Moonmath manual

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Lorem **ipsum** dolor sit amet, consectetur adipiscing elit. Pellentesque semper viverra dictum. Fusce interdum venenatis leo varius vehicula. Etiam ac massa dolor. Quisque vel massa faucibus, facilisis nulla nec, egestas lectus. Sed orci dui, egestas non felis vel, fringilla pretium odio. *Aliquam* vel consectetur felis. Suspendisse justo massa, maximus eget nisi a, maximus gravida mi.

Here is a citation for demonstration: **?**

1 Introduction

This is dump from other papers as inspiration for the intro:

Zero-knowledge proofs (ZKPs) are an important privacy-enhancing tool from cryptography. Theyallow proving the veracity of a statement, related to confidential data, without revealing any in-formation beyond the validity of the statement. ZKPs were initially developed by the academiccommunity in the 1980s, and have seen tremendous improvements since then. They are now ofpractical feasibility in multiple domains of interest to the industry, and to a large community ofdevelopers and researchers. ZKPs can have a positive impact in industries, agencies, and for per-sonal use, by allowing privacy-preserving applications where designated private data can be madeuseful to third parties, despite not being disclosed to them.

ZKP systems involve at least two parties: a prover and a verifier. The goal of the prover is toconvince the verifier that a statement is true, without revealing any additional information. Forexample, suppose the prover holds a birth certificate digitally signed by an authority. In orderto access some service, the prover may have to prove being at least 18 years old, that is, thatthere exists a birth certificate, tied to the identify of the prover and digitally signed by a trustedcertification authority, stating a birthdate consistent with the age claim. A ZKP allows this, withoutthe prover having to reveal the birthdate.

2 The Zoo of Zero-Knowledge Proofs

First, a list of zero-knowledge proof systems:

1. Pinocchio (2013): Paper
   - Notes: trusted setup
2. BCGTV (2013): Paper
   - Notes: trusted setup, implementation
3. BCTV (2013): Paper
   - Notes: trusted setup, implementation
4. Groth16 (2016): Paper
   - Notes: trusted setup
   - Other resources: Talk in 2019 by Georgios Konstantopoulos
5. GM17 (207): Paper
   - Notes: trusted setup
   - Other resources: later Simulation extractability in ROM, 2018
6. Bulletproofs (2017): Paper
   - Notes: no trusted setup
   - Other resources: Polynomial Commitment Scheme on DL, 2016 and KZG10, Polynomial Commitment Scheme on Pairings, 2010
7. Ligero (2017): Paper
   - Notes: no trusted setup
   - Other resources:
8. Hyrax (2017): Paper
   - Notes: no trusted setup
   - Other resources:
9. STARKs (2018): Paper
   - Notes: no trusted setup
   - Other resources:
10. Aurora (2018): Paper
    - Notes: transparent SNARK
    - Other resources:

11. Sonic (2019): Paper
    - Notes: SNORK - SNARK with universal and updateable trusted setup, PCS-based
    - Other resources: Blog post by Mary Maller from 2019 and work on updateable and universal setup from 2018
12. Libra (2019): Paper
    - Notes: trusted setup
    - Other resources:
13. Spartan (2019): Paper
    - Notes: transparent SNARK
    - Other resources:
14. PLONK (2019): Paper
    - Notes: SNORK, PCS-based
    - Other resources: Discussion on Plonk systems and Awesome Plonk list
15. Halo (2019): Paper
    - Notes: no trusted setup, PCS-based, recursive
    - Other resources:
16. Marlin (2019): Paper
    - Notes: SNORK, PCS-based
    - Other resources: Rust Github
17. Fractal (2019): Paper
    - Notes: Recursive, transparent SNARK
    - Other resources:
18. SuperSonic (2019): Paper
    - Notes: transparent SNARK, PCS-based
    - Other resources: Attack on DARK compiler in 2021
19. Redshift (2019): Paper
    - Notes: SNORK, PCS-based
    - Other resources:

Other resources on the zoo: Awesome ZKP list on Github, ZKP community with the reference document

To Do List

- Make table for prover time, verifier time, and proof size
- Think of categories - *Achieved Goals*: Trusted setup or not, Post-quantum or not, ...
- Think of categories - *Mathematical background*: Polynomial commitment scheme, ...
- ... while we discuss the points above, we should also discuss a common notation/language for all these things. (E.g. transparent SNARK/no trusted setup/STARK)

Points to cover while writing

- Make a historical overview over the "discovery" of the different ZKP systems
- Make reader understand what paper is build on what result etc. - the tree of publications!
- Make reader understand the different terminology, e.g. SNARK/SNORK/STARK, PCS, R1CS, updateable, universal, ...
- Make reader understand the mathematical assumptions - and what this means for the zoo.
- Where will the development/evolution go? What are bottlenecks?

Other topics I fell into while compiling this list

- Vector commitments: `https://eprint.iacr.org/2020/527.pdf`
- Snarkl: `http://ace.cs.ohio.edu/~gstewart/papers/snaarkl.pdf`
- Virgo?: `https://people.eecs.berkeley.edu/~kubitron/courses/cs262a-F19/projects/reports/project5_report_ver2.pdf`

3 Preliminaries

3.1 Purpose of the book

The first version of this book is written by security auditors at Least Authority where we audited quite a few snark based systems. Its included "what we have learned" destilate of the time we spend on various audits.

We intend to let illus- trative examples drive the discussion and present the key concepts of pairing computation with as little machinery as possible. For those that are fresh to pairing-based cryptography, it is our hope that this chapter might be particu- larly useful as a first read and prelude to more complete or advanced expositions (e.g. the related chapters in [Gal12]).

On the other hand, we also hope our beginner-friendly intentions do not leave any sophisticated readers dissatisfied by a lack of formality or generality, so in cases where our discussion does sacrifice completeness, we will at least endeavour to point to where a more thorough exposition can be found.

One advantage of writing a survey on pairing computation in 2012 is that, after more than a decade of intense and fast-paced research by mathematicians and cryptographers around the globe, the field is now racing towards full matu- rity. Therefore, an understanding of this text will equip the reader with most of what they need to know in order to tackle any of the vast literature in this remarkable field, at least for a while yet.

Since we are aiming the discussion at active readers, we have matched every example with a corresponding snippet of (hyperlinked) Magma [BCP97] code 1 , where we take inspiration from the helpful Magma pairing tutorial by Dominguez Perez et al. [DKS09].

Early in the book we will develop examples that we then later extend with most of the things we learn in each chapter. This way we incrementally build a few real world snarks but over full fledged cryptographic systems that are nevertheless simple enough to be computed by pen and paper to illustrate all steps in grwat detail.

3.2 How to read this book

Books and papers to read: XXXXXXXXXXX

Software to try: XXXXXXXXXXXXXXXXXX

Correctly prescribing the best reading route for a beginner naturally requires individual diagnosis that depends on their prior knowledge and technical preparation.

3.3 Cryptological Systems

The science of information security is referred to as *cryptology*. In the broadest sense, it deals with encryption and decryption processes, with digital signatures, identification protocols, cryptographic hash functions, secrets sharing, electronic voting procedures and electronic money. EXPAND

3.4 SNARKS

3.5 complexity theory

Before we deal with the mathematics behind zero knowledge proof systems, we must first clarify what is meant by the runtime of an algorithm or the time complexity of an entire mathematical problem. This is particularly important for us when we analyze the various snark systems...

For the reader who is interested in complexity theory, we recommend, or example or , as well as the references contained therein.

3.5.1 Runtime complexity

The runtime complexity of an algorithm describes, roughly speaking, the amount of elementary computation steps that this algorithm requires in order to solve a problem, depending on the size of the input data.

Of course, the exact amount of arithmetic operations required depends on many factors such as the implementation, the operating system used, the CPU and many more. However, such accuracy is seldom required and is mostly meaningful to consider only the asymptotic computational effort.

In computer science, the runtime of an algorithm is therefore not specified in individual calculation steps, but instead looks for an upper limit which approximates the runtime as soon as the input quantity becomes very large. This can be done using the so-called *Landau notation* (also called big -$\mathcal{O}$-notation) A precise definition would, however, go beyond the scope of this work and we therefore refer the reader to .

For us, only a rough understanding of transit times is important in order to be able to talk about the security of crypographic systems. For example, $\mathcal{O}(n)$ means that the running time of the algorithm to be considered is linearly dependent on the size of the input set n, $\mathcal{O}(n^k)$ means that the running time is polynomial and $\mathcal{O}(2^n)$ stands for an exponential running time (chapter 2.4).

An algorithm which has a running time that is greater than a polynomial is often simply referred to as *slow*.

A generalization of the runtime complexity of an algorithm is the so-called *time complexity of a mathematical problem*, which is defined as the runtime of the fastest possible algorithm that can still solve this problem (chapter 3.1).

Since the time complexity of a mathematical problem is concerned with the runtime analysis of all possible (and thus possibly still undiscovered) algorithms, this is often a very difficult and deep-seated question .

For us, the time complexity of the so-called discrete logarithm problem will be important. This is a problem for which we only know slow algorithms on classical computers at the moment, but for which at the same time we cannot rule out that faster algorithms also exist.

3.6 Hash functions

We assume that $H : \{0,1\}^* \to 0,1^k$ is a **hash function** that maps binary strings of arbitrary length onto strings of length k. In addition we define a hash function to be l-bounded if it is only able to map from binary strings of length l to binary strings of length k.

STUFF ON CRYPTOGRAPHIC HASH FUNCTIOND

p&p-hash In this example we define a 16-bounded pen&paper hash function that is simple enough to be computed without a computer. We call it the PaP-Hash and will use it throughout the book as a basic example whenever hashing is involved in other example.

The PaP-Hash $\mathscr{H}_{PaP} : \{0,1\}^{16} \to \{0,1\}^4$ is defined in the following way:

- Decompose the 16-bit preimage $S = (s_0, s_1, \ldots, s_{15})$ into 4 chunks $S_i = (s_{4i+0}, \ldots, s_{4i+3})$ for $i \in \{0,1,2,3\}$.
- For each chunk S_i do a circular bitshift $s_j \to s_{j+1}$ mod 4 for all $s_j \in S_i$
- Xor all four chunks together $S = S_1 \; XOR \; S_2 \; XOR \; S_3 \; XOR \; S_3$
- Compute the result $\mathscr{H}_{PaP}(S) = S \; XOR \; (1001)$

Example 1. *Lets compute our PaP-Hash on a concrete example string* $S = (1110011101110011)$. *Then the decomposition is* $S_0 = (1110)$, $S_1 = (0111)$, $S_2 = (0111)$ *and* $S_3 = (0011)$ *and after a circular bitshift we get* $S'_0 = (0111)$, $S'_1 = (1011)$, $S'_2 = (1011)$ *and* $S'_3 = (1001)$. *Xoring everything together we get* $S = (0111) \; XOR \; (1011) \; XOR \; (1011) \; XOR \; (1001) = (1100) \; XOR \; (0010) = (1110)$. *So we get* $\mathscr{H}_{PaP}(1010011101100011) = (1110)$.

3.7 Software Used in This Book

3.7.1 Sagemath

It order to provide an interactive learning experience, and to allow getting hands-on with the concepts described in this book, we give examples for how to program them in the Sage programming language. Sage is a dialect of the learning-friendly programming language Python, which was extended and optimized for computing with, in and over algebraic objects. Therefore, we recommend installing Sage before diving into the following chapters.

The installation steps for various system configurations are described on the sage websit [1]. Note however that we use Sage version 9, so if you are using Linux and your package manager only contains version 8, you may need to choose a different installation path, such as using prebuilt binaries.

We recommend the interested reader, who is not familiar with sagemath to read on the many tutorial before starting this book. For example

[1] `https://doc.sagemath.org/html/en/installation/index.html`

4 Mathematics

How much mathematics is needed to understand zero knowledge proofs? The answer, of course, depends on many things like the level of detail the reader want to understand them. For example it is possible to describe those proofs not using mathematics at all. However to read a foundational paper like [GROTH16], enough mathematics is needed to at least understand the basic concepts.

Otherwise any student who is interested in learning the concepts, but who has never seen or played with, say, a finite field, or an elliptic curve, may quickly become overwhelmed. This is not so much due to the complexity of the mathematics needs but perhaps more because of the vast amount of technical jargon, of unknown terms, obscure symbols that quickly makes a text ubreadable, desipte the concepts being actually not that hard. As a result the reader might either loose interest, or gain some dangerous smattering that in a worst case scenario materialize in inmature code.

In this chapter we therefore derive the mathematical concepts needed to understand the basic concepts underlying snark development and we encourage the reader who is not familiar with basic number theory and lliptic curves to take the time and read this chapter until they are able to at least solve most of the simple exercises.

If on the other hand the reader is already skilled in elliptic curve cryptography they might skip this section and only come back for reference and comparision. Maybe the most interesing parts are XXX.

We start at a very basic level and only really requie fundamntal concepts like integer arithmetics. At the same time we'll have a focus on teaching the reader how to think mathematically and to understand that there are numbers and methatical structures out there that appear to be very different from the stuff we learned in school and yet on a deeper level they are in deed very similar.

We want to stress however, that our introduction is informal, incomplete and optimized to enable the reader to understand zero knowledge concepts as efficient as possible. Our focus and design choices are so that we give as little theory as necessary but accompanied by a wealth of numerical examples. We found this on the believe, that such an informal, example- driven approach to learning mathematics may ease the beginner's digestion in the initial stages.

For instance, a beginner would be likely to find it more beneficial to first compute a simple toy snark in a pen and paper style all the way through all steps before they dig deeper and actually devop real world production ready systems. Also having already a few simple examples in you head, its likely easier to only then read the actual academic papers.

However in order to be able to derive those toy example, some mathematical groundwork is needed. This chapter therefore will help the reader to focus on what is important, while at the same time serve as first exercises the reader is encouraged to recompute themself. Every section usually then ends with a list of additional exercises in increasing difficulty order, to help the reader memorising and applying the concepts given.

Overall the goal of this chapter is to provide a reader who is starting with nothing more than basic high school algebra, to be able to solve basic tasks in elliptic curve cryptography without the need of a computer.

We start with a brief recapitulation of basic integer aithmetics like long division, the greatest common divisior and Euklids algorithm. After that we introduce modular arithmetics as **the most important** skill to compute our pen and paper examples. We then introduce polynomials, compute their analogs to integer arithmetics and introduce the important concept of Lagrange interpolation.

After this practical warm up, we have to introduce some basic algebraic terms like groups and fields, because those terms are all over the place when reading academic papers in the context of zero knowledge proof. The beginner is good adviced to memorize those terms and think about them. We define these terms in the general abstract way of mathematics, hoping that the non mathematical trained reader will gradually learn to become comfortable with this style. We then give basic examples and do basic computations with these examples to get familiar with the concepts.

4.1 Integer Arithmetics

To some degree, most readers with probably remember integer arithmetics from school. It is however important for the rest of the book to be able to apply those concepts to understand and execute computations in the various pen and paper examples that are the main contribution of the moon math manual. We will therefore recapitulate those concepts filling up some knowledge gaps.

In what follows we applay standard mathematical notations and use the symbol $\mathbb{Z}$ for the set of all integers, that is we write

$$\mathbb{Z} := \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \tag{4.1}$$

So whenever you see the symbol $\mathbb{Z}$, think of the set of all integers. If $a \in \mathbb{Z}$ is an integer, we write $|a|$ for the *absolute value* of a, that is the the non-negative value of a without regard to its sign. In addition we will use the symbol $\mathbb{N}$ for the set of all counting numbers, that is we write

$$\mathbb{N} := \{0, 1, 2, 3, \ldots\} \tag{4.2}$$

including the number 0. So whenever you see the symbol $\mathbb{N}$, think of the set of all non negative integers.

To make it easier to memorize new concepts and symbols, we might frequently link to definitions (See 4.2 for a definition of $\mathbb{Z}$) in the begining, but as to many links render a text unreadable, we will assume the reader will become familiar with definitions as the text proceeds at which point we will not link them anymore.

Both sets $\mathbb{N}$ and $\mathbb{Z}$ have a notion of addition as well as multiplication dedined on them and also most of us are probably able to do many integer computations in their head, we will frequently invoke the sagemath system (3.7.1) for more complicated computations. One way to invoke the integer-type in sage is:

```
sage: ZZ # A sage notation for the integer type
Integer Ring
sage: NN # A sage notation for the counting number type
Non negative integer semiring
sage: ZZ(5) # Get an element from the Ring of integers
5
sage: ZZ(5) + ZZ(3)
```

```
8
sage: ZZ(5) * NN(3)
15
sage: ZZ.random_element(10**50)
23285581981658895627750562815289227207637683248405
sage: ZZ(27713).str(2) # Binary string representation
110110001000001
sage: NN(27713).str(2) # Binary string representation
110110001000001
sage: ZZ(27713).str(16) # Hexadecimal string representation
6c41
```

Of particular interest for us are the so called *prime numbers*, which are counting numbers $p \in \mathbb{N}$ with $p \geq 2$, which are divisible by themself and by 1 only. Prime numbers are called *odd* if they are not the number 2. We write $\mathbb{P}$ for the set of all prime numbers and $\mathbb{P}_{\geq 3}$ for the set of all odd prime numbers. $\mathbb{P}$ is infinite and can be ordered according to size, so that we can writem them as

$$2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,\ldots \tag{4.3}$$

which is sequence $A000040$ in OEIS. In particular, we can talk about small and large prime numbers.

As the *fundamental theorem of arithmetics* tells us, prime numbers are in a certain sense the basic building blocks from which all other natural numbers are composed. To see that, let $n \in \mathbb{N}_{\geq 2}$ be any natural number. Then there are always prime numbers $p_1, p_2, \ldots, p_k \in \mathbb{P}$, such that

$$n = p_1 \cdot p_2 \cdot \ldots \cdot p_k \,. \tag{4.4}$$

This representation is unique, except for permutations in the factors and is called the **prime factorization** of n.

Example 2 (Prime Factorization)**.** *To see what we mean by integer factorization, lets look at the number* 19214758032624000*. To get its prime factors, we can sucessively divide it by all prime numbers in ascending order starting with* 2*. We get*

$$19214758032624000 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 17 \cdot 23 \cdot 43 \cdot 43 \cdot 47$$

We can double check our findings invoking sage, which provides an algorith to factor counting numbers:

```
sage: n = ZZ(19214758032624000)
sage: factor(n)
2^7 * 3^3 * 5^3 * 7 * 11 * 17^2 * 23 * 43^2 * 47
```

Having done the computation from the previous example, reveals an important observation: Computing the factorization was computationally expensive, while on the other hand, giving a string of prime numbers, computing their product is fast.

From this an important question arises: How fast we can compute the prime factorization of a natural number? This is the famous *factorization problem* and as far as we know, there is no method on a classical Turing machine that is able to compute this representation in polynomial time. The fastest algorithms known today run sub-exponentially, with $\mathcal{O}((1+\varepsilon)^n)$ and some $\varepsilon > 0$.

It follows that integer factorization $\Leftrightarrow$ prime number multiplication is an example of, what is called a one-way.function. Something that is easy to compute in one direction, but hard to compute in the other direction. Existence of one way functions like this are basic crytographic assumptions, that the security of many crypto systems is based on.

It should be pointed out however hat the American mathematician Peter Williston Shor developed an algorithm in 1994 which can calculate the prime factor representation of a natural number in polynomial time on a quantum computer. The consequence of this is, of course, that cryosystems, which are based on the time complexity of the prime factor problem, are unsafe as soon as practically usable quantum computers are available.

Exercise 1. *Compute the factorization of* 6469693230 *and double check your results using sage.*

Euklidean Division In general there is no division defined in the usual sense for integers, as for example 7 divided by 3 will not be an integer again. However it is possible to devide any two integers with a remainder. So for example 7 divided by 3 is equal to 2 with a remainder of 1, since $7 = 2 \cdot 3 + 1$.

Doing integer division like this is probably something many of us remember from school. It is usually called *Euclidean division*, or division with remainder and it is an important technique, that every reader must become familiar with to understand many concepts in this book. The precise definition is as follows:

Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ be two integers. Then there is always another integer $m \in \mathbb{Z}$ and a counting number $r \in \mathbb{N}$, with $0 \leq r < |b|$ such that

$$a = m \cdot b + r \tag{4.5}$$

This decomposition of a given b is called *Euklidean division*, where a is called the *divident*, b is called the *divisor*, m is called the *quotient* and r is called the *remainder*.

Notation and Symbols 1. *Suppose that the numbers a, b, m and r satisfy equation (4.5). Then we often write*

$$a \; div \; b := m, \qquad a \; mod \; b := r \tag{4.6}$$

to describe the quotient and the remainder of the Euklidean division. We also say, that an integer a is divisible by another integer b if $a \; mod \; b = 0$ holds. In this case we also write $a|b$.

So in a Nutshell Euclidean division is a process of dividing one integer by another, in a way that produces a quotient and a non negative remainder the latter of which is smaller than the absolute value of the divisor. It can be shown, that both the quotient and the remainder always exist and are unique, as long as the divident is different from 0.

A special situation occures, is the remainder is zero, because in this special case the divident *is* divisible by the divisor. Our notation $a|b$ refelcts that.

Example 3. *Applying Euklidean division and our previously defined notation 4.6 to the divisor* -17 *and the divident* 4*, we get*

$$-17 \; div \; 4 = -5, \qquad -17 \; mod \; 4 = 3$$

because $-17 = -5 \cdot 4 + 3$ *is the Euklidean division of* -17 *and* 4 *(Since the remainder is by definition a non-negative number). In this case* 4 *does not divide* -17 *as the reminder is not zero. Writing* $-17|4$ *therefore has no meaning. On the other hand we can write* $12|4$*, since* 4 *divides* 12*, as* $12 \; mod \; 4 = 0$*. We can invoke sagemath to do the computation for us. We get*

```
sage: ZZ(-17) // ZZ(4) # Integer quotient
-5
sage: ZZ(-17) % ZZ(4) # remainder
3
sage: ZZ(4).divides(ZZ(-17)) # self divides other
False
sage: ZZ(4).divides(ZZ(12))
True
```

Methods to compute Euklidean division for integers are called *integer division algorithms*. Probably the best known algorithm is the so called *long division*, that most of us might have learned in school. It should be noted however that there are faster methods like *Newton–Raphson division*.

As long division is the standard method used for pen-&-paper division of multi-digit numbers expressed in decimal notation, the reader should become familiar with it as we use it all over this book when we do simple pen-and-paper computations. However instead of defining the algorithm formally, we rather give some examples, that hopelly will make the process clear

Example 4 (Integer Long Division)**.** *To give an example of integer long division algorithm, lets divide the integer* $a = 143785$ *by the number* $b = 17$*. Our goal is therfore to find solutions to equation 4.5, that is we need to find the quotient* $m \in \mathbb{Z}$ *and the reminder* $r \in \mathbb{N}$ *such that* $143785 = m \cdot 17 + r$*. Using a notation that is mostly used in Commonwealth countries, we compute*

$$\begin{array}{r} 8457 \\ 17\overline{)\,143785} \\ \underline{136} \\ 77 \\ \underline{68} \\ 98 \\ \underline{85} \\ 135 \\ \underline{119} \\ 16 \end{array} \tag{4.7}$$

We therefore get $m = 8457$ *as well as* $r = 16$ *and indeed we have* $143785 = 8457 \cdot 17 + 16$*, which we can double check invoking sage:*

```
sage: ZZ(143785).quo_rem(ZZ(17)) # Euclidean Division
(8457, 16)
sage: ZZ(143785) == ZZ(8457)*ZZ(17) + ZZ(16) # check
True
```

In a nutshell, the algorithm loops through the digits of the divident from the left to right, subtracting the largest possible multiple of the divisor (at the digit level) at each stage; the multiples then become the digits of the quotient, and the remainder is the first digit of the divident.

Exercise 2 (Integer Long Division)**.** *Find an* $m \in \mathbb{Z}$ *as well as an* $r \in \mathbb{N}$ *such that* $a = m \cdot b + r$ *(See equation 4.5) holds for the folling pairs* $(a,b) = (27,5)$*,* $(a,b) = (27,-5)$*,* $(a,b) = (127,0)$*,* $(a,b) = (-1687,11)$*. In which cases are your solutions unique?*

Exercise 3 (Long Division Algorithm)**.** *Write an algorithm in pseudocode that computes integer long division, handling all edge cases properly.*

The Extended Euklidean Algorithm One of the most critical parts in this book is modular arithmetics XXX and its application in the computations in so called finite fields, as we explain in XXX. In modular arithmetics it is sometimes possible to define actual division and multiplicative inverses of numbers, that is very different from inverses as we know them from other systems like factional numbers.

However, to actually compute those inverses we have to get familar with the so-called *extended Euclidean algorithm*. To recapitulate jargon first, the *greatest common divisor* (GCD) of two nonzero integers a and b is the greatest non-zero counting number d such that d divides both a and b; that is $d|a$ as well as $d|b$. We write $gcd(a,b) := d$ for this number. In addition two counting numbers are called **relative prime**, if their greates common divisor is 1.

The extended Euclidean algorithm is then a method to calculate the greatest common divisor of two counting numbers a and $b \in \mathbb{N}$, as well as two additional integers $s,t \in \mathbb{Z}$, such that the equation

$$gcd(a,b) = s \cdot a + t \cdot b \tag{4.8}$$

holds. The following pseudocode shows in detail how to calculate these numbers with the extended Euclidean algorithm (**?** chapter 2.9):

Algorithm 1 Extended Euklidean Algorithm

Require: $a,b \in \mathbb{N}$ with $a \geq b$
  procedure EXT-EUCLID(a,b)
    $r_0 \leftarrow a$
    $r_1 \leftarrow b$
    $s_0 \leftarrow 1$
    $s_1 \leftarrow 0$
    $k \leftarrow 1$
    while $r_k \neq 0$ **do**
      $q_k \leftarrow r_{k-1}$ div r_k
      $r_{k+1} \leftarrow r_{k-1} - q_k \cdot r_k$
      $s_{k+1} \leftarrow s_{k-1} - q_k \cdot s_k$
      $k \leftarrow k+1$
    end while
    return $gcd(a,b) \leftarrow r_{k-1}$, $s \leftarrow s_{k-1}$ and $t := (r_{k-1} - s_{k-1} \cdot a)$ div b
  end procedure
Ensure: $gcd(a,b) = s \cdot a + t \cdot b$

The algorithm is simple enough to be done effectively in pen-&-paper examples, where it is common to write it as a table where the rows represent the while-loop and the colums represent the values of the the array r, s and t with index k. The following example provides a simple execution:

Example 5. *To illustrate the algorithm, lets apply it to the numbers* $a = 12$ *and* $b = 5$. *Since* $12,5 \in \mathbb{N}$ *as well as* $12 \geq 5$ *all requirements are meat and we compute*

k	r_k	s_k	$t_k = (r_k - s_k \cdot a)$ *div* b
0	*12*	*1*	*0*
1	*5*	*0*	*1*
2	*2*	*1*	*-2*
3	*1*	*-2*	*5*

From this we can see that 12 *and* 5 *are relatively prime (coprime), since their greatest common divisor is* $gcd(12,5) = 1$ *and that the equation* $1 = (-2) \cdot 12 + 5 \cdot 5$ *holds. We can also invoke sage to double check our findings:*

```
sage: ZZ(12).xgcd(ZZ(5)) # (gcd(a,b),s,t)
(1, -2, 5)
```

Exercise 4 (Extended Euklidean Algorithm)**.** *Find integers* $s,t \in \mathbb{Z}$ *such that* $gcd(a,b) = s \cdot a + t \cdot b$ *holds for the folling pairs* $(a,b) = (45,10)$*,* $(a,b) = (13,11)$*,* $(a,b) = (13,12)$*. What pairs* (a,b) *are coprime?*

Exercise 5 (Towards Prime fields)**.** *Let* $n \in \mathbb{N}$ *be a counting number and* p *a prime number, such that* $n < p$*. What is the greatest common divisor* $gcd(p,n)$*?*

4.2 Modular arithmetic

In mathematics, so called *modular arithmetic* is a system of arithmetic for integers, where numbers "wrap around" when reaching a certain value, much like calculations on a clock wrap around whenever the value exceeds the number 12, 24 or 60, depending on your clock. For example if the clock shows that it is 11 o'clock, then 20 hours later it will be 7 o'clock, not 31 o'clock. The letter of which has no meaning on a normal clock that shows hours.

