
Operational notes

Document updated on **June 27, 2022**.

















































The following colors are **not** part of the final product, but serve as highlights in the editing/review process:

- text that needs attention from the Subject Matter Experts: Mirco, Anna,& Jan
- terms that have not yet been defined in the book
- things that need to be checked only at the very final typesetting stage (and it doesn't make sense to do them before)
- text that needs advice from the communications/marketing team: Aaron & Shane
- text that needs to be completed or otherwise edited (by Sylvia)

















































Todo list

12	■ Sylvia: This subsection should be in the introduction of the book as it applies to the	
13	entire math part	12
14	■ zero-knowledge proofs	12
15	■ add references	12
16	■ Sylvia: This subsection should be in the introduction of the book as it applies to the	
17	entire math part	13
18	■ Sylvia: We should put this table into the same introduction as 3.1.1	14
19	■ @jan @anna double check this definition. Is it clear enough? Proper definition re-	
20	quires the concept of equivalence or coprimeness first	14
21	■ Do we even need this quantum computing excursion?	16
22	■ @jan. You wrote: a and b are required to be non-zero in the definition above, so this	
23	can just be deleted. ... a can be zero and existence and uniqueness, non-zeroness	
24	are not obvious. Do you mean something else?	17
25	■ You wrote: if these should only satisfy the equation, why use definition symbols	
26	(:=) and not equality symbols (=)? But this is a definition the symbol a div b IS	
27	DEFINED to be the number b... Is that clear?	17
28	■ check algorithm floating	24
29	■ subtrahend	34
30	■ minuend	34
31	■ algorithm-floating	36
32	■ check algorithm floating	39
33	■ Sylvia: I would like to have a separate counter for definitions	41
34	■ check reference	48
35	■ runtime complexity	48
36	■ add reference	48
37	■ S: what does “efficiently” mean here?	48
38	■ computational hardness assumptions	48
39	■ check reference	48
40	■ check reference	49
41	■ explain last sentence more	49
42	■ “equation”?	51
43	■ check reference	51
44	■ what’s the difference between \mathbb{F}_p^* and \mathbb{Z}_p^* ?	51
45	■ Legendre symbol	51
46	■ Euler’s formular	51
47	■ These are only explained later in the text, ‘4.34’	51
48	■ are these going to be relevant later? yes, they are used in various snark proof systems	51
49	■ Is there a term for this property?	52
50	■ a few examples?	54

















































51	check reference	55
52	TODO: DOUBLE CHECK THIS REASONING.	55
53	Mirco: We can do better than this	57
54	check reference	58
55	add reference	59
56	pseudorandom	59
57	oracle	59
58	check reference	59
59	add reference	63
60	Abelian groups	63
61	codomain	63
62	Check change of wording	63
63	add reference	67
64	check reference	67
65	check reference	71
66	add reference	72
67	TODO: Elliptic Curve asymmetric cryptography examples. Private key, generator,	
68	public key.	74
69	add reference	74
70	maybe remove this sentence?	74
71	affine space	74
72	cusps	75
73	self-intersections	75
74	check reference	76
75	check reference	77
76	jubjub	77
77	check reference	77
78	affine plane	77
79	check reference	78
80	check reference	78
81	check reference	79
82	sign	79
83	more explanation of what the sign is	79
84	check reference	79
85	S: I don't follow this at all	80
86	check reference	80
87	add explanation of how this shows what we claim	80
88	should this def. be moved even earlier?	81
89	chord line	81
90	tangential	81
91	tangent line	81
92	remove Q ?	81
93	where?	82
94	check reference	82
95	check reference	82
96	check reference	82
97	check reference	83
98	check reference	83

99	 check reference	84
100	 check reference	84
101	 check reference	85
102	 add term	85
103	 add term	85
104	 add reference	85
105	 cofactor clearing	85
106	 add reference	85
107	 check reference	85
108	 check reference	86
109	 add reference	86
110	 add reference	86
111	 check reference	86
112	 check reference	86
113	 check reference	87
114	 check reference	87
115	 check reference	87
116	 Explain how	87
117	 write example	88
118	 check reference	88
119	 add reference	88
120	 check reference	88
121	 add reference	89
122	 check reference	89
123	 add reference	89
124	 check reference	89
125	 add reference	89
126	 check reference	89
127	 add reference	89
128	 add reference	89
129	 add reference	89
130	 check reference	89
131	 check reference	89
132	 Check if following Alg is floated too far	90
133	 add reference	90
134	 add reference	90
135	 write up this part	90
136	 is the label in L ^A T _E X correct here?	92
137	 check reference	92
138	 check reference	92
139	 check reference	92
140	 check reference	93
141	 check reference	93
142	 check reference	94
143	 check reference	94
144	 check reference	94
145	 check reference	94
146	 check reference	94

















































147	 add reference	94
148	 check reference	96
149	 check reference	96
150	 check reference	96
151	 check reference	96
152	 check reference	96
153	 check reference	98
154	 check reference	98
155	 check reference	99
156	 either expand on this or delete it	99
157	 add reference	99
158	 check reference	99
159	 check reference	99
160	 check reference	99
161	 check reference	99
162	 check reference	99
163	 check reference	100
164	 check reference	100
165	 check reference	101
166	 check reference	101
167	 add reference	101
168	 add reference	101
169	 This needs to be written (in Algebra)	101
170	 add reference	101
171	 add reference	101
172	 check reference	101
173	 towers of curve extensions	102
174	 check reference	102
175	 check reference	102
176	 check reference	102
177	 check reference	103
178	 add reference	103
179	 check reference	104
180	 S: either add more explanation or move to a footnote	104
181	 type 3 pairing-based cryptography	104
182	 add references?	104
183	 check reference	105
184	 check reference	105
185	 check floating of algorithm	106
186	 add references	106
187	 check reference	107
188	 add reference	107
189	 check reference	107
190	 check reference	107
191	 add reference	108
192	 should all lines of all algorithms be numbered?	108
193	 check reference	109
194	 check reference	109

















195		check reference	109
196		check if the algorithm is floated properly	109
197		check reference	109
198		again?	111
199		check reference	111
200		circuit	111
201		signature schemes	111
202		add reference	111
203		check reference	112
204		check reference	112
205		add references	112
206		add reference	112
207		reference text to be written in Algebra	112
208		check reference	112
209		check reference	112
210		check reference	113
211		add reference	113
212		algebraic closures	113
213		check reference	113
214		check reference	114
215		check reference	114
216		check reference	114
217		check reference	115
218		disambiguate	115
219		add reference	115
220		unify terminology	115
221		check reference	116
222		actually make this a table?	116
223		exercise still to be written?	117
224		add reference	117
225		check reference	117
226		check reference	117
227		add reference	118
228		check reference	119
229		check reference	119
230		check reference	119
231		add reference	120
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233		check reference	120
234		check reference	121
235		what does this mean? Maybe just delete it	122
236		write up this part	123
237		add reference	123
238		check reference	123
239		cyclotomic polynomial	123
240		Pholaard-rho attack	123
241		todo	123
242		why? Because in this book elliptic curves are only defined for fields of chracteristic > 3	124

243	check reference	124
244	check reference	124
245	what does this mean?	124
246	add reference	124
247	add reference	124
248	check reference	124
249	check reference	125
250	add reference	126
251	add exercise	126
252	check reference	127
253	add reference	127
254	add reference	127
255	add reference	127
256	check reference	128
257	check reference	128
258	add reference	128
259	add reference	128
260	add reference	129
261	check reference	129
262	add reference	129
263	add reference	129
264	finish writing this up	130
265	add reference	130
266	correct computations	130
267	fill in missing parts	130
268	add reference	131
269	check equation	131
270	Chapter 1?	132
271	"rigorous"?	132
272	"proving"?	132
273	Add example	133
274	M: 1:1 correspondence might actually be wrong	133
275	binary tuples	133
276	add reference	134
277	add reference	134
278	check reference	134
279	check reference	134
280	Are we using w and x interchangeably or is there a difference between them?	135
281	check reference	135
282	jubjub	135
283	check reference	135
284	check reference	135
285	check wording	135
286	check reference	135
287	check references	136
288	add reference	136
289	add reference	136
290	check reference	137

291	 add reference	137
292	 check reference	138
293	 check reference	138
294	 add reference	138
295	 add reference	139
296	 Schur/Hadamard product	140
297	 add reference	140
298	 check reference	140
299	 check reference	141
300	 add reference	142
301	 check reference	143
302	 check reference	143
303	 check reference	143
304	 check reference	143
305	 check reference	143
306	 add reference	144
307	 add reference	145
308	 check reference	145
309	 check reference	145
310	 check reference	146
311	 check reference	146
312	 add reference	147
313	 check reference	149
314	 add reference	149
315	 check reference	150
316	 check reference	150
317	 check reference	150
318	 Should we refer to R1CS satisfiability (p. 143 here?	151
319	 check reference	152
320	 add reference	152
321	 check reference	152
322	 check reference	153
323	 check reference	153
324	 check reference	154
325	 check reference	156
326	 add reference	157
327	 "by"?	157
328	 check reference	157
329	 check reference	157
330	 add reference	157
331	 add reference	157
332	 check reference	157
333	 add reference	157
334	 clarify language	159
335	 check reference	160
336	 add reference	160
337	 check reference	160
338	 add reference	160

339	check references	163
340	add references to these languages?	163
341	check reference	166
342	check reference	167
343	check reference	167
344	check reference	168
345	check reference	169
346	check reference	169
347	check reference	171
348	check reference	171
349	check reference	172
350	add reference	172
351	check reference	172
352	add reference	172
353	add reference	172
354	check reference	173
355	check reference	173
356	check reference	173
357	check reference	173
358	add reference	173
359	check reference	174
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362	check reference	176
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364	add reference	176
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367	check references	177
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370	can we rotate this by 90°?	178
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372	add reference	179
373	add reference	179
374	shift	181
375	bishift	182
376	add reference	183
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378	Add example	185
379	add reference	186
380	add reference	187
381	check reference	188
382	add reference	188
383	add reference	188
384	check reference	189
385	add reference	189
386	add reference	189

387	 add reference	191
388	 check reference	192
389	 check reference	193
390	 common reference string	193
391	 simulation trapdoor	193
392	 check reference	193
393	 check reference	193
394	 add reference	194
395	 check reference	194
396	 check reference	194
397	 check reference	194
398	 "invariable"?	194
399	 explain why	195
400	 4 examples have the same title. Change it to be distinct	195
401	 check reference	195
402	 add reference	195
403	 check reference	195
404	 add reference	195
405	 add reference	196
406	 add reference	197
407	 check reference	198
408	 add reference	198
409	 add reference	199
410	 check reference	199
411	 check reference	199
412	 add reference	199
413	 add reference	199
414	 check reference	200
415	 add reference	200
416	 add reference	200
417	 add reference	200
418	 check reference	200
419	 add reference	200
420	 add reference	200
421	 add reference	200
422	 add reference	200
423	 add reference	201
424	 add reference	201
425	 add reference	201
426	 add reference	201
427	 check reference	203
428	 check reference	203
429	 add reference	203
430	 add reference	203
431	 add reference	203
432	 add reference	203
433	 add reference	204
434	 add reference	204

435		add reference	204
436		add reference	204
437		fix error	204
438		add reference	204
439		check reference	205
440		add reference	205
441		add reference	205
442		add reference	205
443		add reference	206
444		add reference	206
445		add reference	206
446		add reference	206
447		add reference	206
448		add reference	206
449		add reference	206
450		add reference	207

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MoonMath manual

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TechnoBob and the Least Scruples crew

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June 27, 2022

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Chapter 1

Introduction

This is dump from other papers as inspiration for the intro:

Zero knowledge proofs are a class of cryptographic protocols in which one can prove honest computation without revealing the inputs to that computation. A simple high-level example of a zero-knowledge proof is the ability to prove one is of legal voting age without revealing the respective age. In a typical zero knowledge proof system, there are two participants: a prover and a verifier. A prover will present a mathematical proof of computation to a verifier to prove honest computation. The verifier will then confirm whether the prover has performed honest computation based on predefined methods. Zero knowledge proofs are of particular interest to public blockchain activities as the verifier can be codified in smart contracts as opposed to trusted parties or third-party intermediaries.

Zero-knowledge proofs (ZKPs) are an important privacy-enhancing tool from cryptography. They allow proving the veracity of a statement, related to confidential data, without revealing any information beyond the validity of the statement. ZKPs were initially developed by the academic community in the 1980s, and have seen tremendous improvements since then. They are now of practical feasibility in multiple domains of interest to the industry, and to a large community of developers and researchers. ZKPs can have a positive impact in industries, agencies, and for personal use, by allowing privacy-preserving applications where designated private data can be made useful to third parties, despite not being disclosed to them.

ZKP systems involve at least two parties: a prover and a verifier. The goal of the prover is to convince the verifier that a statement is true, without revealing any additional information. For example, suppose the prover holds a birth certificate digitally signed by an authority. In order to access some service, the prover may have to prove being at least 18 years old, that is, that there exists a birth certificate, tied to the identity of the prover and digitally signed by a trusted certification authority, stating a birthdate consistent with the age claim. A ZKP allows this, without the prover having to reveal the birthdate.

1.1 Target audience

This book is accessible for both beginners and experienced developers alike. Concepts are gradually introduced in a logical and steady pace. Nonetheless, the chapters lend themselves rather well to being read in a different order. More experienced developers might get the most benefit by jumping to the chapters that interest them most. If you like to learn by example, then you should go straight to the chapter on Using Clarity.

It is assumed that you have a basic understanding of programming and the underlying logical concepts. The first chapter covers the general syntax of Clarity but it does not delve into what

programming itself is all about. If this is what you are looking for, then you might have a more difficult time working through this book unless you have an (undiscovered) natural affinity for such topics. Do not let that dissuade you though, find an introductory programming book and press on! The straightforward design of Clarity makes it a great first language to pick up.

1.2 The Zoo of Zero-Knowledge Proofs

First, a list of zero-knowledge proof systems:

1. Pinocchio (2013): Paper

– Notes: trusted setup

2. BCGTV (2013): Paper

– Notes: trusted setup, implementation

3. BCTV (2013): Paper

– Notes: trusted setup, implementation

4. Groth16 (2016): Paper

– Notes: trusted setup

– Other resources: Talk in 2019 by Georgios Konstantopoulos

5. GM17 (2017): Paper

– Notes: trusted setup

– Other resources: later Simulation extractability in ROM, 2018

6. Bulletproofs (2017): Paper

– Notes: no trusted setup

– Other resources: Polynomial Commitment Scheme on DL, 2016 and KZG10, Polynomial Commitment Scheme on Pairings, 2010

7. Ligero (2017): Paper

– Notes: no trusted setup

– Other resources:

8. Hyrax (2017): Paper

– Notes: no trusted setup

– Other resources:

9. STARKs (2018): Paper

– Notes: no trusted setup

– Other resources:

10. Aurora (2018): Paper

- Notes: transparent SNARK

- Other resources:

11. Sonic (2019): Paper

- Notes: SNORK - SNARK with universal and updateable trusted setup, PCS-based

- Other resources: Blog post by Mary Maller from 2019 and work on updateable and universal setup from 2018

12. Libra (2019): Paper

- Notes: trusted setup

- Other resources:

13. Spartan (2019): Paper

- Notes: transparent SNARK

- Other resources:

14. PLONK (2019): Paper

- Notes: SNORK, PCS-based

- Other resources: Discussion on Plonk systems and Awesome Plonk list

15. Halo (2019): Paper

- Notes: no trusted setup, PCS-based, recursive

- Other resources:

16. Marlin (2019): Paper

- Notes: SNORK, PCS-based

- Other resources: Rust Github

17. Fractal (2019): Paper

- Notes: Recursive, transparent SNARK

- Other resources:

18. SuperSonic (2019): Paper

- Notes: transparent SNARK, PCS-based

- Other resources: Attack on DARK compiler in 2021

19. Redshift (2019): Paper

- Notes: SNORK, PCS-based

- Other resources:

Other resources on the zoo: Awesome ZKP list on Github, ZKP community with the reference document

To Do List

- Make table for prover time, verifier time, and proof size
- Think of categories - *Achieved Goals*: Trusted setup or not, Post-quantum or not, ...
- Think of categories - *Mathematical background*: Polynomial commitment scheme, ...
- ... while we discuss the points above, we should also discuss a common notation/language for all these things. (E.g. transparent SNARK/no trusted setup/STARK)

Points to cover while writing

- Make a historical overview over the "discovery" of the different ZKP systems
- Make reader understand what paper is build on what result etc. - the tree of publications!
- Make reader understand the different terminology, e.g. SNARK/SNORK/STARK, PCS, R1CS, updateable, universal, ...
- Make reader understand the mathematical assumptions - and what this means for the zoo.
- Where will the development/evolution go? What are bottlenecks?

Other topics I fell into while compiling this list

- Vector commitments: <https://eprint.iacr.org/2020/527.pdf>
- Snark1: <http://ace.cs.ohio.edu/~gstewart/papers/snaark1.pdf>
- Virgo?: https://people.eecs.berkeley.edu/~kubitron/courses/cs262a-F19/projects/reports/project5_report_ver2.pdf

Chapter 2

Preliminaries

2.1 Preface and Acknowledgements

This book began as a set of lecture and notes accompanying the zk-Summit 0x and 0xx It arose from the desire to collect the scattered information of snarks [] and present them to an audience that does not have a strong background in cryptography []

2.2 Purpose of the book

The first version of this book is written by security auditors at Least Authority where we audited quite a few snark based systems. Its included "what we have learned" destilate of the time we spend on various audits.

We intend to let illustrative examples drive the discussion and present the key concepts of pairing computation with as little machinery as possible. For those that are fresh to pairing-based cryptography, it is our hope that this chapter might be particularly useful as a first read and prelude to more complete or advanced expositions (e.g. the related chapters in [Gal12]).

On the other hand, we also hope our beginner-friendly intentions do not leave any sophisticated readers dissatisfied by a lack of formality or generality, so in cases where our discussion does sacrifice completeness, we will at least endeavour to point to where a more thorough exposition can be found.

One advantage of writing a survey on pairing computation in 2012 is that, after more than a decade of intense and fast-paced research by mathematicians and cryptographers around the globe, the field is now racing towards full maturity. Therefore, an understanding of this text will equip the reader with most of what they need to know in order to tackle any of the vast literature in this remarkable field, at least for a while yet.

Since we are aiming the discussion at active readers, we have matched every example with a corresponding snippet of (hyperlinked) Magma [BCP97] code 1 , where we take inspiration from the helpful Magma pairing tutorial by Dominguez Perez et al. [DKS09].

Early in the book we will develop examples that we then later extend with most of the things we learn in each chapter. This way we incrementally build a few real world snarks but over full fledged cryptographic systems that are nevertheless simple enough to be computed by pen and paper to illustrate all steps in great detail.

2.3 How to read this book

Books and papers to read: XXXXXXXXXXXX

Software to try: XXXXXXXXXXXXXXXXXXXX

Correctly prescribing the best reading route for a beginner naturally requires individual diagnosis that depends on their prior knowledge and technical preparation.

2.4 Cryptological Systems

The science of information security is referred to as *cryptology*. In the broadest sense, it deals with encryption and decryption processes, with digital signatures, identification protocols, cryptographic hash functions, secrets sharing, electronic voting procedures and electronic money.
EXPAND

2.5 SNARKS

2.6 complexity theory

Before we deal with the mathematics behind zero knowledge proof systems, we must first clarify what is meant by the runtime of an algorithm or the time complexity of an entire mathematical problem. This is particularly important for us when we analyze the various snark systems...

For the reader who is interested in complexity theory, we recommend, or example or , as well as the references contained therein.

2.6.1 Runtime complexity

The runtime complexity of an algorithm describes, roughly speaking, the amount of elementary computation steps that this algorithm requires in order to solve a problem, depending on the size of the input data.

Of course, the exact amount of arithmetic operations required depends on many factors such as the implementation, the operating system used, the CPU and many more. However, such accuracy is seldom required and is mostly meaningful to consider only the asymptotic computational effort.

In computer science, the runtime of an algorithm is therefore not specified in individual calculation steps, but instead looks for an upper limit which approximates the runtime as soon as the input quantity becomes very large. This can be done using the so-called *Landau notation* (also called big- \mathcal{O} -notation) A precise definition would, however, go beyond the scope of this work and we therefore refer the reader to .

For us, only a rough understanding of transit times is important in order to be able to talk about the security of cryptographic systems. For example, $\mathcal{O}(n)$ means that the running time of the algorithm to be considered is linearly dependent on the size of the input set n , $\mathcal{O}(n^k)$ means that the running time is polynomial and $\mathcal{O}(2^n)$ stands for an exponential running time (chapter 2.4).

An algorithm which has a running time that is greater than a polynomial is often simply referred to as *slow*.

A generalization of the runtime complexity of an algorithm is the so-called *time complexity of a mathematical problem*, which is defined as the runtime of the fastest possible algorithm that can still solve this problem (chapter 3.1).

Since the time complexity of a mathematical problem is concerned with the runtime analysis of all possible (and thus possibly still undiscovered) algorithms, this is often a very difficult and deep-seated question .

For us, the time complexity of the so-called discrete logarithm problem will be important. This is a problem for which we only know slow algorithms on classical computers at the moment, but for which at the same time we cannot rule out that faster algorithms also exist.

STUFF ON CRYPTOGRAPHIC HASH FUNCTIOND

2.7 Software Used in This Book

2.7.1 Sagemath

It order to provide an interactive learning experience, and to allow getting hands-on with the concepts described in this book, we give examples for how to program them in the Sage programming language. Sage is a dialect of the learning-friendly programming language Python, which was extended and optimized for computing with, in and over algebraic objects. Therefore, we recommend installing Sage before diving into the following chapters.

The installation steps for various system configurations are described on the sage websit ¹. Note however that we use Sage version 9, so if you are using Linux and your package manager only contains version 8, you may need to choose a different installation path, such as using prebuilt binaries.

We recommend the interested reader, who is not familiar with sagemath to read on the many tutorial before starting this book. For example

¹<https://doc.sagemath.org/html/en/installation/index.html>

Chapter 3

arithmetic

3.1 Introduction

3.1.1 Aims and target audience

Sylvia: This subsection should be in the introduction of the book as it applies to the entire math part

How much mathematics do you need to understand **zero-knowledge proofs**? The answer, of course, depends on the level of understanding you aim for. It is possible to describe zero-knowledge proofs without using mathematics at all; however, to read a foundational paper like ? and to understand the internals of snark implementations, some knowledge of mathematics is needed to be able to follow the discussion.

The goal of this chapter is therefore to enable a reader who is starting out with nothing more than basic high school algebra to be able to solve simple tasks in the mathematics underlying most zero knowledge proofs without the need of a computer.

Without a solid grounding in mathematics, someone who is interested in learning the concepts of zero-knowledge proofs, but who has never seen or dealt with, say, a prime field 4.3.1, or an elliptic curve 5, may quickly become overwhelmed. This is not so much due to the complexity of the mathematics needed, but rather because of the vast amount of technical jargon, unknown terms, and obscure symbols that quickly makes a text unreadable, even though the concepts themselves are not actually that complicated. As a result, the reader might either lose interest, or pick up some incoherent bits and pieces of knowledge that, in the worst case scenario, result in immature and unsecure code.

This is why we dedicated this chapter to explaining the mathematical foundations needed to understand the basic concepts underlying snark development. We encourage the reader who is not familiar with basic number theory and elliptic curves to take the time and read this and the following chapters, until they are able to solve at least a few exercises in each chapter.

If, on the other hand, you are already skilled in elliptic curve cryptography, feel free to skip this chapter and only come back to it for reference and comparison. Maybe the most interesting parts are XXX .

We start our explanations at a very basic level, and only assume pre-existing knowledge of fundamental concepts like high school integer arithmetic. At the same time, we'll attempt to teach you to "think mathematically", and to show you that there are numbers and mathematical structures out there that appear to be very different from the things you learned about in high school, but on a deeper level, they are actually quite similar, as we will see in various examples below in this chapter.

Sylvia:
This subsection should be in the introduction of the book as it applies to the entire math part

zero-knowledge proofs

add references

To make it easier to memorize new concepts and symbols, we might frequently link to definitions in the beginning, but as too many links render a text unreadable, we will assume the reader will become familiar with definitions as the text proceeds at which point we will not link them anymore.

We want to stress, however that this introduction is informal, incomplete and optimized to enable the reader to understand zero-knowledge concepts as efficiently as possible. Our focus and design choices are to include as little theory as necessary, focusing on a wealth of numerical examples. We believe that such an informal, example-driven approach to learning mathematics may make it easier for beginners to digest the material in the initial stages.

For instance, as a beginner, you would probably find it more beneficial to first compute a simple toy **snark** with pen and paper all the way through, before actually developing real-world production-ready systems. In addition, it's useful to have a few simple examples in your head before getting started with reading actual academic papers.

However, in order to be able to derive these toy examples, some mathematical groundwork is needed. This chapter therefore will help the inexperienced reader to focus on what we believe is important, accompanied by exercises that you are encouraged to recompute yourself. Every section usually ends with a list of additional exercises in increasing order of difficulty, to help the reader memorize and apply the concepts.

3.1.2 The structure of this chapter

Sylvia: This subsection should be in the introduction of the book as it applies to the entire math part

We start with a brief recapitulation of basic integer arithmetic like long division, the greatest common divisor and Euclid's algorithm. After that, we introduce modular arithmetic as **the most important skill** to compute our pen-and-paper examples. We then introduce polynomials, compute their analogs to integer arithmetic and introduce the important concept of Lagrange interpolation.

After this practical warm up, we introduce some basic algebraic terms like groups and fields, because those terms are used very frequently in academic papers relating to zero-knowledge proofs. The beginner is advised to memorize those terms and think about them: Why are they defined the way they are? Can you come up with a better definition? How and why is every rule important and what happens if that rule is dropped?

We define these terms in the general abstract way of mathematics, hoping that the non mathematical trained reader will gradually learn to become comfortable with this style. We then give basic examples and do basic computations with these examples to get familiar with the concepts.

Sylvia: This subsection should be in the introduction of the book as it applies to the entire math part

3.2 Integer arithmetic

In a sense, integer arithmetic is at the heart of large parts of modern cryptography. Fortunately, most readers will probably remember integer arithmetic from school. It is, however, important that you can confidently apply those concepts to understand and execute computations in the many pen-and-paper examples that form an integral part of the MoonMath Manual. We will therefore recapitulate basic arithmetic concepts to refresh your memory and fill any knowledge gaps.

In what follows, we use many mathematical notations, which we summarized in the following table 3.2:

Symbol	Meaning of Symbol
=	equals
:=	defining the symbol on the r
∈	element from a set
⇔	logical equivalence
$\sum_{j=n}^k a_j$	summation

Sylvia: We should put this table into the same introduction as 3.1.1

In this book, we use the symbol \mathbb{Z} as a short description for the set of all **integers**, that is we write:

$$\mathbb{Z} := \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\} \quad (3.1)$$

Integers are also known as **whole numbers**, that is, numbers that can be written without fractional parts. Examples of numbers that are **not** integers are $\frac{2}{3}$, 1.2 and -1280.006 .

If $a \in \mathbb{Z}$ is an integer, then we write $|a|$ for the **absolute value** of a , that is, the the non-negative value of a without regard to its sign:

$$|4| = 4 \quad (3.2)$$

$$|-4| = 4 \quad (3.3)$$

We use the symbol \mathbb{N} for the set of all positive integers, usually called the set of **natural numbers** and \mathbb{N}_0 for the set of all non negative integers. So whenever you see the symbol \mathbb{N} , think of the set of all positive integers excluding the number 0:

$$\mathbb{N} := \{1, 2, 3, \dots\} \quad \mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$$

Let $n \in \mathbb{N}_0$ be a non negative integer and (b_0, b_1, \dots, b_k) a string of **bits** $b_j \in \{0, 1\} \subset \mathbb{N}_0$ for some non negative integer $k \in \mathbb{N}$, such that the following equation holds:

$$n = \sum_{j=0}^k b_j \cdot 2^j \quad (3.4)$$

In this case, we call (b_0, b_1, \dots, b_k) the **binary representation** of n , say that n is a k -bit number and call k the **bit length** of n . It can be shown, that the binary representation of any non negative integer is unique.

In addition, we use the symbol \mathbb{Q} for the set of all **rational numbers**, which can be represented as the set of all fractions $\frac{n}{m}$, where $n \in \mathbb{Z}$ is an integer and $m \in \mathbb{N}$ is a natural number, such that there is no other fraction $\frac{n'}{m'}$ and natural number $k \in \mathbb{N}$ with $k \neq 1$ and

$$\frac{n}{m} = \frac{k \cdot n'}{k \cdot m'} \quad (3.5)$$

The sets \mathbb{N} , \mathbb{Z} and \mathbb{Q} have a notion of addition and multiplication defined on them. Most of us are probably able to do many integer computations in our head, but this gets more and more difficult as these increase in complexity. We will frequently invoke the SageMath system (2.7.1) for more complicated computations (We define rings and fields later in this book):

Sylvia:
We should
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table into
the same
introduc-
tion as
3.1.1

@jan
@anna
double
check this
definition.
Is it clear
enough?
Proper
definition
requires
the con-
cept of
equiv-
alance or
coprime-
ness first

```

910 sage: ZZ # A sage notation for the integers 1
911 Integer Ring 2
912 sage: NN # A sage notation for the natural numbers 3
913 Non negative integer semiring 4
914 sage: QQ # A sage notation for the rational numbers 5
915 Rational Field 6
916 sage: ZZ(5) # Get an element from the integers 7
917 5 8
918 sage: ZZ(5) + ZZ(3) 9
919 8 10
920 sage: ZZ(5) * NN(3) 11
921 15 12
922 sage: ZZ.random_element(10**50) 13
923 25566069313870694529920886180936545039739782599916 14
924 sage: ZZ(27713).str(2) # Binary string representation 15
925 110110001000001 16
926 sage: NN(27713).str(2) # Binary string representation 17
927 110110001000001 18
928 sage: ZZ(27713).str(16) # Hexadecimal string representation 19
929 6c41 20

```

One set of numbers that is of particular interest to us is the set of **prime numbers**, which are natural numbers $p \in \mathbb{N}$ with $p \geq 2$, which are only divisible by themselves and by 1. All prime numbers apart from the number 2 are called **odd prime numbers**. We write \mathbb{P} for the set of all prime numbers and $\mathbb{P}_{\geq 3}$ for the set of all odd prime numbers. The set of prime numbers \mathbb{P} is an infinite set and can be ordered according to size, which means that for any prime number $p \in \mathbb{P}$ one can always find another prime number $p' \in \mathbb{P}$ with $p < p'$. It follows that there is no largest prime number. Since prime numbers can be ordered by size, we can write them as follows:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, \dots \quad (3.6)$$

As the **fundamental theorem of arithmetic** tells us, prime numbers are, in a certain sense, the basic building blocks from which all other natural numbers are composed. To see that, let $n \in \mathbb{N}$ be any natural number with $n > 1$. Then there are always prime numbers $p_1, p_2, \dots, p_k \in \mathbb{P}$, such that

$$n = p_1 \cdot p_2 \cdot \dots \cdot p_k. \quad (3.7)$$

This representation is unique for each natural number (except for the order of the factors) and is called the **prime factorization** of n .

Example 1 (Prime Factorization). To see what we mean by prime factorization of a number, let's look at the number $504 \in \mathbb{N}$. To get its prime factors, we can successively divide it by all prime numbers in ascending order starting with 2:

$$504 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7$$

We can double check our findings invoking Sage, which provides an algorithm to factor natural numbers:

```

945 sage: n = NN(504) 21
946 sage: factor(n) 22

```

947 $2^3 * 3^2 * 7$

948 The computation from the previous example reveals an important observation: Computing
 949 the factorization of an integer is computationally expensive, because we have to divide repeatedly
 950 by all prime numbers smaller than the number itself until all factors are prime numbers themselves.
 951 From this, an important question arises: How fast can we compute the prime factorization of a
 952 natural number? This question is the famous **integer factorization problem** and, as far as we
 953 know, there is currently no method known that can factor integers much faster than the naive
 954 approach that just divides the given number by all prime numbers in ascending order.

955 On the other hand computing the product of a given set of prime numbers, is fast (just
 956 multiply all factors) and this simple observation implies that the two processes "prime number
 957 multiplication" on the one side and its inverse process "natural number factorization" have very
 958 different computational costs. The factorization problem is therefore an example of a so-called
 959 **one-way function**: An invertible function that is easy to compute in one direction, but hard to
 960 compute in the other direction.

961 It should be pointed out, however, that the American mathematician Peter W. Shor devel-
 962 oped an algorithm in [?] which can calculate the prime factorization of a natural number in
 963 polynomial time on a quantum computer. The consequence of this is that cryptosystems, which
 964 are based on the prime factor problem, are unsafe as soon as practically usable quantum com-
 965 puters become available.

966 *Exercise 1.* What is the absolute value of the integers -123 , 27 and 0 ?

967 *Exercise 2.* Compute the factorization of 30030 and double check your results using Sage.

968 *Exercise 3.* Consider the following equation $4 \cdot x + 21 = 5$. Compute the set of all solutions for
 969 x under the following alternative assumptions:

970 1. The equation is defined over the natural numbers.

971 2. The equation is defined over the integers.

972 *Exercise 4.* Consider the following equation $2x^3 - x^2 - 2x = -1$. Compute the set of all solu-
 973 tions x under the following assumptions:

974 1. The equation is defined over the natural numbers.

975 2. The equation is defined over the integers.

976 3. The equation is defined over the rational numbers.

977 **Euclidean Division** As we know from high school mathematics, integers can be added, sub-
 978 tracted and multiplied and the result is guaranteed to always be an integer again. On the contrary
 979 division in the commonly understood sense is not defined for integers, as, for example, 7 divided
 980 by 3 will not be an integer again. However it is always possible to divide any two integers if we
 981 consider division with remainder. So for example 7 divided by 3 is equal to 2 with a remainder
 982 of 1 , since $7 = 2 \cdot 3 + 1$.

983 It is the content of this section to introduce division with remainder for integers which is
 984 usually called **Euclidean division**. It is an essential technique underlying many concepts in this
 985 book. The precise definition is as follows:

986 Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ be two integers with $b \neq 0$. Then there is always another integer $m \in \mathbb{Z}$
 987 and a natural number $r \in \mathbb{N}$, with $0 \leq r < |b|$ such that

$$a = m \cdot b + r \quad (3.8)$$

Do we
even need
this quan-
tum com-
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excuse?

This decomposition of a given b is called **Euclidean division**, where a is called the **dividend**, b is called the **divisor**, m is called the **quotient** and r is called the **remainder**. It can be shown that both the quotient and the remainder always exist and are unique, as long as the divisor is different from 0.

Notation and Symbols 1. Suppose that the numbers a, b, m and r satisfy equation (3.8). Then we often write

$$a \operatorname{div} b := m, \quad a \operatorname{mod} b := r \quad (3.9)$$

to describe the quotient and the remainder of the Euclidean division. We also say that an integer a is divisible by another integer b if $a \operatorname{mod} b = 0$ holds. In this case we also write $b|a$ and call the integer $a \operatorname{div} b$ the **cofactor** of b in a .

So, in a nutshell Euclidean division is a process of dividing one integer by another in a way that produces a quotient and a non-negative remainder, the latter of which is smaller than the absolute value of the divisor.

A special situation occurs whenever the remainder is zero, because in this case the dividend is divisible by the divisor. Our notation $b|a$ reflects that.

Example 2. Applying Euclidean division and our previously defined notation 3.9 to the dividend -17 and the divisor 4 , we get

$$-17 \operatorname{div} 4 = -5, \quad -17 \operatorname{mod} 4 = 3$$

because $-17 = -5 \cdot 4 + 3$ is the Euclidean division of -17 and 4 (the remainder is, by definition, a non-negative number). In this case 4 does not divide -17 , as the remainder is not zero. The truth value of the expression $4|-17$ therefore is FALSE. On the other hand, the truth value of $4|12$ is TRUE, since 4 divides 12 , as $12 \operatorname{mod} 4 = 0$. We can invoke sage to do the computation for us. We get the following:

```
sage: ZZ(-17) // ZZ(4) # Integer quotient
-5
sage: ZZ(-17) % ZZ(4) # remainder
3
sage: ZZ(4).divides(ZZ(-17)) # self divides other
False
sage: ZZ(4).divides(ZZ(12))
True
```

Remark 1. In 3.9 we defined the notation of $a \operatorname{div} b$ and $a \operatorname{mod} b$, in terms of Euclidean division. It should be noted however that many programming languages like Python and Sage, implement both the operator $(/)$ as well as the operator $(\%)$ differently. Programmers should be aware of this, as the discrepancy between the mathematical notation and the implementation in programming languages might become the source of subtle bugs in implementations of cryptographic primitives.

To give an example consider the the dividend -17 and the divisor -4 . Note that in contrast to the previous example 2, we have a negative divisor. According to our definition we have

$$-17 \operatorname{div} -4 = 5, \quad -17 \operatorname{mod} -4 = 3$$

because $-17 = 5 \cdot (-4) + 3$ is the Euclidean division of -17 and -4 (the remainder is, by definition, a non-negative number). However using the operators $(/)$ and $(\%)$ in Sage we get

@jan.
You wrote: a and b are required to be non-zero in the definition above, so this can just be deleted.
... a can be zero and existence and uniqueness, non-zeroness are not obvious. Do you mean something else?
You wrote: If these should only satisfy the equation, why use definition symbols (:=) and not equality symbols (=)? But this is a definition the symbol a div b IS DEFINED to be the number b... Is that clear?

```

1023 sage: ZZ(-17) // ZZ(-4) # Integer quotient      32
1024 4                                              33
1025 sage: ZZ(-17) % ZZ(-4) # remainder            34
1026 -1                                           35
1027 sage: ZZ(-17).quo_rem(ZZ(-4)) # not Euclidean division 36
1028 (4, -1)                                     37

```

1029 Methods to compute Euclidean division for integers are called **integer division algorithms**.
 1030 Probably the best known algorithm is the so-called **long division**, which most of us might have
 1031 learned in school.

1032 As long division is the standard method used for pen-and-paper division of multi-digit num-
 1033 bers expressed in decimal notation, the reader should become familiar with it as we use it
 1034 throughout this book when we do simple pen-and-paper computations. However, instead of
 1035 defining the algorithm formally, we rather give some examples that will hopefully make the
 1036 process clear.

1037 In a nutshell, the algorithm loops through the digits of the dividend from the left to right,
 1038 subtracting the largest possible multiple of the divisor (at the digit level) at each stage; the
 1039 multiples then become the digits of the quotient, and the remainder is the first digit of the
 1040 dividend.

1041 *Example 3 (Integer Long Division).* To give an example of integer long division algorithm, lets
 1042 divide the integer $a = 143785$ by the number $b = 17$. Our goal is therefore to find solutions
 1043 to equation 3.8, that is, we need to find the quotient $m \in \mathbb{Z}$ and the remainder $r \in \mathbb{N}$ such that
 1044 $143785 = m \cdot 17 + r$. Using a notation that is mostly used in Commonwealth countries, we
 1045 compute as follows:

$$\begin{array}{r}
 8457 \\
 17 \overline{) 143785} \\
 \underline{136} \\
 77 \\
 \underline{68} \\
 98 \\
 \underline{85} \\
 135 \\
 \underline{119} \\
 16
 \end{array}
 \tag{3.10}$$

1046 We therefore get $m = 8457$ as well as $r = 16$ and indeed we have $143785 = 8457 \cdot 17 + 16$,
 1047 which we can double check invoking Sage:

```

1048 sage: ZZ(143785).quo_rem(ZZ(17))              38
1049 (8457, 16)                                    39
1050 sage: ZZ(143785) == ZZ(8457)*ZZ(17) + ZZ(16) # check 40
1051 True                                           41

```

1052 *Exercise 5 (Integer Long Division).* Find an $m \in \mathbb{Z}$ as well as an $r \in \mathbb{N}$ with $0 \leq r < |b|$ such that
 1053 $a = m \cdot b + r$ holds for the following pairs $(a, b) = (27, 5)$, $(a, b) = (27, -5)$, $(a, b) = (127, 0)$,
 1054 $(a, b) = (-1687, 11)$ and $(a, b) = (0, 7)$. In which cases are your solutions unique?

1055 *Exercise 6 (Long Division Algorithm).* Write an algorithm that computes integer long division
 1056 and handling all edge cases properly.

1057 *Exercise 7* (Binary Representation). Write an algorithm that computes the binary representation
 1058 3.4 of any non-negative integer.

1059 **The Extended Euclidean Algorithm** One of the most critical parts in this book is the so
 1060 called modular arithmetic which we will define in 3.3 and its application in the computations of
 1061 **prime fields** as defined in 4.3.1. To be able to do computations in modular arithmetic, we have
 1062 to get familiar with the so-called **extended Euclidean algorithm**. We therefore introduce this
 1063 algorithm here.

1064 The **greatest common divisor** (GCD) of two non-zero integers a and b , is defined as the
 1065 greatest non-zero natural number d such that d divides both a and b , that is, $d|a$ as well as $d|b$.
 1066 We write $\gcd(a, b) := d$ for this number. Since the natural number 1 divides any other integer, 1
 1067 is always a common divisor of any two non-zero integers. However it must not be the greatest.

1068 A common method to compute the greatest common divisor is the so called Eucliden algo-
 1069 rithm. However since we don't need that algorithm in this book, we will introduce the Extended
 1070 Euclidean algorithm which is a method to calculate the greatest common divisor of two natural
 1071 numbers a and $b \in \mathbb{N}$, as well as two additional integers $s, t \in \mathbb{Z}$, such that the following equation
 1072 holds:

$$\gcd(a, b) = s \cdot a + t \cdot b \quad (3.11)$$

1073 The pseudocode in algorithm 1 shows in detail how to calculate the greatest common divisor
 1074 and the numbers s and t with the extended Euclidean algorithm:

Algorithm 1 Extended Euclidean Algorithm

Require: $a, b \in \mathbb{N}$ with $a \geq b$

procedure EXT-EUCLID(a, b)

$r_0 \leftarrow a$ and $r_1 \leftarrow b$

$s_0 \leftarrow 1$ and $s_1 \leftarrow 0$

$t_0 \leftarrow 0$ and $t_1 \leftarrow 1$

$k \leftarrow 2$

while $r_{k-1} \neq 0$ **do**

$q_k \leftarrow r_{k-2} \text{ div } r_{k-1}$

$r_k \leftarrow r_{k-2} \text{ mod } r_{k-1}$

$s_k \leftarrow s_{k-2} - q_k \cdot s_{k-1}$

$t_k \leftarrow t_{k-2} - q_k \cdot t_{k-1}$

$k \leftarrow k + 1$

end while

return $\gcd(a, b) \leftarrow r_{k-2}$, $s \leftarrow s_{k-2}$ and $t \leftarrow t_{k-2}$

end procedure

Ensure: $\gcd(a, b) = s \cdot a + t \cdot b$

1075 The algorithm is simple enough to be done effectively in pen-and-paper examples, where
 1076 it is common to write it as a table where the rows represent the while-loop and the columns
 1077 represent the values of the the array r , s and t with index k . The following example provides a
 1078 simple execution:

1079 *Example 4.* To illustrate algorithm 1, we apply it to the numbers $a = 12$ and $b = 5$. Since
 1080 $12, 5 \in \mathbb{N}$ as well as $12 \geq 5$ all requirements are met and we compute as follows:

k	r_k	s_k	t_k
0	12	1	0
1	5	0	1
2	2	1	-2
3	1	-2	5
4	0		

From this we can see that the greatest common divisor of 12 and 5 is $\gcd(12, 5) = 1$ and that the equation $1 = (-2) \cdot 12 + 5 \cdot 5$ holds. We can also invoke sage to double check our findings:

```
sage: ZZ(12).xgcd(ZZ(5)) # (gcd(a,b), s, t) 42
      (1, -2, 5) 43
```

Exercise 8 (Extended Euclidean Algorithm). Find integers $s, t \in \mathbb{Z}$ such that $\gcd(a, b) = s \cdot a + t \cdot b$ holds for the following pairs $(a, b) = (45, 10)$, $(a, b) = (13, 11)$, $(a, b) = (13, 12)$. What pairs (a, b) are coprime?

Exercise 9 (Towards Prime fields). Let $n \in \mathbb{N}$ be a natural number and p a prime number, such that $n < p$. What is the greatest common divisor $\gcd(p, n)$?

Exercise 10. Find all numbers $k \in \mathbb{N}$ with $0 \leq k \leq 100$ such that $\gcd(100, k) = 5$.

Exercise 11. Show that $\gcd(n, m) = \gcd(n + m, m)$ for all $n, m \in \mathbb{N}$.

Coprime Integers Coprime integers are integers that do not have a common prime number as a factor. As we will see in 3.3 those numbers are important for our purposes because in modular arithmetic, computation that involve coprime numbers are substantially different from computations on non-coprime numbers 3.3.

The naive way to decide if two integers are coprime would be to divide both number suces- sively by all prime numbers smaller then those numbers to see if they share a common prime factor. However two integers are coprime if and only if their greatest common divisor is 1 and hence computing the \gcd is the preferred method.

Example 5. Consider example 4 again. As we have seen, the greatest common divisor of 12 and 5 is 1. This implies that the integers 12 and 5 are coprime, since they share no divisor other then 1, which is not a prime number.

Exercise 12. Consider exercise 8 again. Which pairs (a, b) from that exercise are coprime?

3.3 Modular arithmetic

Modular arithmetic is a system of integer arithmetic, where numbers “wrap around” when reaching a certain value, much like calculations on a clock wrap around whenever the value exceeds the number 12. For example, if the clock shows that it is 11 o’clock, then 20 hours later it will be 7 o’clock, not 31 o’clock. The number 31 has no meaning on a normal clock that shows hours.

The number at which the wrap occurs is called the **modulus**. Modular arithmetic general- izes the clock example to arbitrary moduli and studies equations and phenomena that arise in this new kind of arithmetic. It is of central importance for understanding most modern crypto systems, in large parts because modular arithmetic provides the computational infrastrutute for algebraic types that have cryptographically useful examples of one-way functions.

Although modular arithmetic appears very different from ordinary integer arithmetic that we are all familiar with, we encourage the interested reader to work through the example and to discover that, once they get used to the idea that this is a new kind of calculations, it will seem much less daunting.

Congruence In what follows, let $n \in \mathbb{N}$ with $n \geq 2$ be a fixed natural number that we will call the **modulus** of our modular arithmetic system. With such an n given, we can then group integers into classes, by saying that two integers are in the same class, whenever their Euclidean division 3.2 by n will give the same remainder. We then say that two numbers are **congruent** whenever they are in the same class.

Example 6. If we choose $n = 12$ as in our clock example, then the integers $-7, 5, 17$ and 29 are all congruent with respect to 12 , since all of them have the remainder 5 if we perform Euclidean division on them by 12 . In the picture of an analog 12-hour clock, starting at 5 o'clock, when we add 12 hours we are again at 5 o'clock, representing the number 17 . On the other hand, when we subtract 12 hours, we are at 5 o'clock again, representing the number -7 .

We can formalize this intuition of what congruence should be into a proper definition utilizing Euclidean division (as explained previously in 3.2): Let $a, b \in \mathbb{Z}$ be two integers and $n \in \mathbb{N}$ a natural number, such that $n \geq 2$. Then a and b are said to be **congruent with respect to the modulus n** , if and only if the following equation holds

$$a \bmod n = b \bmod n \quad (3.12)$$

If, on the other hand, two numbers are not congruent with respect to a given modulus n , we call them **incongruent** w.r.t. n .

A **congruence** is then nothing but an equation "up to congruence", which means that the equation only needs to hold if we take the modulus on both sides. In which case we write

$$a \equiv b \pmod{n} \quad (3.13)$$

Exercise 13. Which of the following pairs of numbers are congruent with respect to the modulus 13: $(5, 19), (13, 0), (-4, 9), (0, 0)$.

Exercise 14. Find all integers x , such that the congruence $x \equiv 4 \pmod{6}$ is satisfied.

Computational Rules Having defined the notion of a congruence as an equation "up to a modulus", a follow up question is if we can manipulate a congruence similar to an equation. Indeed we can almost apply the same substitution rules to a congruency then to an equation, with the main difference being that for some non-zero integer $k \in \mathbb{Z}$, the congruence $a \equiv b \pmod{n}$ is equivalent to the congruence $k \cdot a \equiv k \cdot b \pmod{n}$ only, if k and the modulus n are coprime 3.2. The following list gives a set of useful rules:

Suppose that integers $a_1, a_2, b_1, b_2, k \in \mathbb{Z}$ are given. Then the following arithmetic rules hold for congruencies:

- $a_1 \equiv b_1 \pmod{n} \Leftrightarrow a_1 + k \equiv b_1 + k \pmod{n}$ (compatibility with translation)
- $a_1 \equiv b_1 \pmod{n} \Rightarrow k \cdot a_1 \equiv k \cdot b_1 \pmod{n}$ (compatibility with scaling)
- $\gcd(k, n) = 1$ and $k \cdot a_1 \equiv k \cdot b_1 \pmod{n} \Rightarrow a_1 \equiv b_1 \pmod{n}$
- $k \cdot a_1 \equiv k \cdot b_1 \pmod{k \cdot n} \Rightarrow a_1 \equiv b_1 \pmod{n}$

1153 • $a_1 \equiv b_1 \pmod{n}$ and $a_2 \equiv b_2 \pmod{n} \Rightarrow a_1 + a_2 \equiv b_1 + b_2 \pmod{n}$ (compatibil-
 1154 ity with addition)

1155 • $a_1 \equiv b_1 \pmod{n}$ and $a_2 \equiv b_2 \pmod{n} \Rightarrow a_1 \cdot a_2 \equiv b_1 \cdot b_2 \pmod{n}$ (compatibility
 1156 with multiplication)

1157 Other rules, such as compatibility with subtraction, follow from the rules above. For example,
 1158 compatibility with subtraction follows from compatibility with scaling by $k = -1$ and compat-
 1159 ibility with addition.

