Operational notes

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The following colors are **not** part of the final product, but serve as highlights in the editing/review process:

- text that needs attention from the Subject Matter Experts: Mirco, Anna,& Jan
- terms that have not yet been defined in the book
- things that need to be checked only at the very final typesetting stage (and it doesn't make sense to do them before)
- text that needs advice from the communications/marketing team: Aaron & Shane
- text that needs to be completed or otherwise edited (by Sylvia)

NB: This PDF only includes the following chapter(s): Arithmetics, Algebra.

Todo list

MoonMath manual

TechnoBob and the Least Scruples crew

August 17, 2022

Contents

Chapter 3

Arithmetics

S: This chapter talks about different types of arithmetic, so I suggest using "Arithmetics" as the chapter title.

Pluralize chapter title

3.1 Introduction

The goal of this chapter is to bring a reader with only basic school-level algebra up to speed in arithmetics. We start with a brief recapitulation of basic integer arithmetics, discussing long division, the greatest common divisor and Euclidean Division. After that, we introduce modular arithmetics as **the most important** skill to compute our pen-and-paper examples. We then introduce polynomials, compute their analogs to integer arithmetics and introduce the important concept of Lagrange Interpolation.

check if this is already introduced in intro

3.2 Integer arithmetic

In a sense, integer arithmetic is at the heart of large parts of modern cryptography. Fortunately, most readers will probably remember integer arithmetic from school. It is, however, important that you can confidently apply those concepts to understand and execute computations in the many pen-and-paper examples that form an integral part of the MoonMath Manual. We will therefore recapitulate basic arithmetic concepts to refresh your memory and fill any knowledge gaps.

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formality of addressing the reader

Even though the terms and concepts in this chapter might not appear in the literature on zero-knowledge proofs directly, understanding them is necessary to follow subsequent chapters and beyond: terms like **groups** or **fields** also crop up very frequently in academic papers on zero-knowledge cryptography.

3.2.1 Integers, natural numbers and rational numbers

Integers are also known as **whole numbers**, that is, numbers that can be written without fractional parts. Examples of numbers that are **not** integers are $\frac{2}{3}$, 1.2 and -1280.006.

Throughout this book, we use the symbol $\mathbb{Z}$ as a shorthand for the set of all **integers**:

$$\mathbb{Z} := \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\} \tag{3.1}$$

If $a \in \mathbb{Z}$ is an integer, then we write $|a|$ for the **absolute value** of a, that is, the the non-negative value of a without regard to its sign:

$$|4| = 4 \tag{3.2}$$

$$|-4| = 4 \tag{3.3}$$

We use the symbol $\mathbb{N}$ for the set of all positive integers, usually called the set of **natural numbers**. Furthermore, we use $\mathbb{N}_0$ for the set of all non-negative integers. This means that $\mathbb{N}$ does not contain the number 0, while $\mathbb{N}_0$ does:

$$\mathbb{N} := \{1, 2, 3, \ldots\} \qquad \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$$

SB: Talking about the binary representation seems way to complex at this stage, and the concepts introduced here are not used for several chapters. Let $n \in \mathbb{N}_0$ be a non-negative integer and $(b_0, b_1, \ldots b_k)$ a string of **bits** $b_j \in \{0,1\} \subset \mathbb{N}_0$ for some non negative integer $k \in \mathbb{N}$, such that the following equation holds:

Move content on binary representation

$$n = \sum_{j=0}^{k} b_j \cdot 2^j \tag{3.4}$$

In this case, we call $Bits(n) :=< b_0, b_1, \ldots b_k >$ the **binary representation** of n, say that n is a k-bit number and call $k := |n|_2$ the **bit length** of n. It can be shown, that the binary representation of any non negative integer is unique. We call b_0 the **least significant bit** and b_k the **most significant** bit and define the **Hamming weight** of an integer as the number of 1s in its binary representation.

In addition, we use the symbol $\mathbb{Q}$ for the set of all **rational numbers**, which can be represented as the set of all fractions $\frac{n}{m}$, where $n \in \mathbb{Z}$ is an integer and $m \in \mathbb{N}$ is a natural number, such that there is no other fraction $\frac{n'}{m'}$ and natural number $k \in \mathbb{N}$ with $k \neq 1$ and

$$\frac{n}{m} = \frac{k \cdot n'}{k \cdot m'} \tag{3.5}$$

The sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ have a notion of addition anduse multiplication defined on them. Most of us are probably able to do many integer computations in our head, but this gets more and more difficult as these increase in complexity. We will frequently use the SageMath system (2.7.1) for more complicated computations (we define rings and fields later in this book):SB: I would delete lines 12-18 form the Sage example below, unnecessarily confusing at this point

simplify Sage ex.

```
1  sage: ZZ #  Sage notation for the set of integers
2  Integer Ring
3  sage: NN # Sage notation for the set of natural numbers
4  Non negative integer semiring
5  sage: QQ # Sage notation for the set of rational numbers
6  Rational Field
7  sage: ZZ(5) # Get an element from the set of integers
8  5
9  sage: ZZ(5) + ZZ(3)
10 8
11 sage: ZZ(5) * NN(3)
```

```
15
sage: ZZ.random_element(10**50)
41490755353903614474502848705480378778949783554381
sage: ZZ(27713).str(2) # Binary string representation
110110001000001
sage: NN(27713).str(2) # Binary string representation
110110001000001
sage: ZZ(27713).str(16) # Hexadecimal string representation
6c41
```

A set of numbers of particular interest to us is the set of **prime numbers**. A prime number is a natural number $p \in \mathbb{N}$ with $p \geq 2$ that is only divisible by itself and by 1. All prime numbers apart from the number 2 are called **odd** prime numbers. We use $\mathbb{P}$ for the set of all prime numbers and $\mathbb{P}_{\geq 3}$ for the set of all odd prime numbers.

The set of prime numbers $\mathbb{P}$ is an infinite set, and it can be ordered according to size. This means that, for any prime number $p \in \mathbb{P}$, one can always find another prime number $p' \in \mathbb{P}$ with $p < p'$. As a result, there is no largest prime number. Since prime numbers can be ordered by size, we can write them as follows:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, \ldots \tag{3.6}$$

As the **fundamental theorem of arithmetic** tells us, prime numbers are, in a certain sense, the basic building blocks from which all other natural numbers are composed: every natural number can be derived by multiplying prime numbers with each other. To see that, let $n \in \mathbb{N}$ be any natural number with $n > 1$. Then there are always prime numbers $p_1, p_2, \ldots, p_k \in \mathbb{P}$, such that the following equation holds:

$$n = p_1 \cdot p_2 \cdot \ldots \cdot p_k \tag{3.7}$$

This representation is unique for each natural number (except for the order of the **factors** $p_1, p_2, \ldots, p_k$) and is called the **prime factorization** of n.

Example 1 (Prime Factorization). To see what we mean by the prime factorization of a number, let's look at the number $504 \in \mathbb{N}$. To get its prime factors, we can successively divide it by all prime numbers in ascending order starting with 2:

$$504 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7$$

We can double check our findings invoking Sage, which provides an algorithm for factoring natural numbers:

```
sage: n = NN(504)
sage: factor(n)
2^3 * 3^2 * 7
```

The computation from the previous example reveals an important observation: computing the factorization of an integer is computationally expensive, because we have to divide repeatedly by all prime numbers smaller than the number itself until all factors are prime numbers themselves. From this, an important question arises: how fast can we compute the prime factorization of a natural number? This question is the famous **integer factorization problem** and, as far as we know, there is currently no known method that can factor integers much faster than the naive approach of just dividing the given number by all prime numbers in ascending order.

On the other hand, computing the product of a given set of prime numbers is fast: you just multiply all factors. This simple observation implies that the two processes, “prime number multiplication” on the one side and its inverse process “natural number factorization” have very different computational costs. The factorization problem is therefore an example of a so-called **one-way function**: an invertible function that is easy to compute in one direction, but hard to compute in the other direction (see this description of time complexity and polynomial time). [1]

you

a bit overused in the text?

Exercise 1. What is the absolute value of the integers -123, 27 and 0?

Exercise 2. Compute the factorization of 30030 and double check your results using Sage.

Exercise 3. Consider the following equation:

$$4 \cdot x + 21 = 5.$$

Compute the set of all solutions for x under the following alternative assumptions:

1. The equation is defined over the set of natural numbers.
2. The equation is defined over the set of integers.

Exercise 4. Consider the following equation:

$$2x^3 - x^2 - 2x = -1.$$

Compute the set of all solutions x under the following assumptions:

1. The equation is defined over the set ofnatural numbers.
2. The equation is defined over the set of integers.
3. The equation is defined over the set of rational numbers.

3.2.2 Euclidean Division

As we know from high school mathematics, integers can be added, subtracted and multiplied, and the result of these operations is guaranteed to always be an integer as well. On the contrary, division (in the commonly understood sense) is not defined for integers, as, for example, 7 divided by 3 will not result in an integer. However, it is always possible to divide any two integers if we consider **division with a remainder**. For example, 7 divided by 3 is equal to 2 with a remainder of 1, since $7 = 2 \cdot 3 + 1$.

This section introduces division with a remainder for integers, usually called **Euclidean Division**. It is an essential technique underlying many concepts in this book. The precise definition is as follows:

Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ be two integers with $b \neq 0$. Then there is always another integer $m \in \mathbb{Z}$ and a natural number $r \in \mathbb{N}$, with $0 \leq r < |b|$ such that the following holds:

$$a = m \cdot b + r \tag{3.8}$$

[1] It should be noted that what is “easy” and “hard” to compute depends on the computational power available to us. Currently available computers cannot easily compute the prime factorization of natural numbers (in formal terms, the cannot compute it in polynomial time). However, the American mathematician Peter W. Shor developed an algorithm in 1994 which can calculate the prime factorization of a natural number in polynomial time on a quantum computer. The consequence of this is that cryptosystems, which are based on the prime factor problem being computationally hard on currently available computers, become unsafe as soon as practically usable quantum computers become available.

This decomposition of a given b is called **Euclidean Division**, where a is called the **dividend**, b is called the **divisor**, m is called the **quotient** and r is called the **remainder**. It can be shown that both the quotient and the remainder always exist and are unique, as long as the divisor is different from 0.

Notation and Symbols 1. Suppose that the numbers a, b, m and r satisfy equation (3.8). We can then describe the quotient and the remainder of the Euclidean Division as follows:

$$a \text{ div } b := m, \qquad a \bmod b := r \tag{3.9}$$

We also say that an integer a is **divisible** by another integer b if $a \bmod b = 0$ holds. In this case, we write $b|a$, and call the integer a div b the **cofactor** of b in a.

So, in a nutshell, Euclidean Division is the process of dividing one integer by another in a way that produces a quotient and a non-negative remainder, the latter of which is smaller than the absolute value of the divisor.

Example 2. Applying Euclidean Division and the notation defined in 3.9 to the dividend -17 and the divisor 4, we get the following:

$$-17 \text{ div } 4 = -5, \qquad -17 \bmod 4 = 3 \tag{3.10}$$

$-17 = -5 \cdot 4 + 3$ is the Euclidean Division of -17 by 4. The remainder, by definition, is a non-negative number. In this case, 4 does not divide -17, as the remainder is not zero. The truth value of the expression $4|-17$ therefore is FALSE. On the other hand, the truth value of $4|12$ is TRUE, since 4 divides 12, as $12 \bmod 4 = 0$. If we use Sage to do the computation for us, we get the following:

```
sage: ZZ(-17) // ZZ(4) # Integer quotient
-5
sage: ZZ(-17) % ZZ(4) # remainder
3
sage: ZZ(4).divides(ZZ(-17)) # self divides other
False
sage: ZZ(4).divides(ZZ(12))
True
```

Remark 1. In 3.9, we defined the notation of a **div** b and a **mod** b in terms of Euclidean Division. It should be noted, however, that many programing languages (like Python and Sage) implement both the operator $(/)$ amd the operator $(\%)$ differently. Programers should be aware of this, as the discrepancy between the mathematical notation and the implementation in programing languages might become the source of subtle bugs in implementations of cryptographic primitives.

To give an example, consider the the dividend -17 and the divisor -4. Note that, in contrast to the previous example 2, we now have a negative divisor. According to our definition we have the following:

$$-17 \text{ div } -4 = 5, \qquad -17 \bmod -4 = 3 \tag{3.11}$$

$-17 = 5 \cdot (-4) + 3$ is the Euclidean Division of -17 by -4 (the remainder is, by definition, a non-negative number). However, using the operators $(/)$ and $(\%)$ in Sage, we get a different result:

```
sage: ZZ(-17) // ZZ(-4) # Integer quotient
```

```
4
sage: ZZ(-17) % ZZ(-4) # remainder
-1
sage: ZZ(-17).quo_rem(ZZ(-4)) # not Euclidean Division
(4, -1)
```

Methods to compute Euclidean Division for integers are called **integer division algorithms**. Probably the best known algorithm is the so-called **long division**, which most of us might have learned in school.

In a nutshell, the long division algorithm loops through the digits of the dividend from the left to right, subtracting the largest possible multiple of the divisor (at the digit level) at each stage. The multiples then become the digits of the quotient, and the remainder is the first digit of the dividend.

As long division is the standard method used for pen-and-paper division of multi-digit numbers expressed in decimal notation, we use it throughout this book when we do simple pen-and-paper computations, so readers should become familiar with it. However, instead of defining the algorithm formally, we provide some examples instead, as this will hopefully make the process more clear.

readers

Example 3 (Integer Long Division). To give an example of integer long division algorithm, let's divide the integer $a = 143785$ by the number $b = 17$. Our goal is therefore to find solutions to equation 3.8, that is, we need to find the quotient $m \in \mathbb{Z}$ and the remainder $r \in \mathbb{N}$ such that $143785 = m \cdot 17 + r$. Using a notation that is mostly used in English-speaking countries, we compute as follows:SB: I think a more detailed explanation is needed for those unfamiliar with this notation/algorithm

Add explanation

$$
\begin{array}{r}
8457 \\
17\,\overline{)\,143785} \\
\underline{136} \\
77 \\
\underline{68} \\
98 \\
\underline{85} \\
135 \\
\underline{119} \\
16
\end{array}
\tag{3.12}
$$

We calculated $m = 8457$ and $r = 16$, and, indeed, the equation $143785 = 8457 \cdot 17 + 16$ holds. We can double check this invoking Sage:

```
sage: ZZ(143785).quo_rem(ZZ(17))
(8457, 16)
sage: ZZ(143785) == ZZ(8457)*ZZ(17) + ZZ(16) # check
True
```

Exercise 5 (Integer Long Division). Find an $m \in \mathbb{Z}$ and an $r \in \mathbb{N}$ with $0 \le r < |b|$ such that $a = m \cdot b + r$ holds for the following pairs:

- $(a,b) = (27,5)$
- $(a,b) = (27,-5)$
- $(a,b) = (127,0)$

- $(a,b) = (-1687, 11)$
- $(a,b) = (0,7)$

In which cases are your solutions unique?

Exercise 6 (Long Division Algorithm). Using the programming language of your choice, write an algorithm that computes integer long division and handles all edge cases properly.

Exercise 7 (Binary Representation). Using the programming language of your choice, write an algorithm that computes the binary representation 3.4 of any non-negative integer.

3.2.3 The Extended Euclidean Algorithm

One of the most critical parts of this book is the modular arithmetic, defined in section 3.3, and its application in the computations of **prime fields**, defined in section 4.3.1. To be able to do computations in modular arithmetic, we have to get familiar with the so-called **Extended Euclidean Algorithm**, used to calculate the **greatest common divisor** (GCD) of integers.

The greatest common divisor of two non-zero integers a and b is defined as the largest non-zero natural number d such that d divides both a and b, that is, $d|a$ as well as $d|b$ are true. We use the notation $gcd(a,b) := d$ for this number. Since the natural number 1 divides any other integer, 1 is always a common divisor of any two non-zero integers, but it is not necessarily the greatest.

A common method for computing the greatest common divisor is the so-called Euclidean Algorithm. However, since we don't need that algorithm in this book, we will introduce the Extended Euclidean Algorithm, which is a method for calculating the greatest common divisor of two natural numbers a and $b \in \mathbb{N}$, as well as two additional integers $s,t \in \mathbb{Z}$, such that the following equation holds:

$$gcd(a,b) = s \cdot a + t \cdot b \tag{3.13}$$

The pseudocode in algorithm 1 shows in detail how to calculate the greatest common divisor and the numbers s and t with the Extended Euclidean Algorithm:In example 4, the computation stops when $r_k = 0$ (at k_4), not when $r_{k-1} = 0$ (which would be k_5). Also the GCD is $r_3 = r_{k-1} = 1$, not r_{k-2}. Same for s and t.

unify with example 4

The algorithm is simple enough to be used effectively in pen-and-paper examples. It is commonly written as a table where where the rows represent the while-loop and the columns represent the values of the the array r, s and t with index k. The following example provides a simple execution.

Example 4. To illustrate algorithm 1, we apply it to the numbers $a = 12$ and $b = 5$. Since $12, 5 \in \mathbb{N}$ and $12 \geq 5$, all requirements are met, and we compute as follows:

check additional explanation

k	r_k	s_k	t_k
0	12	1	0
1	5	0	1
2	2	1	-2
3	1	-2	5
4	0		

Algorithm 1 Extended Euclidean Algorithm

Require: $a, b \in \mathbb{N}$ with $a \geq b$

 procedure EXT-EUCLID(a, b)
  $r_0 \leftarrow a$ and $r_1 \leftarrow b$
  $s_0 \leftarrow 1$ and $s_1 \leftarrow 0$
  $t_0 \leftarrow 0$ and $t_1 \leftarrow 1$
  $k \leftarrow 2$
  while $r_{k-1} \neq 0$ **do**
   $q_k \leftarrow r_{k-2}$ div r_{k-1}
   $r_k \leftarrow r_{k-2}$ mod r_{k-1}
   $s_k \leftarrow s_{k-2} - q_k \cdot s_{k-1}$
   $t_k \leftarrow t_{k-2} - q_k \cdot t_{k-1}$
   $k \leftarrow k+1$
  end while
  return $gcd(a,b) \leftarrow r_{k-2}$, $s \leftarrow s_{k-2}$ and $t \leftarrow t_{k-2}$
 end procedure

Ensure: $gcd(a,b) = s \cdot a + t \cdot b$

k	r_k	s_k	t_k	q_k				
0	12	1	0	–	$r_0 \leftarrow a = 12$	$s_0 \leftarrow 1$	$t_0 \leftarrow 0$	
1	5	0	1	–	$r_1 \leftarrow b = 5$	$s_1 \leftarrow 0$	$t_1 \leftarrow 1$	
2	2	1	-2	2	$r_2 \leftarrow r_0$ mod $r_1 =$ 12 mod $5 = 2$	$s_2 \leftarrow s_0 - q_2 \cdot s_1 =$ $1 - 2 \cdot 0 = 1$	$t_2 \leftarrow t_0 - q_2 \cdot t_1 =$ $0 - 2 \cdot 1 = -2$	$q_2 \leftarrow r_0$ div $r_1 =$ 12 div $5 = 2$
3	1	-2	5	2	$r_3 \leftarrow r_1$ mod $r_2 =$ 5 mod $2 = 1$	$s_3 \leftarrow s_1 - q_3 \cdot s_2 =$ $0 - 2 \cdot 1 = -2$	$t_3 \leftarrow t_1 - q_3 \cdot t_2 =$ $1 - 2 \cdot -2 = 5$	$q_3 \leftarrow r_1$ div $r_2 =$ 5 div $2 = 2$
4	0				$r_4 \leftarrow r_2$ mod $r_3 =$ 2 mod $1 = 0$			

From this we can see that the greatest common divisor of 12 and 5 is $gcd(12,5) = 1$ and that the equation $1 = (-2) \cdot 12 + 5 \cdot 5$ holds. We can also use Sage to double check our findings:

```
sage: ZZ(12).xgcd(ZZ(5)) # (gcd(a,b),s,t)
(1, -2, 5)
```

Exercise 8 (Extended Euclidean Algorithm). Find integers $s, t \in \mathbb{Z}$ such that $gcd(a,b) = s \cdot a + t \cdot b$ holds for the following pairs:

- $(a,b) = (45,10)$
- $(a,b) = (13,11)$
- $(a,b) = (13,12)$

Exercise 9 (Towards Prime fields). Let $n \in \mathbb{N}$ be a natural number and p a prime number, such that $n < p$. What is the greatest common divisor $gcd(p,n)$?

Exercise 10. Find all numbers $k \in \mathbb{N}$ with $0 \leq k \leq 100$ such that $gcd(100, k) = 5$.

Exercise 11. Show that $gcd(n, m) = gcd(n + m, m)$ for all $n, m \in \mathbb{N}$.

3.2.4 Coprime Integers

Coprime integers are integers that do not share a prime number as a factor. As we will see in 3.3, coprime integers are important for our purposes, because, in modular arithmetic, computations that involve coprime numbers are substantially different from computations on non-coprime numbers 3.3.2.

The naive way to decide if two integers are coprime would be to divide both numbers successively by all prime numbers smaller than those numbers, to see if they share a common prime factor. However, two integers are coprime if and only if their greatest common divisor is 1, which is why computing the gcd is the preferred method.

Example 5. Consider example 4 again. As we have seen, the greatest common divisor of 12 and 5 is 1. This implies that the integers 12 and 5 are coprime, since they share no divisor other than 1, which is not a prime number.

Exercise 12. Consider exercise 8 again. Which pairs (a, b) from that exercise are coprime?

3.3 Modular arithmetic

Modular arithmetic is a system of integer arithmetic where numbers "wrap around" when reaching a certain value, much like calculations on a clock wrap around whenever the value exceeds the number 12. For example, if the clock shows that it is 11 o'clock, then 20 hours later it will be 7 o'clock, not 31 o'clock. The number 31 has no meaning on a normal clock that shows hours.

The number at which the wrap occurs is called the **modulus**. Modular arithmetic generalizes the clock example to arbitrary moduli, and studies equations and phenomena that arise in this new kind of arithmetic. It is of central importance for understanding most modern cryptographic systems, in large parts because modular arithmetic provides the computational infrastructure for algebraic types that have cryptographically useful examples of one-way functions.

Although modular arithmetic appears very different from ordinary integer arithmetic that we are all familiar with, we encourage you to work through the examples and discover that, once they get used to the idea that this is a new kind of calculation, it will seem much less daunting.

3.3.1 Congruence

In what follows, let $n \in \mathbb{N}$ with $n \geq 2$ be a fixed natural number that we will call the **modulus** of our modular arithmetic system. With such an n given, we can then group integers into classes: two integers are in the same class whenever their Euclidean Division (3.2.2) by n will give the same remainder. We two numbers that are in the same class are called **congruent**.

Example 6. If we choose $n = 12$ as in our clock example, then the integers -7, 5, 17 and 29 are all congruent with respect to 12, since all of them have the remainder 5 if we perform Euclidean Division on them by 12. Imagining the picture of an analog 12-hour clock, starting at 5 o'clock and adding 12 hours, we are at 5 o'clock again, representing the number 17. Indeed, in many countries, 5:00 in the afternoon is written as 17:00. On the other hand, when we subtract 12 hours, we are at 5 o'clock again, representing the number -7.

We can formalize this intuition of what congruence should be into a proper definition utilizing Euclidean Division (as explained previously in 3.2). Let a, $b \in \mathbb{Z}$ be two integers, and $n \in \mathbb{N}$ be a natural number such that $n \geq 2$. The integers a and b are said to be **congruent with respect to the modulus** n if and only if the following equation holds:

$$a \bmod n = b \bmod n \tag{3.14}$$

If, on the other hand, two numbers are not congruent with respect to a given modulus n, we call them **incongruent** w.r.t. n.

In other words, **congruence** is an equation "up to congruence", which means that the equation only needs to hold if we take the modulus of both sides. This is expressed with the following notation:

$$a \equiv b \pmod{n} \tag{3.15}$$

Exercise 13. Which of the following pairs of numbers are congruent with respect to the modulus 13:

- $(5,19)$
- $(13,0)$
- $(-4,9)$
- $(0,0)$

Exercise 14. Find all integers x, such that the congruence $x \equiv 4 \pmod{6}$ is satisfied.