The number at which the wrap occures is called the *modulus*. Modular arithmetics generalizes the clock example to arbitrary moduli and studies equations and phenomena that arizes in this new kind of arithmetics. It is of central importance for understanding most modern crypto systems, in large parts because the exponentiation function has an inverse with respect to certain moduli, that is hard to compute. In addition we will see that it provides the foundation of what is called finite fields (See XXX)

Also it will turn out that modular arithmetic appears very different from ordinary integer arithmetic that we are all familiar with, we encouurage the interested reader to work through the example and to discover that, once they accept that this is a new kind of calculations, its actually not that hard.

Congurency In what follows, let $n \in \mathbb{N}$ with $n \geq 2$ be a fixed counting number, that we will call the *modulus* of our modular arithmetics system. With such an n given, we can then group integers into classes, by saying that two integers are in the same class, whenever their Euklidean division 4.1 by n will give the same remainder. We then say that two numbers are *congruent* whenever they are in the same class.

Example 6. *If we choose* $n = 12$ *as in our clock example, then the integers* -7*,* 5*,* 17 *and* 29 *are all congruent with respect to* 12*, since all of them have the remainder* 5 *if we Euklidean divide them by* 12*. In the picture of an analog* 12*-hour clock, starting at* 5 *o'clock, when we add* 12 *hours we are again at* 5 *o'clock, representing the number* 17*. On the other hand when we subtract* 12 *hours, we are at* 5 *o'clock again, representing the number* -7*.*

We can formulize this intuition of what congruency should be into a proper definition utilizing Euklidean division as explained previously 4.1: Let a, $b \in \mathbb{Z}$ be two integers and $n \in \mathbb{N}$ a natural number. Then a and b are said to be **congruent with respect to the modulus** n, if and only if the equation

$$a \bmod n = b \bmod n \tag{4.9}$$

holds. If on the other hand two numbers are not congruent with respect to a given modulus n, we call them *incongruent* w.r.t. n.

A *congruency* is then nothing but an equation "up to congruency", which means that the equation only needs to hold if we take the modulus on both sides. In which case we write

$$a \equiv b \pmod{n} \tag{4.10}$$

Modular Arithmetics On particulary nice thing about congruencies is, that we can do calculations (aithmetics), much like we can with integer equations. That is we can add or multiply numbers on both sides. The main difference is probably that the congruency $a \equiv b \pmod{n}$ is only equivalent to the congruency $k \cdot a \equiv k \cdot b \pmod{n}$ for some non zero integer $k \in \mathbb{Z}$, whenever k and the modulus n are coprime. The following list gives a set of useful rules:

Suppose that the congurencies $a_1 \equiv b_1 \pmod{n}$ as well as $a_2 \equiv b_2 \pmod{n}$ are satisfied for integers $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ and that $k \in \mathbb{Z}$ is another integer. Then:

- $a_1 + k \equiv b_1 + k \pmod{n}$ (compatibility with translation)
- $k \cdot a_1 \equiv k \cdot b_1 \pmod{n}$ (compatibility with scaling)
- $a_1 + a_2 \equiv b_1 + b_2 \pmod{n}$ (compatibility with addition)
- $a_1 \cdot a_2 \equiv b_1 \cdot b_2 \pmod{n}$ (compatibility with multiplication)

Other rules like compatibility with subtraction and exponentiation follow from this rule, as for example compatibility with subtraction is compatibility with scaling by $k = -1$ and compatibility with addition.

Note that the previous rules are implications not equivalences, which means that you can not necessarily reverse those rules. The following rules makes this precise:

- If $a_1 + k \equiv b_1 + k \pmod{n}$, then $a_1 \equiv b_1 \pmod{n}$
- If $k \cdot a_1 \equiv k \cdot b_1 \pmod{n}$ and k is coprime with n, then $a_1 \equiv b_1 \pmod{n}$
- If $k \cdot a_1 \equiv k \cdot b_1 \pmod{k \cdot n}$, then $a_1 \equiv b_1 \pmod{n}$

Another property of congruencies, not known in the traditional arithmetics of integers is the so called *Fermat's Little Theorem*. In simple words, it says that in modular arithmetics every number raised to the power of a prime number modulus is congruent to the number itself. Or, to be more precise, if $p \in \mathbb{P}$ is a prime number and $k \in \mathbb{Z}$ is an integer, then:

$$k^p \equiv k \pmod{p}, \tag{4.11}$$

If k is coprime to p, then we can divide both sides of this congruency by k and rewrite the expression into the equivalent form

$$k^{p-1} \equiv 1 \pmod{p} \tag{4.12}$$

We can invoke sage, to compute examples for both k being coprime and not coprime to p:

```
sage: ZZ(137).gcd(ZZ(64))
1
sage: ZZ(64)** ZZ(137) % ZZ(137) == ZZ(64) % ZZ(137)
True
```

```
sage: ZZ(64)** ZZ(137-1) % ZZ(137) == ZZ(1) % ZZ(137)
True
sage: ZZ(1918).gcd(ZZ(137))
137
sage: ZZ(1918)** ZZ(137) % ZZ(137) == ZZ(1918) % ZZ(137)
True
sage: ZZ(1918)** ZZ(137-1) % ZZ(137) == ZZ(1) % ZZ(137)
False
```

Remark 1. *Congruency 4.12 has a nice interpretation, that gives a first glimpse of the idea of a prime field as we will describe it in XXX: Whenever the modulus is a prime number and $k < p$ then k and p are coprime (Exercise XXX) and hence there is a notion of division by k, that is absent in integer arithmetics. If a is another integer we could define division by k as follows $a/k := a \cdot k^{p-2}$, which is always defined. This makes sense because "division by something" should be defined as "multiplication by the inverse" and since $k \cdot k^{p-2} \equiv 1 \pmod{p}$, the number k^{p-2} behaves like a multiplicative inverse for k in modular arithmetics for prime number moduli and coprime k.*

Now, since this was a lot to digest for a reader who has never encountered modular arithmetics before, lets compute an example that contains most of the stuff we just described:

Example 7. *Assume that we choose the modulus 17 and that our task is to solve the following congruency for $x \in \mathbb{Z}$*

$$7 \cdot (2x+21) + 11 \equiv x - 102 \pmod{17}$$

As many rules for congruencies are more or less same as for integers, we can proceed in a way similar, as we would if we had an equation to solve. The first thing we notice, is that $7 \cdot (2x+21) + 11 = 14x + 158$, since both sides of a congruency contain ordinary integers. We can therefore rewrite the congruency into the equivalent form

$$14x + 158 \equiv x - 102 \pmod{17}$$

In a next step we want to shift all encounters of x to left and every other term to the right. So we applay the "compatibility with translation" rules two times. In a first step we choose $k = -x$ and in a second step we choose $k = -158$. Since "compatibility with translation" transforms a congruency into an equivalent form, the solution set will not change and we get

$$\begin{aligned} 14x + 158 \equiv x - 102 \pmod{17} &\Leftrightarrow \\ 14x - x + 158 - 158 \equiv x - x - 102 - 158 \pmod{17} &\Leftrightarrow \\ 13x \equiv -260 \pmod{17} & \end{aligned}$$

If our congruency would just be a normal integer equation, we would divide both sides by 13 to get $x = -20$ as our solution. However in case of a congruency we need to make sure that the modulus and the number we want to divide by are coprime first. Only then will we get an equivalent expression. So we need to the greatest common divisor $gcd(17,13)$ and since both numbers are prime, we know $gcd(17,13) = 1$, so both numbers are indeed coprime. We therefore compute

$$13x \equiv -260 \pmod{17} \Leftrightarrow x \equiv -20 \pmod{17}$$

Our task is now to find all integers x, such that x is congruent to -20 *with respect to the modulus* 17. *So we have to find all x such*

$$x \text{ mod } 17 = -20 \text{ mod } 17$$

Since $-2 \cdot 17 + 14 = -20$ *we know* -20 *mod* $17 = 14$ *and hence we know that* $x = 14$ *is a solution. However* 31 *is another solution since* 31 *mod* $17 = 14$ *as well and so is* -20. *In fact there are infinite many solutions given by the set*

$$\{\ldots, -20, -3, 14, 31, 48, \ldots\} = \{14 + k \cdot 17 \mid k \in \mathbb{Z}\}$$

Putting all this together we have shown that the every x from the set $\{x = 14 + k \cdot 17 \mid k \in \mathbb{Z}\}$ *is a solution to the congruency* $7 \cdot (2x + 21) + 11 \equiv x - 102 \quad (\text{ mod } 17)$. *We double ckeck for, say,* $x = 14$ *as well as* $x = 14 + 12 \cdot 17 = 218$ *using sage:*

```
sage: (ZZ(7)* (ZZ(2)*ZZ(14) + ZZ(21)) + ZZ(11))  % ZZ(17) == (   48
   ZZ(14) - ZZ(102))  % ZZ(17)
True                                                              49
sage: (ZZ(7)* (ZZ(2)*ZZ(218) + ZZ(21)) + ZZ(11))  % ZZ(17) ==    50
   (ZZ(218) - ZZ(102))  % ZZ(17)
True                                                              51
```

Example 8 (Mudular multiplicative inverse)**.** *In the previous example we encounter the "lucky coincidence", that we could solve the congruency* $13x \equiv -260 \quad (\text{ mod } 17)$ *by division by* 13, *since* 260 *is divisible by* 13 *as integers. But what if this is not the case? So see how this can be solved in modular arithmetics lets consider the following modification of the congruency in the previous example:*

$$7x \equiv -260 \quad (\text{ mod } 17)$$

In this example we can not integer divide both sides of the congruency by 7, *since* $260/7$ *is not an integer. However we recall from Fermats little theorem, that* $k \cdot k^{p-2} \equiv 1 \quad (\text{ mod } p)$ *for every integer k that coprime to every prime modulus p.*

In our example, since 7 *is coprime to* 17 *(both are prime numbers), we can exploit Fermats theorem, in combination with the "compatibility with scaling" rule and multiply both sides of the congruency with* $7^{17-2} = 4747561509943$. *We then get*

$$\begin{aligned} 7x \equiv -260 \quad (\text{ mod } 17) \Leftrightarrow & \\ 7^{17-2} \cdot 7 \cdot x \equiv -260 \cdot 7^{17-2} \quad (\text{ mod } 17) \Leftrightarrow & \\ x \equiv -1234365992585180 \quad (\text{ mod } 17) & \end{aligned}$$

And since -1234365992585180 *mod* $17 = 9$, *we know that the solution is given by the set of all numbers* $x \in \{9 + m \cdot 17 \mid m \in \mathbb{Z}\}$. *We double ckeck for, say,* $x = 9$ *as well as* $x = 9 + 101 \cdot 17 = 1726$ *using sage:*

```
sage: (ZZ(7)* ZZ(9))  % ZZ(17) == (-ZZ(260))  % ZZ(17)            52
True                                                              53
sage: (ZZ(7)* ZZ(1726))  % ZZ(17) == (-ZZ(260))  % ZZ(17)         54
True                                                              55
```

Remark 2. *The discouraged reader, who at this point thinks that modular aithmetics is to complicated, might consider two thinks: First, computing congruencies in modular arithmetics is not really more complicated then computations in more familiar number systems like fractional numbers. Its just a matter of getting used to it. Second, the theory of prime fields (and more general residue class rings) takes a different view on modular rithmetics with the attempt to simplify thinks. In other words, once we understand prime field arithmetics, thinks become conceptually cleaner and more easy to compute.*

The Chinese Remainder Theorem We have seen in the previous paragraph how to solve congruencies in modular arithmetic. However one question that remains is, how to solve systems of congruencies, whith different moduli? The answer is given by the so called *Chinese raimainder theorem*, which tells us, that for any $k \in \mathbb{N}$ and coprime natural numbers $n_1, \ldots n_k \in \mathbb{N}$ as well as integers $a_1, \ldots a_k \in \mathbb{Z}$, the so-called *simultaneous congruency*

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2} \\ &\cdots \\ x &\equiv a_k \pmod{n_k} \end{aligned} \tag{4.13}$$

has a solution and all possible solutions of this congruence system are congruent modulo the product $N = n_1 \cdot \ldots \cdot n_k$. In fact, the following algorithm computes the solution set:

Algorithm 2 Chinese Reminder Theorem

Require: $n_0, \ldots, n_{k-1} \in \mathbb{N}$ coprime
  procedure CONGRUENCY-SYSTEMS-SOLVER($k, a_{0,\ldots,k-1}, n_{0,\ldots,k-1}$)
    $N \leftarrow n_0 \cdot \ldots \cdot n_{k-1}$
    while $j < k$ **do**
      $N_j \leftarrow N/n_j$
      $(\_, s_j, t_j) \leftarrow EXT - EUCLID(N_j, n_j)$ ▷ $1 = s_j \cdot N_j + t_j \cdot n_j$
    end while
    $x' \leftarrow \sum_{j=0}^{k-1} a_j \cdot s_j \cdot N_j$
    $x \leftarrow x' \bmod N$
    return $\{x + m \cdot N \mid m \in \mathbb{Z}\}$
  end procedure
Ensure: $\{x + m \cdot N \mid m \in \mathbb{Z}\}$ is the complete solution set to 4.13.

This is the classical Chinese remainder theorem as it was already known in ancient China. Under certain circumstances, the theorem can be extended to non-coprime moduli $n_1, \ldots, n_k$ but we don't need that extension in the book.

Example 9. *To illustrate how to solve simultaneous congruences using the Chinese remainder theorem, let's look at the following system of congruencies:*

$$\begin{aligned} x &\equiv 4 \quad (\textit{ mod } 7) \\ x &\equiv 1 \quad (\textit{ mod } 3) \\ x &\equiv 3 \quad (\textit{ mod } 5) \\ x &\equiv 0 \quad (\textit{ mod } 11) \end{aligned}$$

Clearly all moduli are coprime and we have $N = 7 \cdot 3 \cdot 5 \cdot 11 = 1155$*, as well as* $N_1 = 165$*,* $N_2 = 385$*,* $N_3 = 231$ *and* $N_4 = 105$*. From this we calculate with the extended Euclidean algorithm*

$$\begin{aligned} 1 &= 2 \cdot 165 + -47 \cdot 7 \\ 1 &= 1 \cdot 385 + -128 \cdot 3 \\ 1 &= 1 \cdot 231 + -46 \cdot 5 \\ 1 &= 2 \cdot 105 + -19 \cdot 11 \end{aligned}$$

so we have $x = 4 \cdot 2 \cdot 165 + 1 \cdot 1 \cdot 385 + 3 \cdot 1 \cdot 231 + 0 \cdot 2 \cdot 105 = 2398$ *as one solution. Because* $2398 \bmod 1155 = 88$ *the set of all solutions is* $\{\ldots, -2222, -1067, 88, 1243, 2398, \ldots\}$*. In particular, there are infinitely many different solutions. We can invoke sage's computation of the Chinese Remainder Theorem (CRT) to double check our findings:*

```
sage: CRT_list([4,1,3,0], [7,3,5,11])
88
```

As we have seen in various examples before, computing congruencies can be cumersome and solution sets are huge in general. It is therefore advantagous to find some kind of simplification for modular arithmetic. Fortunately this is possible if we consider all integers that have the same remainder with respect to a given modulus n to be the same. It then follows from the properties of Euclidean division, that there are exactly n different such sets for every moduls.

If we go a step further and identify each such set (equivalence class) with the corresponding remainder of the Euclidean division, we get a new set, where integer addition and multiplication can be projected to a new kind of addition and multiplication on the equivalence classes.

Roughly speaking the new rules for addition and multiplication are then computed by taking any element of the firsr equivalence class and some element of the second, then add or multiply them in the usual way and see in which equivalence class the result is contained. The following example makes the abstract idea more concrete

Example 10 (Arithmetics modulo 6)**.** *Choosing the modulus* $n = 6$ *we have six equivalence classes of integers which are congruent modulo* 6 *(which have the same remainder when divided by* 6*). We write*

$$\begin{aligned} 0 &:= \{\ldots, -6, 0, 6, 12, \ldots\}, \quad 1 := \{\ldots, -5, 1, 7, 13, \ldots\}, \quad 2 := \{\ldots, -4, 2, 8, 14, \ldots\} \\ 3 &:= \{\ldots, -3, 3, 9, 15, \ldots\}, \quad 4 := \{\ldots, -2, 4, 10, 16, \ldots\}, \quad 5 := \{\ldots, -1, 5, 11, 17, \ldots\} \end{aligned}$$

Now to compute the addition of those equivalence classes, say $2 + 5$*, one chooses arbitrary elements from both sets say* 14 *and* -1*, adds those numbers in the usual way and then looks in which equivalence class the result will be.*

So we have $14 + (-1) = 13$ *and* 13 *is in the equivalence class (of)* 1*. Hence we find that* $2 + 5 = 1$ *in modular* 6 *aithmetics, which is a more readable way to write the congruency* $2 + 5 \equiv 1 \pmod 6$*.*

Applying the same reasoning to all equivalence classes, addition and multiplication can be transferred to the equivalence classes and the results are summarized in the following addition and multiplication tables for modulus 6 *aithmetics:*

$+$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	2	3	2	1

These two tables are all you need to be able to calculate in modular 6 aithmetics.
To see how this simplifies congruency computations, lets look at the congruency

$$7 \cdot (2x+21) + 11 \equiv x - 102 \quad (\textit{ mod } 6\ )$$

from example XXX again (But this time in modular 6 arithmetics to use our addition and multiplication tables). Since 21 mod 6 = 3 and 102 mod 6 == We can rewrite the congruency into the equation $7 \cdot (2x+3) + 11 = x$ AND DO STUFF

For example, to determine the multiplicative inverse of a remainder class, look for the entry that results in 1 in the product table. For example the multiplicative inverse of 5 is 5 itself, since $5 \cdot 5 = 1$. Similar to the integers not all numbers have inverses. For example there is no element, that when multiplied with 4 will give 1. However in contrast to what we know from integers, there are non zero numbers, that, when multiplied gives zero (e.g $4 \cdot 4 = 0$).

```
sage: Z6=Integers(6) # Define integers modulo 6
sage: Z6(2)+Z6(5) # standard representatives of a class
1
sage: Z6(14)+Z6(-1) # different representatives for same class
1
sage: - Z6(2) # additive inverse
4
sage: Z6(5)**(-1) # multiplicative inverse if exists
5
```

TODO:
Barrett reduction
Montgomery modular multiplication (Montgomery domain)

Exercise 6. *Let a,b,k be integers, such that $a \equiv b \quad (\textit{ mod } n\ )$ holds. Show $a^k \equiv b^k \quad (\textit{ mod } n\ )$.*

Exercise 7. *Let a,n be integers, such that a and n are not coprime. For which $b \in \mathbb{Z}$ does the congruency $a \cdot x \equiv b \quad (\textit{ mod } n\ )$ have a solution x and how does the solution set look in that case?*

4.3 Polynomials

A polynomial is an expression consisting of variables (also called indeterminates) and coefficients, that involves only the operations of addition, subtraction, multiplication, and non-negative integer exponentiation of variables. All coefficients of a polynomial must have the same type, e.g. being integers or fractions etc. To be more precise a *polynomial* is an expression

$$P(x) := \sum_{j=0}^{m} a_j t^j = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0\ , \tag{4.14}$$

where x is called the *indeterminate*, each a_j is called a *coefficient*. If R is the type of the coefficients then the set of all **polynomials with coefficients in** R is written as $R[x]$. We often simply write $P(x) \in R[x]$ for a polynomial and denote the constant term as $P(0)$.

A polynomial is called the *zero polynomial* if all coefficients are zero and a polynomial is called the *one polynomial* if the constant term is 1 and all other coefficients are zero.

If a polynomial $P(x) = \sum_{j=0}^{m} a_j x^j$ is given and is not the zero polynomial, we call $deg(P) := m$ the *degree* of P. We moreover define the degree of the zero polynomial to be $-\infty$.

Example 11 (Integer Polyinomials)**.** *The coefficients of a polynomial must all have the same type. The set of polynomials with integer coefficients is written as* $\mathbb{Z}[x]$*. Examples of such polynomialse are:*

- $P_1(x) = 2x^2 - 4x + 17$*, with degree* $deg(P_1) = 2$*.*
- $P_2(x) = x^{23}$*, with degree* $deg(P_2) = 23$*.*
- $P_3(x) = x$*, with degree* $deg(P_3) = 1$*.*
- $P_4(x) = 174$*, with degree* $deg(P_4) = 0$*.*
- $P_5(x) = 1$*, with degree* $deg(P_5) = 0$*.*
- $P_6(x) = 0$*, with degree* $deg(P_5) = -\infty$*.*
- $P_7(x) = (x-1)(x+2)(x-4)$*, with degree* $deg(P_3) = 3$*.*

In particular every integer can be seen as an integer polynomial of degree zero. P_7 *is a polynomial, because we can expand its definition into* $P_4(x) = x^3 - 3x^2 - 4x + 6$*, which is polynomial of degree* 3*. The following expressions are not integer polynomial*

- $Q_1(x) = 2x^2 + 4 + 3x^{-2}$
- $Q_2(x) = 0.5x^4 - 2x$
- $Q_3(x) = 1/x$

We can invoke sage to do computations with polynomials. To do so we have to specify the symbol for the indertimate and the type for the coefficients. Note however that sage defines the degree of the zero polynomial to be -1.

```
sage: Zx = ZZ['x'] # integer polynomials with indeterminate x
sage: Zt.<t> = ZZ[] # integer polynomials with indeterminate t
sage: Zx
Univariate Polynomial Ring in x over Integer Ring
sage: Zt
Univariate Polynomial Ring in t over Integer Ring
sage: p = Zx([1,2,3,4])
sage: q = Zt([1,2,3,4])
sage: p
4*x^3 + 3*x^2 + 2*x + 1
sage: q
4*t^3 + 3*t^2 + 2*t + 1
sage: p.degree()
3
sage: zero = Zx([0])
sage: zero.degree()
-1
```

Given some element from the same type as the coefficients of a polynomial, the polynomial can be evaluated at that element, which means that we insert the element for every occurence of x in the polynomial expression. To be more precise let $P \in R[x]$, with $P(x) = \sum_{j=0} a_j x^j$ be

some polynomial with coefficient of type R and let $b \in R$ be an element of that type. Then the *evaluation* of P at b is given by

$$P(a) = \sum_{j=0} a_j b^j \tag{4.15}$$

Example 12. *We evaluate the integer polynomials from example XXX at certain integers. We get:*

- $P_1(2) = 2 \cdot 2^2 - 4 \cdot 2 + 17 = 17$
- $P_2(3) = 3^{23} = 94143178827$
- $P_3(-4) = -4 = -4$.
- $P_4(15) = 174$.
- $P_5(0) = 1$.
- $P_6(1274) = 0$.
- $P_7(-6) = (-6-1)(-6+2)(-6-4) = -280$.

Note however that is not possible to evaluate any of those polynomial on values of different type. It is for example strictly speaking wrong to write $P_1(0.5)$*, since* 0.5 *is not an integer.*

As we will see in what follows, polynomials behave like integers in many ways. In particular they can be added, subtracted and multiplied. In addition they have their own notion of Euklidean division.

Definition 4.3.0.1. *Let* $\sum_{n=0}^{\infty} a_n x^n$ *and* $\sum_{n=0}^{\infty} b_n x$ *be two polynomials from* $R[x]$*. Then the sum and the product of these polynomials is defined as follows:*

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n \tag{4.16}$$

$$\left(\sum_{n=0}^{\infty} a_n x^n \right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} \sum_{i=0}^{n} a_i b_{ni} x^n \tag{4.17}$$

In the case of polynomials, it is only necessary to note that the degree of the sum is exactly the maximum of the degrees of the summands and that the degree of the product is exactly the sum of the degrees of the factors.

The ring of polynomials shares a lot of properties with the integers. In particular there is also the concept of Euclidean division and the algorithm of long division defined for polynomials.

Definition 4.3.0.2 (Irreducible Polynomial)**.** *TECHNOBOB*

4.3.1 Lagrange interpolation polynomials

4.4 Algebra Structures

The following theorem makes the idea precises

Theorem 4.4.0.1. *Let $n \in \mathbb{N}_{\geq 2}$ be a fixed, natural number and $\mathbb{Z}_n$ the set of equivalence classes of integers with respect to the congruence modulo n relation. Then $\mathbb{Z}_n$ forms a commutative ring with unit with respect to the addition and multiplication defined above.*

Proof. (**?** sentence 1) □

INTO-BLA

When you read papers about cryptography or mathematical papers in general, you will frequently stumble across terms like *fields*, *groups*,*rings* and similar. As we will use these terms repeatedly, we have to start this chapter with a short introduction to those terms.

In a nutshell, those terms define sets that are in some sense analog to numbers, in that you can add, subtract, multiply or divide. Or more abstractly those sets have certain ways to combine two elements into a new element.

We know many example like the natural numbers, the integers, the ratinal or the real numbers from school and they are in some sense already the most fundamental examples

Lets start with the concept of a group. Remember back in school when we learned about addition and subtraction of integers. We learned that we can always add to integers and that the result is an integer again. We also learned that nothing happens, when we add zero to any integer, that it doesn't matter in which order we add a given set of integers and that for every integer there is always a negative counterpart and if we add both together we get zero.

Distilling these rules to the smallest list of properties and make rthem abstract we arrive at the definition of a group:

Definition 4.4.0.2 (Group)**.** *A grou $(\mathbb{G}, \cdot)$ is a set $\mathbb{G}$, together with a map $\cdot : \mathbb{G} \cdot \mathbb{G} \to \mathbb{G}$, called the group law (or addition respectively multiplication), such that the following hold:*

- *(Existence of a neutral element) There is a $e \in \mathbb{G}$ for all $g \in \mathbb{G}$, such that $e \cdot g = g$ as well as $g \cdot e = g$.*
- *(Existence of an inverse) For every $g \in \mathbb{G}$ there is a $g^{-1} \in \mathbb{G}$, such that $g \cdot g^{-1} = e$ as well as $g^{-1} \cdot g = e$.*
- *(Associativity) For every $g_1, g_2, g_3 \in \mathbb{G}$ the equation $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$ holds.*

In addition a group is called commutative if the equation $g_1 \cdot g_2 = g_2 \cdot g_1$ holds for all $g_1, g_2 \in \mathbb{G}$.

A group is called finite, if the underlying set of elements is finite.

*An element $g \in G$ is called a **generator** of that group if every other group element can be generated by combining g with itself only.*

Remark 3. *By definition, a generator is a special element, that can generate the entire group, by adding it repeatedly to itself. For example the number 1 is a generator for the natural numbers (not a group), as you can compute any natural number by adding 1 repeatedly to itself.*

Remark 4. *If not stated otherwise, we will use the $+$ symbol for the group law of a commutative group. In that case we use the symbol 0 for the neutral element and write the inverse as $-x$.*

Sagemath comes with a definition of the category of all groups, which inherits the properties of *Magmas*, which are like groups but without the notion of inverses. For our purposes we only need commutative groups and in particular finite and cyclic groups.

```
sage: Groups()                                                    84
Category of groups                                                85
```

```
sage: CommutativeAdditiveGroups()
Category of commutative additive groups
sage: FiniteGroups()
Category of finite groups
```

Example 13 (Integer Addition and Subtraction)**.** *As previously described, the set of integers together with integer addition is the archetypical example of a group. In contrast to our definition above the group laws is usually not written as $\cdot$, but as $+$ instead. The neutral element is the number 0 and the inverse of a natural number $x \in \mathbb{Z}$ is the negative $-x$. Since $x+y=y+x$ this is an example of a commutative group.*

Example 14 (The trivial group)**.** *The most basic example of a group, is group with just one element $\{\bullet\}$ and the group law $0 \cdot 0 = 0$. This group can be defined in many ways. In sage it is defined as the group of all permutations of a single element (which is just the identity permutation that is doing nothing). Therefor in sage we can write*

```
sage: TrivialGroup = SymmetricGroup(1)
```

Now thinking of integers again, we know, that there are actually two operations addition and multiplication, such that that they interact in a certain way. This is captured by the concept of a ring, where integers are the most basic example in a sense

Definition 4.4.0.3 (Commutative Ring with Unit)**.** *A ring $(R,+,\cdot)$ is a set R, provided with two maps $+ : R \cdot R \to R$ and $\cdot : R \cdot R \to R$, called addition and multiplication, such that the following conditions hold:*

- *$(R,+)$ is a commutative group, where the neutral element is denoted here with 0.*
- *(Existence of a unit) There is an element $1 \in R$ for all $g \in R$,*
- *(Associativity) For every $g_1, g_2, g_3 \in \mathbb{G}$ the equation $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$ holds.*
- *(Distributivity) For all $g_1, g_2, g_3 \in R$ the distributive laws apply:*

$$g_1 \cdot (g_2 + g_3) = g_1 \cdot g_2 + g_1 \cdot g_3 \quad \textit{and} \quad (g_1 + g_2) \cdot g_3 = g_1 \cdot g_3 + g_2 \cdot g_3$$

- *COMMUTATIVITY*

```
sage: CommutativeRings()
Category of commutative rings
sage: CommutativeRings().super_categories()
[Category of rings, Category of commutative monoids]
```

Example 15 (The Ring of Integers)**.** *The set $\mathbb{Z}$ of integers with the usual addition and multiplication is a ring. In sage the ring of integers is called as follows and as we know, not all elements have multiplicative inverses*

Example 16 (Underlying additive group of a ring)**.** *Every ring $(R,+,\cdot)$ gives rise to group, if we just forget about the multiplication*

The following example is more interesting, as it introduces the concept of so called addition and multiplication table. To understand those tables BLA....