1160 Another property of congruencies, not known in the traditional arithmetic of integers is
 1161 **Fermat's Little Theorem**. In simple words, it states that, in modular arithmetic, every number
 1162 raised to the power of a prime number modulus is congruent to the number itself. Or, to be more
 1163 precise, if $p \in \mathbb{P}$ is a prime number and $k \in \mathbb{Z}$ is an integer, then:

$$k^p \equiv k \pmod{p}, \quad (3.14)$$

1164 If k is coprime to p , then we can divide both sides of this congruence by k and rewrite the
 1165 expression into the equivalent form

$$k^{p-1} \equiv 1 \pmod{p} \quad (3.15)$$

1166 The following sage code computes example effects of Fermat's little theorem and highlights the
 1167 effects of the exponent k being coprime and not coprime to p :

```

1168 sage: ZZ(137).gcd(ZZ(64)) 44
1169 1 45
1170 sage: ZZ(64)^ZZ(137) % ZZ(137) == ZZ(64) % ZZ(137) 46
1171 True 47
1172 sage: ZZ(64)^ZZ(137-1) % ZZ(137) == ZZ(1) % ZZ(137) 48
1173 True 49
1174 sage: ZZ(1918).gcd(ZZ(137)) 50
1175 137 51
1176 sage: ZZ(1918)^ZZ(137) % ZZ(137) == ZZ(1918) % ZZ(137) 52
1177 True 53
1178 sage: ZZ(1918)^ZZ(137-1) % ZZ(137) == ZZ(1) % ZZ(137) 54
1179 False 55

```

1180 Let's compute an example that contains most of the concepts described in this section:

Example 7. Assume that we consider the modulus 6 and that our task is to solve the following congruence for $x \in \mathbb{Z}$

$$7 \cdot (2x + 21) + 11 \equiv x - 102 \pmod{6}$$

As many rules for congruencies are more or less same as for integers, we can proceed in a similar way as we would if we had an equation to solve. Since both sides of a congruence contain ordinary integers, we can rewrite the left side as follows: $7 \cdot (2x + 21) + 11 = 14x + 147 + 11 = 14x + 158$. We can therefore rewrite the congruence into the equivalent form

$$14x + 158 \equiv x - 102 \pmod{6}$$

In the next step we want to shift all instances of x to left and every other term to the right. So we apply the “compatibility with translation” rules two times. In a first step we choose $k = -x$

and in a second step we choose $k = -158$. Since “compatibility with translation” transforms a congruence into an equivalent form, the solution set will not change and we get

$$\begin{aligned} 14x + 158 &\equiv x - 102 \pmod{6} \Leftrightarrow \\ 14x - x + 158 - 158 &\equiv x - x - 102 - 158 \pmod{6} \Leftrightarrow \\ 13x &\equiv -260 \pmod{6} \end{aligned}$$

If our congruence would just be a normal integer equation, we would divide both sides by 13 to get $x = -20$ as our solution. However, in case of a congruence, we need to make sure that the modulus and the number we want to divide by are coprime first – only then will we get an equivalent expression (See rule XXX). So we need to find the greatest common divisor $\gcd(13, 6)$. Since 13 is prime and 6 is not a multiple of 13, we know that $\gcd(13, 6) = 1$, so these numbers are indeed coprime. We therefore compute

$$13x \equiv -260 \pmod{6} \Leftrightarrow x \equiv -20 \pmod{6}$$

Our task is now to find all integers x , such that x is congruent to -20 with respect to the modulus 6. So we have to find all x such

$$x \bmod 6 = -20 \bmod 6$$

Since $-4 \cdot 6 + 4 = -20$ we know $-20 \bmod 6 = 4$ and hence we know that $x = 4$ is a solution to this congruence. However, 22 is another solution since $22 \bmod 6 = 4$ as well, and so is -20 . In fact, there are infinitely many solutions given by the set

$$\{\dots, -8, -2, 4, 10, 16, \dots\} = \{4 + k \cdot 6 \mid k \in \mathbb{Z}\}$$

Putting all this together, we have shown that the every x from the set $\{x = 4 + k \cdot 6 \mid k \in \mathbb{Z}\}$ is a solution to the congruence $7 \cdot (2x + 21) + 11 \equiv x - 102 \pmod{6}$. We double check for, say, $x = 4$ as well as $x = 4 + 12 \cdot 6 = 76$ using sage:

```

1184 sage: (ZZ(7) * (ZZ(2) * ZZ(4) + ZZ(21)) + ZZ(11)) % ZZ(6) == (ZZ 56
1185         (4) - ZZ(102)) % ZZ(6)
1186 True 57
1187 sage: (ZZ(7) * (ZZ(2) * ZZ(76) + ZZ(21)) + ZZ(11)) % ZZ(6) == ( 58
1188         ZZ(76) - ZZ(102)) % ZZ(6)
1189 True 59

```

Readers who had not been familiar with modular arithmetic until now and who might be discouraged by how complicated modular arithmetic seems at this point, should keep two things in mind. First, computing congruencies in modular arithmetic is not really more complicated than computations in more familiar number systems (e.g. rational numbers), it is just a matter of getting used to it. Second, once we introduce the idea of remainder class representations 3.3, computations become conceptually cleaner and more easy to handle.

Exercise 15. Consider the modulus 13 and find all solutions $x \in \mathbb{Z}$ to the following congruence $5x + 4 \equiv 28 + 2x \pmod{13}$

Exercise 16. Consider the modulus 23 and find all solutions $x \in \mathbb{Z}$ to the following congruence $69x \equiv 5 \pmod{23}$

Exercise 17. Consider the modulus 23 and find all solutions $x \in \mathbb{Z}$ to the following congruence $69x \equiv 46 \pmod{23}$

- 1202 *Exercise 18.* Let a, b, k be integers, such that $a \equiv b \pmod{n}$ holds. Show $a^k \equiv b^k \pmod{n}$.
 1203 *Exercise 19.* Let a, n be integers, such that a and n are not coprime. For which $b \in \mathbb{Z}$ does the
 1204 congruence $a \cdot x \equiv b \pmod{n}$ have a solution x and how does the solution set look in that
 1205 case?

1206 **The Chinese Remainder Theorem** We have seen how to solve congruencies in modular
 1207 arithmetic. However, one question that remains is how to solve systems of congruencies with
 1208 different moduli? The answer is given by the **Chinese remainder theorem**, which states that
 1209 for any $k \in \mathbb{N}$ and coprime natural numbers $n_1, \dots, n_k \in \mathbb{N}$ as well as integers $a_1, \dots, a_k \in \mathbb{Z}$, the
 1210 so-called **simultaneous congruences**

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2} \\ &\dots \\ x &\equiv a_k \pmod{n_k} \end{aligned} \tag{3.16}$$

1211 has a solution, and all possible solutions of this congruence system are congruent modulo the
 product $N = n_1 \cdot \dots \cdot n_k$.¹ In fact, the following algorithm computes the solution set:

Algorithm 2 Chinese Remainder Theorem

Require: $k \in \mathbb{Z}$, $j \in \mathbb{N}_0$ and $n_0, \dots, n_{k-1} \in \mathbb{N}$ coprime

procedure CONGRUENCE-SYSTEMS-SOLVER(a_0, \dots, a_{k-1})

$N \leftarrow n_0 \cdot \dots \cdot n_{k-1}$

while $j < k$ **do**

$N_j \leftarrow N / n_j$

$(-, s_j, t_j) \leftarrow EXT - EUCLID(N_j, n_j)$ $\triangleright 1 = s_j \cdot N_j + t_j \cdot n_j$

end while

$x' \leftarrow \sum_{j=0}^{k-1} a_j \cdot s_j \cdot N_j$

$x \leftarrow x' \bmod N$

return $\{x + m \cdot N \mid m \in \mathbb{Z}\}$

end procedure

Ensure: $\{x + m \cdot N \mid m \in \mathbb{Z}\}$ is the complete solution set to 3.16.

check
algorithm
floating

1212

Example 8. To illustrate how to solve simultaneous congruences using the Chinese remainder theorem, let's look at the following system of congruencies:

$$\begin{aligned} x &\equiv 4 \pmod{7} \\ x &\equiv 1 \pmod{3} \\ x &\equiv 3 \pmod{5} \\ x &\equiv 0 \pmod{11} \end{aligned}$$

Clearly all moduli are coprime and we have $N = 7 \cdot 3 \cdot 5 \cdot 11 = 1155$, as well as $N_1 = 165$, $N_2 = 385$, $N_3 = 231$ and $N_4 = 105$. From this we calculate with the extended Euclidean algorithm

$$\begin{aligned} 1 &= 2 \cdot 165 + (-47) \cdot 7 \\ 1 &= 1 \cdot 385 + (-128) \cdot 3 \\ 1 &= 1 \cdot 231 + (-46) \cdot 5 \\ 1 &= 2 \cdot 105 + (-19) \cdot 11 \end{aligned}$$

¹This is the classical Chinese remainder theorem as it was already known in ancient China. Under certain circumstances, the theorem can be extended to non-coprime moduli n_1, \dots, n_k but this is beyond the scope of this book. Interested readers should consult XXX [add references](#)

1213 so we have $x = 4 \cdot 2 \cdot 165 + 1 \cdot 1 \cdot 385 + 3 \cdot 1 \cdot 231 + 0 \cdot 2 \cdot 105 = 2398$ as one solution. Because
 1214 $2398 \bmod 1155 = 88$ the set of all solutions is $\{\dots, -2222, -1067, 88, 1243, 2398, \dots\}$. We
 1215 can invoke Sage's computation of the Chinese Remainder Theorem (CRT) to double check our
 1216 findings:

```
1217 sage: CRT_list([4,1,3,0], [7,3,5,11]) 60
1218 88 61
```

1219 **Remainder Class Representation** As we have seen in various examples before, computing
 1220 congruencies can be cumbersome and solution sets are large in general. It is therefore advan-
 1221 taegous to find some kind of simplification for modular arithmetic.

1222 Fortunately, this is possible and relatively straightforward once we identify each set of num-
 1223 bers with equal remainder with that remainder itself and call it the **remainder class** or **residue**
 1224 **class** representation in modulo n arithmetic.

1225 It then follows from the properties of Euclidean division that there are exactly n different
 1226 remainder classes for every modulus n and that integer addition and multiplication can be pro-
 1227 jected to a new kind of addition and multiplication on those classes.

1228 Roughly speaking, the new rules for addition and multiplication are then computed by taking
 1229 any element of the first remainder class and some element of the second, then add or multiply
 1230 them in the usual way and see which remainder class the result is contained in. The following
 1231 example makes this abstract description more concrete:

Example 9 (Arithmetic modulo 6). Choosing the modulus $n = 6$, we have six remainder classes of integers which are congruent modulo 6 (they have the same remainder when divided by 6) and when we identify each of those remainder classes with the remainder, we get the following identification:

$$\begin{aligned} 0 &:= \{\dots, -6, 0, 6, 12, \dots\} \\ 1 &:= \{\dots, -5, 1, 7, 13, \dots\} \\ 2 &:= \{\dots, -4, 2, 8, 14, \dots\} \\ 3 &:= \{\dots, -3, 3, 9, 15, \dots\} \\ 4 &:= \{\dots, -2, 4, 10, 16, \dots\} \\ 5 &:= \{\dots, -1, 5, 11, 17, \dots\} \end{aligned}$$

1232 Now to compute the new addition law of those remainder class representatives, say $2 + 5$, one
 1233 chooses arbitrary elements from both classes, say 14 and -1 , adds those numbers in the usual
 1234 way and then looks at the remainder class of the result.

1235 So we get $14 + (-1) = 13$, and 13 is in the remainder class (of) 1. Hence we find that
 1236 $2 + 5 = 1$ in modular 6 arithmetic, which is a more readable way to write the congruence $2 + 5 \equiv$
 1237 $1 \pmod{6}$.

1238 Applying the same reasoning to all remainder classes, addition and multiplication can be
 1239 transferred to the representatives of the remainder classes. The results for modulus 6 arithmetic
 1240 are summarized in the following addition and multiplication tables:

+	0	1	2	3	4	5	·	0	1	2	3	4	5
0	0	1	2	3	4	5	0	0	0	0	0	0	0
1	1	2	3	4	5	0	1	0	1	2	3	4	5
2	2	3	4	5	0	1	2	0	2	4	0	2	4
3	3	4	5	0	1	2	3	0	3	0	3	0	3
4	4	5	0	1	2	3	4	0	4	2	0	4	2
5	5	0	1	2	3	4	5	0	5	4	3	2	1

1242 This way, we have defined a new arithmetic system that contains just 6 numbers and comes with
 1243 its own definition of addition and multiplication. We call it **modular 6 arithmetic** and write
 1244 the associated type as \mathbb{Z}_6 .

1245 To see why such an identification of a remainder class with its remainder is useful and
 1246 actually simplifies congruence computations a lot, lets go back to the congruence from example
 1247 7 again:

$$7 \cdot (2x + 21) + 11 \equiv x - 102 \pmod{6} \quad (3.17)$$

1248 As shown in example 7, the arithmetic of congruencies can deviate from ordinary arithmetic:
 1249 For example, division needs to check whether the modulus and the dividend are coprimes, and
 1250 solutions are not unique in general.

We can rewrite this congruence as an **equation** over our new arithmetic type \mathbb{Z}_6 by **projecting onto the remainder classes**. In particular, since $7 \bmod 6 = 1$, $21 \bmod 6 = 3$, $11 \bmod 6 = 5$ and $102 \bmod 6 = 0$ we have

$$\begin{aligned} 7 \cdot (2x + 21) + 11 &\equiv x - 102 \pmod{6} \text{ over } \mathbb{Z} \\ &\Leftrightarrow 1 \cdot (2x + 3) + 5 = x \text{ over } \mathbb{Z}_6 \end{aligned}$$

1251 We can use the multiplication and addition table above to solves the equation on the right like
 1252 we would solve normal integer equations:

$$\begin{aligned} 1 \cdot (2x + 3) + 5 &= x \\ 2x + 3 + 5 &= x && \# \text{ addition-table: } 3 + 5 = 2 \\ 2x + 2 &= x && \# \text{ add 4 and } -x \text{ on both sides} \\ 2x + 2 + 4 - x &= x + 4 - x && \# \text{ addition-table: } 2 + 4 = 0 \\ x &= 4 \end{aligned}$$

1253 As we can see, despite the somewhat unfamiliar rules of addition and multiplication, solving
 1254 congruencies this way is very similar to solving normal equations. And, indeed, the solution
 1255 set is identical to the solution set of the original congruence, since 4 is identified with the set
 1256 $\{4 + 6 \cdot k \mid k \in \mathbb{Z}\}$.

1257 We can invoke Sage to do computations in our modular 6 arithmetic type. This is particularly
 1258 useful to double-check our computations:

```
1259 sage: Z6 = Integers(6)                                     62
1260 sage: Z6(2) + Z6(5)                                       63
1261 1                                                         64
1262 sage: Z6(7) * (Z6(2) * Z6(4) + Z6(21)) + Z6(11) == Z6(4) - Z6(102) 65
1263 True                                                       66
```

1264 *Remark 2* (k -bit modulus). In cryptographic papers, we sometimes read phrases like “[...] using
 1265 a 4096-bit modulus”. This means that the underlying modulus n of the modular arithmetic used
 1266 in the system has a binary representation with a length of 4096 bits. In contrast, the number 6
 1267 has the binary representation 110 and hence our example 9 describes a 3-bit modulus arithmetic
 1268 system.

1269 *Exercise 20.* Define \mathbb{Z}_{13} as the the arithmetic modulo 13 analog to example 9. Then consider
 1270 the congruence from exercise 15 and rewrite it into an equation in \mathbb{Z}_{13}

Modular Inverses As we know, integers can be added, subtracted and multiplied so that the result is also an integer, but this is not true for the division of integers in general: for example, $3/2$ is not an integer anymore. To see why this is, from a more theoretical perspective, let us consider the definition of a multiplicative inverse first. When we have a set that has some kind of multiplication defined on it and we have a distinguished element of that set that behaves neutrally with respect to that multiplication (doesn't change anything when multiplied with any other element), then we can define **multiplicative inverses** in the following way:

Let S be our set that has some notion $a \cdot b$ of multiplication and a **neutral element** $1 \in S$, such that $1 \cdot a = a$ for all elements $a \in S$. Then a **multiplicative inverse** a^{-1} of an element $a \in S$ is defined as follows:

$$a \cdot a^{-1} = 1 \quad (3.18)$$

Informally speaking, the definition of a multiplicative inverse means that it “cancels” the original element to give 1 when they are multiplied.

Numbers that have multiplicative inverses are of particular interest, because they immediately lead to the definition of division by those numbers. In fact, if a is number such that the multiplicative inverse a^{-1} exists, then we define **division** by a simply as multiplication by the inverse:

$$\frac{b}{a} := b \cdot a^{-1} \quad (3.19)$$

Example 10. Consider the set of rational numbers, also known as fractions, \mathbb{Q} . For this set, the neutral element of multiplication is 1, since $1 \cdot a = a$ for all rational numbers. For example, $1 \cdot 4 = 4$, $1 \cdot \frac{1}{4} = \frac{1}{4}$, or $1 \cdot 0 = 0$ and so on.

Every rational number $a \neq 0$ has a multiplicative inverse, given by $\frac{1}{a}$. For example, the multiplicative inverse of 3 is $\frac{1}{3}$, since $3 \cdot \frac{1}{3} = 1$, the multiplicative inverse of $\frac{5}{7}$ is $\frac{7}{5}$, since $\frac{5}{7} \cdot \frac{7}{5} = 1$, and so on.

Example 11. Looking at the set \mathbb{Z} of integers, we see that with respect to multiplication the neutral element is the number 1 and we notice that no integer other than 1 or -1 has a multiplicative inverse, since the equation $a \cdot x = 1$ has no integer solutions for $a \neq 1$ or $a \neq -1$.

The definition of multiplicative inverse works verbatim for addition as well where it is called the additive inverse. In the case of integers, the neutral element with respect to addition is 0, since $a + 0 = a$ for all integers $a \in \mathbb{Z}$. The additive inverse always exist and is given by the negative number $-a$, since $a + (-a) = 0$.

Example 12. Looking at the set \mathbb{Z}_6 of residual classes modulo 6 from example 9, we can use the multiplication table to find multiplicative inverses. To do so, we look at the row of the element and then find the entry equal to 1. If such an entry exists, the element of that column is the multiplicative inverse. If, on the other hand, the row has no entry equal to 1, we know that the element has no multiplicative inverse.

For example in \mathbb{Z}_6 the multiplicative inverse of 5 is 5 itself, since $5 \cdot 5 = 1$. We can also see that 5 and 1 are the only elements that have multiplicative inverses in \mathbb{Z}_6 .

Now, since 5 has a multiplicative inverse in modulo 6 arithmetic, we can divide by 5 in \mathbb{Z}_6 , since we have a notation of multiplicative inverse and division is nothing but multiplication by the multiplicative inverse. For example

$$\frac{4}{5} = 4 \cdot 5^{-1} = 4 \cdot 5 = 2$$

From the last example, we can make the interesting observation that while 5 has no multiplicative inverse as an integer, it has a multiplicative inverse in modular 6 arithmetic.

1309 This raises the question which numbers have multiplicative inverses in modular arithmetic.
 1310 The answer is that, in modular n arithmetic, a number r has a multiplicative inverse, if and only
 1311 if n and r are coprime. Since $\gcd(n, r) = 1$ in that case, we know from the extended Euclidean
 1312 algorithm that there are numbers s and t , such that

$$1 = s \cdot n + t \cdot r \quad (3.20)$$

1313 If we take the modulus n on both sides, the term $s \cdot n$ vanishes, which tells us that $t \bmod n$ is the
 1314 multiplicative inverse of r in modular n arithmetic.

1315 *Example 13* (Multiplicative inverses in \mathbb{Z}_6). In the previous example, we looked up multiplica-
 1316 tive inverses in \mathbb{Z}_6 from the lookup-table in Example 9. In real world examples, it is usually
 1317 impossible to write down those lookup tables, as the modulus is way too large, and the sets
 1318 occasionally contain more elements than there are atoms in the observable universe.

1319 Now, trying to determine that $2 \in \mathbb{Z}_6$ has no multiplicative inverse in \mathbb{Z}_6 without using the
 1320 lookup table, we immediately observe that 2 and 6 are not coprime, since their greatest common
 1321 divisor is 2. It follows that equation 3.20 has no solutions s and t , which means that 2 has no
 1322 multiplicative inverse in \mathbb{Z}_6 .

1323 The same reasoning works for 3 and 4, as neither of these are coprime with 6. The case
 1324 of 5 is different, since $\gcd(6, 5) = 1$. To compute the multiplicative inverse of 5, we use the
 1325 extended Euclidean algorithm and compute the following:

k	r_k	s_k	$t_k = (r_k - s_k \cdot a) \div b$
0	6	1	0
1	5	0	1
2	1	1	-1
3	0	.	.

1327 We get $s = 1$ as well as $t = -1$ and have $1 = 1 \cdot 6 - 1 \cdot 5$. From this, it follows that $-1 \bmod 6 =$
 1328 5 is the multiplicative inverse of 5 in modular 6 arithmetic. We can double check using Sage:

1329 **sage:** `ZZ(6).xgcd(ZZ(5))` 67
 1330 `(1, 1, -1)` 68

At this point, the attentive reader might notice that the situation where the modulus is a prime number is of particular interest, because we know from exercise 9 that in these cases all remainder classes must have modular inverses, since $\gcd(r, n) = 1$ for prime n and any $r < n$. In fact, Fermat's little theorem provides a way to compute multiplicative inverses in this situation, since in case of a prime modulus p and $r < p$, we get the following:

$$\begin{aligned} r^p &\equiv r \pmod{p} \Leftrightarrow \\ r^{p-1} &\equiv 1 \pmod{p} \Leftrightarrow \\ r \cdot r^{p-2} &\equiv 1 \pmod{p} \end{aligned}$$

1331 This tells us that the multiplicative inverse of a residue class r in modular p arithmetic is pre-
 1332 cisely r^{p-2} .

Example 14 (Modular 5 arithmetic). To see the unique properties of modular arithmetic when the modulus is a prime number, we will replicate our findings from example 9, but this time for the prime modulus 5. For $n = 5$ we have five equivalence classes of integers which are

congruent modulo 5. We write this as follows:

$$\begin{aligned} 0 &:= \{\dots, -5, 0, 5, 10, \dots\} \\ 1 &:= \{\dots, -4, 1, 6, 11, \dots\} \\ 2 &:= \{\dots, -3, 2, 7, 12, \dots\} \\ 3 &:= \{\dots, -2, 3, 8, 13, \dots\} \\ 4 &:= \{\dots, -1, 4, 9, 14, \dots\} \end{aligned}$$

1333 Addition and multiplication can be transferred to the equivalence classes, in a way exactly
1334 parallel to Example 9. This results in the following addition and multiplication tables:

	+	0	1	2	3	4		·	0	1	2	3	4
	0	0	1	2	3	4		0	0	0	0	0	0
	1	1	2	3	4	0		1	0	1	2	3	4
1335	2	2	3	4	0	1		2	0	2	4	1	3
	3	3	4	0	1	2		3	0	3	1	4	2
	4	4	0	1	2	3		4	0	4	3	2	1

1336 Calling the set of remainder classes in modular 5 arithmetic with this addition and multiplication
1337 \mathbb{Z}_5 , we see some subtle but important differences to the situation in \mathbb{Z}_6 . In particular, we see
1338 that in the multiplication table, every remainder $r \neq 0$ has the entry 1 in its row and therefore
1339 has a multiplicative inverse. In addition, there are no non-zero elements such that their product
1340 is zero.

1341 To use Fermat's little theorem in \mathbb{Z}_5 for computing multiplicative inverses (instead of using
1342 the multiplication table), let's consider $3 \in \mathbb{Z}_5$. We know that the multiplicative inverse is given
1343 by the remainder class that contains $3^{5-2} = 3^3 = 3 \cdot 3 \cdot 3 = 4 \cdot 3 = 2$. And indeed $3^{-1} = 2$, since
1344 $3 \cdot 2 = 1$ in \mathbb{Z}_5 .

1345 We can invoke Sage to do computations in our modular 5 arithmetic type to double-check
1346 our computations:

```

1347 sage: Z5 = Integers(5)                                     69
1348 sage: Z5(3) ** (5-2)                                       70
1349 2                                                         71
1350 sage: Z5(3) ** (-1)                                       72
1351 2                                                         73
1352 sage: Z5(3) ** (5-2) == Z5(3) ** (-1)                   74
1353 True                                                       75

```

Example 15. To understand one of the principal differences between prime number modular arithmetic and non-prime number modular arithmetic, consider the linear equation $a \cdot x + b = 0$ defined over both types \mathbb{Z}_5 and \mathbb{Z}_6 . Since in \mathbb{Z}_5 every non-zero element has a multiplicative inverse, we can always solve these equations in \mathbb{Z}_5 , which is not true in \mathbb{Z}_6 . To see that, consider the equation $3x + 3 = 0$. In \mathbb{Z}_5 we have the following:

$$\begin{aligned} 3x + 3 &= 0 & \# \text{ add 2 and on both sides} \\ 3x + 3 + 2 &= 2 & \# \text{ addition-table: } 2 + 3 = 0 \\ 3x &= 2 & \# \text{ divide by 3 (which equals multiplication by 2)} \\ 2 \cdot (3x) &= 2 \cdot 2 & \# \text{ multiplication-table: } 2 \cdot 2 = 4 \\ x &= 4 \end{aligned}$$

So in the case of our prime number modular arithmetic, we get the unique solution $x = 4$. Now consider \mathbb{Z}_6 :

$$\begin{array}{ll} 3x + 3 = 0 & \# \text{ add 3 and on both sides} \\ 3x + 3 + 3 = 3 & \# \text{ addition-table: } 3 + 3 = 0 \\ 3x = 3 & \# \text{ division not possible (no multiplicative inverse of 3 exists)} \end{array}$$

So, in this case, we cannot solve the equation for x by dividing by 3. And, indeed, when we look at the multiplication table of \mathbb{Z}_6 (Example 9), we find that there are three solutions $x \in \{1, 3, 5\}$, such that $3x + 3 = 0$ holds true for all of them.

Exercise 21. Consider the modulus $n = 24$. Which of the integers 7, 1, 0, 805, -4255 have multiplicative inverses in modular 24 arithmetic? Compute the inverses, in case they exist.

Exercise 22. Find the set of all solutions to the congruence $17(2x + 5) - 4 \equiv 2x + 4 \pmod{5}$. Then project the congruence into \mathbb{Z}_5 and solve the resulting equation in \mathbb{Z}_5 . Compare the results.

Exercise 23. Find the set of all solutions to the congruence $17(2x + 5) - 4 \equiv 2x + 4 \pmod{6}$. Then project the congruence into \mathbb{Z}_6 and try to solve the resulting equation in \mathbb{Z}_6 .

3.4 Polynomial arithmetic

A polynomial is an expression consisting of variables (also called indeterminates) and coefficients that involves only the operations of addition, subtraction and multiplication. All coefficients of a polynomial must have the same type, e.g. being integers or rational numbers etc. To be more precise an *univariate polynomial* is an expression

$$P(x) := \sum_{j=0}^m a_j x^j = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0, \quad (3.21)$$

where x is called the **indeterminate**, each a_j is called a **coefficient**. If R is the type of the coefficients, then the set of all **univariate² polynomials with coefficients in R** is written as $R[x]$. We often simply use **polynomial** instead of univariate polynomial, write $P(x) \in R[x]$ for a polynomial and denote the constant term a_0 as $P(0)$.

A polynomial is called the **zero polynomial** if all coefficients are zero and a polynomial is called the **one polynomial** if the constant term is 1 and all other coefficients are zero.

Given an univariate polynomial $P(x) = \sum_{j=0}^m a_j x^j$ that is not the zero polynomial, we call the non-negative integer $\deg(P) := m$ the *degree* of P and define the degree of the zero polynomial to be $-\infty$, where $-\infty$ (negative infinity) is a symbol with the properties that $-\infty + m = -\infty$ and $-\infty < m$ for all non-negative integers $m \in \mathbb{N}_0$. In addition, we write

$$Lc(P) := a_m \quad (3.22)$$

and call it the **leading coefficient** of the polynomial P . We can restrict the set $R[x]$ of all polynomials with coefficients in R , to the set of all such polynomials that have a degree that does not exceed a certain value. If m is the maximum degree allowed, we write $R_{\leq m}[x]$ for the set of all polynomials with a degree less than or equal to m .

²in our context the term univariate means that the polynomial contains a single variable only

Example 16 (Integer Polynomials). The coefficients of a polynomial must all have the same type. The set of polynomials with integer coefficients is written as $\mathbb{Z}[x]$. Examples of such polynomials are:

$$\begin{array}{ll}
 P_1(x) = 2x^2 - 4x + 17 & \# \text{ with } \deg(P_1) = 2 \text{ and } Lc(P_1) = 2 \\
 P_2(x) = x^{23} & \# \text{ with } \deg(P_2) = 23 \text{ and } Lc(P_2) = 1 \\
 P_3(x) = x & \# \text{ with } \deg(P_3) = 1 \text{ and } Lc(P_3) = 1 \\
 P_4(x) = 174 & \# \text{ with } \deg(P_4) = 0 \text{ and } Lc(P_4) = 174 \\
 P_5(x) = 1 & \# \text{ with } \deg(P_5) = 0 \text{ and } Lc(P_5) = 1 \\
 P_6(x) = 0 & \# \text{ with } \deg(P_6) = -\infty \text{ and } Lc(P_6) = 0 \\
 P_7(x) = (x-2)(x+3)(x-5)
 \end{array}$$

In particular, every integer can be seen as an integer polynomial of degree zero. P_7 is a polynomial, because we can expand its definition into $P_7(x) = x^3 - 4x^2 - 11x + 30$, which is a polynomial of degree 3 and leading coefficient 1. The following expressions are not integer polynomials:

$$\begin{aligned}
 Q_1(x) &= 2x^2 + 4 + 3x^{-2} \\
 Q_2(x) &= 0.5x^4 - 2x \\
 Q_3(x) &= 2^x
 \end{aligned}$$

1382 In particular Q_1 is not an integer polynomial, because the expression x^{-2} has a negative expo-
 1383 nent, Q_2 is not an integer polynomial because the coefficient 0.5 is not an integer and Q_3 is not
 1384 an integer polynomial because the indeterminate appears in the exponent of of a coefficient.

1385 We can invoke Sage to do computations with polynomials. To do so, we have to specify the
 1386 symbol for the indeterminate and the type for the coefficients (For the definition of rings see
 1387 4.2). Note, however that Sage defines the degree of the zero polynomial to be -1 .

```

1388 sage: Zx = ZZ['x'] # integer polynomials with indeterminate x 76
1389 sage: Zt.<t> = ZZ[] # integer polynomials with indeterminate t 77
1390 sage: Zx 78
1391 Univariate Polynomial Ring in x over Integer Ring 79
1392 sage: Zt 80
1393 Univariate Polynomial Ring in t over Integer Ring 81
1394 sage: p1 = Zx([17,-4,2]) 82
1395 sage: p1 83
1396 2*x^2 - 4*x + 17 84
1397 sage: p1.degree() 85
1398 2 86
1399 sage: p1.leading_coefficient() 87
1400 2 88
1401 sage: p2 = Zt(t^23) 89
1402 sage: p2 90
1403 t^23 91
1404 sage: p6 = Zx([0]) 92
1405 sage: p6.degree() 93
1406 -1 94

```

Example 17 (Polynomials over \mathbb{Z}_6). Recall the definition of modular 6 arithmetics \mathbb{Z}_6 as defined in example 9. The set of all polynomials with indeterminate x and coefficients in \mathbb{Z}_6 is symbolized as $\mathbb{Z}_6[x]$. Example of polynomials from $\mathbb{Z}_6[x]$ are:

$$\begin{array}{ll}
 P_1(x) = 2x^2 - 4x + 5 & \# \text{ with } \deg(P_1) = 2 \text{ and } Lc(P_1) = 2 \\
 P_2(x) = x^{23} & \# \text{ with } \deg(P_2) = 23 \text{ and } Lc(P_2) = 1 \\
 P_3(x) = x & \# \text{ with } \deg(P_3) = 1 \text{ and } Lc(P_3) = 1 \\
 P_4(x) = 3 & \# \text{ with } \deg(P_4) = 0 \text{ and } Lc(P_4) = 3 \\
 P_5(x) = 1 & \# \text{ with } \deg(P_5) = 0 \text{ and } Lc(P_5) = 1 \\
 P_6(x) = 0 & \# \text{ with } \deg(P_6) = -\infty \text{ and } Lc(P_6) = 0 \\
 P_7(x) = (x-2)(x+3)(x-5) &
 \end{array}$$

Just like in the previous example, P_7 is a polynomial. However, since we are working with coefficients from \mathbb{Z}_6 now the expansion of P_7 is computed differently, as we have to invoke addition and multiplication in \mathbb{Z}_6 as defined in XXX. We get the following:

$$\begin{aligned}
 (x-2)(x+3)(x-5) &= (x+4)(x+3)(x+1) && \# \text{ additive inverses in } \mathbb{Z}_6 \\
 &= (x^2 + 4x + 3x + 3 \cdot 4)(x+1) && \# \text{ bracket expansion} \\
 &= (x^2 + 1x + 0)(x+1) && \# \text{ computation in } \mathbb{Z}_6 \\
 &= x^3 + x^2 + x^2 + x && \# \text{ bracket expansion} \\
 &= x^3 + 2x^2 + x
 \end{aligned}$$

1407 Again, we can use Sage to do computations with polynomials that have their coefficients in \mathbb{Z}_6
 1408 (For the definition of rings see 4.2). To do so, we have to specify the symbol for the indertemi-
 1409 nate and the type for the coefficients:

```

1410 sage: Z6 = Integers(6)                                     95
1411 sage: Z6x = Z6['x']                                       96
1412 sage: Z6x                                                97
1413 Univariate Polynomial Ring in x over Ring of integers modulo 6 98
1414 sage: p1 = Z6x([5, -4, 2])                                99
1415 sage: p1                                                  100
1416 2*x^2 + 2*x + 5                                           101
1417 sage: p1 = Z6x([17, -4, 2])                              102
1418 sage: p1                                                  103
1419 2*x^2 + 2*x + 5                                           104
1420 sage: Z6x(x-2)*Z6x(x+3)*Z6x(x-5) == Z6x(x^3 + 2*x^2 + x) 105
1421 True                                                    106

```

1422 Given some element from the same type as the coefficients of a polynomial, the poly-
 1423 nomial can be evaluated at that element, which means that we insert the given element for every
 1424 occurrence of the indeterminate x in the polynomial expression.

1425 To be more precise, let $P \in R[x]$, with $P(x) = \sum_{j=0}^m a_j x^j$ be a polynomial with a coefficient
 1426 of type R and let $b \in R$ be an element of that type. Then the **evaluation** of P at b is given as
 1427 follows:

$$P(b) = \sum_{j=0}^m a_j b^j \quad (3.23)$$

Example 18. Consider the integer polynomials from example 16 again. To evaluate them at given points, we have to insert the point for all occurrences of x in the polynomial expression. Inserting arbitrary values from \mathbb{Z} , we get:

$$\begin{aligned} P_1(2) &= 2 \cdot 2^2 - 4 \cdot 2 + 17 = 17 \\ P_2(3) &= 3^{23} = 94143178827 \\ P_3(-4) &= -4 = -4 \\ P_4(15) &= 174 \\ P_5(0) &= 1 \\ P_6(1274) &= 0 \\ P_7(-6) &= (-6-2)(-6+3)(-6-5) = -264 \end{aligned}$$

1428 Note, however, that it is not possible to evaluate any of those polynomial on values of different
1429 type. For example, it is not strictly correct to write $P_1(0.5)$, since 0.5 is not an integer. We can
1430 verify our computations using Sage:

```
1431 sage: Zx = ZZ['x'] 107
1432 sage: p1 = Zx([17, -4, 2]) 108
1433 sage: p7 = Zx(x-2)*Zx(x+3)*Zx(x-5) 109
1434 sage: p1(ZZ(2)) 110
1435 17 111
1436 sage: p7(ZZ(-6)) == ZZ(-264) 112
1437 True 113
```

Example 19. Consider the polynomials with coefficients in \mathbb{Z}_6 from example again. To evaluate them at given values from \mathbb{Z}_6 , we have to insert the point for all occurrences of x in the polynomial expression. We get the following:

$$\begin{aligned} P_1(2) &= 2 \cdot 2^2 - 4 \cdot 2 + 5 = 2 - 2 + 5 = 5 \\ P_2(3) &= 3^{23} = 3 \\ P_3(-4) &= P_3(2) = 2 \\ P_5(0) &= 1 \\ P_6(4) &= 0 \end{aligned}$$

```
1438
1439 sage: Z6 = Integers(6) 114
1440 sage: Z6x = Z6['x'] 115
1441 sage: p1 = Z6x([5, -4, 2]) 116
1442 sage: p1(Z6(2)) == Z6(5) 117
1443 True 118
```

1444 *Exercise 24.* Compare both expansions of P_7 from $\mathbb{Z}[x]$ and from $\mathbb{Z}_6[x]$ in example 16 and
1445 example 19 ,and consider the definition of \mathbb{Z}_6 as given in example 9. Can you see how the
1446 definition of P_7 over \mathbb{Z} projects to the definition over \mathbb{Z}_6 if you consider the residue classes of
1447 \mathbb{Z}_6 ?

Polynomial arithmetic Polynomials behave like integers in many ways. In particular, they can be added, subtracted and multiplied. In addition, they have their own notion of Euclidean division. Informally speaking, we can add two polynomials by simply adding the coefficients of the same index, and we can multiply them by applying the distributive property, that is, by multiplying every term of the left factor with every term of the right factor and adding the results together.

To be more precise let $\sum_{n=0}^{m_1} a_n x^n$ and $\sum_{n=0}^{m_2} b_n x^n$ be two polynomials from $R[x]$. Then the **sum** and the **product** of these polynomials is defined as follows:

$$\sum_{n=0}^{m_1} a_n x^n + \sum_{n=0}^{m_2} b_n x^n = \sum_{n=0}^{\max(\{m_1, m_2\})} (a_n + b_n) x^n \quad (3.24)$$

$$\left(\sum_{n=0}^{m_1} a_n x^n \right) \cdot \left(\sum_{n=0}^{m_2} b_n x^n \right) = \sum_{n=0}^{m_1+m_2} \sum_{i=0}^n a_i b_{n-i} x^n \quad (3.25)$$

A rule for polynomial subtraction can be deduced from these two rules by first multiplying the **subtrahend** with (the polynomial) -1 and then add the result to the **minuend**.

Regarding the definition of the degree of a polynomial, we see that the degree of the sum is always the maximum of the degrees of both summands, and the degree of the product is always the degree of the sum of the factors, since we defined $-\infty + m = -\infty$ for every integer $m \in \mathbb{Z}$.

Example 20. To give an example of how polynomial arithmetic works, consider the following two integer polynomials $P, Q \in \mathbb{Z}[x]$ with $P(x) = 5x^2 - 4x + 2$ and $Q(x) = x^3 - 2x^2 + 5$. The sum of these two polynomials is computed by adding the coefficients of each term with equal exponent in x . This gives the following:

$$\begin{aligned} (P + Q)(x) &= (0 + 1)x^3 + (5 - 2)x^2 + (-4 + 0)x + (2 + 5) \\ &= x^3 + 3x^2 - 4x + 7 \end{aligned}$$

The product of these two polynomials is computed by multiplication of each term in the first factor with each term in the second factor. We get the following:

$$\begin{aligned} (P \cdot Q)(x) &= (5x^2 - 4x + 2) \cdot (x^3 - 2x^2 + 5) \\ &= (5x^5 - 10x^4 + 25x^2) + (-4x^4 + 8x^3 - 20x) + (2x^3 - 4x^2 + 10) \\ &= 5x^5 - 14x^4 + 10x^3 + 21x^2 - 20x + 10 \end{aligned}$$

```

1462
1463 sage: Zx = ZZ['x'] 119
1464 sage: P = Zx([2, -4, 5]) 120
1465 sage: Q = Zx([5, 0, -2, 1]) 121
1466 sage: P+Q == Zx(x^3 +3*x^2 -4*x +7) 122
1467 True 123
1468 sage: P*Q == Zx(5*x^5 -14*x^4 +10*x^3+21*x^2-20*x +10) 124
1469 True 125

```

Example 21. Let us consider the polynomials of the previous example but interpreted in modular 6 arithmetic. So we consider $P, Q \in \mathbb{Z}_6[x]$ again with $P(x) = 5x^2 - 4x + 2$ and $Q(x) = x^3 - 2x^2 +$

5. This time we get the following:

$$\begin{aligned}(P + Q)(x) &= (0 + 1)x^3 + (5 - 2)x^2 + (-4 + 0)x + (2 + 5) \\ &= (0 + 1)x^3 + (5 + 4)x^2 + (2 + 0)x + (2 + 5) \\ &= x^3 + 3x^2 + 2x + 1\end{aligned}$$

$$\begin{aligned}(P \cdot Q)(x) &= (5x^2 - 4x + 2) \cdot (x^3 - 2x^2 + 5) \\ &= (5x^2 + 2x + 2) \cdot (x^3 + 4x^2 + 5) \\ &= (5x^5 + 2x^4 + 1x^2) + (2x^4 + 2x^3 + 4x) + (2x^3 + 2x^2 + 4) \\ &= 5x^5 + 4x^4 + 4x^3 + 3x^2 + 4x + 4\end{aligned}$$

1470

1471	<code>sage: Z6x = Integers(6) ['x']</code>	126
1472	<code>sage: P = Z6x([2, -4, 5])</code>	127
1473	<code>sage: Q = Z6x([5, 0, -2, 1])</code>	128
1474	<code>sage: P+Q == Z6x(x^3 +3*x^2 +2*x +1)</code>	129
1475	<code>True</code>	130
1476	<code>sage: P*Q == Z6x(5*x^5 +4*x^4 +4*x^3+3*x^2+4*x +4)</code>	131
1477	<code>True</code>	132

1478 *Exercise 25.* Compare the sum $P + Q$ and the product $P \cdot Q$ from the previous two examples
 1479 20 and 21 and consider the definition of \mathbb{Z}_6 as given in example 9. How can we derive the
 1480 computations in $\mathbb{Z}_6[x]$ from the computations in $\mathbb{Z}[x]$?

1481 **Euklidean Division** The arithmetic of polynomials share a lot of properties with the arith-
 1482 metic of integers and as a consequence the concept of Euclidean division and the algorithm of
 1483 long division is also defined for polynomials. Recalling the Euclidean division of integers 3.2,
 1484 we know that, given two integers a and $b \neq 0$, there is always another integer m and a natural
 1485 number r with $r < |b|$ such that $a = m \cdot b + r$ holds.

1486 We can generalize this to polynomials whenever the leading coefficient of the dividend
 1487 polynomial has a notion of multiplicative inverse. In fact, given two polynomials A and $B \neq 0$
 1488 from $R[x]$ such that $Lc(B)^{-1}$ exists in R , there exist two polynomials Q (the quotient) and P (the
 1489 remainder), such that the following equation holds:

$$A = Q \cdot B + P \tag{3.26}$$

1490 and $\deg(P) < \deg(B)$. Similarly to integer Euclidean division, both Q and P are uniquely
 1491 defined by these relations.

1492 *Notation and Symbols 2.* Suppose that the polynomials A, B, Q and P satisfy equation 3.26. We
 1493 often use the following notation to describe the quotient and the remainder polynomials of the
 1494 Euclidean division:

$$A \operatorname{div} B := Q, \quad A \operatorname{mod} B := P \tag{3.27}$$

1495 We also say that a polynomial A is divisible by another polynomial B if $A \operatorname{mod} B = 0$ holds. In
 1496 this case, we also write $B|A$ and call B a *factor* of A .

Algorithm 3 Polynomial Euclidean Algorithm

Require: $A, B \in R[x]$ with $B \neq 0$, such that $Lc(B)^{-1}$ exists in R

procedure POLY-LONG-DIVISION(A, B)

$$Q \leftarrow 0$$
$$P \leftarrow A$$
$$d \leftarrow \deg(B)$$
$$c \leftarrow Lc(B)$$
while $\deg(P) \geq d$ **do**
$$S := Lc(P) \cdot c^{-1} \cdot x^{\deg(P)-d}$$
$$Q \leftarrow Q + S$$
$$P \leftarrow P - S \cdot B$$
end while**return** (Q, P)

end procedure

Ensure: $A = Q \cdot B + P$

Analogously to integers, methods to compute Euclidean division for polynomials are called **polynomial division algorithms**. Probably the best known algorithm is the so called **polynomial long division**.

This algorithm works only when there is a notion of division by the leading coefficient of B . It can be generalized, but we will only need this somewhat simpler method in what follows.

Example 22 (Polynomial Long Division). To give an example of how the previous algorithm works, let us divide the integer polynomial $A(x) = x^5 + 2x^3 - 9 \in \mathbb{Z}[x]$ by the integer polynomial $B(x) = x^2 + 4x - 1 \in \mathbb{Z}[x]$. Since B is not the zero polynomial and the leading coefficient of B is 1, which is invertible as an integer, we can apply algorithm 1. Our goal is to find solutions to equation XXX, that is, we need to find the quotient polynomial $Q \in \mathbb{Z}[x]$ and the remainder polynomial $P \in \mathbb{Z}[x]$ such that $x^5 + 2x^3 - 9 = Q(x) \cdot (x^2 + 4x - 1) + P(x)$. Using a notation that is mostly used in anglophone countries, we compute as follows:

$$X^2 + 4X - 1) \overline{\begin{array}{r} X^3 - 4X^2 + 19X - 80 \\ X^5 + 2X^3 - 9 \\ -X^5 - 4X^4 + X^3 \\ \hline -4X^4 + 3X^3 \\ 4X^4 + 16X^3 - 4X^2 \\ \hline 19X^3 - 4X^2 \\ -19X^3 - 76X^2 + 19X \\ \hline -80X^2 + 19X - 9 \\ 80X^2 + 320X - 80 \\ \hline 339X - 89 \end{array}} \quad (3.28)$$

We therefore get $Q(x) = x^3 - 4x^2 + 19x - 80$ as well as $P(x) = 339x - 89$ and indeed we have $x^5 + 2x^3 - 9 = (x^3 - 4x^2 + 19x - 80) \cdot (x^2 + 4x - 1) + (339x - 89)$, which we can double check invoking Sage:

```
sage: Zx = ZZ['x']
```

```
sage: A = Zx([-9, 0, 0, 2, 0, 1])
```

```
sage: B = Zx([-1, 4, 1])
```

algorithm-
floating


```

1515 sage: Q = Zx([-80, 19, -4, 1]) 136
1516 sage: P = Zx([-89, 339]) 137
1517 sage: A == Q*B + P 138
1518 True 139

```

1519 *Example 23.* In the previous example, polynomial division gave a non-trivial (non-vanishing, i.e non-zero) remainder. Of special interest are divisions that don't give a remainder. Such divisors are called factors of the dividend.

1522 For example, consider the integer polynomial P_7 from example 16 again. As we have shown, it can be written both as $x^3 - 4x^2 - 11x + 30$ and as $(x - 2)(x + 3)(x - 5)$. From this, we can see that the polynomials $F_1(x) = (x - 2)$, $F_2(x) = (x + 3)$ and $F_3(x) = (x - 5)$ are all factors of $x^3 - 4x^2 - 11x + 30$, since division of P_7 by any of these factors will result in a zero remainder.

1526 *Exercise 26.* Consider the polynomial expressions $A(x) := -3x^4 + 4x^3 + 2x^2 + 4$ and $B(x) = x^2 - 4x + 2$. Compute the Euclidean division of A by B in the following types:

- 1528 1. $A, B \in \mathbb{Z}[x]$
- 1529 2. $A, B \in \mathbb{Z}_6[x]$
- 1530 3. $A, B \in \mathbb{Z}_5[x]$

1531 Now consider the result in $\mathbb{Z}[x]$ and in $\mathbb{Z}_6[x]$. How can we compute the result in $\mathbb{Z}_6[x]$ from the result in $\mathbb{Z}[x]$?