3.3.2 Computational Rules

Having defined the notion of a congruence as an equation "up to a modulus", a follow-up question is if we can manipulate a congruence similarly to an equation. Indeed, we can almost apply the same substitution rules to a congruency as to an equation, with the main difference being that, for some non-zero integer $k \in \mathbb{Z}$, the congruence $a \equiv b \pmod{n}$ is equivalent to the congruence $k \cdot a \equiv k \cdot b \pmod{n}$ only if k and the modulus n are coprime (see 3.2.4).

Suppose that integers $a_1, a_2, b_1, b_2, k \in \mathbb{Z}$ are given. Then the following arithmetic rules hold for congruences:

- $a_1 \equiv b_1 \pmod{n} \Leftrightarrow a_1 + k \equiv b_1 + k \pmod{n}$ (compatibility with translation)
- $a_1 \equiv b_1 \pmod{n} \Rightarrow k \cdot a_1 \equiv k \cdot b_1 \pmod{n}$ (compatibility with scaling)
- $gcd(k,n) = 1$ and $k \cdot a_1 \equiv k \cdot b_1 \pmod{n} \Rightarrow a_1 \equiv b_1 \pmod{n}$
- $k \cdot a_1 \equiv k \cdot b_1 \pmod{k \cdot n} \Rightarrow a_1 \equiv b_1 \pmod{n}$
- $a_1 \equiv b_1 \pmod{n}$ and $a_2 \equiv b_2 \pmod{n} \Rightarrow a_1 + a_2 \equiv b_1 + b_2 \pmod{n}$ (compatibility with addition)
- $a_1 \equiv b_1 \pmod{n}$ and $a_2 \equiv b_2 \pmod{n} \Rightarrow a_1 \cdot a_2 \equiv b_1 \cdot b_2 \pmod{n}$ (compatibility with multiplication)

Other rules, such as compatibility with subtraction, follow from the rules above. For example, compatibility with subtraction follows from compatibility with scaling by $k = -1$ and compatibility with addition.

Another property of congruences not found in the traditional arithmetic of integers is **Fermat's Little Theorem**. Simply put, it states that, in modular arithmetic, every number raised to the power of a prime number modulus is congruent to the number itself. Or, to be more precise, if $p \in \mathbb{P}$ is a prime number and $k \in \mathbb{Z}$ is an integer, then the following holds:

$$k^p \equiv k \pmod{p} \tag{3.16}$$

If k is coprime to p, then we can divide both sides of this congruence by k and rewrite the expression into the following equivalent form:

$$k^{p-1} \equiv 1 \pmod{p} \tag{3.17}$$

The Sage code below computes examples of Fermat's Little Theorem and highlights the effects of the exponent k being coprime to p (as in the case of 137 and 64) and not coprime to p (as in the case of 1918 and 137):

```
sage: ZZ(137).gcd(ZZ(64))
1
sage: ZZ(64)^ ZZ(137) % ZZ(137) == ZZ(64) % ZZ(137)
True
sage: ZZ(64)^ ZZ(137-1) % ZZ(137) == ZZ(1) % ZZ(137)
True
sage: ZZ(1918).gcd(ZZ(137))
137
sage: ZZ(1918)^ ZZ(137) % ZZ(137) == ZZ(1918) % ZZ(137)
True
sage: ZZ(1918)^ ZZ(137-1) % ZZ(137) == ZZ(1) % ZZ(137)
False
```

The following example contains most of the concepts described in this section.

Example 7. Let us solve the following congruence for $x \in \mathbb{Z}$ in modular 6 arithmetic:

$$7 \cdot (2x + 21) + 11 \equiv x - 102 \pmod{6}$$

As many rules for congruences are more or less same as for equations, we can proceed in a similar way as we would if we had an equation to solve. Since both sides of a congruence contain ordinary integers, we can rewrite the left side as follows:

$$7 \cdot (2x + 21) + 11 = 14x + 147 + 11 = 14x + 158$$

We can therefore rewrite the congruence into the equivalent form:

$$14x + 158 \equiv x - 102 \pmod{6}$$

In the next step, we want to shift all instances of x to the left and every other term to the right. So we apply the"compatibility with translation" rules twice. In the first step, we choose $k = -x$, and in a second step, we choose $k = -158$. separate steps 1 and 2 Since "compatibility with

SB: let's separate these two steps in the equivalence below translation" transforms a congruence into an equivalent form, the solution set will not change, and we get the following:

$$\begin{aligned}14x+158 \equiv x-102 \pmod{6} \Leftrightarrow \\ 14x-x+158-158 \equiv x-x-102-158 \pmod{6} \Leftrightarrow \\ 13x \equiv -260 \pmod{6}\end{aligned}$$

If our congruence was just a regular integer equation, we would divide both sides by 13 to get $x=-20$ as our solution. However, in case of a congruence, we need to make sure that the modulus and the number we want to divide by are coprime to ensure that get an equivalent expression (see rule 3.17). This means that we need to find the greatest common divisor $gcd(13,6)$. Since 13 is prime and 6 is not a multiple of 13, we know that$gcd(13,6)=1$, so these numbers are indeed coprime. We therefore compute as follows:

check reference

$$13x \equiv -260 \pmod{6} \Leftrightarrow x \equiv -20 \pmod{6}$$

Our task now is to find all integers x such that x is congruent to -20 with respect to the modulus 6. In other words, we have to find all x such that the following equation holds:

$$x \bmod 6 = -20 \bmod 6$$

Since $-4\cdot 6+4=-20$, we know that $-20 \bmod 6 = 4$, and hence we know that $x=4$ is a solution to this congruence. However, 22 is another solution, since $22 \bmod 6 = 4$ as well. Another solution is -20. In fact, there are infinitely many solutions given by the following set:

$$\{\ldots,-8,-2,4,10,16,\ldots\} = \{4+k\cdot 6 \mid k\in\mathbb{Z}\}$$

Putting all this together, we have shown that every x from the set $\{x=4+k\cdot 6 \mid k\in\mathbb{Z}\}$ is a solution to the congruence $7\cdot(2x+21)+11 \equiv x-102 \pmod{6}$. We double check for two arbitrary numbers from this set, $x=4$ and $x=4+12\cdot 6=76$ using Sage:

```
sage: (ZZ(7)* (ZZ(2)*ZZ(4) + ZZ(21)) + ZZ(11))  % ZZ(6) == (ZZ
   (4) - ZZ(102))  % ZZ(6)
True
sage: (ZZ(7)* (ZZ(2)*ZZ(76) + ZZ(21)) + ZZ(11))  % ZZ(6) == (
   ZZ(76) - ZZ(102))  % ZZ(6)
True
```

Readers who had not been familiar with modular arithmetic until now and who might be discouraged by how complicated modular arithmetic seems at this point should keep two things in mind. First, computing congruences in modular arithmetic is not really more complicated than computations in more familiar number systems (e.g. rational numbers), it is just a matter of getting used to it. Second, once we introduce the idea of remainder class representations in 3.3.4, computations become conceptually cleaner and easier to handle.

Readers who

Exercise 15. Consider the modulus 13 and find all solutions $x\in\mathbb{Z}$ to the following congruence:

$$5x+4 \equiv 28+2x \pmod{13}$$

Exercise 16. Consider the modulus 23 and find all solutions $x\in\mathbb{Z}$ to the following congruence:

$$69x \equiv 5 \pmod{23}$$

Exercise 17. Consider the modulus 23 and find all solutions $x \in \mathbb{Z}$ to the following congruence:

$$69x \equiv 46 \pmod{23}$$

Exercise 18. Let a, b, k be integers, such that $a \equiv b \pmod{n}$ holds. Show $a^k \equiv b^k \pmod{n}$.

Exercise 19. Let a, n be integers, such that a and n are not coprime. For which $b \in \mathbb{Z}$ does the congruence $a \cdot x \equiv b \pmod{n}$ have a solution x and how does the solution set look in that case?

3.3.3 The Chinese Remainder Theorem

We have seen how to solve congruences in modular arithmetic. In this section, we look at how to solve systems of congruences with different moduli using the **Chinese Remainder Theorem**. This states that, for any $k \in \mathbb{N}$ and coprime natural numbers $n_1, \dots n_k \in \mathbb{N}$, as well as integers $a_1, \dots a_k \in \mathbb{Z}$, the so-called **simultaneous congruences** (in 3.18 below) have a solution, and all possible solutions of this congruence system are congruent modulo the product $N = n_1 \cdot \ldots \cdot n_k$.[2]

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2} \\ &\cdots \\ x &\equiv a_k \pmod{n_k} \end{aligned} \tag{3.18}$$

The following algorithm computes the solution set:

check algorithm floating

Algorithm 2 Chinese Remainder Theorem

Require: $, k \in \mathbb{Z}$, $j \in \mathbb{N}_0$ and $n_0, \dots, n_{k-1} \in \mathbb{N}$ coprime

procedure CONGRUENCE-SYSTEMS-SOLVER$(a_0, \dots, a_{k-1})$

$N \leftarrow n_0 \cdot \ldots \cdot n_{k-1}$

while $j < k$ **do**

$N_j \leftarrow N/n_j$

$(\_, s_j, t_j) \leftarrow EXT - EUCLID(N_j, n_j)$ $\triangleright$ $1 = s_j \cdot N_j + t_j \cdot n_j$

end while

$x' \leftarrow \sum_{j=0}^{k-1} a_j \cdot s_j \cdot N_j$

$x \leftarrow x' \bmod N$

return $\{x + m \cdot N \mid m \in \mathbb{Z}\}$

end procedure

Ensure: $\{x + m \cdot N \mid m \in \mathbb{Z}\}$ is the complete solution set to 3.18.

Example 8. To illustrate how to solve simultaneous congruences using the Chinese Remainder Theorem, let's look at the following system of congruences:

$$\begin{aligned} x &\equiv 4 \pmod{7} \\ x &\equiv 1 \pmod{3} \\ x &\equiv 3 \pmod{5} \\ x &\equiv 0 \pmod{11} \end{aligned}$$

[2] This is the classical Chinese Remainder Theorem as it was already known in ancient China. Under certain circumstances, the theorem can be extended to non-coprime moduli $n_1, \dots, n_k$ but this is beyond the scope of this book. Interested readers should consult XXX add references

Clearly, all moduli are coprime (since they are all prime numbers). Now we calculate as follows:

$$
\begin{aligned}
N &= 7\cdot 3\cdot 5\cdot 11 = 1155\\
N_1 &= 1155/7 = 165\\
N_2 &= 1155/3 = 385\\
N_3 &= 1155/5 = 231\\
N_4 &= 1155/11 = 105
\end{aligned}
$$

From this, we calculate with the Extended Euclidean Algorithm:

add more explanation

$$
\begin{aligned}
1 &= \; 2\cdot 165 &+& \; -47\cdot 7\\
1 &= \; 1\cdot 385 &+& \; -128\cdot 3\\
1 &= \; 1\cdot 231 &+& \; -46\cdot 5\\
1 &= \; 2\cdot 105 &+& \; -19\cdot 11
\end{aligned}
$$

As a result, we get $x = 4\cdot 2\cdot 165 + 1\cdot 1\cdot 385 + 3\cdot 1\cdot 231 + 0\cdot 2\cdot 105 = 2398$ as one solution. Because $2398 \bmod 1155 = 88$, the set of all solutions is $\{\ldots,-2222,-1067,88,1243,2398,\ldots\}$. We can use Sage's computation of the Chinese Remainder Theorem (CRT) to double check our findings:

```
sage: CRT_list([4,1,3,0], [7,3,5,11])
88
```

3.3.4 Remainder Class Representation

As we have seen in various examples before, computing congruences can be cumbersome, and solution sets are large in general. It is therefore advantageous to find some kind of simplification for modular arithmetic.

Fortunately, this is possible and relatively straightforward once we identify each set of numbers that have equal remainders with that remainder itself, and call this set the **remainder class** or **residue class** representation in modulo n arithmetic.

It then follows from the properties of Euclidean Division that there are exactly n different remainder classes for every modulus n, and that integer addition and multiplication can be projected to a new kind of addition and multiplication on those classes.

Informally speaking, the new rules for addition and multiplication are then computed by taking any element of the first remainder class and some element of the second remainder class, then add or multiply them in the usual way and see which remainder class the result is contained in. The following example makes this abstract description more concrete.

Example 9 (Arithmetic modulo 6). Choosing the modulus $n = 6$, we have six remainder classes of integers which are congruent modulo 6, that is, they have the same remainder when divided by 6. When we identify each of those remainder classes with the remainder, we get the following identification:

$$
\begin{aligned}
0 &:= \{\ldots,-6,0,6,12,\ldots\}\\
1 &:= \{\ldots,-5,1,7,13,\ldots\}\\
2 &:= \{\ldots,-4,2,8,14,\ldots\}\\
3 &:= \{\ldots,-3,3,9,15,\ldots\}\\
4 &:= \{\ldots,-2,4,10,16,\ldots\}\\
5 &:= \{\ldots,-1,5,11,17,\ldots\}
\end{aligned}
$$

To compute the new addition law of those remainder class representatives, say $2+5$, we choose an arbitrary element from each class, say 14 and -1, adds those numbers in the usual way, and then looks at the remainder class of the result.

Adding 14 and (-1), we get 13, and 13 is in the remainder class (of) 1. Hence, we find that $2+5=1$ in modular 6 arithmetic, which is a more readable way to write the congruence $2+5 \equiv 1 \pmod 6$.

Applying the same reasoning to all remainder classes, addition and multiplication can be transferred to the representatives of the remainder classes. The results for modulus 6 arithmetic are summarized in the following addition and multiplication tables:

$$\begin{array}{c|cccccc} + & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 2 & 3 & 4 & 5 & 0 \\ 2 & 2 & 3 & 4 & 5 & 0 & 1 \\ 3 & 3 & 4 & 5 & 0 & 1 & 2 \\ 4 & 4 & 5 & 0 & 1 & 2 & 3 \\ 5 & 5 & 0 & 1 & 2 & 3 & 4 \end{array} \qquad \begin{array}{c|cccccc} \cdot & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 0 & 2 & 4 & 0 & 2 & 4 \\ 3 & 0 & 3 & 0 & 3 & 0 & 3 \\ 4 & 0 & 4 & 2 & 0 & 4 & 2 \\ 5 & 0 & 5 & 4 & 3 & 2 & 1 \end{array} \tag{3.19}$$

This way, we have defined a new arithmetic system that contains just 6 numbers and comes with its own definition of addition and multiplication. We call it **modular 6 arithmetic** and write the associated type as $\mathbb{Z}_6$. type

To see why identifying a remainder class with its remainder is useful and actually simplifies congruence computations significantly, let's go back to the congruence from example 7:

$$7 \cdot (2x+21)+11 \equiv x-102 \pmod 6 \tag{3.20}$$

As shown in example 7, the arithmetic of congruences can deviate from ordinary arithmetic: for example, division needs to check whether the modulus and the dividend are coprimes, and solutions are not unique in general.

We can rewrite the congruence in (3.20) as an **equation** over our new arithmetic type $\mathbb{Z}_6$ by **projecting onto the remainder classes**: since $7 \bmod 6 = 1$, $21 \bmod 6 = 3$, $11 \bmod 6 = 5$ and $102 \bmod 6 = 0$, we get the following:

$$\begin{aligned} & 7 \cdot (2x+21)+11 \equiv x-102 \pmod 6 \text{ over } \mathbb{Z} \\ & \qquad \Leftrightarrow\ 1 \cdot (2x+3)+5 = x \text{ over } \mathbb{Z}_6 \end{aligned}$$

We can use the multiplication and addition table in (3.19) above to solve the equation on the right like we would solve normal integer equations:

$$\begin{aligned} 1 \cdot (2x+3)+5 &= x \\ 2x+3+5 &= x && \text{\# addition table: } 3+5=2 \\ 2x+2 &= x && \text{\# add 4 and } -x \text{ on both sides} \\ 2x+2+4-x &= x+4-x && \text{\# addition table: } 2+4=0 \\ x &= 4 \end{aligned}$$

As we can see, despite the somewhat unfamiliar rules of addition and multiplication, solving congruences this way is very similar to solving normal equations. And, indeed, the solution set is identical to the solution set of the original congruence, since 4 is identified with the set $\{4+6 \cdot k \mid k \in \mathbb{Z}\}$.

We can use Sage to do computations in our modular 6 arithmetic type. This is particularly useful to double-check our computations:

```
sage: Z6 = Integers(6)
sage: Z6(2) + Z6(5)
1
sage: Z6(7)*(Z6(2)*Z6(4)+Z6(21))+Z6(11) == Z6(4) - Z6(102)
True
```

Remark 2 (k-bit modulus). In cryptographic papers, we sometimes read phrases like "[...] using a 4096-bit modulus". This means that the underlying modulus n of the modular arithmetic used in the system has a binary representation with a length of 4096 bits. In contrast, the number 6 has the binary representation 110 and hence our example 9 describes a 3-bit modulus arithmetic system.

Exercise 20. Define $\mathbb{Z}_{13}$ as the the arithmetic modulo 13 analogously to example 9. Then consider the congruence from exercise 15 and rewrite it into an equation in $\mathbb{Z}_{13}$.

modulo/ modulus/ modular? unify throughout

3.3.5 Modular Inverses

As we know, integers can be added, subtracted and multiplied so that the result is also an integer, but this is not true for the division of integers in general: for example, $3/2$ is not an integer. To see why this is so from a more theoretical perspective, let us consider the definition of a multiplicative inverse first. When we have a set that has some kind of multiplication defined on it, and we have a distinguished element of that set that behaves neutrally with respect to that multiplication (doesn't change anything when multiplied with any other element), then we can define **multiplicative inverses** in the following way:

Definition 3.3.5.1. Let S be our set that has some notion $a\cdot b$ of multiplication and a **neutral element** $1\in S$, such that $1\cdot a = a$ for all elements $a\in S$. Then a **multiplicative inverse** a^{-1} of an element $a\in S$ is defined as follows:

$$a\cdot a^{-1} = 1 \tag{3.21}$$

Informally speaking, the definition of a multiplicative inverse is means that it "cancels" the original element, so that multiplying the two results in 1.

Numbers that have multiplicative inverses are of particular interest, because they immediately lead to the definition of division by those numbers. In fact, if a is number such that the multiplicative inverse a^{-1} exists, then we define **division** by a simply as multiplication by the inverse:

$$\frac{b}{a} := b\cdot a^{-1} \tag{3.22}$$

Example 10. Consider the set of rational numbers, also known as fractions, $\mathbb{Q}$. For this set, the neutral element of multiplication is 1, since $1\cdot a = a$ for all rational numbers. For example, $1\cdot 4 = 4$, $1\cdot\frac{1}{4} = \frac{1}{4}$, or $1\cdot 0 = 0$ and so on.

Every rational number $a\neq 0$ has a multiplicative inverse, given by $\frac{1}{a}$. For example, the multiplicative inverse of 3 is $\frac{1}{3}$, since $3\cdot\frac{1}{3} = 1$, the multiplicative inverse of $\frac{5}{7}$ is $\frac{7}{5}$, since $\frac{5}{7}\cdot\frac{7}{5} = 1$, and so on.

Example 11. Looking at the set $\mathbb{Z}$ of integers, we see that the neutral element of multiplication is the number 1 We can also see that no integer other than 1 or -1 has a multiplicative inverse, since the equation $a\cdot x = 1$ has no integer solutions for $a\neq 1$ or $a\neq -1$.

The definition of multiplicative inverse has a parallel for addition called the **additive inverse**. In the case of integers, the neutral element with respect to addition is 0, since $a+0=0$ for all integers $a \in \mathbb{Z}$. The additive inverse always exists, and is given by the negative number $-a$, since $a+(-a)=0$.

Example 12. Looking at the set $\mathbb{Z}_6$ of residue classes modulo 6 from example 9, we can use the multiplication table in (3.19) to find multiplicative inverses. To do so, we look at the row of the element and find the entry equal to 1. If such an entry exists, the element of that column is the multiplicative inverse. If, on the other hand, the row has no entry equal to 1, we know that the element has no multiplicative inverse.

For example in, $\mathbb{Z}_6$, the multiplicative inverse of 5 is 5 itself, since $5 \cdot 5 = 1$. We can also see that 5 and 1 are the only elements that have multiplicative inverses in $\mathbb{Z}_6$.

Now, since 5 has a multiplicative inverse in modulo 6 arithmetic, we can divide by 5 in $\mathbb{Z}_6$, since we have a notation of multiplicative inverse and division is nothing but multiplication by the multiplicative inverse:

$$\frac{4}{5} = 4 \cdot 5^{-1} = 4 \cdot 5 = 2$$

From the last example, we can make the interesting observation that, while 5 has no multiplicative inverse as an integer, it has a multiplicative inverse in modular 6 arithmetic.

This raises the question of which numbers have multiplicative inverses in modular arithmetic. The answer is that, in modular n arithmetic, a number r has a multiplicative inverse if and only if n and r are coprime. Since $gcd(n,r)=1$ in that case, we know from the Extended Euclidean Algorithm that there are numbers s and t, such that the following equation holds:

$$1 = s \cdot n + t \cdot r \tag{3.23}$$

If we take the modulus n on both sides, the term $s \cdot n$ vanishes, which tells us that $t \bmod n$ is the multiplicative inverse of r in modular n arithmetic.

expand on this

Example 13 (Multiplicative inverses in $\mathbb{Z}_6$). In the previous example, we looked up multiplicative inverses in $\mathbb{Z}_6$ from the lookup table in (3.19). In real-world examples, it is usually impossible to write down those lookup tables, as the modulus is too large, and the sets occasionally contain more elements than there are atoms in the observable universe.

Now, trying to determine that $2 \in \mathbb{Z}_6$ has no multiplicative inverse in $\mathbb{Z}_6$ without using the lookup table, we immediately observe that 2 and 6 are not coprime, since their greatest common divisor is 2. It follows that equation 3.23 has no solutions s and t, which means that 2 has no multiplicative inverse in Z_6.

The same reasoning works for 3 and 4, as neither of these are coprime with 6. The case of 5 is different, since $gcd(6,5)=1$. To compute the multiplicative inverse of 5, we use the Extended Euclidean Algorithm and compute the following:

k	r_k	s_k	$t_k = (r_k - s_k \cdot a)$ div b
0	6	1	0
1	5	0	1
2	1	1	-1
3	0	.	.

We get $s=1$ as well as $t=-1$ and have $1 = 1 \cdot 6 - 1 \cdot 5$. From this, it follows that $-1 \bmod 6 = 5$ is the multiplicative inverse of 5 in modular 6 arithmetic. We can double check using Sage:

```
sage: ZZ(6).xgcd(ZZ(5))
```

67

```
(1, 1, -1)
```

At this point, the attentive reader might notice that the situation where the modulus is a prime number is of particular interest, because we know from exercise 9 that, in these cases, all remainder classes must have modular inverses, since $gcd(r,n) = 1$ for prime n and any $r < n$. In fact, Fermat's Little Theorem (3.16) provides a way to compute multiplicative inverses in this situation, since, in case of a prime modulus p and $r < p$, we get the following:

$$
\begin{aligned}
r^p &\equiv r \pmod p \Leftrightarrow \\
r^{p-1} &\equiv 1 \pmod p \Leftrightarrow \\
r \cdot r^{p-2} &\equiv 1 \pmod p
\end{aligned}
$$

This tells us that the multiplicative inverse of a residue class r in modular p arithmetic is precisely r^{p-2}.

Example 14 (Modular 5 arithmetic). To see the unique properties of modular arithmetic when the modulus is a prime number, we will replicate our findings from example 9, but this time for the prime modulus 5. For $p = 5$ we have five equivalence classes of integers which are congruent modulo 5. We write this as follows:

$$
\begin{aligned}
0 &:= \{\ldots, -5, 0, 5, 10, \ldots\} \\
1 &:= \{\ldots, -4, 1, 6, 11, \ldots\} \\
2 &:= \{\ldots, -3, 2, 7, 12, \ldots\} \\
3 &:= \{\ldots, -2, 3, 8, 13, \ldots\} \\
4 &:= \{\ldots, -1, 4, 9, 14, \ldots\}
\end{aligned}
$$

Addition and multiplication can be transferred to the equivalence classes, in a way exactly parallel to example 9. This results in the following addition and multiplication tables:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

(3.24)

Calling the set of remainder classes in modular 5 arithmetic with this addition and multiplication $\mathbb{Z}_5$, we see some subtle but important differences to the situation in $\mathbb{Z}_6$. In particular, we see that in the multiplication table, every remainder $r \neq 0$ has the entry 1 in its row and therefore has a multiplicative inverse. In addition, there are no non-zero elements such that their product is zero.