Example 17. *Let $S := \{\bullet, \star, \odot, \otimes\}$ be a set that contains four elements and let adiition and multiplication on S be defined as follows:*

$\cup$	$\bullet$	$\star$	$\odot$	$\otimes$
$\bullet$	$\bullet$	$\star$	$\odot$	$\otimes$
$\star$	$\star$	$\odot$	$\otimes$	$\bullet$
$\odot$	$\odot$	$\otimes$	$\bullet$	$\star$
$\otimes$	$\otimes$	$\bullet$	$\star$	$\odot$

$\circ$	$\bullet$	$\star$	$\odot$	$\otimes$
$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$
$\star$	$\bullet$	$\star$	$\odot$	$\otimes$
$\odot$	$\bullet$	$\odot$	$\bullet$	$\odot$
$\otimes$	$\bullet$	$\otimes$	$\odot$	$\star$

Then $(S, \cup, \circ)$ is a ring with unit $\star$ and zero $\bullet$. It therefore makes sense to ask for solutions to equations like this one: Find $x \in S$ such that

$$\otimes \circ (x \cup \odot) = \star$$

To see how such an "alien equation" can be solved, we have to keep in mind, that rings behaves mostly like normal number when it comes to bracketing and computation rules. The only differences are the symbols and the actual way to add and multiply. With this we solve the equation for x in the "usual way"

$$\begin{aligned}
\otimes \circ (x \cup \odot) &= \star && \text{distributive law} \\
\otimes \circ x \cup \otimes \circ \odot &= \star && \otimes \circ \odot = \odot \\
\otimes \circ x \cup \odot &= \star && \text{add the additive inverse to both sides} \\
\otimes \circ x \cup \odot \cup -\odot &= \star \cup -\odot && \\
\otimes \circ x \cup \bullet &= \star \cup -\odot && \text{inverse property} \\
\otimes \circ x &= \star \cup -\odot && \text{neutral element can be deleted} \\
\otimes \circ x &= \star \cup \odot && -\odot = \odot \text{from addition table} \\
\otimes \circ x &= \otimes && \star \cup \odot = \otimes \text{from addition table} \\
(\otimes)^{-1} \circ \otimes \circ x &= (\otimes)^{-1} \circ \otimes && \text{multiply with the multiplicative inverse} \\
\otimes \circ \otimes \circ x &= \otimes \circ \otimes && \\
\star \circ x &= \star && \\
x &= \star &&
\end{aligned}$$

On the other hand, EQUATION $2x = 3$ has no sultion and equation BLA has more then one solution. So rings are sometimes different from numbers. When we want more we need fields .... BLAHHHH EXPAND

It is a good and fundamental exercise in mathematics to understand examples like this in detail as it helps the reader to loosen the connection to what "should be" numbers and what not.

Definition 4.4.0.4 (Field)**.** *A field $(\mathbb{F}, +, \cdot)$ is a set $\mathbb{F}$, provided with two maps $+ : \mathbb{F} \cdot \mathbb{F} \to \mathbb{F}$ and $\cdot : \mathbb{F} \cdot \mathbb{F} \to \mathbb{F}$, called addition and multiplication, such that the following conditions holds*

- *$(\mathbb{F}, +)$ is a commutative group, where the neutral element is denoted by 0.*
- *$(\mathbb{F} \setminus \{0\}, \cdot)$ is a commutative group, where the neutral element is denoted by 1.*
- *(Distributivity) For all $g_1, g_2, g_3 \in \mathbb{F}$ the distributive laws apply:*

$$g_1 \cdot (g_2 + g_3) = g_1 \cdot g_2 + g_1 \cdot g_3 \quad \text{and} \quad (g_1 + g_2) \cdot g_3 = g_1 \cdot g_3 + g_2 \cdot g_3$$

The characteristic of a field $\mathbb{F}$ is the smallest natural number $n \geq 1$, for which the n -fold sum of 1 equals zero, i.e. for which $\sum_{i=1}^{n} 1 = 0$.

If such a $n > 0$ exists, the field is also called of finite characteristic. If, on the other hand, every finite sum of 1 is not equal to zero, then the field is defined to have characteristic 0.

So a field is a commutative ring with unit, such that every element other the the neutral element of addition has an inverse.

```
sage: Fields()
Category of fields
```

Example 18 (Field of rational numbers)**.** *Probably the best known example of a field is the set of rational numbers $\mathbb{Q}$ together with the usual definition of addition, subtraction, multiplication and division. In sage rational numbers are called like this*

```
sage: QQ
Rational Field
sage: QQ(1/5) # Get an element from the field of rational
   numbers
1/5
sage: QQ(1/5) / QQ(3) # Division
1/15
```

Example 19 (Field with two elements)**.** *It can be shown that in any field, the neutral element 0 of addition must be different from the neutral element 1 of multiplication, that is we always have $0 \neq 1$ in a field. From this follows that the smallest field must contain at least two elements and as the following addition and multiplication tables show, there is indeed a field with two elements, which is usually called $\mathbb{F}_2$:*

Let $\mathbb{F}_2 := \{0,1\}$ be a set that contains two elements and let addition and multiplication on $\mathbb{F}_2$ be defined as follows:

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

For reasons we will understand better in XXX, sage defines this field as a so called Galois field with 2 elements. It is called like this:

```
sage: GF(2)
Finite Field of size 2
sage: GF(2)(1) # Get an element from GF(2)
1
sage: GF(2)(1) + GF(2)(1) # Addition
0
sage: GF(2)(1) / GF(2)(1) # Division
1
```

4.5 Galois fields

As we have seen in the previous section, modular arithmetics behaves in many ways similar to ordinary arithmetics of integers. But in contrast to arithmetics on integers or rational numbers,

we deal with a finite set of elements, which when implemented on computers will not lead to precision problems.

However as we have seen in the last example modular arithmetics is at the same time very different from integer arithmetics as the product of non zero elements can be zero. In addition it is also different from the arithmetics of rational numbers, as there is often no multiplicative inverse, hence no division defined.

In this section we will see that modular arithmetics behaves very nicely, whenever the modulus is a prime number. In that case the rules of modular arithmetics exactly parallels exactly the well know rules of rational arithmetics, despite the fact that the actually computed numbers are very different.

The resulting structures are the so called prime fields and they are the base for many of the contemporary algebra based cryptographic systems.

Since Galois fields are strongly connectedto prime numbers we start with a short overview of prime numbers and provide few basic properties like the fundamental theorem of arithmetic, which says that every natural number can be represented as a finite product of prime numbers.

The key insight here, is that when the modulus is a prime number, modular arithmetic has a well defined division, that is absent for general moduli.

Definition 4.5.0.1 (Prime Fields)**.** *(**?** example 3.4.4 or **?** definition 3.1) Let $p \in \mathbb{P}$ be a prime number. Then we write $(\mathbb{F}, +, \cdot)$ for the set of congruency classes and the induced addition and multiplication as described in theorem (**??**) and call it the **prime field** of characteristic p.*

Remark 5. *We have seen in (4.4.0.1) how do compute addition, subtraction and multiplication in modular arithmetics. AS prime fields are just a special case where the modulus is a prime number, all this stays the same. In addition we have also seen in example (XXX) that division is not always possible in modular arithmetics. However the key insight here is, that division is well defined when the modulus is a prime number. This means that in a prime field we can indeed define devision.*

To be more precise, division is really just multiplication with the so called multiplicative inverse, which is really just another element, such that the product of both elements is equal to 1. This is well known from fractional numbers, where for example the multiplicative element of say 3 is simply $1/3$, since $3 \cdot 1/3 = 1$. Division by 3 is then noth but multiplication by the inverse $1/3$. For example $7/3 = 7 \cdot 1/3$.

We can apply the same reasoning when it comes to prime fields and define division as multiplication with the multiplicative inverse, which leads to the question of how to find the multiplicative inverse of an equivalence class $x \in \mathbb{F}_p$ in a prime field.

*As with all fields, 0 has no inverse, which implies, that division by zero is undefined. So lets assume $x \neq 0$. Then $gcd(x, p) = 1$, since p is a prime number and therefore has no divisors (see **??**).*

So we can use the extended Euclidean algorithm (REF) to compute numbers $x^{-1}, t \in \mathbb{Z}$ with $s \cdot x + t \cdot p = 1$, which gives x^{-1} as the multiplicative inverse of x in $\mathbb{F}_p$, since $x^{-1}x \equiv 1 \pmod p$.

Example 20. *To summarize the basic aspects of computation in prime fields, lets consider the prime field $\mathbb{F}_5$ and simplify the following expression*

$$\left(\frac{2}{3} - 2\right) \cdot 2$$

A first thing to note is that since F_5 is a field all rules like bracketing (distributivity), summing ect. are identical to the rules we learned in school when we where dealing with rational, real or complex numbers.

So we start by evaluating the bracket and get $\left(\frac{2}{3}-2\right)\cdot 2=\frac{2}{3}\cdot 2-2\cdot 2=\frac{2\cdot 2}{3}-2\cdot 2$. *Now we evaluate* $2\cdot 2=4$, *since* $(\ mod\ 4,5)=4$ *and* $-(2\cdot 2)=-4=5-4=1$, *since the negative of a number is just the modulus minus the original number. We therefore get* $\frac{2\cdot 2}{3}-2\cdot 2=\frac{4}{3}+1$.

Now to compute the faction, we need the multiplicative inverse of 3, *which is the number, that when multiplies with* 3 *in* $\mathbb{F}_5$ *gives* 1. *So we use the extended Euclidean algorithm to compute*

$$x^{-1}\cdot 3+t\cdot 5=1$$

Note that in the Euclidean algorithm the compuations of each t_k *is irrelevant here:*

k	r_k	x_k^{-1}	$t_k=(r_k-s_k\cdot a)\ div\ b$
0	3	1	·
1	5	0	·
2	3	1	·
3	2	-1	·
4	1	2	·

So the multiplicative inverse of 3 *in* $\mathbb{Z}_5$ *is* 2 *and indeed if compute* $3\cdot 2$ *we get* 1 *in* $\mathbb{F}_5$.

This implies that we can rewrite our original expression into $\frac{4}{3}+1=4\cdot 2+1=3+1=4$.

The following important property immediately follows from Fermat's little theorem:

Lemma 4.5.0.2. *Let* $p\in\mathbb{P}$ *be a prime number and* $\mathbb{Z}_p$ *be associated prime field of the characteristic p. Then the equations*

$$x^p=x\quad or\quad x^{p-1}=1. \tag{4.18}$$

holds for all elements $x\in\mathbb{Z}_p$ *with* $x\neq 0$.

In order to give the reader an impression of how prime fields can be seen, we give a full computation of a prime field in the following example

Example 21 (The prime field $\mathbb{Z}_5$)**.** *For* $n=5$ *we have five equivalence classes of integers which are congruent modulo* 5. *We write*

$$0:=\{\ldots,-5,0,5,10,\ldots\},\quad 1:=\{\ldots,-4,1,6,11,\ldots\},\quad 2:=\{\ldots,-3,2,7,12,\ldots\}$$
$$3:=\{\ldots,-2,3,8,13,\ldots\},\quad 4:=\{\ldots,-1,4,9,14,\ldots\}$$

Addition and multiplication can now be transferred to the equivalence classes. This results in the following addition and multiplication tables in $\mathbb{Z}_5$*:*

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

These two tables are all you need to be able to calculate in $\mathbb{Z}_5$. *For example, to determine the multiplicative inverse of an element, look for the entry that results in* 1 *in the product table. This is the multiplicative inverse. For example the multiplicative inverse of* 2 *is* 3 *and the multiplicative inverse of* 4 *is* 4 *itself, since* $4\cdot 4=1$.

Hash to Prime fields An important problem in elliptic curve cryptography and in its implementations as a snark is the ability to hash to (various subsets) of elliptic curves. As we will see in XXX those curves are usually defined over prime fields and hashing to a curve often starts with hashing to the prime field. In this paragraph we therefore look at common techniques to hash to a prime field.

In what follows let $\mathbb{F}_q$ be a prime field, such that q is a prime number with m-digits in its binary representation, i.e. $|p_{base_2}| = m$ and let $H : \{0,1\}^* \to \{0,1\}^k$ be a hash function. The methods to map H onto $\mathbb{F}_q$ depend on k.

If $k \leq m-1$, then every image $H(data)$ if interpreted as an integer in its base-2 representation, is smaller then p and hence can directly be interpreted as an element of $\mathbb{F}_q$. So in this case $H : \{0,1\}^* \to \mathbb{F}_q$ can be used unchanged. The drawback of this simple method, is that the bigger the difference between k and m is, the more will the distribution of H deviate from uniformity.

For example in the extreme case $k = 1$, H only maps to $\{0,1\} \subset \mathbb{F}_q$. The best possible case is therefore $k = m-1$. In that case only the highest XXX numbers (DO THE COMPUTATION) are missing from the distribution.

On the other hand if $k \geq m$, then there are basically two commonly used methods to map the output of H onto $\mathbb{F}_q$. The first is to simply forget all leading $k-m+1$-bits from the image of H, which brings you back to our previous consideration.

The second method is to interpret $H(data)$ as an integer and then compute the modulus $H(data)$ mod p. This also introduces a small bias (COMPUTATION FROM THE FORUM ENTRY).

Example 22 (p&p-$\mathbb{F}_{13}$-mod-hash)**.** *Consider our pen&paper hash function from XXX. We want to use this hash function, to define a* 16*-bounded hash function that maps into the prime field* $\mathbb{F}_{13}$*. We define:*

$$\mathscr{H}^{13}_{mod} : \{0,1\}^{16} \to \mathbb{F}_{13} : S \mapsto \mathscr{H}_{PaP}(S) \ mod \ 13$$

Considering the string $S = (1110011101110011)$ *from example XXX again we know* $\mathscr{H}_{PaP}(S) = (1110)$ *and since* $(1110)_{10} = 14$ *and* $14 \ mod \ 13 = 1$ *we get* $\mathscr{H}^{13}_{mod}(S) = 1$*.*

Example 23 (p&p-$\mathbb{F}_{13}$-drop-hash)**.** *We can consider the same pen&paper hash function from XXX and define another hash into* $\mathbb{F}_{13}$*, by deleting the first leading bit from the hash. The result is then a* 3*-digit number and therefore guaranteed to be smaller then* 13*, since* 13 *is equal to* (1101) *in base* 2*.*

Considering the string $S = (1110011101110011)$ *from example XXX again we know* $\mathscr{H}_{PaP}(S) = (1110)$ *and stripping of the leading bit we get* $(110)_{10} = 6$ *as our hash value.*

As we can see this hash function has the drawback of an uneven distribution in $\mathbb{F}_{13}$*. In fact this hash function is unable to map to values from* $\{8,9,10,11,12\}$ *as those numbers have a* 1*-bit in position* 4*. However as we will see in XXX, this hash is cheaper to implement as a circuit as no expensive modulus operation has to be used.*

Definition 4.5.0.3 (The finite field)**.** *Let* $p \in \mathbb{N}$ *be a prime number. Then* $\mathbb{K}_p$ *denotes the remainder class body örper* $\mathbb{Z}/p\mathbb{Z}$ *and* $\mathbb{K}_{p^n}$*, for each* $n \in \mathbb{N}$*, the finite K (unique except for isomorphism) örper with* p^n *elements.*

4.5.1 Square Roots

In this part we deal with *square numbers* and *square roots* in prime fields. To do this, we first define what square roots actually are. We roughly follow Chapter 6.5 in **?**.

Definition 4.5.1.1 (quadratic residue and square roots)**.** *Let $p \in \mathbb{P}$ a prime number and $\mathbb{F}_p$ the associate prime field. Then a number $x \in \mathbb{F}_p$ is called a **square root** of another number $y \in \mathbb{F}_p$, if x is a solution to the equation*

$$x^2 = y \tag{4.19}$$

On the other hand, if y is given and the quadratic equation has no x solution, we call y as quadratic non-residue. For any $y \in \mathbb{F}_p$ we write

$$\sqrt{y}_{|p} := \{x \in \mathbb{F}_p \mid x^2 = y\} \tag{4.20}$$

for the set of all square roots of y in the prime field $\mathbb{F}_n$. (If y is a quadratic non-residue, then $\sqrt{y}_{|p} = \emptyset$ and if $y = 0$, then $\sqrt{y}_{|p} = \{0\}$)

Remark 6. *The notation $\sqrt{y}_{|n}$ for the root of square residues is not found in textbooks, but it is quite practical to clearly distinguish between roots in different prime fields as the symbol $\sqrt{y}$ is ambiguous and it must also be specified in which prime field this root is actually meant.*

Remark 7. *It can be shown that in any prime field every non zero element has either no square root or two of them. We adopt the convention to call the smaller one (when interpreted as an integer) as the **positive** square root and the larger one as the **negative**. This makes sense, as the larger one can always be computed as the modulus minus the smaller one, which is the definition of the negative in prime fields.*

Example 24 (square numbers and roots in $\mathbb{F}_5$)**.** *Let us consider our example (20) again. All square numbers can be found on the main diagonal of the multiplication table. As you can see, in $\mathbb{Z}_5$ only get the square root of 0, 1 and 4 are non empty sets and we get $\sqrt{0}_{|5} = \{0\}$, $\sqrt{1}_{|5} = \{1,4\}$, $\sqrt{2}_{|5} = \emptyset$, $\sqrt{3}_{|5} = \emptyset$ and $\sqrt{4}_{|5} = \{2,3\}$.*

In order to describe whether an element of a prime field is a square number or not, we define (**?** chapter 6.5) the so called Legendre Symbol as its of used in the literature

Definition 4.5.1.2 (Legendre symbol)**.** *Let $p \in \mathbb{P}$ be a prime number and $y \in \mathbb{F}_p$ an element of the associated prime field. Then the so-called Legendre symbol of y is defined as follows:*

$$\left(\frac{y}{p}\right) := \begin{cases} 1 & \text{if } y \text{ has square roots} \\ -1 & \text{if } y \text{ has no square roots} \\ 0 & \text{if } y = 0 \end{cases} \tag{4.21}$$

Example 25. *If we look again at the example (20) we have the following Legendre symbols*

$$\left(\tfrac{0}{5}\right) = 0, \quad \left(\tfrac{1}{5}\right) = 1, \quad \left(\tfrac{2}{5}\right) = -1, \quad \left(\tfrac{3}{5}\right) = -1, \quad \left(\tfrac{4}{5}\right) = 1\,.$$

The following theorem gives a simple criterion for calculating the legendre symbol.

Theorem 4.5.1.3 (Euler criterion)**.** *Let $p \in \mathbb{P}_{\geq 3}$ be an odd Prime number and $y \in \mathbb{F}_p$. Then the Legendre symbol can be computed as*

$$\left(\frac{y}{p}\right) = y^{\frac{p-1}{2}}\,. \tag{4.22}$$

Proof. (**?** proposition 83) □

So the question remains how to actually compute square roots in prime field. The following algorithms give a solution

Definition 4.5.1.4 (Tonelli-Shanks algorithm)**.** *Let p be an odd prime number $p \in \mathbb{P}_{\geq 3}$ and y a quadratic residue in $\mathbb{Z}_p$. Then the so-called Tonneli* **?** *and Shanks* **?** *algorithm computes the two square roots of y. It is defined as follows:*

1. *Find $Q, S \in \mathbb{Z}$ with $p-1 = Q \cdot 2^S$ such that Q is odd.*
2. *Find an arbitrary quadratic non-remainder $z \in \mathbb{Z}_p$.*
3. *The results are then the square roots $r_1 := R$ and $r_2 := p - R$ of y in $\mathbb{F}_p$.*

Remark 8. *The algorithm (4.5.1.4) works in prime fields for any odd prime numbers. From a practical point of view, however, it is efficient only if the prime number is congruent to* 1 *modulo* 4*, since in the other case the formula from the proposition* **??***, which can be calculated more quickly, can be used.*

4.5.2 Exponents and Logarithms

4.5.3 Extension Fields

Eliptic curve pairings often need finite fields that go beyond the finite prime fields.

In fact it is well known, that for every natural number $n \in \mathbb{N}$ there is a field $\mathbb{F}_n$ with n elements, if and only if n is the power of a prime number, i.e. there is a $m \in \mathbb{N}$ with $n = p^m$ for some prime $p \in \mathbb{P}$.

Theorem 4.5.3.1 (Galois Field)**.** *Let $m \in \mathbb{N}$ and $p \in \mathbb{P}$ a prime number. Then there is a field $\mathbb{F}_{p^m}$ of characteristic p, that contains p^n elements.*

We call such a field a **Galois field**. The following algorithm describes the construction of Galois fields (and more general field extensions):

Construction of $\mathbb{F}_{p^m}$

1. Choose a polynomial $P \in \mathbb{F}[t]$ of degree m, that is irreducible.
2. (Field set) of $\mathbb{F}_p$ is the set of all polynomials in $\mathbb{F}[t]$ of degree $< m$, that is
$$\mathbb{F}_{p^m} := \{a_{k-1}t^{k-1} + a_{k-2}t^{k-2} + \ldots + a_1 t + a_0 \mid a_i \in \mathbb{F}_p\}$$
3. (Field Addition) is just addition of the polynomials in the usual way.
4. (Field Multiplication) Multiply the polynomials in the usual way, then compute the long division by P. The remainder is the product.

Remark 9. *In the definition of $\mathbb{F}_{p^m}$, every a_j can have p different values and we have m many of them. This implies that $\mathbb{F}_{p^m}$ contains p^m many elements.*

The construction depends on the choice of an irreducible polynomial, but it can be shown, that all resulting fields are isomorphic, that is they can be transformed into each other, so they are really just different views on the same thing. From an implementations point of view however some choices are better, because they allow for faster computations.

Example 26 (The Galois field $\mathbb{F}_{3^2}$)**.** *In (XXX) we have constructed the prime field $\mathbb{F}_3$. In this example we apply algorithm (XXX) to construct the Galois field $\mathbb{F}_{3^2}$. We start by choosing an irreducibe polynomial of degree 2 with coefficients in $\mathbb{F}_3$. We try $P(t) = t^2 + 1$. Maybe the fastest way to show that P is indeed irreducible is to just insert all elements from $\mathbb{F}_3$ to see if the result is never zero. WE compute*

$$P(0) = 0^2 + 1 = 1$$
$$P(1) = 1^2 + 1 = 2$$
$$P(2) = 2^2 + 1 = 1 + 1 = 2$$

Now the set $\mathbb{F}_{3^2}$ contains all poynomials of degrees lower then two with coefficients in $\mathbb{F}_3$. This is precisely

$$\mathbb{F}_{3^2} = \{0, 1, 2, t, t+1, t+2, 2t, 2t+1, 2t+2\}$$

Addition is then defined as normal addition of polynomials. For example $(t+2)+(2t+2) = (1+2)t+(2+2) = 1$. Doing this computation for all elements give the following addition table

+	0	1	2	t	t+1	t+2	2t	2t+1	2t+2
0	0	1	2	t	t+1	t+2	2t	2t+1	2t+2
1	1	2	0	t+1	t+2	t	2t+1	2t+2	2t
2	2	0	1	r+2	t	t+1	2t+2	2t	2t+1
t	t	t+1	t+2	2t	2t+1	2t+2	0	1	2
t+1	t+1	t+2	t	2t+1	2t+2	2t	1	2	0
t+2	t+2	t	t+1	2t+2	2t	2t+1	2	0	1
2t	2t	2t+1	2t+2	0	1	2	t	t+1	t+2
2t+1	2t+1	2t+2	2t	1	2	0	t+1	t+2	t
2t+2	2t+2	2t	2t+1	2	0	1	t+2	t	t+1

From this table, we can deduce the negative of any element from $\mathbb{F}_{3^2}$. For example in $\mathbb{F}_{3^2}$ we have $-(2t+1) = t+2$, since $(2t+1)+(t+2) = 0$ and the negative of an element is that other element, such that the sum gives the additive neutral element.

Multiplication then needs a bit more computation, as you multiply the polynomials and then divide the result by P and keep the remainder. For example $(t+2)\cdot(2t+2) = 2t^2+2t+t+1 = 2t^2+1$. Long division by $P(t)$ then gives $2t^2+1 : t^2+1 = 2 + \frac{2}{t^2+1}$, so the remainder is 2 and find that the product of $t+2$ and $2t+2$ in $\mathbb{F}_{3^2}$ is 2. Doing this computation for all elements give the following multiplication table (DOTHIS!!! THE TABLE NEEDS AN UPDATE)

·	0	1	2	t	t+1	t+2	2t	2t+1	2t+2
0	0	1	2	t	t+1	t+2	2t	2t+1	2t+2
1	1	2	0	t+1	t+2	t	2t+1	2t+2	2t
2	2	0	1	r+2	t	t+1	2t+2	2t	2t+1
t	t	t+1	t+2	2t	2t+1	2t+2	0	1	2
t+1	t+1	t+2	t	2t+1	2t+2	2t	1	2	0
t+2	t+2	t	t+1	2t+2	2t	2t+1	2	0	1
2t	2t	2t+1	2t+2	0	1	2	t	t+1	t+2
2t+1	2t+1	2t+2	2t	1	2	0	t+1	t+2	t
2t+2	2t+2	2t	2t+1	2	0	1	t+2	t	t+1

From this table, we can deduce the negative of any element from $\mathbb{F}_{3^2}$. For example in $\mathbb{F}_{3^2}$ we have $-(2t+1) = t+2$, since $(2t+1)+(t+2) = 0$ and the negative of an element is that other element, such that the sum gives the additive neutral element.

As we can see from the previous example, it can become quite messy to write elements of extension fields. Especially for larger extension degrees. In the literature we therefore find two ways to have a nicer description.

The most obvious one is to observe that polynomials of maximal degree m are in one to one correspondence with $m+1$-dimensional vectors by

$$\sum_{j=0}^{m} a_j x^j \Leftrightarrow (a_0, a_1, \ldots, a_m)$$

so the coordinate vector associated to a given polynomial is given by its coefficients and vice versa. This way $\mathbb{F}_{q^m}$ can be seen as an m-dimensional vector space over the prime field $\mathbb{F}_q$. Addition is the usual component wise addition of vectors. However multiplication is not obvious and must be computed from the polynomial OR SUBSTITUTION.

Example 27. *In this example the set $\mathbb{F}_{3^2}$ can be written as $\{[0,0],[0,1],[0,2],[1,0],[1,1],[1,2],[2,0],[2,1],[2,2]$ and we get for example $[1,1]+[2,1]=[0,2]$ by adding component wise.*

Power representation

4.6 Epileptic Curves

In this section we introduce epileptic curves as they are used in cryptography, hwhich are certain types of commutative groups basically, well suited for various constructions of various cryptographic primitives.

The eliptic curves we consider are all defined over Galoise fields, so the reader should be familiar with the contend of the previous section.