1533 *Exercise 27.* Show that the polynomial $B(x) = 2x^4 - 3x + 4 \in \mathbb{Z}_5[x]$ is a factor of the polynomial $A(x) = x^7 + 4x^6 + 4x^5 + x^3 + 2x^2 + 2x + 3 \in \mathbb{Z}_5[x]$ that is show $B|A$. What is $B \operatorname{div} A$?

1535 **Prime Factors** Recall that the fundamental theorem of arithmetic 3.7 tells us that every natural number is the product of prime numbers. In this chapter we will see that something similar holds for univariate polynomials $R[x]$, too³.

1538 The polynomial analog to a prime number is a so called an **irreducible polynomial**, which is defined as a polynomial that cannot be factored into the product of two non-constant polynomials using Euclidean division. Irreducible polynomials are for polynomials what prime numbers are for integer: They are the basic building blocks from which all other polynomials can be constructed. To be more precise, let $P \in R[x]$ be any polynomial. Then there are always irreducible polynomials $F_1, F_2, \dots, F_k \in R[x]$, such that the following holds:

$$P = F_1 \cdot F_2 \cdot \dots \cdot F_k . \quad (3.29)$$

1544 This representation is unique, except for permutations in the factors and is called the **prime factorization** of P . Moreover each factor F_i is called a **prime factor** of P .

1546 *Example 24.* Consider the polynomial expression $P = x^2 - 3$. When we interpret P as an integer polynomial $P \in \mathbb{Z}[x]$, we find that this polynomial is irreducible, since any factorization other than $1 \cdot (x^2 - 3)$, must look like $(x - a)(x + a)$ for some integer a , but there is no integers a with $a^2 = 3$.

```

1550 sage: Zx = ZZ['x'] 140
1551 sage: p = Zx(x^2-3) 141

```

³Strictly speaking this is not true for polynomials over arbitrary types R . However in this book we assume R to be a so called unique factorization domain for which the content of this section holds.

```

1552 sage: p.factor() 142
1553 x^2 - 3 143

```

1554 On the other hand interpreting P as a polynomial $P \in \mathbb{Z}_6[x]$ in modulo 6 arithmetic, we see that P
 1555 has two factors $F_1 = (x - 3)$ and $F_2 = (x + 3)$, since $(x - 3)(x + 3) = x^2 - 3x + 3x - 3 \cdot 3 = x^2 - 3$.

1556 Points where a polynomial evaluates to zero are called **roots** of the polynomial. To be more
 1557 precise, let $P \in R[x]$ be a polynomial. Then a root is a point $x_0 \in R$ with $P(x_0) = 0$ and the set
 1558 of all roots of P is defined as follows:

$$R_0(P) := \{x_0 \in R \mid P(x_0) = 0\} \quad (3.30)$$

1559 The roots of a polynomial are of special interest with respect to its prime factorization, since it
 1560 can be shown that for any given root x_0 of P the polynomial $F(x) = (x - x_0)$ is a prime factor of
 1561 P .

1562 Finding the roots of a polynomial is sometimes called **solving the polynomial**. It is a hard
 1563 problem and has been the subject of much research throughout history.

1564 It can be shown that if m is the degree of a polynomial P , then P can not have more than m
 1565 roots. However, in general, polynomials can have less than m roots.

1566 *Example 25.* Consider the integer polynomial $P_7(x) = x^3 - 4x^2 - 11x + 30$ from example 16
 1567 again. We know that its set of roots is given by $R_0(P_7) = \{-3, 2, 5\}$.

1568 On the other hand, we know from example 24 that the integer polynomial $x^2 - 3$ is irre-
 1569 ducible. It follows that it has no roots, since every root defines a prime factor.

1570 *Example 26.* To give another example, consider the integer polynomial $P = x^7 + 3x^6 + 3x^5 +$
 1571 $x^4 - x^3 - 3x^2 - 3x - 1$. We can invoke Sage to compute the roots and prime factors of P :

```

1572 sage: Zx = ZZ['x'] 144
1573 sage: p = Zx(x^7 + 3*x^6 + 3*x^5 + x^4 - x^3 - 3*x^2 - 3*x - 1) 145
1574 )
1575 sage: p.roots() 146
1576 [(1, 1), (-1, 4)] 147
1577 sage: p.factor() 148
1578 (x - 1) * (x + 1)^4 * (x^2 + 1) 149

```

We see that P has the root 1 and that the associated prime factor $(x - 1)$ occurs once in P and that it has the root -1 , where the associated prime factor $(x + 1)$ occurs 4 times in P . This gives the following prime factorization:

$$P = (x - 1)(x + 1)^4(x^2 + 1)$$

1579 *Exercise 28.* Show that if a polynomial $P \in R[x]$ of degree $\deg(P) = m$ has less than m roots, it
 1580 must have a prime factor F of degree $\deg(F) > 1$.

1581 *Exercise 29.* Consider the polynomial $P = x^7 + 3x^6 + 3x^5 + x^4 - x^3 - 3x^2 - 3x - 1 \in \mathbb{Z}_6[x]$.
 1582 Compute the set of all roots of $R_0(P)$ and then compute the prime factorization of P .

1583 **Lagrange interpolation** One particularly useful property of polynomials is that a polynomial
 1584 of degree m is completely determined on $m + 1$ evaluation points, which implies that we can
 1585 uniquely derive a polynomial of degree m from a set S :

$$S = \{(x_0, y_0), (x_1, y_1), \dots, (x_m, y_m) \mid x_i \neq x_j \text{ for all indices } i \text{ and } j\} \quad (3.31)$$

1586 Polynomials therefore have the property that $m + 1$ pairs of points (x_i, y_i) for $x_i \neq x_j$ are enough
 1587 to determine the set of pairs $(x, P(x))$ for all $x \in R$. This “few too many” property of polynomials
 1588 is used in many places, like for example in erasure codes. It is also of importance in snarks and
 1589 we therefore need to understand a method to actually compute a polynomial from a set of points.

1590 If the coefficients of the polynomial we want to find have a notion of multiplicative inverse,
 1591 it is always possible to find such a polynomial using a method called **Lagrange interpolation**,
 1592 which works as follows: Given a set like 3.31, a polynomial P of degree m with $P(x_i) = y_i$ for
 1593 all pairs (x_i, y_i) from S is given by the following algorithm:

check
algorithm
floating

Algorithm 4 Lagrange Interpolation

Require: R must have multiplicative inverses

Require: $S = \{(x_0, y_0), (x_1, y_1), \dots, (x_m, y_m) \mid x_i, y_i \in R, x_i \neq x_j \text{ for all indices } i \text{ and } j\}$

procedure LAGRANGE-INTERPOLATION(S)

for $j \in (0 \dots m)$ **do**

$$l_j(x) \leftarrow \prod_{i=0; i \neq j}^m \frac{x - x_i}{x_j - x_i} = \frac{(x - x_0)}{(x_j - x_0)} \dots \frac{(x - x_{j-1})}{(x_j - x_{j-1})} \frac{(x - x_{j+1})}{(x_j - x_{j+1})} \dots \frac{(x - x_m)}{(x_j - x_m)}$$

end for

$$P \leftarrow \sum_{j=0}^m y_j \cdot l_j$$

return P

end procedure

Ensure: $P \in R[x]$ with $\deg(P) = m$

Ensure: $P(x_j) = y_j$ for all pairs $(x_j, y_j) \in S$

Example 27. Let us consider the set $S = \{(0, 4), (-2, 1), (2, 3)\}$. Our task is to compute a polynomial of degree 2 in $\mathbb{Q}[x]$ with coefficients from the rational numbers \mathbb{Q} . Since \mathbb{Q} has multiplicative inverses, we can use the Lagrange interpolation algorithm from 4, to compute the polynomial.

$$\begin{aligned} l_0(x) &= \frac{x - x_1}{x_0 - x_1} \cdot \frac{x - x_2}{x_0 - x_2} = \frac{x + 2}{0 + 2} \cdot \frac{x - 2}{0 - 2} = -\frac{(x + 2)(x - 2)}{4} \\ &= -\frac{1}{4}(x^2 - 4) \\ l_1(x) &= \frac{x - x_0}{x_1 - x_0} \cdot \frac{x - x_2}{x_1 - x_2} = \frac{x - 0}{-2 - 0} \cdot \frac{x - 2}{-2 - 2} = \frac{x(x - 2)}{8} \\ &= \frac{1}{8}(x^2 - 2x) \\ l_2(x) &= \frac{x - x_0}{x_2 - x_0} \cdot \frac{x - x_1}{x_2 - x_1} = \frac{x - 0}{2 - 0} \cdot \frac{x + 2}{2 + 2} = \frac{x(x + 2)}{8} \\ &= \frac{1}{8}(x^2 + 2x) \\ P(x) &= 4 \cdot \left(-\frac{1}{4}(x^2 - 4)\right) + 1 \cdot \frac{1}{8}(x^2 - 2x) + 3 \cdot \frac{1}{8}(x^2 + 2x) \\ &= -x^2 + 4 + \frac{1}{8}x^2 - \frac{1}{4}x + \frac{3}{8}x^2 + \frac{3}{4}x \\ &= -\frac{1}{2}x^2 + \frac{1}{2}x + 4 \end{aligned}$$

1594 And, indeed, evaluation of P on the x -values of S gives the correct points, since $P(0) = 4$,
 1595 $P(-2) = 1$ and $P(2) = 3$. Sage provides the following function:

```

1596 sage: Qx = QQ['x'] 150
1597 sage: S=[(0,4),(-2,1),(2,3)] 151
1598 sage: Qx.lagrange_polynomial(S) 152
1599 -1/2*x^2 + 1/2*x + 4 153

```

Example 28. To give another example more relevant to the topics of this book, let us consider the same set $S = \{(0,4), (-2,1), (2,3)\}$ as in the previous example. This time, the task is to compute a polynomial $P \in \mathbb{Z}_5[x]$ from this data. Since we know from example 14 that multiplicative inverses exist in \mathbb{Z}_5 , algorithm 4 applies and we can compute a unique polynomial of degree 2 in $\mathbb{Z}_5[x]$ from S . We can use the lookup tables from example 14 for computation in \mathbb{Z}_5 and get the following:

$$\begin{aligned}
l_0(x) &= \frac{x-x_1}{x_0-x_1} \cdot \frac{x-x_2}{x_0-x_2} = \frac{x+2}{0+2} \cdot \frac{x-2}{0-2} = \frac{(x+2)(x-2)}{-4} = \frac{(x+2)(x+3)}{1} \\
&= x^2 + 1 \\
l_1(x) &= \frac{x-x_0}{x_1-x_0} \cdot \frac{x-x_2}{x_1-x_2} = \frac{x-0}{-2-0} \cdot \frac{x-2}{-2-2} = \frac{x}{3} \cdot \frac{x+3}{1} = 2(x^2 + 3x) \\
&= 2x^2 + x \\
l_2(x) &= \frac{x-x_0}{x_2-x_0} \cdot \frac{x-x_1}{x_2-x_1} = \frac{x-0}{2-0} \cdot \frac{x+2}{2+2} = \frac{x(x+2)}{3} = 2(x^2 + 2x) \\
&= 2x^2 + 4x \\
P(x) &= 4 \cdot (x^2 + 1) + 1 \cdot (2x^2 + x) + 3 \cdot (2x^2 + 4x) \\
&= 4x^2 + 4 + 2x^2 + x + x^2 + 2x \\
&= 2x^2 + 3x + 4
\end{aligned}$$

1600 And, indeed, evaluation of P on the x -values of S gives the correct points, since $P(0) = 4$,
1601 $P(-2) = 1$ and $P(2) = 3$. We can doublecheck our findings using Sage:

```

1602 sage: F5 = GF(5) 154
1603 sage: F5x = F5['x'] 155
1604 sage: S=[(0,4),(-2,1),(2,3)] 156
1605 sage: F5x.lagrange_polynomial(S) 157
1606 2*x^2 + 3*x + 4 158

```

1607 *Exercise 30.* Consider modular 5 arithmetic from example 14 and the set $S = \{(0,0), (1,1), (2,2), (3,2)\}$.
1608 Find a polynomial $P \in \mathbb{Z}_5[x]$ such that $P(x_i) = y_i$ for all $(x_i, y_i) \in S$.

1609 *Exercise 31.* Consider the set S from the previous example. Why is it not possible to apply
1610 algorithm 4 to construct a polynomial $P \in \mathbb{Z}_6[x]$, such that $P(x_i) = y_i$ for all $(x_i, y_i) \in S$?

Chapter 4

Algebra

In the previous chapter, we gave an introduction to the basic computational tools needed for a pen-and-paper approach to SNARKs and in this chapter we provide a more abstract clarification of relevant mathematical terminology such as **groups**, **rings** and **fields**.

Scientific literature on cryptography frequently contain such terms, and it is necessary to get at least some understanding of these terms to be able to follow the literature.

4.1 Commutative Groups

Commutative groups are abstractions that capture the essence of mathematical phenomena, like addition and subtraction, or multiplication and division.

To understand commutative groups, let us think back to when we learned about the addition and subtraction of integers in school. We have learned that, whenever we add two integers, the result is guaranteed to be an integer as well. We have also learned that adding zero to any integer means that “nothing happens”, that is, the result of the addition is the same integer we started with. Furthermore, we have learned that the order in which we add two (or more) integers does not matter, that brackets have no influence on the result of addition, and that, for every integer, there is always another integer (the negative) such that we get zero when we add them together.

These conditions are the defining properties of a commutative group, and mathematicians have realized that the exact same set of rules can be found in very different mathematical structures. It therefore makes sense to abstract from integers and to give a formal definition of what a group should be, detached from any concrete examples. This lets us handle entities of very different mathematical origins in a flexible way, while retaining essential structural aspects of many objects in abstract algebra and beyond.

Distilling these rules to the smallest independent list of properties and making them abstract, we arrive at the following definition of a commutative group:

Sylvia: I would like to have a separate counter for definitions

Definition 4.1.0.1. A **commutative group** (\mathbb{G}, \cdot) is a set \mathbb{G} , together with a **map** $\cdot : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G}$ called the **group law**, that combines two elements of the set \mathbb{G} into a third one such that the following properties hold:

- **Commutativity:** For all $g_1, g_2 \in \mathbb{G}$, the equation $g_1 \cdot g_2 = g_2 \cdot g_1$ holds.
- **Associativity:** For every $g_1, g_2, g_3 \in \mathbb{G}$ the equation $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$ holds.
- **Existence of a neutral element:** There is a $e \in \mathbb{G}$ for all $g \in \mathbb{G}$, such that $e \cdot g = g$.

Sylvia: I would like to have a separate counter for definitions

- **Existence of an inverse:** For every $g \in \mathbb{G}$ there is a $g^{-1} \in \mathbb{G}$, such that $g \cdot g^{-1} = e$.

If (\mathbb{G}, \cdot) is a group and $\mathbb{G}' \subset \mathbb{G}$ is a subset of \mathbb{G} , such that the restriction of the group law $\cdot : \mathbb{G}' \times \mathbb{G}' \rightarrow \mathbb{G}'$ is a group law on \mathbb{G}' , then (\mathbb{G}', \cdot) is called a **subgroup** of (\mathbb{G}, \cdot) .

Rephrasing the abstract definition in layman's terms, a group is something where we can do computations in a way that resembles the behavior of the addition of integers. Specifically, this means we can combine some element with another element into a new element in a way that is reversible and where the order of combining elements doesn't matter.

Notation and Symbols 3. Since we are exclusively concerned with commutative groups in this book, we often just call them groups, keeping the notation of commutativity implicit. Commutative groups are also called **Abelian groups**. A set \mathbb{G} with a map \cdot that satisfies all previously mentioned rules, except for the commutativity law is called a non-commutative group.

If there is no risk of ambiguity (about what the group law of that group is), we frequently drop the symbol \cdot and simply write \mathbb{G} as notation for the group, keeping the group law implicit. In this case we also say that \mathbb{G} is of group type, indicating that \mathbb{G} is not simply a set but a set together with a group law.

For commutative groups (\mathbb{G}, \cdot) , we sometimes use the so-called **additive notation** $(\mathbb{G}, +)$, that is, we write $+$ instead of \cdot for the group law, 0 for the neutral element and $-g := g^{-1}$ for the inverse of an element $g \in \mathbb{G}$.

As we will see in the following chapters, groups are heavily used in cryptography and in SNARKs. But let us look at some more familiar examples first:

Example 29 (Integer Addition and Subtraction). The set $(\mathbb{Z}, +)$ of integers with integer addition is the archetypical example of a commutative group, where the group law is traditionally written in additive notation 3.

To compare integer addition against the abstract axioms of a commutative group, we first see that integer addition is commutative and associative, since $a + b = b + a$ as well as $(a + b) + c = a + (b + c)$ for all integers $a, b, c \in \mathbb{Z}$. The neutral element e is the number 0 , since $a + 0 = a$ for all integers $a \in \mathbb{Z}$. Furthermore, the inverse of a number is its negative counterpart, since $a + (-a) = 0$, for all $a \in \mathbb{Z}$. This implies that integers with addition are indeed a commutative group in the abstract sense.

To give an example of a subgroup for the group of integers, consider the set $\mathbb{Z}_{\text{even}} := \{\dots, -4, -2, 0, 2, 4, \dots\}$ of even numbers, including 0 . We can see that this set is a subgroup of $(\mathbb{Z}, +)$, since the sum of two even numbers is always an even number again, since the neutral element 0 is a member of \mathbb{Z}_{even} and since the negative of an even number is itself an even number.

Example 30 (The trivial group). The most basic example of a commutative group is the group with just one element $\{\bullet\}$ and the group law $\bullet \cdot \bullet = \bullet$. We call it the **trivial group**.

The trivial group is a subgroup of any group. To see that let (\mathbb{G}, \cdot) be a group with neutral element $e \in \mathbb{G}$. Then $e \cdot e = e$ as well as $e^{-1} = e$ and it follows that the set $\{e\}$ is a subgroup of \mathbb{G} . In particular $\{0\}$ is a subgroup of $(\mathbb{Z}, +)$, since $0 + 0 = 0$.

Example 31. Consider the addition in modulo 6 arithmetics $(\mathbb{Z}_6, +)$ as defined in in example 9. As we see, the remainder 0 is the neutral element in modulo 6 addition and the inverse of a remainder r is given by $6 - r$, because $r + (6 - r) = 6$, which is congruent to 0 , since $6 \bmod 6 = 0$. Moreover, $r_1 + r_2 = r_2 + r_1$ as well as $(r_1 + r_2) + r_3 = r_1 + (r_2 + r_3)$ are inherited from integer addition. We therefore see that $(\mathbb{Z}_6, +)$ is a group.

The previous example of a commutative group is a very important one for this book. Abstracting from this example and considering residue classes $(\mathbb{Z}_n, +)$ for arbitrary moduli n , it can be shown that $(\mathbb{Z}_n, +)$ is a commutative group with the neutral element 0 and the additive

inverse $n - r$ for any element $r \in \mathbb{Z}_n$. We call such a group the **remainder class group** of modulus n .

Exercise 32. Consider example 14 again and let \mathbb{Z}_5^* be the set of all remainder classes from \mathbb{Z}_5 without the class 0. Then $\mathbb{Z}_5^* = \{1, 2, 3, 4\}$. Show that (\mathbb{Z}_5^*, \cdot) is a commutative group.

Exercise 33. Generalizing the previous exercise, consider the general modulus n , and let \mathbb{Z}_n^* be the set of all remainder classes from \mathbb{Z}_n without the class 0. Then $\mathbb{Z}_n^* = \{1, 2, \dots, n - 1\}$. Provide a counter-example to show that (\mathbb{Z}_n^*, \cdot) is not a group in general.

Find a condition such that (\mathbb{Z}_n^*, \cdot) is a commutative group, compute the neutral element, give a closed form for the inverse of any element and prove the commutative group axioms.

Finite groups As we have seen in the previous examples, groups can either contain infinitely many elements (such as integers) or finitely many elements (as for example the remainder class groups $(\mathbb{Z}_n, +)$). To capture this distinction, a group is called a **finite group** if the underlying set of elements is finite. In that case, the number of elements of that group is called its **order**.

Notation and Symbols 4. Let \mathbb{G} be a finite group. We write $\text{ord}(\mathbb{G})$ or $|\mathbb{G}|$ for the order of \mathbb{G} .

Example 32. Consider the remainder class groups $(\mathbb{Z}_6, +)$ from example 9, the group $(\mathbb{Z}_5, +)$ from example 14, and the group (\mathbb{Z}_5^*, \cdot) from exercise 32. We can easily see that the order of $(\mathbb{Z}_6, +)$ is 6, the order of $(\mathbb{Z}_5, +)$ is 5 and the order of (\mathbb{Z}_5^*, \cdot) is 4.

Exercise 34. Let $n \in \mathbb{N}$ with $n \geq 2$ be some modulus. What is the order of the remainder class group $(\mathbb{Z}_n, +)$.

Generators The set of elements of a group can be complicated and it is not always obvious how to actually compute elements of a given group. From a practical point of view it is therefore desirable, if a group has a small subset of elements, such that all other elements can be generated by applying the group law repeatedly to the elements of that subset or their inverses only. Sets with these properties are called **generator sets**.

Of course, every group \mathbb{G} has a trivial set of generators, when we just consider every element of the group to be in the generator set. The more interesting question is to find smallest possible sets of generators for a given group. Of particular interest in this regard are groups that have a generator set that contains a single element only. In this case, there exists a (not necessarily unique) element $g \in \mathbb{G}$ such that every other element from \mathbb{G} can be computed by the repeated combination of g and its inverse g^{-1} only. Groups with single, not necessarily unique, generators are called **cyclic groups** and any element $g \in \mathbb{G}$ that is able to generate \mathbb{G} is called a **generator**.

Example 33. The most basic example of a cyclic group is the group of integers $(\mathbb{Z}, +)$ with integer addition. In this case, the number 1 is a generator of \mathbb{Z} , since every integer can be obtained by repeatedly adding either 1 or its inverse -1 to itself. For example -4 is generated by 1, since $-4 = -1 + (-1) + (-1) + (-1)$. Another generator of \mathbb{Z} is the number -1 .

Example 34. Consider the group (\mathbb{Z}_5^*, \cdot) from exercise 32. Since $2^1 = 2$, $2^2 = 4$, $2^3 = 3$ and $2^4 = 1$, the element 2 is a generator of (\mathbb{Z}_5^*, \cdot) . Moreover since $3^1 = 3$, $3^2 = 4$, $3^3 = 2$ and $3^4 = 1$, the element 3 is another generator of (\mathbb{Z}_5^*, \cdot) . Cyclic groups can therefore have more than one generator. However since $4^1 = 4$, $4^2 = 1$, $4^3 = 4$ and in general $4^k = 4$ for k odd and $4^k = 1$ for k even the element 4 is not a generator of (\mathbb{Z}_5^*, \cdot) . It follows that in general not every element of a finite cyclic group is a generator.

Example 35. Consider a modulus n and the remainder class groups $(\mathbb{Z}_n, +)$ from exercise 34. These groups are cyclic, with generator 1, since every other element of that group can be constructed by repeatedly adding the remainder class 1 to itself. Since \mathbb{Z}_n is also finite, we know that $(\mathbb{Z}_n, +)$ is a finite cyclic group of order n .

Exercise 35. Consider the group $(\mathbb{Z}_6, +)$ of modular 6 addition from example 9. Show that $5 \in \mathbb{Z}_6$ is a generator and then show that $2 \in \mathbb{Z}_6$ is not a generator.

Exercise 36. Let $p \in \mathbb{P}$ be prime number and (\mathbb{Z}_p^*, \cdot) the finite group from exercise 33. Show that (\mathbb{Z}_p^*, \cdot) is cyclic.

The exponential map Observe that, when \mathbb{G} is a cyclic group of order n and $g \in \mathbb{G}$ is a generator of \mathbb{G} , then there is the following map with respect to the generator g :

$$g^{(\cdot)} : \mathbb{Z}_n \rightarrow \mathbb{G} \quad x \mapsto g^x \quad (4.1)$$

In the map above, g^x means “multiply g x -times by itself” and $g^0 = e_{\mathbb{G}}$. This map, called the **exponential map**, has the remarkable property that it maps the additive group law of the remainder class group $(\mathbb{Z}_n, +)$ in a one-to-one correspondence to the group law of \mathbb{G} .

To see this, first observe that, since $g^0 := e_{\mathbb{G}}$ by definition, the neutral element of \mathbb{Z}_n is mapped to the neutral element of \mathbb{G} , and, since $g^{x+y} = g^x \cdot g^y$, the map respects the group law.

Remark 3 (Scalar multiplication). If a group $(\mathbb{G}, +)$ is written in additive notation 3, then the exponential map is often called **scalar multiplication** and written as

$$(\cdot) \cdot g : \mathbb{Z}_n \rightarrow \mathbb{G} ; x \mapsto x \cdot g \quad (4.2)$$

In this notation the symbol $x \cdot g$ is defined as “add the generator g x -times to itself” and the symbol $0 \cdot g$ is defined to be the neutral element in \mathbb{G} .

Cryptographic applications often utilize finite cyclic groups of very large order n and computing the exponential map by repeated multiplication of the generator with itself is infeasible, for very large remainder classes. The following so called **square and multiply** algorithm solves this problem as it computes the exponential map in approximately k steps, where k is the bit length of the exponent 3.4:

Because the exponential map respects the group law, it doesn’t matter if we do our computation in \mathbb{Z}_n before we write the result into the exponent of g or afterwards: the result will be the same in both cases. This is usually referred to as doing computations “in the exponent”. In cryptography in general, and in SNARK development in particular, we often perform computations “in the exponent” of a generator.

Example 36. Consider the multiplicative group (\mathbb{Z}_5^*, \cdot) from exercise 32. We know from 36 that \mathbb{Z}_5^* is a cyclic group of order 4, and that the element $3 \in \mathbb{Z}_5^*$ is a generator. We then know that the following map respects the group law of addition in \mathbb{Z}_4 and the group law of multiplication in \mathbb{Z}_5^* :

$$3^{(\cdot)} : \mathbb{Z}_4 \rightarrow \mathbb{Z}_5^* ; x \mapsto 3^x$$

To do an example computation “in the exponent” of 3, let’s perform the calculation $1 + 3 + 2$ in the exponent of the generator 3:

$$\begin{aligned} 3^{1+3+2} &= 3^2 \\ &= 4 \end{aligned}$$

Algorithm 5 Cyclic Group Exponentiation**Require:** g group generator of order n **Require:** $x \in \mathbb{Z}_n$ **procedure** EXPONENTIATION(g, x)Let (b_0, \dots, b_k) be a binary representation of x

▷ see example XXX

 $h \leftarrow g$ $y \leftarrow e_{\mathbb{G}}$ **for** $0 \leq j < k$ **do** **if** $b_j = 1$ **then** $y \leftarrow y \cdot h$

▷ multiply

end if $h \leftarrow h \cdot h$

▷ square

end for**return** y **end procedure****Ensure:** $y = g^x$

1760 What we did is, we first performed the computation $1 + 3 + 2 = 1$ in the remainder class group
 1761 $(\mathbb{Z}_4, +)$ and then applied the exponential map $3^{(\cdot)}$ to the result.

1762 However since the exponential map 4.1 "respects the group law" we also could map each
 1763 summand into (\mathbb{F}_5^*, \cdot) first and then apply the group law of (\mathbb{F}_5^*, \cdot) . The result is guranteed to be
 1764 the same:

$$\begin{aligned} 3^1 \cdot 3^3 \cdot 3^2 &= 3 \cdot 2 \cdot 4 \\ &= 1 \cdot 4 \\ &= 4 \end{aligned}$$

1765 Since the exponential map is a one-to-one correspondence that respects the group law, it
 1766 can be shown that this map has an inverse with respect to the base g , called the **base g discrete**
 1767 **logarithm map**:

$$\log_g(\cdot) : \mathbb{G} \rightarrow \mathbb{Z}_n \quad x \mapsto \log_g(x) \quad (4.3)$$

1768 Discrete logarithms are highly important in cryptography, because there are finite cyclic groups
 1769 where the exponential map and its inverse, the discrete logarithm map, are believed to be one-
 1770 way functions, which informally means that computing the exponential map is fast, while com-
 1771 puting the logarithm map is slow (We will look into a more precise definition in 4.1.1).

Example 37. Consider the exponential map $3^{(\cdot)}$ from example 36. Its inverse is the discrete logarithm to the base 3 and it is given by the map

$$\log_3(\cdot) : \mathbb{Z}_5^* \rightarrow \mathbb{Z}_4 \quad x \mapsto \log_3(x)$$

In contrast to the exponential map $3^{(\cdot)}$, we have no way to actually compute this map, other than by trying all elements of the group until we find the correct one. For example in order to compute $\log_3(4)$, we have to find some $x \in \mathbb{Z}_4$, such that $3^x = 4$ and all we can do is repeatedly insert elements x into the exponent of 3 until the result is 4. To do this let's write down all the images of $3^{(\cdot)}$:

$$3^0 = 1, \quad 3^1 = 3, \quad 3^2 = 4, \quad 3^3 = 2$$

Since the discrete logarithm $\log_3(\cdot)$ is defined as the inverse to this function, we can use those images to compute the discrete logarithm:

$$\log_3(1) = 0, \quad \log_3(2) = 3, \quad \log_3(3) = 1, \quad \log_3(4) = 2$$

Note that this computation was only possible, because we were able to write down all images of the exponential map. However, in real world applications the groups in consideration are too large to write down the images of the exponential map.

Exercise 37 (Efficient Scalar Multiplication). Let $(\mathbb{G}, +)$ be a finite cyclic group of order n . Consider algorithm 5 and define it's analog for groups in additive notation.

Factor Groups As we know from the fundamental theorem of arithmetics 3.7, every natural number n is a product of factors, the most basic of which are prime numbers. This reflects into subgroups of a finite cyclic group in an interesting way: If \mathbb{G} , is a finite cyclic group of order n , then every subgroup \mathbb{G}' of \mathbb{G} , is finite and cyclic and the order of \mathbb{G}' , is a factor of n . Moreover for each factor k of n , \mathbb{G} has exactly one subgroup of order k . This is known as the **fundamental theorem of finite cyclic groups**.

Notation and Symbols 5. If \mathbb{G} is a finite cyclic group of order n and k is a factor of n , then we write $\mathbb{G}[k]$ for the unique finite cyclic group, which is the order k subgroup of \mathbb{G} and call it a **factor group** of \mathbb{G} .

One particular interesting situation occurs if the order of a given finite cyclic group is a prime number. As we know from the fundamental theorem of arithmetics 3.7, prime numbers have only two factors, given by the number 1 and the prime number itself. It then follows from the fundamental theorem of finite cyclic groups 4.1, that those groups have no subgroups other than the trivial group 30 and the group itself.

Cryptographic protocols often assume the existence of finite cyclic groups of prime order but sometimes real world implementations of those protocols are not defined on prime order groups, but on groups where the order consist of a (usually large) prime number that has small cofactors 1. In this case a method called **cofactor clearing** has to be applied to ensure that the computations are not done in the group itself but in its (large) prime order subgroup.

To understand cofactor clearing in detail, let \mathbb{G} be a finite cyclic group of order n and let k be a factor of n with associated factor group $\mathbb{G}[k]$. We can project any element $g \in \mathbb{G}[k]$ onto the neutral element e of \mathbb{G} by multiplying g k -times with itself:

$$g^k = e \tag{4.4}$$

From this follows that if $c := n \operatorname{div} k$ is the cofactor 1 of k in n then any element from the full group $g \in \mathbb{G}$ can be projected into the factor group $\mathbb{G}[k]$ by multiplying g c -times with itself. This defines the following map, which is often called **cofactor clearing** in the cryptographic literature:

$$(\cdot)^c : \mathbb{G} \rightarrow \mathbb{G}[k] : g \mapsto g^c \tag{4.5}$$

Example 38. Consider the finite cyclic group (\mathbb{Z}_5^*, \cdot) from example 34. Since the order of \mathbb{Z}_5^* is 4 and 4 has the factors 1, 2 and 4, it follows from the fundamental theorem of finite cyclic groups, that \mathbb{Z}_5^* has 3 unique subgroups. In fact the unique subgroup $\mathbb{Z}_5^*[1]$ of order 1 is given by the trivial group $\{1\}$ that contains only the multiplicative neutral element 1 and the unique subgroup $\mathbb{Z}_5^*[4]$ of order 4 is \mathbb{Z}_5^* itself, since by definition every group is trivially a subgroup of itself. The unique subgroup $\mathbb{Z}_5^*[2]$ of order 2 is more interesting and is given by the set $\mathbb{Z}_5^*[2] = \{1, 4\}$.

Since \mathbb{Z}_5^* is not a prime order group and since the only prime factor of 4 is 2, the "large" prime order subgroup of \mathbb{Z}_5^* is $\mathbb{Z}_5^*[2]$. Moreover since the cofactor of 2 in 4 is also 2, we have the cofactor clearing map $(\cdot)^2 : \mathbb{Z}_5^* \rightarrow \mathbb{Z}_5^*[2]$ and indeed applying this map to all elements from \mathbb{Z}_5^* we see that it maps onto the elements of $\mathbb{Z}_5^*[2]$ only:

$$1^2 = 1, \quad 2^2 = 4, \quad 3^2 = 4, \quad 4^2 = 1$$

We can therefore use this map to "clear the cofactor" of any element from \mathbb{Z}_5^* which means that the element is projected into the "large" prime order subgroup $\mathbb{Z}_5^*[2]$.

Exercise 38. Consider the previous example 38 and show that $\mathbb{Z}_5^*[2]$ is a commutative group.

Exercise 39. Consider the finite cyclic group $(\mathbb{Z}_6, +)$ of modular 6 addition from example 35. Describe all subgroups of $(\mathbb{Z}_6, +)$. Identify the large prime order subgroup of \mathbb{Z}_6 , define its cofactor clearing map and apply that map to all elements of \mathbb{Z}_6 .

Exercise 40. Let (\mathbb{Z}_p^*, \cdot) be the cyclic group from exercise 36. Show that for $p \geq 5$, not every element $x \in \mathbb{F}_p^*$ is a generator of \mathbb{F}_p^* .

Pairings Of particular importance for the development of SNARKs are so-called pairing maps on commutative groups. To see the definition, let $\mathbb{G}_1, \mathbb{G}_2$ and \mathbb{G}_3 be three commutative groups. Then a **pairing map** is a function

$$e(\cdot, \cdot) : \mathbb{G}_1 \times \mathbb{G}_2 \rightarrow \mathbb{G}_3 \tag{4.6}$$

This function takes pairs (g_1, g_2) of elements from \mathbb{G}_1 and \mathbb{G}_2 , and maps them to elements from \mathbb{G}_3 , such that the **bilinearity** property holds, which means that for all $g_1, g'_1 \in \mathbb{G}_1$ and $g_2, g'_2 \in \mathbb{G}_2$ the following two identities are satisfied:

$$e(g_1 \cdot g'_1, g_2) = e(g_1, g_2) \cdot e(g'_1, g_2) \quad \text{and} \quad e(g_1, g_2 \cdot g'_2) = e(g_1, g_2) \cdot e(g_1, g'_2) \tag{4.7}$$

Informally speaking, bilinearity means that it doesn't matter if we first execute the group law on one side and then apply the bilinear map, or if we first apply the bilinear map and then apply the group law in \mathbb{G}_3 .

A pairing map is called **non-degenerate** if, whenever the result of the pairing is the neutral element in \mathbb{G}_3 , one of the input values is the neutral element of \mathbb{G}_1 or \mathbb{G}_2 . To be more precise, $e(g_1, g_2) = e_{\mathbb{G}_3}$ implies $g_1 = e_{\mathbb{G}_1}$ or $g_2 = e_{\mathbb{G}_2}$.

Example 39. One of the most basic examples of a non-degenerate pairing involves $\mathbb{G}_1, \mathbb{G}_2$ and \mathbb{G}_3 all to be the group of integers with addition $(\mathbb{Z}, +)$. Then the following map defines a non-degenerate pairing:

$$e(\cdot, \cdot) : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \quad (a, b) \mapsto a \cdot b$$

Note that bilinearity follows from the distributive law of integers, since for $a, b, c \in \mathbb{Z}$, we have $e(a + b, c) = (a + b) \cdot c = a \cdot c + b \cdot c = e(a, c) + e(b, c)$ and the same reasoning is true for the second argument.

To see that $e(\cdot, \cdot)$ is non-degenerate, assume that $e(a, b) = 0$. Then $a \cdot b = 0$ implies that a or b must be zero.

Exercise 41 (Arithmetic laws for pairing maps). Let $\mathbb{G}_1, \mathbb{G}_2$ as well as \mathbb{G}_3 be finite cyclic groups of the same order n and let $e(\cdot, \cdot) : \mathbb{G}_1 \times \mathbb{G}_2 \rightarrow \mathbb{G}_3$ be a pairing map. Show that for given $g_1 \in \mathbb{G}_1$ and $g_2 \in \mathbb{G}_2$ and all $a, b \in \mathbb{Z}_n$ the following identity holds:

$$e(g_1^a, g_2^b) = e(g_1, g_2)^{a \cdot b} \tag{4.8}$$

Exercise 42. Consider the remainder class groups $(\mathbb{Z}_n, +)$ from example 34 for some modulus n . Show that the map

$$e(\cdot, \cdot) : \mathbb{Z}_n \times \mathbb{Z}_n \rightarrow \mathbb{Z}_n (a, b) \mapsto a \cdot b$$

is a pairing map. Why is the pairing not non-degenerate in general and what condition must be imposed on n , such that the pairing will be non-degenerate?

4.1.1 Cryptographic Groups

In this section, we will look at families of groups that are believed to satisfy certain **computational hardness assumptions**, namely that a particular problem cannot be solved efficiently (where efficiently typically means “in polynomial time of a given security parameter”) in the groups under consideration.

Example 40. To highlight the concept of the computational hardness assumption, consider the group of integers \mathbb{Z} from example 3.6. One of the best known and most researched examples of computational hardness is the assumption that the factorization of integers into prime numbers cannot be solved by any algorithm in polynomial time with respect to the bit-length of the integer.

To be more precise, the computational hardness assumption of integer factorization assumes that, given any integer $z \in \mathbb{Z}$ with bit-length b , there is no integer k and no algorithm with the runtime complexity of $\mathcal{O}(b^k)$ that is able to find the prime numbers $p_1, p_2, \dots, p_j \in \mathbb{P}$, such that $z = p_1 \cdot p_2 \cdot \dots \cdot p_j$.

This hardness assumption was proven to be false, since Shor’s (?) algorithm shows that integer factorization is at least efficiently possible on a quantum computer, since the runtime complexity of this algorithm is $\mathcal{O}(b^3)$. However, no such algorithm is known on a classical computer.

In the realm of classical computers, however, we still have to call the non-existence of such an algorithm an “assumption” because, to date, there is no proof that it is actually impossible to find one. The problem is that it is hard to reason about algorithms that we don’t know.

So, despite the fact that there is currently no known algorithm that can factor integers efficiently on a classical computer, we cannot exclude that such an algorithm might exist in principle, and that someone eventually will discover it in the future.

However, what still makes the assumption plausible, despite the absence of any actual proof, is the fact that, after decades of extensive search, still no such algorithm has been found.

In what follows, we will describe a few computational hardness assumptions that arise in the context of groups in cryptography, because we will refer to them throughout the book.

The discrete logarithm assumption The so-called discrete logarithm problem is one of the most fundamental assumptions in cryptography. To define it, let \mathbb{G} be a finite cyclic group of order r and let g be a generator of \mathbb{G} . We know from 4.1 that there is an exponential map $g^{(\cdot)} : \mathbb{Z}_r \rightarrow \mathbb{G} : x \mapsto g^x$ that maps the residue classes from modulo r arithmetic onto the group in a 1 : 1 correspondence. The **discrete logarithm problem** is the task of finding inverses to this map, that is, to find a solution $x \in \mathbb{Z}_r$ to the following equation for some given $h \in \mathbb{G}$:

$$h = g^x \tag{4.9}$$

In other words, the **discrete logarithm assumption (DL-A)** is the assumption that there exists no algorithm with polynomial running time in the security parameter $\log_2(r)$, that is able to

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compute some x if only h , g and g^x are given in \mathbb{G} . If this is the case for \mathbb{G} , we call \mathbb{G} a **DL-A group**.

Rephrasing the previous definition, DL-A groups are believed to have the property that it is infeasible to compute some number x that solves the equation $h = g^x$ for a given h and g , assuming that the size of the group r is large enough.

Example 41 (Public key cryptography). One the most basic examples of an application for DL-A groups is in public key cryptography, where the parties publicly agree on some pair (\mathbb{G}, g) such that \mathbb{G} is a finite cyclic group of sufficiently large order r , where \mathbb{G} is believed to be a DL-A group, and g is a generator of \mathbb{G} .

In this setting, a secret key is some number $sk \in \mathbb{Z}_r$ and the associated public key pk is the group element $pk = g^{sk}$. Since discrete logarithms are assumed to be hard, it is infeasible for an attacker to compute the secret key from the public key, since it is believed to be hard to find solutions x to the following equation:

$$pk = g^x \quad (4.10)$$

As the previous example shows, identifying DL-A groups is an important practical problem. Unfortunately, it is easy to see that it does not make sense to assume the hardness of the discrete logarithm problem in all finite cyclic groups: Counterexamples are common and easy to construct.

Example 42 (Modular arithmetics for Fermat's primes). It is widely believed that the discrete logarithm problem is hard in multiplicative groups \mathbb{Z}_p^* of prime number modular arithmetics. However, this is not true in general. To see that, consider any so-called Fermat's prime, which is a prime number $p \in \mathbb{P}$, such that $p = 2^n + 1$ for some number n .

We know from exercise 33 that in this case $\mathbb{Z}_p^* = \{1, 2, \dots, p-1\}$ is a group with respect to integer multiplication in modular p arithmetics and since $p = 2^n + 1$, the order of \mathbb{Z}_p^* is 2^n , which implies that the associated security parameter is given by $\log_2(2^n) = n$.

We show that, in this case, \mathbb{Z}_p^* is not a DL-A group, by constructing an algorithm, which is able compute some $x \in \mathbb{Z}_{2^n}$ for any given generator g and arbitrary element h of \mathbb{F}_p^* , such that equation 4.11 holds, and the runtime complexity of the constructed algorithm is $\mathcal{O}(n^2)$, which is quadratic in the security parameter $n = \log_2(2^n)$.

$$h = g^x \quad (4.11)$$

To define such an algorithm, let us assume that the generator g is a public constant and that a group element h is given. Our task is to compute x efficiently.

The first thing to note is that, since x is a number in modular 2^n arithmetic, we can write the binary representation of x as in 4.12, with binary coefficients $c_j \in \{0, 1\}$. In particular, x is an n -bit number if interpreted as an integer.

$$x = c_0 \cdot 2^0 + c_1 \cdot 2^1 + \dots + c_n \cdot 2^n \quad (4.12)$$

We then use this representation to construct an algorithm that computes the bits c_j one after another, starting at c_0 . To see how this can be achieved, observe that we can determine c_0 by raising the input h to the power of 2^{n-1} in \mathbb{F}_p^* . We use the exponential laws and compute as

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follows:

$$\begin{aligned}
 h^{2^{n-1}} &= (g^x)^{2^{n-1}} \\
 &= \left(g^{c_0 \cdot 2^0 + c_1 \cdot 2^1 + \dots + c_n \cdot 2^n} \right)^{2^{n-1}} \\
 &= g^{c_0 \cdot 2^{n-1}} \cdot g^{c_1 \cdot 2^1 \cdot 2^{n-1}} \cdot g^{c_2 \cdot 2^2 \cdot 2^{n-1}} \dots g^{c_n \cdot 2^n \cdot 2^{n-1}} \\
 &= g^{c_0 2^{n-1}} \cdot g^{c_1 \cdot 2^0 \cdot 2^n} \cdot g^{c_2 \cdot 2^1 \cdot 2^n} \dots g^{c_n \cdot 2^{n-1} \cdot 2^n}
 \end{aligned}$$

Now, since g is a generator and \mathbb{F}_p^* is cyclic of order 2^n , we know $g^{2^n} = 1$ and therefore $g^{k \cdot 2^n} = 1^k = 1$. From this, it follows that all but the first factor in the last expression are equal to 1 and we can simplify the expression into the following:

$$h^{2^{n-1}} = g^{c_0 2^{n-1}} \quad (4.13)$$

Now, in case $c_0 = 0$, we get $h^{2^{n-1}} = g^0 = 1$. In case $c_0 = 1$, we get $h^{2^{n-1}} = g^{2^{n-1}} \neq 1$ (To see that $g^{2^{n-1}} \neq 1$, recall that g is a generator of \mathbb{F}_p^* and hence, is \mathbb{F}_p^* a cyclic group of order 2^n , which implies $g^y \neq 1$ for all $y < 2^n$).

Raising h to the power of 2^{n-1} determines c_0 , and we can apply the same reasoning to the coefficient c_1 by raising $h \cdot g^{-c_0 \cdot 2^0}$ to the power of 2^{n-2} . This approach can then be repeated until all the coefficients c_j of x are found.

Assuming that exponentiation in \mathbb{F}_p^* can be done in logarithmic runtime complexity $\log(p)$, it follows that our algorithm has a runtime complexity of $\mathcal{O}(\log^2(p)) = \mathcal{O}(n^2)$, since we have to execute n exponentiations to determine the n binary coefficients of x .

From this, it follows that whenever p is a Fermat's prime, the discrete logarithm assumption does not hold in F_p^* .

The decisional Diffie–Hellman assumption Let \mathbb{G} be a finite cyclic group of order r and let g be a generator of \mathbb{G} . The decisional Diffie–Hellman assumption stipulates that there is no algorithm that has a polynomial runtime complexity in the security parameter $s = \log(r)$ that is able to distinguish the so-called DDH- triple (g^a, g^b, g^{ab}) from any triple (g^a, g^b, g^c) for randomly and independently chosen parameters $a, b, c \in \mathbb{Z}_r$. If this is the case for \mathbb{G} , we call \mathbb{G} a **DDH-A group**.

DDH-A is a stronger assumption than DL-A, in the sense that the discrete logarithm assumption is necessary for the decisional Diffie–Hellman assumption to hold, but not the other way around.

To see why this is the case, assume that the discrete logarithm assumption does not hold. In that case, given a generator g and a group element h , it is easy to compute some residue class $x \in \mathbb{Z}_p$ with $h = g^x$. Then the decisional Diffie–Hellman assumption cannot hold, since given some triple (g^a, g^b, z) , one could efficiently decide whether $z = g^{ab}$ is true by first computing the discrete logarithm b of g^b , then computing $g^{ab} = (g^a)^b$ and deciding whether or not $z = g^{ab}$.

On the other hand, the following example shows that there are groups where the discrete logarithm assumption holds but the decisional Diffie–Hellman assumption does not.

Example 43 (Efficiently computable pairings). Let \mathbb{G} be a finite, cyclic group of order r with generator g , such that the discrete logarithm assumption holds and there is a pairing map $e(\cdot, \cdot) : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G}_T$ for some target group \mathbb{G}_T that is computable in polynomial time of the parameter $\log(r)$.

In a setting like this, it is easy to show that DDH-A cannot hold, since given some triple (g^a, g^b, z) , it is possible to decide in polynomial time w.r.t $\log(r)$ whether $z = g^{ab}$ or not. To see that, check the following equation:

$$e(g^a, g^b) = e(g, z) \quad (4.14)$$

Since the bilinearity properties of $e(\cdot, \cdot)$ imply $e(g^a, g^b) = e(g, g)^{ab} = e(g, g^{ab})$, and $e(g, y) = e(g, y')$ implies $y = y'$ due to the non-degenerate property, the equality means $z = g^{ab}$.

It follows that DDH-A is indeed weaker than DL-A, and groups with efficient pairings cannot be DDH-A groups. The following example shows another important class of groups where DDH-A does not hold: multiplicative groups of prime number residue classes.

Example 44. Let p be a prime number and $\mathbb{Z}_p^* = \{1, 2, \dots, p-1\}$ the multiplicative group of modular p arithmetics as in exercise 33. As we have seen in XXX, this group is finite and cyclic of order $p-1$ and every element $g \neq 1$ is a generator.

To see that \mathbb{F}_p^* cannot be a DDH-A group, recall from XXX that the Legendre symbol $\left(\frac{x}{p}\right)$ of any $x \in \mathbb{F}_p^*$ is efficiently computable by Euler's formular. But the Legendre symbol of g^a reveals whether a is even or odd. Given g^a, g^b and g^{ab} , one can thus efficiently compute and compare the least significant bit of a, b and ab , respectively, which provides a probabilistic method to distinguish g^{ab} from a random group element g^c .

The computational Diffie–Hellman assumption Let \mathbb{G} be a finite cyclic group of order r and let g be a generator of \mathbb{G} . The computational Diffie–Hellman assumption stipulates that, given randomly and independently chosen residue classes $a, b \in \mathbb{Z}_r$, it is not possible to compute g^{ab} if only g, g^a and g^b (but not a and b) are known. If this is the case for \mathbb{G} , we call \mathbb{G} a CDH-A group.