To use Fermat's Little Theorem in $\mathbb{Z}_5$ for computing multiplicative inverses (instead of using the multiplication table), let's consider $3 \in \mathbb{Z}_5$. We know that the multiplicative inverse is given by the remainder class that contains $3^{5-2} = 3^3 = 3 \cdot 3 \cdot 3 = 4 \cdot 3 = 2$. And indeed $3^{-1} = 2$, since $3 \cdot 2 = 1$ in $\mathbb{Z}_5$.

We can use Sage to do computations in our modular 5 arithmetic type to double-check our computations:

```
sage: Z5 = Integers(5)
sage: Z5(3)**(5-2)
```

```
2
sage: Z5(3)**(-1)
2
sage: Z5(3)**(5-2) == Z5(3)**(-1)
True
```

Example 15. To understand one of the principal differences between prime number modular arithmetic and non-prime number modular arithmetic, consider the linear equation $a\cdot x+b=0$ defined over both types $\mathbb{Z}_5$ and $\mathbb{Z}_6$. Since every non-zero element has a multiplicative inverse in $\mathbb{Z}_5$, we can always solve these equations in $\mathbb{Z}_5$, which is not true in $\mathbb{Z}_6$. To see that, consider the equation $3x+3=0$. In $\mathbb{Z}_5$ we have the following:

$$\begin{aligned}
3x+3&=0 && \text{\# add 2 and on both sides}\\
3x+3+2&=2 && \text{\# addition-table: } 2+3=0\\
3x&=2 && \text{\# divide by 3 (which equals multiplication by 2)}\\
2\cdot(3x)&=2\cdot 2 && \text{\# multiplication-table: } 2\cdot 2=4\\
x&=4
\end{aligned}$$

So in the case of our prime number modular arithmetic, we get the unique solution $x=4$. Now consider $\mathbb{Z}_6$:

$$\begin{aligned}
3x+3&=0 && \text{\# add 3 and on both sides}\\
3x+3+3&=3 && \text{\# addition-table: } 3+3=0\\
3x&=3 && \text{\# division not possible (no multiplicative inverse of 3 exists)}
\end{aligned}$$

So, in this case, we cannot solve the equation for x by dividing by 3. And, indeed, when we look at the multiplication table of $\mathbb{Z}_6$ (example 9), we find that there are three solutions $x\in\{1,3,5\}$, such that $3x+3=0$ holds true for all of them.

Exercise 21. Consider the modulus $n=24$. Which of the integers 7, 1, 0, 805, -4255 have multiplicative inverses in modular 24 arithmetic? Compute the inverses, in case they exist.

Exercise 22. Find the set of all solutions to the congruence $17(2x+5)-4\equiv 2x+4 \pmod 5$. Then project the congruence into $\mathbb{Z}_5$ and solve the resulting equation in $\mathbb{Z}_5$. Compare the results.

Exercise 23. Find the set of all solutions to the congruence $17(2x+5)-4\equiv 2x+4 \pmod 6$. Then project the congruence into $\mathbb{Z}_6$ and try to solve the resulting equation in $\mathbb{Z}_6$.

3.4 Polynomial arithmetic

A polynomial is an expression consisting of variables (also-called indeterminates) and coefficients that involves only the operations of addition, subtraction and multiplication. All coefficients of a polynomial must have the same type, e.g. they must all be integers or they must all be rational numbers, etc.

To be more precise, an **univariate**[3] **polynomial** is an expression as shown below:

$$P(x):=\sum_{j=0}^{m}a_jx^j=a_mx^m+a_{m-1}x^{m-1}+\cdots+a_1x+a_0\ , \tag{3.25}$$

[3] In our context, the term univariate means that the polynomial contains a single variable only.

In (3.25) x is called the **variable**, and each a is called a **coefficient**. If R is the type of the coefficients, then the set of all **univariate polynomials with coefficients in** R is written as $R[x]$. Univariate polynomials are often simply called polynomials, and written as $P(x) \in R[x]$. The constant term a_0 as is also written as $P(0)$.

A polynomial is called the **zero polynomial** if all its coefficients are zero. A polynomial is called the **one polynomial** if the constant term is 1 and all other coefficients are zero.

Given a univariate polynomial $P(x) = \sum_{j=0}^{m} a_j x^j$ that is not the zero polynomial, we call the non-negative integer $deg(P) := m$ the *degree* of P, and define the degree of the zero polynomial to be $-\infty$, where $-\infty$ (negative infinity) is a symbol with the properties that $-\infty + m = -\infty$ and $-\infty < m$ for all non-negative integers $m \in \mathbb{N}_0$.

In addition, we denote the coefficient of the term with the highest degree, called **leading coefficient**, of the polynomial P as follows:

$$Lc(P) := a_m \tag{3.26}$$

add explanation on why this is important

We can restrict the set $R[x]$ of **all** polynomials with coefficients in R to the set of all such polynomials that have a degree that does not exceed a certain value. If m is the maximum degree allowed, we write $R_{\leq m}[x]$ for the set of all polynomials with a degree less than or equal to m.

Example 16 (Integer Polynomials). The coefficients of a polynomial must all have the same type. The set of polynomials with integer coefficients is written as $\mathbb{Z}[x]$. Some examples of such polynomials are listed below:

$$\begin{aligned}
P_1(x) &= 2x^2 - 4x + 17 && \# \text{ with } deg(P_1) = 2 \text{ and } Lc(P_1) = 2\\
P_2(x) &= x^{23} && \# \text{ with } deg(P_2) = 23 \text{ and } Lc(P_2) = 1\\
P_3(x) &= x && \# \text{ with } deg(P_3) = 1 \text{ and } Lc(P_3) = 1\\
P_4(x) &= 174 && \# \text{ with } deg(P_4) = 0 \text{ and } Lc(P_4) = 174\\
P_5(x) &= 1 && \# \text{ with } deg(P_5) = 0 \text{ and } Lc(P_5) = 1\\
P_6(x) &= 0 && \# \text{ with } deg(P_6) = -\infty \text{ and } Lc(P_6) = 0\\
P_7(x) &= (x-2)(x+3)(x-5)
\end{aligned}$$

Every integer can be seen as an integer polynomial of degree zero. P_7 is a polynomial, because we can expand its definition into $P_7(x) = x^3 - 4x^2 - 11x + 30$, which is a polynomial of degree 3 and leading coefficient 1.

The following expressions are not integer polynomials:

$$\begin{aligned}
Q_1(x) &= 2x^2 + 4 + 3x^{-2}\\
Q_2(x) &= 0.5x^4 - 2x\\
Q_3(x) &= 2^x
\end{aligned}$$

Q_1 is not an integer polynomial because the expression x^{-2} has a negative exponent. Q_2 is not an integer polynomial because the coefficient 0.5 is not an integer. Q_3 is not an integer polynomial because the variable appears in the exponent of a coefficient.

We can use Sage to do computations with polynomials. To do so, we have to specify the symbol for the variable and the type for the coefficients. (For the definition of rings see 4.2.) Note, however, that Sage defines the degree of the zero polynomial to be -1.

why is this ref here?

```
sage: Zx = ZZ['x'] # integer polynomials with variable x
sage: Zt.<t> = ZZ[] # integer polynomials with variable t
```

76

77 what does this imply?

```
sage: Zx
Univariate Polynomial Ring in x over Integer Ring
sage: Zt
Univariate Polynomial Ring in t over Integer Ring
sage: p1 = Zx([17,-4,2])
sage: p1
2*x^2 - 4*x + 17
sage: p1.degree()
2
sage: p1.leading_coefficient()
2
sage: p2 = Zt(t^23)
sage: p2
t^23
sage: p6 = Zx([0])
sage: p6.degree()
-1
```

Example 17 (Polynomials over $\mathbb{Z}_6$). Recall the definition of modular 6 arithmetics $\mathbb{Z}_6$ from example 9. The set of all polynomials with variable x and coefficients in $\mathbb{Z}_6$ is symbolized as $\mathbb{Z}_6[x]$. Some examples of polynomials from $\mathbb{Z}_6[x]$ are given below:

$$
\begin{aligned}
P_1(x) &= 2x^2-4x+5 && \text{\# with } deg(P_1)=2 \text{ and } Lc(P_1)=2\\
P_2(x) &= x^{23} && \text{\# with } deg(P_2)=23 \text{ and } Lc(P_2)=1\\
P_3(x) &= x && \text{\# with } deg(P_3)=1 \text{ and } Lc(P_3)=1\\
P_4(x) &= 3 && \text{\# with } deg(P_4)=0 \text{ and } Lc(P_4)=3\\
P_5(x) &= 1 && \text{\# with } deg(P_5)=0 \text{ and } Lc(P_5)=1\\
P_6(x) &= 0 && \text{\# with } deg(P_5)=-\infty \text{ and } Lc(P_6)=0\\
P_7(x) &= (x-2)(x+3)(x-5)
\end{aligned}
$$

Just like in the previous example, P_7 is a polynomial. However, since we are working with coefficients from $\mathbb{Z}_6$ now, the expansion of P_7 is computed differently, as we have to use addition and multiplication in $\mathbb{Z}_6$ as defined in (3.19). We get the following:

check reference

$$
\begin{aligned}
(x-2)(x+3)(x-5) &= (x+4)(x+3)(x+1) && \text{\# additive inverses in } \mathbb{Z}_6\\
&= (x^2+4x+3x+3\cdot 4)(x+1) && \text{\# bracket expansion}\\
&= (x^2+1x+0)(x+1) && \text{\# computation in } \mathbb{Z}_6\\
&= x^3+x^2+x^2+x && \text{\# bracket expansion}\\
&= x^3+2x^2+x
\end{aligned}
$$

Again, we can use Sage to do computations with polynomials that have their coefficients in $\mathbb{Z}_6$. (For the definition of rings see 4.2.) To do so, we have to specify the symbol for the variable and the type for the coefficients:

why is this ref here?

```
sage: Z6 = Integers(6)
sage: Z6x = Z6['x']
sage: Z6x
```

```
Univariate Polynomial Ring in x over Ring of integers modulo 6
sage: p1 = Z6x([5,-4,2])
sage: p1
2*x^2 + 2*x + 5
sage: p1 = Z6x([17,-4,2])
sage: p1
2*x^2 + 2*x + 5
sage: Z6x(x-2)*Z6x(x+3)*Z6x(x-5) == Z6x(x^3 + 2*x^2 + x)
True
```

Given some element from the same type as the coefficients of a polynomial, the polynomial can be evaluated at that element, which means that we insert the given element for every occurrence of the variable x in the polynomial expression.

To be more precise, let $P \in R[x]$, with $P(x) = \sum_{j=0}^{m} a_j x^j$ be a polynomial with a coefficient of type R and let $b \in R$ be an element of that type. Then the **evaluation** of P at b is given as follows:

$$P(b) = \sum_{j=0}^{m} a_j b^j \tag{3.27}$$

Example 18. Consider the integer polynomials from example 16 again. To evaluate them at given points, we have to insert the point for all occurences of x in the polynomial expression. Inserting arbitrary values from $\mathbb{Z}$, we get the following:

is this right?

$$\begin{aligned}
P_1(2) &= 2 \cdot 2^2 - 4 \cdot 2 + 17 = 17 \\
P_2(3) &= 3^{23} = 94143178827 \\
P_3(-4) &= -4 = -4 \\
P_4(15) &= 174 \\
P_5(0) &= 1 \\
P_6(1274) &= 0 \\
P_7(-6) &= (-6-2)(-6+3)(-6-5) = -264
\end{aligned}$$

Note, however, that it is not possible to evaluate any of those polynomial on values of different type. For example, it is not strictly correct to write $P_1(0.5)$, since 0.5 is not an integer. We can verify our computations using Sage:

```
sage: Zx = ZZ['x']
sage: p1 = Zx([17,-4,2])
sage: p7 = Zx(x-2)*Zx(x+3)*Zx(x-5)
sage: p1(ZZ(2))
17
sage: p7(ZZ(-6)) == ZZ(-264)
True
```

Example 19. Consider the polynomials with coefficients in $\mathbb{Z}_6$ from example 17 again. To evaluate them at given values from $\mathbb{Z}_6$, we have to insert the point for all occurrences of x in the

check reference

polynomial expression. We get the following:

$$
\begin{aligned}
P_1(2) &= 2\cdot 2^2 - 4\cdot 2 + 5 = 2 - 2 + 5 = 5\\
P_2(3) &= 3^{23} = 3\\
P_3(-4) &= P_3(2) = 2\\
P_5(0) &= 1\\
P_6(4) &= 0
\end{aligned}
$$

```
sage: Z6 = Integers(6)
sage: Z6x = Z6['x']
sage: p1 = Z6x([5,-4,2])
sage: p1(Z6(2)) == Z6(5)
True
```

Exercise 24. Compare both expansions of P_7 from $\mathbb{Z}[x]$ in example 16 and from from $\mathbb{Z}_6[x]$ in example 17 , and consider the definition of $\mathbb{Z}_6$ as given in example 9. Can you see how the definition of P_7 over $\mathbb{Z}$ projects to the definition over $\mathbb{Z}_6$ if you consider the residue classes of $\mathbb{Z}_6$?

the task could be defined more clearly

3.4.1 Polynomial arithmetic

Polynomials behave like integers in many ways. In particular, they can be added, subtracted and multiplied. In addition, they have their own notion of Euclidean Division. Informally speaking, we can add two polynomials by simply adding the coefficients of the same index, and we can multiply them by applying the distributive property, that is, by multiplying every term of the left factor with every term of the right factor and adding the results together.

To be more precise, let $\sum_{n=0}^{m_1} a_n x^n$ and $\sum_{n=0}^{m_2} b_n x^n$ be two polynomials from $R[x]$. Then the **sum** and the **product** of these polynomials is defined as follows:

$$\sum_{n=0}^{m_1} a_n x^n + \sum_{n=0}^{m_2} b_n x^n = \sum_{n=0}^{max(\{m_1,m_2\})} (a_n + b_n)x^n \tag{3.28}$$

$$\left(\sum_{n=0}^{m_1} a_n x^n\right)\cdot\left(\sum_{n=0}^{m_2} b_n x^n\right) = \sum_{n=0}^{m_1+m_2}\sum_{i=0}^{n} a_i b_{n-i} x^n \tag{3.29}$$

A rule for polynomial subtraction can be deduced from these two rules by first multiplying the subtrahend with (the polynomial) -1 and then add the result to the minuend.

subtrahend

minuend

Regarding the definition of the degree of a polynomial, we see that the degree of the sum is always the maximum of the degrees of both summands, and the degree of the product is always the degree of the sum of the factors, since we defined $-\infty + m = -\infty$ for every integer $m \in \mathbb{Z}$.

Example 20. To give an example of how polynomial arithmetic works, consider the following two integer polynomials $P, Q \in \mathbb{Z}[x]$ with $P(x) = 5x^2 - 4x + 2$ and $Q(x) = x^3 - 2x^2 + 5$. The sum of these two polynomials is computed by adding the coefficients of each term with equal exponent in x. This gives the following:

$$
\begin{aligned}
(P+Q)(x) &= (0+1)x^3 + (5-2)x^2 + (-4+0)x + (2+5)\\
&= x^3 + 3x^2 - 4x + 7
\end{aligned}
$$

The product of these two polynomials is computed by multiplying each term in the first factor with each term in the second factor. We get the following:

$$\begin{aligned}(P\cdot Q)(x) &= (5x^2-4x+2)\cdot(x^3-2x^2+5)\\ &= (5x^5-10x^4+25x^2)+(-4x^4+8x^3-20x)+(2x^3-4x^2+10)\\ &= 5x^5-14x^4+10x^3+21x^2-20x+10\end{aligned}$$

```
sage: Zx = ZZ['x']
sage: P = Zx([2,-4,5])
sage: Q = Zx([5,0,-2,1])
sage: P+Q == Zx(x^3 +3*x^2 -4*x +7)
True
sage: P*Q == Zx(5*x^5 -14*x^4 +10*x^3+21*x^2-20*x +10)
True
```

Example 21. Let us consider the polynomials of the previous example 20, but interpreted in modular 6 arithmetic. So we consider $P,Q\in\mathbb{Z}_6[x]$ again with $P(x)=5x^2-4x+2$ and $Q(x)=x^3-2x^2+5$. This time we get the following:

$$\begin{aligned}(P+Q)(x) &= (0+1)x^3+(5-2)x^2+(-4+0)x+(2+5)\\ &= (0+1)x^3+(5+4)x^2+(2+0)x+(2+5)\\ &= x^3+3x^2+2x+1\end{aligned}$$

$$\begin{aligned}(P\cdot Q)(x) &= (5x^2-4x+2)\cdot(x^3-2x^2+5)\\ &= (5x^2+2x+2)\cdot(x^3+4x^2+5)\\ &= (5x^5+2x^4+1x^2)+(2x^4+2x^3+4x)+(2x^3+2x^2+4)\\ &= 5x^5+4x^4+4x^3+3x^2+4x+4\end{aligned}$$

```
sage: Z6x = Integers(6)['x']
sage: P = Z6x([2,-4,5])
sage: Q = Z6x([5,0,-2,1])
sage: P+Q == Z6x(x^3 +3*x^2 +2*x +1)
True
sage: P*Q == Z6x(5*x^5 +4*x^4 +4*x^3+3*x^2+4*x +4)
True
```

Exercise 25. Compare the sum $P+Q$ and the product $P\cdot Q$ from the previous two examples 20 and 21, and consider the definition of $\mathbb{Z}_6$ as given in example 9. How can we derive the computations in $\mathbb{Z}_6[x]$ from the computations in $Z[x]$?

3.4.2 Euclidean Division with polynomials

The arithmetic of polynomials shares a lot of properties with the arithmetic of integers. As a consequence, the concept of Euclidean Division and the algorithm of long division is also defined for polynomials. Recalling the Euclidean Division of integers 3.2.2, we know that, given two integers a and $b \neq 0$, there is always another integer m and a natural number r with $r < |b|$ such that $a = m \cdot b + r$ holds.

We can generalize this to polynomials whenever the leading coefficient of the dividend polynomial has a notion of multiplicative inverse. In fact, given two polynomials A and $B \neq 0$ from $R[x]$ such that $Lc(B)^{-1}$ exists in R, there exist two polynomials Q (the quotient) and P (the remainder), such that the following equation holds and $deg(P) < deg(B)$:

$$A = Q \cdot B + P \tag{3.30}$$

Similarly to integer Euclidean Division, both Q and P are uniquely defined by these relations.

Notation and Symbols 2. Suppose that the polynomials A, B, Q and P satisfy equation 3.30. We often use the following notation to describe the quotient and the remainder polynomials of the Euclidean Division:

$$A \text{ div } B := Q, \qquad A \bmod B := P \tag{3.31}$$

We also say that a polynomial A is divisible by another polynomial B if $A \bmod B = 0$ holds. In this case, we also write $B|A$ and call B a *factor* of A.

Analogously to integers, methods to compute Euclidean Division for polynomials are called **polynomial division algorithms**. Probably the best known algorithm is the so-called **polynomial long division** .

algorithm-floating

Algorithm 3 Polynomial Euclidean Algorithm

Require: $A, B \in R[x]$ with $B \neq 0$, such that $Lc(B)^{-1}$ exists in R
  procedure POLY-LONG-DIVISION(A, B)
    $Q \leftarrow 0$
    $P \leftarrow A$
    $d \leftarrow deg(B)$
    $c \leftarrow Lc(B)$
    while $deg(P) \geq d$ **do**
      $S := Lc(P) \cdot c^{-1} \cdot x^{deg(P)-d}$
      $Q \leftarrow Q + S$
      $P \leftarrow P - S \cdot B$
    end while
    return (Q, P)
  end procedure
Ensure: $A = Q \cdot B + P$

This algorithm works only when there is a notion of division by the leading coefficient of B. It can be generalized, but we will only need this somewhat simpler method in what follows.

Example 22 (Polynomial Long Division). To give an example of how the previous algorithm works, let us divide the integer polynomial $A(x) = x^5 + 2x^3 - 9 \in \mathbb{Z}[x]$ by the integer polynomial $B(x) = x^2 + 4x - 1 \in \mathbb{Z}[x]$. Since B is not the zero polynomial, and the leading coefficient of B is 1, which is invertible as an integer, we can apply algorithm 1. Our goal is to find solutions to equation XXX, that is, we need to find the quotient polynomial $Q \in \mathbb{Z}[x]$ and the remainder polynomial $P \in \mathbb{Z}[x]$ such that $x^5 + 2x^3 - 9 = Q(x) \cdot (x^2 + 4x - 1) + P(x)$. Using a the long

add reference

division notation that is mostly used in anglophone countries, we compute as follows:

$$
\begin{array}{rllllll}
 & X^3 & -4X^2 & +19X & -80 & & \\
\cline{2-6}
X^2+4X-1 \big) & X^5 & & +2X^3 & & & -9 \\
 & -X^5 & -4X^4 & +X^3 & & & \\
\cline{2-4}
 & & -4X^4 & +3X^3 & & & \\
 & & 4X^4 & +16X^3 & -4X^2 & & \\
\cline{3-5}
 & & & 19X^3 & -4X^2 & & \\
 & & & -19X^3 & -76X^2 & +19X & \\
\cline{4-6}
 & & & & -80X^2 & +19X & -9 \\
 & & & & 80X^2 & +320X & -80 \\
\cline{5-7}
 & & & & & 339X & -89
\end{array}
\tag{3.32}
$$

We therefore get $Q(x) = x^3 - 4x^2 + 19x - 80$ and $P(x) = 339x - 89$, and indeed, the equation $A = Q \cdot B + P$ is true with these values, since $x^5 + 2x^3 - 9 = (x^3 - 4x^2 + 19x - 80) \cdot (x^2 + 4x - 1) + (339x - 89)$. We can double check this invoking Sage:

```
sage: Zx = ZZ['x']
sage: A = Zx([-9,0,0,2,0,1])
sage: B = Zx([-1,4,1])
sage: Q = Zx([-80,19,-4,1])
sage: P = Zx([-89,339])
sage: A == Q*B + P
True
```

Example 23. In the previous example, polynomial division gave a non-trivial (non-vanishing, i.e non-zero) remainder. Divisions that don't give a remainder are of special interest. In these cases, divisors are called **factors of the dividend**.

For example, consider the integer polynomial P_7 from example 16 again. As we have shown, it can be written both as $x^3 - 4x^2 - 11x + 30$ and as $(x-2)(x+3)(x-5)$. From this, we can see that the polynomials $F_1(x) = (x-2)$, $F_2(x) = (x+3)$ and $F_3(x) = (x-5)$ are all factors of $x^3 - 4x^2 - 11x + 30$, since division of P_7 by any of these factors will result in a zero remainder.

Exercise 26. Consider the polynomial expressions $A(x) := -3x^4 + 4x^3 + 2x^2 + 4$ and $B(x) = x^2 - 4x + 2$. Compute the Euclidean Division of A by B in the following types:

1. $A, B \in \mathbb{Z}[x]$
2. $A, B \in \mathbb{Z}_6[x]$
3. $A, B \in \mathbb{Z}_5[x]$

Now consider the result in $\mathbb{Z}[x]$ and in $\mathbb{Z}_6[x]$. How can we compute the result in $\mathbb{Z}_6[x]$ from the result in $\mathbb{Z}[x]$?

Exercise 27. Show that the polynomial $B(x) = 2x^4 - 3x + 4 \in \mathbb{Z}_5[x]$ is a factor of the polynomial $A(x) = x^7 + 4x^6 + 4x^5 + x^3 + 2x^2 + 2x + 3 \in \mathbb{Z}_5[x]$, that is, show that $B|A$. What is B div A?

3.4.3 Prime Factors

Recall that the fundamental theorem of arithmetic 3.7 tells us that every natural number is the product of prime numbers. In this chapter, we will see that something similar holds for univariate polynomials $R[x]$, too.[4]

The polynomial analog to a prime number is a so-called **irreducible polynomial**, which is defined as a polynomial that cannot be factored into the product of two non-constant polynomials using Euclidean Division. Irreducible polynomials are to polynomials what prime numbers are to integers: they are the basic building blocks from which all other polynomials can be constructed.