Definition 4.6.0.1 (Short Weierstraß elliptic Curve)**.** *Let $\mathbb{F}_q$ be a Galois field and $a,b \in \mathbb{F}_q$ be two field elements with $4a^3+27b^2 \neq 0$. Then a short Weierstrass elliptic curve E/F_q over F_q is defined as the set*

$$E/\mathbb{F}_q = \{(x,y) \in \mathbb{F}_q \times \mathbb{F}_q \mid y^2 = x^3+ax+b\} \bigcup \{\mathscr{O}\}$$

of all pairs $(x,y) \in \mathbb{F}_q \times \mathbb{F}_q$ of field elements, that satisfy the short Weierstrass equation $y^2 = x^3+ax+b$, together with the "point at infinity $\mathscr{O}$.

If the characteristic of the Galois field is 2 or 3, that is if $\mathbb{F}_q$ is of the form $\mathbb{F}_{2^n}$ or $\mathbb{F}_{3^n}$ for some $n \in \mathbb{N}$, then this is not the most general way to describe an elliptic curve. However for our purposes this is al we need.

An interesting question is: How many elements does a curve over a finite contain? Since the curve consists of pairs of elements from $\mathbb{F}_q$ plus the point at infinity and $\mathbb{F}_q$ contains q elements, the curve can contain at most q^2+1 many elements. However the following estimation is more precise:

Theorem 4.6.0.2 (Hasse bound)**.** *Let $E/\mathbb{F}_q$ be an elliptic curve over a finite field $\mathbb{F}_q$ and let $|E/\mathbb{F}_q|$ be the number of elements in that curve. Then there is a number t called the **trace**, with $|t| \leq 2\sqrt{q}$ and*

$$|E/\mathbb{F}_q| = q+1-t$$

So roughly speaking, the number of elements in an elliptic curve is approximately equal to the sizeq of an underlying field.

Example 28. *Lets consider our prime field $\mathbb{F}_5$ from (XXX). If we choose $a = 1$ and $b = 0$ then $4a^3 + 27b^2 = 4 \neq 0$ and the corresponding elliptic curve $E/\mathbb{F}_5$ is given by all pairs (x,y) from $\mathbb{F}_5$ such that $y^2 = x^3 + x$. We can find this set simply by trying all 25 combinations of pairs. We get*

$$E_1/\mathbb{F}_5 = \{\mathcal{O}, (0,0), (2,0), (3,0)\}$$

So our elliptic curve contains 4 elements and the trace t is therefore 2. Sage code for this curve:

```
sage: EllipticCurve(GF(5),[1,0])
Elliptic Curve defined by y^2 = x^3 + x over Finite Field of
   size 5
sage: EllipticCurve(GF(5),[1,0]).trace_of_frobenius()
2
```

Example 29. *Consider our prime field F_5 from (XXX). If we choose $a = 1$ and $b = 1$ then $4a^3 + 27b^2 = 1 \neq 0$ and the corresponding elliptic curve $E/\mathbb{F}_3$ is given by all pairs (x,y) from $\mathbb{F}_5$ such that $y^2 = x^3 + x + 1$. We can find this set simply by trying all 25 combinations of pairs. We get*

$$E_2/\mathbb{F}_5 = \{\mathcal{O}, (0,1), (2,1), (3,1), (4,2), (4,3), (0,4), (2,4), (3,4)\}$$

So our elliptic curve contains 9 elements and the trace t is therefore -3.

Keys - Keys on elliptic curves - compressed keys

4.7 The group law

One of the key properties of an elliptic curve is that it is possible to define a group law on the set of its points together with the point at infinity, which serves as the neutral element.

The origin of this law is geometric and known as the chord-and-tangent rule. The rule can be described in the following way:

- (Point addition) Let $P, Q \in E/\mathbb{F}_q - \{\mathcal{O}\}$ with $P \neq Q$ be two distinct points on an elliptic curve $E/\mathbb{F}_q - \{\mathcal{O}\}$, that are both not the point at infinity. Then one can define the sum of P and Q as follows: Consider the line l which intersects the curve in P and Q. If l intersects the elliptic curve at a third point R', define the sum $R = P + Q$ of P and Q as the reflection of R' at the x-axis. If it does not intersect the curve at a third point define the sum to be the point at infinity $\mathcal{O}$.

- (Point doubling) Let $P \in E/\mathbb{F}_q$ with $P \neq Q$ be a points on an elliptic curve $E/\mathbb{F}_q - \{\mathcal{O}\}$. Then the double $2P$ of P is defined as follows: Consider the line wich is tangent to the elliptic curve at P. It can be shown, that it intersects the elliptic curve at the second point R'. $2P$ is then the reflection of this point at the x-axis.

- (Point at infinity) We define $P + \mathcal{O} = \mathcal{O}$ for all points of the eliptic curve $P, Q \in E/\mathbb{F}_q$.

It can be shown that the points of an elliptic curve have a group structure with respect to the tangent and chord rule, with $\mathcal{O}$ as the neutral element. The inverse of any element $P \in E/\mathbb{F}_q$ is the reflection of P on the x-axis.

Translating the chord-and-tanent-rule into algebraic expressions gives the following laws for computing the sum of points on an elliptic curve in short Weierstrass form:

- (Commutativity) $P+Q=Q+P$ for all $P,Q\in E/\mathbb{F}_q$.
- (The neutral element) $P+\mathscr{O}=P$ for all $P\in E/\mathbb{F}_q$.
- (Addition rule) For $P,Q\in E/\mathbb{F}_q$ with $P=(x_1,y_1)$ and $Q=(x_2,y_2)$ and $x_1\neq x_2$, the sum $R=P+Q$ is given by $R=(x_3,y_3)$ with

$$x_3=\left(\frac{y_2-y_1}{x_2-x_1}\right)^2-x_1-x_2 \quad y_3=\left(\frac{y_2-y_1}{x_2-x_1}\right)(x_1-x_3)-y_1$$

- (Doubling rule) For $P\in E/\mathbb{F}_q$ with $P=(x,y)$ and $y\neq 0$, the double of P (sum of P with itself) is given by $2P=(x',y')$ with

$$x'=\left(\frac{3x^2+a}{2y}\right)^2-2x \quad y'=\left(\frac{3x^2+a}{2y}\right)^2(x-x')-y$$

- (The inverse element) For $P\in E/\mathbb{F}_q$ with $P=(x,y)$, the inverse (negative) element is given by $-P:=(x,-y)$.

It can be shown, that when $x_1=x_2$ then $y_1=-y_2$. This implies that the previous rules are complete, since in doubling the case $y=0$, means that the point is its own inverse and hence doubling gives the point at infinity.

As we can see, it is very efficient to compute inverses on elliptic curves.

Definition 4.7.0.1 (Elliptic curve exponentiation and generators)**.** *Let $E/\mathbb{F}_q$ an elliptic curve and $P\in E/\mathbb{F}_q$ a point on that curve. Then the **elliptic curve exponentiation** with base P is given by*

$$[\cdot]P:\mathbb{Z}\to E/\mathbb{F}_q; m\mapsto [m]P$$

*where $[m]P=P+P+\ldots+P$ is the m-fold sum of P with itself. Moreover the point P is called a **generator** of the curve, if $[\mathbb{Z}]P=E/\mathbb{F}_q$.*

Remark 10. *The term exponentiation come from the fact, that if the group law is written in multiplicative notation, then an m-fold product $x\cdot x\cdot\ldots\cdot x$ is usually written in as x^m. So our exponential map, is an adoptation of that for groups with an additive notation of the group law.*

Example 30. *Lets consider the curve $E_1/\mathbb{F}_5$ from example XXX again. We have*

$$E_1/\mathbb{F}_5=\{\mathscr{O},(0,0),(2,0),(3,0)\}$$

So as always $\mathscr{O}$ is the neutral element. Since all elements have 0 as their y-coordinate, it follows that all of them are self inverse, that is $-P=P$. To add, say $(2,0)$ and $(3,0)$ we use the addition rule, since their x-coordinates differ, we get

$$(0,0)+(2,0)=(3,0)$$
$$(0,0)+(3,0)=(2,0)$$
$$(2,0)+(3,0)=(0,0)$$

As we can see $(0,2)$ is not a generator of the group since $[1](0,2)=(0,2)$, $[2](0,2)=(0,2)+(0,2)=\mathscr{O}$. HMM I GUESS THERE IS SOMETHING WRONG HERE. THREE TWO ELEMENT SUBGROUPS SHOULDN'T EXIST??

Example 31. *Consider our prime field F_5 from (XXX). If we choose $a = 1$ and $b = 1$ then $4a^3 + 27b^2 = 1 \neq 0$ and the corresponding elliptic curve $E/\mathbb{F}_3$ is given by all pairs (x,y) from $\mathbb{F}_5$ such that $y^2 = x^3 + x + 1$. We can find this set simply by trying all 25 combinations of pairs. We get*

$$E_2/\mathbb{F}_5 = \{\mathcal{O}, (0,1), (2,1), (3,1), (4,2), (4,3), (0,4), (2,4), (3,4)\}$$

So our elliptic curve contains 9 elements and the trace t is therefore -3.

Definition 4.7.0.2 (The elliptic curve discrete logarithm)**.** . *Let $E/\mathbb{F}_q$ be an elliptic curve and $P, Q \in E/\mathbb{F}_q$ be two points. Then the elliptic curve discrete logarithm problem consists of finding a solution $m \in \mathbb{Z}$, such that*

$$P = [m]Q$$

Such an equation is also called a discrete logarithm relation between P and Q

Remark 11. *If neither P nor Q is a generator of a curve, then a discrete logarithm relation does not exists. On the other hand if one of the elements is a generator, then infinite many solutions exists.*

STUFF ON THE INFEASABILITY TO COMPUTE DISCRETE LOGARITHMS

Definition 4.7.0.3 (Cofactor Clearing)**.** *Since $BLS6-6(13)$ is a subgroup on our curve, it is not possible to leave the subgroup using the curves algebraic laws like scalar multiplication or addition. However in applications it often happens that random elements of the curve are generated, while what we really want are points in the subgroup. To get those points we can use cofactor clearing.*

Cryptographically secure elliptic curves Not all elliptic curves satisfy the requirements from applied cryptography .... Here is a list of properties a curve should satisfy:

1. TECHNOBOB

4.7.1 twists

4.8 Pairings

In this section, we discuss *pairings*, which form the basis of several zk-SNARKs and other zero knowledge proof schemes. The SNARKs derived from pairings have the advantage of constant-sized proof sizes, which is crucial to blockchains.

We start out by defining pairings and discussing a simple application which bears some resemblance to the more advanced SNARKs. We then introduce the pairings arising from elliptic curves and describe Miller's algorithm which makes these pairings practical rather than just theoretically interesting.

4.8.1 Hashing to Curve

In various cryptographic primitives the ability to generate elliptic curve points from binary strings that have properties similar to those of cryptographically secure hash functions is desireable. We call this hash-to-curve.

Of particular interest are constructions, that not only hash to curve points, but where the images have spacial properties, like being in the pairing groupd $\mathbb{G}_1$ or $\mathbb{G}_2$.

Thinking about hashing to curves and hashing to certain subgroups in particular maybe the most obvious thing that comes to mind, is to simply hash to the scalar field of the eliptic curve and then use a generator of the subgroup taget and multiply the generator with the hash value in that scalar field. The result is a valid curve point that is guranteed to be in the desired subgroup.

To be more precise given an elliptic curve $E/\mathbb{F}$, a subgroup $\mathbb{G}$ of E of order r, a generator g of $\mathbb{G}$, a bit-string s and a hash function $H : \{0,1\}^* \to \mathbb{F}_r$, then $[H(s)]g$ is an element of $\mathbb{G}$ and hence $H' : \{0,1\}^* \to \mathbb{G}$ with $H'(s) := [H(s)]g$ is a hash function.

This naive approach however is almost never adequate in cryptographic applications as a discrete log relation is always know between any two given hash value $H'(s)$ and $H'(t)$. In fact for $H(s) \neq 0$, we can define $x = H(t)/H(s)$ and then have $[x]H'(s) = H'(t)$.

So we need a different approach:

Try and increment hash functions One of the most straight forward ways to hash a bitstring onto an eliptic curve point, in a secure way, is to use a cryptographic hash function together with one of the methods we described in XXX to hash to the base field of the curve. Ideally the hash function generates an image that is at least one bit longer then the bit representation of the base field modulus.

The image in the base field can then be interpreted as the x-coordinate of the curve point and the two possible y-coordinates are then derived from the curve equation, while one of the bits that exceeded the modulus determines which of the two y-coordinates to choose.

Such an approach would be easy to implement and deterministic and it will conserve the cryptographic properties of the original hash function. However not all x-ccordinates genrated in such a way, will result in quadratic residues, when inserted into the defining equation. It follows that not all field elements give rise to actual curve points. In fact on a prime field, only half of the field elements are quadratic residues and hence assuming an even distribution of the hash values in the field, this method would fail to generate a curve point in about half of the attempts.

One way to account for this problem is the so called *try and increment* method. Its basic assumption is, that hashing different values, the result will eventually lead to a valid curve point.

Therefore instead of simply hashing a string s to the field the concatenation of s with additional bytes is hashed to the field instead. The bytes are initially zero and interpreted as an unsigned integer. If the first *try* of hashing to the field does not result in a valid curve point, the counter is *incremented* and hashed again. This is repeated until a valid curve point is found eventually.

This method has the advantegous that is relatively easy to implement in code and that it preserves the cryptographic properties of the original hash function. However it is not guranteed to find a valid curve point, as there is a chance that all possible values in the choosen size of the counter bytes fail to generate a quadratic residue. Fortunately it is possible to make the probability for this arbitrarily small by choosing large enough counters and relying on the (approximate) uniformity of the hash-to-field function.

One might think that another disadvantage of this method in the context of snarks is that it can not be implemented as a circuit effectively. This however is not fully true, as a circuit/r1cs only needs to enforce the correctness of the computation. Hence for the circuit it is enough to check the hash of the string and the correct counter. It does not need to find that counter.

So to be more formal, we can define a try and increment hash-to-curve like this

DEF

Conidering certain subgroups of the elliptic curve, the usefulness of this methods depends

highly on the actual situation. For example if a hash to the n-torsion subgroup $\mathbb{G}_1$ is desired, there are two possibilities:

First generic try and increment can be used, followed by a cofactor clearing step. This way every hash on the curve is considered valid, but then projected to $\mathbb{G}_1$ afterwards

Second, the try step not only checks if the curve point actually exists, but also if it is a point in $\mathbb{G}_1$. If its not then the increment step is executed until a valid point is found. This is possible in most applications, since $\mathbb{G}_1$ is usually the by fare largest subgroup in E and the probability to find a point in it is large.

The situation for $\mathbb{G}_2$ as defined in XXX is different and the try and increment method usually fails to find hash values in $\mathbb{G}_2$. For once $\mathbb{G}_2$ is defined in E over an extension field, not the prime field itself, so hashing to $\mathbb{F}$ must be extended into a hash to the extension field. This is possible, but not desireable ventually, because even if we find valid curve points in the curve, $\mathbb{G}_2$. We therefore need different approaches for hashing into $\mathbb{G}_2$

Pederson Hashes

Definition 4.8.1.1 (Pedersen's Hash function)**.** *Let $\mathbb{G}$ be a cyclic group of order r and $\{g_1,\ldots,g_k\} \subset \mathbb{G}$ a uniform randomly generated set of generators of $\mathbb{G}$. Then **Pedersen's hash function***

$$H : (\mathbb{F}_r)^k \to \mathbb{G}$$

is defined for any message $M = (M_1,\ldots,M_k) \in \mathbb{F}_r \times \cdots \times \mathbb{F}_r$, by $H(M) = \Pi_{j=1}^{k}[M_j]g_j$.

SIMPLE WORD DESCRIPTION

Remark 12. *Of course Pedersen's hash function can be combined with a hash-to-field function as described in XXX to give a function that maps ordinary bit strings to group points. However for this function to be secure, it is necessary to proof that there is no known discrete log relations between the elements of the generator set.*

It can be shown that collision resistance is equivalent to the hardness of the discrete log problem XXX.

Example 32.

Posaidon Hashes

4.8.2 Special Functions

Theorem 4.8.2.1 (Pseudo-Randomness in cyclic groups)**.** *Let $\mathbb{G}$ be a cyclic group of order r, such that DDT is hard in $\mathbb{G}$, let g be a generator and $\{a_0,a_1,\ldots,a_n\} \subset \mathbb{F}_r^*$ a uniform randomly generated set of invertible field elements, from the scalar field of $\mathbb{G}$. Then the function*

$$H : \{0,1\}^n \to \mathbb{G}$$

defined for any binary string $s = (s_1,\ldots,s_n)$ by $H(s) = [a_0 \cdot \Pi_{j=1}^{n} a_j^{s_j}]g$ is a pseudo-random function.

4.9 Constructing Elliptic curves

4.9.1 The Complex Multiplication Method

- Choose a prime number $p \in \mathbb{P}$ and integers $t, D \in \mathbb{Z}$, such that the equation $-Dv^2 = 4p - t^2$ has solutions $\pm v \in \mathbb{Z}$.
- If one of the values $p+1-t$ or $p+1+t$ a prime number, then proceed to the next steps, otherwise we go back to step 1.
- Compute the set $CL(Dv^2) = \{(a,b,c) \mid a,b,c \in \mathbb{Z}, |b| \leq a \leq \sqrt{\frac{Dv^2}{3}}, a \leq c, b^2 - 4ac = -Dv^2, (a,b,c) = 1 WHATISTHIS?\}$. If $|b| = a$ or $a = c$, then $b \geq 0$.
- Compute $H_D(x) = \Pi_{(a,b,c) \in CL(D)}(x - j(\frac{-b+\sqrt{-Dv^2}}{2a}))$
- Round the coefficients of H_D to the closed integers.
- Compute $H_{D,p} = H_D \, mod_p$
- Find a root j of $H_{D,p}$
- If $j \neq 0$ or $j \neq 1728$, then choose $c \in \mathbb{F}_p$ and define the elliptic curve $E/\mathbb{F}_p$ defined by $y^2 = x^3 + 3kx + 2k$ for $k = \frac{j}{1728-j}$.
- Compute the order of $E/\mathbb{F}_p$. If it divides either $p+1-u$ or $p+1-u$, then $E/\mathbb{F}_p$ is the result.
- Otherwise choose $c \in \mathbb{F}_p$ with $c \neq 1$ and $c \neq 0$ and define the elliptic curve $E'/\mathbb{F}_p$ defined by $y^2 = x^3 + 3kc^2x + 2kc^3$ for $k = \frac{j}{1728-j}$.

4.10 Classes of elliptic curves

In this section we describes ways to describe elliptic curves different from the general Weierstrass form. Alternative descriptions are sometimes useful because of DIFFERENT ways to express the group laws

4.10.1 Montgomery Form

Let $\mathbb{F}$ be a finite field with characteristic $\neq 2$. A Montgomery curve over $\mathbb{F}$ is defined by the equation

$$M : By^2 = x^3 + Ax^2 + x$$

for certain $A, B \in \mathbb{F}$, with $A \neq \pm 2$, $b \neq 0$ and $B(A^2 - 4) \neq 0$.

The Montgomery form is just another form to define an elliptic curve. In fact every curve in Montgomery form can be transformed into an elliptic curve in Weierstrass form and vice versa. To see that assume that a curve in Montgomery form $By^2 = x^3 + Ax^2 + x$ is given. The associated Weierstrass form is then

$$y^2 = x^3 + \frac{3 - A^2}{3B^2} \cdot x + \frac{2A^3 - 9A}{27B^3}$$

On the other hand, an elliptic curve over base field $\mathbb{F}$ in Weierstrass form $E : v^2 = t^3 + at + b$ can be converted to Montgomery form if and only if the following conditions hold:

- E has order divisible by 4
- The polynomial $z^3 + az + b \in \mathbb{F}[]$ has at least one root $\alpha \in \mathbb{F}$
- $3\alpha^2 + a$ is a quadratic residue in $\mathbb{F}$.

When these conditions are satisfied, then for $s = (\sqrt{3\alpha^2 + a})^{-1}$ the equivalent Montgomery curve is given by

$$sy^2 = x^3 + (3\alpha s)x^2 + x$$

4.10.2 Twisted Edwards Form

In this section we describe curves, that are defined over Galois fields of characteristics $\neq 2$. A **twisted Edwards curve** over a Galois field $\mathbb{F}$ is a curve, defined by the equation

$$E/\mathbb{F} : a \cdot x^2 + y^2 = 1 + d \cdot x^2 y^2$$

where $x, y \in \mathbb{F}$ and $d \in \mathbb{F}\backslash\{0,1\}$ and $d \in \mathbb{F}\backslash\{1\}$ with $a \neq d$. Such a curve is called an Edwards curve (non twisted), if $a = 1$.

On of our main goals is do implement example snarks defined over the scalar field of our BLS6 curve. In particular we want to do elliptic curve cryptography inside of circuits. To do so we need an elliptic curve that we can implement as a circuit over the scalar field of BLS6.

As we have seen in XXX Weierstrass curve have somewhat complicated addition and doubling laws as many cases have to be distinguished. Those cases translate to branches in computer programs, which are costly when implemented in circuits/r1cs. It is therefore advantageous to look for curves with a more simple addition/doubling rule.

Edwards curves are particular useful here as they have a compact addition law that works for all point. Implementing that rule therefore needs no branching.

The most remarkable fact about twisted Edwards curves is their simple addition law. Assuming that a has a root in $\mathbb{F}$ and d has no root in $\mathbb{F}$, the sum of any two points (x_1, y_1), (x_2, y_2) on such an Edwards curve E is given by

$$(x_1, y_1) + (x_2, y_2) = \left(\frac{x_1 y_2 + y_1 x_2}{1 + d x_1 x_2 y_1 y_2}, \frac{y_1 y_2 - a x_1 x_2}{1 - d x_1 x_2 y_1 y_2} \right)$$

The point $(0,1)$ is the neutral element of the addition law. The inverse of a point (x_1, y_1) on E is $(-x_1, y1)$. The addition law is very simple to implement and it can also be used for point doubling. It also works for the neutral element, for inverses..

As Edwards curves have such simple non branching addition laws that work for all points, including the neutral element and inverses, they are a interesting for snarks as they can be implemented with only a few constraints in circuits.

(0,1)

The twisted Edwards form is just another form to define an elliptic curve. In fact every curve in twisted Edwards form can be transformed into an elliptic curve in Montgomery form and vice versa. To see that assume that a curve in twisted Edwards form $a \cdot x^2 + y^2 = 1 + d \cdot x^2 y^2$ is given. The associated Montgomery form is then

$$\frac{4}{a-d} y^2 = x^3 + \frac{2(a+d)}{a-d} \cdot x^2 + x$$

On the other hand a Montgomery curve $By^2 = x^3 + Ax^2 + x$ (WITH STUFF) can be transformed into a twisted Edwards curve

$$(\frac{A+2}{B})x^2 + y^2 = 1 + (\frac{A-2}{B})x^2 y^2$$

4.11 Pend and Paper example curves

4.11.1 BLS6-6 – our pen& paper curve

In this example we want to use the complex multiplication method, to derive a pairing friendly elliptic curve that has similar properties to curves that are used in actual cryptographic protocols. However we design the curve specifically to be useful in pen&paper examples, which mostly means that the curve should contain only a few points, such that we are able to derive exhaustive addition and pairing tables.

A well understood family of pairing friendly curves are the BLS curves (STUFF ABOUT THE HISTORY AND THE NAMING CONVENTION)

BLS curves are particular useful in our case if the embedding degree k satisfies $k \equiv 6 \pmod 0$. In this case the system of polynomials from section XXX parameterizes these curves.

Of course the smallest embedding degree k that satisfies the congruency, is $k = 6$. We therefore aim for a BLS6 curve as our main pen&paper example.

As explained in XXX, the defining polynomials for any BLS6 curve are given by

$$\begin{aligned} r(x) &= \Phi_6(x) \\ t(x) &= x+1 \\ q(x) &= \frac{1}{3}(x-1)^2(x^2-x+1)+x \end{aligned}$$

where Φ_6 is the 6-th cyclotomic polynomial. For any $x \in \mathbb{N}$, where $r(x), t(x), q(x)$ are natural numbers, with $q(x) > 3$ and $r(x) > 3$ those values describe elliptic curves with discriminant $D = 3$, characteristic $p(x)$, prime order subgroup $r(x)$ and Frobenious trace $t(x)$.

We start by looking-up the 6-th cyclotomic polynimial which is $\Phi_6 = x^2 - x + 1$ and then insert small values for x into the defining polynomials r, t, q. This gives the following results:

$$\begin{array}{lll} x = 1 & (r(x), t(x), q(x)) & (1,2,1) \\ x = 2 & (r(x), t(x), q(x)) & (3,3,3) \\ x = 3 & (r(x), t(x), q(x)) & (7,4,\frac{37}{3}) \\ x = 4 & (r(x), t(x), q(x)) & (13,5,43) \end{array}$$

Since $q(1) = 1$ is not a prime number, the first x that gives a proper curve is $x = 2$. However such a curve would be defined over a base field of characteristic 3 and we would rather like to avoid that. We therefore use $x = 4$, which defines a curve of fields of characteristic 43. Since the prime field $\mathbb{F}_{43}$ has 43 elements and 43 has binary representation 101011, which are 6 digits, the name of our pen&paper curve should be $BLS6-6$.

We can check that the embedding degree is indeed 6, since $k = 6$ is the smallest number k such that $r = 13$ divides $43^k - 1$.

Strictly speaking BLS6-6 is not pairing friendly according to the definition in XXX, since indeed $r = 13 > \sqrt{43}$, but the second requirement is not satisfied. This however is irrelevant as the hole point of constructing this curve is to have a "large" prime oder subgroup that is as small as possible.

From the the defining equations of BLS curves, we can immediately deduce that BLS6-6 has a "large" subgroup of prime order 13, which is well suited for our purposes as 13 elements can be easily handled in the associated addition, scalar multiplication and pairing tables.

To see how the rest of the curve will look, we use XX to compute the number of rational points on the curve which is either $q+1-t$ or $q+1+t$. We get $43+1-5 = 39$ or $43+1+5 = 49$. Since our subgroup order $r = 13$ must divide the number of points, it follows that our curve has

39 point and hence a cofactor of 3, which implies that there is a single non trivial "small" order subgroup that contains three elements.

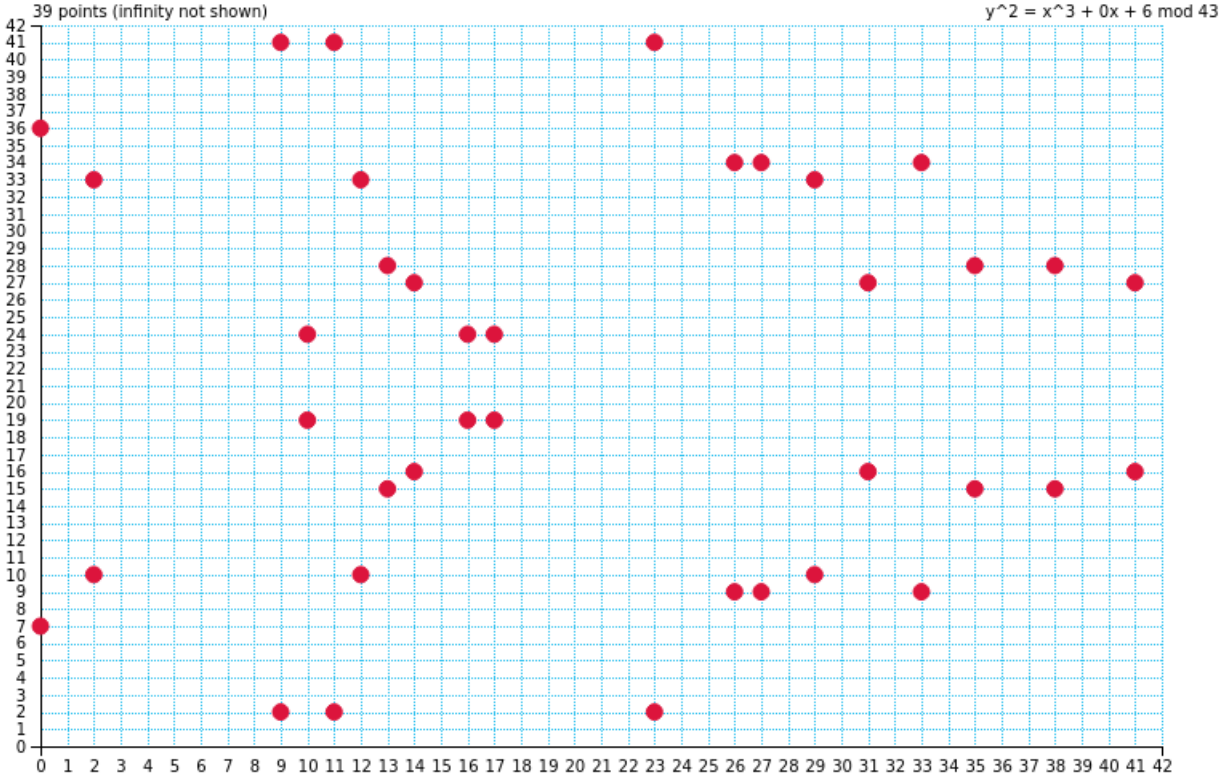
To compute the defining equation $y^2 = x^3 + ax + b$ of BLS6-6, we use the complex multiplication algorithm as described above in XXX. The goal is to find $a, b \in \mathbb{F}_{43}$ representations, that are particulary nice to work with. As shown for example in XXX the discriminant D of all BLS curves is -3, which gives them the general form $y^2 = x^3 + b$.

