4q = t^2 + |D|v^2$ holds. We get

$$\begin{aligned} 4q &= t^2 + |D|v^2 && \Rightarrow \\ 20 &= (-3)^2 + |D|v^2 && \Leftrightarrow \\ 11 &= |D|v^2 \end{aligned}$$

Now, since 11 is a prime number, the only solution is $|D| = 11$ and $v = 1$ here. So $D = -11$ and since the Euclidean division of -11 by 4 is $-11 = -3 \cdot 4 + 1$ we have $-11 \bmod 4 = 1$, which shows that $D = -11$ is a proper choice.

In the next step, we have to compute the Hilbert class polynomial H_{-11} and to do so, we first have to find the set $ICG(D)$. To compute that set, observe, that since $\sqrt{\frac{|D|}{3}} \approx 1.915 < 2$, we know from $A \leq \sqrt{\frac{|D|}{3}}$ and $A \in \mathbb{Z}$ that A must be either 0 or 1.

For $A = 0$, we know $B = 0$ from the constraint $|B| \leq A$, but in this case there can be no C satisfying $-11 = B^2 - 4AC$. So we try $A = 1$ and deduce $B \in \{-1, 0, 1\}$ from the constraint $|B| \leq A$. The case $B = -1$ can be excluded since then $B < 0$ has to imply $|B| < A$. In addition, the case $B = 0$ can be exclude as there can be integer C with $-11 = -4C$ since 11 is a prime number.

This leaves the case $B = 1$ and we compute $C = 3$ from the equation $-11 = 1^2 - 4C$, which gives the solution $(A, B, C) = (1, 1, 3)$ and we get

$$ICG(D) = \{(1, 1, 3)\}$$

With the set $ICG(D)$ at hand we can compute the Hilbert class polynomial of $D = -11$. To do so, we have to insert the term $\frac{-1+\sqrt{-11}}{2\cdot 1}$ into the j-function. To do so first observe that $\sqrt{-11} = i\sqrt{11}$, where i is the imaginary unit, defined by $i^2 = -1$. Using this, we can invoke SageMath to compute the j-invariant and get

$$H_{-11}(x) = x - j\left(\frac{-1 + i\sqrt{11}}{2}\right) = x + 32768$$

So, as we can see, in this particular case, the Hilbert class polynomial is a linear function with a single integer coefficient. In the next step we have to project it onto a polynomial from $\mathbb{F}_5[x]$, by computing the modular 5 remainder of the coefficients 1 and 32768. We get $32768 \bmod 5 = 3$ from which follows that the projected Hilbert class polynomial is

$$H_{-11,5}(x) = x + 3$$

considered as a polynomial from $\mathbb{F}_5[x]$. As we can see the only root of this polynomial is $j = 2$, since $H_{-11,5}(2) = 2 + 3 = 0$. We therefore have a situation with $j \neq 0$ and $j \neq 1728$, which tells us that we have to compute the parameter

$$c_1 = \frac{2}{1728 - 2}$$

in modular 5 arithmetics. Since 1728 mod $5 = 3$, we get $c_1 = 2$. Then we have to check if the curve $E(\mathbb{F}_5)$ defined by the Weierstraß $y^2 = x^3 + 3\cdot 2x + 2\cdot 2$ has the correct order. We invoke Sage and find that the order is indeed 9, so it is a curve with the required parameters and we are done.

Note however, that in real-world applications, it might be useful to choose parameters a and b that have certain properties, e.g. to be a small as possible. As we know from XXX, choosing any quadratic residue $d \in \mathbb{F}_5$ gives a curve of the same order defined by $y^2 = x^2 + ak^2x + bk^3$. Since 4 is a quadratic residue in $\mathbb{F}_4$, we can transform the curve defined by $y^2 = x^3 + x + 4$ into the curve $y^2 = x^3 + 4^2 + 4\cdot 4^3$ which gives

add reference

$$y^2 = x^3 + x + 1$$

which is the curve $E_1(\mathbb{F}_5)$, that we used extensively throughout this book. So using the complex multiplication method, we were able to derive a curve with specific properties from scratch.

Example 101. Consider the tiny jubjub curve *TJJ_13* from example XXX. We want to use the complex multiplication method to derive that curve from scratch. Since *TJJ_13* is a curve of order $r = 20$ over the prime field of order $q = 13$, we know from example XX that its trace of Frobenius is $t = -6$, which also implies that q and $|t|$ are coprime.

add reference

We then have to find parameters $D, v \in \mathbb{Z}$ with $D < 0$ and D mod $4 = 0$ or D mod $4 = 1$, such that $4q = t^2 + |D|v^2$ holds. We get

$$\begin{aligned} 4q &= t^2 + |D|v^2 && \Rightarrow \\ 4\cdot 13 &= (-6)^2 + |D|v^2 && \Rightarrow \\ 52 &= 36 + |D|v^2 && \Leftrightarrow \\ 16 &= |D|v^2 \end{aligned}$$

This equation has two solutions for (D, v), given by $(-4, \pm 2)$ and $(-16, \pm 1)$. Now looking at the first solution, we know that the case $D = -4$ implies $j = 1728$ and the constructed curve is defined by a Weierstraß equation XXX that has a vanishing parameter $b = 0$. We can therefore conclude that choosing $D = -4$ will not help us reconstructing *TJJ_13*. It will produce curves with order 20, just not the one we are looking for.