To be more precise, let $P \in R[x]$ be any polynomial. Then there always exist irreducible polynomials $F_1, F_2, \ldots, F_k \in R[x]$, such that the following holds:

$$P = F_1 \cdot F_2 \cdot \ldots \cdot F_k \,. \tag{3.33}$$

This representation is unique (except for permutations in the factors) and is called the **prime factorization** of P. Moreover, each factor F_i is called a **prime factor** of P.

Example 24. Consider the polynomial expression $P = x^2 - 3$. When we interpret P as an integer polynomial $P \in \mathbb{Z}[x]$, we find that this polynomial is irreducible, since any factorization other then $1 \cdot (x^2 - 3)$, must look like $(x-a)(x+a)$ for some integer a, but there is no integers a with $a^2 = 3$.

```
sage: Zx = ZZ['x']
sage: p = Zx(x^2-3)
sage: p.factor()
x^2 - 3
```

On the other hand, interpreting P as a polynomial $P \in \mathbb{Z}_6[x]$ in modulo 6 arithmetic, we see that P has two factors $F_1 = (x-3)$ and $F_2 = (x+3)$, since $(x-3)(x+3) = x^2 - 3x + 3x - 3 \cdot 3 = x^2 - 3$.

Points where a polynomial evaluates to zero are called **roots** of the polynomial. To be more precise, let $P \in R[x]$ be a polynomial. Then a root is a point $x_0 \in R$ with $P(x_0) = 0$ and the set of all roots of P is defined as follows:

$$R_0(P) := \{x_0 \in R \mid P(x_0) = 0\} \tag{3.34}$$

The roots of a polynomial are of special interest with respect to its prime factorization, since it can be shown that, for any given root x_0 of P, the polynomial $F(x) = (x - x_0)$ is a prime factor of P.

Finding the roots of a polynomial is sometimes called **solving the polynomial**. It is a difficult problem that has been the subject of much research throughout history.

It can be shown that if m is the degree of a polynomial P, then P cannot have more than m roots. However, in general, polynomials can have less than m roots.

Example 25. Consider the integer polynomial $P_7(x) = x^3 - 4x^2 - 11x + 30$ from example 16 again. We know that its set of roots is given by $R_0(P_7) = \{-3, 2, 5\}$.

On the other hand, we know from example 24 that the integer polynomial $x^2 - 3$ is irreducible. It follows that it has no roots, since every root defines a prime factor.

[4]Strictly speaking, this is not true for polynomials over arbitrary types R. However, in this book, we assume R to be a so-called unique factorization domain for which the content of this section holds.

Example 26. To give another example, consider the integer polynomial $P = x^7 + 3x^6 + 3x^5 + x^4 - x^3 - 3x^2 - 3x - 1$. We can use Sage to compute the roots and prime factors of P:

```
sage: Zx = ZZ['x']
sage: p = Zx(x^7 + 3*x^6 + 3*x^5 + x^4 - x^3 - 3*x^2 - 3*x - 1
   )
sage: p.roots()
[(1, 1), (-1, 4)]
sage: p.factor()
(x - 1) * (x + 1)^4 * (x^2 + 1)
```

We see that P has the root 1, and that the associated prime factor $(x-1)$ occurs once in P. We can also see that P has the root -1, where the associated prime factor $(x+1)$ occurs 4 times in P. This gives the following prime factorization:

$$P = (x-1)(x+1)^4(x^2+1)$$

Exercise 28. Show that if a polynomial $P \in R[x]$ of degree $deg(P) = m$ has less than m roots, it must have a prime factor F of degree $deg(F) > 1$.

Exercise 29. Consider the polynomial $P = x^7 + 3x^6 + 3x^5 + x^4 - x^3 - 3x^2 - 3x - 1 \in \mathbb{Z}_6[x]$. Compute the set of all roots of $R_0(P)$ and then compute the prime factorization of P.

3.4.4 Lagrange Interpolation

One particularly useful property of polynomials is that a polynomial of degree m is completely determined on $m+1$ evaluation points, which implies that we can uniquely derive a polynomial of degree m from a set S:

$$S = \{(x_0, y_0), (x_1, y_1), \ldots, (x_m, y_m) \mid x_i \neq x_j \text{ for all indices i and j}\} \tag{3.35}$$

Polynomials therefore have the property that $m+1$ pairs of points (x_i, y_i) for $x_i \neq x_j$ are enough to determine the set of pairs $(x, P(x))$ for all $x \in R$. This "few too many" property of polynomials is widely used, including in SNARKs. Therefore, we need to understand the method to actually compute a polynomial from a set of points.

If the coefficients of the polynomial we want to find have a notion of multiplicative inverse, it is always possible to find such a polynomial using a method called **Lagrange Interpolation**, which works as follows. Given a set like 3.35, a polynomial P of degree m with $P(x_i) = y_i$ for all pairs (x_i, y_i) from S is given by the following algorithm:

check algorithm floating

Example 27. Let us consider the set $S = \{(0,4), (-2,1), (2,3)\}$. Our task is to compute a polynomial of degree 2 in $\mathbb{Q}[x]$ with coefficients from the set of rational numbers $\mathbb{Q}$. Since $\mathbb{Q}$ has multiplicative inverses, we can use method of Lagrange Interpolation from Algorithm 4 to

Algorithm 4 Lagrange Interpolation

Require: R must have multiplicative inverses

Require: $S = \{(x_0,y_0),(x_1,y_1),\ldots,(x_m,y_m) \mid x_i,y_i \in R, x_i \neq x_j$ for all indices i and j$\}$

  procedure LAGRANGE-INTERPOLATION(S)

    for $j \in (0\ldots m)$ **do**

      $l_j(x) \leftarrow \Pi^m_{i=0;i\neq j}\frac{x-x_i}{x_j-x_i} = \frac{(x-x_0)}{(x_j-x_0)}\cdots\frac{(x-x_{j-1})}{(x_j-x_{j-1})}\frac{(x-x_{j+1})}{(x_j-x_{j+1})}\cdots\frac{(x-x_m)}{(x_j-x_m)}$

    end for

    $P \leftarrow \sum^m_{j=0} y_j \cdot l_j$

    return P

  end procedure

Ensure: $P \in R[x]$ with $deg(P) = m$

Ensure: $P(x_j) = y_j$ for all pairs $(x_j,y_j) \in S$

compute the polynomial:

$$\begin{aligned}
l_0(x) &= \frac{x-x_1}{x_0-x_1}\cdot\frac{x-x_2}{x_0-x_2} = \frac{x+2}{0+2}\cdot\frac{x-2}{0-2} = -\frac{(x+2)(x-2)}{4}\\
&= -\frac{1}{4}(x^2-4)\\
l_1(x) &= \frac{x-x_0}{x_1-x_0}\cdot\frac{x-x_2}{x_1-x_2} = \frac{x-0}{-2-0}\cdot\frac{x-2}{-2-2} = \frac{x(x-2)}{8}\\
&= \frac{1}{8}(x^2-2x)\\
l_2(x) &= \frac{x-x_0}{x_2-x_0}\cdot\frac{x-x_1}{x_2-x_1} = \frac{x-0}{2-0}\cdot\frac{x+2}{2+2} = \frac{x(x+2)}{8}\\
&= \frac{1}{8}(x^2+2x)\\
P(x) &= 4\cdot(-\frac{1}{4}(x^2-4)) + 1\cdot\frac{1}{8}(x^2-2x) + 3\cdot\frac{1}{8}(x^2+2x)\\
&= -x^2+4+\frac{1}{8}x^2-\frac{1}{4}x+\frac{3}{8}x^2+\frac{3}{4}x\\
&= -\frac{1}{2}x^2+\frac{1}{2}x+4
\end{aligned}$$

And, indeed, evaluation of P on the x-values of S gives the correct points, since $P(0) = 4$, $P(-2) = 1$ and $P(2) = 3$. Sage confirms this result:

```
sage: Qx = QQ['x']
sage: S=[(0,4),(-2,1),(2,3)]
sage: Qx.lagrange_polynomial(S)
-1/2*x^2 + 1/2*x + 4
```

Example 28. To give another example more relevant to the topics of this book, let us consider the same set as in the previous example, $S = \{(0,4),(-2,1),(2,3)\}$. This time, the task is to compute a polynomial $P \in \mathbb{Z}_5[x]$ from this data. Since we know from example 14 that multiplicative inverses exist in $\mathbb{Z}_5$, algorithm 4 is applicable and we can compute a unique polynomial of degree 2 in $\mathbb{Z}_5[x]$ from S. We can use the lookup tables from (3.24) for computations in $\mathbb{Z}_5$

and get the following:

$$
\begin{aligned}
l_0(x) &= \frac{x-x_1}{x_0-x_1}\cdot\frac{x-x_2}{x_0-x_2} = \frac{x+2}{0+2}\cdot\frac{x-2}{0-2} = \frac{(x+2)(x-2)}{-4} = \frac{(x+2)(x+3)}{1} \\
&= x^2+1 \\
l_1(x) &= \frac{x-x_0}{x_1-x_0}\cdot\frac{x-x_2}{x_1-x_2} = \frac{x-0}{-2-0}\cdot\frac{x-2}{-2-2} = \frac{x}{3}\cdot\frac{x+3}{1} = 2(x^2+3x) \\
&= 2x^2+x \\
l_2(x) &= \frac{x-x_0}{x_2-x_0}\cdot\frac{x-x_1}{x_2-x_1} = \frac{x-0}{2-0}\cdot\frac{x+2}{2+2} = \frac{x(x+2)}{3} = 2(x^2+2x) \\
&= 2x^2+4x \\
P(x) &= 4\cdot(x^2+1)+1\cdot(2x^2+x)+3\cdot(2x^2+4x) \\
&= 4x^2+4+2x^2+x+x^2+2x \\
&= 2x^2+3x+4
\end{aligned}
$$

And, indeed, evaluation of P on the x-values of S gives the correct points, since $P(0)=4$, $P(-2)=1$ and $P(2)=3$. We can double check our findings using Sage:

```
sage: F5 = GF(5)
sage: F5x = F5['x']
sage: S=[(0,4),(-2,1),(2,3)]
sage: F5x.lagrange_polynomial(S)
2*x^2 + 3*x + 4
```

Exercise 30. Consider modular 5 arithmetic from example 14, and the set $S=\{(0,0),(1,1),(2,2),(3,2)\}$. Find a polynomial $P\in\mathbb{Z}_5[x]$ such that $P(x_i)=y_i$ for all $(x_i,y_i)\in S$.

Exercise 31. Consider the set S from the previous example. Why is it not possible to apply algorithm 4 to construct a polynomial $P\in\mathbb{Z}_6[x]$ such that $P(x_i)=y_i$ for all $(x_i,y_i)\in S$?

Chapter 4

Algebra

In the previous chapter, we gave an introduction to the basic computational tools needed for a pen-and-paper approach to SNARKs. In this chapter, we provide a more abstract clarification of relevant mathematical terminology such as **groups**, **rings** and **fields**.

Scientific literature on cryptography frequently contains such terms, and it is necessary to get at least some understanding of these terms to be able to follow the literature.

4.1 Commutative Groups

Commutative groups are abstractions that capture the essence of mathematical phenomena, like addition and subtraction, or multiplication and division.

To understand commutative groups, let us think back to when we learned about the addition and subtraction of integers in school. We have learned that, whenever we add two integers, the result is guaranteed to be an integer as well. We have also learned that adding zero to any integer means that "nothing happens" since the result of the addition is the same integer we started with. Furthermore, we have learned that the order in which we add two (or more) integers does not matter, that brackets have no influence on the result of addition, and that, for every integer, there is always another integer (the negative) such that we get zero when we add them together.

These conditions are the defining properties of a commutative group, and mathematicians have realized that the exact same set of rules can be found in very different mathematical structures. It therefore makes sense to give an abstract, formal definition of what a group should be, detached from any concrete examples such as integers. This lets us handle entities of very different mathematical origins in a flexible way, while retaining essential structural aspects of many objects in abstract algebra and beyond.

Distilling these rules to the smallest independent list of properties and making them abstract, we arrive at the following definition of a commutative group:

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Sylvia: I would like to have a separate counter for definitions

Definition 4.1.0.1. A **commutative group** $(\mathbb{G}, \cdot)$ consists of a set $\mathbb{G}$ and a **map** $\cdot : \mathbb{G} \times \mathbb{G} \to \mathbb{G}$. The map is called the **group law**, and it combines two elements of the set $\mathbb{G}$ into a third one such that the following properties hold:

- **Commutativity**: For all $g_1, g_2 \in \mathbb{G}$, the equation $g_1 \cdot g_2 = g_2 \cdot g_1$ holds.
- **Associativity**: For every $g_1, g_2, g_3 \in \mathbb{G}$ the equation $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$ holds.
- **Existence of a neutral element**: For every $g \in \mathbb{G}$, there is an $e \in \mathbb{G}$ such that $e \cdot g = g$.

- **Existence of an inverse**: For every $g \in \mathbb{G}$, there is a $g^{-1} \in \mathbb{G}$ such that $g \cdot g^{-1} = e$.

If $(\mathbb{G}, \cdot)$ is a group, and $\mathbb{G}' \subset \mathbb{G}$ is a subset of $\mathbb{G}$ such that the **restriction** of the group law $\cdot : \mathbb{G}' \times \mathbb{G}' \to \mathbb{G}'$ is a group law on $\mathbb{G}'$, then $(\mathbb{G}', \cdot)$ is called a **subgroup** of $(\mathbb{G}, \cdot)$.

Rephrasing the abstract definition in layman's terms, a group is something where we can do computations in a way that resembles the behavior of the addition of integers. Specifically, this means we can combine some element with another element into a new element in a way that is reversible and where the order of combining elements doesn't matter.

Notation and Symbols 3. Since we are exclusively concerned with commutative groups in this book, we often just call them groups, keeping the notation of commutativity implicit.[1]

If there is no risk of ambiguity (about what the group law of a group is), we frequently drop the symbol $\cdot$ and simply write $\mathbb{G}$ as notation for the group, keeping the group law implicit. In this case we also say that $\mathbb{G}$ is of group type, indicating that $\mathbb{G}$ is not simply a set but a set together with a group law.

Notation and Symbols 4 (**Additive notation**). For commutative groups $(\mathbb{G}, \cdot)$, we sometimes use the so-called **additive notation** $(\mathbb{G}, +)$, that is, we write $+$ instead of $\cdot$ for the group law, 0 for the neutral element and $-g := g^{-1}$ for the inverse of an element $g \in \mathbb{G}$.

As we will see in the following chapters, groups are heavily used in cryptography and in SNARKs. But let us look at some more familiar examples fist.

Example 29 (Integer Addition and Subtraction). The set $(\mathbb{Z}, +)$ of integers with integer addition is the archetypical example of a commutative group, where the group law is traditionally written in additive notation (notation 4).

To compare integer addition against the abstract axioms of a commuative group, we first note that integer addition is **commutative and associative**, since $a+b = b+a$ as well as $(a+b)+c = a+(b+c)$ for all integers $a, b, c \in \mathbb{Z}$. The **neutral element** e is the number 0, since $a+0 = a$ for all integers $a \in \mathbb{Z}$. Furthermore, the **inverse** of a number is its negative counterpart, since $a+(-a) = 0$ for all $a \in \mathbb{Z}$. This implies that integers with addition are indeed a commutative group in the abstract sense.

To give an example of a subgroup of the group of integers, consider the set of even numbers, including 0.

$$\mathbb{Z}_{even} := \{\ldots, -4, -2, 0, 2, 4, \ldots\}$$

We can see that this set is a subgroup of $(\mathbb{Z}, +)$, since the sum of two even numbers is always an even number again, since the neutral element 0 is a member of $\mathbb{Z}_{even}$ and sice the negative of an even number is itself an even number.

Example 30 (The trivial group). The most basic example of a commutative group is the group with just one element $\{\bullet\}$ and the group law $\bullet \cdot \bullet = \bullet$. We call it the **trivial group**.

The trivial group is a subgroup of any group. To see that, let $(\mathbb{G}, \cdot)$ be a group with the neutral element $e \in \mathbb{G}$. Then $e \cdot e = e$ as well as $e^{-1} = e$ both hold. Consequently, the set $\{e\}$ is a subgroup of $\mathbb{G}$. In particular, $\{0\}$ is a subgroup of $(\mathbb{Z}, +)$, since $0+0 = 0$.

Example 31. Consider addition in modulo 6 arithmetics $(\mathbb{Z}_6, +)$, as defined in in example 9. As we see, the remainder 0 is the neutral element in modulo 6 addition, and the inverse of a remainder r is given by $6-r$, because $r+(6-r) = 6$. 6 is congruent to 0 since $6 \bmod 6 = 0$. Moreover, $r_1 + r_2 = r_2 + r_1$ as well as $(r_1+r_2)+r_3 = r_1+(r_2+r_3)$ are inherited from integer addition. We therefore see that $(\mathbb{Z}_6, +)$ is a group.

[1]Commutative groups are also called **Abelian groups**. A set $\mathbb{G}$ with a map $\cdot$ that satisfies all previously mentioned rules except for the commutativity law is called a **non-commutative group.**

The previous example of a commutative group is a very important one for this book. Abstracting from this example and considering residue classes $(\mathbb{Z}_n, +)$ for arbitrary moduli n, it can be shown that $(\mathbb{Z}_n, +)$ is a commutative group with the neutral element 0 and the additive inverse $n-r$ for any element $r \in \mathbb{Z}_n$. We call such a group the **remainder class group** of modulus n.

Exercise 32. Consider example 14 again, and let $\mathbb{Z}_5^*$ be the set of all remainder classes from $\mathbb{Z}_5$ without the class 0. Then $\mathbb{Z}_5^* = \{1,2,3,4\}$. Show that $(\mathbb{Z}_5^*, \cdot)$ is a commutative group.

Exercise 33. Generalizing the previous exercise, consider the general modulus n, and let $\mathbb{Z}_n^*$ be the set of all remainder classes from $\mathbb{Z}_n$ without the class 0. Then $\mathbb{Z}_n^* = \{1,2,\ldots,n-1\}$. Provide a counter-example to show that $(\mathbb{Z}_n^*, \cdot)$ is not a group in general.

Find a condition such that $(\mathbb{Z}_n^*, \cdot)$ is a commutative group, compute the neutral element, give a closed form for the inverse of any element and prove the commutative group axioms.

4.1.1 Finite groups

As we have seen in the previous examples, groups can either contain infinitely many elements (such as integers) or finitely many elements (as for example the remainder class groups $(\mathbb{Z}_n, +)$). To capture this distinction, a group is called a **finite group** if the underlying set of elements is finite. In that case, the number of elements of that group is called its **order**.

Notation and Symbols 5. Let $\mathbb{G}$ be a finite group. We write $ord(\mathbb{G})$ or $|\mathbb{G}|$ for the order of $\mathbb{G}$.

Example 32. Consider the remainder class groups $(\mathbb{Z}_6, +)$ from example 9, the group $(\mathbb{Z}_5, +)$ from example 14, and the group $(\mathbb{Z}_5^*, \cdot)$ from exercise 32. We can easily see that the order of $(\mathbb{Z}_6, +)$ is 6, the order of $(\mathbb{Z}_5, +)$ is 5 and the order of $(\mathbb{Z}_5^*, \cdot)$ is 4.

Exercise 34. Let $n \in \mathbb{N}$ with $n \geq 2$ be some modulus. What is the order of the remainder class group $(\mathbb{Z}_n, +)$?

4.1.2 Generators

The set of elements of a group can be complicated, and it is not always obvious how to actually compute elements of a given group. From a practical point of view, it is therefore desirable to have groups with a **generator set**. This is a small subset of elements from which all other elements can be generated by applying the group law repeatedly to only the elements of the generator set or their inverses.

complicated

and/or

Of course, every group $\mathbb{G}$ has a trivial set of generators, when we just consider every element of the group to be in the generator set. The more interesting question is to find smallest possible generator set for a given group. Of particular interest in this regard are groups that have a generator set that contains a single element only. In this case, there exists a (not necessarily unique) element $g \in \mathbb{G}$ such that every other element from $\mathbb{G}$ can be computed by the repeated combination of g and its inverse g^{-1} only.

Definition 4.1.2.1 (**Cyclic groups**). Groups with single, not necessarily unique, generators are called **cyclic groups** and any element $g \in \mathbb{G}$ that is able to generate $\mathbb{G}$ is called a **generator**.

Example 33. The most basic example of a cyclic group is the group of integers with integer addition $(\mathbb{Z}, +)$. In this case, the number 1 is a generator of $\mathbb{Z}$, since every integer can be obtained by repeatedly adding either 1 or its inverse -1 to itself. For example, -4 is generated by 1, since $-4 = -1 + (-1) + (-1) + (-1)$. Another generator of $\mathbb{Z}$ is the number -1.

Example 34. Consider the group $(\mathbb{Z}_5^*, \cdot)$ from exercise 32. Since $2^1 = 2$, $2^2 = 4$, $2^3 = 3$ and $2^4 = 1$, the element 2 is a generator of $(\mathbb{Z}_5^*, \cdot)$. Moreover, since $3^1 = 3$, $3^2 = 4$, $3^3 = 2$ and $3^4 = 1$, the element 3 is another generator of $(\mathbb{Z}_5^*, \cdot)$. Cyclic groups can therefore have more than one generator. However since $4^1 = 4$, $4^2 = 1$, $4^3 = 4$ and in general $4^k = 4$ for k odd and $4^k = 1$ for k even the element 4 is not a generator of $(\mathbb{Z}_5^*, \cdot)$. It follows that in general not every element of a finite cyclic group is a generator.

Example 35. Consider a modulus n and the remainder class groups $(\mathbb{Z}_n, +)$ from exercise 34. These groups are cyclic, with generator 1, since every other element of that group can be constructed by repeatedly adding the remainder class 1 to itself. Since $\mathbb{Z}_n$ is also finite, we know that $(\mathbb{Z}_n, +)$ is a finite cyclic group of order n.

Exercise 35. Consider the group $(\mathbb{Z}_6, +)$ of modular 6 addition from example 9. Show that $5 \in \mathbb{Z}_6$ is a generator, and then show that $2 \in \mathbb{Z}_6$ is not a generator.

Exercise 36. Let $p \in \mathbb{P}$ be prime number and $(\mathbb{Z}_p^*, \cdot)$ the finite group from exercise 33. Show that $(\mathbb{Z}_p^*, \cdot)$ is cyclic.

4.1.3 The exponential map

Observe that, when $\mathbb{G}$ is a cyclic group of order n and $g \in \mathbb{G}$ is a generator of $\mathbb{G}$, then there exists a so-called **exponential map**, which maps the additive group law of the remainder class group $(\mathbb{Z}_n, +)$ onto the group law of $\mathbb{G}$ in a one-to-one correspondence. The exponential map can be formalized as in (4.1) below (where g^x means "multiply g by itself x times" and $g^0 = e_{\mathbb{G}}$).

$$g^{(\cdot)} : \mathbb{Z}_n \to \mathbb{G}\; x \mapsto g^x \tag{4.1}$$

To see how the exponential map works, first observe that, since $g^0 := e_{\mathbb{G}}$ by definition, the neutral element of $\mathbb{Z}_n$ is mapped to the neutral element of $\mathbb{G}$. Furthermore, since $g^{x+y} = g^x \cdot g^y$, the map respects the group law.

Notation and Symbols 6 (**Scalar multiplication**). If a group $(\mathbb{G}, +)$ is written in additive notation (notation 4), then the exponential map is often called **scalar multiplication**, and written as follows:

$$(\cdot) \cdot g : \mathbb{Z}_n \to \mathbb{G}\,;\, x \mapsto x \cdot g \tag{4.2}$$

In this notation, the symbol $x \cdot g$ is defined as "add the generator g to itself x times" and the symbol $0 \cdot g$ is defined to be the neutral element in $\mathbb{G}$.