This is because the Hilbert class polynomial $H_3(x) = x$, since $CL(3) = \{[1,1,1]\}$ and in this case $j(\frac{-1+i\sqrt{3}}{1}) = 0$. It follows that the general curve equation is given by $y^2 = x^3 + b$ and it only remains to find b, such that the curve has the correct number of points which is 39. Since $b \in \mathbb{F}_{43}$, we can just put values for b into the equation and count points. The smallest value then is b and we get

$$BLS6-6: \quad y^2 = x^3 + 6 \quad \text{for all } x, y \in \mathbb{F}_{43}$$

There are other choice for b like $b = 10$ or $b = 23$, but all these curves are isomorphic and hence represent the same thing really but in different way only.

Since BLS6-6 only contains 39 points it is possible to give a visual impression of the curve:



As we can see our curve is somewhat nice, as it does not contain self inverse points that is points with $y = 0$. It follows that the addition law can be optimized, since the branch for those cases can be eliminated.

Note: Is there a way to printe the entire addition table from https://graui.de/code/elliptic2/ here? Would be nice to have but is a bit large.

Since the order of BLS6-6 is $39 = 3 \cdot 13$, we know that it has a "large" subgroup of order 13 and small subgroup of order 3. We can use XXX to find those groups. We have $BLS6-6(3) = \{\mathcal{O}, (0,7), (0,36)\}$.

In addition we have the generator $g_{BLS6} := (13,15)$ that generates

$$BLS6-6(13) = \{(13,15) \to (33,34) \to (38,15) \to (35,28) \to (26,34) \to (27,34) \to (27,9) \to (26,9) \to (35,15) \to (38,28) \to (33,9) \to (13,28) \to \mathscr{O}\} \quad (4.23)$$

Computations "in the exponent": In cryptography and in particular in snarks a lot HAPPENS IN THE EXPONENT...

To use our example to explain what this means observe that from this representation, we can deduce a map from the scalar field $\mathbb{F}_{13}$ to $BLS6-6(13)$ with respect to our generator. WE have

$$[\cdot]_{(13,15)} : \mathbb{F}_{13} \to BLS6-6(13) \; ; \; x \mapsto [x](13,15)$$

So for example we have $[1]_{(13,15)} = (13,15)$, $[7]_{(13,15)} = (27,9)$ and $[0]_{(13,15)} = \mathscr{O}$. In particular this map is a homomorphism of groups from the additive group $\mathbb{F}_{13}$ to $BLS6-6(13)$. This means in particular, the the additive neutral element from $\mathbb{F}_{13}$ is mapped to $\mathscr{O}$ and negatives are mapped to inverses. For example $[-2]_{(13,15)} = -[2]_{(13,15)}$, since $[-2]_{(13,15)} = [11]_{(13,15)} = (33,9) = (33,-34) = -(33,34) = -[2]_{(13,15)}$

The map also give a visualization of the ECDL problem in $BLS6-6(13)$, which is concerned with finding solutions $x \in \mathbb{F}_{13}$ for the equation $[x]_{(13,15)} = (x,y)$ for any $(x,y) \in BLS6-6(13)$. Of course ECDL is not hard in $BLS6-6(13)$, since we can deduce the solutions easily from XXX. For example the solution to $[x]_{(13,15)} = (35,15)$ is $x = 9$, since $[9](13,15) = (35,15)$.

Since $[0]_{(13,15)}$ maps the group of cyclic integers modulo 13 onto our group $BLS6-6(13)$, we can use this to write down the group law in the following way:

·	$\mathscr{O}$	(13,15)	(33,34)	(38,15)	(35,28)	(26,34)	(27,34)	(27,9)	(26,9)	(35,15)	(38,28)	(33,9)	(13,28)
$\mathscr{O}$	$\mathscr{O}$	(13,15)	(33,34)	(38,15)	(35,28)	(26,34)	(27,34)	(27,9)	(26,9)	(35,15)	(38,28)	(33,9)	(13,28)
(13,15)	(13,15)	(33,34)	(38,15)	(35,28)	(26,34)	(27,34)	(27,9)	(26,9)	(35,15)	(38,28)	(33,9)	(13,28)	$\mathscr{O}$
(33,34)	(33,34)	(38,15)	(35,28)	(26,34)	(27,34)	(27,9)	(26,9)	(35,15)	(38,28)	(33,9)	(13,28)	$\mathscr{O}$	(13,15)
(38,15)	(38,15)	(35,28)	(26,34)	(27,34)	(27,9)	(26,9)	(35,15)	(38,28)	(33,9)	(13,28)	$\mathscr{O}$	(13,15)	(33,34)
(35,28)	(35,28)	(26,34)	(27,34)	(27,9)	(26,9)	(35,15)	(38,28)	(33,9)	(13,28)	$\mathscr{O}$	(13,15)	(33,34)	(38,15)
(26,34)	(26,34)	(27,34)	(27,9)	(26,9)	(35,15)	(38,28)	(33,9)	(13,28)	$\mathscr{O}$	(13,15)	(33,34)	(38,15)	(35,28)
(27,34)	(27,34)	(27,9)	(26,9)	(35,15)	(38,28)	(33,9)	(13,28)	$\mathscr{O}$	(13,15)	(33,34)	(38,15)	(35,28)	(26,34)
(27,9)	(27,9)	(26,9)	(35,15)	(38,28)	(33,9)	(13,28)	$\mathscr{O}$	(13,15)	(33,34)	(38,15)	(35,28)	(26,34)	(27,34)
(26,9)	(26,9)	(35,15)	(38,28)	(33,9)	(13,28)	$\mathscr{O}$	(13,15)	(33,34)	(38,15)	(35,28)	(26,34)	(27,34)	(27,9)
(35,15)	(35,15)	(38,28)	(33,9)	(13,28)	$\mathscr{O}$	(13,15)	(33,34)	(38,15)	(35,28)	(26,34)	(27,34)	(27,9)	(26,9)
(38,28)	(38,28)	(33,9)	(13,28)	$\mathscr{O}$	(13,15)	(33,34)	(38,15)	(35,28)	(26,34)	(27,34)	(27,9)	(26,9)	(35,15)
(33,9)	(33,9)	(13,28)	$\mathscr{O}$	(13,15)	(33,34)	(38,15)	(35,28)	(26,34)	(27,34)	(27,9)	(26,9)	(35,15)	(38,28)
(13,28)	(13,28)	$\mathscr{O}$	(13,15)	(33,34)	(38,15)	(35,28)	(26,34)	(27,34)	(27,9)	(26,9)	(35,15)	(38,28)	(33,9)

Cofactor clearing:

Given an arbitrary point on the curve that is not in any of our two subgroups like $(2,33)$, we can project it on both subgroups $BLS6-6(3)$ and $BLS6-6(13)$ respectively, by *multiplication with the cofactor*. Since $39 = 3 \cdot 13$, we have to multiply $(2,33)$ with 13 to map it onto $BLS6-6(3)$ and we have to multiply $(2,33)$ with 3 to map it onto $BLS6-6(13)$. Indeed we get $[13](2,33) = (0,36)$ which is an element of $BLS6-6(3)$ and $[3](2,33) = (35,15)$ which is an element of $BLS6-6(13)$

In what follows we want to compute type 2 pairings on our BLS6 curve. We therefore need to extract the subgroup $\mathbb{G}_1$ as well as $\mathbb{G}_2$ from the full 13-torsion group. We already know from XXX that $\mathbb{G}_1$ is given by

$$\mathbb{G}_1 = \{(13,15) \to (33,34) \to (38,15) \to (35,28) \to (26,34) \to (27,34) \to (27,9) \to (26,9) \to (35,15) \to (38,28) \to (33,9) \to (13,28) \to \mathscr{O}\}$$

In type 2 pairings, the group $\mathbb{G}_2$ is defined by those elements P of the full 13-torsion group, that are mapped to $43 \cdot P$ under the Frobenius endomorphism XXX. Since $BLS6/\mathbb{F}_{13^6}$ contains 6321251664 elements, we can not simply loop through all elements, to find the full 13-torsion group and extract all elements from $\mathbb{G}_2$. However we can derive the full 13-torsion as the set of all 13-division points and then extract G_2 from this

```
sage: F43 = GF(43)
sage: F43t.<t> = F43[]
sage: F43_6.<v> = GF(43^6, name='v', modulus=t^6+6) # t^6+6
   irreducible
sage: BLS6 = EllipticCurve (F43_6,[0 ,6])
sage: INF = BLS6(0) # point at infinity
sage: for P in INF.division_points(13): # PI(P) == [q]P
....:     if P.order() == 13: # exclude point at infinity
....:         PiP = BLS6([a.frobenius() for a in P])
....:         qP = 43*P
....:         if PiP == qP:
....:             print(P.xy())
```

Choose $g_2 = (7v^2, 16v^3)$ as generator of $\mathbb{G}_2$, we get

$$\begin{aligned}\mathbb{G}_2 = \{&(7v^2,16v^3) \to (10v^2,28v^3) \to (42v^2,16v^3) \to (37v^2,27v^3) \to \\ &(16v^2,28v^3) \to (17v^2,28v^3) \to (17v^2,15v^3) \to (16v^2,15v^3) \to \\ &(37v^2,16v^3) \to (42v^2,27v^3) \to (10v^2,15v^3) \to (7v^2,27v^3) \to \mathcal{O}\}\end{aligned}$$

e.g. $[3]g_2 = (42v^2, 16v^3)$.

Having those groups we can do pairings. We choose the Weil pairing and invoke sagemath. For example the Weil pairing between our two generators is

$$e(g_1,g_2) = 5v^5 + 16v^4 + 16v^3 + 15v^2 + 3v + 41$$

```
sage: g1 = BLS6([13,15])
sage: g2 = BLS6([7*v^2, 16*v^3])
sage: g1.weil_pairing(g2,13)
5*v^5 + 16*v^4 + 16*v^3 + 15*v^2 + 3*v + 41
```

As we have seen, $\mathbb{G}_2$ needs quite a bit more storage space then $\mathbb{G}_1$, since elements in $\mathbb{G}_2$ are pairs of polynomials of degree < 6 with coefficients in $\mathbb{F}_{43}$, while elements from $\mathbb{G}_1$ are just pairs of elements from $\mathbb{F}_{43}$.

As we know from XXX it is possible to reduce the space needed to store $\mathbb{G}_2$ by using the concept of a twist. In our case $BLS6$ has embedding degree 6 and the curve parameter a in $y^2 = x^3 + ax + b$ is zero. We therefore know from XXX, that BLS6 has three different twist: A quadratic twist, a cubic twist and a sextic twist. We want to compute all of these twist:

The quadratic twisted BLS6-6 curve: Consider our BLS6-6 curve $BLS6-6/\mathbb{F}_{43^6}$. A quadratic twist is then another curve $BLS6-6_{2-twist}$ over $\mathbb{F}_{43^3}$ isomorphic to the original curve. We use XXX. The task is to find an $\omega \in \mathbb{F}_{43^6}$, such that $\omega^2 \in \mathbb{F}_{43^3}$. We choose $\omega = x^4 + 7x^3 + 9x^2 + 11x + 8$. Then we interpret $\delta = \omega^2 = 27x^2 + 17x + 35$ as an element from $\mathbb{F}_{43^3}$. So our twisted curve is $y^2 = x^3 + a\delta^2 x + b\delta^3 = x^3 + 6 \cdot (27t^2 + 17t + 35)$ so we get

$$BLS6-6_{2-twist}/\mathbb{F}_{43^3} : y^2 = x^3 + (10t^2 + 14t + 15)$$

Baby JubJub

To give an understanding what the Baby-JubJub curve is, we want to parallel its development here to find a Baby-Jubjub like curve for pen and paper.

As with the original large Baby-JubJub curve we apply the method from to define a pen& paper Baby-JubJub-like curve over the scalar field of the "large" BLS6 prime order subgroup, which is $\mathbb{F}_{13}$.

Since 13 mod $4 = 1$ we would go with A.1. As we will only find a few curves, we will tweak the algorithm and run

```
sage: F13 = GF(13)
sage: for A in xrange(3, 13):
....:     if (A-2) % 4 != 0:
....:         continue
....:     try:
....:         E = EllipticCurve(F13, [0, A, 0, 1, 0]) # Montgomery form
....:         E
....:         E.order()
....:     except:
....:         continue
```

So we get two curves in Montgomery form $y^2 = x^3 + 6 * x^2 + x$ which has order 8 and $y^2 = x^3 + 10 * x^2 + x$, which has order 16. We could transform one of them into an Edwards curve, however

So to find our Edwards curve, we will do exhaustive search rather

```
sage: for d in F13:
....:     j= ZZ(0)
....:     for x in F13:
....:         for y in F13:
....:             if x^2+y^2 == 1+d*x^2*y^2:
....:                 j=j+1
....:     print('d=',d)
....:     print('order=',j)
```

and get $x^2 + y^2 = 1 + 7 \cdot x^2 y^2$ which has 20 points. The associated Montgomery curve is then using XXX given by $8y^2 = x^3 + 6 \cdot x^2 + x$.

So we define our Baby-JubJub Edwards curve to be

$$EdBJJ/\mathbb{F}_{13} : x^2 + y^2 = 1 + 7 \cdot x^2 y^2$$

with associated Montgomery form to be

$$MBJJ/\mathbb{F}_{13} : 8y^2 = x^3 + 6 \cdot x^2 + x$$

As $20 = 2 \cdot 2 \cdot 5$, we have a "large" prime order subgroup of order 5 and a cofactor 4. The group of rational points is

```
sage: for x in F13:
....:     for y in F13:
....:         if x^2+y^2 == F13(1)+F13(7)*x^2*y^2:
....:             print(x,y)
```

$(0,1),(0,12),(1,0),(2,4),(2,9),(4,2),(4,11),(5,6),(5,7),(6,5),(6,8),(7,5),(7,8),(8,6),(8,7),(9,2),(9,11$

with neutral element $(0,12)$

As expected we have a prime order subgroup of size 5, which can be generated by $(11,9)$. We get $\{(11,9) \to (6,8) \to (7,8) \to (2,9) \to (0,1))\}$.

```
sage: def Edwards_add((x1,y1),(x2,y2),d):                                    152
....:     x3 = F13((F13(x1)*F13(y2)+F13(y1)*F13(x2))/((F13(1)+               153
   F13(d)*F13(x1)*F13
....: (x2)*F13(y1)*F13(y2))))                                                154
....:     y3 = F13((F13(y1)*F13(y2)-F13(x1)*F13(x2))/((F13(1)-               155
   F13(d)*F13(x1)*F13
....: (x2)*F13(y1)*F13(y2))))                                                156
....:     return (x3,y3)                                                     157
```

Hashing to the pairing groups We give various constructions to hash into $\mathbb{G}_1$ and $\mathbb{G}_2$.

We start with hashing to the scalar field... TO APPEAR

Non of these techniques work for hashing into $\mathbb{G}_2$. We therefore implement Pederson's Hash for BLS6.

We start with $\mathbb{G}_1$. Our goal is to define an 12-bit bounded hash function

$$H_1 : \{0,1\}^{12} \to \mathbb{G}_1$$

Since $12 = 3 \cdot 4$ we "randomly" select 4 uniformly distributed generators $\{(38,15),(35,28),(27,34),(38,28)\}$ from $\mathbb{G}_1$ and use the pseudo-random function from XXX. For every genrator we therefore have to choose a set of 4 randomly generated invertible elements from $\mathbb{F}_{13}$. We choose

$$\begin{array}{lcl} (38,15) & : & \{2,7,5,9\} \\ (35,28) & : & \{11,4,7,7\} \\ (27,34) & : & \{5,3,7,12\} \\ (38,28) & : & \{6,5,1,8\} \end{array}$$

So our hash function is computed like this:

$$H_1(x_{11},x_1,\ldots,x_0) = [2 \cdot 7^{x_{11}} \cdot 5^{x_{10}} \cdot 9^{x_9}](38,15) + [11 \cdot 4^{x_8} \cdot 7^{x_7} \cdot 7^{x_6}](35,28) + [5 \cdot 3^{x_5} \cdot 7^{x_4} \cdot 12^{x_3}](27,34) + [6 \cdot 5^{x_2} \cdot 1^{x_1} \cdot 8^{x_0}](38,28)$$

Note that $a^x = 1$ whe $x = 0$ and hence those terms can be omitted in the computation. In particular the hash of the 12-bit zero string is given by

$$WRONG-ORDERING-REDOH_1(0) = [2](38,15) + [11](35,28) + [5](27,34) + [6](38,28) = (27,34) + (26,34) + (35,28) + (26,9) = (33,9) + (13,28) = (38,28)$$

The hash of 011010101100 is given by

$$H_1(011010101100) = WRONG - ORDERING - REDO$$
$$[2 \cdot 7^0 \cdot 5^1 \cdot 9^1](38,15) + [11 \cdot 4^0 \cdot 7^1 \cdot 7^0](35,28) + [5 \cdot 3^1 \cdot 7^0 \cdot 12^1](27,34) + [6 \cdot 5^1 \cdot 1^0 \cdot 8^0](38,28) =$$
$$[2 \cdot 5 \cdot 9](38,15) + [11 \cdot 7](35,28) + [5 \cdot 3 \cdot 12](27,34) + [6 \cdot 5](38,28) =$$
$$[12](38,15) + [12](35,28) + [11](27,34) + [4](38,28) =$$
$$TO_APPEAR$$

We can use the same technique to define a 12-bit bounded hash function in $\mathbb{G}_2$:

$$H_2 : \{0,1\}^{12} \to \mathbb{G}_2$$

Again we "randomly" select 4 uniformly distributed generators $\{(7v^2, 16v^3), (42v^2, 16v^3), (17v^2, 15v^3), (10v^2, 1$ from $\mathbb{G}_2$ and use the pseudo-random function from XXX. For every genrator we therefore have to choose a set of 4 randomly generated invertible elements from $\mathbb{F}_{13}$. We choose

$$\begin{array}{lcl} (7v^2, 16v^3) & : & \{8,4,5,7\} \\ (42v^2, 16v^3) & : & \{12,1,3,8\} \\ (17v^2, 15v^3) & : & \{2,3,9,11\} \\ (10v^2, 15v^3) & : & \{3,6,9,10\} \end{array}$$

So our hash function is computed like this:

$$H_1(x_{11}, x_{10}, \ldots, x_0) = [8 \cdot 4^{x_{11}} \cdot 5^{x_{10}} \cdot 7^{x_9}](7v^2, 16v^3) + [12 \cdot 1^{x_8} \cdot 3^{x_7} \cdot 8^{x_6}](42v^2, 16v^3) + [2 \cdot 3^{x_5} \cdot 9^{x_4} \cdot 11^{x_3}](17v^2, 15v^3) + [3 \cdot 6^{x_2} \cdot 9^{x_1} \cdot 10^{x_0}](10v^2, 15v^3)$$

We extend this to a hash function that maps unbounded bitstring to $\mathbb{G}_2$ by precomposing with an actual haah function like *MD5* and feet the first 12 bits of its outcome into our previously defined hash function.

$$TinyMD5_{\mathbb{G}_2} : \{0,1\}^* \to \mathbb{G}_2$$

with $TinyMD5_{\mathbb{G}_2}(s) = H_2(MD5(s)_0, \ldots MD5(s)_{11})$. For example, since $MD5(\text{""}) = 0xd41d8cd98f00b204e98$ and the binary representation of the hexadecimal number $0x27e$ is 001001111110 we compute $TinyMD5_{\mathbb{G}_2}$ of the empty string as $TinyMD5_{\mathbb{G}_2}(\text{""}) = H_2(MD5(s)_{11}, \ldots MD5(s)_0) = H2(001001111110) =$

Baby-JubJub-2

To give an understanding what the Baby-JubJub curve is, we want to parallel its development here to find a Baby-Jubjub like curve for pen and paper.

The original Baby-JubJub is a twisted Edwards curve over $\mathbb{F}_?$ with $a = -1$ and $d = ?$.

As with the original large Baby-JubJub curve we apply the method from to define a pen& paper Baby-JubJub-like curve over the scalar field of the "large" BLS6 prime order subgroup, which is $\mathbb{F}_{13}$.

Since 13 mod 4 = 1 we would go with A.1. As we will only find a few curves, we will tweak the algorithm and run

```
sage: F13 = GF(13)
sage: for A in xrange(3, 13):
....:     if (A-2) % 4 != 0:
....:         continue
....:     try:
....:         E = EllipticCurve(F13, [0, A, 0, 1, 0]) # Montgomery form
....:         E
....:         E.order()
....:     except:
....:         continue
```

So we get two curves in Montgomery form $y^2 = x^3 + 6 * x^2 + x$ which has order 8 and $y^2 = x^3 + 10 * x^2 + x$, which has order 16. We could transform one of them into an Edwards curve, however

So to find our Edwards curve, we will do exhaustive search rather

```
sage: j = ZZ(0)
sage: for a in F13:
....:     for d in F13:
....:         j = 0
....:         for x in F13:
....:             for y in F13:
....:                 if a*x^2 + y^2 == 1+d*x^2*y^2:
....:                     j=j+1
....:         print('curve: a=',a,'d=',d,'order:',j)
```

We want to choose a curve that has a large prime order subgroup and a small cofactor. So we go with $2x^2 + y^2 = 1 + 3 \cdot x^2 y^2$ which has order 14.

The associated Montgomery curve is then using XXX given by $9y^2 = x^3 + 2x^2 + x$.

So we define our Baby-JubJub Edwards curve to be

$$EdBJJ/\mathbb{F}_{13} : 2x^2 + y^2 = 1 + 3 \cdot x^2 y^2$$

with associated Montgomery form to be

$$MBJJ/\mathbb{F}_{13} : 9y^2 = x^3 + 2x^2 + x$$

As $14 = 2 \cdot 7$, we have a "large" prime order subgroup of order 7 and a cofactor 2. The group of rational points is

```
sage: for x in F13:
....:     for y in F13:
....:         if F13(2)*x^2+y^2 == F13(1)+F13(11)*x^2*y^2:
....:             print(x,y)
```

$$(0,1),(0,12),(2,4),(2,9),(4,5),(4,8),(5,2),(5,11),(8,2),(8,11),(9,5),(9,8),(11,4),(11,9)$$

with neutral element $(0,1)$

As expected we have a prime order subgroup of size 5, which can be generated by $(11,9)$. We get $\{(11,9) \to (6,8) \to (7,8) \to (2,9) \to (0,1))\}$.

```
sage: def Edwards_add((x1,y1),(x2,y2),a,d):
....:     x3 = F13((F13(x1)*F13(y2)+F13(y1)*F13(x2))/((F13(1)+
   F13(d)*F13(x1)*F13(x2)*F13(y1)*F13(y2))))
....:     y3 = F13((F13(y1)*F13(y2)-F13(a)*F13(x1)*F13(x2))/((
   F13(1)-F13(d)*F13(x1)*F13(x2)*F13(y1)*F13(y2))))
....:     return (x3,y3)
```

4.11.2 MNT4 MNT6 Cycles

Theorem 4.11.2.1. *Let q be a prime and $E/\mathbb{F}_q$ be an ordinary elliptic curve such that $r = |E(Fq)|$ is a prime greater than 3.*

- *E has embedding degree $k = 4$ if and only if there exists $x \in \mathbb{Z}$ such that $t = -x$ or $t = x+1$, and $q = x^2 + x + 1$.*
- *E has embedding degree $k = 6$ if and only if there exists $x \in \mathbb{Z}$ such that $t = 1 \pm 2x$ and $q = 4x^2 + 1$.*
- *There is an elliptic curve $E/\mathbb{F}_q$ with embedding degree 6, discriminant D, and $|E(Fq)| = r$ if and only if there is an elliptic curve $E'/\mathbb{F}_r$ with embedding degree 4, discriminant D, and $|E'(\mathbb{F}_r)| = q$.*

We can use this theorem to find an MNT6-MNT4 cycle over very small prime fields with characteristics > 3:

MNT4 For our MNT4 curve, we can choose $x = 2$. Then $q = 7$ and if we choose $t = x+1$ then $r = q + 1 - t = 7 + 1 - 3 = 5$. Therefore our MNT4 curve is a curve $y^2 = x^3 + ax + b$ defined over $\mathbb{F}_7$ that consists of 5 points.

To construct the actual curve we could use the complex multiplication method again, but since the parameters a and b are from $\mathbb{F}_7$ there are only 48 possibilities so we simply loop through all possible a's and b's and count the curve points until we find a curve that has 5 rational points. We get

$$y^2 = x^3 + 4x + 1$$

defined over $\mathbb{F}_7$, with scalar field $\mathbb{F}_5$. Since $7 = 2^2 + 2 + 1$, we know from theorem XXX, that this curve has embedding degree 4 and hence qualifies as a pen&paper pairing friendly elliptic curve. Since the curve's order is a prime and therefore has no non trivial factors, it has no non trivial subgroups. The curve has the following set of elements

$$MNT4 = \{(0,1) \to (0,6) \to (4,2) \to (4,5) \to \mathscr{O}\}$$

```
sage: F7 = GF(7)                                                    185
sage: MNT4 = EllipticCurve (F7,[4 ,1])                             186
sage: [P.xy() for P in MNT4.points() if P.order() > 1]              187
[(0, 1), (0, 6), (4, 2), (4, 5)]                                    188
```

The multiplication table is

$\cdot$	$\mathscr{O}$	$(0,1)$	$(4,5)$	$(4,2)$	$(0,6)$
$\mathscr{O}$	$\mathscr{O}$	$(0,1)$	$(4,5)$	$(4,2)$	$(0,6)$
$(0,1)$	$(0,1)$	$(4,5)$	$(4,2)$	$(0,6)$	$\mathscr{O}$
$(4,5)$	$(4,5)$	$(4,2)$	$(0,6)$	$\mathscr{O}$	$(0,1)$
$(4,2)$	$(4,2)$	$(0,6)$	$\mathscr{O}$	$(0,1)$	$(4,5)$
$(0,6)$	$(0,6)$	$\mathscr{O}$	$(0,1)$	$(4,5)$	$(4,2)$

In what follows we choose our generator to be $g_{MNT4} = (0,1)$.