add reference

So we choose the second solution $D = -16$ and in the next step, we have to compute the Hilbert class polynomial H_{-16}. To do so, we first have to find the set $ICG(D)$. To compute that set, observe, that since $\sqrt{\frac{|-16|}{3}} \approx 2.31 < 3$, we know from $A \leq \sqrt{\frac{|-16|}{3}}$ and $A \in \mathbb{Z}$ that A must be in the range 0..2. So we loop through all possible values of A and through all possible values of B under the constraints $|B| \leq A$ and if $B < 0$ then $|B| < A$ and the compute potential C's from $-16 = B^2 - 4AC$. We get the following two solution $(1,0,4)$ and $(2,0,2)$, giving we get

$$ICG(D) = \{(1,0,4), (2,0,2)\}$$

With the set $ICG(D)$ at hand we can compute the Hilbert class polynomial of $D = -16$. We can invoke Sagemath to compute the j-invariant and get

$$\begin{aligned} H_{-16}(x) &= \left(x - j\left(\frac{i\sqrt{16}}{2}\right)\right)\left(x - j\left(\frac{i\sqrt{16}}{4}\right)\right) \\ &= (x - 287496)(x - 1728) \end{aligned}$$

So as we can see, in this particular case, the Hilbert class polynomial is a quadratic function with two integer coefficient. In the next step we have to project it onto a polynomial from

$\mathbb{F}_5[x]$, by computing the modular 5 remainder of the coefficients 1, 287496 and 1728. We get 287496 mod 13 = 1 and 1728 mod 13 = 2 from which follows that the projected Hilbert class polynomial is

$$H_{-11,5}(x) = (x-1)(x-12) = (x+12)(x+1)$$

considered as a polynomial from $\mathbb{F}_5[x]$. So we have two roots given by $j = 1$ and $j = 12$. We already know that $j = 12$ is the wrong root to construct the tiny jubjub curve, since 1728 mod 13 = 2 and that case can not construct a curve with $b \neq 0$. So we choose $j = 1$.

Another way to decide the proper root, is to compute the j-invariant of the tiny-jubjub curve. We get

$$\begin{aligned} j(TJJ\_13) &= 12\frac{4\cdot 8^3}{4\cdot 8^3 + 1\cdot 8^2} \\ &= 12\frac{4\cdot 5}{4\cdot 5+12} \\ &= 12\frac{7}{7+12} \\ &= 12\frac{7}{7+12} \\ &= 1 \end{aligned}$$

which is equal to the root $j = 1$ of the Hilbert class polynomial $H_{-16,13}$ as expected. We therefore have a situation with $j \neq 0$ and $j \neq 1728$, which tells us that we have to compute the parameter

$$c_1 = \frac{1}{12-1} = 6$$

in modular 5 arithmetics. Since 1728 mod 13 = 12, we get $c_1 = 6$. Then we have to check if the curve $E(\mathbb{F}_5)$ defined by the Weierstraß $y^2 = x^3 + 3\cdot 6x + 2\cdot 6$ which is equivalent to

$$y^2 = x^3 + 5x + 12$$

has the correct order. We invoke Sage and find that the order is 8, which implies that the trace of this curve is 6 not −6 as required. So we have to consider the second possibility and choose some quadratic non-residue $c_2 \in \mathbb{F}_{13}$. We choose $c_2 = 5$ and compute the Weierstraß equation $y^2 = x^3 + 5c_2^2 + 12c_2^3$ as

$$y^2 = x^3 + 8x + 5$$

We invoke Sage and find that the order is 20, which is indeed the correct one. As we know from XXX, choosing any quadratic residue $d \in \mathbb{F}_5$ gives a curve of the same order defined by $y^2 = x^2 + ad^2x + bd^3$. Since 12 is a quadratic residue in $\mathbb{F}_{13}$, we can transform the curve defined by $y^2 = x^3 + 8x + 5$ into the curve $y^2 = x^3 + 12^2\cdot 8 + 5\cdot 12^3$ which gives

add reference

$$y^2 = x^3 + 8x + 8$$

which is the tiny jubjub curve, that we used extensively throughout this book. So using the complex multiplication method, we were able to derive a curve with specific properties from scratch.

Example 102. To consider a real-world example, we want to use the complex multiplication method in combination with Sage to compute Secp256k1 from scratch. So by example XXX,

add reference

we decided to compute an elliptic curve over a prime field $\mathbb{F}_p$ of order r for the security parameters

$$p = 115792089237316195423570985008687907853269984665640564039457584007908834671663$$

$$r = 115792089237316195423570985008687907852837564279074904382605163141518161494337$$

which, according to example XXX, gives the trace of Frobenius $t = 432420386565659656852420866390673177327$. We also decided that we want a curve of the form $y^2 = x^3 + b$, that is we want the parameter a to be zero. This implies, the j-invariant of our curve must be zero.