Cryptographic applications often utilize finite cyclic groups of a very large order n, which means that computing the exponential map by repeated multiplication of the generator with itself is infeasible for very large remainder classes. Algorithm 5, called **square and multiply**, solves this problem by computing the exponential map in approximately k steps, where k is the bit length of the exponent (3.4):SB: I think moving the explanation of bit length here would work better

unify title of Alg with text

move 3.4 here

Because the exponential map respects the group law, it doesn't matter if we do our computation in $\mathbb{Z}_n$ before we write the result into the exponent of g or afterwards: the result will be the same in both cases. The latter mehtod is usually referred to as doing computations "in the exponent". In cryptography in general, and in SNARK development in particular, we often perform computations "in the exponent" of a generator.

check algorithm floating

Algorithm 5 Cyclic Group Exponentiation

Require: g group generator of order n
Require: $x \in \mathbb{Z}_n$
  procedure EXPONENTIATION(g, x)
    Let $(b_0, \ldots, b_k)$ be a binary representation of x ▷ see example XXX
    $h \leftarrow g$
    $y \leftarrow e_{\mathbb{G}}$
    for $0 \leq j < k$ **do**
      if $b_j = 1$ **then**
        $y \leftarrow y \cdot h$ ▷ multiply
      end if
      $h \leftarrow h \cdot h$ ▷ square
    end for
    return y
  end procedure
Ensure: $y = g^x$

Example 36. Consider the multiplicative group $(\mathbb{Z}_5^*, \cdot)$ from exercise 32. We know from 36 that $\mathbb{Z}_5^*$ is a cyclic group of order 4, and that the element $3 \in \mathbb{Z}_5^*$ is a generator. This means that we also know that the following map respects the group law of addition in $\mathbb{Z}_4$ and the group law of multiplication in $\mathbb{Z}_5^*$:

$$3^{(\cdot)} : \mathbb{Z}_4 \to \mathbb{Z}_5^* \; ; \; x \mapsto 3^x$$

To do an example computation "in the exponent" of 3 , let's perform the calculation $1+3+2$ in the exponent of the generator 3:

$$3^{1+3+2} = 3^2 \tag{4.3}$$
$$= 4 \tag{4.4}$$

In (4.3) above, we first performed the computation $1+3+2 = 1$ in the remainder class group $(\mathbb{Z}_4, +)$ and then applied the exponential map $3^{(\cdot)}$ to the result in (4.4). should be 2?

However, since the exponential map (4.1) respects the group law, we also could map each summand into $(\mathbb{Z}_5^*, \cdot)$ first and then apply the group law of $(\mathbb{Z}_5^*, \cdot)$. The result is guranteed to be the same:

$$\begin{aligned} 3^1 \cdot 3^3 \cdot 3^2 &= 3 \cdot 2 \cdot 4 \\ &= 1 \cdot 4 \\ &= 4 \end{aligned}$$

Since the exponential map (4.1) is a one-to-one correspondence that respects the group law, it can be shown that this map has an inverse with respect to the base g, called the **base g discrete logarithm map**:

$$log_g(\cdot) : \mathbb{G} \to \mathbb{Z}_n \; x \mapsto log_g(x) \tag{4.5}$$

Discrete logarithms are highly important in cryptography, because there are finite cyclic groups where the exponential map and its inverse, the discrete logarithm map, are believed to be one-way functions, which informally means that computing the exponential map is fast, while computing the logarithm map is slow (We will look into a more precise definition in 4.1.6).

Example 37. Consider the exponential map $3^{(\cdot)}$ from example 36. Its inverse is the discrete logarithm to the base 3, given by the map below:

$$log_3(\cdot): \mathbb{Z}_5^* \to \mathbb{Z}_4 \; x \mapsto log_3(x)$$

In contrast to the exponential map $3^{(\cdot)}$, we have no way to actually compute this map, other than by trying all elements of the group until we find the correct one. For example, in order to compute $log_3(4)$, we have to find some $x \in \mathbb{Z}_4$ such that $3^x = 4$, and all we can do is repeatedly insert elements x into the exponent of 3 until the result is 4. To do this, let's write down all the images of $3^{(\cdot)}$:

images

$$3^0 = 1, \quad 3^1 = 3, \quad 3^2 = 4, \quad 3^3 = 2$$

Since the discrete logarithm $log_3(\cdot)$ is defined as the inverse to this function, we can use those images to compute the discrete logarithm:

$$log_3(1) = 0, \quad log_3(2) = 3, \quad log_3(3) = 1, \quad log_3(4) = 2$$

Note that this computation was only possible because we were able to write down all images of the exponential map. However, in real world applications the groups in consideration are too large to write down the images of the exponential map.

Exercise 37 (Efficient Scalar Multiplication). Let $(\mathbb{G}, +)$ be a finite cyclic group of order n. Consider algorithm 5 and define its analog for groups in additive notation.

4.1.4 Factor Groups

As we know from the fundamental theorem of arithmetic (3.7), every natural number n is a product of factors, the most basic of which are prime numbers. This parallels subgroups of finite cyclic groups in an interesting way.

Definition 4.1.4.1 (**The fundamental theorem of finite cyclic groups**). If $\mathbb{G}$ is a finite cyclic group of order n, then every subgroup $\mathbb{G}'$ of $\mathbb{G}$ is finite and cyclic, and the order $\mathbb{G}'$ is a factor of n. Moreover for each factor k of n, $\mathbb{G}$ has exactly one subgroup of order k. This is known as the **fundamental theorem of finite cyclic groups**.

Notation and Symbols 7. If $\mathbb{G}$ is a finite cyclic group of order n and k is a factor of n, then we write $\mathbb{G}[k]$ for the unique finite cyclic group which is the order k subgroup of $\mathbb{G}$, and call it a **factor group** of $\mathbb{G}$.

One particularly interesting situation occurs if the order of a given finite cyclic group is a prime number. As we know from the fundamental theorem of arithmetics (3.7), prime numbers have only two factors: the number 1 and the prime number itself. It then follows from the fundamental theorem of finite cyclic groups (definition 4.1.4.1) that those groups have no subgroups other than the trivial group (example 30) and the group itself.

Cryptographic protocols often assume the existence of finite cyclic groups of prime order. However some real-world implementations of those protocols are not defined on prime order groups, but on groups where the order consist of a (usually large) prime number that has small cofactors (see notation 1). In this case, a method called **cofactor clearing** has to be applied to ensure that the computations are not done in the group itself but in its (large) prime order subgroup.

To understand cofactor clearing in detail, let $\mathbb{G}$ be a finite cyclic group of order n, and let k be a factor of n with associated factor group $\mathbb{G}[k]$. We can project any element $g \in \mathbb{G}[k]$ onto the neutral element e of $\mathbb{G}$ by multiplying g k-times with itself:

$$g^k = e \tag{4.6}$$

Consequently, if $c := n \text{ div } k$ is the cofactor of k in n, then any element from the full group $g \in \mathbb{G}$ can be projected into the factor group $\mathbb{G}[k]$ by multiplying g c-times with itself. This defines the following map, which is often called **cofactor clearing** in cryptographic literature:

$$(\cdot)^c : \mathbb{G} \to \mathbb{G}[k] \; : \; g \mapsto g^c \tag{4.7}$$

Example 38. Consider the finite cyclic group $(\mathbb{Z}_5^*, \cdot)$ from example 34. Since the order of $\mathbb{Z}_5^*$ is 4, and 4 has the factors 1, 2 and 4, it follows from the fundamental theorem of finite cyclic groups (Definition 4.1.4.1) that $\mathbb{Z}_5^*$ has 3 unique subgroups. In fact, the unique subgroup $\mathbb{Z}_5^*[1]$ of order 1 is given by the trivial group $\{1\}$ that contains only the multiplicative neutral element 1. The unique subgroup $\mathbb{Z}_5^*[4]$ of order 4 is $\mathbb{Z}_5^*$ itself, since, by definition, every group is trivially a subgroup of itself. The unique subgroup $\mathbb{Z}_5^*[2]$ of order 2 is more interesting, and is given by the set $\mathbb{Z}_5^*[2] = \{1, 4\}$.

Since $\mathbb{Z}_5^*$ is not a prime order group, and, since the only prime factor of 4 is 2, the "large" prime order subgroup of $\mathbb{Z}_5^*$ is $\mathbb{Z}_5^*[2]$. Moreover, since the cofactor of 2 in 4 is also 2, we get the cofactor clearing map $(\cdot)^2 : \mathbb{Z}_5^* \to \mathbb{Z}_5^*[2]$. As expected, when we apply this map to all elements of $\mathbb{Z}_5^*$, we see that it maps onto the elements of $\mathbb{Z}_5^*[2]$ only:

$$1^2 = 1 \quad 2^2 = 4 \quad 3^2 = 4 \quad 4^2 = 1 \tag{4.8}$$

We can therefore use this map to "clear the cofactor" of any element from $\mathbb{Z}_5^*$, which means that the element is projected onto the "large" prime order subgroup $\mathbb{Z}_5^*[2]$.

Exercise 38. Consider the previous example 38, and show that $\mathbb{Z}_5^*[2]$ is a commutative group.

Exercise 39. Consider the finite cyclic group $(\mathbb{Z}_6, +)$ of modular 6 addition from example 35. Describe all subgroups of $(\mathbb{Z}_6, +)$. Identify the large prime order subgroup of $\mathbb{Z}_6$, define its cofactor clearing map and apply that map to all elements of $\mathbb{Z}_6$.

Exercise 40. Let $(\mathbb{Z}_p^*, \cdot)$ be the cyclic group from exercise 36. Show that, for $p \geq 5$, not every element $x \in \mathbb{F}_p^*$ is a generator of $\mathbb{F}_p^*$.

4.1.5 Pairings

Of particular importance for the development of SNARKs are so-called **pairing maps** on commutative groups, defined below.

Definition 4.1.5.1 (**Pairing map**). Let $\mathbb{G}_1$, $\mathbb{G}_2$ and $\mathbb{G}_3$ be three commutative groups. Then a **pairing map** is a function

$$e(\cdot, \cdot) : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_3 \tag{4.9}$$

This function takes pairs (g_1, g_2) of elements from $\mathbb{G}_1$ and $\mathbb{G}_2$, and maps them to elements from $\mathbb{G}_3$ such that the **bilinearity** property holds, which means that for all $g_1, g_1' \in \mathbb{G}_1$ and $g_2, g_2' \in \mathbb{G}_2$ the following two identities are satisfied:

$$e(g_1 \cdot g_1', g_2) = e(g_1, g_2) \cdot e(g_1', g_2) \quad \text{and} \quad e(g_1, g_2 \cdot g_2') = e(g_1, g_2) \cdot e(g_1, g_2') \tag{4.10}$$

Informally speaking, bilinearity means that it doesn't matter if we first execute the group law on one side and then apply the bilinear map, or if we first apply the bilinear map and then apply the group law in $\mathbb{G}_3$.

A pairing map is called **non-degenerate** if, whenever the result of the pairing is the neutral element in $\mathbb{G}_3$, one of the input values is the neutral element of $\mathbb{G}_1$ or $\mathbb{G}_2$. To be more precise, $e(g_1, g_2) = e_{\mathbb{G}_3}$ implies $g_1 = e_{\mathbb{G}_1}$ or $g_2 = e_{\mathbb{G}_2}$.

Example 39. One of the most basic examples of a non-degenerate pairing involves the groups $\mathbb{G}_1$, $\mathbb{G}_2$ and $\mathbb{G}_3$ all being groups of integers with addition $(\mathbb{Z},+)$. In this case, the following map defines a non-degenerate pairing:

$$e(\cdot,\cdot):\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}\;(a,b)\mapsto a\cdot b \tag{4.11}$$

Note that bilinearity follows from the distributive law of integers, meaning that, for $a,b,c\in\mathbb{Z}$, the equation $e(a+b,c)=(a+b)\cdot c=a\cdot c+b\cdot c=e(a,c)+e(b,c)$ holds (and the same reasoning is true for the second argument b). b

To see that $e(\cdot,\cdot)$ is non-degenerate, assume that $e(a,b)=0$. Then $a\cdot b=0$ implies that a or b must be zero.

Exercise 41 (Arithmetic laws for pairing maps). Let $\mathbb{G}_1$, $\mathbb{G}_2$ and $\mathbb{G}_3$ be finite cyclic groups of the same order n, and let $e(\cdot,\cdot):\mathbb{G}_1\times\mathbb{G}_2\to\mathbb{G}_3$ be a pairing map. Show that, for given $g_1\in\mathbb{G}_1$, $g_2\in\mathbb{G}_2$ and all $a,b\in\mathbb{Z}_n$, the following identity holds:

$$e(g_1^a,g_2^b)=e(g_1,g_2)^{a\cdot b} \tag{4.12}$$

Exercise 42. Consider the remainder class groups $(\mathbb{Z}_n,+)$ from example 34 for some modulus n. Show that the following map is a pairing map.

$$e(\cdot,\cdot):\mathbb{Z}_n\times\mathbb{Z}_n\to\mathbb{Z}_n\;(a,b)\mapsto a\cdot b \tag{4.13}$$

Why is the pairing not non-degenerate in general, and what condition must be imposed on n such that the pairing will be non-degenerate?

4.1.6 Cryptographic Groups

In this section, we look at classes of groups that are believed to satisfy certain **computational hardness assumptions**, meaning that it is not feasible to compute them in polynomial time.

Example 40. To give an example for a well-known computational hardness assumption, consider the problem of factorization, i.e. computing the prime factors of a composite integer (see example 1). If the prime factors are very large, this is infeasible to do, and is expected to remain infeasible. We assume the problem is **computationally hard** or **infeasible**.

Note that, in example 40, we say that the problem is infeasible to solve **if the prime factors are large enough**. we say that Naturally, this is made more precise in the cryptographic standard model, where we have a security parameter, and we say that "there exists a security parameter such that it is not feasible to compute a solution to the problem". In the following examples, the security parameter roughly correlates with the order of the group in consideration. In this book, we do not include the security parameter in our definitions, since we only aim to provide an intuitive understanding of the cryptographic assumptions, not teach the ability to perform rigorous analysis.

Furthermore, understand that these are **assumptions**. Academics have been looking for efficient prime factorization algorithms for a long time, and they have been getting better and better while computers have become faster and faster – but there always was a higher security parameter for which the problem still was infeasible.

In what follows, we describe a few problems arising in the context of groups in cryptography that are assumed to be infeasible. We will refer to them throughout the book.

4.1.6.1 The Discrete Logarithm Problem

The so-called **Discrete Logarithm Problem (DLP)**, also called the **Discrete Logarithm Assumption**, is one of the most fundamental assumptions in cryptography.

Definition 4.1.6.1. Let $\mathbb{G}$ be a finite cyclic group of order r and let g be a generator of $\mathbb{G}$. We know from (4.1) that there is an exponential map $g^{(\cdot)} : \mathbb{Z}_r \to \mathbb{G}$; $x \mapsto g^x$ that maps the residue classes from modulo r arithmetic onto the group in a 1 : 1 correspondence. The **Discrete Logarithm Problem** is the task of finding an inverse to this map, that is, to find a solution $x \in \mathbb{Z}_r$ to the following equation for some given $h, g \in \mathbb{G}$:

$$h = g^x \tag{4.14}$$

There are groups in which the DLP is assumed to be infeasible to solve, and there are groups in which it isn't. We call the former group **DL-secure** groups.

Rephrasing the previous definition, it is believed that, in DL-secure groups, there is a number n such that it is infeasible to compute some number x that solves the equation $h = g^x$ for a given h and g, assuming that the order of the group n is large enough. The number n here corresponds to the security parameter discussed above.

Example 41 (Public key cryptography). One the most basic examples of an application for DL-secure groups is in public key cryptography, where the parties publicly agree on some pair $(\mathbb{G}, g)$ such that $\mathbb{G}$ is a finite cyclic group of appropriate order n, believed to be a DL-secure group, and g is a generator of $\mathbb{G}$.

In this setting, a secret key is some number $sk \in \mathbb{Z}_r$ and the associated public key pk is the group element $pk = g^{sk}$. Since discrete logarithms are assumed to be hard, it is infeasible for an attacker to compute the secret key from the public key, as this would involve finding solutions x to the following equation (which is believed to be infeasible):

$$pk = g^x \tag{4.15}$$

As example 41 shows, identifying DL-secure groups is an important practical problem. Unfortunately, it is easy to see that it does not make sense to assume the hardness of the Discrete Logarithm Problem in all finite cyclic groups: counterexamples are common and easy to construct.

mention a few examples

4.1.6.2 The decisional Diffie–Hellman assumption

Definition 4.1.6.2. Let $\mathbb{G}$ be a finite cyclic group of order n and let g be a generator of $\mathbb{G}$. The decisional Diffie–Hellman (DDH) problem is to distinguish (g^a, g^b, g^{ab}) from the triple (g^a, g^b, g^c) for uniformly random values $a, b, c \in \mathbb{Z}_r$.

If we assume the DDH problem is infeasible to solve in $\mathbb{G}$, we call $\mathbb{G}$ a **DDH-secure** group.

DDH-security is a stronger assumption than DL-security (4.1.6.1), in the sense that if the DDH problem is infeasible, so is the DLP, but not necessarily the other way around.

To see why this is the case, assume that the discrete logarithm assumption does not hold. In that case, given a generator g and a group element h, it is easy to compute some element $x \in \mathbb{Z}_p$ with $h = g^x$. Then the decisional Diffie–Hellman assumption cannot hold, since given some triple (g^a, g^b, z), one could efficiently decide whether $z = g^{ab}$ is true by first computing the discrete logarithm b of g^b, then computing $g^{ab} = (g^a)^b$ and deciding whether or not $z = g^{ab}$.

On the other hand, the following example shows that there are groups where the discrete logarithm assumption holds but the Decisional Diffie–Hellman Assumption does not.

Example 42 (Efficiently computable bilinear pairings). Let $\mathbb{G}$ be a DL-secure, finite, cyclic group of order r with generator g, and $\mathbb{G}_T$ another group such that there is an efficiently computable pairing map $e(\cdot,\cdot) : \mathbb{G} \times \mathbb{G} \to \mathbb{G}_T$ that is bilinear and non degenerate (4.9).

In a setting like this, it is easy to show that solving DDH cannot be infeasible, since, given some triple (g^a, g^b, z), it is possible to efficiently check whether $z = g^{ab}$ by making use of the following pairing:

$$e(g^a, g^b) \stackrel{?}{=} e(g,z) \tag{4.16}$$

Since the bilinearity properties of $e(\cdot,\cdot)$ imply $e(g^a, g^b) = e(g,g)^{ab} = e(g, g^{ab})$, and $e(g,y) = e(g,y')$ implies $y = y'$ due to the non-degenerate property, the equality means $z = g^{ab}$.

It follows that the DDH assumption is indeed stronger than the discrete log assumption, and groups with efficient pairings cannot be DDH-secure groups.

4.1.6.3 The Computational Diffie–Hellman Assumption

Definition 4.1.6.3. Let $\mathbb{G}$ be a finite cyclic group of order n and let g be a generator of $\mathbb{G}$. The **computational Diffie–Hellman assumption** stipulates that, given randomly and independently chosen elements $a, b \in \mathbb{Z}_r$, it is not possible to compute g^{ab} if only g, g^a and g^b (but not a and b) are known. If this is the case for $\mathbb{G}$, we call $\mathbb{G}$ a **CDH-secure** group.

In general, we don't know if CDH-security is a stronger assumption than DL-security, or if both assumptions are equivalent. We know that DL-security is necessary for CDH-security, but the other direction is currently not well understood. In particular, there are no known DL-secure groups that are not also CDH-secure.

To see why the discrete logarithm assumption is necessary, assume that it does not hold. Then, given a generator g and a group element h, it is easy to compute some element $x \in \mathbb{Z}_p$ with $h = g^x$. In that case, the computational Diffie–Hellman assumption cannot hold, since, given g, g^a and g^b, it is possible to efficiently compute b, meaning that $g^{ab} = (g^a)^b$ can be computed from this data.

The computational Diffie–Hellman assumption is a weaker assumption than the Decisional Diffie–Hellman Assumption. This means that there are groups where CDH holds and DDH does not hold, while there cannot be groups in which DDH holds but CDH does not hold. To see that, assume that it is efficiently possible to compute g^{ab} from g, g^a and g^b. Then, given (g^a, g^b, z) it is easy to decide whether $z = g^{ab}$ holds or not.

Several variations and special cases of CDH exist. For example, the **square Computational Diffie–Hellman Assumption** assumes that, given g and g^x, it is computationally hard to compute g^{x^2}. The **inverse Computational Diffie–Hellman Assumption** assumes that, given g and g^x, it is computationally hard to compute $g^{x^{-1}}$.

4.1.7 Hashing to Groups

4.1.7.1 Hash functions

Generally speaking, a hash function is any function that can be used to map data of arbitrary size to fixed-size values. Since binary strings of arbitrary length are a way to represent data in general, we can understand a **hash function** as the following map where $\{0,1\}^*$ represents the set of all binary strings of arbitrary but finite length and $\{0,1\}^k$ represents the set of all binary strings that have a length of exactly k bits:

$$H : \{0,1\}^* \to \{0,1\}^k \tag{4.17}$$

The **images** of H, that is, the values returned by the hash function H, are called **hash values**, **digests**, or simply **hashes**.

Notation and Symbols 8. In what follows, we call an element $b \in \{0,1\}$ a **bit**. If $s \in \{0,1\}^*$ is a binary string, we write $|s| = k$ for its **length**, that is, for the number of bits in s. We write $<>$ for the empty binary string, and $s =< b_1, b_2, \ldots, b_k >$ for a binary string of length k.[2] check footnote

If two binary strings $s =< b_1, b_2, \ldots, b_k >$ and $s' =< b'_1, b'_2, \ldots, b'_l >$ are given, then we write $s||s'$ for the **concatenation** that is the string $s||s' =< b_1, b_2, \ldots, b_k, b'_1, b'_2, \ldots, b'_l >$.

If H is a hash function that maps binary strings of arbitrary length onto binary strings of length k, and $s \in \{0,1\}^*$ is a binary string, we write $H(s)_j$ for the bit at position j in the image $H(s)$.

Example 43 (k-truncation hash). One of the most basic hash functions $H_k : \{0,1\}^* \to \{0,1\}^k$ is given by simply truncating every binary string s of size $|s| > k$ to a string of size k and by filling any string s' of size $|s'| < k$ with zeros. To make this hash function deterministic, we define that both truncation and filling should happen "on the left".

For example, if the parameter k is given by $k = 3$, $s_1 =< 0,0,0,0,1,0,1,0,1,1,1,0 >$ and $s_2 = 1$, then $H_3(x_1) =< 1,1,0 >$ and $H_3(x_2) =< 0,0,1 >$. should be $H_3(s_1)$ and $H_3(s_2)$?

A desirable property of a hash function is **uniformity**, which means that it should map input values as evenly as possible over its output range. In mathematical terms, every string of length k from $\{0,1\}^k$ should be generated with roughly the same probability.

Of particular interest are so-called **cryptographic** hash functions, which are hash functions that are also **one-way functions**, which essentially means that, given a string y from $\{0,1\}^k$ it is infeasible to find a string $x \in \{0,1\}^*$ such that $H(x) = y$ holds. This property is usually called **preimage-resistance**.

Moreover, if a string $x_1 \in \{0,1\}^*$ is given, then it should be infeasible to find another string $x_2 \in \{0,1\}^*$ with $x_1 \neq x_2$ and $H(x_1) = H_(x_2)$

In addition, it should be infeasible to find two strings $x_1, x_2 \in \{0,1\}^*$ such that $H(x_1) = H(x_2)$, which is called **collision resistance**. It is important to note, though, that collisions always exist, since a function $H : \{0,1\}^* \to \{0,1\}^k$ inevitably maps infinitely many values onto the same hash. In fact, for any hash function with digests of length k, finding a preimage to a given digest can always be done using a brute force search in 2^k evaluation steps. It should just be practically impossible to compute those values, and statistically very unlikely to generate two of them by chance.

A third property of a cryptographic hash function is that small changes in the input string, like changing a single bit, should generate hash values that look completely different from each other. This is called **diffusion** or the avalanche effect.

Because cryptographic hash functions map tiny changes in input values onto large changes in the output, implementation errors that change the outcome are usually easy to spot by comparing them to expected output values. The definitions of cryptographic hash functions are therefore usually accompanied by some test vectors of common inputs and expected digests. Since the empty string $<>$ is the only string of length 0, a common test vector is the expected digest of the empty string.