In what follows we want to compute type 2 pairings on our MNT4 curve. We therefore need to extract the subgroup $\mathbb{G}_1$ as well as $\mathbb{G}_2$ from the full 5-torsion group. Since the order of MNT4 is a prime number, we already know from XXX that $\mathbb{G}_1$ is given by

$$\mathbb{G}_1 = \{(0,1) \to (0,6) \to (4,2) \to (4,5) \to \mathscr{O}\}$$

In type 2 pairings, the group $\mathbb{G}_2$ is defined by those elements P of the full 5-torsion group, that are mapped to $7 \cdot P$ under the Frobenius endomorphism XXX. Since $MNT4/\mathbb{F}_{7^4}$ only contains 2475 elements, we can loop through all elements, to find the full 5-torsion group and extract all elements from $\mathbb{G}_2$:

```
sage: F7t.<t> = F7[]
sage: F7_4.<u> = GF(7^4, name='u', modulus=t^4+t+1) #
   embedding degree is 4
sage: MNT4 = EllipticCurve (F7_4,[4 ,1])
sage: INF = MNT4(0) # point at infinity
sage: for P in INF.division_points(5): # PI(P) == [q]P
....:     if P.order() == 5: # exclude point at infinity
....:         PiP = MNT4([a.frobenius() for a in P])
....:         qP = 7*P
....:         if PiP == qP:
....:             print(P.xy())
```

Choose $g_2 = (2u^3+5u^2+4u+2, 2u^3+3u+5)$ as generator of $\mathbb{G}_2$, we get

$$\mathbb{G}_2 = \{(2u^3+5u^2+4u+2, 2u^3+3u+5) \to (5u^3+2u^2+3u+6, 2u^2+3u) \to (5u^3+2u^2+3u+6, 5u^2+4u) \to (2u^3+5u^2+4u+2, 5u^3+4u+2) \to \mathcal{O}\}$$

e.g. $[3]g_2 = (5u^3+2u^2+3u+6, 5u^2+4u)$.

Having those groups we can do pairings. We choose the Weil pairing and invoke sagemath. For example the Weil pairing between our two generators is

$$e(g_1, g_2) = 5u^3+2u^2+6u$$

```
sage: g1 = MNT4([0,1])
sage: g2 = MNT4(2*u^3 + 5*u^2 + 4*u + 2, 2*u^3 + 3*u + 5)
sage: g1.weil_pairing(g2,5)
5*u^3 + 2*u^2 + 6*u
```

The full pairing table can the be written as

$e(\cdot,\cdot)$	$\mathcal{O}$	g_1	$[2]g_1$	$[3]g_1$	$[4]g_1$
$\mathcal{O}$	1	1	1	1	1
g_2	1	$5u^3+2u^2+6u$	$6u^3+5u^2+6$	$2u^3+u^2+2u+3$	u^3+6u^2+6u+4
$[2]g_2$	1	$6u^3+5u^2+6$	u^3+6u^2+6u+4	$5u^3+2u^2+6u$	$2u^3+u^2+2u+3$
$[3]g_2$	1	$2u^3+u^2+2u+3$	$5u^3+2u^2+6u$	u^3+6u^2+6u+4	$6u^3+5u^2+6$
$[4]g_2$	1	u^3+6u^2+6u+4	$2u^3+u^2+2u+3$	$6u^3+5u^2+6$	$5u^3+2u^2+6u$

MNT6 For our MNT6 curve, we can choose $x = 1$. Then $q = 5$ and if we choose $t = 1+2x$ then $r = q+1-t = 5+1+1 = 7$. Therefore our MNT6 curve is a curve $y^2 = x^3+ax+b$ defined over $\mathbb{F}_5$ that consists of 7 points.

To construct the actual curve we could use the complex multiplication method again, but since the parameters a and b are from $\mathbb{F}_5$ there are only 24 possibilities, we simply loop through

all possible a's and b's and count the curve points until we find a curve that has 7 rational points. We get

$$y^2 = x^3 + 2x + 1$$

defined over $\mathbb{F}_5$. Since $5 = 4 \cdot 1 + 1$, we know from theorem XXX, that this curve has embedding degree 6 and hence qualifies as a pen&paper pairing friendly elliptic curve.

The curve has the following set of elements

$$MNT6 = \{(1,2) \to (3,3) \to (0,1) \to (0,4) \to (3,2) \to (1,3) \to \mathscr{O}\}$$

The multiplication table is

$\cdot$	$\mathscr{O}$	$(1,2)$	$(3,3)$	$(0,1)$	$(0,4)$	$(3,2)$	$(1,3)$
$\mathscr{O}$	$\mathscr{O}$	$(1,2)$	$(3,3)$	$(0,1)$	$(0,4)$	$(3,2)$	$(1,3)$
$(1,2)$	$(1,2)$	$(3,3)$	$(0,1)$	$(0,4)$	$(3,2)$	$(1,3)$	$\mathscr{O}$
$(3,3)$	$(3,3)$	$(0,1)$	$(0,4)$	$(3,2)$	$(1,3)$	$\mathscr{O}$	$(1,2)$
$(0,1)$	$(0,1)$	$(0,4)$	$(3,2)$	$(1,3)$	$\mathscr{O}$	$(1,2)$	$(3,3)$
$(0,4)$	$(0,4)$	$(3,2)$	$(1,3)$	$\mathscr{O}$	$(1,2)$	$(3,3)$	$(0,1)$
$(3,2)$	$(3,2)$	$(1,3)$	$\mathscr{O}$	$(1,2)$	$(3,3)$	$(0,1)$	$(0,4)$
$(1,3)$	$(1,3)$	$\mathscr{O}$	$(1,2)$	$(3,3)$	$(0,1)$	$(0,4)$	$(3,2)$

In what follows we choose our generator to be $g_{MNT6} = (1,2)$.

In what follows we want to compute type 2 pairings on our MNT6 curve. We therefore need to extract the subgroup $\mathbb{G}_1$ as well as $\mathbb{G}_2$ from the full 7-torsion group. Since the order of MNT6 is a prime number, we already know from XXX that $\mathbb{G}_1$ is given by

$$\mathbb{G}_1 = \{(1,2) \to (3,3) \to (0,1) \to (0,4) \to (3,2) \to (1,3) \to \mathscr{O}\}$$

In type 2 pairings, the group $\mathbb{G}_2$ is defined by those elements P of the full 7-torsion group, that are mapped to $5 \cdot P$ under the Frobenius endomorphism XXX. Since $MNT6/\mathbb{F}_{56}$ contains 15680 elements, we can still loop through all elements, to find the full 7-torsion group and extract all elements from $\mathbb{G}_2$

```
sage: G.<x> = GF(5^6) # embedding degree is 6
sage: MNT6 = EllipticCurve (G,[2 ,1])
sage: INF = MNT6(0) # point at infinity
sage: for P in INF.division_points(7): # PI(P) == [q]P
....:     if P.order() == 7: # exclude point at infinity
....:         PiP = MNT6([a.frobenius() for a in P])
....:         qP = 5*P
....:         if PiP == qP:
....:             print(P.xy())
```

$$\mathbb{G}_2 = \{(x^3+2x^2+4x, x^5+2x^4+4x^3+3x^2+3) \to (x^5+4x^4+2x^3+3x^2+x+2, 3x^4+2x^3+x) \to$$
$$(4x^5+x^4+2x^3, 3x^5+x^4+x^3+4x+4) \to (4x^5+x^4+2x^3, 2x^5+4x^4+4x^3+x+1) \to$$
$$(x^5+4x^4+2x^3+3x^2+x+2, 2x^4+3x^3+4x) \to (x^3+2x^2+4x, 4x^5+3x^4+x^3+2x^2+2) \to \mathscr{O}\}$$

We choose the generator $g_2 = (x^3+2x^2+4x, x^5+2x^4+4x^3+3x^2+3)$

Remark 13. *Note however that our MNT6 curve discriminant* $D = -16(4a^3 + 27b^2) = -16(4 \cdot 2^3 + 27 \cdot 1^2) = -944$*, while our MNT4 curve has discriminsnt XXX. Hence our example curves are not those guranteed by theorem XXX. Those curve are both given by* $y^2 = x^3 + 2x + 1$ *over* $\mathbb{F}_5$ *and* $\mathbb{F}_7$*, respectively. However as both curves have the same defining equation, we rather choose examples that are visually distinguishable by their defining equations.*

4.11.3 Edwards curve cycles

5 Zk-Proof Systems

Some philosophical stuff about compuational models for snarks. Bounded computability...

5.1 Computational Models

Proofs are the evidence of correctness of the assertions, and people can verify the cor-rectness by reading the proof. However, we obtain much more than the correctness itself:After you read one proof of an assertion, you know not only the correctness, but also why itis correct. Is it possible to solely show the correctness of an assertion without revealing theknowledge of proofs? It turns out that it is indeed possible, and this is the topic of today'slecture: Zero Knowledge Systems.

Example 33 (Generalized factorization snark)**.** *As one of our major running examples we want to derive a zk-SNARK for the following generalized factorization problem:*

Given two numbers $a,b \in \mathbb{F}_{13}$, find two additional numbers $x,y \in \mathbb{F}_{13}$, such that

$$(x \cdot y) \cdot a = b$$

and proof knowledge of those numbers, without actually revealing them.

Of course this example reduces to the classic factorization problem (over $\mathbb{F}_{13}$ by setting $y = 1$)

This zero knowledge system deals with the following situation: "Given two publicly known numbers $a,b \in \mathbb{F}_{13}$ a proofer can show that they know two additional numbers $x,y \in \mathbb{F}_{13}$, such that $(x \cdot y) \cdot a = b$, without actually revealing x or y."

Of course our choice of what information to hide and what to reveal was completely arbitrary. Every other split would also be possible, but eventually gives a different problem.

For example the task could be to not hide any of the variables. Such a system has no zero knowledge and deals with verifiable computations: "A worker can proof that they multiplied three publicly known numbers $a,b,x \in \mathbb{F}_{13}$ and that the result is $z \in \mathbb{F}_{13}$, in such a way that no verifier has to repeat the computation."

5.1.1 Formal Languages

Roughly speaking a formal language is nothing but a set of words, that are strings of letters taken from some alphabet and formed according to some defining rules of that language.

In computer science, formal languages are used for defining the grammar of programming languages in which the words of the language represent concepts that are associated with particular meanings or semantics. In computational complexity theory, decision problems are typically defined as formal languages, and complexity classes are defined as the sets of the formal languages that can be parsed by machines with limited computational power.

Definition 5.1.1.1 (Formal Language)**.** *Let Σ be a set and Σ^* the set of all finite strings of elements from Σ. Then a **formal language** L is a subset of Σ^*. The set Σ is called the **alphabet***

*of L and elements from L are called **words**. The rules that specify which strings from Σ^* belong to L are called the **grammar** of L.*

*In the context of proofing systems we often call words **statements**.*

Example 34 (Generalized factorization snark)**.** *Consider example 32 again. Definition 5.1.1.1 is not quite suitable yet to define the example, since there is not distinction between public input and private input.*

However if we assume for the moment that the task in example 32 is to simply find $a,b,x,y \in \mathbb{F}_{13}$ such that that $x \cdot y \cdot a \cdot = b$, then we can define the entire solution set as a language L_{factor} over the alphabet $\Sigma = \mathbb{F}_{13}$. We then say that a string $w \in \Sigma^*$ is a statement in our language L_{factor} if and only if w consists of 4 letters w_1, w_2, w_3, w_4 that satisfy the equation $w_1 \cdot w_2 \cdot w_3 = w_4$.

Example 35 (Binary strings)**.** *If we take the set $\{0,1\}$ as our alphabet Σ and imply no rules at all to form words in this set. Then our language L is the set $\{0,1\}^*$ of all finite binary strings. So for example $(0,0,1,0,1,0,1,1,0)$ is a word in this language.*

Example 36 (Programing Language)**.**

Example 37 (Compiler)**.**

As we have seen in general not all strings from an alphabet are words in a language. So an important question is, weather a given string belongs to a language or not.

Definition 5.1.1.2 (Relation, Statement, Instance and Witness)**.** *Let Σ_I and Σ_W be two alphabets. Then the binary relation $R \subset \Sigma_I^* \times \Sigma_W^*$ is called a **checking relation** for the language*

$$L_R := \{(i,w) \in \Sigma_I^* \times \Sigma_W^* \mid R(i,w) \}$$

*of all **instances** $i \in \Sigma_I^*$ and **witnesses** $i \in \Sigma_I^*$, such that the **statement** (i,w) satisfies the checking relation.*

Remark 14. *To summarize the definition, a statement is nothing but a membership claim of the form $x \in L$. So statements are really nothing but strings in an alphabet that adhere to the rules of a language.*

However in the context of checking relations, there is another interpretations in terms of a knowledge claim of the form "In the scope of relation R, I know a witness for instance x." This is of particular importance in the context of zero knowledge proofing systems, where the instance represents public knowledge, while the witness represents the data that is hidden (the zero-knowledge part).

For some cases, the knowledge and membership types of statements can be informally considered interchangeable, but formally there are technical reasons to distinguish between the two notions (See for example XXX)

Example 38 (Generalized factorization snark)**.** *Consider example 32 and our associate formal language 33. We can define another language $L_{zk-factor}$ for that example by defining the alphabet $\Sigma_I \times \Sigma_W$ to be $\mathbb{F}_{13} \times \mathbb{F}_{13}$ and the checking relation $R_{zk-factor}$ such that $R(i,w)$ holds if and only if instance i is a two letter string $i = (a,b)$ and witness w is a two letter string $w = (x,y)$, such that the equation $x \cdot y \cdot a = b$ holds.*

So to summarize four elements $x,y,a,b \in \mathbb{F}_{13}$ form a statement $((x,y),(a,b))$ in $L_{zk-factor}$ with instance (a,b) and witness x, y, precisely if, given a and b, the values x and y are a solution to the generalized factorization problem $x \cdot y \cdot a = b$.

Example 39 (SHA256 relation)**.** *ssss*

As the following example shows checking relations and their languages are quite general and able to express in particular the class of all terminating computer programs:

Example 40 (Computer Program)**.** *Let A be a terminating algorithm that transforms a binary string of inputs in finite execution steps into a binary output string. We can then interpret A as a map*

$$A : \{0,1\}^* \to \{0,1\}^*$$

Algorithm A then defines a relation $R \subset \{0,1\}^* \times \{0,1\}^*$ in the following way: instance string $i \in \{0,1\}^*$ and witness string $w \in \{0,1\}^*$ satisfy the relation R, that is $R(i,w)$, if and only if w is the result of algorithm A executed on input instance i.

5.1.2 Circuits

Definition 5.1.2.1 (Circuits)**.** *Let Σ_I and Σ_W be two alphabets. Then a directed, acyclic graph C is called a **circuit** over $\Sigma_I \times \Sigma_W$, if the graph has an ordering and every node has a label in the following way:*

- *Every source node (called input) has a letter from $\Sigma_I \times \Sigma_W$ as label.*
- *Every sink node (called output) has a letter from $\Sigma_I \times \Sigma_W$ as label.*
- *Every other node (called gate) with j incoming edges has a label that consist of a function $f : (\Sigma_I \times \Sigma_W)^j \to \Sigma_I \times \Sigma_W$.*

Remark 15 (Circuit-SAT)**.** *Every circuit with n input nodes and m output nodes can be seen a function that transforms strings of size n from $\Sigma_I \times \Sigma_W$ into strings of size m over the same alphabet. The transformation is done by sending the strings from a node along the outgoing edges to other nodes. If those nodes are gates, then the string is transformed according to the label.*

By executing the previous transformation, every node of a circuit has an associated letter from $\Sigma_I \times \Sigma_W$ and this defines a checking relation over $\Sigma_I^* \times \Sigma_W^*$. To be more precise, let C be a circuit with n nodes and $(i,w) \in \Sigma_I^j \times \Sigma_W^k$ a string. Then $R_C(i,w)$ iff THE CIRCUIT IS SATISFIED WHEN ALL LABELS ARE ASSOCIATED TO ALL NODES IN THE CIRCUIT.... BUT MORE PRECISE

MODULO ERRORS. TO BE CONTINUED.....

An Assignment associates field elements to all edges (indices) in an algebraic circuit. An Assignment is valid, if the field element arise from executing the circuit. Every other assignment is invalid.

The checking relation for circuit-SAT then is satidfied if valid asignment (TODO: THE WITNESS/INSTANCE SPLITTING)

Valid assignments are proofs for proper circuit execution.

So to summarize, algebraic circuits (over a field $\mathbb{F}$) are directed acyclic graphs, that express arbitrary, but bounded computation. Vertices with only outgoing edges (leafs, sources) represent inputs to the computation, vertices with only ingoing edges (roots, sinks) represent outputs from the computation and internal vertices represent field operations (Either addition or multiplication). It should be noted however that there are many circuits that can represent the same laguage...

Circuits have a notion of execution, where input values are send from leafs along edges, through internal vertices to roots.

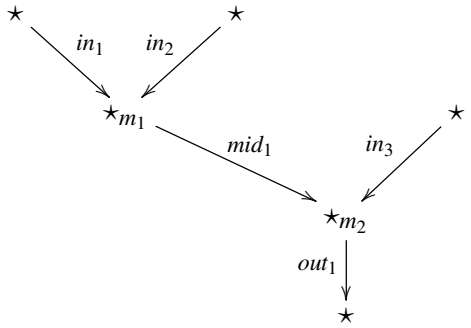
Remark 16. *Algebraic circuits are usually derived by Compilers, that transform higher languages to circuits. An example of such a compiler is XXX. Note: Different Compiler give very different circuit representations and Compiler optimization is important.*

Example 41 (Generalized factorization snark)**.** *Consider our generalized factorization example 32 with associated language 37.*

To write this example in circuit-SAT, consider the following function

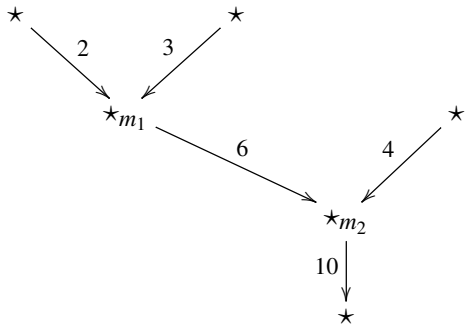
$$f : \mathbb{F}_{13} \times \mathbb{F}_{13} \times \mathbb{F}_{13} \to \mathbb{F}_{13}; (x_1, x_2, x_3) \mapsto (x_1 \cdot x_2) \cdot x_3$$

A valid circuit for $f : \mathbb{F}_{11} \times \mathbb{F}_{11} \times \mathbb{F}_{11} \to \mathbb{F}_{11}; (x_1, x_2, x_3) \mapsto (x_1 \cdot x_2) \cdot x_3$ is given by:



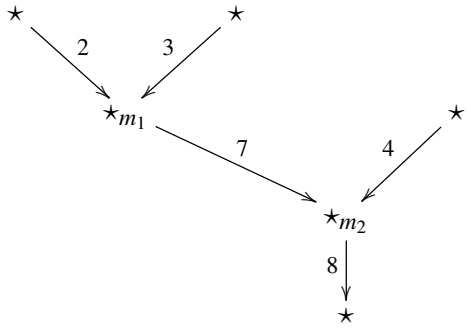
with edge-index set $I := \{in_1, in_2, in_3, mid_1, out_1\}$.

To given a valid assignment, consider the set $I_{valid} := \{in_1, in_2, in_3, mid_1, out_1\} = \{2,3,4,6,10\}$



Appears from multiplying the input values at m_1, m_2 in $\mathbb{F}_{13}$, hence by executing the circuit.

Non valid assignment: $I_{err} := \{in_1, in_2, in_3, mid_1, out_1\} = \{2,3,4,7,8\}$



Can not appear from multiplying the input values at m_1, m_2 in $\mathbb{F}_{13}$

To match the requirements of the inital task 32, we have to split the statement into instance and witness. So given index set $I := \{in_1, in_2, in_3, mid_1, out_1\}$, we assume that every step in the computation other then in_3 and out_1 are part of the witness. So we choose:

- *Instance* $S = \{in_3, out_1\}$.
- *Witness* $W = \{in_1, in_2, mid_1\}$.

Example 42 (Baby JubJub for BLS6-6)**.**

Example 43 (ECDH as a circuit)**.** *over BLS6*

Example 44 (BLS Signature)**.** *example of one layer recursion over MNT4 and MNT6*

Example 45 (Boolean Circuits)**.**

Example 46 (Algebraic (Aithmetic) Circuits)**.**

Any program can be reduced to an arithmetic circuit (a circuit that contains only addition and multiplication gates). A particular reduction can be found for example in [BSCG+13]

5.1.3 Rank-1 Constraint Systems

Definition 5.1.3.1 (Rank-1 Constraint system)**.** *Let* $\mathbb{F}$ *be a Galois field,* i, j, k *three numbers and* A*,* B *and* C *three* $(i+j+1) \times k$ *matrices with coefficients in* $\mathbb{F}$*. Then any vector* $x = (1, \phi, w) \in \mathbb{F}^{1+i+j}$ *that satisfies the* ***rank-1 constraint system*** *(R1CS)*

$$Ax \odot Bx = Cx$$

(where $\odot$ *is the Hadamard/Schur product) is called a* ***statement*** *of that system, with* ***instance*** ϕ *and* ***witness*** w*.*

We call k *the* ***number of constraints****,* i *the* ***instance*** *size and* j *the* ***witness*** *size.*

Remark 17. *Any Rank-1 constraint system defines a formal language in the following way: Consider the alphabets* $\Sigma_I := \mathbb{F}$ *and* $\Sigma_W : \mathbb{F}$*. Then a checking relation* $R_{R1CS} \subset \Sigma_I^i \times \Sigma_W^j \subset \Sigma_I^* \times \Sigma_W^*$ *is defined by*

$$R_{R1CS}(i, w) \Leftrightarrow (i, w) \text{ satisfies the R1CS}$$

As shown in XXX such a checking relation defines a formal language. We call this language ***R1CS satisfiability****.*

Example 47 (Generalized factorization snark)**.** *Defining the 5-dimensional affine vector* $w = (1, in_1, in_2, in_3, m_1, out_1)$ *for* $in_1, in_2, in_3, m_1, out_1 \in \mathbb{F}_{13}$ *and the* $6 \times ?$*-matrices*

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We can instantiate the general R1CS equation $Aw \odot Bw = Cw$ *as*

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ in_1 \\ in_2 \\ in_3 \\ m_1 \\ out_1 \end{pmatrix} \odot \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ in_1 \\ in_2 \\ in_3 \\ m_1 \\ out_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ in_1 \\ in_2 \\ in_3 \\ m_1 \\ out_1 \end{pmatrix}$$

So evaluating all three matrix products and the Hadarmat prodoct we get two constraint equations

$$\begin{aligned} in_1 \cdot in_2 &= m_1 \\ m_1 \cdot in_3 &= out_1 \end{aligned}$$

So from the way this R1CS is constructed, we know that whatever the underlying field $\mathbb{F}$ is, the only solutions to this equations are

$$\{(0,0,0), (0,1,0), (1,0,0), (1,1,1)\}$$

Gadgets Rank 1 contraints systems can become very large ....

Boolean Algebra

Sometimes it is necessary to assume that a statement describes boolean variables. However by definition the alphabet of a statement is a finite field, which is often the scalar field of a large prime order cyclic group. So developers need a way to simulate boolean algebra inside other finite fields.

The most common way to do this, is to interpret the additive and multiplicate neutral element $\{0,1\} \subset F$ as boolean values. This is convinient because they are defined in any field.

In what follows we will define a few of the most basic R1CS to check boolean expressions in R1CS satidfyability. We will leave other basic constructions as exercises to the reader.

We start with actually constraining field elements to boolean values then Once field elements are boolean constraint, we need constraints that are able to enforce boolean algebra on them. We therefore give constraints for the functionally complete set of Boolean operators give by *AND* and *NOT*. As all other boolean operations can be constructed from *AND* and *NOT*, this sufficies. However in actual implementations it is of high importance to limit the number of constraints as much as possible. In reality it is therefor advantageous to implement all logic operators in constraints.

Boolean Constraint So when a developer needs boolean variables as part of their statement, a R1CS is required on those variables, that enforces the variable to be either 1 or 0. So to "constrain a field element $x \in \mathbb{F}$ to be 1 or 0 what we need is a system of equation $(A_i x) \cdot (B_i x) = C_i x$ for some $A_i, B_i, C_i \in \mathbb{F}$, such that the only possible solutions for x are 0 or 1. As it turns out such a system can be realized by a single equation $x \cdot (1-x) = 0$ We see that indeed 0 and 1 are the only solutions here, since for the right side to be zero, at least one factor on the left side needs to be zero and this only happens for 0 and 1.

So now that we have found a correct equation for a boolean constrain, we have to translate it into the associated R1CS format, which is given by

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} \odot \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}$$

So we get the following statement $\phi = (1, i, w) = (1, x)$, with instance (public input) $i = x$ and now witness (private input) w. In addition we get the matrices $A = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 \end{pmatrix}$.

To make those constraints easily accesable for R1CS developers, a gadget is convinient:

AND-constraints Given three field elements $x, y, z \in \mathbb{F}$ that represent boolean variables, we want to find a R1CS, such that $w = (1, x, y, z)$ satisfies the constraint system if and only if $x \, AND \, y = z$.

So first we have to constrain x, y and z to be boolean as explained in XXX. The next thin is we need to find a R1CS that enforces the *AND* logic. We can simply choose $x \cdot y = z$, since (for boolean constraint values) $x \cdot y$ equals 1 if and only if both x and y are 1.

Now that we have found a correct equation for a boolean constrain, we have to translate it

into the associated R1CS format, which is given by

$$\begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\x\\y\\z\end{pmatrix}\odot\begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\x\\y\\z\end{pmatrix}=\begin{pmatrix}0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\x\\y\\z\end{pmatrix}$$

Combining this R1CS with the required fthree boolean constraints for x, y and z we get

$$\left(\begin{array}{cccc}0 & 1 & 0 & 0\\ \hline 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right)\begin{pmatrix}1\\x\\y\\z\end{pmatrix}\odot\left(\begin{array}{cccc}0 & 0 & 1 & 0\\ \hline 1 & -1 & 0 & 0\\ 1 & 0 & -1 & 0\\ 1 & 0 & 0 & -1\end{array}\right)\begin{pmatrix}1\\x\\y\\z\end{pmatrix}=\left(\begin{array}{cccc}0 & 0 & 0 & 1\\ \hline 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right)\begin{pmatrix}1\\x\\y\\z\end{pmatrix}$$

So from the way this R1CS is constructed, we know that whatever the underlying field $\mathbb{F}$ is, the only solutions to this equations are

$$\{(0,0,0),(0,1,0),(1,0,0),(1,1,1)\}$$

which is the set of all $(x,y,z)\in\{0,1\}^3$ such that $x\,AND\,y=z$.

NOT constraint Given two field elements $x,y\in\mathbb{F}$ that represent boolean variables, we want to find a R1CS, such that $w=(1,x,y)$ satisfies the constraint system if and only if $x=\neg y$.

So again we have to constrain x and y to be boolean as explained in XXX. The next think is we need to find a R1CS that enforces the *NOT* logic. We can simply choose $(1-x)=y$, since (for boolean constraint values) this enforces that y is always the boolean opposite of x.

Now that we have found a correct equation for a boolean constrain, we have to translate it into the associated R1CS format, which is given by

$$\begin{pmatrix}1 & -1 & 0\end{pmatrix}\begin{pmatrix}1\\x\\y\end{pmatrix}\odot\begin{pmatrix}1 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\x\\y\end{pmatrix}=\begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\x\\y\end{pmatrix}$$

So actually we wrote the linear equation $1-x=y$ like $(1-x)\cdot 1=y$ and translated that into the matrix equation.

Combining this R1CS with the required fthree boolean constraints for x, y and z we get

$$\left(\begin{array}{ccc}1 & -1 & 0\\ \hline 0 & 1 & 0\\ 0 & 0 & 1\end{array}\right)\begin{pmatrix}1\\x\\y\end{pmatrix}\odot\left(\begin{array}{ccc}1 & 0 & 0\\ \hline 1 & -1 & 0\\ 1 & 0 & -1\end{array}\right)\begin{pmatrix}1\\x\\y\end{pmatrix}=\left(\begin{array}{ccc}0 & 0 & 1\\ \hline 0 & 0 & 0\\ 0 & 0 & 0\end{array}\right)\begin{pmatrix}1\\x\\y\end{pmatrix}$$

So from the way this R1CS is constructed, we know that whatever the underlying field $\mathbb{F}$ is, the only solutions to this equations are

$$\{(0,1),(1,0)\}$$

which is the set of all $(x,y)\in\{0,1\}^2$ such that $x=\neg y$.