In a first step we have to find a CM-discriminant D and some integer v, such that the equation

$$4p = t^2 + |D|v^2$$

is satisfied. Since we aim for a vanishing j-invariant, the first thing to try is $D = -3$. In this case we can compute $v^2 = (4p - t^2)$ and if v^2 happens to be an integers that has a square root v, we are done. Invoking Sage we compute

```
sage: D = -3
sage: p = 115792089237316195423570985008687907853269984665640
   564039457584007908834671663
sage: r = 115792089237316195423570985008687907852837564279074
   904382605163141518161494337
sage: t = p+1-r
sage: v_sqr = (4*p - t^2)/abs(D)
sage: v_sqr.is_integer()
True
sage: v = sqrt(v_sqr)
sage: v.is_integer()
True
sage: 4*p == t^2 + abs(D)*v^2
True
sage: v
303414439467246543595250775667605759171
```

So indeed the pair $(D, v) = (-3, 303414439467246543595250775667605759171)$ solves the equation, which tells us that there is a curve of order r over a prime field of order p, defined by a Weierstraß equation $y^2 = x^3 + b$ for some $b \in \mathbb{F}_p$. So we need to compute b.

Now for $D = -3$ we already know that the associated Hilbert class polynomial is given by $H_{-3}(x) = x$, which gives the projected Hilbert class polynomial as $H_{-3,p} = x$ and the j-invariant of our curve is guaranteed to be $j = 0$. Now looking into table XXX, we see that there are 6 possible cases to construct a curve with the correct order r. In order to construct the curves of those case we have to choose some arbitrary quadratic and cubic non residue. So we loop through $\mathbb{F}_p$ to find them, invoking Sage:

add reference

```
sage: F = GF(p)
sage: for c2 in F:
....:     try: # quadratic residue
....:         _ = c2.nth_root(2)
....:     except ValueError: # quadratic non residue
....:         break
```

```
sage: c2
3
sage: for c3 in F:
....:     try:
....:         _ = c3.nth_root(3)
....:     except ValueError:
....:         break
sage: c3
2
```

So we found the quadratic non residue $c_2 = 3$ and the cubic non residue $c_3 = 2$. Using those numbers we check the six cases against the the expected order r of the curve we want to syntesize:

```
sage: C1 = EllipticCurve(F,[0,1])
sage: C1.order() == r
False
sage: C2 = EllipticCurve(F,[0,c2^3])
sage: C2.order() == r
False
sage: C3 = EllipticCurve(F,[0,c3^2])
sage: C3.order() == r
False
sage: C4 = EllipticCurve(F,[0,c3^2*c2^3])
sage: C4.order() == r
False
sage: C5 = EllipticCurve(F,[0,c3^(-2)])
sage: C5.order() == r
False
sage: C6 = EllipticCurve(F,[0,c3^(-2)*c2^3])
sage: C6.order() == r
True
```

So, as expected, we found an elliptic curve of the correct order r over a prime field of size p. So in principal we are done, as we have found a curve with the same basic properties as Secp256k1. However, the curve is defined by the equation

$$y^2=x^3+86844066927987146567678238756515930889952488499230423029593188005931626003754$$

that use a very large parameter b_1, which might perform slow in certain algorithms. It is also not very elegant to be written down by hand. It might therefore be advantageous to find an isomorphic curve with the smallest possible parameter b_2. So in order to find such a b_2, we have to choose a quadratic residue d, such that $b_2 = b_1 \cdot d^3$ is as small as possible. To do so we rewrite the last equation into

$$d = \sqrt[3]{\frac{b_2}{b_1}}$$

and then invoke Sage to loop through values $b_2 \in \mathbb{F}_p$ until it finds some number such that the quotient $\frac{b_2}{b_1}$ has a cube root d and this cube root itself is a quadratic residue.

```
sage: b1=86844066927987146567678238756515930889952488499230423
    029593188005931626003754
```

```
sage: for b2 in F:
....:     try:
....:         d = (b2/b1).nth_root(3)
....:         try:
....:             _ = d.nth_root(2)
....:             if d != 0:
....:                 break
....:         except ValueError:
....:             pass
....:     except ValueError:
....:         pass
sage: b2
7
```

So, indeed, the smallest possible value is $b_2 = 7$ and the defining Weierstraß equation of a curve over $\mathbb{F}_p$ with prime order r is

$$y^2 = x^3 + 7$$

which we might call secp256k1. As we have seen the complex multiplication method is powerful enough to derive cryptographically secure curves like Secp256k1 from scratch.

The ***BLS*****6_6 pen& paper curve** In this paragraph we summarize our understanding of elliptic curves to derive our main pen & paper example for the rest of the book. To do so, we want to use the complex multiplication method, to derive a pairing friendly elliptic curve that has similar properties to curves that are used in actual cryptographic protocols. However, we design the curve specifically to be useful in pen&paper examples, which mostly means that the curve should contain only a few points, such that we are able to derive exhaustive addition and pairing tables.

A well-understood family of pairing-friendly curves is the the group of BLS curves (STUFF ABOUT THE HISTORY AND THE NAMING CONVENTION), which are derived in [XXX]. add reference BLS curves are particular useful in our case if the embedding degree k satisfies $k \equiv 6 \pmod 0$. Of course the smallest embedding degree k that satisfies this congruency, is $k = 6$ and we therefore aim for a BLS6 curve as our main pen&paper example.

To apply the complex multiplication method from XXX, recall that this method starts with add reference a definition of the base field $\mathbb{F}_{p^m}$ as well as the trace of Frobenius t and the order of the curve. If the order $p^m + 1 - t$ is not a prime number, then what is necessary to control is the order r of the largest prime factor group.