Example 44 (k-truncation hash). Consider the k-truncation hash from example 43. Since the empty string has length 0, it follows that the digest of the empty string is the string of length k

[2]The difference between the notations $b \in \{0,1\}$ and $s \in \{0,1\}^*$ is the following: $b \in \{0,1\}$ means that b is equal to either 0 or 1, whereas s is a string composed of an arbitrary number of 0s and 1s (and s can also be an empty string).

that only contains 0s:

$$H_k(<>) = < 0,0,\ldots,0,0 > \tag{4.18}$$

It is pretty obvious from the definition of H_k that this simple hash function is not a cryptographic hash function. In particular, every digest is its own preimage, since $H_k(y) = y$ for every string of size exactly k. Finding preimages is therefore easy, so the property of preimage resistance does not hold.

In addition, it is easy to construct collisions, as all strings s of size $|s| > k$ that share the same k-bits "on the right" are mapped to the same hash value. This means that this function is not collision resistant, either.

Finally, this hash function does not have a lot of diffusion, as changing bits that are not part of the k right-most bits won't change the digest at all.

Computing cryptographically secure hash functions in pen-and-paper style is possible but tedious. Fortunately, Sage can import the **hashlib** library, which is intended to provide a reliable and stable base for writing Python programs that require cryptographic functions. The following examples explain how to use `hashlib` in Sage.

Example 45. An example of a hash function that is generally believed to be a cryptographically secure hash function is the so-called **SHA256** hash, which, in our notation, is a function that maps binary strings of arbitrary length onto binary strings of length 256:

$$SHA256 : \{0,1\}^* \to \{0,1\}^{256} \tag{4.19}$$

To evaluate a proper implementation of the $SHA256$ hash function, the digest of the empty string is supposed to be the following:

$$SHA256(<>) = e3b0c44298fc1c149afbf4c8996fb92427ae41e4649b934ca495991b7852b855 \tag{4.20}$$

For better human readability, it is common practice to represent the digest of a string not in its binary form, but in a hexadecimal representation. We can use Sage to compute $SHA256$ and freely transit between binary, hexadecimal and decimal representations. To do so, we import `hashlib`'s implementation of SHA256:

```
sage: import hashlib
sage: test = 'e3b0c44298fc1c149afbf4c8996fb92427ae41e4649b934
    ca495991b7852b855'
sage: empty_string = ""
sage: binary_string = empty_string.encode()
sage: hasher = hashlib.sha256(binary_string)
sage: result = hasher.hexdigest()
sage: type(result) # Sage represents digests as strings
<class 'str'>
sage: d = ZZ('0x'+ result) # conversion to an integer
sage: d.str(16) == test # hash is equal to test vector
True
sage: d.str(16) # hexadecimal representation
e3b0c44298fc1c149afbf4c8996fb92427ae41e4649b934ca495991b7852b8
    55
sage: d.str(2) # binary representation
```

```
1110001110110000110001000100001010011000111111000001110000010
   1001001101011111011111101001100100010011001011011111101110010
   0100100000100111101011100100000111100100011001001001101111001
   0011010011001010010010010101100110010001101101111000010100 1
   01011100001010101
sage: d.str(10) # decimal representation
1029873362495540970295352123225813227897999006481980349933793 9
   7001115665086549
```

4.1.7.2 Hashing to cyclic groups

As we have seen in the previous section, general hash functions map binary strings of arbitrary length onto binary strings of some fixed length. However, it is desirable in various cryptographic primitives to not simply hash to binary strings of fixed length, but to hash into algebraic structures like groups, while keeping (some of) the properties of the hash function, like preimage resistance or collision resistance.

Hash functions like this can be defined for various algebraic structures, but, in a sense, the most fundamental ones are hash functions that map into groups, because they can be easily extended to map into other structures like rings or fields.

To give a more precise definition, let $\mathbb{G}$ be a group and $\{0,1\}^*$ the set of all finite, binary strings, then a **hash-to-group** function is a deterministic map

$$H : \{0,1\}^* \to \mathbb{G} \tag{4.21}$$

As the following example shows, hashing to finite cyclic groups can be trivially achieved for the price of some undesirable properties of the hash function:

Example 46 (Naive cyclic group hash). Let $\mathbb{G}$ be a finite cyclic group of order n. If the task is to implement a hash-to-group function, one immediate approach can be based on the observation that binary strings of size k can be interpreted as integers $z \in \mathbb{Z}$ in the range $0 \leq z < 2^k$ using equation 3.4.

To be more precise, let $H : \{0,1\}^* \to \{0,1\}^k$ be a hash function for some parameter k, g a generator of $\mathbb{G}$, and $s \in \{0,1\}^*$ a binary string. Using equation 3.4 and notation 8, the following expression is a non-negative integer:

$$z_{H(s)} = H(s)_0 \cdot 2^0 + H(s)_1 \cdot 2^1 + \ldots + H(s)_k \cdot 2^k \tag{4.22}$$

A hash-to-group function for the group $\mathbb{G}$ can then be defined as a composition of the exponential map $g^{(\cdot)}$ of g with the interpretation of $H(s)$ as an integer:

$$H_g : \{0,1\}^* \to \mathbb{G} : \; s \mapsto g^{z_{H(s)}} \tag{4.23}$$

Constructing a hash-to-group function like this is easy for cyclic groups, and it might be good enough in certain applications. It is, however, almost never adequate in cryptographic applications, as a discrete log relation might be constructible between some hash values $H_g(s)$ and $H_g(t)$, regardless of whether or not $\mathbb{G}$ is DL-secure (see section4.1.6.1).

a few examples?

To be more precise, a discrete log relation between the group elements $H_g(s)$ and $H_g(t)$ is any element $x \in \mathbb{Z}_n$ such that $H_g(s) = H_g(t)^x$. To see how such an x can be constructed, assume that $z_{H(s)}$ has a multiplicative inverse in $\mathbb{Z}_n$. In this case, the element $x = z_{H(t)} \cdot z_{H(s)}^{-1}$ from $\mathbb{Z}_n$ is a discrete log relation between $H_g(s)$ and $H_g(t)$:

$$
\begin{aligned}
g^{z_{H(t)}} &= g^{z_{H(t)}} && \Leftrightarrow \\
g^{z_{H(t)}} &= g^{z_{H(t)} \cdot z_{H(s)} \cdot z_{H(s)}^{-1}} && \Leftrightarrow \\
g^{z_{H(t)}} &= g^{z_{H(s)} \cdot x} && \Leftrightarrow \\
H_g(t) &= (H_g(s))^x
\end{aligned}
$$

Therefore, applications where discrete log relations between hash values are undesirable need different approaches. Many of these approaches start with a way to hash into the set $\mathbb{Z}_r$ of modular r arithmetics.

4.1.7.3 Pedersen Hashes

The so-called **Pedersen Hash Function** [Pedersen, 1992] provides a way to map fixed size tuples of elements from modular arithmetics onto elements of finite cyclic groups in such a way that discrete log relations (see example 46) between different images are avoidable. Compositions of a Pedersen Hash with a general hash function (4.17) then provide hash-to-group functions that map strings of arbitrary length onto group elements.

To be more precise, let j be an integer, $\mathbb{G}$ a finite cyclic group of order n, and $\{g_1,\ldots,g_j\} \subset \mathbb{G}$ a uniform and randomly generated set of generators of $\mathbb{G}$. Then **Pedersen's hash function** is defined as follows:

$$H_{\{g_1,\ldots,g_j\}} : (\mathbb{Z}_r)^j \to \mathbb{G} : \; (x_1,\ldots,x_j) \mapsto \Pi_{i=1}^k g_j^{x_j} \tag{4.24}$$

It can be shown that Pedersen's hash function is collision-resistant under the assumption that $\mathbb{G}$ is DL-secure (see section4.1.6.1). It is important to note though, that the following family of functions does not qualify as a pseudorandom function family.

pseudorandom function family

$$\{H_{\{g_1,\ldots,g_j\}} \mid g_1,\ldots,g_j \in \mathbb{G}\} \tag{4.25}$$

From an implementation perspective, it is important to derive the set of generators $\{g_1,\ldots,g_k\}$ in such a way that they are as uniform and random as possible. In particular, any known discrete log relation between two generators, that is, any known $x \in \mathbb{Z}_n$ with $g_h = (g_i)^x$, must be avoided.

Example 47. To compute an actual Pedersen's hash, consider the cyclic group $\mathbb{Z}_5^*$ from example 34. We know from example 38 that the elements 2 and 3 are generators of $\mathbb{Z}_5^*$, and it follows that the following map is a Pedersen's hash function:

$$H_{\{2,3\}} : \mathbb{Z}_4 \times Z_4 \to \mathbb{Z}_5^* \; ; \; (x,y) \mapsto 2^x \cdot 3^y \tag{4.26}$$

To see how this map can be calculated, we choose the input value $(1,3)$ from $\mathbb{Z}_4 \times Z_4$. Then, using the multiplication table from (3.24), we calculate $H_{\{2,3\}}(1,3) = 2^1 \cdot 3^3 = 2 \cdot 2 = 4$.

To see how the composition of a hash function with $H_{\{2,3\}}$ defines a hash-to-group function, consider the $SHA256$ hash function from example 45. Given some binary string $s \in \{0,1\}^*$, we can insert the two least significant bits $SHA256(s)_0$ and $SHA256(s)_1$ from the image $SHA256(s)$ into $H_{\{2,3\}}$ to get an element in $\mathbb{F}_5^*$. This defines the following hash-to-group function

$$SHA256\_H_{\{2,3\}} : \{0,1\}^* \to \mathbb{Z}_5^* \; ; \; s \mapsto 2^{SHA256(s)_0} \cdot 3^{SHA256(s)_1}$$

To see how this hash function can be calculated, consider the empty string $<>$. Since we know from the Sage computation in example 45, that $SHA256(<>)_0 = 1$ and that $SHA256(<>)_1 = 0$, we get $SHA\_256H_{\{2,3\}}(<>) = 2^1 \cdot 3^0 = 2$.

Of course, computing $SHA256\_H_{\{2,3\}}$ in a pen-and-paper style is difficult. However, we can easily implement this function in Sage in the following way:

```
sage: import hashlib
sage: def SHA256_H(x):
....:     Z5 = Integers(5) # define the group type
....:     hasher = hashlib.sha256(x) # compute SHA256
....:     digest = hasher.hexdigest()
....:     z = ZZ(digest, 16) # cast into integer
....:     z_bin = z.digits(base=2, padto=256) # cast to 256
   bits
....:     return Z5(2)^z_bin[0] * Z5(3)^z_bin[1]
sage: SHA256_H(b"") # evaluate on empty string
2
sage: SHA256_H(b"SHA") # possible images are {1,2,3}
3
sage: SHA256_H(b"Math")
1
```

Exercise 43. Consider the multiplicative group $\mathbb{Z}_{13}^*$ of modular 13 arithmetic from example 33. Choose a set of 3 generators of $\mathbb{Z}_{13}^*$, define its associated Pedersen Hash Function, and compute the Pedersen Hash of $(3,7,11) \in \mathbb{Z}_{12}$.

Exercise 44. Consider the Pedersen Hash from exercise 43. Compose it with the *SHA*256 hash function from example 45 to define a hash-to-group function. Implement that function in Sage.

4.1.7.4 Pseudorandom Function Families in DDH-secure groups

As noted in 4.24, the family of Pederson's hash functions, parameterized by a set of generators $\{g_1,\dots,g_j\}$ does not qualify as a family of pseudorandom functions, and should therefore not be instantiated as such. To see an example of a proper family of pseudorandom functions in groups where the decisional Diffie–Hellman assumption (see section4.1.6.2) is assumed to hold true, let $\mathbb{G}$ be a DDH-secure cyclic group of order n with generator g, and let $\{a_0,a_1,\dots,a_k\} \subset \mathbb{Z}_n^*$ be a uniform randomly generated set of numbers invertible in modular n arithmetics. Then a family of pseudorandom functions, parameterized by the seed $\{a_0,a_1,\dots,a_k\}$ is given as follows:

$$F_{\{a_0,a_1,\dots,a_k\}} : \{0,1\}^{k+1} \to \mathbb{G} : \ (b_0,\dots,b_k) \mapsto g^{b_0 \cdot \Pi_{i=1}^k a_i^{b_i}} \tag{4.27}$$

Exercise 45. Consider the multiplicative group $\mathbb{Z}_{13}^*$ of modular 13 arithmetic from example 33 and the parameter $k = 3$. Choose a generator of $\mathbb{Z}_{13}^*$, a seed and instantiate a member of the family given in (4.27) for that seed. Evaluate that member on the binary string $< 1,0,1 >$.

4.2 Commutative Rings

In the previous section, we have seen that integers are a commutative group with respect to integer addition. However, as we know, there are two arithmetic operations defined on integers: addition and multiplication. However, in contrast to addition, multiplication does not define a group structure, given that integers generally don't have multiplicative inverses. Configurations like these constitute so-called **commutative rings with unit**, and are defined as follows:

Definition 4.2.0.1 (Commutative ring with unit). A **commutative ring with unit** $(R,+,\cdot,1)$ is a set R with two maps, $+ : R\times R \to R$ and $\cdot : R\times R \to R$, called **addition** and **multiplication**, and an element $1 \in R$, called the **unit**, such that the following conditions hold:

- $(R,+)$ is a commutative group where the neutral element is denoted with 0.
- **Commutativity of multiplication**: $r_1\cdot r_2 = r_2\cdot r_1$ for all $r_1,r_2 \in R$.
- **Multiplicative neutral unit** : $1\cdot g = g$ for all $g \in R$.
- **Associativity**: For every $g_1,g_2,g_3 \in \mathbb{G}$, the equation $g_1\cdot(g_2\cdot g_3) = (g_1\cdot g_2)\cdot g_3$ holds.
- **Distributivity**: For all $g_1,g_2,g_3 \in R$, the distributive law $g_1\cdot(g_2+g_3) = g_1\cdot g_2 + g_1\cdot g_3$ holds.

what are the gs here? copy+paste error from 4.1.0.1?

If $(R,+,\cdot,1)$ is a commutative ring with unit, and $R' \subset R$ is a subset of R such that the restriction of addition and multiplication to R' define a commutative ring with addition $+ : R'\times R' \to R'$, multiplication $\cdot : R'\times R' \to R'$ and unit 1 on R', then $(R',+,\cdot,1)$ is called a **subring** of $(R,+,\cdot,1)$.

Notation and Symbols 9. Since we are exclusively concerned with commutative rings in this book, we often just call them rings, keeping the notation of commutativity implicit. A set R with two maps, $+$ and $\cdot$, which satisfies all previously mentioned rules except for the commutativity law of multiplication, is called a non-commutative ring.

If there is no risk of ambiguity (about what the addition and multiplication maps of a ring are), we frequently drop the symbols $+$ and $\cdot$ and simply write R as notation for the ring, keeping those maps implicit. In this case we also say that R is of ring type, indicating that R is not simply a set but a set together with an addition and a multiplication map.

Example 48 (The ring of integers). The set $\mathbb{Z}$ of integers with the usual addition and multiplication is the archetypical example of a commutative ring with unit 1.

```
sage: ZZ
Integer Ring
```

Example 49 (Underlying commutative group of a ring). Every commutative ring with unit $(R,+,\cdot,1)$ gives rise to a group, if we disregard multiplication.

The following example is somewhat unusual, but we encourage you to think through it because it helps to detach the mind from familiar styles of computation, and concentrate on the abstract algebraic explanation.

Example 50. Let $S := \{\bullet,\star,\odot,\otimes\}$ be a set that contains four elements, and let addition and multiplication on S be defined as follows:

$\cup$	$\bullet$	$\star$	$\odot$	$\otimes$
$\bullet$	$\bullet$	$\star$	$\odot$	$\otimes$
$\star$	$\star$	$\odot$	$\otimes$	$\bullet$
$\odot$	$\odot$	$\otimes$	$\bullet$	$\star$
$\otimes$	$\otimes$	$\bullet$	$\star$	$\odot$

$\circ$	$\bullet$	$\star$	$\odot$	$\otimes$
$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
$\star$	$\bullet$	$\star$	$\odot$	$\otimes$
$\odot$	$\bullet$	$\odot$	$\bullet$	$\odot$
$\otimes$	$\bullet$	$\otimes$	$\odot$	$\star$

(4.28)

Then $(S,\cup,\circ,\star)$ is a ring with unit $\star$ and zero $\bullet$. It therefore makes sense to ask for solutions to equations like the following one:

$$\otimes\circ(x\cup\odot) = \star \tag{4.29}$$

The task here is to find $x \in S$ such that (4.29) holds. To see how such a "moonmath equation" can be solved, we have to keep in mind that rings behave mostly like normal numbers when it comes to bracketing and computation rules. The only differences are the symbols, and the actual way to add and multiply them. With this in mind, we solve the equation for x in the "usual way": [3]

$$
\begin{aligned}
\otimes \circ (x \cup \odot) &= \star && \text{\# apply the distributive law}\\
\otimes \circ x \cup \otimes \circ \odot &= \star && \text{\# } \otimes \circ \odot = \odot\\
\otimes \circ x \cup \odot &= \star && \text{\# concatenate the } \cup \text{ inverse of } \odot \text{ to both sides}\\
\otimes \circ x \cup \odot \cup -\odot &= \star \cup -\odot && \text{\# } \odot \cup -\odot = \bullet\\
\otimes \circ x \cup \bullet &= \star \cup -\odot && \text{\# } \bullet \text{ is the } \cup \text{ neutral element}\\
\otimes \circ x &= \star \cup -\odot && \text{\# for } \cup \text{ we have } -\odot = \odot\\
\otimes \circ x &= \star \cup \odot && \text{\# } \star \cup \odot = \otimes\\
\otimes \circ x &= \otimes && \text{\# concatenate the } \circ \text{ inverse of } \otimes \text{ to both sides}\\
(\otimes)^{-1} \circ \otimes \circ x &= (\otimes)^{-1} \circ \otimes && \text{\# multiply with the multiplicative inverse}\\
\star \circ x &= \star\\
x &= \star
\end{aligned}
$$

So, even though this equation looked really alien at first glance, we could solve it basically exactly the way we solve "normal" equations containing numbers.

So,

Note, however, that whenever a multiplicative inverse is needed to solve an equation in the usual way in a ring, things can be very different than most of us are used to. For example, the following equation cannot be solved for x in the usual way, since there is no multiplicative inverse for $\odot$ in our ring.

$$\odot \circ x = \otimes \tag{4.30}$$

We can confirm this by looking at the multiplication table in (4.28) to see that no such x exits.

As another example, the following equation does not have a single solution but two: $x \in \{\star, \otimes\}$.

$$\odot \circ x = \odot \tag{4.31}$$

Having no solution or two solutions is certainly not something we are used to from types like the rational numbers $\mathbb{Q}$.

Example 51 (Ring of Polynomials). Considering the definition of polynomials from section 3.4 again, we notice that what we have informally called the type R of the coefficients must in fact be a commutative ring with a unit, since we need addition, multiplication, commutativity and the existence of a unit for $R[x]$ to have the properties we expect.

In fact, if we consider R to be a ring and we define addition and multiplication of polynomials as in (3.28), the set $R[x]$ is a commutative ring with a unit, where the polynomial 1 is the multiplicative unit. We call this ring the **ring of polynomials with coefficients in** R.

```
sage: ZZ['x']
```
192

[3]Note that there are more efficient ways to solve this equation. The point of our computation is to show how the axioms of a ring can be used to solve the equation.

```
Univariate Polynomial Ring in x over Integer Ring
```

Example 52 (Ring of modular n arithmetic). Let n be a modulus and $(\mathbb{Z}_n, +, \cdot)$ the set of all remainder classes of integers modulo n, with the projection of integer addition and multiplication as defined in section3.3.4. Then $(\mathbb{Z}_n, +, \cdot)$ is a commutative ring with unit 1.

```
sage: Integers(6)
Ring of integers modulo 6
```

Example 53 (Binary Representations in Modular Arithmetic). TODO (Non unique)

add example

Example 54 (Polynomial evaluation in the exponent of group generators). As we show in section 6.2.3, a key insight in many zero-knowlege protocols is the ability to encode computations as polynomials and then to hide the information of that computation by evaluating the polynomial "in the exponent" of certain cryptographic groups (section 8.2).

To understand the underlying principle of this idea, consider the exponential map (4.1) again. If $\mathbb{G}$ is a finite cyclic group of order n with generator $g \in \mathbb{G}$, then the ring structure of $(\mathbb{Z}_n, +, \cdot)$ corresponds to the group structure of $\mathbb{G}$ in the following way:

$$g^{x+y} = g^x \cdot g^y \qquad g^{x\cdot y} = (g^x)^y \qquad \text{for all } x, y \in \mathbb{Z}_n \tag{4.32}$$

This correspondense allows polynomials with coefficients in $\mathbb{Z}_n$ to be evaluated "in the exponent" of a group generator. To see what this means, let $p \in \mathbb{Z}_n[x]$ be a polynomial with $p(x) = a_m \cdot x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0$, and let $s \in \mathbb{Z}_n$ be an evaluation point. Then the previously defined exponential laws 4.32 imply the following identity:

$$\begin{aligned} g^{p(s)} &= g^{a_m \cdot s^m + a_{m-1}s^{m-1} + \ldots + a_1 s + a_0} \\ &= \left(g^{s^m}\right)^{a_m} \cdot \left(g^{s^{m-1}}\right)^{a_{m-1}} \cdot \ldots \cdot (g^s)^{a_1} \cdot g^{a_0} \end{aligned} \tag{4.33}$$

Utilizing these identities, it is possible to evaluate any polynomial p of degree $deg(p) \leq m$ at a "secret" evaluation point s in the exponent of g without any knowledge about s, assuming that $\mathbb{G}$ is a DL-group. To see this, assume that the set $\{g, g^s, g^{s^2}, \ldots, g^{s^m}\}$ is given, but s is unknown. Then $g^{p(s)}$ can be computed using (4.33), but it is not feasible to compute s.

Example 55. To give an example of the evaluation of a polynomial in the exponent of a finite cyclic group, consider the exponential map from example 36:

$$3^{(\cdot)} : \mathbb{Z}_4 \to \mathbb{Z}_5^* \,;\, x \mapsto 3^x \tag{4.34}$$

Choosing the polynomial $p(x) = 2x^2 + 3x + 1$ from $\mathbb{Z}_4[x]$, we first evaluate the polynomial at the point $s = 2$, and then write the result into the exponent 3 as follows:

$$\begin{aligned} 3^{p(2)} &= 3^{2\cdot 2^2 + 3\cdot 2 + 1} \\ &= 3^{2\cdot 0 + 2 + 1} \\ &= 3^3 \\ &= 2 \end{aligned}$$

This was possible because we had access to the evaluation point 2. On the other hand, if we only had access to the set $\{3, 4, 1\}$ and we knew that this set represents the set $\{3, 3^s, 3^{s^2}\}$ for some secret value s, we could evaluate p at the point s in the exponent of 3 as follows:

$$\begin{aligned} 3^{p(s)} &= 1^2 \cdot 4^3 \cdot 3^1 \\ &= 1 \cdot 4 \cdot 3 \\ &= 2 \end{aligned}$$

Both computations agree, since the secret point s was equal to 2 in this example. However the second evaluation was possible without any knowledge about s.

agree

4.2.1 Hashing into Modular Arithmetic

As we have seen in section 4.1.7, various constructions for hashing to groups are known and used in applications. As commutative rings are commutative groups when we disregard the multiplication, hash-to-group constructions can be applied for hashing into commutative rings. We review some frequently used applications below.

put sub-subsection title here

One of the most widely used applications of hash-into-ring constructions are hash functions that map into the ring $\mathbb{Z}_n$ of modular n arithmetics for some modulus n. Different approaches of constructing such a function are known, but probably the most widely used ones are based on the insight that the images of general hash functions can be interpreted as binary representations of integers, as explained in example 46.