In general, we don't know if CDH-A is a stronger assumption than DL-A, or if both assumptions are equivalent. We know that DL-A is necessary for CDH-A, but the other direction is currently not well understood. In particular, there are no groups known where DL-A holds but CDH-A does not hold [?].

To see why the discrete logarithm assumption is necessary, assume that it does not hold. So, given a generator g and a group element h , it is easy to compute some residue class $x \in \mathbb{Z}_p$ with $h = g^x$. In that case, the computational Diffie–Hellman assumption cannot hold, since, given g, g^a and g^b , one can efficiently compute b and hence is able to compute $g^{ab} = (g^a)^b$ from this data.

The computational Diffie–Hellman assumption is a weaker assumption than the decisional Diffie–Hellman assumption, which means that there are groups where CDH-A holds and DDH-A does not hold, while there cannot be groups such that DDH-A holds but CDH-A does not hold. To see that, assume that it is efficiently possible to compute g^{ab} from g, g^a and g^b . Then, given (g^a, g^b, z) it is easy to decide if $z = g^{ab}$ holds or not.

Several variations and special cases of the CDH-A exist. For example, the **square computational Diffie–Hellman assumption** assumes that, given g and g^x , it is computationally hard to compute g^{x^2} . The **inverse computational Diffie–Hellman assumption** assumes that, given g and g^x , it is computationally hard to compute $g^{x^{-1}}$.

4.1.2 Hashing to Groups

Hash functions Generally speaking, a hash function is any function that can be used to map data of arbitrary size to fixed-size values. Since binary strings of arbitrary length are a general

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vant later?
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way to represent arbitrary data, we can understand a general **hash function** as the following map where $\{0,1\}^*$ represents the set of all binary strings of arbitrary but finite length and $\{0,1\}^k$ represents the set of all binary strings that have a length of exactly k bits:

$$H : \{0,1\}^* \rightarrow \{0,1\}^k \quad (4.15)$$

In our definition, a hash function maps binary strings of arbitrary size onto binary strings of size exactly k . The **images** of H , that is, the values returned by the hash function H , are called **hash values**, **digests**, or simply **hashes**.

A hash function must be deterministic, that is, when we insert the same input x into H , the image $H(x)$ must always be the same. In addition, a hash function should be as uniform as possible, which means that it should map input values as evenly as possible over its output range. In mathematical terms, every string of length k from $\{0,1\}^k$ should be generated with roughly the same probability.

Example 45 (k -truncation hash). One of the most basic hash functions $H_k : \{0,1\}^* \rightarrow \{0,1\}^k$ is given by simply truncating every binary string s of size $s.len() > k$ to a string of size k and by filling any string s' of size $s'.len() < k$ with zeros. To make this hash function deterministic, we define that both truncation and filling should happen “on the left”.

For example, if $k = 3$, $x_1 = (0000101011101010011101010101)$ and $x_2 = 1$, then $H(x_1) = (101)$ and $H(x_2) = (001)$. It is easy to see that this hash function is deterministic and uniform.

Of particular interest are so-called **cryptographic** hash functions, which are hash functions that are also **one-way functions**, which essentially means that, given a string y from $\{0,1\}^k$ it is practically infeasible to find a string $x \in \{0,1\}^*$ such that $H(x) = y$ holds. This property is usually called **preimage-resistance**.

In addition, it should be infeasible to find two strings $x_1, x_2 \in \{0,1\}^*$, such that $H(x_1) = H(x_2)$, which is called **collision resistance**. It is important to note, though, that collisions always exist, since a function $H : \{0,1\}^* \rightarrow \{0,1\}^k$ inevitably maps infinitely many values onto the same hash. In fact, for any hash function with digests of length k , finding a preimage to a given digest can always be done using a brute force search in 2^k evaluation steps. It should just be practically impossible to compute those values, and statistically very unlikely to generate two of them by chance.

A third property of a cryptographic hash function is that small changes in the input string, like changing a single bit, should generate hash values that look completely different from each other.

Because cryptographically secure hash functions map tiny changes in input values onto large changes in the output, implementation errors that change the outcome are usually easy to spot by comparing them to expected output values. The definitions of cryptographically secure hash functions are therefore usually accompanied by some test vectors of common inputs and expected digests. Since the empty string $''$ is the only string of length 0, a common test vector is the expected digest of the empty string.

Example 46 (k -truncation hash). Consider the k -truncation hash from example 45. Since the empty string has length 0, it follows that the digest of the empty string is string of length k that only contains 0's:

$$H_k('') = (000\dots 000) \quad (4.16)$$

It is pretty obvious from the definition of H_k that this simple hash function is not a cryptographic hash function. In particular, every digest is its own preimage, since $H_k(y) = y$ for every string of size exactly k . Finding preimages is therefore easy, so the property of preimage resistance does not hold.

Is there a term for this property?

In addition, it is easy to construct collisions as all strings of size $> k$ that share the same k -bits “on the right” are mapped to the same hash value, so this function is not collision resistant, either.

Finally, this hash function is not very chaotic, as changing bits that are not part of the k right-most bits don’t change the digest at all.

Computing cryptographically secure hash functions in pen-and-paper style is possible but tedious. Fortunately, Sage can import the **hashlib** library, which is intended to provide a reliable and stable base for writing Python programs that require cryptographic functions. The following examples explain how to use hashlib in Sage.

Example 47. An example of a hash function that is generally believed to be a cryptographically secure hash function is the so-called **SHA256** hash, which, in our notation, is a function that maps binary strings of arbitrary length onto binary strings of length 256:

$$SHA256 : \{0,1\}^* \rightarrow \{0,1\}^{256} \quad (4.17)$$

To evaluate a proper implementation of the *SHA256* hash function, the digest of the empty string is supposed to be

$$SHA256("") = e3b0c44298fc1c149afbf4c8996fb92427ae41e4649b934ca495991b7852b855 \quad (4.18)$$

For better human readability, it is common practice to represent the digest of a string not in its binary form, but in a hexadecimal representation. We can use Sage to compute *SHA256* and freely transit between binary, hexadecimal and decimal representations. To do so, we import hashlib’s implementation of SHA256:

```
sage: import hashlib 159
sage: test = 'e3b0c44298fc1c149afbf4c8996fb92427ae41e4649b934 160
ca495991b7852b855'
sage: hasher = hashlib.sha256(b'') 161
sage: str = hasher.hexdigest() 162
sage: type(str) 163
<class 'str'> 164
sage: d = ZZ('0x'+ str) # conversion to integer type 165
sage: d.str(16) == str 166
True 167
sage: d.str(16) == test 168
True 169
sage: d.str(16) 170
e3b0c44298fc1c149afbf4c8996fb92427ae41e4649b934ca495991b7852b8 171
55
sage: d.str(2) 172
11100011101100001100010001000010100110001111110000011100000101 173
001001101011111011111010011001000100110010110111101110010
01001000010011110101110010000011110010001100100100110111001
00110100110010100100100101011001100100011011011110000101001
01011100001010101
sage: d.str(10) 174
10298733624955409702953521232258132278979990064819803499337939 175
7001115665086549
```

Hashing to cyclic groups As we have seen in the previous paragraph, general hash functions map binary strings of arbitrary length onto binary strings of length k . However, it is desirable in various cryptographic primitives to not simply hash to binary strings of fixed length but to hash into algebraic structures like groups, while keeping (some of) the properties like preimage resistance or collision resistance.

Hash functions like this can be defined for various algebraic structures, but, in a sense, the most fundamental ones are hash functions that map into groups, because they can be easily extended to map into other structures like rings or fields.

To give a more precise definition, let \mathbb{G} be a group and $\{0, 1\}^*$ the set of all finite, binary strings, then a **hash-to-group** function is a deterministic map

$$H : \{0, 1\}^* \rightarrow \mathbb{G} \quad (4.19)$$

Common properties of hash functions, like uniformity, are desirable but not always realized in real-world instantiations of hash-to-group functions, so we skip those requirements for now and keep the definition very general.

As the following example shows, hashing to finite cyclic groups can be trivially achieved for the price of some undesirable properties of the hash function:

Example 48 (Naive cyclic group hash). Let \mathbb{G} be a finite cyclic group. If the task is to implement a hash-to-group function, one immediate approach can be based on the observation that binary strings of size k can be interpreted as integers $z \in \mathbb{Z}$ in the range $0 \leq z < 2^k$.

To be more precise, choose an ordinary hash function $H : \{0, 1\}^* \rightarrow \{0, 1\}^k$ for some parameter k and a generator g of \mathbb{G} . Then the expression below is a positive integer (where $H(s)_j$ means the bit at the j -th position of $H(s)$):

$$z_{H(s)} = H(s)_0 \cdot 2^0 + H(s)_1 \cdot 2^1 + \dots + H(s)_k \cdot 2^k \quad (4.20)$$

A hash-to-group function for the group \mathbb{G} can then be defined as a concatenation of the exponential map $g^{(\cdot)}$ of g with the interpretation of $H(s)$ as an integer:

$$H_g : \{0, 1\}^* \rightarrow \mathbb{G} : s \mapsto g^{z_{H(s)}} \quad (4.21)$$

Constructing a hash-to-group function like this is easy for cyclic groups, and it might be good enough in certain applications. It is, however, almost never adequate in cryptographic applications, as discrete log relations might be constructible between two given hash values $H_g(s)$ and $H_g(t)$.

a few examples?

To see that, assume that \mathbb{G} is of order r and that $z_{H(s)}$ has a multiplicative inverse in modular r arithmetics. In that case, we can compute $x = z_{H(t)} \cdot z_{H(s)}^{-1}$ in \mathbb{Z}_r and find a discrete log relation between the group hash values, that is, find some x with $H_g(t) = (H_g(s))^x$:

$$\begin{aligned} H_g(t) &= (H_g(s))^x && \Leftrightarrow \\ g^{z_{H(t)}} &= g^{z_{H(s)} \cdot x} && \Leftrightarrow \\ g^{z_{H(t)}} &= g^{z_{H(s)} \cdot x} \end{aligned}$$

Therefore applications where discrete log relations between hash values are undesirable need different approaches. Many of these approaches start with a way to hash into the set \mathbb{Z}_r of modular r arithmetics.

Hashing to modular arithmetics One of the most widely used applications of hash-into-group functions are hash functions that map into the set \mathbb{Z}_r of modular r arithmetics for some modulus r . Different approaches to construct such a function are known, but probably the most widely used ones are based on the insight that the images of arbitrary hash functions can be interpreted as binary representations of integers, as explained in example 48.

It follows from this interpretation that one simple method of hashing into \mathbb{Z}_r is constructed by observing that if r is a modulus with a bit-length of $k = r.\text{nbits}()$, then every binary string $(b_0, b_1, \dots, b_{k-2})$ of length $k - 1$ defines an integer z in the range $0 \leq z < 2^{k-1} \leq r$, by defining z :

$$z = b_0 \cdot 2^0 + b_1 \cdot 2^1 + \dots + b_{k-2} \cdot 2^{k-2} \quad (4.22)$$

Now, since $z < r$, we know that z is guaranteed to be in the set $\{0, 1, \dots, r-1\}$, and hence it can be interpreted as an element of \mathbb{Z}_r . From this it follows that if $H : \{0, 1\}^* \rightarrow \{0, 1\}^{k-1}$ is a hash function, then a hash-to-group function can be constructed as follows (where $H(s)_j$ means the j -th bit of the image binary string $H(s)$ of the original binary hash function):

$$H_{r.\text{nbits}()-1} : \{0, 1\}^* \rightarrow \mathbb{Z}_r : s \mapsto H(s)_0 \cdot 2^0 + H(s)_1 \cdot 2^1 + \dots + H(s)_{k-2} \cdot 2^{k-2} \quad (4.23)$$

A drawback of this hash function is that the distribution of the hash values in \mathbb{Z}_r is not necessarily uniform. In fact, if $r - 2^{k-1} \neq 0$, then by design $H_{r.\text{nbits}()-1}$ will never hash onto values $z \geq 2^{k-1}$. Good moduli r are therefore as close to 2^{k-1} as possible, while less good moduli are closer to 2^k . In the worst case, when $r = 2^k - 1$, it misses $2^{k-1} - 1$, that is, almost half of all elements, from \mathbb{Z}_r .

An advantage of this approach is that properties like preimage resistance or collision resistance of the original hash function $H(\cdot)$ are preserved.

Example 49. To give an implementation of the $H_{r.\text{nbits}()-1}$ hash function, we use a 5-bit truncation of the *SHA256* hash from example 47 and define a hash into \mathbb{Z}_{16} as follows:

$$H_{16.\text{nbits}()-1} : \{0, 1\}^* \rightarrow \mathbb{Z}_{16} : s \mapsto \text{SHA256}(s)_0 \cdot 2^0 + \text{SHA256}(s)_1 \cdot 2^1 + \dots + \text{SHA256}(s)_4 \cdot 2^4$$

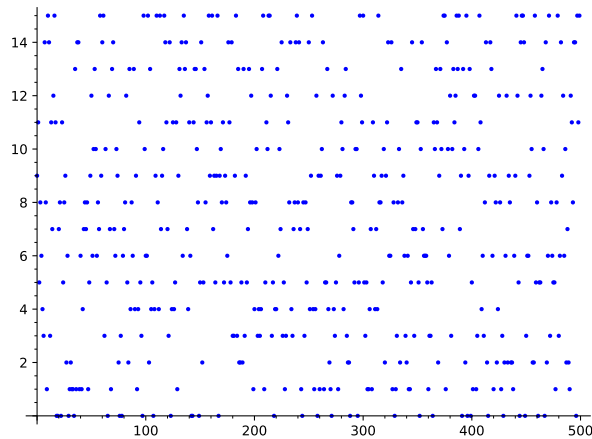
Since $k = 16.\text{nbits}() = 5$ and $16 - 2^{k-1} = 0$, this hash maps uniformly onto \mathbb{Z}_{16} . We can invoke Sage to implement it:

```
sage: import hashlib 176
sage: def Hash5(x): 177
.....:     hasher = hashlib.sha256(x) 178
.....:     digest = hasher.hexdigest() 179
.....:     d = ZZ(digest, base=16) 180
.....:     d = d.str(2)[-4:] 181
.....:     return ZZ(d, base=2) 182
sage: Hash5(b' ') 183
5 184
```

We can then use Sage to apply this function to a large set of input values in order to plot a visualization of the distribution over the set $\{0, \dots, 15\}$. Executing over 500 input values gives the following plot:

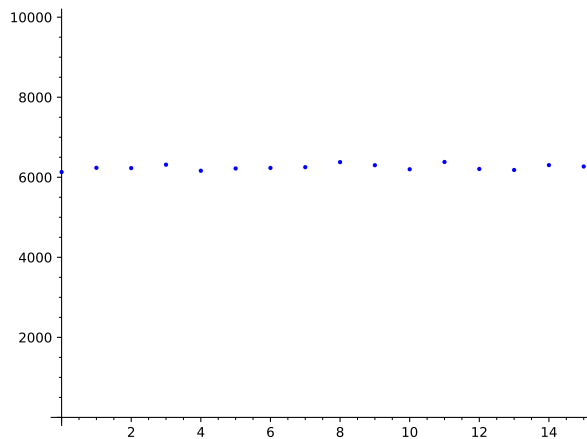
check
reference

TODO:
DOUBLE
CHECK
THIS
REA-
SONING.



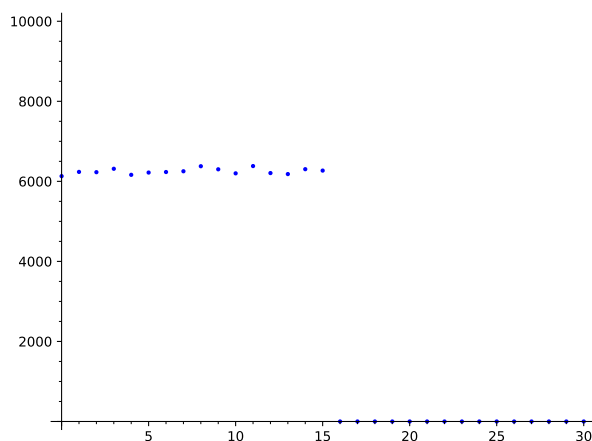
2133

2134 To get an intuition of uniformity, we can count the number of times the hash function $H_{16.nb\text{its}()}-1$
 2135 maps onto each number in the set $\{0, 1, \dots, 15\}$ in a loop of 100000 hashes, and compare that
 2136 to the ideal uniform distribution, which would map exactly 6250 samples to each element. This
 2137 gives the following result:



2138

2139 The uniformity of distribution problem becomes apparent if we want to construct a similar hash
 2140 function for \mathbb{Z}_r for any r in the range $17 \leq r \leq 31$. In this case, the definition of the hash
 2141 function is exactly the same as for \mathbb{Z}_{16} , and hence, the images will not exceed the value 16.
 2142 So, for example, even in the case of hashing to \mathbb{Z}_{31} , the hash function never maps to any value
 2143 larger than 16, leaving almost half of all numbers out of the image range.



2144

Mirco:
We can
do better
than this

The second widely used method of hashing into \mathbb{Z}_r is constructed by observing the following: If r is a modulus with a bit-length of $r.bits() = k_1$ and $H : \{0, 1\}^* \rightarrow \{0, 1\}^{k_2}$ is a hash function that produces digests of size k_2 , with $k_2 \geq k_1$, then a hash-to-group function can be constructed by interpreting the image of H as a binary representation of an integer and then taking the modulus by r to map into \mathbb{Z}_r . This is formalized in the equation below, where $H(s)_j$ means the j 'th bit of the image binary string $H(s)$ of the original binary hash function.

$$H'_{mod_r} : \{0, 1\}^* \rightarrow \mathbb{Z}_r : s \mapsto \left(H(s)_0 \cdot 2^0 + H(s)_1 \cdot 2^1 + \dots + H(s)_{k_2} \cdot 2^{k_2} \right) \bmod r \quad (4.24)$$

A drawback of this hash function is that computing the modulus requires some computational effort. In addition, the distribution of the hash values in \mathbb{Z}_r might not be even, depending on the difference $2^{k_2+1} - r$. An advantage of it is that potential properties of the original hash function $H(\cdot)$ (like preimage resistance or collision resistance) are preserved, and the distribution can be made almost uniform, with only negligible bias depending on what modulus r and images size k_2 are chosen.

Example 50. To give an implementation of the H_{mod_r} hash function, we use k_2 -bit truncation of the *SHA256* hash from example 47, and define a hash into \mathbb{Z}_{23} as follows:

$$H_{mod_{23}, k_2} : \{0, 1\}^* \rightarrow \mathbb{Z}_{23} : \\ s \mapsto \left(SHA256(s)_0 \cdot 2^0 + SHA256(s)_1 \cdot 2^1 + \dots + SHA256(s)_{k_2} \cdot 2^{k_2} \right) \bmod 23$$

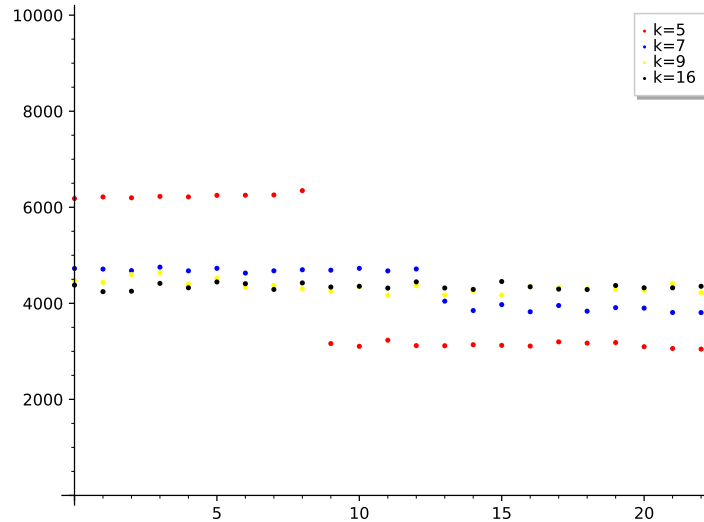
We want to use various instantiations of k_2 to visualize the impact of truncation length on the distribution of the hashes in \mathbb{Z}_{23} . We can invoke Sage to implement it as follows:

```

sage: import hashlib                                     185
sage: Z23 = Integers(23)                                 186
sage: def Hash_mod23(x, k2):                             187
.....:     hasher = hashlib.sha256(x.encode('utf-8'))    188
.....:     digest = hasher.hexdigest()                   189
.....:     d = ZZ(digest, base=16)                        190
.....:     d = d.str(2)[-k2:]                             191
.....:     d = ZZ(d, base=2)                              192
.....:     return Z23(d)                                  193

```

We can then use Sage to apply this function to a large set of input values in order to plot visualizations of the distribution over the set $\{0, \dots, 22\}$ for various values of k_2 , by counting the number of times it maps onto each number in a loop of 100000 hashes. We get the following plot:



2173

2174 A third method that can sometimes be found in implementations is the so-called “**try-and-**
 2175 **increment**” method. To understand this method, we define an integer $z \in \mathbb{Z}$ from any hash
 2176 value $H(s)$ as we did in the previous methods, that is, we define $z = H(s)_0 \cdot 2^0 + H(s)_1 \cdot 2^1 +$
 2177 $\dots + H(s)_{k-1} \cdot 2^k$.

2178 Hashing into \mathbb{Z}_r is then achievable by first computing z , and then trying to see if $z \in \mathbb{Z}_r$. If
 2179 it is, then the hash is done; if not, the string s is modified in a deterministic way and the process
 2180 is repeated until a suitable number z is found. A suitable, deterministic modification could be
 2181 to concatenate the original string by some bit counter. A “try-and-increment” algorithm would
 then work like in algorithm 6.

check
reference

Algorithm 6 Hash-to- \mathbb{Z}_n

Require: $r \in \mathbb{Z}$ with $r.\text{nbits}() = k$ and $s \in \{0, 1\}^*$

procedure TRY-AND-INCREMENT(r, k, s)

$c \leftarrow 0$

repeat

$s' \leftarrow s || c_bits()$

$z \leftarrow H(s')_0 \cdot 2^0 + H(s')_1 \cdot 2^1 + \dots + H(s')_k \cdot 2^k$

$c \leftarrow c + 1$

until $z < r$

return x

end procedure

Ensure: $z \in \mathbb{Z}_r$

2182

2183 Depending on the parameters, this method can be very efficient. In fact, if k is sufficiently
 2184 large and r is close to 2^{k+1} , the probability for $z < r$ is very high and the repeat loop will almost
 2185 always be executed a single time only. A drawback is, however, that the probability of having
 2186 to execute the loop multiple times is not zero.

2187 Once some hash function into modular arithmetics is found, it can often be combined with
 2188 additional techniques to hash into more general finite cyclic groups. The following paragraphs
 2189 describe a few of those methods widely adopted in SNARK development.

2190 **Pedersen Hashes** The so-called **Pedersen hash function** [?] provides a way to map binary
 2191 inputs of fixed size k onto elements of finite cyclic groups that avoids discrete log relations

between the images as they occur in the naive approach XXX. Combining it with a classical hash function provides a hash function that maps strings of arbitrary length onto group elements.

To be more precise, let j be an integer, \mathbb{G} a finite cyclic group of order r and $\{g_1, \dots, g_j\} \subset \mathbb{G}$ a uniform randomly generated set of generators of \mathbb{G} . Then **Pedersen's hash function** is defined as follows:

$$H_{Ped} : (\mathbb{Z}_r)^j \rightarrow \mathbb{G} : (x_1, \dots, x_j) \mapsto \prod_{i=1}^j g_i^{x_i} \quad (4.25)$$

It can be shown that Pedersen's hash function is collision-resistant under the assumption that \mathbb{G} is a DL-A group. However, it is important to note that Pedersen hashes cannot be assumed to be **pseudorandom** and should therefore not be used where a hash function serves as an approximation of a random **oracle**. will these be explained in the initial chapters?

From an implementation perspective, it is important to derive the set of generators $\{g_1, \dots, g_j\}$ in such a way that they are as uniform and random as possible. In particular, any known discrete log relation between two generators, that is, any known $x \in \mathbb{Z}_r$ with $g_h = (g_i)^x$ must be avoided.

To see how Pedersen hashes can be used to define an actual hash-to-group function according to our definition, we can use any of the hash-to- \mathbb{Z}_r functions as we have derived them in equation 4.23.

MimC Hashes [?]

Pseudorandom Functions in DDH-A groups As noted in above, Pederson's hash function does not have the properties a random function and should therefore not be instantiated as such. To see an example of a random oracle function in groups where the decisional Diffie–Hellman construction is assumed to hold true, let \mathbb{G} be a DDH-A group of order r with generator g and $\{a_0, a_1, \dots, a_k\} \subset \mathbb{Z}_r^*$ a uniform randomly generated set of numbers invertible in modular r arithmetics. Then a pseudo-random function is given by the as follows:

$$F_{rand} : \{0, 1\}^{k+1} \rightarrow \mathbb{G} : (b_0, \dots, b_k) \mapsto g^{b_0 \cdot \prod_{i=1}^k a_i^{b_i}} \quad (4.26)$$

Of course, if $H : \{0, 1\}^* \rightarrow \{0, 1\}^{k+1}$ is a random oracle, then the concatenation of F_{rand} and H also defines a random oracle

$$H_{rand, \mathbb{G}} : \{0, 1\}^* \rightarrow \mathbb{G} : s \mapsto F_{rand}(H(s)) \quad (4.27)$$

4.2 Commutative Rings

In the previous section we have seen that integers are a commutative group with respect to integer addition, but as we know there are in fact two arithmetic operations defined on integers: addition and multiplication. However, in contrast to addition, multiplication does not define a group structure, given that integers generally don't have multiplicative inverses. Configurations like these constitute so-called **commutative rings with unit** and the following definition will make the structure explicit:

Definition 4.2.0.1 (Commutative ring with unit). A **commutative ring with unit** $(R, +, \cdot, 1)$ is a set R provided with two maps $+: R \times R \rightarrow R$ and $\cdot: R \times R \rightarrow R$, called **addition** and **multiplication** and an element $1 \in R$, called the **unit**, such that the following conditions hold:

- $(R, +)$ is a commutative group, where the neutral element is denoted with 0.
- **Commutativity of multiplication:** $r_1 \cdot r_2 = r_2 \cdot r_1$ for all $r_1, r_2 \in R$.

• **Multiplicative neutral unit** : $1 \cdot g = g$ for all $g \in R$.

• **Associativity**: For every $g_1, g_2, g_3 \in \mathbb{G}$ the equation $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$ holds.

• **Distributivity**: For all $g_1, g_2, g_3 \in R$ the distributive law $g_1 \cdot (g_2 + g_3) = g_1 \cdot g_2 + g_1 \cdot g_3$ holds.

If $(R, +, \cdot, 1)$ is a commutative ring with unit and $R' \subset R$ is a subset of R , such that the restriction of addition and multiplication to R' define a commutative ring with addition $+: R' \times R' \rightarrow R'$, multiplication $\cdot: R' \times R' \rightarrow R'$ and unit 1 on R' , then $(R', +, \cdot, 1)$ is called a **subring** of $(R, +, \cdot, 1)$.

Notation and Symbols 6. Since we are exclusively concerned with commutative rings in this book, we often just call them rings, keeping the notation of commutativity implicit. A set R with two maps $+$ and \cdot that satisfies all previously mentioned rules, except for the commutativity law of the multiplication is called a non-commutative ring.

If there is no risk of ambiguity (about what the addition and multiplication maps of a ring are), we frequently drop the symbols $+$ and \cdot and simply write R as notation for the ring, keeping those maps implicit. In this case we also say that R is of ring type, indicating that R is not simply a set but a set together with an addition and a multiplication map.

Example 51 (The ring of integers). The set \mathbb{Z} of integers with the usual addition and multiplication is the archetypical example of a commutative ring with unit 1.

sage: ZZ

194

Integer Ring

195

Example 52 (Underlying commutative group of a ring). Every commutative ring with unit $(R, +, \cdot, 1)$ gives rise to a group, if we disregard multiplication.

The following example is somewhat unusual, but we encourage you to think through it because it helps to detach the mind from familiar styles of computation and concentrate on the abstract algebraic explanation.

Example 53. Let $S := \{\bullet, \star, \odot, \otimes\}$ be a set that contains four elements, and let addition and multiplication on S be defined as follows:

\cup	\bullet	\star	\odot	\otimes
\bullet	\bullet	\star	\odot	\otimes
\star	\star	\odot	\otimes	\bullet
\odot	\odot	\otimes	\bullet	\star
\otimes	\otimes	\bullet	\star	\odot

\circ	\bullet	\star	\odot	\otimes
\bullet	\bullet	\bullet	\bullet	\bullet
\star	\bullet	\star	\odot	\otimes
\odot	\bullet	\odot	\bullet	\odot
\otimes	\bullet	\otimes	\odot	\star

Then (S, \cup, \circ, \star) is a ring with unit \star and zero \bullet . It therefore makes sense to ask for solutions to equations like this one: Find $x \in S$ such that

$$\otimes \circ (x \cup \odot) = \star$$

To see how such a “moonmath equation” can be solved, we have to keep in mind that rings behaves mostly like normal numbers when it comes to bracketing and computation rules. The only differences are the symbols, and the actual way to add and multiply them. With this in

mind, we solve the equation for x in the “usual way”¹:

$$\begin{array}{ll}
 \otimes \circ (x \cup \odot) = \star & \# \text{ apply the distributive law} \\
 \otimes \circ x \cup \otimes \circ \odot = \star & \# \otimes \circ \odot = \odot \\
 \otimes \circ x \cup \odot = \star & \# \text{ concatenate the } \cup \text{ inverse of } \odot \text{ to both sides} \\
 \otimes \circ x \cup \odot \cup -\odot = \star \cup -\odot & \# \odot \cup -\odot = \bullet \\
 \otimes \circ x \cup \bullet = \star \cup -\odot & \# \bullet \text{ is the } \cup \text{ neutral element} \\
 \otimes \circ x = \star \cup -\odot & \# \text{ for } \cup \text{ we have } -\odot = \odot \\
 \otimes \circ x = \star \cup \odot & \# \star \cup \odot = \otimes \\
 \otimes \circ x = \otimes & \# \text{ concatenate the } \circ \text{ inverse of } \otimes \text{ to both sides} \\
 (\otimes)^{-1} \circ \otimes \circ x = (\otimes)^{-1} \circ \otimes & \# \text{ multiply with the multiplicative inverse} \\
 \star \circ x = \star & \\
 x = \star &
 \end{array}$$

2255 So, even though this equation looked really alien at first glance, we could solve it basically
 2256 exactly the way we solve “normal” equations containing numbers.

2257 Note, however, that whenever a multiplicative inverse would be needed to solve an equation
 2258 in the usual way in a ring, things can be very different than most of us are used to. For example,
 2259 the following equation cannot be solved for x in the usual way, since there is no multiplicative
 2260 inverse for \odot in our ring.

$$\odot \circ x = \otimes \quad (4.28)$$

2261 We can confirm this by looking at the multiplication table to see that no such x exists.

2262 As another example, the following equation does not have a single solution but two: $x \in$
 2263 $\{\star, \otimes\}$.

$$\odot \circ x = \odot \quad (4.29)$$

2264 Having no solution or two solutions is certainly not something to expect from types like the
 2265 rational numbers \mathbb{Q} .

2266 *Example 54* (Ring of Polynomials). Considering the definition of polynomials from 3.4 again,
 2267 we notice that what we have informally called the type R of the coefficients must in fact be a
 2268 commutative ring with a unit, since we need addition, multiplication, commutativity and the
 2269 existence of a unit for $R[x]$ to have the properties we expect.

2270 In fact if we consider R to be a ring and we define addition and multiplication of polyno-
 2271 mials as in 3.24, the set $R[x]$ is a commutative ring with a unit, where the polynomial 1 is the
 2272 multiplicative unit. We call this ring the **ring of polynomials with coefficients in R** .

2273 **sage: ZZ['x']**

196

2274 **Univariate Polynomial Ring in x over Integer Ring**

197

2275 *Example 55* (Ring of modular n arithmetic). Let n be a modulus and $(\mathbb{Z}_n, +, \cdot)$ the set of all re-
 2276 mainder classes of integers modulo n , with the projection of integer addition and multiplication
 2277 as defined in 3.3. Then $(\mathbb{Z}_n, +, \cdot)$ is a commutative ring with unit 1.

¹Note that there are more efficient ways to solve this equation. The point of our computation is to show how the axioms of a ring can be used to solve the equation

2278 **sage: Integers(6)**
 2279 **Ring of integers modulo 6**

198

199

2280 *Example 56* (Binary Representations in Modular Arithmetic). TODO (Non unique)

2281 *Example 57* (Polynomial evaluation in the exponent of group generators). As we show in 6.2.3,
 2282 a key insights in many zero knowledge protocols is the ability to encode computations as poly-
 2283 nomials and then to hide the information of that computation by evaluating the polynomial "in
 2284 the exponent" of certain cryptographic groups 8.2.

To understand the underlying principle of this idea, consider the exponential map 44 again. If \mathbb{G} is a finite cyclic group of order n with generator $g \in \mathbb{G}$, then the ring structure of $(\mathbb{Z}_n, +, \cdot)$ corresponds to the group structure of \mathbb{G} in the following way:

$$g^{x+y} = g^x \cdot g^y \quad g^{x \cdot y} = (g^x)^y \quad \text{for all } x, y \in \mathbb{Z}_n \quad (4.30)$$

This correspondence allows polynomials with coefficients in \mathbb{Z}_n to be evaluated "in the exponent" of a group generator. To see what this means, let $p \in \mathbb{Z}_n[x]$ be a polynomial with $p(x) = a_m \cdot x^m + a_{m-1}x^{m-1} + \dots + a_1x + a_0$ and let $s \in \mathbb{Z}_n$ be an evaluation point. Then the previously defined exponential laws 4.30 imply the following identity:

$$\begin{aligned} g^{p(s)} &= g^{a_m \cdot s^m + a_{m-1}s^{m-1} + \dots + a_1s + a_0} \\ &= \left(g^{s^m}\right)^{a_m} \cdot \left(g^{s^{m-1}}\right)^{a_{m-1}} \cdot \dots \cdot (g^s)^{a_1} \cdot g^{a_0} \end{aligned} \quad (4.31)$$

2285 Utilizing these identities, it is possible to evaluate any polynomial p of degree $\deg(p) \leq m$ at a
 2286 "secret" evaluation point s in the exponent of g without any knowledge about s , assuming that
 2287 \mathbb{G} is a DL-group. To see this assume that the set $\{g, g^s, g^{s^2}, \dots, g^{s^m}\}$ is given, but s is unknown.
 2288 Then $g^{p(s)}$ can be computed using 4.31, however it is not feasible to compute s .

Example 58. To give an example of the evaluation of a polynomial in the exponent of a finite cyclic group, consider the exponential map from example 36:

$$3^{(\cdot)} : \mathbb{Z}_4 \rightarrow \mathbb{Z}_5^* \quad x \mapsto 3^x$$

Choosing the polynomial $p(x) = 2x^2 + 3x + 1$ from $\mathbb{Z}_4[x]$, we first evaluate the polynomial at the point $s = 2$ and then write the result into the exponent 3 as follows:

$$\begin{aligned} 3^{p(2)} &= 3^{2 \cdot 2^2 + 3 \cdot 2 + 1} \\ &= 3^{2 \cdot 0 + 2 + 1} \\ &= 3^3 \\ &= 2 \end{aligned}$$

This was possible, because we had access to the evaluation point 2. On the other hand, if we only had access to the set $\{3, 4, 1\}$ and we knew that this set represents the set $\{3, 3^s, 3^{s^2}\}$ for some secret value s , we could evaluate p at the point s in the exponent of 3 as

$$\begin{aligned} 3^{p(s)} &= 1^2 \cdot 4^3 \cdot 3^1 \\ &= 1 \cdot 4 \cdot 3 \\ &= 2 \end{aligned}$$

2289 Both computations agree, since the secret point s was equal to 2 in this example. However the
 2290 second evaluation was possible without any knowledge about s .

Hashing to Commutative Rings As we have seen in XXX, various constructions for hashing-to-groups are known and used in applications. As commutative rings are **Abelian groups**, when we disregard the multiplicative structure, hash-to-group constructions can be applied for hashing into commutative rings, too. This is possible in general, as the **codomain** of a general hash function $\{0, 1\}^*$ is just the set of binary strings of arbitrary but finite length, which has no algebraic structure that the hash function must respect.

add reference

codomain groups

4.3 Fields

We started this chapter with the definition of a group (section 4.1), which we then expanded into the definition of a commutative ring with a unit (section 4.2). Such rings generalize the behavior of integers. In this section, we will look at those special cases of commutative rings where every element other than the neutral element of addition has a multiplicative inverse. Those structures behave very much like the rational numbers \mathbb{Q} . Rational numbers are, in a sense, an extension of the ring of integers, that is, they are constructed by including newly defined multiplicative inverses (fractions) to the integers.

Definition 4.3.0.1 (Field). A **field** $(\mathbb{F}, +, \cdot, 1)$ is a set \mathbb{F} provided with two maps $+: \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ and $\cdot: \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$, called **addition** and **multiplication**, such that the following conditions hold:

- $(\mathbb{F}, +)$ is a commutative group, where the neutral element is denoted by 0.
- $(\mathbb{F} \setminus \{0\}, \cdot)$ is a commutative group, where the neutral element is denoted by 1.
- (Distributivity) For all $g_1, g_2, g_3 \in \mathbb{F}$ the distributive law $g_1 \cdot (g_2 + g_3) = g_1 \cdot g_2 + g_1 \cdot g_3$ holds.

If $(\mathbb{F}, +, \cdot)$ is a field and $\mathbb{F}' \subset \mathbb{F}$ is a subset of \mathbb{F} , such that the restriction of addition and multiplication to \mathbb{F}' define a field with addition $+: \mathbb{F}' \times \mathbb{F}' \rightarrow \mathbb{F}'$ and multiplication $\cdot: \mathbb{F}' \times \mathbb{F}' \rightarrow \mathbb{F}'$ on \mathbb{F}' , then $(\mathbb{F}', +, \cdot)$ is called a **subfield** of $(\mathbb{F}, +, \cdot)$ and $(\mathbb{F}, +, \cdot)$ is called an **extension field** of $(\mathbb{F}', +, \cdot)$.

Notation and Symbols 7. If there is no risk of ambiguity (about what the addition and multiplication maps of a ring are), we frequently drop the symbols $+$ and \cdot and simply write \mathbb{F} as notation for the field, keeping those maps implicit. In this case we also say that \mathbb{F} is of field type, indicating that \mathbb{F} is not simply a set but a set together with an addition and a multiplication map.

Moreover, we use the symbol \mathbb{F}^* to describe the multiplicative group of the field, that is, the set of elements with multiplication as the group law, excluding the neutral element of addition.

The **characteristic** of a field \mathbb{F} , represented as $\text{char}(\mathbb{F})$, is the smallest natural number $n \geq 1$ for which the n -fold sum of 1 equals zero, i.e. for which $\sum_{i=1}^n 1 = 0$. If such an $n > 0$ exists, the field is also said to have a **finite characteristic**. If, on the other hand, every finite sum of 1 is such that it is not equal to zero, then the field is defined to have characteristic 0. **S: Tried to disambiguate the scope of negation between 1. “It is true of every finite sum of 1 that it is not equal to 0” and 2. “It is not true of every finite sum of 1 that it is equal to 0” From the example below, it looks like 1. is the intended meaning here, correct?**

Check change of wording

Example 59 (Field of rational numbers). Probably the best known example of a field is the set of rational numbers \mathbb{Q} together with the usual definition of addition, subtraction, multiplication and division. Since there is no natural number $n \in \mathbb{N}$, such that $\sum_{j=0}^n 1 = 0$ in the set of rational numbers, the characteristic of the field \mathbb{Q} is given by $\text{char}(\mathbb{Q}) = 0$.

2333 **sage:** QQ 200
 2334 **Rational Field** 201

2335 *Example 60* (Field with two elements). It can be shown that, in any field, the neutral element
 2336 of addition 0 must be different from the neutral element of multiplication 1, that is, $0 \neq 1$
 2337 always holds in a field. From this, it follows that the smallest field must contain at least two
 2338 elements. As the following addition and multiplication tables show, there is indeed a field with
 2339 two elements, which is usually called \mathbb{F}_2 :

2340 Let $\mathbb{F}_2 := \{0, 1\}$ be a set that contains two elements and let addition and multiplication on
 2341 \mathbb{F}_2 be defined as follows:

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}
 \qquad
 \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

2343 Since $1 + 1 = 0$ in the field \mathbb{F}_2 , we know that the characteristic of \mathbb{F}_2 given by $\text{char}(\mathbb{F}_2) = 2$.

2344 **sage:** F2 = GF(2) 202
 2345 **sage:** F2(1) # Get an element from GF(2) 203
 2346 1 204
 2347 **sage:** F2(1) + F2(1) # Addition 205
 2348 0 206
 2349 **sage:** F2(1) / F2(1) # Division 207
 2350 1 208

2351 *Exercise 43.* Consider the ring of modular 5 arithmetics $(\mathbb{Z}_5, +, \cdot)$ from example 14. Show that
 2352 $(\mathbb{Z}_5, +, \cdot)$ is a field. What is the characteristic of \mathbb{Z}_5 ? Proof that the equation $a \cdot x = b$ has only a
 2353 single solution $x \in \mathbb{Z}_5$ for any given $a, b \in \mathbb{Z}_5^*$.

2354 *Exercise 44.* Consider the ring of modular 6 arithmetics $(\mathbb{Z}_6, +, \cdot)$ from example 9. Show that
 2355 $(\mathbb{Z}_6, +, \cdot)$ is not a field.

2356 4.3.1 Prime fields

2357 As we have seen in the various examples of the previous sections, modular arithmetics behaves
 2358 similarly to the ordinary arithmetics of integers in many ways. This is due to the fact that
 2359 remainder class sets \mathbb{Z}_n are commutative rings with units 55.

2360 However, we have also seen in 36 that, whenever the modulus is a prime number, every
 2361 remainder class other than the zero class has a modular multiplicative inverse. This is an impor-
 2362 tant observation, since it immediately implies that, in case the modulus is a prime number, the
 2363 remainder class set \mathbb{Z}_n is not just a ring but actually a **field**. Moreover, since $\sum_{j=0}^n 1 = 0$ in \mathbb{Z}_n ,
 2364 we know that those fields have the finite characteristic n .

2365 *Notation and Symbols 8* (Prime Fields). Let $p \in \mathbb{P}$ be a prime number and $(\mathbb{Z}_p, +, \cdot)$ the ring
 2366 of modular p arithmetics 55. To distinguish prime fields from arbitrary modular arithmetic
 2367 rings, we write $(\mathbb{F}_p, +, \cdot)$ for the ring of modular p arithmetics and call it the **prime field** of
 2368 characteristic p .

2369 Prime fields are the foundation for many of the contemporary algebra-based cryptographic
 2370 systems, as they have many desirable properties. One of them is that any prime field of char-
 2371 acteristic p contain exactly p elements which can be represented on a computer with not more

then $\log_2(p)$ many bits. On the other hand fields like the rational numbers, require a potentially unbounded amount of bits for any full-precision representation.

Since prime fields are special cases of modular arithmetic rings, addition and multiplication can be computed by first doing normal integer addition and multiplication, and then considering the remainder in Euclidean division by p as the result. For any prime field element $x \in \mathbb{F}_p$, its additive inverse (the negative) is given by $-x = p - x \bmod p$. For $x \neq 0$ the multiplicative inverse always exists and is given by $x^{-1} = x^{p-2}$. Division is then defined by multiplication with the multiplicative inverse as explained in 3.3. Alternatively the multiplicative inverse can be computed using the Extended Euclidean Algorithm as explained in 3.20.

Example 61. The smallest field is the field \mathbb{F}_2 of characteristic 2 as we have seen in example 60. It is the prime field of the prime number 2.

Example 62. To summarize the basic aspects of computation in prime fields, let us consider the prime field \mathbb{F}_5 and simplify the following expression:

$$\left(\frac{2}{3} - 2\right) \cdot 2$$

A first thing to note is that since \mathbb{F}_5 is a field, all rules are identical to the rules we learned in school when we were dealing with rational, real or complex numbers. This means we can use e.g. bracketing (distributivity) or addition as usual:

$$\begin{aligned} \left(\frac{2}{3} - 2\right) \cdot 2 &= \frac{2}{3} \cdot 2 - 2 \cdot 2 && \# \text{ distributive law} \\ &= \frac{2 \cdot 2}{3} - 2 \cdot 2 && 4 \bmod 5 = 4 \\ &= \frac{4}{3} - 4 && \# \text{ multiplicative inverse of 3 is } 3^{5-2} \bmod 5 = 2 \\ &= 4 \cdot 2 - 4 && \# \text{ additive inverse of 4 is } 5 - 4 = 1 \\ &= 4 \cdot 2 + 1 && 8 \bmod 5 = 3 \\ &= 3 + 1 && 4 \bmod 5 = 4 \\ &= 4 \end{aligned}$$

In this computation, we computed the multiplicative inverse of 3 using the identity $x^{-1} = x^{p-2}$ in a prime field. This is impractical for large prime numbers. Recall that another way of computing the multiplicative inverse is the Extended Euclidean Algorithm (see 3.11). To refresh our memory, the algorithm solves the equation $x^{-1} \cdot 3 + t \cdot 5 = 1$, for x^{-1} , even though in this case t is irrelevant. We get:

k	r_k	x_k^{-1}	t_k
0	3	1	.
1	5	0	.
2	3	1	.
3	2	-1	.
4	1	2	.

So the multiplicative inverse of 3 in \mathbb{Z}_5 is 2, and, indeed, if we compute the product of 3 with its multiplicative inverse 2 we get the neutral element 1 in \mathbb{F}_5 .

Exercise 45 (Prime field \mathbb{F}_3). Construct the addition and multiplication table of the prime field \mathbb{F}_3 .

Exercise 46. Consider the prime field \mathbb{F}_{13} . Find the set of all pairs $(x, y) \in \mathbb{F}_{13} \times \mathbb{F}_{13}$ that satisfy the equation

$$x^2 + y^2 = 1 + 7 \cdot x^2 \cdot y^2$$

Square Roots As we know from integer arithmetics some integers like 4 or 9 are squares of other integers, as for example $4 = 2^2$ and $9 = 3^2$. However we also know that not all integers are squares of other integers, as for example there is no integers $x \in \mathbb{Z}$ such that $x^2 = 2$. If an integer a is square of another integer b , then it make sense to define the square-root of a to be b .

In the context of prime fields an element that is a square of another element is also called a **quadratic residue** and an element that is not a square of another element is called a **quadratic non-residue**. Those notations are of particular importance in our studies on elliptic curves in chapter 5, as only square numbers can actually be points on an elliptic curve.

To make the intuition of quadratic residues and their roots precise, let $p \in \mathbb{P}$ be a prime number and \mathbb{F}_p its associated prime field. Then a number $x \in \mathbb{F}_p$ is called a **square root** of another number $y \in \mathbb{F}_p$, if x is a solution to the following equation:

$$x^2 = y \tag{4.32}$$

In this case, y is called a **quadratic residue**. On the other hand, if y is given and the quadratic equation has no solution x , we call y a **quadratic non-residue**. For any $y \in \mathbb{F}_p$, we denote the set of all square roots of y in the prime field \mathbb{F}_p as follows:

$$\sqrt{y} := \{x \in \mathbb{F}_p \mid x^2 = y\} \tag{4.33}$$

Informally speaking, quadratic residues are numbers such that we can take the square root of them, while quadratic non-residues are numbers that don't have square roots. The situation therefore parallels the familiar case of integers, where some integers like 4 or 9 have square roots and others like 2 or 3 don't (as integers).

If y is a quadratic non-residue, then $\sqrt{y} = \emptyset$ (an empty set), and if $y = 0$, then $\sqrt{y} = \{0\}$. Moreover if $y \neq 0$ is a quadratic residue then it has precisely two roots $\sqrt{y} = \{x, p - x\}$ for some $x \in \mathbb{F}_p$. We adopt the convention to call the smaller one (when interpreted as an integer) as the **positive** square root and the larger one as the **negative**. If $p \in \mathbb{P}_{\geq 3}$ is an odd prime number with associated prime field \mathbb{F}_p , then there are precisely $(p + 1)/2$ many quadratic residues and $(p - 1)/2$ quadratic non residues.

Example 63 (Quadratic residues and roots in \mathbb{F}_5). Let us consider the prime field \mathbb{F}_5 14 again. All square numbers can be found on the main diagonal of the multiplication table in example 14. As you can see, in \mathbb{F}_5 only the numbers 0, 1 and 4 have square roots and we get $\sqrt{0} = \{0\}$, $\sqrt{1} = \{1, 4\}$, $\sqrt{2} = \emptyset$, $\sqrt{3} = \emptyset$ and $\sqrt{4} = \{2, 3\}$. The numbers 0, 1 and 4 are therefore quadratic residues, while the numbers 2 and 3 are quadratic non-residues.