In the case of BLS_6 curves, the parameter m is chosen to be 1, which means that the curves are defined over prime fields. All relevant parameters p, t and r are then themselfs parameterized by the following three polynomials

$$\begin{aligned} r(x) &= \Phi_6(x) \\ t(x) &= x + 1 \\ q(x) &= \frac{1}{3}(x-1)^2(x^2 - x + 1) + x \end{aligned}$$

where Φ_6 is the 6-th cyclotomic polynomial and $x \in \mathbb{N}$ is a parameter that the designer has to choose in such a way that the evaluation of p, t and r at the point x gives integers that have the proper size to meet the security requirements of the curve that they want to design. It is then guaranteed that the complex multiplication method can be used in combination with those parameters to define an elliptic curve with CM-discriminant $D = -3$ and embedding degree $k = 6$ and curve equation $y^2 = x^3 + b$ for some $b \in \mathbb{F}_p$.

For example if the curve should target the 128-bit security level, due to the Pholaard-rho attack (TODO) the parameter r should be prime number of at least 256 bits.

In order to design the smallest, most insecure BLS_6 curve, we therefore have to find a parameter x, such that $r(x)$, $t(x)$ and $q(x)$ are the smallest natural numbers, that satisfy $q(x) > 3$ and $r(x) > 3$.

We therefore initiate the design process of our $BLS6$ curve by looking-up the 6-th cyclotomic polynimial which is $\Phi_6 = x^2 - x + 1$ and then insert small values for x into the defining polynomials r,t,q. We get the following results:

$$\begin{array}{lll} x=1 & (r(x),t(x),q(x)) & (1,2,1) \\ x=2 & (r(x),t(x),q(x)) & (3,3,3) \\ x=3 & (r(x),t(x),q(x)) & (7,4,\frac{37}{3}) \\ x=4 & (r(x),t(x),q(x)) & (13,5,43) \end{array}$$

Since $q(1) = 1$ is not a prime number, the first x that gives a proper curve is $x = 2$. However, such a curve would be defined over a base field of characteristic 3 and we would rather like to avoid that. We therefore find $x = 4$, which defines a curve over the prime field of characteristic 43, that has a trace of Frobenius $t = 5$ and a larger order prime group of size $r = 13$.

Since the prime field $\mathbb{F}_{43}$ has 43 elements and 43's binary representation is $43_2 = 101011$, which are 6 digits, the name of our pen&paper curve should be $BLS6\_6$, since its is common behaviour to name a BLS curve by its embedding degree and the bit-length of the modulus in the base field. We call $BLS6\_6$ the **moon-math-curve**.

Recalling from XXX, we know that the Hasse bound implies that $BLS6\_6$ will contain exactly 39 elements. Since the prime factorization of 39 is $39 = 3 \cdot 13$, we have a "large" prime factor group of size 13 as expected and a small cofactor group of size 3. Fortunately a subgroup of order 13 is well suited for our purposes as 13 elements can be easily handled in the associated addition, scalar multiplication and pairing tables in a pen-and-paper style.

add reference

We can check that the embedding degree is indeed 6 as expected, since $k = 6$ is the smallest number k such that $r = 13$ divides $43^k - 1$.

```
sage: for k in range(1,42): # Fermat's little theorem
....:     if (43^k-1)%13 == 0:
....:         break
sage: k
6
```

In order to compute the defining equation $y^2 = x^3 + ax + b$ of BLS6-6, we use the complex multiplication method as described in XXX. The goal is to find $a, b \in \mathbb{F}_{43}$ representations, that are particularly nice to work with. The authors of XXX showed that the CM-discriminant of every BLS curve is $D = -3$ and indeed the equation

add reference

add reference

$$\begin{aligned} 4p &= t^2 + |D|v^2 && \Rightarrow \\ 4\cdot 43 &= 5^2 + |D|v^2 && \Rightarrow \\ 172 &= 25 + |D|v^2 && \Leftrightarrow \\ 49 &= |D|v^2 \end{aligned}$$

has the four solutions $(D,v) \in \{(-3,-7),(-3,7),(-49,-1),(-49,1)\}$ if D is required to be negative, as expected. So $D = -3$ is indeed a proper CM-discriminant and we can deduce that the parameter a has to be 0 and that the Hilbert class polynomial is given by

$$H_{-3,43}(x) = x$$

This implies that the j-invariant of $BLS6\_6$ is given by $j(BLS6\_6) = 0$. We therefore have to look at case XXX in table XXX to derive a parameter b. To decide the proper case for $j_0 = 0$ and $D = -3$, we therefore have to choose some arbitrary quadratic non residue c_2 and cubic non residue c_3 in $\mathbb{F}_{43}$. We choose $c_2 = 5$ and $c_3 = 36$. We check

add reference

add reference

```
sage: F43 = GF(43)
sage: c2 = F43(5)
....: try: # quadratic residue
....:     c2.nth_root(2)
....: except ValueError: # quadratic non residue
....:     c2
sage: c3 =F43(36)
....: try:
....:     c3.nth_root(3)
....: except ValueError:
....:     c3
```

Using those numbers we check the six possible cases from XXX against the the expected order 39 of the curve we want to syntesize:

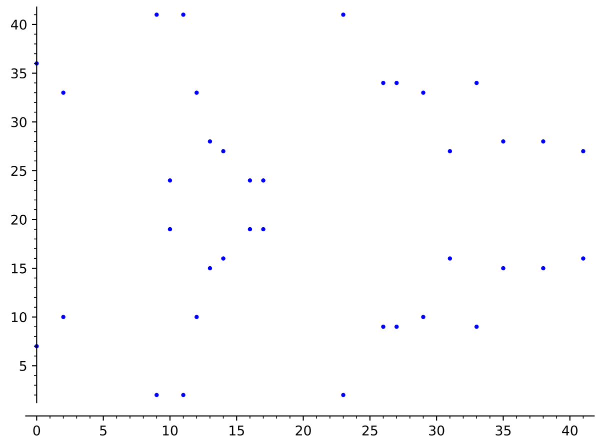
add reference

```
sage: BLS61 = EllipticCurve(F43,[0,1])
sage: BLS61.order() == 39
False
sage: BLS62 = EllipticCurve(F43,[0,c2^3])
sage: BLS62.order() == 39
False
sage: BLS63 = EllipticCurve(F43,[0,c3^2])
sage: BLS63.order() == 39
True
sage: BLS64 = EllipticCurve(F43,[0,c3^2*c2^3])
sage: BLS64.order() == 39
False
sage: BLS65 = EllipticCurve(F43,[0,c3^(-2)])
sage: BLS65.order() == 39
False
sage: BLS66 = EllipticCurve(F43,[0,c3^(-2)*c2^3])
sage: BLS66.order() == 39
False
sage: BLS6 = BLS63 # our BLS6 curve in the book
```

So, as expected we found an elliptic curve of the correct order 39 over a prime field of size 43, defined by the equation

$$BLS6\_6 := \{(x,y) \mid y^2 = x^3 + 6 \text{ for all } x,y \in \mathbb{F}_{43}\} \tag{5.26}$$

There are other choice for b like $b = 10$ or $b = 23$, but all these curves are isomorphic and hence represent the same curve really but in a different way only. Since BLS6-6 only contains 39 points it is possible to give a visual impression of the curve:



As we can see our curve is somewhat nice, as it does not contain self inverse points that is points with $y = 0$. It follows that the addition law can be optimized, since the branch for those cases can be eliminated.

Summarizing the previous procedure, we have used the method of Barreto, Lynn and Scott to construct a pairing friendly elliptic curve of embedding degree 6. However, in order to do elliptic curve cryptography on this curve note that since the order of $BLS6\_6$ is 39 its group of rational points is not a finite cyclic group of prime order. We therefore have to find a suitable subgroup as our main target and since $39 = 13 \cdot 3$, we know that the curve must contain a "large" prime order group of size 13 and a small cofactor group of order 3.

It is the content of the following step to construct this group. One way to do so is to find a generator. We can achieve this by choosing an arbitrary element of the group that is not the point at infinity and then multiply that point with the cofactor of the groups order. If the result is not the point at infinity, the result will be a generator and if it is the point at infinity we have to choose a different element.

So in order to find a generator for the large order subgroup of size 13, we first notice that the cofactor of 13 is 3, since $39 = 3 \cdot 13$. We then need to construct an arbitrary element from $BLS6\_6$. To do so in a pen-and-paper style, we can choose some $arbitrary x \in \mathbb{F}_{43}$ and see if there is some ssolution $y \in \mathbb{F}_{43}$ that satisfies the defining Weierstraß equation $y^2 = x^3 + 6$. We choose $x = 9$. Then $y = 2$ is a proper solution, since

$$\begin{aligned}
y^2 &= x^3 + 6 && \Rightarrow \\
2^2 &= 9^3 + 6 && \Leftrightarrow \\
4 &= 4
\end{aligned}$$

and this implies that $P = (9,2)$ is therefore a point on $BLS6\_6$. To see if we can project this point onto a generator of the large order prim group $BLS6\_6[13]$, we have to multiply P with the cofactor, that is we have to compute $[3](9,2)$. After some computation (EXERCISE) we get $[3](9,2) = (13,15)$ and since this is not the point at infinity we know that $(13,15)$ must be a generator of $BLS6\_6[13]$. We write

$$g_{BLS6\_6[13]} = (13,15) \tag{5.27}$$

as we will need this generator in pairing computations all over the book. Since $g_{BLS6\_6[13]}$ is a generator, we can use it to construct the subgoup $BLS6\_6[13]$, by repeatedly adding the generator to itself. We use Sage and get

```
sage: P = BLS6(9,2)
```
594

```
sage: Q = 3*P
sage: Q.xy()
(13, 15)
sage: BLS6_13 = []
sage: for x in range(0,13): # cyclic of order 13
....:     P = x*Q
....:     BLS6_13.append(P)
```

Repeatedly adding a generator to itself, as we just did, will generate small groups in logarithmic order with respect to the generator as explained in XXX. We therefore get the following description of the large prime order subgroup of $BLS6\_6$:

add reference

$$
\begin{aligned}
BLS6\_6[13] = & \\
& \{(13,15) \to (33,34) \to (38,15) \to (35,28) \to (26,34) \to (27,34) \to \\
& \quad (27,9) \to (26,9) \to (35,15) \to (38,28) \to (33,9) \to (13,28) \to \mathcal{O}\}
\end{aligned} \tag{5.28}
$$