It follows from this interpretation that one simple method of hashing into $\mathbb{Z}_n$ is constructed by observing that if n is a modulus with a bit length (3.4) of $k = |n|$, then every binary string $< b_0, b_1, \ldots, b_{k-2} >$ of length $k-1$ defines an integer z in the rage $0 \le z \le 2^{k-1} - 1 < n$:

$$z = b_0 \cdot 2^0 + b_1 \cdot 2^1 + \ldots + b_{k-2} \cdot 2^{k-2} \tag{4.35}$$

Now, since $z < n$, we know that z is guaranteed to be in the set $\{0, 1, \ldots, n-1\}$, and hence it can be interpreted as an element of $\mathbb{Z}_n$. Consequently, if $H : \{0,1\}^* \to \{0,1\}^{k-1}$ is a hash function, then a hash-to-ring function can be constructed as follows:

$$H_{|n|_2 - 1} : \{0,1\}^* \to \mathbb{Z}_r : \; s \mapsto H(s)_0 \cdot 2^0 + H(s)_1 \cdot 2^1 + \ldots + H(s)_{k-2} \cdot 2^{k-2} \tag{4.36}$$

A drawback of this hash function is that the distribution of the hash values in $\mathbb{Z}_n$ is not necessarily uniform. In fact, if n is larger than 2^{k-1}, then by design $H_{|n|_2-1}$ will never hash onto values $z \ge 2^{k-1}$. Using this hashing method therefore generates approximately uniform hashes only if n is very close to 2^{k-1}. In the worst case, when $n = 2^k - 1$, it misses almost half of all elements from $\mathbb{Z}_n$.

An advantage of this approach is that properties like preimage resistance or collision resistance (see section 4.1.7.1) of the original hash function $H(\cdot)$ are preserved.

Example 56. To analyze a particular implementation of a $H_{|n|_2-1}$ hash function, we use a 5-bit truncation of the *SHA*256 hash from example 45 and define a hash into $\mathbb{Z}_{16}$ as follows:

$$H_{|16|_2-5} : \{0,1\}^* \to \mathbb{Z}_{16} : \; s \mapsto SHA256(s)_0 \cdot 2^0 + SHAH256(s)_1 \cdot 2^1 + \ldots + SHA256(s)_4 \cdot 2^4$$

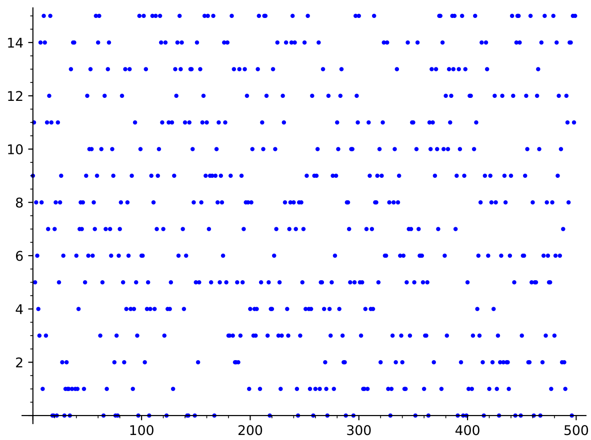
Since $k = |16|_2 = 5$ and $16 - 2^{k-1} = 0$, this hash maps uniformly onto $\mathbb{Z}_{16}$. We can use Sage to implement it:

```
sage: import hashlib
```

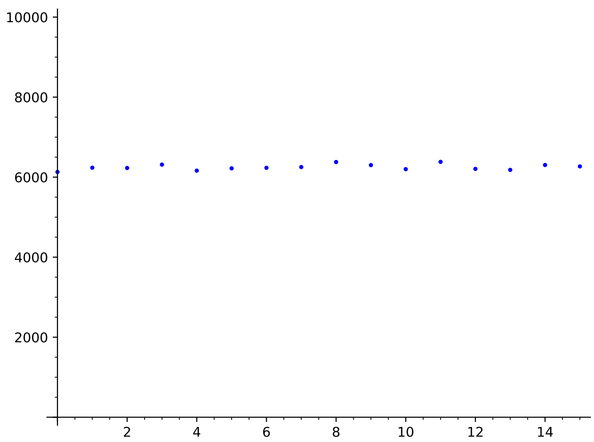
196

```
sage: def Hash5(x):
....:     Z16 = Integers(16)
....:     hasher = hashlib.sha256(x) # compute SHA56
....:     digest = hasher.hexdigest()
....:     d = ZZ(digest, base=16) # cast to integer
....:     d = d.str(2)[-4:] # keep 5 least significant bits
....:     d = ZZ(d, base=2) # cast to integer
....:     return Z16(d) # cast to Z16
sage: Hash5(b'')
5
```

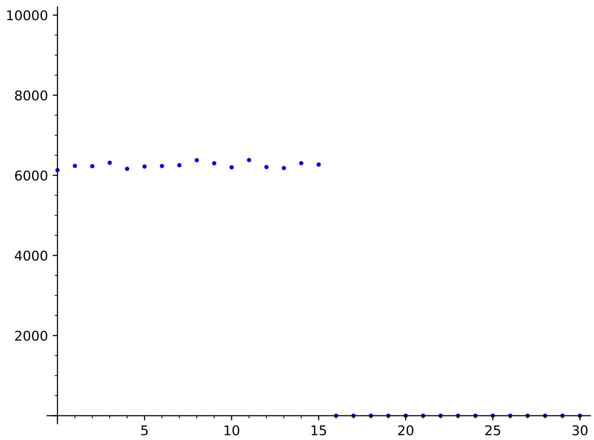
We can then use Sage to apply this function to a large set of input values in order to plot a visualization of the distribution over the set $\{0,\dots,15\}$. Executing over 500 input values gives the following plot:



To get an intuition of uniformity, we can count the number of times the hash function $H_{|16|_2-1}$ maps onto each number in the set $\{0,1,\dots,15\}$ in a loop of 100000 hashes, and compare that to the ideal uniform distribution, which would map exactly 6250 samples to each element. This gives the following result:



The lack of uniformity becomes apparent if we want to construct a similar hash function for $\mathbb{Z}_n$ for any other 5 bit integer n in the range $17 \le n \le 31$. In this case, the definition of the hash function is exactly the same as for $\mathbb{Z}_{16}$, and hence, the images will not exceed the value 15. So, for example, even in the case of hashing to $\mathbb{Z}_{31}$, the hash function never maps to any value larger than 15, leaving almost half of all numbers out of the image range.



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A second widely used method of hashing into $\mathbb{Z}_n$ is constructed by observing the following: If n is a modulus with a bit-length of $|n|_2 = k_1$, and $H : \{0,1\}^* \to \{0,1\}^{k_2}$ is a hash function that produces digests of size k_2, and $k_2 \geq k_1$, then a hash-to-ring function can be constructed by interpreting the image of H as a binary representation of an integer, and then taking the modulus by n to map into $\mathbb{Z}_n$:.

$$H'_{mod_n} : \{0,1\}^* \to \mathbb{Z}_n : \; s \mapsto \left(H(s)_0 \cdot 2^0 + H(s)_1 \cdot 2^1 + \ldots + H(s)_{k_2} \cdot 2^{k_2} \right) \bmod n \tag{4.37}$$

A drawback of this hash function is that computing the modulus requires some computational effort. In addition, the distribution of the hash values in $\mathbb{Z}_n$ might not be uniform, depending on the number $2^{k_2+1} \bmod n$. An advantage of this function is that potential properties of the original hash function $H(\cdot)$ (like preimage resistance or collision resistance) are preserved, and the distribution can be made almost uniform, with only negligible bias depending on what modulus n and images size k_2 are chosen.

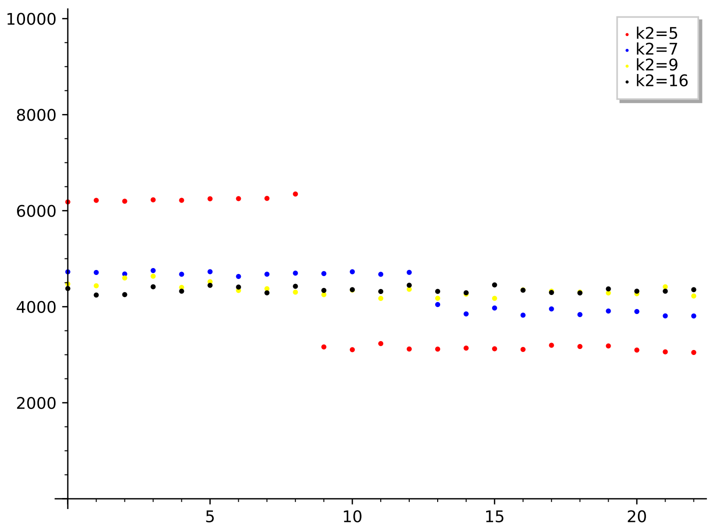
Example 57. To give an implementation of the H_{mod_n} hash function, we use k_2-bit truncation of the $SHA256$ hash from example 45, and define a hash into $\mathbb{Z}_{23}$ as follows:

$$\begin{aligned} &H_{mod_{23},k_2} : \{0,1\}^* \to \mathbb{Z}_{23} : \\ &\qquad s \mapsto \left(SHA256(s)_0 \cdot 2^0 + SHAH256(s)_1 \cdot 2^1 + \ldots + SHA256(s)_{k_2} \cdot 2^{k_2} \right) \bmod 23 \end{aligned}$$

We want to use various instantiations of k_2 to visualize the impact of truncation length on the distribution of the hashes in $\mathbb{Z}_{23}$. We can use Sage to implement it as follows:

```
sage: import hashlib
sage: Z23 = Integers(23)
sage: def Hash_mod23(x, k2):
....:     hasher = hashlib.sha256(x.encode('utf-8')) # Compute
    SHA256
....:     digest = hasher.hexdigest()
....:     d = ZZ(digest, base=16) # cast to integer
....:     d = d.str(2)[-k2:] # keep k2+1 LSB
....:     d = ZZ(d, base=2) # cast to integer
....:     return Z23(d) # cast to Z23
```

We can then use Sage to apply this function to a large set of input values in order to plot visualizations of the distribution over the set $\{0,\ldots,22\}$ for various values of k_2, by counting the number of times it maps onto each number in a loop of 100000 hashes. We get the following plot:



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A third method that can sometimes be found in implementations is the so-called **"try-and-increment" method**. To understand this method, we define an integer $z \in \mathbb{Z}$ from any hash value $H(s)$ as we did in the previous methods:

$$z = H(s)_0 \cdot 2^0 + H(s)_1 \cdot 2^1 + \ldots + H(s)_{k-1} \cdot 2^k \tag{4.38}$$

Hashing into $\mathbb{Z}_n$ is then achievable by first computing z, and then trying to see if $z \in \mathbb{Z}_n$. If it is, then the hash is done; if not, the string s is modified in a deterministic way and the process is repeated until a suitable element $z \in \mathbb{Z}_n$ is found. A suitable, deterministic modification could be to concatenate the original string by some bit counter. A "try-and-increment" algorithm would then work like in algorithm 6.

Algorithm 6 Hash-to-$\mathbb{Z}_n$

Require: $n \in \mathbb{Z}$ with $|n|_2 = k$ and $s \in \{0,1\}^*$
  procedure TRY-AND-INCREMENT(n,k,s)
    $c \leftarrow 0$
    repeat
      $s' \leftarrow s||c\_bits()$
      $z \leftarrow H(s')_0 \cdot 2^0 + H(s')_1 \cdot 2^1 + \ldots + H(s')_k \cdot 2^k$
      $c \leftarrow c+1$
    until $z < n$
    return x
  end procedure
Ensure: $z \in \mathbb{Z}_n$

Depending on the parameters, this method can be very efficient. In fact, if k is sufficiently large and n is close to 2^{k+1}, the probability for $z < n$ is very high, and the repeat loop will almost always be executed a single time only. A drawback is, however, that the probability of having to execute the loop multiple times is not zero.

4.3 Fields

We started this chapter with the definition of a group (section 4.1), which we then expanded into the definition of a commutative ring with a unit (section4.2). These types of rings generalize the behavior of integers. In this section, we look at those special cases of commutative rings where every element other than the neutral element of addition has a multiplicative inverse. Those structures behave very much like the set of rational numbers $\mathbb{Q}$. Rational numbers are, in a sense, an extension of the ring of integers, that is, they are constructed by including newly defined multiplicative inverses (fractions) to the integers. Fields are defined as follows:

Definition 4.3.0.1 (Field). A **field** $(\mathbb{F},+,\cdot)$ is a set $\mathbb{F}$ with two maps $+ : \mathbb{F}\times\mathbb{F}\to\mathbb{F}$ and $\cdot : \mathbb{F}\times\mathbb{F}\to\mathbb{F}$ called **addition** and **multiplication**, such that the following conditions hold:

- $(\mathbb{F},+)$ is a commutative group, where the neutral element is denoted by 0.
- $(\mathbb{F}\setminus\{0\},\cdot)$ is a commutative group, where the neutral element is denoted by 1.
- (Distributivity) The equation $g_1\cdot(g_2+g_3) = g_1\cdot g_2 + g_1\cdot g_3$ holds for all $g_1,g_2,g_3\in\mathbb{F}$.

If $(\mathbb{F},+,\cdot)$ is a field and $\mathbb{F}'\subset\mathbb{F}$ is a subset of $\mathbb{F}$ such that the restriction of addition and multiplication to $\mathbb{F}'$ define a field with addition $+ : \mathbb{F}'\times\mathbb{F}'\to\mathbb{F}'$ and multiplication $\cdot : \mathbb{F}'\times\mathbb{F}'\to\mathbb{F}'$ on $\mathbb{F}'$, then $(\mathbb{F}',+,\cdot)$ is called a **subfield** of $(\mathbb{F},+,\cdot)$ and $(\mathbb{F},+,\cdot)$ is called an **extension field** of $(\mathbb{F}',+,\cdot)$.

Notation and Symbols 10. If there is no risk of ambiguity (about what the addition and multiplication maps of a field are), we frequently omit the symbols $+$ and $\cdot$, and simply write $\mathbb{F}$ as notation for a field, keeping maps implicit. In this case, we also say that $\mathbb{F}$ is of field type, indicating that $\mathbb{F}$ is not simply a set but a set with an addition and a multiplication map that satisfies the definition of a field (4.3.0.1).

We call $(\mathbb{F},+)$ the **additive group** of the field. We use the notation $\mathbb{F}^* := \mathbb{F}\setminus\{0\}$ for the set of all elements excluding the neutral element of addition, called $(\mathbb{F}^*,\cdot)$ the **multiplicative group** of the field.

The **characteristic** of a field $\mathbb{F}$, represented as $char(\mathbb{F})$, is the smallest natural number $n\geq 1$ for which the n-fold sum of the multiplicative neutral element 1 equals zero, i.e. for which $\sum_{i=1}^{n} 1 = 0$. If such an $n>0$ exists, the field is said to have a **finite characteristic**. If, on the other hand, every finite sum of 1 is such that it is not equal to zero, then the field is defined to have characteristic 0. S: Tried to disambiguate the scope of negation between 1. "It is true of every finite sum of 1 that it is not equal to 0" and 2. "It is not true of every finite sum of 1 that it is equal to 0" From the example below, it looks like 1. is the intended meaning here, correct?

Check change of wording

Example 58 (Field of rational numbers). Probably the best known example of a field is the set of rational numbers $\mathbb{Q}$ together with the usual definition of addition, subtraction, multiplication and division. Since there is no natural number $n\in\mathbb{N}$ such that $\sum_{j=0}^{n} 1 = 0$ in the set of rational numbers, the characteristic of the field $\mathbb{Q}$ is given by $char(\mathbb{Q}) = 0$.

```
sage: QQ
Rational Field
```

Example 59 (Field with two elements). It can be shown that, in any field, the neutral element of addition 0 must be different from the neutral element of multiplication 1, that is, $0\neq 1$ always holds in a field. This means that the smallest field must contain at least two elements. As the following addition and multiplication tables show, there is indeed a field with two elements, which is usually called $\mathbb{F}_2$:

Let $\mathbb{F}_2 := \{0,1\}$ be a set that contains two elements, and let addition and multiplication on $\mathbb{F}_2$ be defined as follows:

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array} \qquad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \tag{4.39}$$

Since $1+1=0$ in the field $\mathbb{F}_2$, we know that the characteristic of $\mathbb{F}_2$ given by $char(\mathbb{F}_2)=2$. The multiplicative subgroup $\mathbb{F}_2^*$ of $\mathbb{F}_2$ is given by the trivial group $\{1\}$.

```
sage: F2 = GF(2)
sage: F2(1) # Get an element from GF(2)
1
sage: F2(1) + F2(1) # Addition
0
sage: F2(1) / F2(1) # Division
1
```

Exercise 46. Consider the ring of modular 5 arithmetics $(\mathbb{Z}_5,+,\cdot)$ from example 14. Show that $(\mathbb{Z}_5,+,\cdot)$ is a field. What is the characteristic of $\mathbb{Z}_5$? Prove that the equation $a\cdot x=b$ has only a single solution $x\in\mathbb{Z}_5$ for any given $a,b\in\mathbb{Z}_5^*$.

Exercise 47. Consider the ring of modular 6 arithmetics $(\mathbb{Z}_6,+,\cdot)$ from example 9. Show that $(\mathbb{Z}_6,+,\cdot)$ is not a field.

4.3.1 Prime fields

As we have seen in many of the examples in previous sections, modular arithmetic behaves similarly to the ordinary arithmetics of integers in many ways. This is due to the fact that remainder class sets $\mathbb{Z}_n$ are commutative rings with units (see example 52).

However, we have also seen in example 36 that, whenever the modulus is a prime number, every remainder class other than the zero class has a modular multiplicative inverse. This is an important observation, since it immediately implies that, in case the modulus is a prime number, the remainder class set $\mathbb{Z}_n$ is not just a ring but actually a **field**. Moreover, since $\sum_{j=0}^{n} 1 = 0$ in $\mathbb{Z}_n$, we know that those fields have the finite characteristic n.

Notation and Symbols 11 (Prime Fields). Let $p\in\mathbb{P}$ be a prime number and $(\mathbb{Z}_p,+,\cdot)$ the ring of modular p arithmetics (see example 52). To distinguish prime fields from arbitrary modular arithmetic rings, we write $(\mathbb{F}_p,+,\cdot)$ for the ring of modular p arithmetics and call it the **prime field** of characteristic p.

Prime fields are the foundation of many of the contemporary algebra-based cryptographic systems, as they have a number of desirable properties. One of these is that any prime field of characteristic p contains exactly p elements, which can be represented on a computer with not more than $log_2(p)$ many bits. On the other hand, fields like rational numbers require a potentially unbounded amount of bits for any full-precision representation.

Since prime fields are special cases of modular arithmetic rings, addition and multiplication can be computed by first doing normal integer addition and multiplication, and then considering the remainder in Euclidean division by p as the result. For any prime field element $x\in\mathbb{F}_p$, its additive inverse (the negative) is given by $-x=p-x \bmod p$. For $x\neq 0$, the multiplicative inverse always exists, and is given by $x^{-1}=x^{p-2}$. Division is then defined by multiplication with the multiplicative inverse, as explained in section 3.3.5. Alternatively, the multiplicative inverse can be computed using the Extended Euclidean Algorithm as explained in (3.23).

Example 60. The smallest possible field is the field $\mathbb{F}_2$ of characteristic 2, as we have seen in example 59. It is the prime field of the prime number 2.

Example 61. The field $\mathbb{F}_5$ from example 14 is a prime field, as defined by its addition and multiplication table (3.24).

Example 62. To summarize the basic aspects of computation in prime fields, let us consider the prime field $\mathbb{F}_5$ (example 14) and simplify the following expression:

$$\left(\frac{2}{3}-2\right)\cdot 2 \tag{4.40}$$

The first thing to note is that, since $\mathbb{F}_5$ is a field, all rules are identical to the rules we learned in school when we where dealing with rational, real or complex numbers. This means we can use methods like bracketing (distributivity) or addition as usual. For ease of computation, we can consult the addition and multiplication tables in (3.24).

$$\begin{aligned}
\left(\frac{2}{3}-2\right)\cdot 2 &= \frac{2}{3}\cdot 2 - 2\cdot 2 && \text{\# distributive law} \\
&= \frac{2\cdot 2}{3} - 2\cdot 2 && 4 \bmod 5 = 4 \\
&= \frac{4}{3} - 4 && \text{\# multiplicative inverse of 3 is } 3^{5-2} \bmod 5 = 2 \\
&= 4\cdot 2 - 4 && \text{\# additive inverse of 4 is } 5-4=1 \\
&= 4\cdot 2 + 1 && 8 \bmod 5 = 3 \\
&= 3+1 && 4 \bmod 5 = 4 \\
&= 4
\end{aligned}$$

In this example, we computed the multiplicative inverse of 3 using the identity $x^{-1} = x^{p-2}$ in a prime field. This is impractical for large prime numbers. Recall that another way of computing the multiplicative inverse is the Extended Euclidean Algorithm (see 3.13). To refresh our memory, the algorithm solves the equation $x^{-1}\cdot 3 + t\cdot 5 = 1$, for x^{-1} (even though t is irrelevant in this case). We get the following:

$$\begin{array}{c|ccc}
\text{k} & r_k & x_k^{-1} & t_k \\
\hline
0 & 3 & 1 & \cdot \\
1 & 5 & 0 & \cdot \\
2 & 3 & 1 & \cdot \\
3 & 2 & -1 & \cdot \\
4 & 1 & 2 & \cdot
\end{array} \tag{4.41}$$

So the multiplicative inverse of 3 in $\mathbb{Z}_5$ is 2, and, indeed, if we compute the product of 3 with its multiplicative inverse 2, we get the neutral element 1 in $\mathbb{F}_5$.

Exercise 48 (Prime field $\mathbb{F}_3$). Construct the addition and multiplication table of the prime field $\mathbb{F}_3$.

Exercise 49 (Prime field $\mathbb{F}_{13}$). Construct the addition and multiplication table of the prime field $\mathbb{F}_{13}$.

Exercise 50. Consider the prime field $\mathbb{F}_{13}$ from exercise 49. Find the set of all pairs $(x,y) \in \mathbb{F}_{13}\times\mathbb{F}_{13}$ that satisfy the following equation:

$$x^2 + y^2 = 1 + 7\cdot x^2 \cdot y^2 \tag{4.42}$$

4.3.2 Square Roots

As we know from integer arithmetics, some integers, like 4 or 9, are squares of other integers: for example, $4 = 2^2$ and $9 = 3^2$. However, we also know that not all integers are squares of other integers: for example, there is no integers $x \in \mathbb{Z}$ such that $x^2 = 2$. If an integer a is square of another integer b, then it make sense to define the square root of a to be b.

In the context of prime fields, an element that is a square of another element is also called a **quadratic residue**, and an element that is not a square of another element is called a **quadratic non-residue**. This distinction is of particular importance in our studies on elliptic curves (chapter 5), as only square numbers can actually be points on an elliptic curve.

To make the intuition of quadratic residues and their roots precise, we give the following definition:

Definition 4.3.2.1. let $p \in \mathbb{P}$ be a prime number and $\mathbb{F}_p$ its associated prime field. Then a number $x \in \mathbb{F}_p$ is called a **square root** of another number $y \in \mathbb{F}_p$, if x is a solution to the following equation:

$$x^2 = y \tag{4.43}$$

In this case, y is called a **quadratic residue**. On the other hand, if y is given and the quadratic equation has no solution x , we call y a **quadratic non-residue**.

For any $y \in \mathbb{F}_p$, we denote the set of all square roots of y in the prime field $\mathbb{F}_p$ as follows:

$$\sqrt{y} := \{x \in \mathbb{F}_p \mid x^2 = y\} \tag{4.44}$$

Informally speaking, quadratic residues are numbers that have a square root, while quadratic non-residues are numbers that don't have square roots. The situation therefore parallels the familiar case of integers, where some integers like 4 or 9 have a square root, and others like 2 or 3 don't (within the set of integers).

If y is a quadratic non-residue, then $\sqrt{y} = \emptyset$ (an empty set), and if $y = 0$, then $\sqrt{y} = \{0\}$.

Moreover if $y \neq 0$ is a quadratic residue, then it has precisely two roots $\sqrt{y} = \{x, p-x\}$ for some $x \in \mathbb{F}_p$. We adopt the convention to call the smaller one (when interpreted as an integer) the **positive square root** and the larger one the **negative square root**.

If $p \in \mathbb{P}_{\geq 3}$ is an odd prime number with associated prime field $\mathbb{F}_p$, then there are precisely $(p+1)/2$ many quardratic residues and $(p-1)/2$ quadratic non-residues.