In order to describe whether an element of a prime field is a square number or not, the so-called **Legendre symbol** can sometimes be found in the literature, which is why we will summerize it here:

Let $p \in \mathbb{P}$ be a prime number and $y \in \mathbb{F}_p$ an element from the associated prime field. Then the *Legendre symbol* of y is defined as follows:

$$\left(\frac{y}{p}\right) := \begin{cases} 1 & \text{if } y \text{ has square roots} \\ -1 & \text{if } y \text{ has no square roots} \\ 0 & \text{if } y = 0 \end{cases} \tag{4.34}$$

2427 *Example 64.* Looking at the quadratic residues and non residues in \mathbb{F}_5 from example 14 again,
 2428 we can deduce the following Legendre symbols, from example 63.

$$\left(\frac{0}{5}\right) = 0, \quad \left(\frac{1}{5}\right) = 1, \quad \left(\frac{2}{5}\right) = -1, \quad \left(\frac{3}{5}\right) = -1, \quad \left(\frac{4}{5}\right) = 1.$$

2429 The Legendre symbol provides a criterion to decide whether or not an element from a prime
 2430 field has a quadratic root or not. This, however, is not just of theoretical use: The so-called
 2431 **Euler criterion** provides a compact way to actually compute the Legendre symbol. To see that,
 2432 let $p \in \mathbb{P}_{\geq 3}$ be an odd prime number and $y \in \mathbb{F}_p$. Then the Legendre symbol can be computed
 2433 as follows:

$$\left(\frac{y}{p}\right) = y^{\frac{p-1}{2}}. \quad (4.35)$$

Example 65. Looking at the quadratic residues and non residues in \mathbb{F}_5 from example 63 again,
 we can compute the following Legendre symbols using the Euler criterion:

$$\begin{aligned} \left(\frac{0}{5}\right) &= 0^{\frac{5-1}{2}} = 0^2 = 0 \\ \left(\frac{1}{5}\right) &= 1^{\frac{5-1}{2}} = 1^2 = 1 \\ \left(\frac{2}{5}\right) &= 2^{\frac{5-1}{2}} = 2^2 = 4 = -1 \\ \left(\frac{3}{5}\right) &= 3^{\frac{5-1}{2}} = 3^2 = 4 = -1 \\ \left(\frac{4}{5}\right) &= 4^{\frac{5-1}{2}} = 4^2 = 1 \end{aligned}$$

2434 Hashing into prime fields

2435 An important problem in SNARK development is the ability to hash to (various subsets) of
 2436 elliptic curves. As we will see in XXX, those curves are often defined over prime fields, and
 2437 hashing to a curve might start with hashing to the prime field. It is therefore important to
 2438 understand how to hash into prime fields.

add refer-
ence

2439 On pages 55–58, we looked at a few methods of hashing into the residue class rings \mathbb{Z}_n for
 2440 arbitrary $n > 1$. As prime fields are just special instances of those rings, all methods for hashing
 2441 into \mathbb{Z}_n functions can be used for hashing into prime fields, too.

check
reference

2442 **MiMC Hash functions** As we will see in XXX,

2443 To define a MiMC hash function, let \mathbb{F}_p be a prime field with prime modulus $p \in \mathbb{P}$, let n be
 2444 the smallest natural number, such that $\gcd(n, p-1) = 1$. Let r be the smallest integer greater
 2445 than or equal to $\frac{\log(p)}{\log_2(3)}$ and let $C = \{c_i \in \mathbb{F}_p \mid 0 \leq i \leq r\}$ be a set of randomly generated field
 2446 elements. The definition of the MiMC hash function then starts with an invertible map

$$E(\cdot) : \mathbb{F}_p \rightarrow \mathbb{F}_p \quad x \mapsto F_{r-1}(\cdots F_1(F_0(x)) \cdots) \quad (4.36)$$

2447 where $F_i(x) = (x + c_i)^n$

4.3.2 Prime Field Extensions

Prime fields, as defined in the previous section, are basic building blocks in cryptography. However, as we will see in chapter 8 so-called pairing-based SNARK systems are crucially dependent on certain group pairings 4.6 defined on elliptic curves over **prime field extensions**. In this section we therefore introduce those extensions.

Given some prime number $p \in \mathbb{P}$, a natural number $m \in \mathbb{N}$ and an irreducible polynomial $P \in \mathbb{F}_p[x]$ of degree m with coefficients from the prime field \mathbb{F}_p , then a prime field extension $(\mathbb{F}_{p^m}, +, \cdot)$ is defined as follows:

The set \mathbb{F}_{p^m} of the prime field extension is given by the set of all polynomials with a degree less than m :

$$\mathbb{F}_{p^m} := \{a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \dots + a_1x + a_0 \mid a_i \in \mathbb{F}_p\} \quad (4.37)$$

The addition law of the prime field extension \mathbb{F}_{p^m} is given by the usual addition of polynomials as defined in 3.24:

$$+ : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \rightarrow \mathbb{F}_{p^m}, (\sum_{j=0}^m a_j x^j, \sum_{j=0}^m b_j x^j) \mapsto \sum_{j=0}^m (a_j + b_j) x^j \quad (4.38)$$

The multiplication law of the prime field extension \mathbb{F}_{p^m} is given by first multiplying the two polynomials as defined in 3.25. The result is then divided by the result by the irreducible polynomial P and the remainder of that division is the result:

$$\cdot : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \rightarrow \mathbb{F}_{p^m}, (\sum_{j=0}^m a_j x^j, \sum_{j=0}^m b_j x^j) \mapsto (\sum_{n=0}^{2m} \sum_{i=0}^n a_i b_{n-i} x^n) \bmod P \quad (4.39)$$

The neutral element of the additive group $(\mathbb{F}_{p^m}, +)$ is given by the polynomial 0 and the additive inverse is given by the polynomial with all negative coefficients. The neutral element of the multiplicative group $(\mathbb{F}_{p^m}^*, \cdot)$ is given by the polynomial 1 and the multiplicative inverse can be computed by the extended Euclidean algorithm.

We can easily see from the definition of \mathbb{F}_{p^m} that the field is of characteristic p , since the multiplicative neutral element 1 is equivalent to the multiplicative element 1 from the underlying prime field, and hence $\sum_{j=0}^p 1 = 0$. Moreover, \mathbb{F}_{p^m} is finite and contains p^m many elements, since elements are polynomials of degree $< m$, and every coefficient a_j can have p many different values. In addition, we see that the prime field \mathbb{F}_p is a subfield of \mathbb{F}_{p^m} that occurs when we restrict the elements of \mathbb{F}_{p^m} to polynomials of degree zero.

One key point is that the construction of \mathbb{F}_{p^m} depends on the choice of an irreducible polynomial, and, in fact, different choices will give different multiplication tables, since the remainders from dividing a product by P will be different.

It can, however, be shown that the fields for different choices of P are **isomorphic**, which means that there is a one-to-one correspondence between all of them. Consequently, from an abstract point of view, they are the same thing. From an implementations point of view, however, some choices are preferable to others because they allow for faster computations.

Remark 4. Similar to the way prime fields \mathbb{F}_p are generated by starting with the ring of integers and then dividing by a prime number p and keeping the remainder, prime field extensions \mathbb{F}_{p^m} are generated by starting with the ring $\mathbb{F}_p[x]$ of polynomials and then dividing them by an irreducible polynomial of degree m and keeping the remainder.

In fact it can be shown that \mathbb{F}_{p^m} is the set of all remainders when dividing any polynomial $Q \in \mathbb{F}_p[x]$ by the reducible polynomial P . This is analogous to how \mathbb{F}_p is the set of all remainders when dividing integers by p .

Any field \mathbb{F}_{p^m} constructed in the above manner is a field extension of \mathbb{F}_p . To be more general, a field $\mathbb{F}_{p^{m_2}}$ is a field extension of a field $\mathbb{F}_{p^{m_1}}$, if and only if m_1 divides m_2 . From

2489 this, we can deduce that, for any given fixed prime number, there are nested sequences of fields
 2490 whenever the power m_j divides the power m_{j+1} , such that $\mathbb{F}_{p^{m_j}}$ is a subfield of $\mathbb{F}_{p^{m_{j+1}}}$:

$$\mathbb{F}_p \subset \mathbb{F}_{p^{m_1}} \subset \cdots \subset \mathbb{F}_{p^{m_k}} \quad (4.40)$$

2491 To get a more intuitive picture of this, we construct an extension field of the prime field \mathbb{F}_3
 2492 in the following example, and we can see how \mathbb{F}_3 sits inside that extension field.

Example 66 (The Extension field \mathbb{F}_{3^2}). In exercise 45 we have constructed the prime field \mathbb{F}_3 . In this example, we apply the definition of a field extension 4.37 to construct the extension field \mathbb{F}_{3^2} . We start by choosing an irreducible polynomial of degree 2 with coefficients in \mathbb{F}_3 . We try $P(t) = t^2 + 1$. Possibly the fastest way to show that P is indeed irreducible is to just insert all elements from \mathbb{F}_3 to see if the result is ever zero. We compute as follows:

$$P(0) = 0^2 + 1 = 1$$

$$P(1) = 1^2 + 1 = 2$$

$$P(2) = 2^2 + 1 = 1 + 1 = 2$$

This implies that P is irreducible. The set \mathbb{F}_{3^2} contains all polynomials of degrees lower than two, with coefficients in \mathbb{F}_3 , which are precisely as listed below:

$$\mathbb{F}_{3^2} = \{0, 1, 2, t, t+1, t+2, 2t, 2t+1, 2t+2\}$$

2493 As expected, our extension field contains 9 elements. Addition is defined as addition of poly-
 2494 nomials; for example $(t+2) + (2t+2) = (1+2)t + (2+2) = 1$. Doing this computation for all
 2495 elements gives the following addition table

+	0	1	2	t	t+1	t+2	2t	2t+1	2t+2
0	0	1	2	t	t+1	t+2	2t	2t+1	2t+2
1	1	2	0	t+1	t+2	t	2t+1	2t+2	2t
2	2	0	1	t+2	t	t+1	2t+2	2t	2t+1
t	t	t+1	t+2	2t	2t+1	2t+2	0	1	2
t+1	t+1	t+2	t	2t+1	2t+2	2t	1	2	0
t+2	t+2	t	t+1	2t+2	2t	2t+1	2	0	1
2t	2t	2t+1	2t+2	0	1	2	t	t+1	t+2
2t+1	2t+1	2t+2	2t	1	2	0	t+1	t+2	t
2t+2	2t+2	2t	2t+1	2	0	1	t+2	t	t+1

2497 As we can see, the group $(\mathbb{F}_3, +)$ is a subgroup of the group $(\mathbb{F}_{3^2}, +)$, obtained by only consid-
 2498 ering the first three rows and columns of this table.

2499 As it was the case in previous examples, we can use the table to deduce the negative of any
 2500 element from \mathbb{F}_{3^2} . For example, in \mathbb{F}_{3^2} we have $-(2t+1) = t+2$, since $(2t+1) + (t+2) = 0$

Multiplication needs a bit more computation, as we first have to multiply the polynomials, and whenever the result has a degree ≥ 2 , we have to divide it by P and keep the remainder. To see how this works, let us compute the product of $t+2$ and $2t+2$ in \mathbb{F}_{3^2} :

$$\begin{aligned} (t+2) \cdot (2t+2) &= (2t^2 + 2t + t + 1) \bmod (t^2 + 1) \\ &= (2t^2 + 1) \bmod (t^2 + 1) & \# 2t^2 + 1 : t^2 + 1 &= 2 + \frac{2}{t^2 + 1} \\ &= 2 \end{aligned}$$

2501 This means that the product of $t+2$ and $2t+2$ in \mathbb{F}_{3^2} is 2. Performing this computation for all
 2502 elements gives the following multiplication table:

·	0	1	2	t	t+1	t+2	2t	2t+1	2t+2
0	0	0	0	0	0	0	0	0	0
1	0	1	2	t	t+1	t+2	2t	2t+1	2t+2
2	0	2	1	2t	2t+2	2t+1	t	t+2	t+1
t	0	t	2t	2	t+2	2t+2	1	t+1	2t+1
t+1	0	t+1	2t+2	t+2	2t	1	2t+1	2	t
t+2	0	t+2	2t+1	2t+2	1	t	t+1	2t	2
2t	0	2t	t	1	2t+1	t+1	2	2t+2	t+2
2t+1	0	2t+1	t+2	t+1	2	2t	2t+2	t	1
2t+2	0	2t+2	t+1	2t+1	t	2	t+2	1	2t

As it was the case in previous examples, we can use the table to deduce the multiplicative inverse of any non-zero element from \mathbb{F}_{3^2} . For example, in \mathbb{F}_{3^2} we have $(2t+1)^{-1} = 2t+2$, since $(2t+1) \cdot (2t+2) = 1$.

From the multiplication table, we can also see that the only quadratic residues in \mathbb{F}_{3^2} are from the set $\{0, 1, 2, t, 2t\}$, with $\sqrt{0} = \{0\}$, $\sqrt{1} = \{1, 2\}$, $\sqrt{2} = \{t, 2t\}$, $\sqrt{t} = \{t+2, 2t+1\}$ and $\sqrt{2t} = \{t+1, 2t+2\}$.

Since \mathbb{F}_{3^2} is a field, we can solve equations as we would for other fields, (such as rational numbers). To see that, let us find all $x \in \mathbb{F}_{3^2}$ that solve the quadratic equation $(t+1)(x^2 + (2t+2)) = 2$. We compute as follows:

$$\begin{aligned}
 (t+1)(x^2 + (2t+2)) &= 2 && \# 2 \text{ distributive law} \\
 (t+1)x^2 + (t+1)(2t+2) &= 2 \\
 (t+1)x^2 + (t) &= 2 && \# 2 \text{ add the additive inverse of } t \\
 (t+1)x^2 + (t) + (2t) &= (2) + (2t) \\
 (t+1)x^2 &= 2t+2 && \# \text{ multiply with the multiplicative invers of } t+1 \\
 (t+2)(t+1)x^2 &= (t+2)(2t+2) && \# \text{ multiply with the multiplicative invers of } t+1 \\
 x^2 &= 2 && \# 2 \text{ is quadratic residue. Take the roots.} \\
 x &\in \{t, 2t\}
 \end{aligned}$$

Computations in extension fields are arguably on the edge of what can reasonably be done with pen and paper. Fortunately, Sage provides us with a simple way to do the computations.

```

sage: Z3 = GF(3) # prime field
sage: Z3t.<t> = Z3[] # polynomials over Z3
sage: P = Z3t(t^2+1)
sage: P.is_irreducible()
True
sage: F3_2.<t> = GF(3^2, name='t', modulus=P)
sage: F3_2
Finite Field in t of size 3^2
sage: F3_2(t+2)*F3_2(2*t+2) == F3_2(2)
True
sage: F3_2(2*t+2)^(-1) # multiplicative inverse
2*t + 1
sage: # verify our solution to (t+1)(x^2 + (2t+2)) = 2
sage: F3_2(t+1)*(F3_2(t)**2 + F3_2(2*t+2)) == F3_2(2)
True

```

```

2527 sage: F3_2(t+1)*(F3_2(2*t)**2 + F3_2(2*t+2)) == F3_2(2)      224
2528 True                                                            225

```

2529 *Exercise 47.* Consider the extension field \mathbb{F}_{32} from the previous example and find all pairs of
 2530 elements $(x, y) \in \mathbb{F}_{32}$, for which the following equation holds:

$$y^2 = x^3 + 4 \quad (4.41)$$

2531 *Exercise 48.* Show that the polynomial $P = x^3 + x + 1$ from $\mathbb{F}_5[x]$ is irreducible. Then consider
 2532 the extension field \mathbb{F}_{5^3} defined relative to P . Compute the multiplicative inverse of $(2t^2 + 4) \in$
 2533 \mathbb{F}_{5^3} using the extended Euclidean algorithm. Then find all $x \in \mathbb{F}_{5^3}$ that solve the following
 2534 equation:

$$(2t^2 + 4)(x - (t^2 + 4t + 2)) = (2t + 3) \quad (4.42)$$

2535 **Hashing into extension fields** On page 67, we have seen how to hash into prime fields. As
 2536 elements of extension fields can be seen as polynomials over prime fields, hashing into extension
 2537 fields is therefore possible if every coefficient of the polynomial is hashed independently.

check
reference

2538 4.4 Projective Planes

2539 Projective planes are particular geometric constructs defined over a given field. In a sense,
 2540 projective planes extend the concept of the ordinary Euclidean plane by including “points at
 2541 infinity.”

2542 Such an inclusion of infinity points makes projective planes particularly useful in the de-
 2543 scription of elliptic curves, as the description of such a curve in an ordinary plane needs an
 2544 additional symbol “the point at infinity” to give the set of points on the curve the structure of
 2545 a group. Translating the curve into projective geometry includes this “point at infinity” more
 2546 naturally into the set of all points on a projective plane.

2547 To understand the idea of constructing of projective planes, note that in an ordinary Eu-
 2548 clidean plane, two lines either intersect in a single point or are parallel. In the latter case, both
 2549 lines are either the same, that is, they intersect in all points, or do not intersect at all. A projec-
 2550 tive plane can then be thought of as an ordinary plane, but equipped with additional “point at
 2551 infinity” such that two different lines always intersect in a single point. Parallel lines intersect
 2552 “at infinity”.

2553 To be more precise, let \mathbb{F} be a field, $\mathbb{F}^3 := \mathbb{F} \times \mathbb{F} \times \mathbb{F}$ the set of all three tuples over \mathbb{F} and
 2554 $x \in \mathbb{F}^3$ with $x = (X, Y, Z)$. Then there is exactly one *line* in \mathbb{F}^3 that intersects both $(0, 0, 0)$ and
 2555 x . This line is given as follows:

$$[X : Y : Z] := \{(k \cdot X, k \cdot Y, k \cdot Z) \mid k \in \mathbb{F}\} \quad (4.43)$$

2556 A **point** in the **projective plane** over \mathbb{F} is defined as such a **line**, and the projective plane is the
 2557 set of all such points:

$$\mathbb{FP}^2 := \{[X : Y : Z] \mid (X, Y, Z) \in \mathbb{F}^3 \text{ with } (X, Y, Z) \neq (0, 0, 0)\} \quad (4.44)$$

2558 It can be shown that a projective plane over a finite field \mathbb{F}_{p^m} contains $p^{2m} + p^m + 1$ number of
 2559 elements.

2560 To understand why $[X : Y : Z]$ is called a line, consider the situation where the underlying
 2561 field \mathbb{F} is the set of real numbers \mathbb{R} . In this case, \mathbb{R}^3 can be seen as the three-dimensional space,

and $[X : Y : Z]$ is an ordinary line in this 3-dimensional space that intersects zero and the point with coordinates X, Y and Z .

The key observation here is that points in the projective plane are lines in the 3-dimensional space \mathbb{F}^3 . Additionally, for finite fields, the terms **space** and **line** share very little visual similarity with their counterparts over the set of real numbers.

It follows from this that points $[X : Y : Z] \in \mathbb{F}\mathbb{P}^2$ are not simply described by fixed coordinates (X, Y, Z) , but by **sets of coordinates**, where two different coordinates (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) describe the same point if and only if there is some field element k such that $(X_1, Y_1, Z_1) = (k \cdot X_2, k \cdot Y_2, k \cdot Z_2)$. Points $[X : Y : Z]$ are called **projective coordinates**.

Notation and Symbols 9 (Projective coordinates). Projective coordinates of the form $[X : Y : 1]$ are descriptions of so-called **affine points**. Projective coordinates of the form $[X : Y : 0]$ are descriptions of so-called **points at infinity**. In particular, the projective coordinate $[1 : 0 : 0]$ describes the so-called **line at infinity**.

Example 67. Consider the field \mathbb{F}_3 from example XXX. As this field only contains three elements, it does not take too much effort to construct its associated projective plane $\mathbb{F}_3\mathbb{P}^2$, as we know that it only contains 13 elements.

To find $\mathbb{F}_3\mathbb{P}^2$, we have to compute the set of all lines in $\mathbb{F}_3 \times \mathbb{F}_3 \times \mathbb{F}_3$ that intersect $(0, 0, 0)$.

add reference

Since those lines are parameterized by tuples (x_1, x_2, x_3) , we compute as follows:

$$\begin{aligned}
[0 : 0 : 1] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(0, 0, 1), (0, 0, 2)\} \\
[0 : 0 : 2] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(0, 0, 2), (0, 0, 1)\} = [0 : 0 : 1] \\
[0 : 1 : 0] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(0, 1, 0), (0, 2, 0)\} \\
[0 : 1 : 1] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(0, 1, 1), (0, 2, 2)\} \\
[0 : 1 : 2] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(0, 1, 2), (0, 2, 1)\} \\
[0 : 2 : 0] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(0, 2, 0), (0, 1, 0)\} = [0 : 1 : 0] \\
[0 : 2 : 1] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(0, 2, 1), (0, 1, 2)\} = [0 : 1 : 2] \\
[0 : 2 : 2] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(0, 2, 2), (0, 1, 1)\} = [0 : 1 : 1] \\
[1 : 0 : 0] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(1, 0, 0), (2, 0, 0)\} \\
[1 : 0 : 1] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(1, 0, 1), (2, 0, 2)\} \\
[1 : 0 : 2] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(1, 0, 2), (2, 0, 1)\} \\
[1 : 1 : 0] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(1, 1, 0), (2, 2, 0)\} \\
[1 : 1 : 1] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(1, 1, 1), (2, 2, 2)\} \\
[1 : 1 : 2] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(1, 1, 2), (2, 2, 1)\} \\
[1 : 2 : 0] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(1, 2, 0), (2, 1, 0)\} \\
[1 : 2 : 1] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(1, 2, 1), (2, 1, 2)\} \\
[1 : 2 : 2] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(1, 2, 2), (2, 1, 1)\} \\
[2 : 0 : 0] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(2, 0, 0), (1, 0, 0)\} = [1 : 0 : 0] \\
[2 : 0 : 1] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(2, 0, 1), (1, 0, 2)\} = [1 : 0 : 2] \\
[2 : 0 : 2] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(2, 0, 2), (1, 0, 1)\} = [1 : 0 : 1] \\
[2 : 1 : 0] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(2, 1, 0), (1, 2, 0)\} = [1 : 2 : 0] \\
[2 : 1 : 1] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(2, 1, 1), (1, 2, 2)\} = [1 : 2 : 2] \\
[2 : 1 : 2] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(2, 1, 2), (1, 2, 1)\} = [1 : 2 : 1] \\
[2 : 2 : 0] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(2, 2, 0), (1, 1, 0)\} = [1 : 1 : 0] \\
[2 : 2 : 1] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(2, 2, 1), (1, 1, 2)\} = [1 : 1 : 2] \\
[2 : 2 : 2] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(2, 2, 2), (1, 1, 1)\} = [1 : 1 : 1]
\end{aligned}$$

These lines define the 13 points in the projective plane $\mathbb{F}_3\mathbb{P}$:

$$\begin{aligned}
\mathbb{F}_3\mathbb{P} = \{ & [0 : 0 : 1], [0 : 1 : 0], [0 : 1 : 1], [0 : 1 : 2], [1 : 0 : 0], [1 : 0 : 1], \\
& [1 : 0 : 2], [1 : 1 : 0], [1 : 1 : 1], [1 : 1 : 2], [1 : 2 : 0], [1 : 2 : 1], [1 : 2 : 2] \}
\end{aligned}$$

2578 This projective plane contains 9 affine points, three points at infinity and one line at infinity.

2579 To understand the ambiguity in projective coordinates a bit better, let us consider the point
 2580 $[1 : 2 : 2]$. As this point in the projective plane is a line in \mathbb{F}_3^3 , it has the projective coordinates
 2581 $(1, 2, 2)$ as well as $(2, 1, 1)$, since the former coordinate gives the latter when multiplied in \mathbb{F}_3
 2582 by the factor 2. In addition, note that, for the same reasons, the points $[1 : 2 : 2]$ and $[2 : 1 : 1]$
 2583 are the same, since their underlying sets are equal.

2584 *Exercise 49.* Construct the so-called **Fano plane**, that is, the projective plane over the finite
 2585 field \mathbb{F}_2 .

Chapter 5

Elliptic Curves

Generally speaking, elliptic curves are “curves” defined in geometric planes like the Euclidean or the projective plane over some given field. One of the key features of elliptic curves over finite fields from the point of view of cryptography is that their set of points has a group law such that the resulting group is finite and cyclic, and it is believed that the discrete logarithm problem on these groups is hard.

A special class of elliptic curves are so-called **pairing-friendly curves**, which have a notation of a group pairing as defined in XXX. This pairing has cryptographically advantageous properties. Those curve are useful in the development of SNARKs, since they allow to compute so-called R1CS-satisfiability “in the exponent” **MIRCO: (THIS HAS TO BE REWRITTEN WITH WAY MORE DETAIL)**

In this chapter, we introduce elliptic curves as they are used in pairing-based approaches to the construction of SNARKs. The elliptic curves we consider are all defined over prime fields or prime field extensions and the reader should be familiar with the content of the previous section on those fields.

In its most generality elliptic curves are defined as a smooth projective curve of genus 1 defined over some field \mathbb{F} with a distinguished \mathbb{F} -rational point, but this definition is not very useful for the introductory character of this book. We will therefore look at 3 more practical definitions in the following sections, by introducing Weierstraß, Montgomery and Edwards curves. All of them are widely used in cryptography, and understanding them is crucial to being able to follow the rest of this book.

5.1 Elliptic Curve Arithmetics

5.1.1 Short Weierstraß Curves

In this section, we introduce **short Weierstraß** curves, which are the most general types of curves over finite fields of characteristic greater than 3.

We start with their representation in **affine space**. This representation has the advantage that affine points correspond to pairs of numbers, which makes it more accessible for beginners. However, it has the disadvantage that a special “point at infinity”, that is not a point on the curve, is necessary to describe the group structure. We introduce the elliptic curve group law and describe elliptic curve scalar multiplication, which is an instantiation of the exponential map from general cyclic groups.

Then we look at the projective representation of short Weierstraß curves. This has the advantage that no special symbol is necessary to represent the point at infinity but comes with

TODO:
Elliptic
Curve
asymmet-
ric cryp-
tography
examples.
Private
key, gen-
erator,
public
key.

add refer-
ence

maybe re-
move this
sentence?

affine
space

the drawback that projective points are classes of numbers, which might be a bit unusual for a beginner.

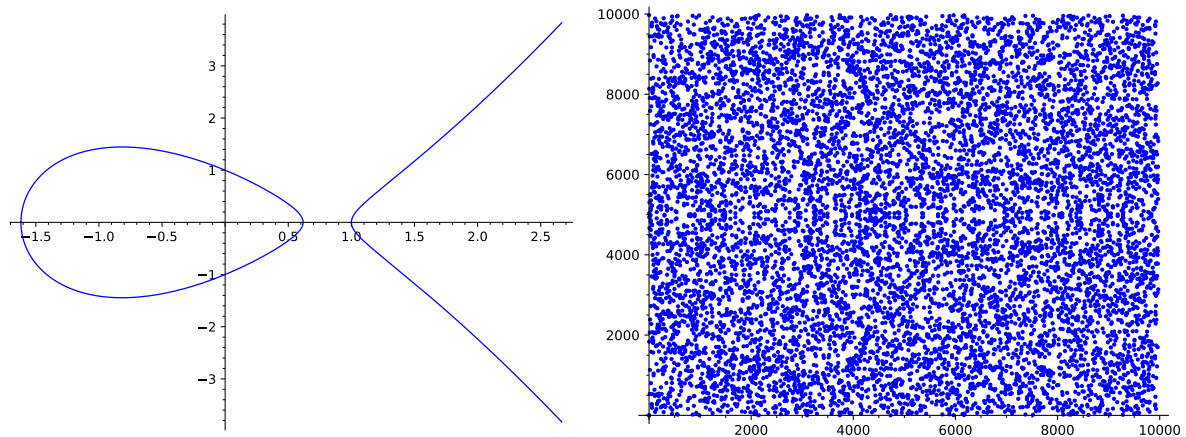
We finish this section with an explicit equivalence that transforms affine representations into projective ones and vice versa.

Affine short Weierstraß form Probably the least abstract and most straight-forward way to introduce elliptic curves for non-mathematicians and beginners is the so-called affine representation of a short Weierstraß curve. To see what this is, let \mathbb{F} be a finite field of order q and $a, b \in \mathbb{F}$ two field elements such that $4a^3 + 27b^2 \bmod q \neq 0$. Then a **short Weierstraß elliptic curve** $E(\mathbb{F})$ over \mathbb{F} in its affine representation is the set of all pairs of field elements $(x, y) \in \mathbb{F} \times \mathbb{F}$ that satisfy the short Weierstraß cubic equation $y^2 = x^3 + a \cdot x + b$, together with a distinguished symbol \mathcal{O} , called the **point at infinity**:

$$E(\mathbb{F}) = \{(x, y) \in \mathbb{F} \times \mathbb{F} \mid y^2 = x^3 + a \cdot x + b\} \cup \{\mathcal{O}\} \quad (5.1)$$

Notation and Symbols 10. In the literature, the set $E(\mathbb{F})$, which includes the symbol \mathcal{O} , is often called the set of **rational points** of the elliptic curve, in which case the curve itself is usually written as E/\mathbb{F} . However, in what follows, we will frequently identify an elliptic curve with its set of rational points and therefore use the notation $E(\mathbb{F})$ instead. This is possible in our case, since we only the group structure of the curve in consideration is relevant for us.

The term “curve” is used here because, in the ordinary 2 dimensional plane \mathbb{R}^2 , the set of all points (x, y) that satisfy $y^2 = x^3 + a \cdot x + b$ looks like a curve. We should note however that visualizing elliptic curves over finite fields as “curves” has its limitations, and we will therefore not stress the geometric picture too much, but focus on the computational properties instead. To understand the visual difference, consider the following two elliptic curves:



Both elliptic curves are defined by the same short Weierstraß equation $y^2 = x^3 - 2x + 1$, but the first curve is defined in the real affine plane \mathbb{R}^2 , that is, the pair (x, y) contains real numbers, while the second one is defined in the affine plane \mathbb{F}_{9973}^2 , which means that both x and y are from the prime field \mathbb{F}_{9973} . Every blue dot represents a pair (x, y) , that is a solution to $y^2 = x^3 - 2x + 1$. As we can see, the second curve hardly looks like a geometric structure one would naturally call a curve. This shows that our geometric intuitions from \mathbb{R}^2 are obfuscated in curves over finite fields.

The identity $6 \cdot (4a^3 + 27b^2) \bmod q \neq 0$ ensures that the curve is non-singular, which basically means that the curve has no **cusps** or **self-intersections**.

cusps

self-intersections

Throughout this book, the reader is advised to do as many computations in a pen-and-paper fashion as possible, as this helps getting a deeper understanding of the details. However, when dealing with elliptic curves, computations can quickly become cumbersome and tedious, and one might get lost in the details. Fortunately, Sage is very helpful in dealing with elliptic curves. This book introduces the reader to the great elliptic curve capabilities of Sage. The following snippet shows a way to define elliptic curves and work with them in Sage:

```

sage: F5 = GF(5) # define the base field
sage: a = F5(2) # parameter a
sage: b = F5(4) # parameter b
sage: # check non-singularity
sage: F5(6)*(F5(4)*a^3+F5(27)*b^2) != F5(0)
True
sage: # short Weierstrass curve
sage: E = EllipticCurve(F5,[a,b]) # y^2 == x^3 + ax +b
sage: P = E(0,2) # 2^2 == 0^3 + 2*0 + 4
sage: P.xy() # affine coordinates
(0, 2)
sage: INF = E(0) # point at infinity
sage: try: # point at infinity has no affine coordinates
.....:     INF.xy()
.....: except ZeroDivisionError:
.....:     pass
sage: P = E.plot() # create a plotted version

```

The following three examples give a more practical understanding of what an elliptic curve is and how we can compute it. The reader is advised to read them carefully, and ideally, to also carry out the computation themselves. We will repeatedly build on these examples in this chapter, and use the second example throughout this book.

Example 68. To provide the reader with an example of a small elliptic curve where all computation can be done with pen and paper, consider the prime field \mathbb{F}_5 from example 62 (page 65), which should be quite familiar to readers who had worked through the examples and exercises in the previous chapter.

To define an elliptic curve over \mathbb{F}_5 , we have to choose two numbers a and b from that field. Assuming we choose $a = 1$ and $b = 1$ then $4a^3 + 27b^2 \equiv 1 \pmod{5}$ from which follows that the corresponding elliptic curve $E_1(\mathbb{F}_5)$ is given by the set of all pairs (x, y) from \mathbb{F}_5 that satisfy the equation $y^2 = x^3 + x + 1$, together with the special symbol \mathcal{O} , which represents the “point at infinity”.

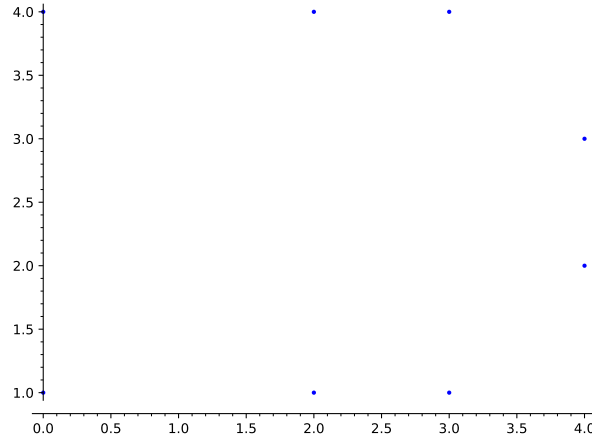
To get a better understanding of that curve, observe that if we choose arbitrarily the pair $(x, y) = (1, 1)$, we see that $1^2 \neq 1^3 + 1 + 1$ and hence $(1, 1)$ is not an element of the curve $E_1(\mathbb{F}_5)$. On the other hand choosing for example $(x, y) = (2, 1)$ gives $1^2 = 2^3 + 2 + 1$ and hence the pair $(2, 1)$ is an element of $E_1(\mathbb{F}_5)$ (Remember that all computations are done in modulo 5 arithmetics).

Now since the set $\mathbb{F}_5 \times \mathbb{F}_5$ of all pairs (x, y) from \mathbb{F}_5 contains only $5 \cdot 5 = 25$ pairs, we can compute the curve, by just inserting every possible pair (x, y) into the short Weierstrass equation $y^2 = x^3 + x + 1$. If the equation holds, the pair is a curve point, if not that means that the point is not on the curve. Combining the result of this computation with the point at infinity gives the curve as follows:

$$E_1(\mathbb{F}_5) = \{\mathcal{O}, (0, 1), (2, 1), (3, 1), (4, 2), (4, 3), (0, 4), (2, 4), (3, 4)\}$$

check
reference

2692 This means that our elliptic curve is a set of 9 elements, 8 of which are pairs of numbers and
 2693 one special symbol \mathcal{O} . Visualizing E_1 gives the following plot:



2695 In the development of SNARKs, it is sometimes necessary to do elliptic curve cryptography
 2696 “in a circuit”, which basically means that the elliptic curves need to be implemented in a certain
 2697 SNARK-friendly way. We will look at what this means in chapter 7. To be able to do this
 2698 efficiently, it is desirable to have curves with special properties. The following example is a
 2699 pen-and-paper version of such a curve, called **Baby-jubjub**, which resembles cryptographically
 2700 secure curves extensively used in real-world SNARKs. The interested reader is advised to study
 2701 this example carefully, as we will use it and build on it in various places throughout the book. S:
 2702 I feel like a lot of people won’t get the Lewis Carroll reference unless we make it more explicit.
 2703 M: The term Baby-JubJub is actually the name of a curve used in zCash and Ethereum a lot.
 2704 **IDK why they choosed that name.**

check
reference

jubjub

2705 *Example 69 (Tiny-Jubjub).* Consider the prime field \mathbb{F}_{13} from exercise ?? (page ??). If we
 2706 choose $a = 8$ and $b = 8$, then $4a^3 + 27b^2 \equiv 6 \pmod{13}$ and the corresponding elliptic curve
 2707 is given by all pairs (x, y) from \mathbb{F}_{13} such that $y^2 = x^3 + 8x + 8$ holds. We call this curve the
 2708 **Tiny-jubjub** curve, or TJJ_13 for short.

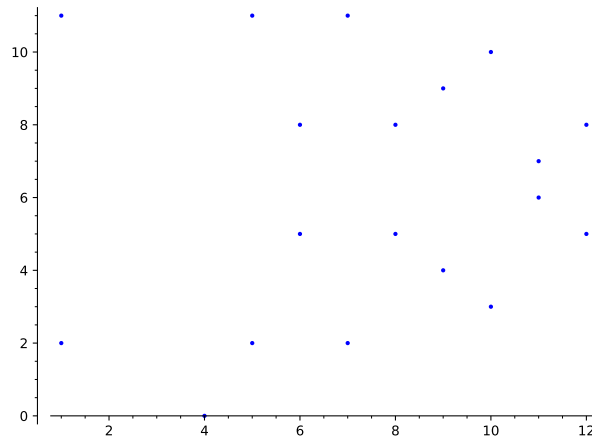
check
reference

2709 Now, since the set $\mathbb{F}_{13} \times \mathbb{F}_{13}$ of all pairs (x, y) from \mathbb{F}_{13} contains only $13 \cdot 13 = 169$ pairs,
 2710 we can compute the curve by just inserting every possible pair (x, y) into the short Weierstraß
 2711 equation $y^2 = x^3 + 8x + 8$. We get the following result:

$$TJJ_13 = \{\mathcal{O}, (1, 2), (1, 11), (4, 0), (5, 2), (5, 11), (6, 5), (6, 8), (7, 2), (7, 11), (8, 5), (8, 8), \\ (9, 4), (9, 9), (10, 3), (10, 10), (11, 6), (11, 7), (12, 5), (12, 8)\} \quad (5.2)$$

2712 As we can see, the curve consists of 20 points; 19 points from the affine plane and the point at
 2713 infinity. To get a visual impression of the TJJ_13 curve, we might plot all of its points (except
 2714 the point at infinity) in the $\mathbb{F}_{13} \times \mathbb{F}_{13}$ affine plane. We get the following plot:

affine
plane



2715

2716 As we will see in what follows, this curve is rather special, as it is possible to represent it in two
 2717 alternative forms called the **Montgomery** and the **twisted Edwards form** (See sections 5.1.2
 2718 and 5.1.3, respectively).

check
reference

2719 Now that we have seen two pen-and-paper friendly elliptic curves, let us look at a curve, that
 2720 is used in actual cryptography. Cryptographically secure elliptic curves are not **qualitatively**
 2721 different from the curves we looked at so far, but the prime number modulus of their prime field
 2722 is much larger. Typical examples use prime numbers that have binary representations in the
 2723 magnitude of more than double the size of the desired security level. If, for example, a security
 2724 of 128 bits is desired, a prime modulus of binary size ≥ 256 is chosen. The following example
 2725 provides such a curve.

check
reference

Example 70 (Bitcoin's Secp256k1 curve). To give an example of a real-world, cryptographically secure curve, let us look at curve Secp256k1, which is famous for being used in the public key cryptography of Bitcoin. The prime field \mathbb{F}_p of Secp256k1 is defined by the following prime number:

$$p = 115792089237316195423570985008687907853269984665640564039457584007908834671663$$

2726 The binary representation of this number needs 256 bits, which implies that the prime field
 2727 \mathbb{F}_p contains approximately 2^{256} many elements, which is considered quite large. To get a better
 2728 impression of how large the base field is, consider that the number 2^{256} is approximately in the
 2729 same order of magnitude as the estimated number of atoms in the observable universe.

The curve Secp256k1 is defined by the parameters $a, b \in \mathbb{F}_p$ with $a = 0$ and $b = 7$. Since $4 \cdot 0^3 + 27 \cdot 7^2 \bmod p = 1323$, those parameters indeed define an elliptic curve given as follows:

$$\text{Secp256k1} = \{(x, y) \in \mathbb{F}_p \times \mathbb{F}_p \mid y^2 = x^3 + 7\}$$

Clearly, the Secp256k1 curve is too large to do computations by hand, since it can be shown that the number of its elements is a prime number r that also has a binary representation of 256 bits:

$$r = 115792089237316195423570985008687907852837564279074904382605163141518161494337$$

2730 Cryptographically secure elliptic curves are therefore not useful in pen-and-paper computations.
 2731 Fortunately, Sage handles large curves efficiently:

2732 **sage: p = 1157920892373161954235709850086879078532699846656405** 243
 2733 **64039457584007908834671663**

```

2734 sage: # Hexadecimal representation
2735 sage: p.str(16)
2736 ffffffffffffffffffffffffffffffffffffffffffffffffffefffffc
2737 2f
2738 sage: p.is_prime()
2739 True
2740 sage: p.nbits()
2741 256
2742 sage: Fp = GF(p)
2743 sage: Secp256k1 = EllipticCurve(Fp, [0, 7])
2744 sage: r = Secp256k1.order() # number of elements
2745 sage: r.str(16)
2746 fffffffffffffffffffffffffffffffebaaedce6af48a03bbfd25e8cd03641
2747 41
2748 sage: r.is_prime()
2749 True
2750 sage: r.nbits()
2751 256

```

2752 *Exercise 50.* Look up the definition of curve BLS12-381, implement it in Sage and compute its
 2753 order.

2754 **Affine compressed representation** As we have seen in example 70, cryptographically secure
 2755 elliptic curves are defined over large prime fields, where elements of those fields typically need
 2756 more than 255 bits of storage on a computer. Since elliptic curve points consist of pairs of those
 2757 field elements, they need double that amount of storage.

2758 However, we can reduce the amount of space needed to represent a curve point by using
 2759 a technique called **point compression**. Note that, up to a sign, the y coordinate of a curve
 2760 point can be computed from the x coordinate by simply inserting x into the Weierstraß equation
 2761 and then computing the roots of the result. This gives two results, and it means that we can
 2762 represent a curve point in **compressed form** by simply storing the x coordinate together with
 2763 a single sign bit only, the latter of which deterministically decides which of the two roots to
 2764 choose. One convention could be to always choose the root closer to 0 when the sign bit is 0,
 2765 and the root closer to the order of \mathbb{F} when the sign bit is 1. In case the y coordinate is zero, both
 2766 sign bits give the same result.

Example 71 (Tiny-jubjub). To understand the concept of compressed curve points a bit better,
 consider the *TJJ_13* curve from example 69 again. Since this curve is defined over the prime
 field \mathbb{F}_{13} , and numbers between 0 and 13 need approximately 4 bits to be represented, each
TJJ_13 point on this curve needs 8 bits of storage in uncompressed form. The following set
 represents the uncompressed form of the points on this curve:

$$TJJ_13 = \{\mathcal{O}, (1, 2), (1, 11), (4, 0), (5, 2), (5, 11), (6, 5), (6, 8), (7, 2), (7, 11), \\ (8, 5), (8, 8), (9, 4), (9, 9), (10, 3), (10, 10), (11, 6), (11, 7), (12, 5), (12, 8)\}$$

Using the technique of point compression, we can reduce the bits needed to represent the points
 on this curve to 5 per point. To achieve this, we can replace the y coordinate in each (x, y) pair
 by a sign bit indicating whether or not y is closer to 0 or to 13. As a result y values in the range
 $[0, \dots, 6]$ will have the sign bit 0, while y -values in the range $[7, \dots, 12]$ will have the sign bit 1.

check
reference

sign

more ex-
planation
of what
the sign is

check
reference

Applying this to the points in *TJJ_13* gives the compressed representation as follows:

$$TJJ_{13} = \{\mathcal{O}, (1,0), (1,1), (4,0), (5,0), (5,1), (6,0), (6,1), (7,0), (7,1), (8,0), (8,1), (9,0), (9,1), (10,0), (10,1), (11,0), (11,1), (12,0), (12,1)\}$$

Note that the numbers $7, \dots, 12$ are the negatives (additive inverses) of the numbers $1, \dots, 6$ in modular 13 arithmetics and that $-0 = 0$. Calling the compression bit a “sign bit” therefore makes sense.

To recover the uncompressed counterpart of, say, the compressed point $(5, 1)$, we insert the x coordinate 5 into the Weierstraß equation and get $y^2 = 5^3 + 8 \cdot 5 + 8 = 4$. As expected, 4 is a quadratic residue in \mathbb{F}_{13} with roots $\sqrt{4} = \{2, 11\}$. Since the sign bit of the point is 1, we have to choose the root closer to the modulus 13, which is 11. The uncompressed point is therefore $(5, 11)$.

Looking at the previous examples, the compression rate does not look very impressive. However, looking at the real-life example of the Secp256k1 curve shows that compression is has significant practical advantages.

Example 72. Consider the Secp256k1 curve from example 70 again. The following code invokes Sage to generate a random affine curve point, then applies our compression method to it:

```
sage: P = Secp256k1.random_point().xy()
sage: P
(1031516485458923212431072684560215298570682976470118124170755
 31297966297499499, 9654376537625196092932975916044184443049
 5341599136462789254941180034691599894)
sage: # uncompressed affine point size
sage: ZZ(P[0]).nbits()+ZZ(P[1]).nbits()
512
sage: # compute the compression
sage: if P[1] > Fp(-1)/Fp(2):
.....:     PARITY = 1
.....: else:
.....:     PARITY = 0
sage: PCOMPRESSED = [P[0], PARITY]
sage: PCOMPRESSED
[1031516485458923212431072684560215298570682976470118124170755
 31297966297499499, 1]
sage: # compressed affine point size
sage: ZZ(PCOMPRESSED[0]).nbits()+ZZ(PCOMPRESSED[1]).nbits()
257
```

Affine group law One of the key properties of an elliptic curve is that it is possible to define a group law on the set of its rational points such that the point at infinity serves as the neutral element and inverses are reflections on the x -axis.

The origin of this law can be understood in a geometric picture and is known as the **chord-and-tangent rule**. In the affine representation of a short Weierstraß curve, the rule can be described in the following way:

S: I don't follow this at all

check reference

add explanation of how this shows what we claim

Definition 5.1.1.1. Chord-and-tangent rule

- (Point at infinity) We define the point at infinity \mathcal{O} as the neutral element of addition, that is, we define $P + \mathcal{O} = P$ for all points $P \in E(\mathbb{F})$.
- (Point addition) Let $P, Q \in E(\mathbb{F}) \setminus \{\mathcal{O}\}$ with $P \neq Q$ be two distinct points on an elliptic curve, neither of them the point at infinity. The sum of P and Q is defined as follows:
Consider the line l which intersects the curve in P and Q . If l intersects the elliptic curve at a third point R' , define the sum $R = P \oplus Q$ of P and Q as the reflection of R' at the x -axis. If the line l does not intersect the curve at a third point, define the sum to be the point at infinity \mathcal{O} . It can be shown that no such **chord line** will intersect the curve in more than three points, so addition is not ambiguous.
- (Point doubling) Let $P \in E(\mathbb{F}) \setminus \{\mathcal{O}\}$ be a point on an elliptic curve, that is not the point at infinity. The sum of P with itself (the doubling of P) is defined as follows:
Consider the line which is **tangential** to the elliptic curve at P . If this line intersects the elliptic curve at a second point R' , the sum $2P = P + P$ is the reflection of R' at the x -axis. If it does not intersect the curve at a third, point define the sum to be the point at infinity \mathcal{O} . It can be shown that no such **tangent line** will intersect the curve in more than two points, so addition is not ambiguous.

It can be shown that the points of an elliptic curve form a commutative group with respect to the tangent-and-chord rule such that \mathcal{O} acts the neutral element, and the inverse of any element $P \in E(\mathbb{F})$ is the reflection of P on the x -axis.

To translate the geometric description into algebraic equations, first observe that, for any two given curve points $(x_1, y_1), (x_2, y_2) \in E(\mathbb{F})$, it can be shown that the identity $x_1 = x_2$ implies $y_2 = \pm y_1$, which shows that the following rules are a complete description of the affine addition law.

Definition 5.1.1.2. Chord-and-tangent rule: algebraic equations

- (Neutral element) The point at infinity \mathcal{O} is the neutral element.
- (Additive inverse) The additive inverse of \mathcal{O} is \mathcal{O} . For any other curve point $(x, y) \in E(\mathbb{F}) \setminus \{\mathcal{O}\}$, the additive inverse is given by $(x, -y)$.
- (Addition rule) For any two curve points $P, Q \in E(\mathbb{F})$, addition is defined by one of the following three cases:
 1. (Adding the neutral element) If $Q = \mathcal{O}$, then the sum is defined as $P \oplus Q = P$.
 2. (Adding inverse elements) If $P = (x, y)$ and $Q = (x, -y)$, the sum is defined as $P \oplus Q = \mathcal{O}$.
 3. (Adding non-self-inverse equal points) If $P = (x, y)$ and $Q = (x, y)$ with $y \neq 0$, the sum $2P = (x', y')$ is defined as follows: **We only referred to P in the definition of point doubling above so Q seems a bit confusing here even though it's defined as equal to P**
 4. (Adding non-inverse different points) If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ such that $x_1 \neq x_2$, the sum $R = P + Q$ with $R = (x_3, y_3)$ is defined as follows:

$$x' = \left(\frac{3x^2 + a}{2y} \right)^2 - 2x \quad , \quad y' = \left(\frac{3x^2 + a}{2y} \right)^2 (x - x') - y$$

$$x_3 = \left(\frac{y_2 - y_1}{x_2 - x_1} \right)^2 - x_1 - x_2 \quad , \quad y_3 = \left(\frac{y_2 - y_1}{x_2 - x_1} \right) (x_1 - x_3) - y_1$$

2841 Note that short Weierstraß curve points P with $P = (x, 0)$ are inverses of themselves, which
 2842 implies $2P = \mathcal{O}$ in this case.