Having a logarithmic description of this group is tremendously helpful in pen-and-paper computations. To see that, observe that we know fromXXX that there is an exponential map from the scalar field $\mathbb{F}_{13}$ to $BLS6\_6[13]$ with respect to our generator

add reference

$$[\cdot]_{(13,15)} : \mathbb{F}_{13} \to BLS6\_6[13] \; ; \; x \mapsto [x](13,15)$$

which generates the group in logarithmic order. So for example we have $[1]_{(13,15)} = (13,15)$, $[7]_{(13,15)} = (27,9)$ and $[0]_{(13,15)} = \mathcal{O}$ and so on. The point for our purposes is, that we can use this representation to do computations in $BLS6\_6[13]$ efficiently in our head using XXX. For example

add reference

$$
\begin{aligned}
(27,34) \oplus (33,9) &= [6](13,15) \oplus [11](13,15) \\
&= [6+11](13,15) \\
&= [4](13,15) \\
&= (35,28)
\end{aligned}
$$

So XXX is really all we need to do computations in $BLS6\_6[13]$ in this book efficiently. However, out of convenience, the following picture lists the entire addition table of that group. It might be useful in pen-and-paper computations:

add reference

$\oplus$	$\mathcal{O}$	(13,15)	(33,34)	(38,15)	(35,28)	(26,34)	(27,34)	(27,9)	(26,9)	(35,15)	(38,28)	(33,9)	(13,28)
$\mathcal{O}$	$\mathcal{O}$	(13,15)	(33,34)	(38,15)	(35,28)	(26,34)	(27,34)	(27,9)	(26,9)	(35,15)	(38,28)	(33,9)	(13,28)
(13,15)	(13,15)	(33,34)	(38,15)	(35,28)	(26,34)	(27,34)	(27,9)	(26,9)	(35,15)	(38,28)	(33,9)	(13,28)	$\mathcal{O}$
(33,34)	(33,34)	(38,15)	(35,28)	(26,34)	(27,34)	(27,9)	(26,9)	(35,15)	(38,28)	(33,9)	(13,28)	$\mathcal{O}$	(13,15)
(38,15)	(38,15)	(35,28)	(26,34)	(27,34)	(27,9)	(26,9)	(35,15)	(38,28)	(33,9)	(13,28)	$\mathcal{O}$	(13,15)	(33,34)
(35,28)	(35,28)	(26,34)	(27,34)	(27,9)	(26,9)	(35,15)	(38,28)	(33,9)	(13,28)	$\mathcal{O}$	(13,15)	(33,34)	(38,15)
(26,34)	(26,34)	(27,34)	(27,9)	(26,9)	(35,15)	(38,28)	(33,9)	(13,28)	$\mathcal{O}$	(13,15)	(33,34)	(38,15)	(35,28)
(27,34)	(27,34)	(27,9)	(26,9)	(35,15)	(38,28)	(33,9)	(13,28)	$\mathcal{O}$	(13,15)	(33,34)	(38,15)	(35,28)	(26,34)
(27,9)	(27,9)	(26,9)	(35,15)	(38,28)	(33,9)	(13,28)	$\mathcal{O}$	(13,15)	(33,34)	(38,15)	(35,28)	(26,34)	(27,34)
(26,9)	(26,9)	(35,15)	(38,28)	(33,9)	(13,28)	$\mathcal{O}$	(13,15)	(33,34)	(38,15)	(35,28)	(26,34)	(27,34)	(27,9)
(35,15)	(35,15)	(38,28)	(33,9)	(13,28)	$\mathcal{O}$	(13,15)	(33,34)	(38,15)	(35,28)	(26,34)	(27,34)	(27,9)	(26,9)
(38,28)	(38,28)	(33,9)	(13,28)	$\mathcal{O}$	(13,15)	(33,34)	(38,15)	(35,28)	(26,34)	(27,34)	(27,9)	(26,9)	(35,15)
(33,9)	(33,9)	(13,28)	$\mathcal{O}$	(13,15)	(33,34)	(38,15)	(35,28)	(26,34)	(27,34)	(27,9)	(26,9)	(35,15)	(38,28)
(13,28)	(13,28)	$\mathcal{O}$	(13,15)	(33,34)	(38,15)	(35,28)	(26,34)	(27,34)	(27,9)	(26,9)	(35,15)	(38,28)	(33,9)

Now that we have constructed a “large” cyclic prime order subgroup of $BLS6\_6$ sutable for many pen and paper computations in elliptic curve cryptography, we have to look at how to do pairings in this context. We know that $BLS6\_6$ is a pairing-friendly curve by design, since it has a small embedding degree $k = 6$. It is therefore possible to compute Weil pairings efficiently. However in order to do so, we have to decides the groups $\mathbb{G}_1$ and $\mathbb{G}_2$ as explained in XXX.

Since $BLS6\_6$ has two non-trivial subgroups, it would be possible to use any of them as the n-torsion group. However, in cryptography, the only secure choice is to use the large prime order subgroup, which in our case is $BLS6\_6[13]$. we therefore decide to consider the 13-torsion and define

$$\mathbb{G}_1[13] = \{(13,15) \to (33,34) \to (38,15) \to (35,28) \to (26,34) \to (27,34) \to (27,9) \to (26,9) \to (35,15) \to (38,28) \to (33,9) \to (13,28) \to \mathcal{O}\}$$

as the first argument for the Weil pairing function.