Example 63 (Quadratic residues and roots in $\mathbb{F}_5$). Let us consider the prime field $\mathbb{F}_5$ from example 14 again. All square numbers can be found on the main diagonal of the multiplication table in (3.24). As you can see, in $\mathbb{F}_5$, only the numbers 0, 1 and 4 have square roots: $\sqrt{0} = \{0\}$, $\sqrt{1} = \{1,4\}$, $\sqrt{2} = \emptyset$, $\sqrt{3} = \emptyset$ and $\sqrt{4} = \{2,3\}$. The numbers 0, 1 and 4 are therefore quadratic residues, while the numbers 2 and 3 are quadratic non-residues.

In order to describe whether an element of a prime field is a square number or not, the so-called **Legendre symbol** can sometimes be found in the literature, defined as follows:

Let $p \in \mathbb{P}$ be a prime number and $y \in \mathbb{F}_p$ an element from the associated prime field. Then the *Legendre symbol* of y is defined as follows:

$$\left(\frac{y}{p}\right) := \begin{cases} 1 & \text{if } y \text{ has square roots} \\ -1 & \text{if } y \text{ has no square roots} \\ 0 & \text{if } y = 0 \end{cases} \tag{4.45}$$

Example 64. Looking at the quadratic residues and non-residues in $\mathbb{F}_5$ from example 14 again, we can deduce the following Legendre symbols based on example 63.

$$\left(\tfrac{0}{5}\right) = 0, \quad \left(\tfrac{1}{5}\right) = 1, \quad \left(\tfrac{2}{5}\right) = -1, \quad \left(\tfrac{3}{5}\right) = -1, \quad \left(\tfrac{4}{5}\right) = 1\ .$$

The Legendre symbol provides a criterion to decide whether or not an element from a prime field has a quadratic root or not. This, however, is not just of theoretical use: the so-called **Euler criterion** provides a compact way to actually compute the Legendre symbol. To see that, let $p \in \mathbb{P}_{\geq 3}$ be an odd prime number and $y \in \mathbb{F}_p$. Then the Legendre symbol can be computed as follows:

$$\left(\frac{y}{p}\right) = y^{\frac{p-1}{2}} \tag{4.46}$$

Example 65. Looking at the quadratic residues and non-residues in $\mathbb{F}_5$ from example 63 again, we can compute the following Legendre symbols using the Euler criterion:

$$\begin{aligned}
\left(\frac{0}{5}\right) &= 0^{\frac{5-1}{2}} = 0^2 = 0 \\
\left(\frac{1}{5}\right) &= 1^{\frac{5-1}{2}} = 1^2 = 1 \\
\left(\frac{2}{5}\right) &= 2^{\frac{5-1}{2}} = 2^2 = 4 = -1 \\
\left(\frac{3}{5}\right) &= 3^{\frac{5-1}{2}} = 3^2 = 4 = -1 \\
\left(\frac{4}{5}\right) &= 4^{\frac{5-1}{2}} = 4^2 = 1
\end{aligned}$$

Exercise 51. Consider the prime field $\mathbb{F}_{13}$ from exercise 49. Compute the Legendre symbol $\left(\frac{x}{13}\right)$ and the set of roots $\sqrt{x}$ for all elements $x \in \mathbb{F}_{13}$.

4.3.2.1 Hashing into prime fields

An important problem in cryptography is the ability to hash to (various subsets) of elliptic curves. As we will see in chapter 5, those curves are often defined over prime fields, and hashing to a curve might start with hashing to the prime field. It is therefore important to understand how to hash into prime fields.

In section 4.2.1, we looked at a few methods of hashing into the modular arithmetic rings $\mathbb{Z}_n$ for arbitrary $n > 1$. As prime fields are just special instances of those rings, all methods for hashing into $\mathbb{Z}_n$ functions can be used for hashing into prime fields, too.

4.3.3 Prime Field Extensions

Prime fields, as defined in the previous section, are basic building blocks of cryptography. However, as we will see in chapter 8, so-called pairing-based SNARK systems are crucially dependent on certain group pairings (4.9) defined on elliptic curves over **prime field extensions**. In this section, we therefore introduce those extensions.

Given some prime number $p \in \mathbb{P}$, a natural number $m \in \mathbb{N}$, and an irreducible polynomial $P \in \mathbb{F}_p[x]$ of degree m with coefficients from the prime field $\mathbb{F}_p$, a prime field extension $(\mathbb{F}_{p^m}, +, \cdot)$ is defined as follows.

The set $\mathbb{F}_{p^m}$ of the prime field extension is given by the set of all polynomials with a degree less than m:

$$\mathbb{F}_{p^m} := \{a_{m-1}x^{m-1} + a_{k-2}x^{k-2} + \ldots + a_1 x + a_0 \mid a_i \in \mathbb{F}_p\} \tag{4.47}$$

The addition law of the prime field extension $\mathbb{F}_{p^m}$ is given by the usual addition of polynomials as defined in (3.28):

$$+ : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{F}_{p^m} \, , (\textstyle\sum_{j=0}^{m} a_j x^j, \sum_{j=0}^{m} b_j x^j) \mapsto \sum_{j=0}^{m} (a_j + b_j) x^j \tag{4.48}$$

The multiplication law of the prime field extension $\mathbb{F}_{p^m}$ is given by first multiplying the two polynomials as defined in (3.29), then dividing the result by the irreducible polynomial p and keeping the remainder:

$$\cdot : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \to \mathbb{F}_{p^m} \, , \, (\textstyle\sum_{j=0}^{m} a_j x^j, \sum_{j=0}^{m} b_j x^j) \mapsto \left(\sum_{n=0}^{2m} \sum_{i=0}^{n} a_i b_{n-i} x^n \right) \bmod P \tag{4.49}$$

The neutral element of the additive group $(\mathbb{F}_{p^m}, +)$ is given by the zero polynomial 0. The additive inverse is given by the polynomial with all negative coefficients. The neutral element of the multiplicative group $(\mathbb{F}^*_{p^m}, \cdot)$ is given by the unit polynomial 1. The multiplicative inverse can be computed by the Extended Euclidean Algorithm (see 3.13).

check reference

We can see from the definition of $\mathbb{F}_{p^m}$ that the field is of characteristic p, since the multiplicative neutral element 1 is equivalent to the multiplicative element 1 from the underlying prime field, and hence $\sum_{j=0}^{p} 1 = 0$. Moreover, $\mathbb{F}_{p^m}$ is finite and contains p^m many elements, since elements are polynomials of degree $< m$, and every coefficient a_j can have p many different values. In addition, we see that the prime field $\mathbb{F}_p$ is a subfield of $\mathbb{F}_{p^m}$ that occurs when we restrict the elements of $\mathbb{F}_{p^m}$ to polynomials of degree zero.

One key point is that the construction of $\mathbb{F}_{p^m}$ depends on the choice of an irreducible polynomial, and, in fact, different choices will give different multiplication tables, since the remainders from dividing a polynomial product by those polynomials will be different.

It can, however, be shown that the fields for different choices of P are **isomorphic**, which means that there is a one-to-one correspondence between all of them. As a result, from an abstract point of view, they are the same thing. From an implementations point of view, however, some choices are preferable to others because they allow for faster computations.

Remark 3. Similarly to the way prime fields $\mathbb{F}_p$ are generated by starting with the ring of integers and then dividing by a prime number p and keeping the remainder, prime field extensions $\mathbb{F}_{p^m}$ are generated by starting with the ring $\mathbb{F}_p[x]$ of polynomials and then dividing them by an irreducible polynomial of degree m and keeping the remainder.

In fact, it can be shown that $\mathbb{F}_{p^m}$ is the set of all remainders when dividing any polynomial $Q \in \mathbb{F}_p[x]$ by an irreducible polynomial P of degree m. This is analogous to how $\mathbb{F}_p$ is the set of all remainders when dividing integers by p.

Any field $\mathbb{F}_{p^m}$ constructed in the above manner is a field extension of $\mathbb{F}_p$. To be more general, a field $\mathbb{F}_{p^{m_2}}$ is a field extension of a field $\mathbb{F}_{p^{m_1}}$ if and only if m_1 divides m_2. From this, we can deduce that, for any given fixed prime number, there are nested sequences of subfields whenever the power m_j divides the power m_{j+1}:

$$\mathbb{F}_p \subset \mathbb{F}_{p^{m_1}} \subset \cdots \subset \mathbb{F}_{p^{m_k}} \tag{4.50}$$

To get a more intuitive picture of this, we construct an extension field of the prime field $\mathbb{F}_3$ in the following example, and we can see how $\mathbb{F}_3$ sits inside that extension field.

Example 66 (The Extension field $\mathbb{F}_{3^2}$). In exercise 48, we have constructed the prime field $\mathbb{F}_3$. In this example, we apply the definition of a field extension (4.47) to construct the extension field $\mathbb{F}_{3^2}$. We start by choosing an irreducible polynomial of degree 2 with coefficients in $\mathbb{F}_3$.

We try $P(t) = t^2 + 1$. Possibly the fastest way to show that P is indeed irreducible is to just insert all elements from $\mathbb{F}_3$ to see if the result is ever zero. We compute as follows:

$$\begin{aligned}
P(0) &= 0^2 + 1 = 1 \\
P(1) &= 1^2 + 1 = 2 \\
P(2) &= 2^2 + 1 = 1 + 1 = 2
\end{aligned}$$

This implies that P is irreducible, since all factors must be of the form $(t - a)$ for a being a root of P. The set $\mathbb{F}_{3^2}$ contains all polynomials of degrees lower than two with coefficients in $\mathbb{F}_3$, which are precisely as listed below:

$$\mathbb{F}_{3^2} = \{0, 1, 2, t, t+1, t+2, 2t, 2t+1, 2t+2\} \tag{4.51}$$

As expected, our extension field contains 9 elements. Addition is defined as addition of polynomials; for example $(t + 2) + (2t + 2) = (1 + 2)t + (2 + 2) = 1$. Doing this computation for all elements gives the following addition table

+	0	1	2	t	t+1	t+2	2t	2t+1	2t+2
0	0	1	2	t	t+1	t+2	2t	2t+1	2t+2
1	1	2	0	t+1	t+2	t	2t+1	2t+2	2t
2	2	0	1	r+2	t	t+1	2t+2	2t	2t+1
t	t	t+1	t+2	2t	2t+1	2t+2	0	1	2
t+1	t+1	t+2	t	2t+1	2t+2	2t	1	2	0
t+2	t+2	t	t+1	2t+2	2t	2t+1	2	0	1
2t	2t	2t+1	2t+2	0	1	2	t	t+1	t+2
2t+1	2t+1	2t+2	2t	1	2	0	t+1	t+2	t
2t+2	2t+2	2t	2t+1	2	0	1	t+2	t	t+1

(4.52)

As we can see, the group $(\mathbb{F}_3, +)$ is a subgroup of the group $(\mathbb{F}_{3^2}, +)$, obtained by only considering the first three rows and columns of this table.

We can use the addition table (4.52) to deduce the additive inverse (the negative) of any element from $\mathbb{F}_{3^2}$. For example, in $\mathbb{F}_{3^2}$ we have $-(2t + 1) = t + 2$, since $(2t + 1) + (t + 2) = 0$.

Multiplication needs a bit more computation, as we first have to multiply the polynomials, and whenever the result has a degree ≥ 2, we have to apply a polynomial division algorithm (algorithm 3) to divide the product by the polynomial P and keep the remainder. To see how this works, let us compute the product of $t + 2$ and $2t + 2$ in $\mathbb{F}_{3^2}$:

$$\begin{aligned}
(t+2) \cdot (2t+2) &= (2t^2 + 2t + t + 1) \bmod (t^2 + 1) \\
&= (2t^2 + 1) \bmod (t^2 + 1) && \# \, 2t^2 + 1 : t^2 + 1 = 2 + \frac{2}{t^2 + 1} \\
&= 2
\end{aligned}$$

This means that the product of $t + 2$ and $2t + 2$ in $\mathbb{F}_{3^2}$ is 2. Performing this computation for all elements gives the following multiplication table:

·	0	1	2	t	t+1	t+2	2t	2t+1	2t+2
0	0	0	0	0	0	0	0	0	0
1	0	1	2	t	t+1	t+2	2t	2t+1	2t+2
2	0	2	1	2t	2t+2	2t+1	t	t+2	t+1
t	0	t	2t	2	t+2	2t+2	1	t+1	2t+1
t+1	0	t+1	2t+2	t+2	2t	1	2t+1	2	t
t+2	0	t+2	2t+1	2t+2	1	t	t+1	2t	2
2t	0	2t	t	1	2t+1	t+1	2	2t+2	t+2
2t+1	0	2t+1	t+2	t+1	2	2t	2t+2	t	1
2t+2	0	2t+2	t+1	2t+1	t	2	t+2	1	2t

(4.53)

As was the case in previous examples, we can use the table (4.53) to deduce the multiplicative inverse of any non-zero element from $\mathbb{F}_{3^2}$. For example, in $\mathbb{F}_{3^2}$ we have $(2t+1)^{-1} = 2t+2$, since $(2t+1)\cdot(2t+2) = 1$.

Looking at the multiplication table (4.53), we can also see that the only quadratic residues in $\mathbb{F}_{3^2}$ are from the set $\{0,1,2,t,2t\}$, with $\sqrt{0} = \{0\}$, $\sqrt{1} = \{1,2\}$, $\sqrt{2} = \{t,2t\}$, $\sqrt{t} = \{t+2,2t+1\}$ and $\sqrt{2t} = \{t+1,2t+2\}$.

Since $\mathbb{F}_{3^2}$ is a field, we can solve equations as we would for other fields (such as rational numbers). To see that, let us find all $x \in \mathbb{F}_{3^2}$ that solve the quadratic equation $(t+1)(x^2+(2t+2)) = 2$. We compute as follows:

$$
\begin{aligned}
(t+1)(x^2+(2t+2)) &= 2 && \text{\# 2 distributive law}\\
(t+1)x^2+(t+1)(2t+2) &= 2\\
(t+1)x^2+(t) &= 2 && \text{\# 2 add the additive inverse of } t\\
(t+1)x^2+(t)+(2t) &= (2)+(2t)\\
(t+1)x^2 &= 2t+2 && \text{\# multiply with the multiplicative invers of } t+1\\
(t+2)(t+1)x^2 &= (t+2)(2t+2) && \text{\# multiply with the multiplicative invers of } t+1\\
x^2 &= 2 && \text{\# 2 is quadratic residue. Take the roots.}\\
x &\in \{t,2t\}
\end{aligned}
$$

Computations in extension fields are arguably on the edge of what can reasonably be done with pen and paper. Fortunately, Sage provides us with a simple way to do these computations.

```
sage: Z3 = GF(3) # prime field
sage: Z3t.<t> = Z3[] # polynomials over Z3
sage: P = Z3t(t^2+1)
sage: P.is_irreducible()
True
sage: F3_2.<t> = GF(3^2, name='t', modulus=P) # Extension
   field F_3^2
sage: F3_2
Finite Field in t of size 3^2
sage: F3_2(t+2)*F3_2(2*t+2) == F3_2(2)
True
sage: F3_2(2*t+2)^(-1) # multiplicative inverse
2*t + 1
```

```
sage: # verify our solution to (t+1)(x^2 + (2t+2)) = 2
sage: F3_2(t+1)*(F3_2(t)**2 + F3_2(2*t+2)) == F3_2(2)
True
sage: F3_2(t+1)*(F3_2(2*t)**2 + F3_2(2*t+2)) == F3_2(2)
True
```

Exercise 52. Consider the extension field $\mathbb{F}_{3^2}$ from the previous example and find all pairs of elements $(x,y) \in \mathbb{F}_{3^2}$, for which the following equation holds:

$$y^2 = x^3 + 4 \tag{4.54}$$

Exercise 53. Show that the polynomial $Q = x^2 + x + 2$ from $\mathbb{F}_3[x]$ is irreducible. Construct the multiplication table of $\mathbb{F}_{3^2}$ with respect to Q and compare it to the multiplication table of $\mathbb{F}_{3^2}$ from example 66.

Exercise 54. Show that the polynomial $P = t^3 + t + 1$ from $\mathbb{F}_5[t]$ is irreducible. Then consider the extension field $\mathbb{F}_{5^3}$ defined relative to P. Compute the multiplicative inverse of $(2t^2 + 4) \in \mathbb{F}_{5^3}$ using the Extended Euclidean Algorithm. Then find all $x \in \mathbb{F}_{5^3}$ that solve the following equation:

$$(2t^2 + 4)(x - (t^2 + 4t + 2)) = (2t + 3) \tag{4.55}$$

Exercise 55. Consider the prime field $\mathbb{F}_5$. Show that the polynomial $P = x^2 + 2$ from $\mathbb{F}_5[x]$ is irreducible. Implement the finite field $\mathbb{F}_{5^2}$ in Sage.

4.4 Projective Planes

Projective planes are particular geometric constructs defined over a given field. In a sense, projective planes extend the concept of the ordinary Euclidean plane by including "points at infinity."

To understand the idea of constructing of projective planes, note that, in an ordinary Euclidean plane, two lines either intersect in a single point or are parallel. In the latter case, both lines are either the same, that is, they intersect in all points, or do not intersect at all. A projective plane can then be thought of as an ordinary plane, but equipped with an additional "point at infinity" such that two different lines always intersect in a single point. Parallel lines intersect "at infinity".

Such an inclusion of infinity points makes projective planes particularly useful in the description of elliptic curves, as the description of such a curve in an ordinary plane needs an additional symbol for "the point at infinity" to give the set of points on the curve the structure of a group 5.1. Translating the curve into projective geometry includes this "point at infinity" more naturally into the set of all points on a projective plane.

To be more precise, let $\mathbb{F}$ be a field, $\mathbb{F}^3 := \mathbb{F} \times \mathbb{F} \times \mathbb{F}$ the set of all tuples of three elements over $\mathbb{F}$ and $x \in \mathbb{F}^3$ with $x = (X,Y,Z)$. Then there is exactly one *line* L_x in $\mathbb{F}^3$ that intersects both $(0,0,0)$ and x, given by the set $L_x = \{(k \cdot X, k \cdot Y, k \cdot Z) \mid k \in \mathbb{F}\}$. A point in the **projective plane** over $\mathbb{F}$ can then be defined as such a **line** if we exclude the intersection of that line with $(0,0,0)$. This leads to the following definition of a **point** in projective geometry:

$$[X : Y : Z] := \{(k \cdot X, k \cdot Y, k \cdot Z) \mid k \in \mathbb{F}^*\} \tag{4.56}$$

Points in projective geometry are therefore lines in $\mathbb{F}^3$ where the intersection with $(0,0,0)$ is excluded. Given a field $\mathbb{F}$, the **projective plane** of that field is then defined as the set of all points excluding the point $[0:0:0]$:

$$\mathbb{FP}^2 := \{[X:Y:Z] \mid (X,Y,Z) \in \mathbb{F}^3 \text{ with } (X,Y,Z) \neq (0,0,0)\} \tag{4.57}$$

It can be shown that a projective plane over a finite field $\mathbb{F}_{p^m}$ contains $p^{2m} + p^m + 1$ number of elements.

To understand why the projective point $[X:Y:Z]$ is also a line, consider the situation where the underlying field $\mathbb{F}$ is the set of rational numbers $\mathbb{Q}$. In this case, $\mathbb{Q}^3$ can be seen as the three-dimensional space, and $[X:Y:Z]$ is an ordinary line in this 3-dimensional space that intersects zero and the point with coordinates X, Y and Z such that the intersection with zero is excluded.

The key observation here is that points in the projective plane $\mathbb{FP}^2$ are lines in the 3-dimensional space $\mathbb{F}^3$. However, it should be kept in mind that, for finite fields, the terms **space** and **line** share very little visual similarity with their counterparts over the set of rational numbers.

It follows from this that points $[X:Y:Z] \in \mathbb{FP}^2$ are not simply described by fixed coordinates (X,Y,Z), but by **sets of coordinates**, where two different coordinates (X_1,Y_1,Z_1) and (X_2,Y_2,Z_2) describe the same point if and only if there is some non-zero field element $k \in \mathbb{F}^*$ such that $(X_1,Y_1,Z_1) = (k \cdot X_2, k \cdot Y_2, k \cdot Z_2)$. Points $[X:Y:Z]$ are called **projective coordinates**.

Notation and Symbols 12 (Projective coordinates). Projective coordinates of the form $[X:Y:1]$ are descriptions of so-called **affine points**. Projective coordinates of the form $[X:Y:0]$ are descriptions of so-called **points at infinity**. In particular, the projective coordinate $[1:0:0]$ describes the so-called **line at infinity**.

Example 67. Consider the field $\mathbb{F}_3$ from exercise 48 . As this field only contains three elements, it does not take too much effort to construct its associated projective plane $\mathbb{F}_3\mathbb{P}^2$, which we know only contains 13 elements.

To find $\mathbb{F}_3\mathbb{P}^2$, we have to compute the set of all lines in $\mathbb{F}_3 \times \mathbb{F}_3 \times \mathbb{F}_3$ that intersect $(0,0,0)$, excluding their intersection with $(0,0,0)$. Since those lines are parameterized by tuples (x_1,x_2,x_3), we compute as follows:

$$
\begin{aligned}
[0:0:1] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(0,0,1),(0,0,2)\}\\
[0:0:2] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(0,0,2),(0,0,1)\} = [0:0:1]\\
[0:1:0] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(0,1,0),(0,2,0)\}\\
[0:1:1] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(0,1,1),(0,2,2)\}\\
[0:1:2] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(0,1,2),(0,2,1)\}\\
[0:2:0] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(0,2,0),(0,1,0)\} = [0:1:0]\\
[0:2:1] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(0,2,1),(0,1,2)\} = [0:1:2]\\
[0:2:2] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(0,2,2),(0,1,1)\} = [0:1:1]\\
[1:0:0] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(1,0,0),(2,0,0)\}\\
[1:0:1] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(1,0,1),(2,0,2)\}\\
[1:0:2] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(1,0,2),(2,0,1)\}\\
[1:1:0] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(1,1,0),(2,2,0)\}\\
[1:1:1] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(1,1,1),(2,2,2)\}\\
[1:1:2] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(1,1,2),(2,2,1)\}\\
[1:2:0] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(1,2,0),(2,1,0)\}\\
[1:2:1] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(1,2,1),(2,1,2)\}\\
[1:2:2] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(1,2,2),(2,1,1)\}\\
[2:0:0] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(2,0,0),(1,0,0)\} = [1:0:0]\\
[2:0:1] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(2,0,1),(1,0,2)\} = [1:0:2]\\
[2:0:2] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(2,0,2),(1,0,1)\} = [1:0:1]\\
[2:1:0] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(2,1,0),(1,2,0)\} = [1:2:0]\\
[2:1:1] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(2,1,1),(1,2,2)\} = [1:2:2]\\
[2:1:2] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(2,1,2),(1,2,1)\} = [1:2:1]\\
[2:2:0] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(2,2,0),(1,1,0)\} = [1:1:0]\\
[2:2:1] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(2,2,1),(1,1,2)\} = [1:1:2]\\
[2:2:2] &= \{(k\cdot x_1, k\cdot x_2, k\cdot x_3) \mid k \in \mathbb{F}_3^*\} = \{(2,2,2),(1,1,1)\} = [1:1:1]
\end{aligned}
$$

These lines define the 13 points in the projective plane $\mathbb{F}_3\mathbb{P}$:

$$
\begin{aligned}
\mathbb{F}_3\mathbb{P} = \{&[0:0:1],[0:1:0],[0:1:1],[0:1:2],[1:0:0],[1:0:1],\\
&[1:0:2],[1:1:0],[1:1:1],[1:1:2],[1:2:0],[1:2:1],[1:2:2]\}
\end{aligned}
$$

This projective plane contains 9 affine points, three points at infinity and one line at infinity.

To understand the ambiguity in projective coordinates a bit better, let us consider the point $[1:2:2]$. As this point in the projective plane is a line in $\mathbb{F}_3^3\backslash\{(0,0,0)\}$, it has the projective coordinates $(1,2,2)$ as well as $(2,1,1)$, since the former coordinate gives the latter when multiplied in $\mathbb{F}_3$ by the factor 2. In addition, note that, for the same reasons, the points $[1:2:2]$ and $[2:1:1]$ are the same, since their underlying sets are equal.

Exercise 56. Construct the so-called **Fano plane**, that is, the projective plane over the finite field $\mathbb{F}_2$.

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