2843 *Notation and Symbols 11.* Let \mathbb{F} be a field and $E(\mathbb{F})$ be an elliptic curve over \mathbb{F} . We write \oplus for
 2844 the group law on $E(\mathbb{F})$ and $(E(\mathbb{F}), \oplus)$ for the group of rational points.

2845 As we can see, it is very efficient to compute inverses on elliptic curves. However, com-
 2846 puting the addition of elliptic curve points in the affine representation needs to consider many
 2847 cases and involves extensive finite field divisions. As we will see in the next paragraph, this can
 2848 be simplified in projective coordinates. where?

2849 To get some practical impression of how the group law on an elliptic curve is computed,
 2850 let's look at some actual cases:

Example 73. Consider the elliptic curve $E_1(\mathbb{F}_5)$ from example 68 again. As we have seen, the
 set of rational points contains 9 elements: check
reference

$$E_1(\mathbb{F}_5) = \{\mathcal{O}, (0, 1), (2, 1), (3, 1), (4, 2), (4, 3), (0, 4), (2, 4), (3, 4)\}$$

2851 We know that this set defines a group, so we can add any two elements from $E_1(\mathbb{F}_5)$ to get a
 2852 third element.

To give an example, consider the elements $(0, 1)$ and $(4, 2)$. Neither of these elements is
 the neutral element \mathcal{O} , and since, the x coordinate of $(0, 1)$ is different from the x coordinate of
 $(4, 2)$, we know that we have to use the chord rule, that is, rule number 4 from definition 5.1.1.2,
 to compute the sum $(0, 1) \oplus (4, 2)$: check
reference

$$\begin{aligned} x_3 &= \left(\frac{y_2 - y_1}{x_2 - x_1} \right)^2 - x_1 - x_2 && \# \text{ insert points} \\ &= \left(\frac{2 - 1}{4 - 0} \right)^2 - 0 - 4 && \# \text{ simplify in } \mathbb{F}_5 \\ &= \left(\frac{1}{4} \right)^2 + 1 = 4^2 + 1 = 1 + 1 = 2 \end{aligned}$$

$$\begin{aligned} y_3 &= \left(\frac{y_2 - y_1}{x_2 - x_1} \right) (x_1 - x_3) - y_1 && \# \text{ insert points} \\ &= \left(\frac{2 - 1}{4 - 0} \right) (0 - 2) - 1 && \# \text{ simplify in } \mathbb{F}_5 \\ &= \left(\frac{1}{4} \right) \cdot 3 + 4 = 4 \cdot 3 + 4 = 2 + 4 = 1 \end{aligned}$$

So, in our elliptic curve $E_1(\mathbb{F}_5)$ we get $(0, 1) \oplus (4, 2) = (2, 1)$, and, indeed, the pair $(2, 1)$ is an
 element of $E_1(\mathbb{F}_5)$ as expected. On the other hand, $(0, 1) \oplus (0, 4) = \mathcal{O}$, since both points have
 equal x coordinates and inverse y coordinates, rendering them inverses of each other. Adding
 the point $(4, 2)$ to itself, we have to use the tangent rule, that is, rule 3 from definition 5.1.1.2: check
reference

$$\begin{aligned}
 x' &= \left(\frac{3x^2 + a}{2y} \right)^2 - 2x && \# \text{ insert points} \\
 &= \left(\frac{3 \cdot 4^2 + 1}{2 \cdot 2} \right)^2 - 2 \cdot 4 && \# \text{ simplify in } \mathbb{F}_5 \\
 &= \left(\frac{3 \cdot 1 + 1}{4} \right)^2 + 3 \cdot 4 = \left(\frac{4}{4} \right)^2 + 2 = 1 + 2 = 3
 \end{aligned}$$

$$\begin{aligned}
 y' &= \left(\frac{3x^2 + a}{2y} \right)^2 (x - x') - y && \# \text{ insert points} \\
 &= \left(\frac{3 \cdot 4^2 + 1}{2 \cdot 2} \right)^2 (4 - 3) - 2 && \# \text{ simplify in } \mathbb{F}_5 \\
 &= 1 \cdot 1 + 3 = 4
 \end{aligned}$$

2853 So, in our elliptic curve $E_1(\mathbb{F}_5)$, we get the doubling of $(4, 2)$, that is, $(4, 2) \oplus (4, 2) = (3, 4)$,
 2854 and, indeed the pair $(3, 4)$ is an element of $E_1(\mathbb{F}_5)$ as expected. The group $E_1(\mathbb{F}_5)$ has no self-
 2855 inverse points other than the neutral element \mathcal{O} , since no point has 0 as its y coordinate. We can
 2856 invoke Sage to double-check the computations.

```

2857 sage: F5 = GF(5)                                     277
2858 sage: E1 = EllipticCurve(F5, [1, 1])                 278
2859 sage: INF = E1(0) # point at infinity                 279
2860 sage: P1 = E1(0, 1)                                   280
2861 sage: P2 = E1(4, 2)                                   281
2862 sage: P3 = E1(0, 4)                                   282
2863 sage: R1 = E1(2, 1)                                   283
2864 sage: R2 = E1(3, 4)                                   284
2865 sage: R1 == P1+P2                                     285
2866 True                                                  286
2867 sage: INF == P1+P3                                    287
2868 True                                                  288
2869 sage: R2 == P2+P2                                     289
2870 True                                                  290
2871 sage: R2 == 2*P2                                       291
2872 True                                                  292
2873 sage: P3 == P3 + INF                                   293
2874 True                                                  294

```

Example 74 (Tiny-jubjub). Consider the *TJJ_13*-curve from example 69 again and recall that its group of rational points is given as follows:

check
reference

$$\begin{aligned}
 TJJ_{13} = \{ &\mathcal{O}, (1, 2), (1, 11), (4, 0), (5, 2), (5, 11), (6, 5), (6, 8), (7, 2), (7, 11), \\
 &(8, 5), (8, 8), (9, 4), (9, 9), (10, 3), (10, 10), (11, 6), (11, 7), (12, 5), (12, 8) \}
 \end{aligned}$$

2875 In contrast to the group from the previous example, this group contains a self-inverse point,
 2876 which is different from the neutral element, defined by $(4, 0)$. To see what this means, observe
 2877 that we cannot add $(4, 0)$ to itself using the tangent rule 3 from definition 5.1.1.2, as the y
 2878 coordinate is zero. Instead, we have to use rule 2, since $0 = -0$. We therefore get $(4, 0) \oplus$

check
reference

2879 $(4,0) = \mathcal{O}$ in *TJJ_13*. The point $(4,0)$ is therefore the inverse of itself, as adding it to itself
 2880 results in the neutral element.

```

2881 sage: F13 = GF(13)                                295
2882 sage: MJJ = EllipticCurve(F13, [8, 8])              296
2883 sage: P = MJJ(4, 0)                                297
2884 sage: INF = MJJ(0) # Point at infinity              298
2885 sage: INF == P+P                                    299
2886 True                                                300
2887 sage: INF == 2*P                                    301
2888 True                                                302

```

2889 *Example 75.* Consider the Secp256k1 curve from example 70 again. The following code in-
 2890 vokes Sage to generate a random affine curve point, then applies our compression method:

check
reference

```

2891 sage: P = Secp256k1.random_point()                  303
2892 sage: Q = Secp256k1.random_point()                  304
2893 sage: INF = Secp256k1(0)                             305
2894 sage: R1 = -P                                         306
2895 sage: R2 = P + Q                                     307
2896 sage: R3 = Secp256k1.order()*P                       308
2897 sage: P.xy()                                          309
2898 (4693324551387010171607236610287446477446880864085889415738070 310
2899      8221482004401226, 50977877985656086694891507925675689060732
2900      157246333726365319478278495727506720)
2901 sage: Q.xy()                                          311
2902 (4165936974647719035556980504277562089799429426081463940794235 312
2903      6958526858675225, 22652021841062826376995594023289180635768
2904      52666383787557950579915185027860314)
2905 sage: (ZZ(R1[0]).str(16), ZZ(R1[1]).str(16))          313
2906 ('67c3468d5154a95a876a7c6e23d9cb6a47771773a8f8fb834506441f4fbc 314
2907      4c4a', '8f4b8b26442024e01d847fd0dfcd5998b432dbb5184dc0b296
2908      05a2520b8270f')
2909 sage: R2.xy()                                          315
2910 (1084222454095274307638493649009502123041150439478734138946697 316
2911      56066597672279285, 1180069212255859673575499876395978437957
2912      2883996627759636219136100352930888591)
2913 sage: R3 == INF                                       317
2914 True                                                  318
2915 sage: P[1]+R1[1] == Fp(0) # -(x,y) = (x,-y)         319
2916 True                                                  320

```

2917 *Exercise 51.* Consider the *TJJ_13*-curve from example 69.

check
reference

- 2918 1. Compute the inverse of $(10,10)$, \mathcal{O} , $(4,0)$ and $(1,2)$.
- 2919 2. Compute the expression $3 \cdot (1,11) - (9,9)$.
- 2920 3. Solve the equation $x + 2(9,4) = (5,2)$ for some $x \in TJJ_{13}$
- 2921 4. Solve the equation $x \cdot (7,11) = (8,5)$ for $x \in \mathbb{Z}$

Scalar multiplication As we have seen in the previous section, elliptic curves $E(\mathbb{F})$ have the structure of a commutative group associated to them. Moreover, It can moreover be shown that this group is finite and cyclic whenever the field is finite.

To understand elliptic curve scalar multiplication, recall from page 43 that every finite cyclic group of order q has a generator g and an associated exponential map $g^{(\cdot)} : \mathbb{Z}_q \rightarrow \mathbb{G}$, where g^n is the n -fold product of g with itself.

check
reference

Elliptic curve scalar multiplication is the exponential map written in additive notation. To be more precise, let \mathbb{F} be a finite field, $E(\mathbb{F})$ an elliptic curve of order r , and P a generator of $E(\mathbb{F})$. Then the **elliptic curve scalar multiplication** with base P is defined as follows (where $[0]P = \mathcal{O}$ and $[m]P = P + P + \dots + P$ is the m -fold sum of P with itself):

$$[\cdot]P : \mathbb{Z}_r \rightarrow E(\mathbb{F}); m \mapsto [m]P$$

therefore, elliptic curve scalar multiplication is an instantiation of the general exponential map using additive instead of multiplicative notation. This map is a homomorph of groups, which means that $[n + m]P = [n]P \oplus [m]P$.

As with all finite, cyclic groups, the inverse of the exponential map exists and is usually called the **elliptic curve discrete logarithm map**. However, elliptic curves are believed to be XXX-groups, which means that we don't know of any efficient way to actually compute this map.

add term

Scalar multiplication and its inverse, the elliptic curve discrete logarithm, define the **elliptic curve discrete logarithm problem**, which consists of finding solutions $m \in \mathbb{Z}_r$ such that the following equation holds:

$$P = [m]Q \tag{5.3}$$

Any solution m is usually called a **discrete logarithm relation** between P and Q . If Q is a generator of the curve, then there is a discrete logarithm relation between Q and any other point, since Q generates the group by repeatedly adding Q to itself. Therefore, we know that some discrete logarithm relation exists for generator Q and point P . However, since elliptic curves are believed to be XXX-groups, finding actual relations m is computationally hard, with runtimes being approximately the size of the order of the group. In practice, we often need the assumption that a discrete logarithm relation exists, while the relation itself is not known.

add term

One useful property of the exponential map in regard to the examples in this book is that it can be used to greatly simplify pen-and-paper computations. As we have seen in example XXX, computing the elliptic curve addition law takes quite a bit of effort when done without a computer. However, when g is a generator of a small pen-and-paper elliptic curve group of order r , we can use the exponential map to write the group using cofactor clearing, which implies that $[r]g = \mathcal{O}$:

add refer-
encecofactor
clearing

$$\mathbb{G} = \{[1]g \rightarrow [2]g \rightarrow [3]g \rightarrow \dots \rightarrow [r-1]g \rightarrow \mathcal{O}\} \tag{5.4}$$

“Logarithmic ordering” like this greatly simplifies complicated elliptic curve addition to the much simpler case of modular r addition. In order to add two curve points P and Q , we only have to look up their discrete log relations with the generator, say $P = [n]g$ and $Q = [m]g$, and compute the sum as $P \oplus Q = [n + m]g$. This is, of course, only possible for small groups where we can keep a clear overview, such as XXX.

add refer-
ence

In the following example, we will look at some implications of the fact that elliptic curves are finite cyclic groups. We will apply the fundamental theorem of finite cyclic groups and look how it reflects on the curves in consideration.

Example 76. Consider the elliptic curve group $E_1(\mathbb{F}_5)$ from example 68. Since it is a finite cyclic group of order 9, and the prime factorization of 9 is $3 \cdot 3$, we can use the fundamental

check
reference

theorem of finite cyclic groups to reason about all its subgroups. In fact, since the only prime factor of 9 is 3, we know that $E_1(\mathbb{F}_5)$ has the following subgroups:

- $\mathbb{G}_1 = E_1(\mathbb{F}_5)$ is a subgroup of order 9. By definition, any group is a subgroup of itself.
- $\mathbb{G}_2 = \{(2, 1), (2, 4), \mathcal{O}\}$ is a subgroup of order 3. This is the subgroup associated to the prime factor 3.
- $\mathbb{G}_3 = \{\mathcal{O}\}$ is a subgroup of order 1. This is the trivial subgroup.

Moreover, since $E_1(\mathbb{F}_5)$ and all its subgroups are cyclic, we know from page 43 that they must have generators. For example, the curve point $(2, 1)$ is a generator of the order 3 subgroup \mathbb{G}_2 , since every element of \mathbb{G}_2 can be generated by repeatedly adding $(2, 1)$ to itself:

check
reference

$$\begin{aligned}[1](2, 1) &= (2, 1) \\ [2](2, 1) &= (2, 4) \\ [3](2, 1) &= \mathcal{O}\end{aligned}$$

Since $(2, 1)$ is a generator, we know from XXX that it gives rise to an exponential map from the finite field \mathbb{F}_3 onto \mathbb{G}_2 defined by scalar multiplication:

add refer-
ence

$$[\cdot](2, 1) : \mathbb{F}_3 \rightarrow \mathbb{G}_2 : x \mapsto [x](2, 1)$$

To give an example of a generator that generates the entire group $E_1(\mathbb{F}_5)$, consider the point $(0, 1)$. Applying the tangent rule repeatedly, we compute as follows:

$$\begin{array}{ll} [0](0, 1) = \mathcal{O} & [1](0, 1) = (0, 1) \\ [2](0, 1) = (4, 2) & [3](0, 1) = (2, 1) \\ [4](0, 1) = (3, 4) & [5](0, 1) = (3, 1) \\ [6](0, 1) = (2, 4) & [7](0, 1) = (4, 3) \\ [8](0, 1) = (0, 4) & [9](0, 1) = \mathcal{O} \end{array}$$

Again, since $(2, 1)$ is a generator, we know from XXX that it gives rise to an exponential map. However, since the group order is not a prime number, the exponential map does not map from any field, but from the residue class ring \mathbb{Z}_9 only:

add refer-
ence

$$[\cdot](0, 1) : \mathbb{Z}_9 \rightarrow \mathbb{G}_1 : x \mapsto [x](0, 1)$$

Using the generator $(0, 1)$ and its associated exponential map, we can write $E(\mathbb{F}_1)$ i logarithmic order with respect to $(0, 1)$ as explained in equation 5.4. We get the following:

check
reference

$$E_1(\mathbb{F}_5) = \{(0, 1) \rightarrow (4, 2) \rightarrow (2, 1) \rightarrow (3, 4) \rightarrow (3, 1) \rightarrow (2, 4) \rightarrow (4, 3) \rightarrow (0, 4) \rightarrow \mathcal{O}\}$$

This indicates that the first element is a generator, and the n -th element is the scalar product of n and the generator. To see how logarithmic orders like this simplify the computations in small elliptic curve groups, consider example 73 again. In that example, we use the chord-and-tangent rule to compute $(0, 1) \oplus (4, 2)$. Now, in the logarithmic order of $E_1(\mathbb{F})$, we can compute that sum much easier, since we can directly see that $(0, 1) = [1](0, 1)$ and $(4, 2) = [2](0, 1)$. We can then deduce $(0, 1) \oplus (4, 2) = (2, 1)$ immediately, since $[1](0, 1) \oplus [2](0, 1) = [3](0, 1) = (2, 1)$.

check
reference

To give another example, we can immediately see that $(3, 4) \oplus (4, 3) = (4, 2)$, without doing any expensive elliptic curve addition, since we know $(3, 4) = [4](0, 1)$ as well as $(4, 3) =$

2975 $[7](0, 1)$ from the logarithmic representation of $E_1(\mathbb{F}_5)$. Since $4 + 7 = 2$ in \mathbb{Z}_9 , the result must
 2976 be $[2](0, 1) = (4, 2)$.

2977 Finally we can use $E_1(\mathbb{F}_5)$ as an example to understand the concept of cofactor clearing from
 2978 5.4. Since the order of $E_1(\mathbb{F}_5)$ is 9, we only have a single factor, which happen to be the cofactor
 2979 as well. Cofactor clearing then implies that we can map any element from $E_1(\mathbb{F}_5)$ onto its prime
 2980 factor group \mathbb{G}_2 by scalar multiplication with 3. For example, taking the element $(3, 4)$, which
 2981 is not in \mathbb{G}_2 , and multiplying it with 3, we get $[3](3, 4) = (2, 1)$, which is an element of \mathbb{G}_2 as
 2982 expected.

check
reference

2983 In the following example, we will look at the subgroups of our tiny-jubjub curve, define
 2984 generators, and compute the logarithmic order for pen-and-paper computations. Then we take
 2985 another look at the principle of cofactor clearing.

2986 *Example 77.* Consider the tiny-jubjub curve TJJ_13 from example 69 again. Since the order of
 2987 TJJ_13 is 20, and the prime factorization of 20 is $2^2 \cdot 5$, we know that the TJJ_13 contains a
 2988 “large” prime-order subgroup of size 5 and a small prime order subgroup of size 2.

check
reference

2989 To compute those groups, we can apply the technique of cofactor clearing in a try-and-repeat
 2990 loop. We start the loop by arbitrarily choosing an element $P \in TJJ_13$, then multiplying that
 2991 element with the cofactor of the group that we want to compute. If the result is \mathcal{O} , we try a
 2992 different element and repeat the process until the result is different from the point at infinity \mathcal{O} .

2993 To compute a generator for the small prime-order subgroup $(TJJ_13)_2$, first observe that the
 2994 cofactor is 10, since $20 = 2 \cdot 10$. We then arbitrarily choose the curve point $(5, 11) \in TJJ_13$
 2995 and compute $[10](5, 11) = \mathcal{O}$. Since the result is the point at infinity, we have to try another
 2996 curve point, say $(9, 4)$. We get $[10](9, 4) = (4, 0)$ and we can deduce that $(4, 0)$ is a generator
 2997 of $(TJJ_13)_2$. Logarithmic order then gives $(TJJ_13)_2 = \{(4, 0) \rightarrow \mathcal{O}\}$ as expected, since we
 2998 know from example 74 that $(4, 0)$ is self-inverse, with $(4, 0) \oplus (4, 0) = \mathcal{O}$. We double check the
 2999 computations using Sage:

check
reference

3000	sage: <code>F13 = GF(13)</code>	321
3001	sage: <code>TJJ = EllipticCurve(F13, [8, 8])</code>	322
3002	sage: <code>P = TJJ(5, 11)</code>	323
3003	sage: <code>INF = TJJ(0)</code>	324
3004	sage: <code>10*P == INF</code>	325
3005	True	326
3006	sage: <code>Q = TJJ(9, 4)</code>	327
3007	sage: <code>R = TJJ(4, 0)</code>	328
3008	sage: <code>10*Q == R</code>	329
3009	True	330

We can apply the same reasoning to the “large” prime-order subgroup $(TJJ_13)_5$, which contains 5 elements. To compute a generator for this group, first observe that the associated cofactor is 4, since $20 = 5 \cdot 4$. We choose the curve point $(9, 4) \in TJJ_13$ again, and compute $[4](9, 4) = (7, 11)$. We can deduce that $(7, 11)$ is a generator of $(TJJ_13)_5$. Using the generator $(7, 11)$, we compute the exponential map $[\cdot](7, 11) : \mathbb{F}_5 \rightarrow TJJ_13$ and get the following:

Explain
how

$$\begin{aligned}
 [0](7, 11) &= \mathcal{O} \\
 [1](7, 11) &= (7, 11) \\
 [2](7, 11) &= (8, 5) \\
 [3](7, 11) &= (8, 8) \\
 [4](7, 11) &= (7, 2)
 \end{aligned}$$

We can use this computation to write the large-order prime group $(TJJ_13)_5$ of the tiny-jubjub curve in logarithmic order, which we will use quite frequently in what follows. We get the following:

$$(TJJ_13)_5 = \{(7, 11) \rightarrow (8, 5) \rightarrow (8, 8) \rightarrow (7, 2) \rightarrow \mathcal{O}\} \quad (5.5)$$

From this, we can immediately see, for example that $(8, 8) \oplus (7, 2) = (8, 5)$, since $3 + 4 = 2$ in \mathbb{F}_5 .

From the previous two examples, the reader might get the impression that elliptic curve computation can be largely replaced by modular arithmetics. This however, is not true in general, but only an artifact of small groups, where it is possible to write the entire group in a logarithmic order. The following example gives some understanding of why this is not possible in cryptographically secure groups.

Example 78. SEKTP BICOIN. DISCRETE LOG HARDNESS PROHIBITS ADDITION IN THE FIELD...

write example

Projective short Weierstraß form As we have seen in the previous section, describing elliptic curves as pairs of points that satisfy a certain equation is relatively straight-forward. However, in order to define a group structure on the set of points, we had to add a special point at infinity to act as the neutral element.

Recalling from the definition of projective planes (section 4.4), we know that points at infinity are handled as ordinary points in projective geometry. Therefore, it makes sense to look at the definition of a short Weierstraß curve in projective geometry.

check reference

To see what a short Weierstraß curve in projective coordinates is, let \mathbb{F} be a finite field of order q and characteristic > 3 , let $a, b \in \mathbb{F}$ be two field elements such that $4a^3 + 27b^2 \bmod q \neq 0$ and let \mathbb{FP}^2 be the projective plane over \mathbb{F} . Then a **short Weierstraß elliptic curve** over \mathbb{F} in its projective representation is the set of all points $[X : Y : Z] \in \mathbb{FP}^2$ from the projective plane that satisfy the **homogenous** cubic equation $Y^2 \cdot Z = X^3 + a \cdot X \cdot Z^2 + b \cdot Z^3$:

$$E(\mathbb{FP}^2) = \{[X : Y : Z] \in \mathbb{FP}^2 \mid Y^2 \cdot Z = X^3 + a \cdot X \cdot Z^2 + b \cdot Z^3\} \quad (5.6)$$

To understand how the point at infinity is unified in this definition, recall from XXX that, in projective geometry, points at infinity are given by homogeneous coordinates $[X : Y : 0]$. Inserting representatives $(x_1, y_1, 0) \in [X : Y : 0]$ from those classes into the defining homogenous cubic equations gives the following:

add reference

$$\begin{aligned} y_1^2 \cdot 0 &= x_1^3 + a \cdot x_1 \cdot 0^2 + b \cdot 0^3 \\ 0 &= x_1^3 \end{aligned} \quad \Leftrightarrow$$

This shows that the only point at infinity, that is also a point on a projective short Weierstraß curve is the class $[0, 1, 0] = \{(0, y, 0) \mid y \in \mathbb{F}\}$.

This point is the projective representation of \mathcal{O} . The projective representation of a short Weierstraß curve, therefore, has the advantage that it does not need a special symbol to represent the point at infinity \mathcal{O} from the affine definition.

Example 79. To get an intuition of how an elliptic curve in projective geometry looks, consider curve $E_1(\mathbb{F}_5)$ from example (68). We know that, in its affine representation, the set of rational points is given as follows:

check reference

$$E_1(\mathbb{F}_5) = \{\mathcal{O}, (0, 1), (2, 1), (3, 1), (4, 2), (4, 3), (0, 4), (2, 4), (3, 4)\} \quad (5.7)$$

3042 This is defined as the set of all pairs $(x, y) \in \mathbb{F}_5 \times \mathbb{F}_5$ such that the affine short Weierstraß
 3043 equation $y^2 = x^3 + ax + b$ with $a = 1$ and $b = 1$ is satisfied.

3044 To find the projective representation of a short Weierstraß curve with the same parameters
 3045 $a = 1$ and $b = 1$, we have to compute the set of projective points $[X : Y : Z]$ from the projec-
 3046 tive plane $\mathbb{F}_5\mathbb{P}^2$ that satisfy the following homogenous cubic equation for any representative
 3047 $(x_1, y_1, z_1) \in [X : Y : Z]$:

$$y_1^2 z_1 = x_1^3 + 1 \cdot x_1 z_1^2 + 1 \cdot z_1^3 \quad (5.8)$$

3048 We know from XXX that the projective plane $\mathbb{F}_5\mathbb{P}^2$ contains $5^2 + 5 + 1 = 31$ elements, so we
 3049 can take the effort and insert all elements into equation 5.8 and see if both sides match.

For example, consider the projective point $[0 : 4 : 1]$. We know from XXX that this point in
 the projective plane represents the following line in the three-dimensional space \mathbb{F}^3 :

$$[0 : 4 : 1] = \{(0, 0, 0), (0, 4, 1), (0, 3, 2), (0, 2, 3), (0, 1, 4)\}$$

To check whether or not $[0 : 4 : 1]$ satisfies 5.8, we can insert any representative, in other words,
 any element from XXX. Each element satisfies the equation if and only if all other elements
 satisfy the equation. So, we insert $(0, 4, 1)$ and get the following result:

$$1^2 \cdot 1 = 0^3 + 1 \cdot 0 \cdot 1^2 + 1 \cdot 1^3$$

This tells us that the affine point $[0 : 4 : 1]$ is indeed a solution to the equation 5.8, but we
 could just as well have inserted any other representative. For example, inserting $(0, 3, 2)$ also
 satisfies 5.8:

$$3^2 \cdot 2 = 0^3 + 1 \cdot 0 \cdot 2^2 + 1 \cdot 2^3$$

3050 To find the projective representation of E_1 , we first observe that the projective line at infinity
 3051 $[1 : 0 : 0]$ is not a curve point on any projective short Weierstraß curve, since it cannot satisfy
 3052 XXX for any parameter a and b . Therefore, we can exclude it from our consideration.

3053 Moreover, a point at infinity $[X : Y : 0]$ can only satisfy equation XXX for any a and b , if
 3054 $X = 0$, which implies that the only point at infinity relevant for short Weierstraß elliptic curves
 3055 is $[0 : 1 : 0]$, since $[0 : k : 0] = [0 : 1 : 0]$ for all k from the finite field. Therefore, we can exclude
 3056 all points at infinity except the point $[0 : 1 : 0]$.

3057 All points that remain are the affine points $[X : Y : 1]$. Inserting all of them into XXX, we
 3058 get the set of all projective curve points as follows:

$$E_1(\mathbb{F}_5\mathbb{P}^2) = \{[0 : 1 : 0], [0 : 1 : 1], [2 : 1 : 1], [3 : 1 : 1], \\ [4 : 2 : 1], [4 : 3 : 1], [0 : 4 : 1], [2 : 4 : 1], [3 : 4 : 1]\}$$

3059 If we compare this with the affine representation, we see that there is a 1:1 correspondence
 3060 between the points in the affine representation in 5.7 and the affine points in projective geometry,
 3061 and that the point $[0 : 1 : 0]$ represents the additional point \mathcal{O} in the projective representation.

3062 *Exercise 52.* Compute the projective representation of the tiny-jubjub curve and the logarithmic
 3063 order of its large prime-order subgroup with respect to the generator $(7, 11)$.

3064 **Projective Group law** As we have seen on page 74, one of the key properties of an elliptic
 3065 curve is that it comes with a definition of a group law on the set of its rational points, described
 3066 geometrically by the chord-and-tangent rule (definition 5.1.1.1). This rule was kind of intuitive,

with the exception of the distinguished point at infinity, which appeared whenever the chord or the tangent did not have a third intersection point with the curve.

One of the key features of projective coordinates is that, in projective space, it is guaranteed that any chord will always intersect the curve in three points, and any tangent will intersect it in two points including the tangent point. So, the geometric picture simplifies, as we don't need to consider external symbols and associated cases.

Again, it can be shown that the points of an elliptic curve in projective space form a commutative group with respect to the tangent-and-chord rule such that the projective point $[0 : 1 : 0]$ is the neutral element, and the additive inverse of a point $[X : Y : Z]$ is given by $[X : -Y : Z]$. The addition law is usually described by the following algorithm, minimizing the number of necessary additions and multiplications in the base field.

Exercise 53. Compare the affine addition law for short Weierstraß curves with the projective addition rule. Which branch in the projective rule corresponds to which case in the affine law?

Check if following Alg is floated too far

Coordinate Transformations As we have seen in example XXX, there was a close relation between the affine and the projective representation of a short Weierstraß curve. This was not a coincidence. In fact, from a mathematical point of view, projective and affine short Weierstraß curves describe the same thing, as there is a one-to-one correspondence (an isomorphism) between both representations for any arbitrary parameters a and b .

add reference

To specify the isomorphism, let $E(\mathbb{F})$ and $E(\mathbb{FP}^2)$ be an affine and a projective short Weierstraß curve defined for the same parameters a and b . Then the map in 5.9 maps points from the affine representation to points from the projective representation of a short Weierstraß curve. In other words, if the pair of points (x, y) satisfies the affine equation $y^2 = x^3 + ax + b$, then all homogeneous coordinates $(x_1, y_1, z_1) \in [x : y : 1]$ satisfy the projective equation $y_1^2 \cdot z_1 = x_1^3 + ay_1 \cdot z_1^2 + b \cdot z_1^3$.

$$\Phi : E(\mathbb{F}) \rightarrow E(\mathbb{FP}^2) : \begin{array}{ll} (x, y) & \mapsto [x : y : 1] \\ \mathcal{O} & \mapsto [0 : 1 : 0] \end{array} \quad (5.9)$$

The inverse is given by the following map:

$$\Phi^{-1} : E(\mathbb{FP}^2) \rightarrow E(\mathbb{F}) : [X : Y : Z] \mapsto \begin{cases} (\frac{X}{Z}, \frac{Y}{Z}) & \text{if } Z \neq 0 \\ \mathcal{O} & \text{if } Z = 0 \end{cases} \quad (5.10)$$

Note that the only projective point $[X : Y : Z]$ with $Z \neq 0$ that satisfies XXX is given by the class $[0 : 1 : 0]$.

add reference

One key feature of Φ and its inverse is that it respects the group structure, which means that $\Phi((x_1, y_1) \oplus (x_2, y_2))$ is equal to $\Phi(x_1, y_1) \oplus \Phi(x_2, y_2)$. The same holds true for the inverse map Φ^{-1} .

Maps with these properties are called **group isomorphisms**, and, from a mathematical point of view, the existence of Φ implies that these two definitions are equivalent, and implementations can choose freely between these representations.

5.1.2 Montgomery Curves

History and use of them (optimized scalar multiplication)

write up this part

Algorithm 7 Projective Weierstraß Addition Law

Require: $[X_1 : Y_1 : Z_1], [X_2 : Y_2 : Z_2] \in E(\mathbb{F}_{\mathbb{P}}^2)$

procedure ADD-RULE($[X_1 : Y_1 : Z_1], [X_2 : Y_2 : Z_2]$)

if $[X_1 : Y_1 : Z_1] == [0 : 1 : 0]$ **then**

$[X_3 : Y_3 : Z_3] \leftarrow [X_2 : Y_2 : Z_2]$

else if $[X_2 : Y_2 : Z_2] == [0 : 1 : 0]$ **then**

$[X_3 : Y_3 : Z_3] \leftarrow [X_1 : Y_1 : Z_1]$

else

$U_1 \leftarrow Y_2 \cdot Z_1$

$U_2 \leftarrow Y_1 \cdot Z_2$

$V_1 \leftarrow X_2 \cdot Z_1$

$V_2 \leftarrow X_1 \cdot Z_2$

if $V_1 == V_2$ **then**

if $U_1 \neq U_2$ **then** $[X_3 : Y_3 : Z_3] \leftarrow [0 : 1 : 0]$

else

if $Y_1 == 0$ **then** $[X_3 : Y_3 : Z_3] \leftarrow [0 : 1 : 0]$

else

$W \leftarrow a \cdot Z_1^2 + 3 \cdot X_1^2$

$S \leftarrow Y_1 \cdot Z_1$

$B \leftarrow X_1 \cdot Y_1 \cdot S$

$H \leftarrow W^2 - 8 \cdot B$

$X' \leftarrow 2 \cdot H \cdot S$

$Y' \leftarrow W \cdot (4 \cdot B - H) - 8 \cdot Y_1^2 \cdot S^2$

$Z' \leftarrow 8 \cdot S^3$

$[X_3 : Y_3 : Z_3] \leftarrow [X' : Y' : Z']$

end if

end if

else

$U = U_1 - U_2$

$V = V_1 - V_2$

$W = Z_1 \cdot Z_2$

$A = U^2 \cdot W - V^3 - 2 \cdot V^2 \cdot V_2$

$X' = V \cdot A$

$Y' = U \cdot (V^2 \cdot V_2 - A) - V^3 \cdot U_2$

$Z' = V^3 \cdot W$

$[X_3 : Y_3 : Z_3] \leftarrow [X' : Y' : Z']$

end if

end if

return $[X_3 : Y_3 : Z_3]$

end procedure

Ensure: $[X_3 : Y_3 : Z_3] == [X_1 : Y_1 : Z_1] \oplus [X_2 : Y_2 : Z_2]$

Affine Montgomery Form To see what a Montgomery curve in affine coordinates is, let \mathbb{F} be a finite field of characteristic > 2 , and let $A, B \in \mathbb{F}$ be two field elements such that $B \neq 0$ and $A^2 \neq 4$. A **Montgomery elliptic curve** $M(\mathbb{F})$ over \mathbb{F} in its affine representation is the set of all pairs of field elements $(x, y) \in \mathbb{F} \times \mathbb{F}$ that satisfy the Montgomery cubic equation $B \cdot y^2 = x^3 + A \cdot x^2 + x$, together with a distinguished symbol \mathcal{O} , called the **point at infinity**.

$$M(\mathbb{F}) = \{(x, y) \in \mathbb{F} \times \mathbb{F} \mid B \cdot y^2 = x^3 + A \cdot x^2 + x\} \cup \{\mathcal{O}\} \quad (5.11)$$

Despite the fact that Montgomery curves look different from short Weierstraß curves, they are just a special way to describe certain short Weierstraß curves. In fact, every curve in affine Montgomery form can be transformed into an elliptic curve in Weierstraß form. To see that, assume that a curve is given in Montgomery form $By^2 = x^3 + Ax^2 + x$. The associated Weierstraß form is then as follows:

$$y^2 = x^3 + \frac{3 - A^2}{3B^2} \cdot x + \frac{2A^3 - 9A}{27B^3} \quad (5.12)$$

On the other hand, an elliptic curve $E(\mathbb{F})$ over base field \mathbb{F} in Weierstraß form $y^2 = x^3 + ax + b$ can be converted to Montgomery form if and only if the following conditions hold:

Definition 5.1.2.1. Requirements for Montgomery curves

- The number of points on $E(F)$ is divisible by 4
- The polynomial $z^3 + az + b \in \mathbb{F}[z]$ has at least one root $z_0 \in \mathbb{F}$
- $3z_0^2 + a$ is a quadratic residue in \mathbb{F} .

When these conditions are satisfied, then for $s = (\sqrt{3z_0^2 + a})^{-1}$, the equivalent Montgomery curve is defined by the following equation:

$$sy^2 = x^3 + (3z_0s)x^2 + x \quad (5.13)$$

In the following example we will look at our tiny-jubjub curve again, and show that it is actually a Montgomery curve.

Example 80. Consider the prime field \mathbb{F}_{13} and the tiny-jubjub curve TJJ_13 from example 69. To see that it is a Montgomery curve, we have to check the requirements from 5.1.2.1:

Since the order of TJJ_13 is 20, which is divisible by 4, the first requirement is met.

Next, since $a = 8$ and $b = 8$, we have to check if the polynomial $P(z) = z^3 + 8z + 8$ has a root in \mathbb{F}_{13} . We simply evaluate P at all numbers $z \in \mathbb{F}_{13}$, and find that $P(4) = 0$, so a root is given by $z_0 = 4$.

In the last step, we have to check that $3 \cdot z_0^2 + a$ has a root in \mathbb{F}_{13} . We compute as follows:

$$\begin{aligned} 3z_0^2 + a &= 3 \cdot 4^2 + 8 \\ &= 3 \cdot 3 + 8 \\ &= 9 + 8 \\ &= 4 \end{aligned}$$

To see if 4 is a quadratic residue, we can use Euler's criterion (4.35) to compute the Legendre symbol of 4. We get the following:

is the label in L^AT_EX correct here?

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$$\left(\frac{4}{13}\right) = 4^{\frac{13-1}{2}} = 4^6 = 1$$

3131 This means that 4 does have a root in \mathbb{F}_{13} . In fact, computing a root of 4 in \mathbb{F}_{13} is easy, since
 3132 the integer root 2 of 4 is also one of its roots in \mathbb{F}_{13} . The other root is given by $13 - 4 = 9$.

Since all requirements are met, we have now shown that *TJJ_13* is indeed a Montgomery curve, and we can use 5.13 to compute its associated Montgomery form. We compute as follows:

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$$\begin{aligned} s &= \left(\sqrt{3 \cdot z_0^2 + 8} \right)^{-1} \\ &= 2^{-1} && \# \text{ Fermat's little theorem} \\ &= 2^{13-2} && \# 2048 \bmod 13 = 7 \\ &= 7 \end{aligned}$$

The defining equation for the Montgomery form of our tiny-jubjub curve is then given by the following equation:

$$\begin{aligned} sy^2 &= x^3 + (3z_0s)x^2 + x && \Rightarrow \\ 7 \cdot y^2 &= x^3 + (3 \cdot 4 \cdot 7)x^2 + x && \Leftrightarrow \\ 7 \cdot y^2 &= x^3 + 6x^2 + x \end{aligned}$$

3133 So, we get the defining parameters as $B = 7$ and $A = 6$, and we can write the tiny-jubjub curve
 3134 in its affine Montgomery representation as follows:

$$TJJ_13 = \{(x, y) \in \mathbb{F}_{13} \times \mathbb{F}_{13} \mid 7 \cdot y^2 = x^3 + 6x^2 + x\} \cup \{\mathcal{O}\} \quad (5.14)$$

Now that we have the abstract definition of our tiny-jubjub curve in Montgomery form, we can compute the set of points by inserting all pairs $(x, y) \in \mathbb{F}_{13} \times \mathbb{F}_{13}$ similarly to how we computed the curve points in its Weierstraß representation. We get the following:

$$TJJ_13 = \{\mathcal{O}, (0, 0), (1, 4), (1, 9), (2, 4), (2, 9), (3, 5), (3, 8), (4, 4), (4, 9), (5, 1), (5, 12), (7, 1), (7, 12), (8, 1), (8, 12), (9, 2), (9, 11), (10, 3), (10, 10)\}$$

3135

```

3136 sage: F13 = GF(13)                                     331
3137 sage: L_MTJJ = []                                       332
3138 .....: for x in F13:                                    333
3139 .....:     for y in F13:                                334
3140 .....:         if F13(7)*y^2 == x^3 + F13(6)*x^2 + x:   335
3141 .....:             L_MTJJ.append((x, y))                336
3142 sage: MTJJ = Set(L_MTJJ)                                337
3143 sage: # does not compute the point at infinity          338

```

3144 **Affine Montgomery coordinate transformation** Comparing the Montgomery representa-
 3145 tion of the previous example (equation 5.14) with the Weierstraß representation of the same
 3146 curve (equation 5.2), we see that there is a 1:1 correspondence between the curve points in both

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examples. This is no accident. In fact, if $M_{A,B}$ is a Montgomery curve, and $E_{a,b}$ a Weierstraß curve with $a = \frac{3-A^2}{3B^2}$ and $b = \frac{2A^2-9A}{27B^3}$ then the following function maps all points in Montgomery representation onto the points in Weierstraß representation:

$$\Phi : M_{A,B} \rightarrow E_{a,b} : (x,y) \mapsto \left(\frac{3x+A}{3B}, \frac{y}{B} \right) \quad (5.15)$$

This map is a 1:1 correspondence (an isomorphism), and its inverse map is given by the following equation (where z_0 is a root of the polynomial $z^3 + az + b \in \mathbb{F}[z]$ and $s = (\sqrt{3z_0^2 + a})^{-1}$).

$$\Phi^{-1} : E_{a,b} \rightarrow M_{A,B} : (x,y) \mapsto (s \cdot (x - z_0), s \cdot y) \quad (5.16)$$

Using this map, it is therefore possible for implementations of Montgomery curves to freely transit between the Weierstraß and the Montgomery representation. However, as we saw in definition 5.1.2.1, not every Weierstraß curve is a Montgomery curve, as all criteria in 5.1.2.1 have to be satisfied. This means that the map Φ^{-1} does not always exist.

Example 81. Consider our tiny-jubjub curve again. In equation 5.2 we derived its Weierstraß representation and in example 5.14, we derived its Montgomery representation.

To see how coordinate transformation Φ works in this example, let's map points from the Montgomery representation onto points from the Weierstraß representation. Inserting, for example, the point $(0,0)$ from the Montgomery representation 5.14 into Φ gives the following:

$$\begin{aligned} \Phi(0,0) &= \left(\frac{3 \cdot 0 + A}{3B}, \frac{0}{B} \right) \\ &= \left(\frac{3 \cdot 0 + 6}{3 \cdot 7}, \frac{0}{7} \right) \\ &= \left(\frac{6}{8}, 0 \right) \\ &= (4,0) \end{aligned}$$

As we can see, the Montgomery point $(0,0)$ maps to the self-inverse point $(4,0)$ of the Weierstraß representation. On the other hand, we can use our computations of $s = 7$ and $z_0 = 4$ from XXX to compute the inverse map Φ^{-1} , which maps points on the Weierstraß representation to points on the Montgomery form. Inserting, for example, $(4,0)$ we get the following:

$$\begin{aligned} \Phi^{-1}(4,0) &= (s \cdot (4 - z_0), s \cdot 0) \\ &= (7 \cdot (4 - 4), 0) \\ &= (0,0) \end{aligned}$$

As expected, the inverse map maps the Weierstraß point back to where it originated in the Montgomery form. We can invoke Sage to check that our computation of Φ is correct:

```
sage: # Compute PHI of Montgomery form: 339
sage: L_PHI_MTJJ = [] 340
sage: for (x,y) in L_MTJJ: # LMJJ as defined previously 341
....:     v = (F13(3)*x + F13(6)) / (F13(3)*F13(7)) 342
....:     w = y/F13(7) 343
....:     L_PHI_MTJJ.append((v,w)) 344
```

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ence

```

3166 sage: PHI_MTJJ = Set(L_PHI_MTJJ) 345
3167 sage: # Computation Weierstrass form 346
3168 sage: C_WTJJ = EllipticCurve(F13, [8, 8]) 347
3169 sage: L_WTJJ = [P.xy() for P in C_WTJJ.points() if P.order() > 348
3170 1]
3171 sage: WTJJ = Set(L_WTJJ) 349
3172 sage: # check PHI(Montgomery) == Weierstrass 350
3173 sage: WTJJ == PHI_MTJJ 351
3174 True 352
3175 sage: # check the inverse map PHI^(-1) 353
3176 sage: L_PHIINV_WTJJ = [] 354
3177 sage: for (v,w) in L_WTJJ: 355
3178 .....:     x = F13(7)*(v-F13(4)) 356
3179 .....:     y = F13(7)*w 357
3180 .....:     L_PHIINV_WTJJ.append((x,y)) 358
3181 sage: PHIINV_WTJJ = Set(L_PHIINV_WTJJ) 359
3182 sage: MTJJ == PHIINV_WTJJ 360
3183 True 361

```

3184 **Montgomery group law** We have seen that Montgomery curves special cases of short Weier-
3185 straß curves. As such, they have a group structure defined on the set of their points, which can
3186 also be derived from the chord-and-tangent rule. In accordance with short Weierstraß curves, it
3187 can be shown that the identity $x_1 = x_2$ implies $y_2 = \pm y_1$, meaning that the following rules are a
3188 complete description of the affine addition law.

3189 *Definition 5.1.2.2. Montgomery group law*

- 3190 • (Neutral element) Point at infinity \mathcal{O} is the neutral element.
- 3191 • (Additive inverse) The additive inverse of \mathcal{O} is \mathcal{O} . For any other curve point $(x,y) \in$
3192 $M(\mathbb{F}_q) \setminus \{\mathcal{O}\}$, the additive inverse is given by $(x, -y)$.
- 3193 • (Addition rule) For any two curve points $P, Q \in M(\mathbb{F}_q)$, addition is defined by one of the
3194 following cases:
 - 3195 1. (Adding the neutral element) If $Q = \mathcal{O}$, then the sum is defined as $P + Q = P$.
 - 3196 2. (Adding inverse elements) If $P = (x,y)$ and $Q = (x, -y)$, the sum is defined as $P +$
3197 $Q = \mathcal{O}$.
 3. (Adding non-self-inverse equal points) If $P = (x,y)$ and $Q = (x,y)$ with $y \neq 0$, the
sum $2P = (x', y')$ is defined as follows:

$$x' = \left(\frac{3x_1^2 + 2Ax_1 + 1}{2By_1} \right)^2 \cdot B - (x_1 + x_2) - A, \quad y' = \frac{3x_1^2 + 2Ax_1 + 1}{2By_1} (x_1 - x') - y_1$$

4. (Adding non-inverse different points) If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ such that $x_1 \neq$
 x_2 , the sum $R = P + Q$ with $R = (x_3, y_3)$ is defined as follows:

$$x' = \left(\frac{y_2 - y_1}{x_2 - x_1} \right)^2 B - (x_1 + x_2) - A, \quad y' = \frac{y_2 - y_1}{x_2 - x_1} (x_1 - x') - y_1$$

5.1.3 Twisted Edwards Curves

As we have seen in 5.1.2.2 both Weierstraß and Montgomery curves have somewhat complicated addition and doubling laws, as many cases have to be distinguished. Those various cases translate to branches in computer programs.

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In the context of SNARK development, two computational models for bounded computations are used, called **circuits** and **rank-1 constraint systems**. Program branches are undesirably costly when implemented in those models. It is therefore advantageous to look for curves with an addition/doubling rule that requires no branches and as few field operations as possible.

Twisted Edwards curves are particularly useful here, as a subclass of these curves has a compact and easily implementable addition law that works for all points including the point at infinity. Implementing this law needs no branching.

Twisted Edwards Form To see what an affine **twisted Edwards curve** looks like, let \mathbb{F} be a finite field of characteristic > 2 , and let $a, d \in \mathbb{F} \setminus \{0\}$ be two non-zero field elements with $a \neq d$. A **twisted Edwards elliptic curve** in its affine representation is the set of all pairs (x, y) from $\mathbb{F} \times \mathbb{F}$ that satisfy the twisted Edwards equation $a \cdot x^2 + y^2 = 1 + d \cdot x^2 y^2$, given below:

$$E(\mathbb{F}) = \{(x, y) \in \mathbb{F} \times \mathbb{F} \mid a \cdot x^2 + y^2 = 1 + d \cdot x^2 y^2\} \quad (5.17)$$

A twisted Edwards curve is called an **Edwards curve (non-twisted)**, if the parameter a is equal to 1, and it is called a **SNARK-friendly twisted Edwards curve** if the parameter a is a quadratic residue and the parameter d is a quadratic non-residue.

As we can see from the definition, affine twisted Edwards curves look somewhat different from Weierstraß curves, as their affine representation does not need a special symbol to represent the point at infinity. In fact, we will see that the pair $(0, 1)$ is always a point on any twisted Edwards curve, and that it takes the role of the point at infinity.