In order to construct the domain for the second argument, we need to construct $\mathbb{G}_2[13]$, which, according to the general theory should be defined by those elements P of the full 13-torsion group $BLS6\_6[13]$, that are mapped to $43 \cdot P$ under the Frobenius endomorphism XXX. add reference

To compute $\mathbb{G}_2[13]$ we therefore have to find the full 13-torsion group first. To do so, we use the technique from XXX, which tells us, that the full 13-torsion can be found in the curve extension

$$BLS6\_6 := \{(x,y) \mid y^2 = x^3 + 6 \text{ for all } x,y \in \mathbb{F}_{43^6}\} \tag{5.29}$$

over the extension field $\mathbb{F}_{43^6}$, since the embedding degree of $BLS6\_6$ is 6. So we have to construct $\mathbb{F}_{43^6}$, a field that contains 6321363049 many elements. In order to do so we use the procedure of XXX and start by choosing a non-reducible polynomial of degree 6 from the ring of polynomials $\mathbb{F}_{43}[t]$. We choose $p(t) = t^6 + 6$. Using Sage we get add reference

```
sage: F43 = GF(43)
sage: F43t.<t> = F43[]
sage: p = F43t(t^6+6)
sage: p.is_irreducible()
True
sage: F43_6.<v> = GF(43^6, name='v', modulus=p)
```

Recall from XXX that elements $x \in \mathbb{F}_{43^6}$ can be seen as polynomials $a_0 + a_1 v + a_2 v^2 + \ldots + a_5 v^5$ with the usual addition of polynomials and multiplication modulo $t^6 + 6$. add reference

In order to compute $\mathbb{G}_2[13]$ we first have to extend $BLS6\_6$ to $\mathbb{F}_{43^6}$, that is we keep the defining equation but expend the domain from $\mathbb{F}_{43}$ to $\mathbb{F}_{43^6}$. After that we have to find at least one element P from that curve, that is not the point at infinity, that is in the full 13-torsion and that satisfies the identity $\pi(P) = [43]P$. We can then use this element as our generator of $\mathbb{G}_2[13]$ and construct all other elements by repeated addition to itself.

Since $BLS6(\mathbb{F}_{43^6})$ contains 6321251664 elements, it's not a good strategy to simply loop through all elements. Fortunately Sage has a way to loop through elements from the torsion group directly. We get

```
sage: BLS6 = EllipticCurve (F43_6,[0 ,6]) # curve extension
sage: INF = BLS6(0) # point at infinity
sage: for P in INF.division_points(13): # full 13-torsion
....: # PI(P) == [q]P
....:     if P.order() == 13: # exclude point at infinity
....:         PiP = BLS6([a.frobenius() for a in P])
....:         qP = 43*P
```

```
....:         if PiP == qP:
....:             break
sage: P.xy()
(7*v^2, 16*v^3)
```

So we found an element from the full 13-torsion, that is in the Eigenspace of the Eigenvalue 43, which implies that it is an element of $\mathbb{G}_2[13]$. As $\mathbb{G}_2[13]$ is cyclic of prime order this element must be a generator and we write

$$g_{\mathbb{G}_2[13]} = (7v^2, 16v^3) \tag{5.30}$$

We can use this generator to compute $\mathbb{G}_2$ is logarithmic order with respect to $g_{\mathbb{G}_[13]}$. Using Sage we get

```
sage: Q = BLS6(7*v^2,16*v^3)
sage: BLS6_13_2 = []
sage: for x in range(0,13):
....:     P = x*Q
....:     BLS6_13_2.append(P)
```

$$\begin{aligned}\mathbb{G}_2 = \{(7v^2,16v^3) \to (10v^2,28v^3) \to (42v^2,16v^3) \to (37v^2,27v^3) \to \\ (16v^2,28v^3) \to (17v^2,28v^3) \to (17v^2,15v^3) \to (16v^2,15v^3) \to \\ (37v^2,16v^3) \to (42v^2,27v^3) \to (10v^2,15v^3) \to (7v^2,27v^3) \to \mathcal{O}\}\end{aligned}$$

Again, having a logarithmic description of $\mathbb{G}_2[13]$ is tremendously helpful in pen-and-paper computations, as it reduces complicated computation in the extended curve to modular 13 arithmetics. For example

$$\begin{aligned}(17v^2,28v^3) \oplus (10v^2,15v^2) &= [6](7v^2,16v^3) \oplus [11](7v^2,16v^3)\\ &= [6+11](7v^2,16v^3)\\ &= [4](7v^2,16v^3)\\ &= (37v^2,27v^3)\end{aligned}$$

So XXX is really all we need to do computations in $\mathbb{G}_2[13]$ in this book efficiently.

add reference

To summarize the previous steps, we have found two subgroups $\mathbb{G}_1[13]$ as well as $\mathbb{G}_2[13]$ suitable to do Weil pairings on *BLS*6_6 as explained in XXX. Using the logarithmic order XXX of $\mathbb{G}_1[13]$, the logarithmic order XXX of $\mathbb{G}_2[13]$ and the bilinearity

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$$e([k_1]g_{BLS6\_6[13]}, [k_2]g_{\mathbb{G}_2[13]}) = e(g_{BLS6\_6[13]}, g_{\mathbb{G}_2[13]})^{k_1 \cdot k_2}$$

we can do Weil pairings on *BLS*6_6 in a pen-and-paper style, observing that the Weil pairing between our two generators is given by the identity

$$e(g_{BLS6\_6[13]}, g_{\mathbb{G}_2[13]}) = 5v^5 + 16v^4 + 16v^3 + 15v^2 + 3v + 41$$

```
sage: g1 = BLS6([13,15])
sage: g2 = BLS6([7*v^2, 16*v^3])
sage: g1.weil_pairing(g2,13)
5*v^5 + 16*v^4 + 16*v^3 + 15*v^2 + 3*v + 41
```

Hashing to the pairing groups We give various constructions to hash into $\mathbb{G}_1$ and $\mathbb{G}_2$.