EXERCISE: DO OR; XOR; NAND

More complicated logical constraints can then be optained by combining all sub-R1CS together. For example if the task is to enforce $(in_1\,AND\neg in_2)ANDin_3=out_1$ we first apply the FLATTENING technique from XXX, which gives is

$$\begin{array}{rcl}\neg in_2 & = & mid_1\\ in_1\,AND\,mid_1 & = & mid_2\\ mid_2\,AND\,in_3 & = & out_1\end{array}$$

So we have the statement $w=(1,in_1,in_2,in_3,mid_1,mid_2,out_1)$, 6 boolean constraints for the variables, 2 constraints for the 2 *AND* operations and 1 constraint for the *NOT* operation.

Binary representations

In circuit computations its is often necessary to use the binary representation of a prime field element. Binary representations of prime field elements work execactly like binary representations of ordinary unsigned integers. Only the algebraic operations are different. To compute the binary representation of some number $x \in \mathbb{F}_p$ we need to know the number of bits in the binary representation of p first. We write this as $m = |p_{bin}|$.

Then a bitstring $(b_0, \ldots, b_m) \in \{0,1\}^m$ is the binary representation of the field element x, if and only if

$$x = b_0 \cdot 2^0 + b_1 \cdot 2^1 + \ldots + b_m \cdot 2^m$$

Note that, since p is a prime number that has a leading bit 1 at position m. Moreover every prime number $p > 2$ is odd and hence has least significat bit set to 1. Hence all numbers 2^j for $0 \leq j \leq m$ are elements of $\mathbb{F}_p$ and the equation is well defined. We can therefore enforce this equation as a R1CS, by flattening the equation:

$$\begin{array}{lcl} b_0 \cdot 1 & = & mid_0 \\ b_1 \cdot 2 & = & mid_1 \\ \cdots & = & \cdots \\ b_m \cdot 2^m & = & mid_m \\ (mid_0 + mid_1 + \ldots + mid_m) \cdot 1 & = & x \end{array}$$

So we have the statement $w = (1, x, b_0, \ldots, b_m, mid_0, \ldots, mid_m)$ and we need $(m+1)$ constraints to enforce the binary representation in addition to the m constraints that enforce booleanness.

At this point we see, that writing more complex R1CS becomes clumbsy and in actual implementations people therefore use languages to makes the constraint system more readable. In this example we could write for example something like this:

keeping in mind that this is a meta level algorith to **generate** the R1CS, not the R1CS itself, as constructs like for loops have not direct meening on the level of the R1CS itself.

Example 48. *Considering the prime field $\mathbb{F}_{13}$, we want to enforce the binary representation of $7 \in \mathbb{F}_{13}$. To find the number of bits that we need to consider in our R1Cs, we start with the binary representation of 13, which is $(1,0,1,1)$ since $13 = 1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3$. So $m = 4$ and we have to enforce a 4-bit representation for 7, which is $(1,1,1,0)$, since $7 = 1 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2 + 0 \cdot 2^3$.*

A valid statement is then given by $w = (1,7,1,1,1,0,1,2,4,0)$ and indeed we satify the 9 required constraints

$$\begin{array}{lcll} 1 \cdot (1-1) & = & 0 & \textit{// boolean constraints} \\ 1 \cdot (1-1) & = & 0 & \\ 1 \cdot (1-1) & = & 0 & \\ 0 \cdot (1-0) & = & 0 & \\ & & & \\ 1 \cdot 1 & = & 1 & \\ 1 \cdot 2 & = & 2 & \\ 1 \cdot 4 & = & 4 & \\ 0 \cdot 8 & = & 0 & \\ (1+2+4+0) \cdot 1 & = & 7 & \end{array}$$

Conditional (ternary) operator

It is often required to implement the terniary conditional operator ? : as a R1CS. In general this operator takes three arguments, a boolean value b and two expressions if_true and if_false, usually written as $b\ ?\ c\ :\ d$ and executes c and d according to the value of b.

If we assume all three arguments to be values from a finite field, such that b is boolean constraint (XXX), we can enforce a field element x to be the result of the conditional operator as

$$x = b \cdot c + (1-b) \cdot d$$

Flattening the code gives

$$\begin{array}{lcl} b \cdot c & = & mid_0 \\ (1-b) \cdot d & = & mid_1 \\ (mid_0 + mid_1) \cdot 1 & = & x \end{array}$$

So we have the statement $w = (1, x, b, c, d, mid_0, mid_1)$ and we need 3 constraints to enforce the conditional operator in addition to 1 constraint that enforces booleanness of b.

NOTE: THERE WAS THIS PODCAST WITH ANNA AND THE GUY JAN TALKE TO WHERE HE SAID; CONDITIONALS CAN BE IMPLEMENTED SUCH THAT NOT BOTH BRANCHES ARE EXCUTED: LOOK THAT UP

Range Proofs

$x > 5$...

UintN

STUFF ABOUT HOW UINTN COMPUTATIONS ARE NOT STANDARDIZED AND THAT THERE ARE IMPLEMENTATIONS OTHER THEN MOD-N.... WE FIX ON MOD-N. WHAT DO ZEXE CIRCOM ECT FIX ON?

As we know circuits are not defined over integers but over finite fields instead. We therefore have no notation of integers in circuits. However on computers we also not use integers natively but Uint's instead.

As we know a UintN type is a representation of integers in the range of $0 \ldots 2^N$ with the exception that algebraic operations like addition and multiplication deviate from actual integers, whenever the result exceeds the largest representable number $2^N - 1$.

In circuit design it is therefore important to distinguish between various things tht might look like integers, but are actually not. For example Haskells type NAT is an actual implementation of natural numbers. In particular this means ....

Example 49 (Uint8)**.** *What is* $0xFFF0 + 0xFFF0$ *and so on...*

Bit constraints In prime fields, addition and multiplication behaves exactly like addition and multiplication with integers as long as the result does not exceed the modulus.

This makes the representation of UintNs in a prime field $\mathbb{F}_p$ potentially ambigious, as there are two possible representations, whenever $2^N - 1 < p$. In that case any element of $UintN$ could be interpreted as an element of $\mathbb{F}_p$. This however is dangerous as the algebraic laws like addition and multiplication behave very different in general.

It is therefore common to represent UintN types in circuits as binary constraints strings of field elements of length N.

Example 50. *Consider the Uint4 type over the prime field $\mathbb{F}_{17}$. Since $2^4 = 16$, Uint4 can represent the numbers $0,\ldots,15$ and it would be possible to interpret them as elements in $\mathbb{F}_{17}$. However addition*

Twisted Edwards curves

Sometimes it required to do elliptic curve cryptography "inside of a circuit". This means that we have to implement the algebraic operations (addition, scalar multiplication) of an elliptic curve as a R1CS. To do this efficiently the curve that we want to implement must be defined over the same base field as the field that is used in the R1CS.

Example 51. *So for example when we consider an R1CS over the field $\mathbb{F}_{13}$ as we did in example XXX, then we need a curve that is also defined over $\mathbb{F}_{13}$. Moreover it is advantegous to use a (twisted) Edwards curve inside a circuit, as the addition law contains no branching (See XXX). As we have seen in XXX our Baby-Jubjub curve is an Edwards curve defined over $\mathbb{F}_{13}$. So it is well suited for elliptic curve cryptography in our pend and paper examples*

Twisted Edwards curves constraints As we have seen in XXX, an Edwards curve over a finite field F is the set of all pairs of points $(x,y) \in \mathbb{F} \times \mathbb{F}$, such that x and y satisfy the equation $a \cdot x^2 + y^2 = 1 + d \cdot x^2 y^2$.

We can interpret this equation as a constraint on x and y and rewrite it as a R1CS by applying the flattenin technique from XXX.

$$\begin{array}{lcl} x \cdot x & = & x\_sq \\ y \cdot y & = & y\_sq \\ x\_sq \cdot y\_sq & = & xy\_sq \\ (a \cdot x\_sq + y\_sq) \cdot 1 & = & 1 + d \cdot xy\_sq \end{array}$$

So we have the statement $w = (1, x, y, x\_sq, y\_sq, xy\_sq)$ and we need 4 constraints to enforce that x and y are points on the Edwards curve $x^2 + y^2 = 1 + d \cdot x^2 y^2$. Writing the constraint system in matrix form, we get:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \\ x\_sq \\ y\_sq \\ xy\_sq \end{pmatrix} \odot \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \\ x\_sq \\ y\_sq \\ xy\_sq \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & d \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \\ x\_sq \\ y\_sq \\ xy\_sq \end{pmatrix}$$

EXERCISE: WRITE THE R1CS FOR WEIERSTRASS CURVE POINTS

Example 52 (Baby-JubJub)**.** *Considering our pen and paper Baby JubJub curve over from XXX, we know that the curve is defined over $\mathbb{F}_{13}$ and that $(11,9)$ is a curve point, while $(2,3)$ is not a curve point.*

Starting with $(11,9)$, we can compute the statement $w = (1,11,9,4,3,12)$. Substituting this into the constraints we get

$$\begin{array}{lcl} 11 \cdot 11 & = & 4 \\ 9 \cdot 9 & = & 3 \\ 4 \cdot 3 & = & 12 \\ (1 \cdot 4 + 3) \cdot 1 & = & 1 + 7 \cdot 12 \end{array}$$

which is true in $\mathbb{F}_{13}$. So our statement is indeed a valid assignment to the twisted Edwards curve constraining system.

Now considering the non valid point $(2,3)$*, we can still come up with some kind of statement* w *that will satisfy some of the constraints. But fixing* $x = 2$ *and* $y = 3$*, we can never satisfy all constraints. For example* $w = (1,2,3,4,9,10)$ *will satisfy the first three constraints, but the last constrain can not be satisfied. Or* $w = (1,2,3,4,3,12)$ *will satisfy the first and the last constrain, but not the others.*

Twisted Edwards curves addition As we have seen in XXX one the major advantages of working with (twisted) Edwards curves is the existence of an addition law, that contains no branching and is valid for all curve points. Moreover the neutral element is not "at infinity" but the actual curve poin $(0,1)$.

As we know from XXX, give two points (x_1,y_1) and (x_2,y_2) on a twisted Edwards curve their sum is given by

$$(x_3,y_3) = \left(\frac{x_1y_2 + y_1x_2}{1 + d \cdot x_1x_2y_1y_2}, \frac{y_1y_2 - a \cdot x_1x_2}{1 - d \cdot x_1x_2y_1y_2}\right)$$

We can realize this equation as a R1CS as follows: First not that we can rewrite the addition law as

$$\begin{aligned}
x_1 \cdot x_2 &= x_{12} \\
y_1 \cdot y_2 &= y_{12} \\
x_1 \cdot y_2 &= xy_{12} \\
y_1 \cdot x_2 &= yx_{12} \\
x_{12} \cdot y_{12} &= xy_{1212} \\
x_3 \cdot (1 + d \cdot xy_{1212}) &= xy_{12} + yx_{12} \\
y_3 \cdot (1 - d \cdot xy_{1212}) &= y_{12} - a \cdot x_{12}
\end{aligned}$$

So we have the statement $w = (1,x_1,y_1,x_2,y_2,x_3,y_3,x_{12},y_{12},xy_{12},yx_{12},xy_{1212})$ and we need 7 constraints to enforce that $(x_1,y_1) + (x_2,y_2) = (x_3,y_3)$

Example 53 (Baby-JubJub)**.** *Considering our pen and paper Baby JubJub curve over from XXX. We recall from XXX that* $(11,9)$ *is a generator for the large prime order subgroup. We therefor already know from XXX that* $(11,9) + (7,8) = (11,9) + [3](11,9) = [4](11,9) = (2,9)$*. So we compute a valid statement as* $w = (1,11,9,7,8,2,9,12,7,10,11,6)$*. Indeed*

$$\begin{aligned}
11 \cdot 7 &= 12 \\
9 \cdot 8 &= 7 \\
11 \cdot 8 &= 10 \\
9 \cdot 7 &= 11 \\
10 \cdot 11 &= 6 \\
2 \cdot (1 + 7 \cdot 6) &= 10 + 11 \\
9 \cdot (1 - 7 \cdot 6) &= 7 - 1 \cdot 12
\end{aligned}$$

There are optimizations for this using only 6 constraints, available:

Twisted Edwards curves inversion Similar to elliptic curves in Weierstrass form, inversion is cheap on Edwards curve as the negative of a curve point $-(x,y)$ is given by $(-x,y)$. So a curve point (x_2,y_2) is the additive inverse of another curve point (x_1,y_1) precisely if the equation $(x_1,y_1) = (-x_2,y_2)$ holds. We can write this as

$$\begin{aligned}
x_1 \cdot 1 &= -x_2 \\
y_1 \cdot 1 &= y_2
\end{aligned}$$

We therefor have a statement of the form $w = (1, x_1, y_1, x_2, y_2)$ and can write the constraints into a matrix equation as

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} \odot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}$$

In addition we need the following constraints:

$$\begin{aligned} x_1 \cdot 1 &= -x_2 \\ y_1 \cdot 1 &= y_2 \end{aligned}$$

Twisted Edwards curves scalar multiplication Although there are highly optimzed R1CS implementations for scal multiplication on elliptic curves, the basic idea is somewhat simple: Given an elliptic curve $E/\mathbb{F}_r$, a scalar $x \in \mathbb{F}_r$ with binary representation $(b_0, \ldots, b_m)$ and a curve point $P \in E/\mathbb{F}_r$, the scalar multiplication $[x]P$ can be written as

$$[x]P = [b_0]P + [b_1]([2]P) + [b_2]([4]P) + \ldots + [b_m]([2^m]P)$$

and since b_j is either 0 or 1, $[b_j](kP)$ is either the neutral element of the curve or $[2^j]P$. However $[2^j]P$ can be computed inductively by curve point doubling, since $[2^j]P = [2]([2^{j-1}]P)$.

So scalar multiplication can be reduced to a loop of length m, where the original curve point is repeadedly douled and added to the result, whenever the appropriate bit in the scalar is equal to one.

So to enforce that a curve point (x_2, y_2) is the scalar product $[k](x_1, y_1)$ of a scalar $x \in F_r$ and a curve point (x_1, y_1), we need an R1CS the defines point doubling on the curve (XXX) and an R1CS that enforces the binary representation of x (XXX).

In case of twisted Edwards curve, we can use ordinary addition for doubling, as the constraints works for both cases (doublin is addition, where both arguments are equal). Moreover $[b](x, y) = (b \cdot x, b \cdot y)$ for boolean b. Hence flattening equation XXX gives

$$\begin{aligned} b_0 \cdot x_1 &= x_{0,1} \quad //[b_0]P \\ b_0 \cdot y_1 &= y_{0,1} \end{aligned}$$

In addition we need to constrain $(b_0, \ldots, b_N)$ to be the binary representation of x and we need to constrain each b_j to be boolean.

As we can see a R1CS for scalar multiplication utilizes many R1CS that we have introduced before. For efficiency and readability it is therefore useful to apply the concept of a gadget (XXX). A pseudocode method to derive the associated R1CS could look like this:

Curve Cycles A particulary interesting case with far reaching implication is the situation when we have two curve E_1 and E_2, such that the scalar field of curve E_1 is the base field of curve E_2 and vice versa. In that case it is possible to implement the group laws of one curve in circuits defined over the scalar field of the other curve.

The RAM Model

FROM THE PODCAST WITH ANNA R. AND THE GUY FROM JAN....

Generalizations

many circuits can be found here:

5.1.4 Quadratic Arithmetic Programs

As shown by [Pinocchio] rank-1 constraint systems can be transformed into so called quadratic arithmetic programs assuming $\mathbb{F}$.

taken from the pinocchio paper. For proving arithmetic circuit-sat. Given a R1CS QAPs transform potential solution vectors into two polynomials p and t, such that p is divisible by t if and only if the vector is a solution to the R1CS.

They are major building blocks for **succinct** proofs, since with high probability, the divisibility check can be performed in a single point of those polynomials. So computationally expensive polynomial division check is reduced TO WHAT? (IN FIELDS THERE IS ALWAYS DIVISIBILITY)

Definition 5.1.4.1 (Quadratic Arithmetic Program)**.** *Assume we have a Galois field $\mathbb{F}$, three numbers i, j, k as well as three $(i+j+1) \times k$ matrices A, B and C with coefficients in $\mathbb{F}$ that define the R1CS $Ax \odot Bx = Cx$ for some statement $x = (1, i, w)$ and let $m_1, \ldots, m_k \in \mathbb{F}$ be arbitrary field elements.*

*Then a **quadratic arithmetic program** of the R1CS is the following set of polynomials over $\mathbb{F}$*

$$QAP = \left\{ t \in \mathbb{F}[x], \{a_h, b_h, c_h \in \mathbb{F}[x]\}_{h=1}^{i+j+1} \right\}$$

*where $t(x) := \Pi_{l=1}^{k}(x - m_l)$ is a polynomial f degree k, called the **target polynomial** of the QAP and $a_h(x)$, $b_h(x)$ as well as $c_h(x)$ are the unique degree $k-1$ polynomials that are defined by the equations*

$$a_h(m_l) = A_{h,l} \quad b_h(m_l) = B_{h,l} \quad c_h(m_l) = C_{h,l} \quad h = 1, \ldots, i+j+1, l = 1, \ldots, k$$

The major point is that R1CS-sat can be reformulated into the divisibility of a polynomials defined by any QAP.

Theorem 5.1.4.2. *Assume that an R1CS and an associated QAP as defined in XXX are given. Then the affine vector $y = (1, i, w)$ is a solution to the R1CS, if and only if the polynomial*

$$p(x) = \left(\sum y_h \cdot a_h(x)\right) \cdot \left(\sum y_h \cdot b_h(x)\right) - \sum y_h \cdot c_h(x)$$

is divisible by the target polynomial t.

The polynomials a_h, b_h and c_h are uniquely defined by the equations in XXX. However to actually compute them we need some algorithm like the Langrange XXX from XXX.

Example 54 (Generalized factorization snark)**.** *In this example we want to transform the R1CS from example 37 into an associated QAP.*

We start by choosing an arbitrary field element for every constraint in the R1CS, since we have 2 constraints we choose $m_1 = 5$ and $m_2 = 7$

With this choice we get the target polynomial $t(x) = (x - m_1)(x - m_2) = (x-5)(x-7) = (x+8)(x+6) = x^2 + x + 9$.

Since our statement has structure $w = (1, in_1, in_2, in_3, m_1, out_1)$ we have to compute the following degree 1 polynomials

$\{a_c, a_{in_1}, a_{in_2}, a_{in_3}, a_{mid_1}, a_{out}\}$ $\{b_c, b_{in_1}, b_{in_2}, b_{in_3}, b_{mid_1}, b_{out}\}$ $\{c_c, c_{in_1}, c_{in_2}, c_{in_3}, c_{mid_1}, c_{out}\}$

Apply QAP rule XXX to the $a_{k\in I}$ polynomials gives

$$\begin{array}{llllll} a_c(5)=0, & a_{in_1}(5)=1, & a_{in_2}(5)=0, & a_{in_3}(5)=0, & a_{mid_1}(5)=0, & a_{out}(5)=0 \\ a_c(7)=0, & a_{in_1}(7)=0, & a_{in_2}(7)=0, & a_{in_3}(7)=0, & a_{mid_1}(7)=1, & a_{out}(7)=0 \end{array}$$

$$\begin{array}{llllll} b_c(5)=0, & b_{in_1}(5)=0, & b_{in_2}(5)=1, & b_{in_3}(5)=0, & b_{mid_1}(5)=0, & b_{out}(5)=0 \\ b_c(7)=0, & b_{in_1}(7)=0, & b_{in_2}(7)=0, & b_{in_3}(7)=1, & b_{mid_1}(7)=0, & b_{out}(7)=0 \end{array}$$

$$\begin{array}{llllll} c_c(5)=0, & c_{in_1}(5)=0, & c_{in_2}(5)=0, & c_{in_3}(5)=0, & c_{mid_1}(5)=1, & c_{out}(5)=0 \\ c_c(7)=0, & c_{in_1}(7)=0, & c_{in_2}(7)=0, & c_{in_3}(7)=0, & c_{mid_1}(7)=0, & c_{out}(7)=1 \end{array}$$

Since our polynomials are of degree 1 only we don't have to invoke Langrange method but can deduce the solutions right away.

Polynomials are defined on the two values 5 and 7 here. Linear Polynomial $f(x)=m\cdot x+b$ is fully determined by this. Derive the general equation:

- $5m+b=f(5)$ *and* $7m+b=f(7)$
- $b=f(5)-5m$ *and* $b=f(7)-7m$
- $b=f(5)+8m$ *and* $b=f(7)+6m$
- $f(5)+8m=f(7)+6m$
- $8m-6m=f(7)-f(5)$
- $2m=f(7)+12f(5)$
- $7\cdot 2m=7(f(7)+12f(5))$
- $m=7(f(7)+12f(5))$
-
- $b=f(5)+8m$
- $b=f(5)+8\cdot(7(f(7)+12f(5)))$
- $b=f(5)+4(f(7)+12f(5))$
- $b=f(5)+4f(7)+9f(5)$
- $b=10f(5)+4f(7)$

Gives the general equation: $f(x)=7(f(7)+12f(5))x+10f(5)+4f(7)$

For a_{in_1} the computation looks like this:

- $a_{in_1}(x)=7(a_{in_1}(7)+12a_{in_1}(5))x+10a_{in_1}(5)+4a_{in_1}(7)=$
- $7(0+12\cdot 1)x+10\cdot 1+4\cdot 0=$
- $7\cdot 12x+10=$
- $6x+10$

- $a_{mid_1}(x) = 7(a_{mid_1}(7) + 12a_{mid_1}(5))x + 10a_{mid_1}(5) + 4a_{mid_1}(7) =$
- $7(1 + 12 \cdot 0)x + 10 \cdot 0 + 4 \cdot 1 =$
- $7 \cdot 1x + 4 =$
- $7x + 4$

$a_c(x) = 0$	$b_c(x) = 0$	$c_c(x) = 0$
$a_{in_1}(x) = 6x + 10$	$b_{in_1}(x) = 0$	$c_{in_1}(x) = 0$
$a_{in_2}(x) = 0$	$b_{in_2}(x) = 6x + 10$	$c_{in_2}(x) = 0$
$a_{in_3}(x) = 0$	$b_{in_3}(x) = 7x + 4$	$c_{in_3}(x) = 0$
$a_{mid_1}(x) = 7x + 4$	$b_{mid_1}(x) = 0$	$c_{mid_1}(x) = 6x + 10$
$a_{out}(x) = 0$	$b_{out}(x) = 0$	$c_{out}(x) = 7x + 4$

This gives the quadratic arithmetic program for our generalized factorization snark as

$$QAP = \{x^2 + x + 9, \{0, 6x + 10, 0, 0, 7x + 4, 0\}, \{0, 0, 6x + 10, 7x + 4, 0, 0\}, \{0, 0, 0, 0, 6x + 10, 7x + 4\}\}$$

Now as we recall, the main point for using QAPs in snarks is the fact, that solutions to R1CS are in 1:1 correspondence to the divisibility of a polynomial p, constructed from a R1CS solution and the polynomials of the QAP and the target polynomial.

So lets see this in our example. We already know from example XXX, that $I = \{1, 2, 3, 4, 6, 11\}$ is a solution to the R1CS XXX of our problem. To see how this translates to polyinomial divisibility we compute the polynomial p_I by

$$\begin{aligned}
p_I(x) &= (\sum_{h \in |I|} I_h \cdot a_h(x)) \cdot (\sum_{h \in |I|} I_h \cdot b_h(x)) - (\sum_{h \in |I|} I_h \cdot c_h(x)) \\
&= (2(6x + 10) + 6(7x + 4)) \cdot (3(6x + 10) + 4(7x + 4)) - (6(6x + 10) + 11(7x + 4)) \\
&= ((12x + 7) + (3x + 11)) \cdot ((5x + 4) + (2x + 3)) - ((10x + 8) + (12x + 5)) \\
&= (2x + 5) \cdot (7x + 7) - (9x) \\
&= (x^2 + 2 \cdot 7x + 5 \cdot 7x + 5 \cdot 7) - (9x) \\
&= (x^2 + x + 9x + 9) - (9x) \\
&= x^2 + x + 9
\end{aligned}$$

And as we can see in this particular example $p_I(x)$ is equal to the target polynomial $t(x)$ and hence it is divisible by t with $p/t = 1$.

To give a counter example we already know from XXX that $I = \{1, 2, 3, 4, 8, 2\}$ is not a solution to our R1CS. To see how this translates to polyinomial divisibility we compute the polynomial p_I by

$$\begin{aligned}
p_I(x) &= (\sum_{h \in |I|} I_h \cdot a_h(x)) \cdot (\sum_{h \in |I|} I_h \cdot b_h(x)) - (\sum_{h \in |I|} I_h \cdot c_h(x)) \\
&= (2(6x + 10) + 6(7x + 4)) \cdot (3(6x + 10) + 4(7x + 4)) - (6(6x + 10) + 11(7x + 4)) \\
&= 8x^2 + 11x + 3
\end{aligned}$$

This polynomial is not divisible by the target polynomial t since Not divisible by t: $(8x^2 + 11x + 3)/(x^2 + x + 9) = 8 + \frac{3x+8}{x^2+x+9}$

5.1.5 Quadratic span programs

5.2 proof system

Now a *proof system* is nothing but a game between two parties, where one parties task is to convince the other party, that a given string over some alphabet is a statement is some agreed on language. To be more precise. Such a system is more over *zero knowledge* if this possible without revealing any information about the (parts of) that string.

Definition 5.2.0.1 ((Interactive) Proofing System)**.** *Let L be some formal language over an alphabet Σ. Then an **interactive proof system** for L is a pair (P,V) of two probabilistic interactive algorithms, where P is called the **prover** and V is called the **verifier**.*

Both algorithms are able to send messages to one another. Each algorithm only sees its own state, some shared initial state and the communication messages.

The verifier is bounded to a number of steps which is polynomial in the size of the shared initial state, after which it stops in an accept state or in a reject state. We impose no restrictions on the local computation conducted by the prover.

We require that, whenever the verifier is executed the following two conditions hold:

- *(Completeness) If a string $x \in \Sigma^*$ is a member of language L, that is $x \in L$ and both prover and verifier follow the protocol; the verifier will accept.*
- *(Soundness) If a string $x \in \Sigma^*$ is not a member of language L, that is $x \notin L$ and the verifier follows the protocol; the verifier will not be convinced.*
- *(Zero-knowledge) If a string $x \in \Sigma^*$ is a member of language L, that is $x \in L$ and the prover follows the protocol; the verifier will not learn anything about x but $x \in L$.*

In the context of zero knowledge proving systems definition XXX gets a slight adaptation:

- Instance: Input commonly known to both prover (P) and verifier (V), and used to support the statement of what needs to be proven. This common input may either be local to the prover-verifier interaction, or public in the sense of being known by external parties (Some scientific articles use "instance" and "statement" interchangeably, but we distinguish between the two.).
- Witness: Private input to the prover. Others may or may not know something about the witness.
- Relation: Specification of relationship between instances and witness. A relation can be viewed as a set of permissible pairs (instance, witness).
- Language: Set of statements that appear as a permissible pair in the given relation.
- Statement:Defined by instance and relation. Claims the instance has a witness in the relation(which is either true or false).

The following subsections define ways to describe checking relations that are particularly useful in the context of zero knowledge proofing systems

5.2.1 Succinct NIZK

Preprocessing style: trusted setup, multi party ceremony

Blum, Feldman and Micali extended the notion tonon-interactivezero-knowledge(NIZK) proofs in the common reference string model. NIZK proofs are useful in theconstruction of non-interactive cryptographic schemes, e.g., digital signatures and CCA-secure public key encryption.