Despite their different appearances however, twisted Edwards curves are equivalent to Montgomery curves in the sense that, for every twisted Edwards curve, there is a Montgomery curve, and a way to map the points of one curve in a 1:1 correspondence onto the other and vice versa. To see that, assume that a curve in twisted Edwards form $a \cdot x^2 + y^2 = 1 + d \cdot x^2 y^2$ is given. The associated Montgomery curve is then defined by the Montgomery equation:

$$\frac{4}{a-d} y^2 = x^3 + \frac{2(a+d)}{a-d} \cdot x^2 + x \quad (5.18)$$

On the other hand, a Montgomery curve $By^2 = x^3 + Ax^2 + x$ with $B \neq 0$ and $A^2 \neq 4$ can give rise to a twisted Edwards curve defined by the following equation:

$$\left(\frac{A+2}{B}\right)x^2 + y^2 = 1 + \left(\frac{A-2}{B}\right)x^2 y^2 \quad (5.19)$$

As we have seen in equation 5.12 and the following discussion, Montgomery curves are just a special class of Weierstraß curves. Furthermore we now know that twisted Edwards curves are special Weierstraß curves too. This means that the more general way to describe elliptic curves is as Weierstraß curves.

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Example 82. Consider the tiny-jubjub curve from example 69 again. We know from example 80 that it is a Montgomery curve, and, since Montgomery curves are equivalent to twisted Edwards curves, we want to write this curve in twisted Edwards form. We use equation 5.19,

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and compute the parameters a and d as follows:

$$\begin{aligned}
 a &= \frac{A+2}{B} && \# \text{ insert } A=6 \text{ and } B=7 \\
 &= \frac{8}{7} = 3 && \# 7^{-1} = 2 \\
 \\
 d &= \frac{A-2}{B} \\
 &= \frac{4}{7} = 8
 \end{aligned}$$

Thus, we get the defining parameters as $a = 3$ and $d = 8$. Since our goal is to use this curve later on in implementations of pen-and-paper SNARKs, let us show that tiny-jubjub is also a **SNARK-friendly** twisted Edwards curve. To see that, we have to show that a is a quadratic residue and d is a quadratic non-residue. We therefore compute the Legendre symbols of a and d using Euler's criterion. We get the following:

$$\begin{aligned}
 \left(\frac{3}{13} \right) &= 3^{\frac{13-1}{2}} \\
 &= 3^6 = 1
 \end{aligned}$$

$$\begin{aligned}
 \left(\frac{8}{13} \right) &= 8^{\frac{13-1}{2}} \\
 &= 8^6 = 12 = -1
 \end{aligned}$$

3231 This proves that tiny-jubjub is SNARK-friendly. We can write the tiny-jubjub curve in its
3232 affine twisted Edwards representation as follows:

$$TJJ_{13} = \{(x, y) \in \mathbb{F}_{13} \times \mathbb{F}_{13} \mid 3 \cdot x^2 + y^2 = 1 + 8 \cdot x^2 \cdot y^2\} \quad (5.20)$$

3233 Now that we have the abstract definition of our tiny-jubjub curve in twisted Edwards form,
3234 we can compute the set of points by inserting all pairs $(x, y) \in \mathbb{F}_{13} \times \mathbb{F}_{13}$, similarly to how we
3235 computed the curve points in its Weierstraß or Edwards representation. We get the following:

$$\begin{aligned}
 TJJ_{13} = \{ &(0, 1), (0, 12), (1, 2), (1, 11), (2, 6), (2, 7), (3, 0), (5, 5), (5, 8), (6, 4), \\
 &(6, 9), (7, 4), (7, 9), (8, 5), (8, 8), (10, 0), (11, 6), (11, 7), (12, 2), (12, 11) \}
 \end{aligned} \quad (5.21)$$

3236

```

3237 sage: F13 = GF(13)                                     362
3238 sage: L_ETJJ = []                                       363
3239 .....: for x in F13:                                     364
3240 .....:     for y in F13:                                 365
3241 .....:         if F13(3)*x^2 + y^2 == 1+ F13(8)*x^2*y^2: 366
3242 .....:             L_ETJJ.append((x, y))                 367
3243 sage: ETJJ = Set(L_ETJJ)                                368

```

Twisted Edwards group law As we have seen, twisted Edwards curves are equivalent to Montgomery curves, and, as such, also have a group law. However, in contrast to Montgomery and Weierstraß curves, the group law of SNARK-friendly twisted Edwards curves can be described by a single computation that works in all cases, no matter if we add the neutral element, the inverse, or if we have to double a point. To see what the group law looks like, first observe that the point $(0, 1)$ is a solution to $a \cdot x^2 + y^2 = 1 + d \cdot x^2 \cdot y^2$ for any curve. The sum of any two points $(x_1, y_1), (x_2, y_2)$ on an Edwards curve $E(\mathbb{F})$ is then given by the following equation:

$$(x_1, y_1) \oplus (x_2, y_2) = \left(\frac{x_1 y_2 + y_1 x_2}{1 + d x_1 x_2 y_1 y_2}, \frac{y_1 y_2 - a x_1 x_2}{1 - d x_1 x_2 y_1 y_2} \right) \quad (5.22)$$

and it can be shown that the point $(0, 1)$ serves as the neutral element and the inverse of a point (x_1, y_1) is given by $(-x_1, y_1)$.

Example 83. Lets look at the tiny-jubjub curve in Edwards form from example 5.20 again. As we have seen, this curve is given by

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$$TJJ_13 = \{(0, 1), (0, 12), (1, 2), (1, 11), (2, 6), (2, 7), (3, 0), (5, 5), (5, 8), (6, 4), (6, 9), (7, 4), (7, 9), (8, 5), (8, 8), (10, 0), (11, 6), (11, 7), (12, 2), (12, 11)\}$$

To get an understanding of the twisted Edwards addition law, let's first add the neutral element $(0, 1)$ to itself. We apply the group law 5.22 and get the following:

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$$\begin{aligned} (0, 1) \oplus (0, 1) &= \left(\frac{0 \cdot 1 + 1 \cdot 0}{1 + 8 \cdot 0 \cdot 0 \cdot 1 \cdot 1}, \frac{1 \cdot 1 - 3 \cdot 0 \cdot 0}{1 - 8 \cdot 0 \cdot 0 \cdot 1 \cdot 1} \right) \\ &= (0, 1) \end{aligned}$$

So, as expected, the neutral element added to itself gives the neutral element again. Now let's add the neutral element to some other curve point. We get the following:

$$\begin{aligned} (0, 1) \oplus (8, 5) &= \left(\frac{0 \cdot 5 + 1 \cdot 8}{1 + 8 \cdot 0 \cdot 8 \cdot 1 \cdot 5}, \frac{1 \cdot 5 - 3 \cdot 0 \cdot 8}{1 - 8 \cdot 0 \cdot 8 \cdot 1 \cdot 5} \right) \\ &= (8, 5) \end{aligned}$$

Again, as expected, adding the neutral element to any element will result in that element again. Given any curve point (x, y) , we know that its inverse is given by $(-x, y)$. To see how the addition of a point to its inverse works, we compute as follows:

$$\begin{aligned} (5, 5) \oplus (8, 5) &= \left(\frac{5 \cdot 5 + 5 \cdot 8}{1 + 8 \cdot 5 \cdot 8 \cdot 5 \cdot 5}, \frac{5 \cdot 5 - 3 \cdot 5 \cdot 8}{1 - 8 \cdot 5 \cdot 8 \cdot 5 \cdot 5} \right) \\ &= \left(\frac{12 + 1}{1 + 5}, \frac{12 - 3}{1 - 5} \right) \\ &= \left(\frac{0}{6}, \frac{12 + 10}{1 + 8} \right) \\ &= \left(0, \frac{9}{9} \right) \\ &= (0, 1) \end{aligned}$$

Adding a curve point to its inverse gives the neutral element, as expected. As we have seen from these examples, the twisted Edwards addition law handles edge cases particularly well and in a unified way.

5.2 Elliptic Curve Pairings

As we have seen in equation 4.6, some groups come with the notation of a so-called pairing map, which is a non-degenerate bilinear map from two groups into another group.

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In this section, we discuss **pairings on elliptic curves**, which form the basis of several zk-SNARKs and other zero-knowledge proof schemes. The SNARKs derived from pairings have the advantage of constant proof sizes, which is crucial to blockchains.

We start out by defining elliptic curve pairings and discussing a simple application which bears some resemblance to more advanced SNARKs. We then introduce the pairings arising from elliptic curves and describe Miller's algorithm, which makes these pairings practical rather than just theoretically interesting.

Elliptic curves have a few structures, like the Weil or the Tate map that qualifies as pairing.

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Embedding Degrees As we will see in what follows, every elliptic curve gives rise to a pairing map. However, we will also see in example XXX that not every such pairing can be efficiently computed. In order to distinguish curves with efficiently computable pairings from the rest, we need to start with an introduction to the so-called **embedding degree** of a curve.

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ence

Definition 5.2.0.1. Embedding degree

Let \mathbb{F} be a finite field, let $E(\mathbb{F})$ be an elliptic curve over \mathbb{F} , and let n be a prime number that divides the order of $E(\mathbb{F})$. The embedding degree of $E(\mathbb{F})$ with respect to n is then the smallest integer k such that n divides $q^k - 1$.

Fermat's little theorem (page 22 ff.) implies that every curve has at least **some** embedding degree k , since at least $k = n - 1$ is always a solution to the congruency $q^k \equiv 1 \pmod{n}$. This implies that the remainder of the integer division of $q^k - 1$ by n is 0.

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Example 84. To get a better intuition of the embedding degree, let's consider the elliptic curve $E_1(\mathbb{F}_5)$ from example 68. We know from 68 that the order of $E_1(\mathbb{F}_5)$ is 9, and, since the only prime factor of 9 is 3, we compute the embedding degree of $E_1(\mathbb{F}_5)$ with respect to 3.

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To find the embedding degree, we have to find the smallest integer k such that 3 divides $q^k - 1 = 5^k - 1$. We try and increment until we find a proper k .

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$$k = 1: 5^1 - 1 = 4$$

not divisible by 3

$$k = 2: 5^2 - 1 = 24$$

divisible by 3

Now we know that the embedding degree of $E_1(\mathbb{F}_5)$ is 2 relative to the prime factor 3.

Example 85. Let us consider the tiny-jubjub curve TJJ_13 from example 69. We know from 69 that the order of TJJ_13 is 20, and that the order therefore has two prime factors. A "large" prime factor 5 and a small prime factor 2.

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We start by computing the embedding degree of TJJ_13 with respect to the large prime factor 5. To find that embedding degree, we have to find the smallest integer k such that 5 divides $q^k - 1 = 13^k - 1$. We try and increment until we find a proper k .

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$$k = 1: 13^1 - 1 = 12$$

not divisible by 5

$$k = 2: 13^2 - 1 = 168$$

not divisible by 5

$$k = 3: 13^3 - 1 = 2196$$

not divisible by 5

$$k = 4: 13^4 - 1 = 28560$$

divisible by 5

Now we know that the embedding degree of TJJ_13 is 4 relative to the prime factor 5.

In real-world applications, like on pairing-friendly elliptic curves such as BLS_12-381, usually only the embedding degree of the large prime factor is relevant, which in the case of our tiny-jubjub curve is represented by 5. It should be noted, however that every prime factor of a curve's order has its own notation of embedding degree despite the fact that this is mostly irrelevant in applications.

To find the embedding degree of the small prime factor 2, we have to find the smallest integer k such that 2 divides $q^k - 1 = 13^k - 1$. We try and increment until we find a proper k .

$$k = 1: 13^1 - 1 = 12 \quad \text{divisible by 2}$$

Now we know that the embedding degree of TJJ_13 is 1 relative to the prime factor 2. As we have seen, different prime factors can have different embedding degrees in general.

```

sage: p = 13
sage: # large prime factor
sage: n = 5
sage: for k in range(1,5): # Fermat's little theorem
.....:     if (p^k-1)%n == 0:
.....:         break
sage: k
4
sage: # small prime factor
sage: n = 2
sage: for k in range(1,2): # Fermat's little theorem
.....:     if (p^k-1)%n == 0:
.....:         break
sage: k
1

```

Example 86. To give an example of a cryptographically secure real-world elliptic curve that does not have a small embedding degree, let's look at curve Secp256k1 again. We know from 70 that the order of this curve is a prime number, so we only have a single embedding degree.

To test potential embedding degrees k , say, in the range $1 \dots 1000$, we can invoke Sage and compute as follows:

```

sage: p = 1157920892373161954235709850086879078532699846656405
      64039457584007908834671663
sage: n = 1157920892373161954235709850086879078528375642790749
      04382605163141518161494337
sage: for k in range(1,1000):
.....:     if (p^k-1)%n == 0:
.....:         break
sage: k
999

```

We see that Secp256k1 has at least no embedding degree $k < 1000$, which renders Secp256k1 a curve that has no small embedding degree. This property will be of importance later on.

Elliptic Curves over extension fields Suppose that p is a prime number, and \mathbb{F}_p its associated prime field. We know from equation 4.37 that the fields \mathbb{F}_{p^m} are extensions of \mathbb{F}_p in the sense

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3328 that \mathbb{F}_p is a subfield of \mathbb{F}_{p^m} . This implies that we can extend the affine plane that an elliptic
 3329 curve is defined on by changing the base field to any extension field. To be more precise, let
 3330 $E(\mathbb{F}) = \{(x, y) \in \mathbb{F} \times \mathbb{F} \mid y^2 = x^3 + a \cdot x + b\}$ be an affine short Weierstraß curve, with parameters
 3331 a and b taken from \mathbb{F} . If \mathbb{F}' is an extension field of \mathbb{F} , then we extend the domain of the curve
 3332 by defining $E(\mathbb{F}')$ as follows:

$$E(\mathbb{F}') = \{(x, y) \in \mathbb{F}' \times \mathbb{F}' \mid y^2 = x^3 + a \cdot x + b\} \quad (5.23)$$

3333 While we did not change the defining parameters, we consider curve points from the affine
 3334 plane over the extension field now. Since $\mathbb{F} \subset \mathbb{F}'$, it can be shown that the original elliptic curve
 3335 $E(\mathbb{F})$ is a sub-curve of the extension curve $E(\mathbb{F}')$.

Example 87. Consider the prime field \mathbb{F}_5 from example 62 and the elliptic curve $E_1(\mathbb{F}_5)$ from
 example 68. Since we know from XXX that \mathbb{F}_{5^2} is an extension field of \mathbb{F}_5 , we can extend the
 definition of $E_1(\mathbb{F}_5)$ to define a curve over \mathbb{F}_{5^2} :

$$E_1(\mathbb{F}_{5^2}) = \{(x, y) \in \mathbb{F} \times \mathbb{F} \mid y^2 = x^3 + x + 1\}$$

3336 Since \mathbb{F}_{5^2} contains 25 points, in order to compute the set $E_1(\mathbb{F}_{5^2})$, we have to try $25 \cdot 25 = 625$
 3337 pairs, which is probably a bit too much for the average motivated reader. Instead, we invoke
 3338 Sage to compute the curve for us. To do, we so choose the representation of \mathbb{F}_{5^2} from XXX. We
 3339 get:

3340	<code>sage: F5= GF(5)</code>	391
3341	<code>sage: F5t.<t> = F5[]</code>	392
3342	<code>sage: P = F5t(t^2+2)</code>	393
3343	<code>sage: P.is_irreducible()</code>	394
3344	<code>True</code>	395
3345	<code>sage: F5_2.<t> = GF(5^2, name='t', modulus=P)</code>	396
3346	<code>sage: E1F5_2 = EllipticCurve(F5_2, [1, 1])</code>	397
3347	<code>sage: E1F5_2.order()</code>	398
3348	<code>27</code>	399

The curve $E_1(\mathbb{F}_{5^2})$ consist of 27 points, in contrast to curve $E_1(\mathbb{F}_5)$, which consists of 9 points.
 Printing the points gives the following:

$$\begin{aligned} E_1(\mathbb{F}_{5^2}) = \{ & \mathcal{O}, (0, 4), (0, 1), (3, 4), (3, 1), (4, 3), (4, 2), (2, 4), (2, 1), \\ & (4t + 3, 3t + 4), (4t + 3, 2t + 1), (3t + 2, t), (3t + 2, 4t), \\ & (2t + 2, t), (2t + 2, 4t), (2t + 1, 4t + 4), (2t + 1, t + 1), \\ & (2t + 3, 3), (2t + 3, 2), (t + 3, 2t + 4), (t + 3, 3t + 1), \\ & (3t + 1, t + 4), (3t + 1, 4t + 1), (3t + 3, 3), (3t + 3, 2), (1, 4t) \} \end{aligned}$$

3349 As we can see, curve $E_1(\mathbb{F}_5)$ sits inside curve $E(\mathbb{F}_{5^2})$, which is implied by \mathbb{F}_5 being a subfield
 3350 of \mathbb{F}_{5^2} .

3351 **Full torsion groups** The fundamental theorem of finite cyclic groups XXX implies that every
 3352 prime factor n of a cyclic group's order defines a subgroup of the size of the prime factor. Such
 3353 a subgroup is called an n -torsion group. We have seen many of those subgroups in the examples
 3354 XXX and XXX.

3355 When we consider elliptic curve extensions as defined in 5.23, we could ask what happens
 3356 to the n -torsion groups in the extension. One might intuitively think that their extension just

parallels the extension of the curve. For example, when $E(\mathbb{F}_p)$ is a curve over prime field \mathbb{F}_p , with some n -torsion group \mathbb{G} and when we extend the curve to $E(\mathbb{F}_{p^m})$, then there is a bigger n -torsion group such that \mathbb{G} is a subgroup. This might make intuitive sense, as $E(\mathbb{F}_p)$ is a sub-curve of $E(\mathbb{F}_{p^m})$.

However, the actual situation is a bit more surprising than that. To see that, let \mathbb{F}_p be a prime field and let $E(\mathbb{F}_p)$ be an elliptic curve of order r , with embedding degree k and n -torsion group $E(\mathbb{F}_p)[n]$ for the same prime factor n of r . Then it can be shown that the n -torsion group $E(\mathbb{F}_{p^m})[n]$ of a curve extension is equal to $E(\mathbb{F}_p)[n]$, as long as the power m is less than the embedding degree k of $E(\mathbb{F}_p)$.

However, for the prime power p^m , for any $m \geq k$, $E(\mathbb{F}_{p^m})[n]$ is strictly larger than $E(\mathbb{F}_p)[n]$ and contains $E(\mathbb{F}_p)[n]$ as a subgroup. We call the n -torsion group $E(\mathbb{F}_{p^k})[n]$ of the extension of E over \mathbb{F}_{p^k} the **full n -torsion group** of that elliptic curve. It can be shown that it contains n^2 many elements and consists of $n + 1$ subgroups, one of which is $E(\mathbb{F}_p)[n]$.

So, roughly speaking, when we consider **towers of curve extensions** $E(\mathbb{F}_{p^m})$ ordered by the prime power m , then the n -torsion group stays constant for every level m , that is smaller than the embedding degree, while it suddenly blossoms into a larger group on level k with $n + 1$ subgroups, and then stays like that for any level m larger than k . In other words, once the extension field is big enough to find one more point of order n (that is not defined over the base field), then we actually find all of the points in the full torsion group.

Example 88. Consider curve $E_1(\mathbb{F}_5)$ again. We know that it contains a 3-torsion group and that the embedding degree of 3 is 2. From this we can deduce that we can find the full 3-torsion group $E_1[3]$ in the curve extension $E_1(\mathbb{F}_{5^2})$, the latter of which we computed in example 87.

Since that curve is small, in order to find the full 3-torsion, we can loop through all elements of $E_1(\mathbb{F}_{5^2})$ and check the defining equation $[3]P = \mathcal{O}$. Invoking Sage, we compute as follows:

```
sage: INF = E1F5_2(0) # Point at infinity          400
sage: L_E1_3 = []                                     401
sage: for p in E1F5_2:                               402
.....:     if 3*p == INF:                           403
.....:         L_E1_3.append(p)                       404
sage: E1_3 = Set(L_E1_3) # Full 3-torsion set        405
```

We get the following result:

$$E_1[3] = \{\mathcal{O}, (1, t), (1, 4t), (2, 1), (2, 4), (2t + 1, t + 1), (2t + 1, 4t + 4), (3t + 1, t + 4), (3t + 1, 4t + 1)\}$$

Example 89. Consider the tiny-jubjub curve from example 69. We know from example 85 that it contains a 5-torsion group and that the embedding degree of 5 is 4. This implies that we can find the full 5-torsion group $TJJ_13[5]$ in the curve extension $TJJ_13(\mathbb{F}_{13^4})$.

To compute the full torsion, first observe that, since \mathbb{F}_{13^4} contains 28561 elements, computing $TJJ_13(\mathbb{F}_{13^4})$ means checking $28561^2 = 815730721$ elements. From each of these curve points P , we then have to check the equation $[5]P = \mathcal{O}$. Doing this for 815730721 is a bit too slow even on a computer.

Fortunately, Sage has a way to loop through points of a given order efficiently. The following Sage code provides a way to compute the full torsion group:

```
sage: # define the extension field                    406
sage: F13= GF(13) # prime field                      407
sage: F13t.<t> = F13[] # polynomials over t           408
```

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```

3400 sage: P = F13t(t^4+2) # irreducible polynomial of degree 4 409
3401 sage: P.is_irreducible() 410
3402 True 411
3403 sage: F13_4.<t> = GF(13^4, name='t', modulus=P) # F_{13^4} 412
3404 sage: TJJF13_4 = EllipticCurve(F13_4,[8,8]) # tiny-jubjub 413
3405     extension
3406 sage: # compute the full 5-torsion 414
3407 sage: L_TJJF13_4_5 = [] 415
3408 sage: INF = TJJF13_4(0) 416
3409 sage: for P in INF.division_points(5): # [5]P == INF 417
3410     ....:     L_TJJF13_4_5.append(P) 418
3411 sage: len(L_TJJF13_4_5) 419
3412 25 420
3413 sage: TJJF13_4_5 = Set(L_TJJF13_4_5) 421

```

3414 As expected, we get a group that contains $5^2 = 25$ elements. As it's rather tedious to write
 3415 this group down, and as we don't need it in what follows, we forgo doing this. To see that the
 3416 embedding degree 4 is actually the smallest prime power to find the full 5-torsion group, let's
 3417 compute the 5-torsion group over of the tiny-jubjub curve of the extension field \mathbb{F}_{13^3} . We get
 3418 the following:

```

3419 sage: # define the extension field 422
3420 sage: P = F13t(t^3+2) # irreducible polynomial of degree 3 423
3421 sage: P.is_irreducible() 424
3422 True 425
3423 sage: F13_3.<t> = GF(13^3, name='t', modulus=P) # F_{13^3} 426
3424 sage: TJJF13_3 = EllipticCurve(F13_3,[8,8]) # tiny-jubjub 427
3425     extension
3426 sage: # compute the 5-torsion 428
3427 sage: L_TJJF13_3_5 = [] 429
3428 sage: INF = TJJF13_3(0) 430
3429 sage: for P in INF.division_points(5): # [5]P == INF 431
3430     ....:     L_TJJF13_3_5.append(P) 432
3431 sage: len(L_TJJF13_3_5) 433
3432 5 434
3433 sage: TJJF13_3_5 = Set(L_TJJF13_3_5) # full 5-torsion 435

```

3434 As we can see, the 5-torsion group of tiny-jubjub over \mathbb{F}_{13^3} is equal to the 5-torsion group
 3435 of tiny-jubjub over \mathbb{F}_{13} itself.

3436 *Example 90.* Let's look at the curve Secp256k1. We know from example 70 that the curve is of
 3437 some prime order r . Because of this, the only n -torsion group to consider is the curve itself, so
 3438 the curve group is the r -torsion.

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However, in order to find the full r -torsion of Secp256k1, we need to compute the embed-
 ding degree k . And as we have seen in XXX it is at least not small. However, we know from
 Fermat's little theorem (page 22 ff.) that a finite embedding degree must exist. It can be shown
 that it is given by the following 256-bit number:

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$$k = 192986815395526992372618308347813175472927379845817397100860523586360249056$$

3439 This means that the embedding degree is **very large**, which implies that the field extension \mathbb{F}_{p^k} is
 3440 very large too. To understand how big \mathbb{F}_{p^k} is, recall that an element of \mathbb{F}_{p^m} can be represented as

3441 a string $[x_0, \dots, x_m]$ of m elements, each containing a number from the prime field \mathbb{F}_p . Now, in
 3442 the case of Secp256k1, such a representation has k -many entries, each of them 256 bits in size.
 3443 So, without any optimizations, representing such an element would need $k \cdot 256$ bits, which is
 3444 too much to be represented in the observable universe.

3445 **Torsion subgroups** As we have stated above, any full n -torsion group contains $n + 1$ cyclic
 3446 subgroups, two of which are of particular interest in pairing-based elliptic curve cryptography.
 3447 To characterize these groups, we need to consider the so-called **Frobenius endomorphism** of
 3448 an elliptic curve $E(\mathbb{F})$ over some finite field \mathbb{F} of characteristic p :

$$\pi : E(\mathbb{F}) \rightarrow E(\mathbb{F}) : \begin{array}{ccc} (x, y) & \mapsto & (x^p, y^p) \\ \mathcal{O} & \mapsto & \mathcal{O} \end{array} \quad (5.24)$$

3449 It can be shown that π maps curve points to curve points. The first thing to note is that, in case
 3450 \mathbb{F} is a prime field, the Frobenius endomorphism acts trivially, since $(x^p, y^p) = (x, y)$ on prime
 3451 fields due to Fermat's little theorem (page 22 ff.). This means that the Frobenius map is more
 3452 interesting over prime field extensions.

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3453 With the Frobenius map at hand, we can characterize two important subgroups of the full
 3454 n -torsion. The first subgroup is the n -torsion group that already exists in the curve over the
 3455 base field. In pairing-based cryptography, this group is usually written as \mathbb{G}_1 , assuming that the
 3456 prime factor n in the definition is implicitly given. Since we know that the Frobenius map acts
 3457 trivially on curves over the prime field, we can define \mathbb{G}_1 as follows:

$$\mathbb{G}_1[n] := \{(x, y) \in E[n] \mid \pi(x, y) = (x, y)\} \quad (5.25)$$

3458 In more mathematical terms, this definition means that \mathbb{G}_1 is the **Eigenspace** of the Frobenius
 3459 map with respect to the **Eigenvalue 1**.

3460 It can be shown that there is another subgroup of the full n -torsion group that can be char-
 3461 acterized by the Frobenius map. In the context of so-called **type 3 pairing-based cryptography**,
 3462 this subgroup is usually called \mathbb{G}_2 and it is defined as follows:

$$\mathbb{G}_2[n] := \{(x, y) \in E[n] \mid \pi(x, y) = [p](x, y)\} \quad (5.26)$$

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ation or
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3463 In mathematical terms, \mathbb{G}_2 is the **Eigenspace** of the Frobenius map with respect to the
 3464 **Eigenvalue p** .

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phy

3465 *Notation and Symbols 12.* If the prime factor n of a curve's order is clear from the context, we
 3466 sometimes simply write \mathbb{G}_1 and \mathbb{G}_2 to mean $\mathbb{G}_1[n]$ and $\mathbb{G}_2[n]$, respectively.

3467 It should be noted, however that other definitions of \mathbb{G}_2 also exists in the literature. However,
 3468 in the context of pairing-based cryptography, this is the most common one. It is particularly
 3469 useful because we can define hash functions that map into \mathbb{G}_2 , which is not possible for all
 3470 subgroups of the full n -torsion.

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ences?

3471 *Example 91.* Consider the curve $E_1(\mathbb{F}_5)$ from example 68 again. As we have seen, this curve
 3472 has the embedding degree $k = 2$, and a full 3-torsion group is given as follows:

$$\begin{aligned} E_1[3] = \{ & \mathcal{O}, (2, 1), (2, 4), (1, t), (1, 4t), (2t + 1, t + 1), \\ & (2t + 1, 4t + 4), (3t + 1, t + 4), (3t + 1, 4t + 1) \} \end{aligned} \quad (5.27)$$

3473 According to the general theory, $E_1[3]$ contains 4 subgroups, and we can characterize the
 3474 subgroups \mathbb{G}_1 and \mathbb{G}_2 using the Frobenius endomorphism. Unfortunately, at the time of writing,

Sage does not have a predefined Frobenius endomorphism for elliptic curves, so we have to use the Frobenius endomorphism of the underlying field as a temporary workaround. We compute as follows:

```

3478 sage: L_G1 = []
3479 sage: for P in E1_3:
3480     ....:     PiP = E1F5_2([a.frobenius() for a in P]) # pi(P)
3481     ....:     if P == PiP:
3482     ....:         L_G1.append(P)
3483 sage: G1 = Set(L_G1)

```

As expected, the group $\mathbb{G}_1 = \{\mathcal{O}, (2, 4), (2, 1)\}$ is identical to the 3-torsion group of the (unextended) curve over the prime field $E_1(\mathbb{F}_5)$. We can use almost the same algorithm to compute the group \mathbb{G}_2 and get the following:

```

3487 sage: L_G2 = []
3488 sage: for P in E1_3:
3489     ....:     PiP = E1F5_2([a.frobenius() for a in P]) # pi(P)
3490     ....:     pP = 5*P # [5]P
3491     ....:     if pP == PiP:
3492     ....:         L_G2.append(P)
3493 sage: G2 = Set(L_G2)

```

Thus, we have computed the the second subgroup of the full 3-torsion group of curve E_1 as the set $\mathbb{G}_2 = \{\mathcal{O}, (1, t), (1, 4t)\}$.

Example 92. Consider the tiny-jubjub curve *TJJ_13* from example 69. In example 89, we computed its full 5 torsion, which is a group that has 6 subgroups. We compute G_1 using Sage as follows:

```

3499 sage: L_TJJ_G1 = []
3500 sage: for P in TJJF13_4_5:
3501     ....:     PiP = TJJF13_4([a.frobenius() for a in P]) # pi(P)
3502     ....:     if P == PiP:
3503     ....:         L_TJJ_G1.append(P)
3504 sage: TJJ_G1 = Set(L_TJJ_G1)

```

We get $\mathbb{G}_1 = \{\mathcal{O}, (7, 2), (8, 8), (8, 5), (7, 11)\}$

```

3506 sage: L_TJJ_G1 = []
3507 sage: for P in TJJF13_4_5:
3508     ....:     PiP = TJJF13_4([a.frobenius() for a in P]) # pi(P)
3509     ....:     pP = 13*P # [5]P
3510     ....:     if pP == PiP:
3511     ....:         L_TJJ_G1.append(P)
3512 sage: TJJ_G1 = Set(L_TJJ_G1)

```

$\mathbb{G}_2 = \{\mathcal{O}, (9t^2 + 7, t^3 + 11t), (9t^2 + 7, 12t^3 + 2t), (4t^2 + 7, 5t^3 + 10t), (4t^2 + 7, 8t^3 + 3t)\}$

Example 93. Consider Bitcoin's curve *Secp256k1* again. Since the group \mathbb{G}_1 is identical to the torsion group of the unextended curve, and since *Secp256k1* has prime order, we know that, in this case, \mathbb{G}_1 is identical to *Secp256k1*. It is however, infeasible not to compute not only \mathbb{G}_2 itself, but to even compute an average element of \mathbb{G}_2 , as elements need too much storage to be

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3518 representable in this universe.

3519 **The Weil pairing** In this part, we consider a pairing function defined on the subgroups $\mathbb{G}_1[r]$
 3520 and $\mathbb{G}_2[r]$ of the full r -torsion $E[r]$ of a short Weierstraß elliptic curve. To be more precise, let
 3521 $E(\mathbb{F}_p)$ be an elliptic curve of embedding degree k such that r is a prime factor of its order. Then
 3522 the **Weil pairing** is a bilinear, non-degenerate map:

$$e(\cdot, \cdot) : \mathbb{G}_1[r] \times \mathbb{G}_2[r] \rightarrow \mathbb{F}_{p^k} ; (P, Q) \mapsto (-1)^r \cdot \frac{f_{r,P}(Q)}{f_{r,Q}(P)} \quad (5.28)$$

The extension field elements $f_{r,P}(Q), f_{r,Q}(P) \in \mathbb{F}_{p^k}$ are computed by **Miller's algorithm**:

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Algorithm 8 Miller's algorithm for short Weierstraß curves $y^2 = x^3 + ax + b$

Require: $r > 3, P \in E[r], Q \in E[r]$ and

$b_0, \dots, b_t \in \{0, 1\}$ with $r = b_0 \cdot 2^0 + b_1 \cdot 2^1 + \dots + b_t \cdot 2^t$ and $b_t = 1$

procedure MILLER'S ALGORITHM(P, Q)

if $P = \mathcal{O}$ or $Q = \mathcal{O}$ or $P = Q$ **then**

return $f_{r,P}(Q) \leftarrow (-1)^r$

end if

$(x_T, y_T) \leftarrow (x_P, y_P)$

$f_1 \leftarrow 1$

$f_2 \leftarrow 1$

for $j \leftarrow t - 1, \dots, 0$ **do**

$m \leftarrow \frac{3 \cdot x_T^2 + a}{2 \cdot y_T}$

$f_1 \leftarrow f_1^2 \cdot (y_Q - y_T - m \cdot (x_Q - x_T))$

$f_2 \leftarrow f_2^2 \cdot (x_Q + 2x_T - m^2)$

$x_{2T} \leftarrow m^2 - 2x_T$

$y_{2T} \leftarrow -y_T - m \cdot (x_{2T} - x_T)$

$(x_T, y_T) \leftarrow (x_{2T}, y_{2T})$

if $b_j = 1$ **then**

$m \leftarrow \frac{y_T - y_P}{x_T - x_P}$

$f_1 \leftarrow f_1 \cdot (y_Q - y_T - m \cdot (x_Q - x_T))$

$f_2 \leftarrow f_2 \cdot (x_Q + (x_P + x_T) - m^2)$

$x_{T+P} \leftarrow m^2 - x_T - x_P$

$y_{T+P} \leftarrow -y_T - m \cdot (x_{T+P} - x_T)$

$(x_T, y_T) \leftarrow (x_{T+P}, y_{T+P})$

end if

end for

$f_1 \leftarrow f_1 \cdot (x_Q - x_T)$

return $f_{r,P}(Q) \leftarrow \frac{f_1}{f_2}$

end procedure

3523 Understanding how the algorithm works in detail requires the concept of **divisors**, which is
 3524 outside of the scope this book. The interested reader might look at XXX.

3525 In real-world applications of pairing-friendly elliptic curves, the embedding degree is usu-
 3526 ally a small number like 2, 4, 6 or 12, and the number r is the largest prime factor of the curve's
 3527 order.
 3528

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3529 *Example 94.* Consider curve $E_1(\mathbb{F}_5)$ from example 68. Since the only prime factor of the
 3530 group's order is 3, we cannot compute the Weil pairing on this group using our definition of
 3531 Miller's algorithm. In fact, since \mathbb{G}_1 is of order 3, executing the if statement on line XXX will
 3532 lead to a "division by zero" error in the computation of the slope m .

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Example 95. Consider the tiny-jubjub curve $TJJ_13(\mathbb{F}_{13})$ from example 69 again. We want to
 instantiate the general definition of the Weil pairing for this example. To do so, recall that, as we
 have see in example 85, its embedding degree is 4, and that we have the following type-3 pairing
 groups (where \mathbb{G}_1 and \mathbb{G}_2 are subgroups of the full 5-torsion found in the curve $TJJ_13(\mathbb{F}_{13^4})$):

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$$\mathbb{G}_1 = \{\mathcal{O}, (7, 2), (8, 8), (8, 5), (7, 11)\}$$

$$\mathbb{G}_2 = \{\mathcal{O}, (9t^2 + 7, t^3 + 11t), (9t^2 + 7, 12t^3 + 2t), (4t^2 + 7, 5t^3 + 10t), (4t^2 + 7, 8t^3 + 3t)\}$$

3533 The type-3 Weil pairing is a map $e(\cdot, \cdot) : \mathbb{G}_1 \times \mathbb{G}_2 \rightarrow \mathbb{F}_{13^4}$. From the first if-statement in
 3534 Miller's algorithm, we can deduce that $e(\mathcal{O}, Q) = 1$ as well as $e(P, \mathcal{O}) = 1$ for all arguments
 3535 $P \in \mathbb{G}_1$ and $Q \in \mathbb{G}_2$. In order to compute a non-trivial Weil pairing, we choose the arguments
 3536 $P = (7, 2)$ and $Q = (9t^2 + 7, 12t^3 + 2t)$.

3537 To compute the pairing $e((7, 2), (9t^2 + 7, 12t^3 + 2t))$, we have to compute the extension field
 3538 elements $f_{5,P}(Q)$ and $f_{5,Q}(P)$ by applying Miller's algorithm. Do do so, observe that we have
 3539 $5 = 1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2$, so we get $t = 2$ as well as $b_0 = 1, b_1 = 0$ and $b_2 = 1$. The loop therefore
 3540 needs to be executed two times.

Computing $f_{5,P}(Q)$, we initiate $(x_T, y_T) = (7, 2)$ as well as $f_1 = 1$ and $f_2 = 1$. Then we
 proceed as follows:

j	b_j	m	f_1	f_2	x_{2T}	y_{2T}	x_{T+P}	y_{T+P}
1	.							

$$\begin{aligned}
m &= \frac{3 \cdot x_T^2 + a}{2 \cdot y_T} \\
&= \frac{3 \cdot 2^2 + 1}{2 \cdot 4} = \frac{3}{3} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
f_1 &= f_1^2 \cdot (y_Q - y_T - m \cdot (x_Q - x_T)) \\
&= 1^2 \cdot (t - 4 - 1 \cdot (1 - 2)) = t - 4 + 1 \\
&= t + 2
\end{aligned}$$

$$\begin{aligned}
f_2 &= f_2^2 \cdot (x_Q + 2x_T - m^2) \\
&= 1^2 \cdot (1 + 2 \cdot 2 - 1^2) = (1 + 4 - 1) \\
&= 4
\end{aligned}$$

$$\begin{aligned}
x_{2T} &= m^2 - 2x_T \\
&= 1^2 - 2 \cdot 2 = -3 \\
&= 2
\end{aligned}$$

$$\begin{aligned}
y_{2T} &= -y_T - m \cdot (x_{2T} - x_T) \\
&= -4 - 1 \cdot (2 - 2) = -4 \\
&= 1
\end{aligned}$$

We update $(x_T, y_T) = (2, 1)$ and, since $b_0 = 1$, we have to execute the if statement on line XXX in the **for** loop. However, since we only loop a single time, we don't need to compute y_{T+P} , since we only need the updated x_T in the final step. We get:

$$\begin{aligned}
m &= \frac{y_T - y_P}{x_T - x_P} \\
&= \frac{1 - 4}{2 - x_P}
\end{aligned}$$

$$f_1 = f_1 \cdot (y_Q - y_T - m \cdot (x_Q - x_T))$$

$$f_2 = f_2 \cdot (x_Q + (x_P + x_T) - m^2)$$

$$x_{T+P} = m^2 - x_T - x_P$$

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should all lines of all algorithms be numbered?

3541 5.3 Hashing to Curves

3542 Elliptic curve cryptography frequently requires the ability to hash data onto elliptic curves. If
 3543 the order of the curve is not a prime number, hashing to prime number subgroups is also of

importance. In the context of pairing-friendly curves, it is also sometimes necessary to hash specifically onto the group \mathbb{G}_1 or \mathbb{G}_2 .

As we have seen in section 4.1.2, many general methods are known for hashing into groups in general, and finite cyclic groups in particular. As elliptic groups are cyclic, those methods can be utilized in this case, too. However, in what follows we want to describe some methods specific to elliptic curves that are frequently used in real-world applications.

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Try-and-increment hash functions One of the most straight-forward ways of hashing a bit-string onto an elliptic curve point in a secure way is to use a cryptographic hash function together with one of the methods we described in section 4.1.2 to hash to the modular arithmetic base field of the curve. Ideally, the hash function generates an image that is at least one bit longer than the bit representation of the base field modulus.

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The image in the base field can then be interpreted as the x coordinate of the curve point, and the two possible y coordinates are derived from the curve equation, while one of the bits that exceeded the modulus determines which of the two y coordinates to choose.

Such an approach would be deterministic and easy to implement, and it would conserve the cryptographic properties of the original hash function. However, not all x coordinates generated in such a way will result in quadratic residues when inserted into the defining equation. It follows that not all field elements give rise to actual curve points. In fact, on a prime field, only half of the field elements are quadratic residues. Hence, assuming an even distribution of the hash values in the field, this method would fail to generate a curve point in about half of the attempts.

One way to account for this problem is the so-called **try-and-increment** method. Its basic assumption is that, when hashing different values, the result will eventually lead to a valid curve point.

Therefore, instead of simply hashing a string s to the field, we hash the concatenation of s with additional bytes to the field instead. In other words, we use a try-and-increment hash as described in 6. If the first try of hashing to the field does not result in a valid curve point, the counter is incremented, and the hashing is repeated again. This is done until a valid curve point is found.

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This method has a number of advantages: It is relatively easy to implement in code, and it maintains the cryptographic properties of the original hash function. However, it is not guaranteed to find a valid curve point, as there is a chance that all possible values in the chosen size of the counter will fail to generate a quadratic residue. Fortunately, it is possible to make the probability for this arbitrarily small by choosing large enough counters and relying on the (approximate) uniformity of the hash-to-field function.

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If the curve is not of prime order, the result will be a general curve point that might not be in the “large” prime-order subgroup. In this case, a **cofactor clearing** step is then necessary to project the curve point onto the subgroup. This is done by scalar multiplication with the cofactor of prime order with respect to the curves order.

Example 96. Consider the tiny-jubjub curve from example 69. We want to construct a try-and-increment hash function that hashes a binary string s of arbitrary length onto the large prime-order subgroup of size 5.

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Since the curve, as well as our targeted subgroup, is defined over the field \mathbb{F}_{13} , and the binary representation of 13 is $13.bits() = 1101$, we apply SHA256 from Sage’s hashlib library on the concatenation $s||c$ for some binary counter string, and use the first 4 bits of the image to try to hash into \mathbb{F}_{13} . In case we are able to hash to a value z such that $z^3 + 8 \cdot z + 8$ is a quadratic residue in \mathbb{F}_{13} , we use the 5-th bit to decide which of the two possible roots of $z^3 + 8 \cdot z + 8$ we

Algorithm 9 Hash-to- $E(\mathbb{F}_r)$ **Require:** $r \in \mathbb{Z}$ with $r.\text{nbits}() = k$ and $s \in \{0, 1\}^*$ **Require:** Curve equation $y^2 = x^3 + ax + b$ over \mathbb{F}_r **procedure** TRY-AND-INCREMENT(r, k, s) $c \leftarrow 0$ **repeat** $s' \leftarrow s || c.\text{bits}()$ $z \leftarrow H(s')_0 \cdot 2^0 + H(s')_1 \cdot 2^1 + \dots + H(s')_k \cdot 2^k$ $x \leftarrow z^3 + a \cdot z + b$ $c \leftarrow c + 1$ **until** $z < r$ and $x^{\frac{r-1}{2}} \bmod r = 1$ **if** $H(s')_{k+1} == 0$ **then** $y \leftarrow \sqrt{x}$ # (root in \mathbb{F}_r)**else** $y \leftarrow r - \sqrt{x}$ # (root in \mathbb{F}_r)**end if****return** (x, y) **end procedure****Ensure:** $(x, y) \in E(\mathbb{F}_r)$

will choose as the y coordinate. The result is a curve point different from the point at infinity. To project it to a point of \mathbb{G}_1 , we multiply it with the cofactor 4. If the result is still not the point at infinity, it is the result of the hash.

To make this concrete, let $s = '10011001111010110100000111'$ be our binary string that we want to hash onto \mathbb{G}_1 . We use a 4-bit binary counter starting at zero, that is, we choose $c = 0000$. Invoking Sage, we define the try-hash function as follows:

```

sage: import hashlib                                462
sage: def try_hash(s, c):                            463
.....:     s_1 = s+c                                464
.....:     hasher = hashlib.sha256(s_1.encode('utf-8')) 465
.....:     digest = hasher.hexdigest()                466
.....:     d = Integer(digest, base=16)               467
.....:     sign = d.str(2)[-5:-4]                     468
.....:     d = d.str(2)[-4:]                           469
.....:     z = Integer(d, base=2)                     470
.....:     return (z, sign)                           471
sage: try_hash('10011001111010110100000111', '0000') 472
(15, '1')                                           473

```

As we can see, our first attempt to hash into \mathbb{F}_{13} was not successful, as 15 is not a number in \mathbb{F}_{13} , so we increment the binary counter by 1 and try again:

```

sage: try_hash('10011001111010110100000111', '0001') 474
(3, '0')                                           475

```

With this try, we found a hash into \mathbb{F}_{13} . However, this point is not guaranteed to define a curve point. To see that, we insert $z = 3$ into the right side of the Weierstraß equation of the tiny-jubjub curve, and compute $3^3 + 8 * 3 + 8 = 7$. However, 7 is not a quadratic residue in

3616 \mathbb{F}_{13} , since $7^{\frac{13-1}{2}} = 7^6 = 12 = -1$. This means that 3 is not a suitable point, and we have to
 3617 increment the counter two more times:

```

3618 sage: try_hash('10011001111010110100000111', '0010') 476
3619 (3, '0') 477
3620 sage: try_hash('10011001111010110100000111', '0011') 478
3621 (6, '1') 479

```

Since $6^3 + 8 \cdot 6 + 8 = 12$, and we have $\sqrt{12} \in \{5, 8\}$, we finally found the valid x coordinate $x = 6$ for the curve point hash. Now, since the sign bit of this hash is 1, we choose the larger root $y = 8$ as the y coordinate and get the following hash which is a valid curve point point on the tiny-jubjub curve:

$$H('10011001111010110100000111') = (6, 8)$$

In order to project this onto the “large” prime-order subgroup, we have to do cofactor clearing, that is, we have to multiply the point with the cofactor 4. We get the following:

$$[4](6, 8) = \mathcal{O}$$

3622 This means that the hash value is still not right. We therefore have to increment the counter
 3623 two more times again, until we finally find a correct hash to \mathbb{G}_1 :

```

3624 sage: try_hash('10011001111010110100000111', '0100') 480
3625 (0, '1') 481
3626 sage: try_hash('10011001111010110100000111', '0101') 482
3627 (12, '0') 483

```

Since $12^3 + 8 \cdot 12 + 8 = 12$, and we have $\sqrt{12} \in \{5, 8\}$, we found another valid x coordinate $x = 12$ for the curve point hash. Since the sign bit of this hash is 0, we choose the smaller root $y = 5$ as the y coordinate, and get the following hash, which is a valid curve point point on the tiny-jubjub curve:

$$H('10011001111010110100000111') = (12, 5)$$

In order to project this onto the “large” prime-order subgroup we have to do cofactor clearing, again? that is, we have to multiply the point with the cofactor 4. We get the following:

$$[4](12, 5) = (8, 5)$$

3628 So, hashing the binary string '10011001111010110100000111' onto \mathbb{G}_1 gives the hash value
 3629 (8, 5) as a result.

3630 5.4 Constructing elliptic curves

3631 Cryptographically secure elliptic curves like Secp256k1 from example 70 have been known for
 3632 quite some time. Given the latest advancements of cryptography, however, it is often necessary
 3633 to design and instantiate elliptic curves from scratch that satisfy certain very specific properties.

3634 For example, in the context of SNARK development, it was necessary to design a curve that
 3635 can be efficiently implemented inside of a so-called **circuit** in order to enable primitives like
 3636 elliptic curve **signature schemes** in a zero-knowledge proof. Such a curve is given by the Baby-
 3637 jubjub curve in XXX, and we have paralleled its definition by introducing the tiny-jubjub curve

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from example 69. As we have seen, those curves are instances of so-called twisted Edwards curves, and as such have easy to implement addition laws that work without branching. However, we introduced the tiny-jubjub curve out of thin air, as we just gave the curve parameters without explaining how we came up with them.

Another requirement in the context of many so-called **pairing-based zero-knowledge proofing systems** is the existence of a suitable, pairing-friendly curve with a specified security level and a low embedding degree as defined in 5.2.0.1. Famous examples are the BLS_12 and the NMT curves.

The major goal of this section is to explain the most important method of designing elliptic curves with predefined properties from scratch, called the **complex multiplication method**. We will apply this method in section XXX to synthesize a particular BLS_6 curve, which is one of the most insecure curves, but it will serve as the main curve to build our pen-and-paper SNARKs on. As we will see, this curve has a “large” prime factor subgroup of order 13, which implies that we can use our tiny-jubjub curve to implement certain elliptic curve cryptographic primitives in circuits over that BLS_6 curve.

Before we introduce the complex multiplication method, we have to explain a few properties of elliptic curves that are of key importance in understanding the complex multiplication method.

The Trace of Frobenius To understand the complex multiplication method of elliptic curves, we have to define the so-called **trace** of an elliptic curve first.

We know from XXX that elliptic curves over finite fields are products of cyclic groups of finite order. Therefore, an interesting question is whether it is possible to estimate the number of elements that this curve contains. Since an affine short Weierstraß curve consists of pairs (x, y) of elements from a finite field \mathbb{F}_q plus the point at infinity, and the field \mathbb{F}_q contains q elements, the number of curve points cannot be arbitrarily large, since it can contain at most $q^2 + 1$ many elements.

There is however, a more precise estimation, usually called the **Hasse bound**. To understand it, let $E(\mathbb{F}_q)$ be an affine short Weierstraß curve over a finite field \mathbb{F}_w of order q , and let $|E(\mathbb{F}_q)|$ be the order of the curve. Then there is an integer $t \in \mathbb{Z}$, called the **trace of Frobenius** of the curve, such that $|t| \leq 2\sqrt{q}$ and the following equation holds:

$$|E(\mathbb{F})| = q + 1 - t \quad (5.29)$$

A positive trace, therefore, implies that the curve contains less points than the underlying field, whereas a negative trace means that the curve contains more points. However, the estimation $|t| \leq 2\sqrt{q}$ implies that the difference is not very large in either direction, and the number of elements in an elliptic curve is always approximately in the same order of magnitude as the size of the curve’s base field.

Example 97. Consider the elliptic curve $E_1(\mathbb{F}_5)$ from example 68. We know that it contains 9 curve points. Since the order of \mathbb{F}_5 is 5, we compute the trace of $E_1(\mathbb{F})$ to be $t = -3$, since the Hasse bound is given by the following equation:

$$9 = 5 + 1 - (-3)$$

Indeed, we have $|t| \leq 2\sqrt{q}$, since $\sqrt{5} > 2.23$ and $|-3| = 3 \leq 4.46 = 2 \cdot 2.23 < 2 \cdot \sqrt{5}$.

Example 98. To compute the trace of the tiny-jubjub curve, recall from example 77 that the order of TJJ_13 is 20. Since the order of \mathbb{F}_{13} is 13, we can therefore use the Hasse bound and compute the trace as $t = -6$:

$$20 = 13 + 1 - (-6) \quad (5.30)$$

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3677 Again, we have $|t| \leq 2\sqrt{q}$, since $\sqrt{13} > 3.60$ and $|-6| = 6 \leq 7.20 = 2 \cdot 3.60 < 2 \cdot \sqrt{13}$.

Example 99. To compute the trace of Secp256k1, recall from example 70 [that this curve is](#) defined over a prime field with p elements, and that the order of that group is given by r :

check
reference

```
p = 115792089237316195423570985008687907853269984665640564039457584007908834671663
r = 115792089237316195423570985008687907852837564279074904382605163141518161494337
```

Using the Hesse bound $r = p + 1 - t$, we therefore compute $t = p + 1 - r$, which gives the trace of curve Secp256k1 as follows:

```
t = 432420386565659656852420866390673177327
```

3678 As we can see, Secp256k1 contains less elements than its underlying field. However, the
3679 difference is tiny, since the order of Secp256k1 is in the same order of magnitude as the order
3680 of the underlying field. Compared to p and r , t is tiny.