We start with hashing to the scalar field... TO APPEAR

Non of these techniques work for hashing into $\mathbb{G}_2$. We therefore implement Pederson's Hash for BLS6.

We start with $\mathbb{G}_1$. Our goal is to define an 12-bit bounded hash function

$$H_1 : \{0,1\}^{12} \to \mathbb{G}_1$$

Since $12 = 3 \cdot 4$ we "randomly" select 4 uniformly distributed generators $\{(38,15),(35,28),(27,34),(38,28)\}$ from $\mathbb{G}_1$ and use the pseudo-random function from XXX. For every generator we therefore have to choose a set of 4 randomly generated invertible elements from $\mathbb{F}_{13}$. We choose

add reference

$$\begin{array}{lcl}
(38,15) & : & \{2,7,5,9\} \\
(35,28) & : & \{11,4,7,7\} \\
(27,34) & : & \{5,3,7,12\} \\
(38,28) & : & \{6,5,1,8\}
\end{array}$$

So our hash function is computed like this:

$$\begin{aligned}
H_1(x_{11},x_1,\ldots,x_0) = [2\cdot 7^{x_{11}}\cdot 5^{x_{10}}\cdot 9^{x_9}](38,15) + [11\cdot 4^{x_8}\cdot 7^{x_7}\cdot 7^{x_6}](35,28) + \\
[5\cdot 3^{x_5}\cdot 7^{x_4}\cdot 12^{x_3}](27,34) + [6\cdot 5^{x_2}\cdot 1^{x_1}\cdot 8^{x_0}](38,28)
\end{aligned}$$

Note that $a^x = 1$ whe $x = 0$ and hence those terms can be omitted in the computation. In particular the hash of the 12-bit zero string is given by

$$\begin{aligned}
WRONG-ORDERING-REDOH_1(0) = [2](38,15)+[11](35,28)+[5](27,34)+[6](38,28) = \\
(27,34)+(26,34)+(35,28)+(26,9) = (33,9)+(13,28) = (38,28)
\end{aligned}$$

The hash of 011010101100 is given by

$$\begin{aligned}
H_1(011010101100) = WRONG-ORDERING-REDO \\
[2\cdot 7^0\cdot 5^1\cdot 9^1](38,15)+[11\cdot 4^0\cdot 7^1\cdot 7^0](35,28)+[5\cdot 3^1\cdot 7^0\cdot 12^1](27,34)+[6\cdot 5^1\cdot 1^0\cdot 8^0](38,28) = \\
[2\cdot 5\cdot 9](38,15)+[11\cdot 7](35,28)+[5\cdot 3\cdot 12](27,34)+[6\cdot 5](38,28) = \\
[12](38,15)+[12](35,28)+[11](27,34)+[4](38,28) = \\
TO_APPEAR
\end{aligned}$$

We can use the same technique to define a 12-bit bounded hash function in $\mathbb{G}_2$:

$$H_2 : \{0,1\}^{12} \to \mathbb{G}_2$$

Again we "randomly" select 4 uniformly distributed generators $\{(7v^2,16v^3),(42v^2,16v^3),(17v^2,15v^3),(10v^2,15v^3)\}$ from $\mathbb{G}_2$ and use the pseudo-random function from XXX. For every genrator we therefore have to choose a set of 4 randomly generated invertible elements from $\mathbb{F}_{13}$. We choose

add reference

$$\begin{array}{lcl}
(7v^2,16v^3) & : & \{8,4,5,7\} \\
(42v^2,16v^3) & : & \{12,1,3,8\} \\
(17v^2,15v^3) & : & \{2,3,9,11\} \\
(10v^2,15v^3) & : & \{3,6,9,10\}
\end{array}$$

So our hash function is computed like this:

$$H_1(x_{11}, x_{10}, \ldots, x_0) = [8 \cdot 4^{x_{11}} \cdot 5^{x_{10}} \cdot 7^{x_9}](7v^2, 16v^3) + [12 \cdot 1^{x_8} \cdot 3^{x_7} \cdot 8^{x_6}](42v^2, 16v^3) + \\ [2 \cdot 3^{x_5} \cdot 9^{x_4} \cdot 11^{x_3}](17v^2, 15v^3) + [3 \cdot 6^{x_2} \cdot 9^{x_1} \cdot 10^{x_0}](10v^2, 15v^3)$$

We extend this to a hash function that maps unbounded bitstring to $\mathbb{G}_2$ by precomposing with an actual hash function like $MD5$ and feet the first 12 bits of its outcome into our previously defined hash function.

$$TinyMD5_{\mathbb{G}_2} : \{0,1\}^* \to \mathbb{G}_2$$

with $TinyMD5_{\mathbb{G}_2}(s) = H_2(MD5(s)_0, \ldots MD5(s)_{11})$. For example, since $MD5(\text{''''}) = 0xd41d8cd98f00b204e9800998ecf8427e$ and the binary representation of the hexadecimal number $0x27e$ is 001001111110 we compute $TinyMD5_{\mathbb{G}_2}$ of the empty string as $TinyMD5_{\mathbb{G}_2}(\text{''''}) = H_2(MD5(s)_{11}, \ldots MD5(s)_0) = H2(001001111110) =$

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