Definition 5.2.1.1. *Let $\mathscr{R}$ be a relation generator that given a security parameter λ in unary returns a polynomial time decidable binary relation R. For pairs $(i,w) \in R$ we call i the instance[1] and w the witness. We define R_λ to be the set of possible relations R the relation generator may output given 1^λ. We will in the following for notational simplicity assume λ can be deduced from the description of R. The relation generator may also output some side information, an auxiliary input z, which will be given to the adversary. An efficient prover publicly verifiable non-interactive argument for R is a quadruple of probabilistic polynomial algorithms* $(\text{SETUP}, \text{PROVE}, \text{VFY}, \text{SIM})$ *such*

- *Setup:* $(CRS, \tau) \rightarrow Setup(R)$*: The setup produces a common reference string CRS and a simulation trapdoor τ for the relation R.*
- *Proof:* $\pi \rightarrow Prove(R, CRS, i, w)$*: The prover algorithm takes as input a common reference string CRS and a statement $(i,w) \in R$ and returns an argument π.*
- *Verify:* $0/1 \rightarrow Vfy(R, CRS, i, \pi)$*: The verification algorithm takes as input a common reference string CRS, an instance i and an argument π and returns 0 (reject) or 1 (accept).*
- $\pi \rightarrow Sim(R, \tau, i)$*: The simulator takes as input a simulation trapdoor τ and instance i and returns an argument π.*

Common Reference String Generation Also called trusted setup phase. The field elements needed in this step are called toxic waste ...

Trusted third party The most simple approach to generate a common reference string is a so called *trusted third party*. By assumption the entire systems trusts this party to generate the common reference string exactly accoring to the rules and the party will delete all traces of the toxic waste after CRS generation.

Player exchangeable Multi Party Ceremonies Achive soundness if only a single party is honest and correctly deletes toxix waste. Is always zero knowledge.

State of the art works in the random beacon model.

A random beacon produces publicly available and verifyable random values at fixed intervals. The difference between random beacons and random oracles, is that random beacons are not available until certain time slots. Random beacons can be instanciated for example by evaluation of say 2^{40} iterations of SHA256 on some high entropy, publically available data like the closing value of the stock market on a certain date, the output of a selected set of national lotteries and so on.

The assumption is that any given random beacon value contains large amounts of entropy that is independent from the influence of an adversary in previous time slots.

[1] Note that in Groth16 this is called the statement. We think the term instance is more consistent with SOMETHING.

Groth16

Groth's constant size NIZK argument is based on constructing a set of polynomial equations and using pairings to efficiently verify these equations. Gennaro, Gentry,Parno and Raykova [Pinocchio] found an insightful construction of polynomial equations based on Lagrange interpolation polynomials yielding a pairing-based NIZK argumentwith a common reference string size proportional to the size of the statement and wit-ness.

It constructs a snark for arithmetic circuit satisfiability, where a proof consists of only 3 group elements. In addition to being small, the proof is also easy to verify. The verifier just needs to compute a number of exponentiations proportional to the instance size and check a single pairing product equation, which only has 3 pairings.

The construction can be instantiated with any type of pairings including Type III pairings, which are the most efficient pairings. The argument has perfect completeness and perfect zero-knowledge. For soundness ??

In the common reference string model.

Setup:

- random elements $\alpha, \beta, \gamma, \delta, s \in \mathbb{F}_{scalar}$
- Common reference string CRS_{QAP}, specific to the QAP and the choice of statement and witness $CRS_{QAP} = (CRS_{\mathbb{G}_1}, CRS_{\mathbb{G}_2})$, with $n = deg(t)$:

$$CRS_{\mathbb{G}_1} = \left\{ \begin{array}{c} [\alpha]g, [\beta]g, [\delta]g, \left\{[s^k]g\right\}_{k=0}^{n-1}, \left\{[\frac{\beta a_k(s)+\alpha b_k(s)+c_k(s)}{\gamma}]g\right\}_{k\in I} \\ \left\{[\frac{\beta a_k(s)+\alpha b_k(s)+c_k(s)}{\delta}]g\right\}_{k\in W}, \left\{[\frac{s^k t(s)}{\delta}]g\right\}_{k=0}^{n-2} \end{array} \right\}$$

$$CRS_{\mathbb{G}_2} = \left\{ [\beta]h, [\gamma]h, [\delta]h, \left\{[s^k]h\right\}_{k=0}^{n-1} \right\}$$

- Toxic waste: Must delete random elements after CRS_{QAP} generation.

Example 55 (Generalized factorization snark)**.** *In this example we want to compile our main example in Groth16. Input is the R1CS from example 46. We choose the following global parameters*

$$\textit{curve = BLS6-6} \quad \mathbb{G}_1 = \textit{BLS6-6(13)} \quad g = (13,15) \quad \mathbb{G}_2 = \quad h = (7v^2, 16v^3) \textit{ and } \mathbb{G}_T = \mathbb{F}^*_{43^6}.$$

Example 56 (Trusted third party for the factorization snark)**.** *We consider ourself as a trusted third part to generate the common reference string for our generalized factorization snark. We therefore choose the following secret field elements $\alpha = 6$, $\beta = 5$, $\gamma = 4$, $\delta = 3$, $s = 2$ from $\mathbb{F}_{13}$ and are very careful to hide them from anyone how hasn't read this book. From those values we can then instantiate the common reference string XXX:*

$$CRS_{\mathbb{G}_1} = \left\{ \begin{array}{c} [6](13,15), [5](13,15), [3](13,15), \left\{[s^k](13,15)\right\}_{k=0}^{1}, \left\{[\frac{5a_k(2)+6b_k(2)+c_k(2)}{4}](13,15)\right\}_{k\in S} \\ \left\{[\frac{5a_k(2)+6b_k(2)+c_k(2)}{3}](13,15)\right\}_{k\in W}, \left\{[\frac{s^k t(2)}{3}](13,15)\right\}_{k=0}^{0} \end{array} \right\}$$

Since we have instance indices $I = \{1, in_1, in_2\}$ and witness indices $W = \{in_3, mid_1, out_1\}$ we have The instance parts.

$$\left[\frac{5a_c(2)+6b_c(2)+c_c(2)}{4}\right](13,15) \;=\; \left[\frac{5\cdot 0+6\cdot 0+0}{4}\right](13,15) \;=\; [0](13,15) \;=\; \mathscr{O}$$

$$\left[\frac{5a_{in_3}(2)+6b_{in_3}(2)+c_{in_3}(2)}{4}\right](13,15) = [(5\cdot 0+6\cdot(7\cdot 2+4)+0)\cdot 10](13,15) =$$
$$[(6\cdot 5)\cdot 10](13,15) = [1](13,15) = (13,15)$$

$$\left[\frac{5a_{out}(2)+6b_{out}(2)+c_{out}(2)}{4}\right](13,15) = [(5\cdot 0+6\cdot 0+(7\cdot 2+4))\cdot 10](13,15) =$$
$$[5\cdot 10](13,15) = [11](13,15) = (33,9)$$

Witness part:

$$\left[\frac{5a_{in_1}(2)+6b_{in_1}(2)+c_{in_1}(2)}{3}\right](13,15) = [(5\cdot(6\cdot 2+10)+6\cdot 0+0)\cdot 9](13,15) =$$
$$[(5\cdot 9)\cdot 9](13,15) = [2](13,15) = (33,34)$$

$$\left[\frac{5a_{in_2}(2)+6b_{in_2}(2)+c_{in_2}(2)}{3}\right](13,15) = [(5\cdot 0+6\cdot(6\cdot 2+10)+0)\cdot 9](13,15) =$$
$$[(6\cdot 9)\cdot 9](13,15) = [5](13,15) = (26,34)$$

$$\left[\frac{5a_{mid_1}(2)+6b_{mid_1}(2)+c_{mid_1}(2)}{3}\right](13,15) = [(5\cdot(7\cdot 2+4)+6\cdot 0+0)\cdot 9](13,15) =$$
$$[(5\cdot 5)\cdot 9](13,15) = [4](13,15) = (35,28)$$

For $\left\{\left[\frac{s^k t(2)}{3}\right](13,15)\right\}_{k=0}^{0}$ *we get*

$$\left[\frac{2^0 t(2)}{3}\right](13,15) = [t(2)\cdot 9](13,15) = [(2^2+2+9)\cdot 9](13,15) = [5](13,15) = (26,34)$$

All together, the $\mathbb{G}_1$ *part of the CRS is:*

$$CRS_{\mathbb{G}_1} = \left\{ \begin{array}{c} (27,34),(26,34),(38,15),\{(13,15),(33,34)\},\{\mathscr{O},(13,15),(33,9)\} \\ \{(33,34),(26,34),(35,28)\},\{(26,34)\} \end{array} \right\}$$

To compute the $\mathbb{G}_2$ *part*

$$CRS_{\mathbb{G}_2} = \left\{ [5](7v^2,16v^3),[4](7v^2,16v^3),[3](7v^2,16v^3),\left\{[2^k](7v^2,16v^3)\right\}_{k=0}^{1} \right\}$$

$$CRS_{\mathbb{G}_2} = \left\{ [5](7v^2,16v^3),[4](7v^2,16v^3),[3](7v^2,16v^3),\left\{[1](7v^2,16v^3),[2](7v^2,16v^3)\right\} \right\}$$
$$CRS_{\mathbb{G}_2} = \left\{ (16v^2,28v^3),(37v^2,27v^3),(42v^2,16v^3),\left\{(7v^2,16v^3),(10v^2,28v^3)\right\} \right\}$$

So alltogether our common reference string is

$$\left(\begin{array}{c} \left\{ \begin{array}{c} (27,34),(26,34),(38,15),\{(13,15),(33,34)\},\{\mathscr{O},(13,15),(33,9)\} \\ \{(33,34),(26,34),(35,28)\},\{(26,34)\} \end{array} \right\} \\ \left\{ (16v^2,28v^3),(37v^2,27v^3),(42v^2,16v^3),\left\{(7v^2,16v^3),(10v^2,28v^3)\right\} \right\} \end{array} \right)$$

Example 57 (Player exchangeable multi party ceremony for the factorization snark)**.** *In this example we want to simulate a real world player exchangeable multi party ceremony for our factorization snark XXX as explained in XXX.*

We use our TinyMD5 hash function XXX to hash to $\mathbb{G}_2$.

We assume that we have a coordinator Alice together with three parties Bob, Carol and Dave that want to contribute their randomness to the protocol. Since the degree n of the target polynomial is 2*, we need to compute the common reference string*

$$CRS = \{\}$$

For contributer $j > 0$ *in phase l to compute the proof of knowledge XXX, we need to define the* $transcript_{l,j-1}$ *of the previous round. We define it as sha256 of* $MPC_{l,j-1}$*. To be more precise we define*

$$transcript_{1,j-1} = MD5('[s]g_1[s]g_2[s^2]g_1[\alpha]g_1[\alpha \cdot s]g_1[\beta]g_1[\beta]g_2[\beta \cdot s]g_1')$$

The only thing actually important about the transcript, is that it is publically available data that is not accesable for anyone before the MPC-data of round $j-1$ *in phase l exists.*

We start with the first round usually called the 'powers of tau' EXPLAIN THAT TERM... The computation is initialized With $s = 1,\ \alpha = 1,\ \beta = 1$*. Hence the computation starts with the following data*

$$MPC_{1,0} = \left\{ \begin{array}{lcl} ([s]g_1, [s]g_2) & = & ((13,15),(7v^2,16v^3)) \\ [s^2]g_1 & = & (13,15) \\ [\alpha]g_1 & = & (13,15) \\ [\alpha \cdot s]g_1 & = & (13,15) \\ ([\beta]g_1, [\beta]g_2) & = & ((13,15),(7v^2,16v^3)) \\ [\beta \cdot s]g_1 & = & (13,15) \end{array} \right\}$$

Then

$$\begin{aligned} &transcript_{1,0} = \\ &\quad MD5('(13,15)(7v^2,16v^3)(13,15)(13,15)(13,15)(13,15)(7v^2,16v^3)(13,15)') = \\ &\qquad\qquad f2baea4d3dba5eef5c63bb210920e7d9 \end{aligned}$$

We obtain that hash by computing

printf'%s'"(13,15)(7v^2,16v^3)(13,15)(13,15)(13,15)(13,15)(7v^2,16v^3)(13,15)"|md5sum

Everyone agreed, that the MPC starts on the 21.03.2020 and everyone can contribute for exactly a year until the 20.03.2021.

It then proceeds in a round robin style, starting with Bob, who optains that data in $MPC_{1,0}$ *and then computes his contribution. Lets assume that Bob is honest and that bought a 13-sided dice (PICTURE OF 13-SIDED DICE) to randomly find three secret field values from our prime field* $\mathbb{F}_{13}$*. He though the dice and got* $\alpha = 4$*,* $\beta = 8$ *and* $s = 2$*. He then updates* $MPC_{1,0}$*:*

$$MPC_{1,1} = \left\{ \begin{array}{lcll} ([s]g_1, [s]g_2) & = & ([2](13,15),[2](7v^2,16v^3)) & = & ((33,34),(10v^2,28v^3)) \\ [s^2]g_1 & = & [4](13,15) & = & (35,28) \\ [\alpha]g_1 & = & [4](13,15) & = & (35,28) \\ [\alpha \cdot s]g_1 & = & [8](13,15) & = & (26,9) \\ ([\beta]g_1, [\beta]g_2) & = & ([8](13,15),[8](7v^2,16v^3)) & = & ((26,9),(16v^2,15v^3)) \\ [\beta \cdot s]g_1 & = & [3](13,15) & = & (38,15) \end{array} \right\}$$

In addition he compute

$$POK_{1,1}\left\{\begin{array}{rcl} y_s & = & POK(2, f2baea4d3dba5eef5c63bb210920e7d9) = ((33,34),(16v^2,28v^3)) \\ y_\alpha & = & POK(4, f2baea4d3dba5eef5c63bb210920e7d9) = ((35,28),(10v^2,15v^3)) \\ y_\beta & = & POK(8, f2baea4d3dba5eef5c63bb210920e7d9) = ((26,9),(16v^2,28v^3)) \end{array}\right\}$$

since $[s]g_1 = (33,34)$, $[\alpha]g_1 = (35,28)$ *and* $[\beta]g_1 = (26,9)$. *as well as*

$$\begin{aligned} TinyMD5_2('(33,34)f2baea4d3dba5eef5c63bb210920e7d9') = \\ H_2(MD5('(33,34)f2baea4d3dba5eef5c63bb210920e7d9').trunc(3)) = \\ H_2(2066b3b6b6d97c46c3ac6ee2ccd23ad9.trunc(3)) = H_2(ad9) = \\ H_2(101011011001) = \\ [8\cdot 4^1\cdot 5^0\cdot 7^1](7v^2,16v^3)+[12\cdot 1^0\cdot 3^1\cdot 8^1](42v^2,16v^3)+ \\ [2\cdot 3^0\cdot 9^1\cdot 11^1](17v^2,15v^3)+[3\cdot 6^0\cdot 9^0\cdot 10^1](10v^2,15v^3) = \\ [8\cdot 4\cdot 7](7v^2,16v^3)+[12\cdot 3\cdot 8](42v^2,16v^3)+[2\cdot 9\cdot 11](17v^2,15v^3)+[3\cdot 10](10v^2,15v^3) = \\ [8\cdot 4\cdot 7](7v^2,16v^3)+[12\cdot 3\cdot 8](42v^2,16v^3)+[2\cdot 9\cdot 11](17v^2,15v^3)+[3\cdot 10](10v^2,15v^3) = \\ [3](7v^2,16v^3)+[2](42v^2,16v^3)+[3](17v^2,15v^3)+[4](10v^2,15v^3) = \\ [3](7v^2,16v^3)+[2*3](7v^2,16v^3)+[3*7](7v^2,16v^3)+[4*11](7v^2,16v^3) = \\ (42v^2,16v^3)+(17v^2,28v^3)+(16v^2,15v^3)+(16v^2,28v^3) = \\ [3](7v^2,16v^3)+[6](7v^2,16v^3)+[8](7v^2,16v^3)+[5](7v^2,16v^3) = \\ [3+6+8+5](7v^2,16v^3) = (37v^2,16v^3) \end{aligned}$$

So we get $[2](37v^2,16v^3) = (16v^2,28v^3)$

==========

$$\begin{aligned} TinyMD5_2('(35,28)f2baea4d3dba5eef5c63bb210920e7d9') = \\ H_2(MD5('(35,28)f2baea4d3dba5eef5c63bb210920e7d9').trunc(3)) = \\ H_2(ad54fa3674f6a84fab9208d7a94c9163.trunc(3)) = H_2(163) = \\ H_2(000101100011) = \\ [8\cdot 4^0\cdot 5^0\cdot 7^0](7v^2,16v^3)+[12\cdot 1^1\cdot 3^0\cdot 8^1](42v^2,16v^3)+ \\ [2\cdot 3^1\cdot 9^0\cdot 11^0](17v^2,15v^3)+[3\cdot 6^0\cdot 9^1\cdot 10^1](10v^2,15v^3) = \\ [8](7v^2,16v^3)+[12\cdot 8](42v^2,16v^3)+[2\cdot 3](17v^2,15v^3)+[3\cdot 9\cdot 10](10v^2,15v^3) = \\ [8](7v^2,16v^3)+[5](42v^2,16v^3)+[6](17v^2,15v^3)+[10](10v^2,15v^3) = \\ [8](7v^2,16v^3)+[5*3](7v^2,16v^3)+[6*7](7v^2,16v^3)+[10*11](7v^2,16v^3) = \\ (16v^2,15v^3)+(10v^2,28v^3)+(42v^2,16v^3)+(17v^2,28v^3) = \\ [8](7v^2,16v^3)+[2](7v^2,16v^3)+[3](7v^2,16v^3)+[6](7v^2,16v^3) = \\ [8+2+3+6](7v^2,16v^3) = (17v^2,28v^3) \end{aligned}$$

So we get $[4](17v^2,28v^3) = (10v^2,15v^3)$

$$
\begin{aligned}
TinyMD5_2('(26,9)f2baea4d3dba5eef5c63bb210920e7d9') &= \\
H_2(MD5('(26,9)f2baea4d3dba5eef5c63bb210920e7d9').trunc(3)) &= \\
H_2(b87b632f7027ad78cadc2452beb30e9a.trunc(3)) = H_2(e9a) &= \\
H_2(111010011010) &= \\
[8\cdot 4^1\cdot 5^1\cdot 7^1](7v^2,16v^3)+[12\cdot 1^0\cdot 3^1\cdot 8^0](42v^2,16v^3)+ &\\
[2\cdot 3^0\cdot 9^1\cdot 11^1](17v^2,15v^3)+[3\cdot 6^0\cdot 9^1\cdot 10^0](10v^2,15v^3) &= \\
[8\cdot 4\cdot 5\cdot 7](7v^2,16v^3)+[12\cdot 3](42v^2,16v^3)+[2\cdot 9\cdot 11](17v^2,15v^3)+[3\cdot 9](10v^2,15v^3) &= \\
[2](7v^2,16v^3)+[10](42v^2,16v^3)+[3](17v^2,15v^3)+[1](10v^2,15v^3) &= \\
[2](7v^2,16v^3)+[10*3](7v^2,16v^3)+[3*7](7v^2,16v^3)+[1*11](7v^2,16v^3) &= \\
(10v^2,28v^3)+(37v^2,27v^3)+(16v^2,15v^3)+(10v^2,15v^3) &= \\
[2](7v^2,16v^3)+[4](7v^2,16v^3)+[8](7v^2,16v^3)+[11](7v^2,16v^3) &= \\
[2+4+8+11](7v^2,16v^3) = (7v^2,27v^3) &
\end{aligned}
$$

So we get $[8](17v^2,28v^3) = (16v^2,28v^3)$

So Bob publishes $MPC_{1,1}$ *as well as* $POK_{1,1}$ *and after that its Carols turn. Lets also assume that Carrol is honest. So Carol looks at Bobs data and compute the transcript according to our rules*

$$
\begin{aligned}
transcript_{1,1} = & \\
MD5('(33,34)(10v^2,28v^3)(35,28)(35,28)(26,9)(26,9)(16v^2,15v^3)(38,15)') = & \\
fe72e18b90014062682a77136944e362 &
\end{aligned}
$$

We obtain that hash by computing

print f'%s'"(33,34)$(10v^2,28v^3)$*(35,28)(35,28)(26,9)(26,9)*$(16v^2,15v^3)$*(38,15)"|md5sum*

Carol then computes here contribution. Since she is honest she chooses randomly three secret field values from our prime field $\mathbb{F}_{13}$*, by invoking her compter. She found* $\alpha = 3$*,* $\beta = 4$ *and* $s = 9$ *and updates* $MPC_{1,1}$*:*

$$
MPC_{1,2} = \left\{ \begin{array}{lcll}
([s]g_1,[s]g_2) & = & ([9](33,34),[9](10v^2,28v^3)) & = & ((26,34),(16v^2,28v^3)) \\
[s^2]g_1 & = & [9\cdot 9](35,28) & = & (13,28) \\
[\alpha]g_1 & = & [3](35,28) & = & (13,28) \\
[\alpha\cdot s]g_1 & = & [3\cdot 9](26,9) & = & (26,9) \\
([\beta]g_1,[\beta]g_2) & = & ([4](26,9),[4](16v^2,15v^3)) & = & ((27,34),(17v^2,28v^3)) \\
[\beta\cdot s]g_1 & = & [4\cdot 9](38,15) & = & (35,28)
\end{array} \right\}
$$

In addition he compute

$$
POK_{1,2} \left\{ \begin{array}{lcl}
y_s & = & POK(9, fe72e18b90014062682a77136944e362) = ((35,15),(17v^2,28v^3)) \\
y_\alpha & = & POK(3, fe72e18b90014062682a77136944e362) = ((38,15),(17v^2,15v^3)) \\
y_\beta & = & POK(4, fe72e18b90014062682a77136944e362) = ((35,28),(42v^2,27v^3))
\end{array} \right\}
$$

$$TinyMD5_2('(35,15)fe72e18b90014062682a77136944e362') =$$
$$H_2(MD5('(35,15)fe72e18b90014062682a77136944e362').trunc(3)) =$$
$$H_2(115f145ceffdda73e916dc5ba8ae7354.trunc(3)) = H_2(354) =$$
$$H_2(001101010100) =$$
$$[8\cdot 4^0\cdot 5^0\cdot 7^1](7v^2,16v^3)+[12\cdot 1^1\cdot 3^0\cdot 8^1](42v^2,16v^3)+$$
$$[2\cdot 3^0\cdot 9^1\cdot 11^0](17v^2,15v^3)+[3\cdot 6^1\cdot 9^0\cdot 10^0](10v^2,15v^3) =$$
$$[8\cdot 7](7v^2,16v^3)+[12\cdot 8](42v^2,16v^3)+[2\cdot 9](17v^2,15v^3)+[3\cdot 6](10v^2,15v^3) =$$
$$[4](7v^2,16v^3)+[5](42v^2,16v^3)+[5](17v^2,15v^3)+[5](10v^2,15v^3) =$$
$$[4](7v^2,16v^3)+[5*3](7v^2,16v^3)+[5*7](7v^2,16v^3)+[5*11](7v^2,16v^3) =$$
$$(37v^2,27v^3)+(10v^2,28v^3)+(37v^2,16v^3)+(42v^2,16v^3) =$$
$$[4](7v^2,16v^3)+[2](7v^2,16v^3)+[9](7v^2,16v^3)+[3](7v^2,16v^3) =$$
$$[4+2+9+3](7v^2,16v^3) = (16v^2,28v^3)$$

So we get $[9](16v^2,28v^3) = (17v^2,28v^3)$

$$TinyMD5_2('(38,15)fe72e18b90014062682a77136944e362') =$$
$$H_2(MD5('(38,15)fe72e18b90014062682a77136944e362').trunc(3)) =$$
$$H_2(cc4da0c02c4c1b15e72d6cc6430206ab.trunc(3)) = H_2(6ab) =$$
$$H_2(011010101011) =$$
$$[8\cdot 4^0\cdot 5^1\cdot 7^1](7v^2,16v^3)+[12\cdot 1^0\cdot 3^1\cdot 8^0](42v^2,16v^3)+$$
$$[2\cdot 3^1\cdot 9^0\cdot 11^1](17v^2,15v^3)+[3\cdot 6^0\cdot 9^1\cdot 10^1](10v^2,15v^3) =$$
$$[8\cdot 5\cdot 7](7v^2,16v^3)+[12\cdot 3](42v^2,16v^3)+[2\cdot 3\cdot 11](17v^2,15v^3)+[3\cdot 9\cdot 10](10v^2,15v^3) =$$
$$[7](7v^2,16v^3)+[10](42v^2,16v^3)+[1](17v^2,15v^3)+[10](10v^2,15v^3) =$$
$$[7](7v^2,16v^3)+[10*3](7v^2,16v^3)+[1*7](7v^2,16v^3)+[10*11](7v^2,16v^3) =$$
$$(17v^2,15v^3)+(17v^2,28v^3)+(17v^2,15v^3)+(17v^2,28v^3) =$$
$$[7](7v^2,16v^3)+[4](7v^2,16v^3)+[7](7v^2,16v^3)+[6](7v^2,16v^3) =$$
$$[7+4+7+6](7v^2,16v^3) = (10v^2,15v^3)$$

So we get $[3](10v^2,15v^3) = (17v^2,15v^3)$

$$
\begin{aligned}
TinyMD5_2('(35,28)fe72e18b90014062682a77136944e362') = \\
H_2(MD5('(35,28)fe72e18b90014062682a77136944e362').trunc(3)) = \\
H_2(502323bc55c75f7189fad7999c9f1708.trunc(3)) = H_2(708) = \\
H_2(011100001000) = \\
[8\cdot 4^0\cdot 5^1\cdot 7^1](7v^2,16v^3)+[12\cdot 1^1\cdot 3^0\cdot 8^0](42v^2,16v^3)+ \\
[2\cdot 3^0\cdot 9^0\cdot 11^1](17v^2,15v^3)+[3\cdot 6^0\cdot 9^0\cdot 10^0](10v^2,15v^3) = \\
[8\cdot 5\cdot 7](7v^2,16v^3)+[12](42v^2,16v^3)+[2\cdot 11](17v^2,15v^3)+[3](10v^2,15v^3) = \\
[7](7v^2,16v^3)+[12](42v^2,16v^3)+[9](17v^2,15v^3)+[3](10v^2,15v^3) = \\
[7](7v^2,16v^3)+[12*3](7v^2,16v^3)+[9*7](7v^2,16v^3)+[3*11](7v^2,16v^3) = \\
(17v^2,15v^3)+(42v^2,27v^3)+(10v^2,15v^3)+(17v^2,15v^3) = \\
[7](7v^2,16v^3)+[10](7v^2,16v^3)+[11](7v^2,16v^3)+[7](7v^2,16v^3) = \\
[7+10+11+7](7v^2,16v^3) = (37v^2,16v^3)
\end{aligned}
$$

So we get $[4](37v^2,16v^3) = (42v^2,27v^3)$

Dave thinks he can outsmart the syste, Since he is the last to contribute, he just makes up an entirely new MPC, that does not contain any randomness from the previous contributers. He thinks he can do that because, no one can distinguish his $MPC_{1,3}$ *from a correct one. If this is done in a smart way, he will even be able to compute the correct POKs.*

So Dave choses $s = 12$, $\alpha = 11$ *and* $\beta = 10$ *and he will keep those values, hoping to be able to use them later to forge false proofs in the factorization snark. He then compute*

$$
MPC_{1,3} = \left\{ \begin{array}{lcl}
([s]g_1,[s]g_2) & = & ((13,28),(7v^2,27v^3)) \\
[s^2]g_1 & = & (13,15) \\
[\alpha]g_1 & = & (33,9) \\
[\alpha\cdot s]g_1 & = & (33,34) \\
([\beta]g_1,[\beta]g_2) & = & ((38,28),(42v^2,27v^3)) \\
[\beta\cdot s]g_1 & = & (38,15)
\end{array} \right\}
$$

Dave does not delete s, α *and* β*, because if this is accepted as phase one of the common reference string computation, Dave controls already* $3/4$*-th of the cheating key to forge proofs. So Dave is careful to get the proofs of knowledge right. He computes the transcript of Carols contribution as*

$$
\begin{aligned}
transcript_{1,2} = \\
MD5('(26,34)(16v^2,28v^3)(13,28)(13,28)(26,9)(27,34)(17v^2,28v^3)(35,28)') = \\
c8e6308fffd47009f5f65e773ae4b499
\end{aligned}
$$

We obtain that hash by computing
printf'%s'"(26,34)$(16v^2,28v^3)$*(13,28)(13,28)(26,9)(27,34)*$(17v^2,28v^3)$*(35,28)"|md5sum*

6 Exercises and Solutions

TODO: All exercises we provided should have a solution, which we give here in all detail.