```
3681 sage: p = 1157920892373161954235709850086879078532699846656405 484
3682         64039457584007908834671663
3683 sage: r = 1157920892373161954235709850086879078528375642790749 485
3684         04382605163141518161494337
3685 sage: t = p + 1 - r 486
3686 sage: t.nbits() 487
3687 129 488
3688 sage: abs(RR(t)) <= 2*sqrt(RR(p)) 489
3689 True 490
```

3690 **The j -invariant** As we have seen in XXX, [two elliptic curves \$E_1\(\mathbb{F}\)\$ defined by \$y^2 = x^3 + ax + b\$ and \$E_2\(\mathbb{F}\)\$ defined by \$y^2 + a'x + b'\$ are strictly isomorphic if and only if there is a quadratic](#)
3691 [residue \$d \in \mathbb{F}\$ such that \$a' = ad^2\$ and \$b' = bd^3\$.](#)

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ence

3693 There is, however, a more general way to classify elliptic curves over finite fields \mathbb{F}_q , based
3694 on the so-called **j -invariant** of an elliptic curve with $j(E(\mathbb{F}_q)) \in \mathbb{F}_q$, as defined below:

$$j(E(\mathbb{F}_q)) = (1728 \bmod q) \frac{4 \cdot a^3}{4 \cdot a^3 + (27 \bmod q) \cdot b^2} \quad (5.31)$$

3695 A detailed description of the j -invariant is beyond the scope of this book. For our present
3696 purposes, it is sufficient to note that two elliptic curves $E_1(\mathbb{F})$ and $E_2(\mathbb{F}')$ are isomorphic over
3697 the **algebraic closures** of \mathbb{F} and \mathbb{F}' , if and only if $\overline{\mathbb{F}} = \overline{\mathbb{F}'}$ and $j(E_1) = j(E_2)$.

algebraic
closures

3698 So, the j -invariant is an important tool to classify elliptic curves and it is needed in the com-
3699 plex multiplication method to decide on an actual curve instantiation that implements abstractly
3700 chosen properties.

Example 100. Consider the elliptic curve $E_1(\mathbb{F}_5)$ from example 68. [We compute its \$j\$ -invariant](#) as follows:

check
reference

$$\begin{aligned} j(E_1(\mathbb{F}_5)) &= (1728 \bmod 5) \frac{4 \cdot 1^3}{4 \cdot 1^3 + (27 \bmod 5) \cdot 1^2} \\ &= 3 \frac{4}{4 + 2} \\ &= 3 \cdot 4 = 2 \end{aligned}$$

Example 101. Consider the elliptic curve *TJJ_13* from example 69. We compute its *j*-invariant as follows:

check
reference

$$\begin{aligned}
 j(E_1(\mathbb{F}_5)) &= (1728 \bmod 13) \frac{4 \cdot 8^3}{4 \cdot 8^3 + (27 \bmod 13) \cdot 8^2} \\
 &= 12 \cdot \frac{4 \cdot 5}{4 \cdot 5 + 1 \cdot 12} \\
 &= 12 \cdot \frac{7}{7 + 12} \\
 &= 12 \cdot 7 \cdot 6^{-1} \\
 &= 12 \cdot 7 \cdot 11 \\
 &01
 \end{aligned}$$

3701 *Example 102.* Consider Secp256k1 from example Secp256k1. We compute its *j*-invariant using
3702 Sage:

check
reference

```

3703 sage: p = 1157920892373161954235709850086879078532699846656405 491
3704       64039457584007908834671663
3705 sage: F = GF(p) 492
3706 sage: j = F(1728) * ( (F(4) * F(0) ^3) / (F(4) * F(0) ^3 + F(27) * F(7) ^2) ) 493
3707 sage: j == F(0) 494
3708 True 495

```

3709 **The Complex Multiplication Method** As we have seen in the previous sections, elliptic
3710 curves have various defining properties, like their order, their prime factors, the embedding de-
3711 gree, or the cardinality (number of elements) of the base field. The **complex multiplication**
3712 (CM) method provides a practical way of constructing elliptic curves with pre-defined restric-
3713 tions on the order and the base field.

3714 The method usually starts by choosing a base field \mathbb{F}_q of the curve $E(\mathbb{F}_q)$ we want to con-
3715 struct such that $q = p^m$ for some prime number p , and “ $m \in \mathbb{N}$ with $m \geq 1$. We assume $p > 3$
3716 to simplify things in what follows.

3717 Next, the trace of Frobenius $t \in \mathbb{Z}$ of the curve is chosen such that p and t are coprime, that
3718 is, $\gcd(p, t) = 1$ holds true. The choice of t also defines the curve’s order r , since $r = p + 1 - t$
3719 by the Hasse bound (equation 5.29), so choosing t will define the large order subgroup as well
3720 as all small cofactors. r has to be defined in such a way that the elliptic curve meets the security
3721 requirements of the application it is designed for.

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reference

3722 Note that the choice of p and t also determines the embedding degree k of any prime-order
3723 subgroup of the curve, since k is defined as the smallest number such that the prime order n
3724 divides the number $q^k - 1$.

$$\begin{aligned}
 D &< 0 \\
 D \bmod 4 &= 0 \text{ or } D \bmod 4 = 1 \\
 4q &= t^2 + |D|v^2
 \end{aligned} \tag{5.32}$$

3725 In order for the complex multiplication method to work, neither q nor t can be arbitrary,
3726 but must be chosen in such a way that two additional integers $D \in \mathbb{Z}$ and $v \in \mathbb{Z}$ exist and the
3727 following conditions hold:

If such numbers exist, we call D the **CM-discriminant**, and we know that we can construct a curve $E(\mathbb{F}_q)$ over a finite field \mathbb{F}_q such that the order of the curve is $|E(\mathbb{F}_q)| = q + 1 - t$.

It is the content of the complex multiplication method to actually construct such a curve, that is finding the parameters a and b from \mathbb{F}_q in the defining Weierstraß equation such that the curve has the desired order r .

Finding solutions to equation 5.29, can be achieved in different ways, but we will forego the fine detail here. In general, it can be said that there are well-known constraints for elliptic curve families (e.g. the BLS (ECT) families) that provides families of solutions. In what follows, we will look at one type curve in the BLS-family, which gives an entire range of solutions. Are we looking at a subtype of BLS or is BLS the specific type we're referring to?

Assuming that the proper parameters q , t , D and v are found, we have to compute the so-called **Hilbert class polynomial** $H_D \in \mathbb{Z}[x]$ of the CM-discriminant D , which is a polynomial with integer coefficients. To do so, we first have to compute the following set:

$$ICG(D) = \{(A, B, C) \mid A, B, C \in \mathbb{Z}, D = B^2 - 4AC, \gcd(A, B, C) = 1,$$

$$|B| \leq A \leq \sqrt{\frac{|D|}{3}}, A \leq C, \text{ if } B < 0 \text{ then } |B| < A < C\}$$

One way to compute this set is to first compute the integer $A_{max} = \text{Floor}(\sqrt{\frac{|D|}{3}})$, then loop through all the integers A in the range $[0, \dots, A_{max}]$, as well as through all the integers B in the range $[-A_{max}, \dots, A_{max}]$, then see if there is an integer C that satisfies $D = B^2 - 4AC$ and the rest of the requirements in XXX.

To compute the Hilbert class polynomial, the so-called ***j*-function** (or *j*-invariant) is needed, which is a complex function defined on the upper half \mathbb{H} of the complex plane \mathbb{C} , usually written as follows:

$$j: \mathbb{H} \rightarrow \mathbb{C} \quad (5.33)$$

Roughly speaking, what this means is that the j -functions takes complex numbers $(x + i \cdot y)$ with a positive imaginary part $y > 0$ as inputs and returns a complex number $j(x + i \cdot y)$ as a result.

For the purposes of this book, it is not important to understand the j -function in detail, and we can use Sage to compute it in a similar way that we would use Sage to compute any other well-known function. It should be noted, however, that the computation of the j -function in Sage is sometimes prone to precision errors. For example, the j -function has a root in $\frac{-1+iv\sqrt{3}}{2}$, which Sage only approximates. Therefore, when using Sage to compute the j -function, we need to take precision loss into account and possibly round to the nearest integer.

```
sage: z = ComplexField(100) (0,1) 496
sage: z # (0+1i) 497
1.00000000000000000000000000000000*I 498
sage: elliptic_j(z) 499
1728.000000000000000000000000000000 500
sage: # j-function only defined for positive imaginary 501
      arguments
sage: z = ComplexField(100) (1,-1) 502
sage: try: 503
.....:     elliptic_j(z) 504
.....: except PariError: 505
```

```

3766     ....:         pass
3767 sage: # root at (-1+i sqrt(3))/2
3768 sage: z = ComplexField(100)(-1, sqrt(3))/2
3769 sage: elliptic_j(z)
3770 -2.6445453750358706361219364880e-88
3771 sage: elliptic_j(z).imag().round()
3772 0
3773 sage: elliptic_j(z).real().round()
3774 0

```

With a way to compute the j -function and the precomputed set $ICG(D)$ at hand, we can now compute the Hilbert class polynomial as follows:

$$H_D(x) = \prod_{(A,B,C) \in ICG(D)} \left(x - j \left(\frac{-B + \sqrt{D}}{2A} \right) \right) \quad (5.34)$$

In other words, we loop over all elements (A, B, C) from the set $ICG(D)$ and compute the j -function at the point $\frac{-B + \sqrt{D}}{2A}$, where D is the CM-discriminant that we chose in a previous step. The result defines a factor of the Hilbert class polynomial and all factors are multiplied together.

It can be shown that the Hilbert class polynomial is an integer polynomial, but actual computations need high-precision arithmetics to avoid approximation errors that usually occur in computer approximations of the j -function (as shown above). So, in case the calculated Hilbert class polynomial does not have integer coefficients, we need to round the result to the nearest integer. Given that the precision we used was high enough, the result will be correct.

In the next step, we use the Hilbert class polynomial $H_D \in \mathbb{Z}[x]$, and project it to a polynomial $H_{D,q} \in \mathbb{F}_q[x]$ with coefficients in the base field \mathbb{F}_q as chosen in the first step. We do this by simply computing the new coefficients as the old coefficients modulus p , that is, if $H_D(x) = a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0$, we compute the q -modulus of each coefficient $\tilde{a}_j = a_j \bmod p$, which defines the **projected Hilbert class polynomial** as follows:

$$H_{D,p}(x) = \tilde{a}_m x^m + \tilde{a}_{m-1} x^{m-1} + \dots + \tilde{a}_1 x + \tilde{a}_0$$

We then search for roots of $H_{D,p}$, since every root j_0 of $H_{D,p}$ defines a family of elliptic curves over \mathbb{F}_q , which all have a j -invariant 5.31 or 5.33 equal to j_0 . We can pick any root, since all of them will lead to proper curves eventually.

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However, some of the curves with the correct j -invariant might have an order different from the one we initially decided on. Therefore, we need a way to decide on a curve with the correct order.

To compute such a curve, we have to distinguish a few different cases based on our choice of the root j_0 and of the CM-discriminant D . If $j_0 \neq 0$ or $j_0 \neq 1728 \bmod q$, we compute $c_1 = \frac{j_0}{(1728 \bmod q) - j_0}$, then we chose some arbitrary quadratic non-residue $c_2 \in \mathbb{F}_q$, and some arbitrary cubic non-residue $c_3 \in \mathbb{F}_q$.

The following table is guaranteed to define a curve with the correct order $r = q + 1 - t$ for the trace of Frobenius t we initially decided on:

actually
make this
a table?

Definition 5.4.0.1. • Case $j_0 \neq 0$ and $j_0 \neq 1728 \bmod q$. A curve with the correct order is defined by one of the following equations:

$$y^2 = x^3 + 3c_1 x + 2c_1 \quad \text{or} \quad y^2 = x^3 + 3c_1 c_2^2 x + 2c_1 c_2^3 \quad (5.35)$$

- Case $j_0 = 0$ and $D \neq -3$. A curve with the correct order is defined by one of the following equations:

$$y^2 = x^3 + 1 \quad \text{or} \quad y^2 = x^3 + c_2^3 \quad (5.36)$$

- Case $j_0 = 0$ and $D = -3$. A curve with the correct order is defined by one of the following equations:

$$\begin{aligned} y^2 &= x^3 + 1 \quad \text{or} \quad y^2 = x^3 + c_2^3 \quad \text{or} \\ y^2 &= x^3 + c_3^2 \quad \text{or} \quad y^2 = c_3^2 c_2^3 \quad \text{or} \\ y^2 &= x^3 + c_3^{-2} \quad \text{or} \quad y^2 = x^3 + c_3^{-2} c_2^3 \end{aligned}$$

- Case $j_0 = 1728 \bmod q$ and $D \neq -4$. A curve with the correct order is defined by one of the following equations:

$$y^2 = x^3 + x \quad \text{or} \quad y^2 = x^3 + c_2^2 x \quad (5.37)$$

- Case $j_0 = 1728 \bmod q$ and $D = -4$. A curve with the correct order is defined by one of the following equations:

$$\begin{aligned} y^2 &= x^3 + x \quad \text{or} \quad y^2 = x^3 + c_2 x \quad \text{or} \\ y^2 &= x^3 + c_2^2 x \quad \text{or} \quad y^2 = x^3 + c_2^3 x \end{aligned}$$

To decide the proper defining Weierstraß equation, we therefore have to compute the order of any of the potential curves above, and then choose the one that fits our initial requirements. Since it can be shown that the Hilbert class polynomials for the CM-discriminants $D = -3$ and $D = -4$ are given by $H_{-3,q}(x) = x$ and $H_{-4,q}(x) = x - (1728 \bmod q)$ (EXERCISE), the previous cases are exhaustive.

To summarize, using the complex multiplication method, it is possible to synthesize elliptic curves with predefined order over predefined base fields from scratch. However, the curves that are constructed this way are just some representatives of a larger class of curves, all of which have the same order. Therefore, in real-world applications, it is sometimes more advantageous to choose a different representative from that class. To do so recall from XXX that any curve defined by the Weierstraß equation $y^2 = x^3 + ax + b$ is isomorphic to a curve of the form $y^2 = x^3 + ad^2x + bd^3$ for some quadratic residue $d \in \mathbb{F}_q$.

In order to find a suitable representative (e.g. with small parameters a and b) in the last step, the curve designer might choose a quadratic residue d such that the transformed curve has the properties they wanted.

Example 103. Consider curve $E_1(\mathbb{F}_5)$ from example 68. We want to use the complex multiplication method to derive that curve from scratch. Since $E_1(\mathbb{F}_5)$ is a curve of order $r = 9$ over the prime field of order $q = 5$, we know from example 97 that its trace of Frobenius is $t = -3$, which also implies that q and $|t|$ are coprime.

We then have to find parameters $D, v \in \mathbb{Z}$ such that the criteria in 5.32 hold. We get the following:

$$\begin{aligned} 4q &= t^2 + |D|v^2 && \Rightarrow \\ 20 &= (-3)^2 + |D|v^2 && \Leftrightarrow \\ 11 &= |D|v^2 \end{aligned}$$

exercise
still to be
written?

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Now, since 11 is a prime number, the only solution is $|D| = 11$ and $v = 1$ here. With $D = -11$ and the Euclidean division of -11 by 4 being $-11 = -3 \cdot 4 + 1$, we have $-11 \bmod 4 = 1$, which shows that $D = -11$ is a proper choice.

In the next step, we have to compute the Hilbert class polynomial H_{-11} . To do so, we first have to find the set $ICG(D)$. To compute that set, observe that, since $\sqrt{\frac{|D|}{3}} \approx 1.915 < 2$, we know from $A \leq \sqrt{\frac{|D|}{3}}$ and $A \in \mathbb{Z}$ that A must be either 0 or 1.

For $A = 0$, we know $B = 0$ from the constraint $|B| \leq A$. However, in this case, there could be no C satisfying $-11 = B^2 - 4AC$. So we try $A = 1$ and deduce $B \in \{-1, 0, 1\}$ from the constraint $|B| \leq A$. The case $B = -1$ can be excluded, since then $B < 0$ has to imply $|B| < A$. The case $B = 0$ can also be excluded, as there cannot be an integer C with $-11 = -4C$, since 11 is a prime number.

This leaves the case $B = 1$, and we compute $C = 3$ from the equation $-11 = 1^2 - 4C$, which gives the solution $(A, B, C) = (1, 1, 3)$:

$$ICG(D) = \{(1, 1, 3)\}$$

With the set $ICG(D)$ at hand, we can compute the Hilbert class polynomial of $D = -11$. To do so, we have to insert the term $\frac{-1+\sqrt{-11}}{2}$ into the j -function. To do so, first observe that $\sqrt{-11} = i\sqrt{11}$, where i is the imaginary unit, defined by $i^2 = -1$. Using this, we can invoke Sage to compute the j -invariant and get the following:

$$H_{-11}(x) = x - j\left(\frac{-1+i\sqrt{11}}{2}\right) = x + 32768$$

As we can see, in this particular case, the Hilbert class polynomial is a linear function with a single integer coefficient. In the next step, we have to project it onto a polynomial from $\mathbb{F}_5[x]$ by computing the modular 5 remainder of the coefficients 1 and 32768. We get $32768 \bmod 5 = 3$, from which it follows that the projected Hilbert class polynomial is considered a polynomial from $\mathbb{F}_5[x]$:

$$H_{-11,5}(x) = x + 3$$

As we can see, the only root of this polynomial is $j = 2$, since $H_{-11,5}(2) = 2 + 3 = 0$. We therefore have a situation with $j \neq 0$ and $j \neq 1728$, which tells us that we have to compute the parameter c_1 in modular 5 arithmetics:

$$c_1 = \frac{2}{1728 - 2}$$

Since $1728 \bmod 5 = 3$, we get $c_1 = 2$.

Next, we have to check if the curve $E(\mathbb{F}_5)$ defined by the Weierstraß equation $y^2 = x^3 + 3 \cdot 2x + 2 \cdot 2$ has the correct order. We invoke Sage, and find that the order is indeed 9, so it is a curve with the required parameters. Thus, we have successfully constructed the curve with the desired properties.

Note, however, that in real-world applications, it might be useful to choose parameters a and b that have certain properties, e.g. to be as small as possible. As we know from XXX, choosing any quadratic residue $d \in \mathbb{F}_5$ gives a curve of the same order defined by $y^2 = x^2 + ak^2x + bk^3$. Since 4 is a quadratic residue in \mathbb{F}_4 , we can transform the curve defined by $y^2 = x^3 + x + 4$ into the curve $y^2 = x^3 + 4^2 + 4 \cdot 4^3$ which gives the following:

$$y^2 = x^3 + x + 1$$

add reference

3839 This is the curve $E_1(\mathbb{F}_5)$ that we used extensively throughout this book. Thus, using the
 3840 complex multiplication method, we were able to derive a curve with specific properties from
 3841 scratch.

3842 *Example 104.* Consider the tiny-jubjub curve TJJ_13 from example 69. We want to use the
 3843 complex multiplication method to derive that curve from scratch. Since TJJ_13 is a curve of
 3844 order $r = 20$ over the prime field of order $q = 13$, we know from example 98 that its trace of
 3845 Frobenius is $t = -6$, which also implies that q and $|t|$ are coprime.

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We then have to find parameters $D, v \in \mathbb{Z}$ such that 5.32 holds. We get the following:

$$\begin{aligned} 4q &= t^2 + |D|v^2 && \Rightarrow \\ 4 \cdot 13 &= (-6)^2 + |D|v^2 && \Rightarrow \\ 52 &= 36 + |D|v^2 && \Leftrightarrow \\ 16 &= |D|v^2 \end{aligned}$$

3846 This equation has two solutions for (D, v) , namely $(-4, \pm 2)$ and $(-16, \pm 1)$. Looking at the
 3847 first solution, we know that $D = -4$ implies $j = 1728$, and the constructed curve is defined by
 3848 a Weierstraß equation 5.1 that has a vanishing parameter $b = 0$. We can therefore conclude that
 3849 choosing $D = -4$ will not help us reconstructing TJJ_13 . It will produce curves with order 20,
 3850 just not the one we are looking for.

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So we choose the second solution $D = -16$. In the next step, we have to compute the Hilbert class polynomial H_{-16} . To do so, we first have to find the set $ICG(D)$. To compute that set, observe that since $\sqrt{\frac{|-16|}{3}} \approx 2.31 < 3$, we know from $A \leq \sqrt{\frac{|-16|}{3}}$ and $A \in \mathbb{Z}$ that A must be in the range $0..2$. So we loop through all possible values of A and through all possible values of B under the constraints $|B| \leq A$, and if $B < 0$ then $|B| < A$. Then we compute potential C 's from $-16 = B^2 - 4AC$. We get the following two solutions for $ICG(D)$: we get

$$ICG(D) = \{(1, 0, 4), (2, 0, 2)\}$$

With the set $ICG(D)$ at hand, we can compute the Hilbert class polynomial of $D = -16$. We can invoke Sage to compute the j -invariant and get the following:

$$\begin{aligned} H_{-16}(x) &= \left(x - j \left(\frac{i\sqrt{16}}{2} \right) \right) \left(x - j \left(\frac{i\sqrt{16}}{4} \right) \right) \\ &= (x - 287496)(x - 1728) \end{aligned}$$

As we can see, in this particular case, the Hilbert class polynomial is a quadratic function with two integer coefficients. In the next step, we have to project it onto a polynomial from $\mathbb{F}_5[x]$ by computing the modular 5 remainder of the coefficients 1, 287496 and 1728. We get $287496 \bmod 13 = 1$ and $1728 \bmod 13 = 2$, which means that the projected Hilbert class polynomial is as follows:

$$H_{-11,5}(x) = (x - 1)(x - 12) = (x + 12)(x + 1)$$

3851 This is considered a polynomial from $\mathbb{F}_5[x]$. Thus, we have two roots, namely $j = 1$ and $j =$
 3852 12 . We already know that $j = 12$ is the wrong root to construct the tiny-jubjub curve, since
 3853 $1728 \bmod 13 = 2$, and that case is not compatible with a curve with $b \neq 0$. So we choose $j = 1$.

Another way to decide the proper root is to compute the j -invariant of the tiny-jubjub curve. We get the following:

$$\begin{aligned}
 j(TJJ_13) &= 12 \frac{4 \cdot 8^3}{4 \cdot 8^3 + 1 \cdot 8^2} \\
 &= 12 \frac{4 \cdot 5}{4 \cdot 5 + 12} \\
 &= 12 \frac{7}{7 + 12} \\
 &= 12 \frac{7}{7 + 12} \\
 &= 1
 \end{aligned}$$

This is equal to the root $j = 1$ of the Hilbert class polynomial $H_{-16,13}$ as expected. We therefore have a situation with $j \neq 0$ and $j \neq 1728$, which tells us that we have to compute the parameter c_1 in modular 5 arithmetics:

$$c_1 = \frac{1}{12 - 1} = 6$$

Since $1728 \bmod 13 = 12$, we get $c_1 = 6$. Then we have to check if the curve $E(\mathbb{F}_5)$ defined by the Weierstraß equation $y^2 = x^3 + 3 \cdot 6x + 2 \cdot 6$, which is equivalent to $y^2 = x^3 + 5x + 12$, has the correct order. We invoke Sage and find that the order is 8, which implies that the trace of this curve is 6, not -6 as required. So we have to consider the second possibility, and choose some quadratic non-residue $c_2 \in \mathbb{F}_{13}$. We choose $c_2 = 5$ and compute the Weierstraß equation $y^2 = x^3 + 5c_2^2 + 12c_2^3$ as follows:

$$y^2 = x^3 + 8x + 5$$

We invoke Sage and find that the order is 20, which is indeed the correct one. As we know from XXX, choosing any quadratic residue $d \in \mathbb{F}_5$ gives a curve of the same order defined by $y^2 = x^2 + ad^2x + bd^3$. Since 12 is a quadratic residue in \mathbb{F}_{13} , we can transform the curve defined by $y^2 = x^3 + 8x + 5$ into the curve $y^2 = x^3 + 12^2 \cdot 8 + 5 \cdot 12^3$ which gives the following:

$$y^2 = x^3 + 8x + 8$$

add reference

3854 This is the tiny-jubjub curve that we used extensively throughout this book. So using the
3855 complex multiplication method, we were able to derive a curve with specific properties from
3856 scratch.

Example 105. To consider a real-world example, we want to use the complex multiplication method in combination with Sage to compute Secp256k1 from scratch. So based on example 70, we decided to compute an elliptic curve over a prime field \mathbb{F}_p of order r for the following security parameters:

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$$\begin{aligned}
 p &= 115792089237316195423570985008687907853269984665640564039457584007908834671663 \\
 r &= 115792089237316195423570985008687907852837564279074904382605163141518161494337
 \end{aligned}$$

According to example 99, this gives the following trace of Frobenius:

check reference

$$t = 4324203865659659652420866390673177327$$

3857 We also decided that we want a curve of the form $y^2 = x^3 + b$, that is, we want the parameter
3858 a to be zero. This implies that the j -invariant of our curve must be zero.

In a first step, we have to find a CM-discriminant D and some integer v such that the equation $4p = t^2 + |D|v^2$ is satisfied. Since we aim for a vanishing j -invariant, the first thing to try is $D = -3$. In this case, we can compute $v^2 = (4p - t^2)$, and if v^2 happens to be an integer that has a square root v , we are done. Invoking Sage we compute as follows:

```

3863 sage: D = -3
3864 sage: p = 1157920892373161954235709850086879078532699846656405
3865         64039457584007908834671663
3866 sage: r = 1157920892373161954235709850086879078528375642790749
3867         04382605163141518161494337
3868 sage: t = p+1-r
3869 sage: v_sqr = (4*p - t^2)/abs(D)
3870 sage: v_sqr.is_integer()
3871 True
3872 sage: v = sqrt(v_sqr)
3873 sage: v.is_integer()
3874 True
3875 sage: 4*p == t^2 + abs(D)*v^2
3876 True
3877 sage: v
3878 303414439467246543595250775667605759171

```

The pair $(D, v) = (-3, 303414439467246543595250775667605759171)$ does indeed solve the equation, which tells us that there is a curve of order r over a prime field of order p , defined by a Weierstraß equation $y^2 = x^3 + b$ for some $b \in \mathbb{F}_p$. Now we need to compute b .

For $D = -3$, we already know that the associated Hilbert class polynomial is given by $H_{-3}(x) = x$, which gives the projected Hilbert class polynomial as $H_{-3,p} = x$ and the j -invariant of our curve is guaranteed to be $j = 0$. Now, looking at 5.4.0.1, we see that there are 6 possible cases to construct a curve with the correct order r . In order to construct the curves in question, we have to choose some arbitrary quadratic and cubic non-residue. So we loop through \mathbb{F}_p to find them, invoking Sage:

```

3888 sage: F = GF(p)
3889 sage: for c2 in F:
3890     ....:     try: # quadratic residue
3891     ....:         _ = c2.nth_root(2)
3892     ....:     except ValueError: # quadratic non-residue
3893     ....:         break
3894 sage: c2
3895 3
3896 sage: for c3 in F:
3897     ....:     try:
3898     ....:         _ = c3.nth_root(3)
3899     ....:     except ValueError:
3900     ....:         break
3901 sage: c3
3902 2

```

We found the quadratic non-residue $c_2 = 3$ and the cubic non-residue $c_3 = 2$. Using those numbers, we check the six cases against the the expected order r of the curve we want to

check
reference

3905 synthesize:

```

3906 sage: C1 = EllipticCurve(F, [0, 1]) 544
3907 sage: C1.order() == r 545
3908 False 546
3909 sage: C2 = EllipticCurve(F, [0, c2^3]) 547
3910 sage: C2.order() == r 548
3911 False 549
3912 sage: C3 = EllipticCurve(F, [0, c3^2]) 550
3913 sage: C3.order() == r 551
3914 False 552
3915 sage: C4 = EllipticCurve(F, [0, c3^2*c2^3]) 553
3916 sage: C4.order() == r 554
3917 False 555
3918 sage: C5 = EllipticCurve(F, [0, c3^(-2)]) 556
3919 sage: C5.order() == r 557
3920 False 558
3921 sage: C6 = EllipticCurve(F, [0, c3^(-2)*c2^3]) 559
3922 sage: C6.order() == r 560
3923 True 561

```

As expected, we found an elliptic curve of the correct order r over a prime field of size p . In principle, we are done, as we have found a curve with the same basic properties as Secp256k1. However, the curve is defined by the following equation, which uses a very large parameter b_1 , and so it might perform too slowly in certain algorithms.

$$y^2 = x^3 + 86844066927987146567678238756515930889952488499230423029593188005931626003754x$$

It is also not very elegant to be written down by hand. It might therefore be advantageous to find an isomorphic curve with the smallest possible parameter b_2 . In order to find such a b_2 , we have to choose a quadratic residue d such that $b_2 = b_1 \cdot d^3$ is as small as possible. To do so, we rewrite the last equation into the following form:

$$d = \sqrt[3]{\frac{b_2}{b_1}}$$

what does this mean? Maybe just delete it

3924 Then we invoke Sage to loop through values $b_2 \in \mathbb{F}_p$ until it finds some number such that
 3925 the quotient $\frac{b_2}{b_1}$ has a cube root d and this cube root itself is a quadratic residue.

```

3926 sage: b1=86844066927987146567678238756515930889952488499230423 562
3927 029593188005931626003754
3928 sage: for b2 in F: 563
3929 ....:     try: 564
3930 ....:         d = (b2/b1).nth_root(3) 565
3931 ....:         try: 566
3932 ....:             _ = d.nth_root(2) 567
3933 ....:             if d != 0: 568
3934 ....:                 break 569
3935 ....:         except ValueError: 570
3936 ....:             pass 571
3937 ....:     except ValueError: 572
3938 ....:         pass 573

```


3939 **sage: b2**
 3940 **7**

574

575

3941 Indeed, the smallest possible value is $b_2 = 7$ and the defining Weierstraß equation of a curve
 3942 over \mathbb{F}_p with prime order r is $y^2 = x^3 + 7$, which we might call Secp256k1. As we have just
 3943 seen, the complex multiplication method is powerful enough to derive cryptographically secure
 3944 curves like Secp256k1 from scratch.

3945 **The BLS6_6 pen-and-paper curve** In this paragraph, we summarize our understanding of
 3946 elliptic curves to derive our main pen-and-paper example for the rest of the book. To do so, we
 3947 want to use the complex multiplication method to derive a pairing-friendly elliptic curve that
 3948 has similar properties to curves that are used in actual cryptographic protocols. However, we
 3949 design the curve specifically to be useful in pen-and-paper examples, which mostly means that
 3950 the curve should contain only a few points so that we are able to derive exhaustive addition and
 3951 pairing tables.

3952 A well-understood family of pairing-friendly curves is the the group of BLS curves (STUFF
 3953 ABOUT THE HISTORY AND THE NAMING CONVENTION), which are derived in [XXX].
 3954 BLS curves are particularly useful in our case if the embedding degree k satisfies $k \equiv 6 \pmod{0}$.
 3955 Of course, the smallest embedding degree k that satisfies this congruency is $k = 6$ and we there-
 3956 fore aim for a BLS6 curve as our main pen-and-paper example.

write up
this part

add refer-
ence

3957 To apply the complex multiplication method from page 114 ff., recall that this method starts
 3958 with a definition of the base field \mathbb{F}_{p^m} , as well as the trace of Frobenius t and the order of the
 3959 curve. If the order $p^m + 1 - t$ is not a prime number, then the order r of the largest prime factor
 3960 group needs to be controlled.

check
reference

3961 In the case of BLS_6 curves, the parameter m is chosen to be 1, which means that the
 3962 curves are defined over prime fields. All relevant parameters p , t and r are then themselves
 3963 parameterized by the following three polynomials:

$$\begin{aligned} r(x) &= \Phi_6(x) \\ t(x) &= x + 1 \\ q(x) &= \frac{1}{3}(x-1)^2(x^2 - x + 1) + x \end{aligned} \tag{5.38}$$

3964 In the equations above, Φ_6 is the 6-th cyclotomic polynomial and $x \in \mathbb{N}$ is a parameter
 3965 that the designer has to choose in such a way that the evaluation of p , t and r at the point x
 3966 gives integers that have the proper size to meet the security requirements of the curve that they
 3967 want to design. It is then guaranteed that the complex multiplication method can be used in
 3968 combination with those parameters to define an elliptic curve with CM-discriminant $D = -3$,
 3969 embedding degree $k = 6$, and curve equation $y^2 = x^3 + b$ for some $b \in \mathbb{F}_p$.

cyclotomic
polyno-
mial

3970 For example, if the curve should target the 128-bit security level, due to the Pholaard-rho
 3971 attack (TODO) the parameter r should be prime number of at least 256 bits.

Pholaard-
rho attack

3972 In order to design the smallest BLS_6 curve, we therefore have to find a parameter x such
 3973 that $r(x)$, $t(x)$ and $q(x)$ are the smallest natural numbers that satisfy $q(x) > 3$ and $r(x) > 3$.¹

todo

We therefore initiate the design process of our BLS6 curve by looking up the 6-th cyclo-
 tomic polynomial, which is $\Phi_6 = x^2 - x + 1$, and then insert small values for x into the defining

¹The smallest BLS curve will also be the most insecure BLS curve. However, since our goal with this curve is ease of pen-and-paper computation rather than security, it fits the purposes of this book.

polynomials r, t, q . We get the following results:

$$\begin{array}{lll} x = 1 & (r(x), t(x), q(x)) & (1, 2, 1) \\ x = 2 & (r(x), t(x), q(x)) & (3, 3, 3) \\ x = 3 & (r(x), t(x), q(x)) & (7, 4, \frac{37}{3}) \\ x = 4 & (r(x), t(x), q(x)) & (13, 5, 43) \end{array}$$

3974 Since $q(1) = 1$ is not a prime number, the first x that gives a proper curve is $x = 2$. However,
3975 such a curve would be defined over a base field of characteristic 3, and **we would rather like to**
3976 **avoid that**. We therefore find $x = 4$, which defines a curve over the prime field of characteristic
3977 43 that has a trace of Frobenius $t = 5$ and a larger order prime group of size $r = 13$.

3978 Since the prime field \mathbb{F}_{43} has 43 elements and 43's binary representation is $43_2 = 101011$,
3979 which consists of 6 digits, the name of our pen-and-paper curve should be *BLS6_6*, since its is
3980 common to name a BLS curve by its embedding degree and the bit-length of the modulus in the
3981 base field. We call *BLS6_6* the **moon-math-curve**.

3982 Based on 5.29, we know that the Hasse bound implies that *BLS6_6* will contain exactly 39
3983 elements. Since the prime factorization of 39 is $39 = 3 \cdot 13$, we have a “large” prime factor
3984 group of size 13, as expected, and a small cofactor group of size 3. Fortunately, a subgroup of
3985 order 13 is well suited for our purposes, as 13 elements can be easily handled in the associated
3986 addition, scalar multiplication and pairing tables in a pen-and-paper style.

3987 We can check that the embedding degree is indeed 6 as expected, since $k = 6$ is the smallest
3988 number k such that $r = 13$ divides $43^k - 1$.

```
3989 sage: for k in range(1, 42): # Fermat's little theorem           576
3990 .....:     if (43^k-1)%13 == 0:                               577
3991 .....:         break                                           578
3992 sage: k                                                         579
3993 6                                                                580
```

3994 In order to compute the defining equation $y^2 = x^3 + ax + b$ of *BLS6_6*, we use the complex
3995 multiplication method as described in 5.4. The goal is to find $a, b \in \mathbb{F}_{43}$ representations that
3996 are particularly **nice to work with**. The authors of XXX showed that the CM-discriminant of
3997 every BLS curve is $D = -3$ and, indeed, the following equation has the four solutions $(D, v) \in$
3998 $\{(-3, -7), (-3, 7), (-49, -1), (-49, 1)\}$ if D is required to be negative, as expected:

$$\begin{array}{ll} 4p = t^2 + |D|v^2 & \Rightarrow \\ 4 \cdot 43 = 5^2 + |D|v^2 & \Rightarrow \\ 172 = 25 + |D|v^2 & \Leftrightarrow \\ 49 = |D|v^2 & \end{array}$$

3999 This means that $D = -3$ is indeed a proper CM-discriminant, and we can deduce that the
4000 parameter a has to be 0, and that the Hilbert class polynomial is given by $H_{-3,43}(x) = x$.

4001 This implies that the j -invariant of *BLS6_6* is given by $j(\text{BLS6_6}) = 0$. We therefore have
4002 to look at case XXX in table 5.4.0.1 to derive a parameter b . To decide the proper case for
4003 $j_0 = 0$ and $D = -3$, we therefore have to choose some arbitrary quadratic non-residue c_2 and
4004 cubic non-residue c_3 in \mathbb{F}_{43} . We choose $c_2 = 5$ and $c_3 = 36$. We check these with Sage:

```
4005 sage: F43 = GF(43)
```

why? Be-
cause in
this book
elliptic
curves are
only de-
fined for
fields of
characteris-
tic > 3

check
reference

check
reference

what
does this
mean?

add refer-
ence

add refer-
ence

check
reference


```

4006 sage: c2 = F43(5) 582
4007 .....: try: # quadratic residue 583
4008 .....:     c2.nth_root(2) 584
4009 .....: except ValueError: # quadratic non-residue 585
4010 .....:     c2 586
4011 sage: c3 = F43(36) 587
4012 .....: try: 588
4013 .....:     c3.nth_root(3) 589
4014 .....: except ValueError: 590
4015 .....:     c3 591

```

4016 Using those numbers we check the six possible cases from 5.4.0.1 against the the expected
 4017 order 39 of the curve we want to synthesize:

check
reference

```

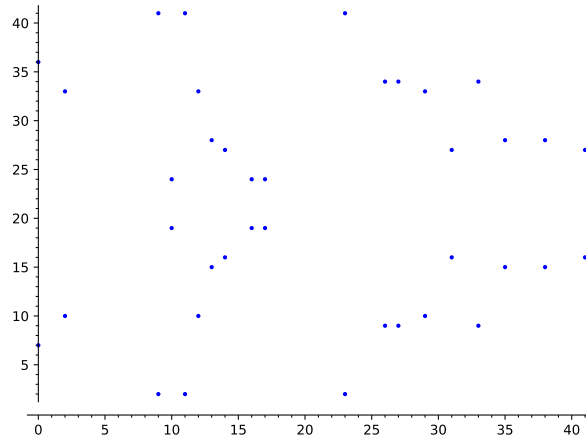
4018 sage: BLS61 = EllipticCurve(F43, [0, 1]) 592
4019 sage: BLS61.order() == 39 593
4020 False 594
4021 sage: BLS62 = EllipticCurve(F43, [0, c2^3]) 595
4022 sage: BLS62.order() == 39 596
4023 False 597
4024 sage: BLS63 = EllipticCurve(F43, [0, c3^2]) 598
4025 sage: BLS63.order() == 39 599
4026 True 600
4027 sage: BLS64 = EllipticCurve(F43, [0, c3^2*c2^3]) 601
4028 sage: BLS64.order() == 39 602
4029 False 603
4030 sage: BLS65 = EllipticCurve(F43, [0, c3^(-2)]) 604
4031 sage: BLS65.order() == 39 605
4032 False 606
4033 sage: BLS66 = EllipticCurve(F43, [0, c3^(-2)*c2^3]) 607
4034 sage: BLS66.order() == 39 608
4035 False 609
4036 sage: BLS6 = BLS63 # our BLS6 curve in the book 610

```

4037 As expected, we found an elliptic curve of the correct order 39 over a prime field of size 43,
 4038 defined by the following equation:

$$BLS6_6 := \{(x, y) \mid y^2 = x^3 + 6 \text{ for all } x, y \in \mathbb{F}_{43}\} \quad (5.39)$$

4039 There are other choices for b , such as $b = 10$ or $b = 23$, but all these curves are isomorphic,
 4040 and hence represent the same curve in a different way. Since BLS6-6 only contains 39 points, it
 4041 is possible to give a visual impression of the curve:



As we can see, our curve has some desirable properties: it does not contain self-inverse points, that is, points with $y = 0$. It follows that the addition law can be optimized, since the branch for those cases can be eliminated.

Summarizing the previous procedure, we have used the method of Barreto, Lynn and Scott to construct a pairing-friendly elliptic curve of embedding degree 6. However, in order to do elliptic curve cryptography on this curve, note that, since the order of $BLS6_6$ is 39, its group of rational points is not a finite cyclic group of prime order. We therefore have to find a suitable subgroup as our main target. Since $39 = 13 \cdot 3$, we know that the curve must contain a “large” prime-order group of size 13 and a small cofactor group of order 3.

The following step is to construct this group. One way to do so is to find a generator. We can achieve this by choosing an arbitrary element of the group that is not the point at infinity, and then multiply that point with the cofactor of the group’s order. If the result is not the point at infinity, the result will be a generator. If it is the point at infinity we have to choose a different element.

In order to find a generator for the large order subgroup of size 13, we first notice that the cofactor of 13 is 3, since $39 = 3 \cdot 13$. We then need to construct an arbitrary element from $BLS6_6$. To do so in a pen-and-paper style, we can choose some *arbitrary* $x \in \mathbb{F}_{43}$ and see if there is some solution $y \in \mathbb{F}_{43}$ that satisfies the defining Weierstraß equation $y^2 = x^3 + 6$. We choose $x = 9$, and check that $y = 2$ is a proper solution:

$$\begin{aligned} y^2 &= x^3 + 6 && \Rightarrow \\ 2^2 &= 9^3 + 6 && \Leftrightarrow \\ 4 &= 4 \end{aligned}$$

This implies that $P = (9, 2)$ is therefore a point on $BLS6_6$. To see if we can project this point onto a generator of the large order prime group $BLS6_6[13]$, we have to multiply P with the cofactor, that is, we have to compute $[3](9, 2)$. After some computation (EXERCISE) we get $[3](9, 2) = (13, 15)$. Since this is not the point at infinity, we know that $(13, 15)$ must be a generator of $BLS6_6[13]$. The generator $g_{BLS6_6[13]}$, which we will use in pairing computations in the remainder of this book, is given as follows:

$$g_{BLS6_6[13]} = (13, 15) \tag{5.40}$$

Since $g_{BLS6_6[13]}$ is a generator, we can use it to construct the subgroup $BLS6_6[13]$ by repeatedly adding the generator to itself. Using Sage, we get the following:

```
sage: P = BLS6(9, 2)
```

add reference

add exercise

```

4066 sage: Q = 3*P 612
4067 sage: Q.xy() 613
4068 (13, 15) 614
4069 sage: BLS6_13 = [] 615
4070 sage: for x in range(0,13): # cyclic of order 13 616
4071     ....:     P = x*Q 617
4072     ....:     BLS6_13.append(P) 618

```

Repeatedly adding a generator to itself, as we just did, will generate small groups in logarithmic order with respect to the generator as, explained on page 43 ff. We therefore get the following description of the large prime-order subgroup of $BLS6_6$:

check
reference

$$\begin{aligned}
 BLS6_6[13] = \\
 \{ (13, 15) \rightarrow (33, 34) \rightarrow (38, 15) \rightarrow (35, 28) \rightarrow (26, 34) \rightarrow (27, 34) \rightarrow \\
 (27, 9) \rightarrow (26, 9) \rightarrow (35, 15) \rightarrow (38, 28) \rightarrow (33, 9) \rightarrow (13, 28) \rightarrow \mathcal{O} \} \quad (5.41)
 \end{aligned}$$

Having a logarithmic description of this group is tremendously helpful in pen-and-paper computations. To see that, observe that we know from XXX that there is an exponential map from the scalar field \mathbb{F}_{13} to $BLS6_6[13]$ with respect to our generator, which generates the group in logarithmic order:

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ence

$$[\cdot]_{(13,15)} : \mathbb{F}_{13} \rightarrow BLS6_6[13] ; x \mapsto [x](13, 15)$$

So, for example, we have $[1]_{(13,15)} = (13, 15)$, $[7]_{(13,15)} = (27, 9)$ and $[0]_{(13,15)} = \mathcal{O}$ and so on. The relevant point here is that we can use this representation to do computations in $BLS6_6[13]$ efficiently in our head using XXX, as in the following example:

add refer-
ence

$$\begin{aligned}
 (27, 34) \oplus (33, 9) &= [6](13, 15) \oplus [11](13, 15) \\
 &= [6 + 11](13, 15) \\
 &= [4](13, 15) \\
 &= (35, 28)
 \end{aligned}$$

So XXX is really all we need to do computations in $BLS6_6[13]$ in this book efficiently. However, out of convenience, the following picture lists the entire addition table of that group, as it might be useful in pen-and-paper computations:

add refer-
ence

\oplus	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)	(35, 28)	(26, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)	(13, 28)
\mathcal{O}	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)	(35, 28)	(26, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)	(13, 28)
(13, 15)	(13, 15)	(33, 34)	(38, 15)	(35, 28)	(26, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)	(13, 28)	\mathcal{O}
(33, 34)	(33, 34)	(38, 15)	(35, 28)	(26, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)	(13, 28)	\mathcal{O}	(13, 15)
(38, 15)	(38, 15)	(35, 28)	(26, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)	(13, 28)	\mathcal{O}	(13, 15)	(33, 34)
(35, 28)	(35, 28)	(26, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)	(13, 28)	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)
(26, 34)	(26, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)	(13, 28)	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)	(35, 28)
(27, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)	(13, 28)	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)	(35, 28)	(26, 34)
(27, 9)	(27, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)	(13, 28)	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)	(35, 28)	(26, 34)	(27, 34)
(26, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)	(13, 28)	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)	(35, 28)	(26, 34)	(27, 34)	(27, 9)
(35, 15)	(35, 15)	(38, 28)	(33, 9)	(13, 28)	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)	(35, 28)	(26, 34)	(27, 34)	(27, 9)	(26, 9)
(38, 28)	(38, 28)	(33, 9)	(13, 28)	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)	(35, 28)	(26, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)
(33, 9)	(33, 9)	(13, 28)	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)	(35, 28)	(26, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)	(38, 28)
(13, 28)	(13, 28)	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)	(35, 28)	(26, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)

Now that we have constructed a “large” cyclic prime-order subgroup of $BLS6_6$ suitable for many pen-and-paper computations in elliptic curve cryptography, we have to look at how to do pairings in this context. We know that $BLS6_6$ is a pairing-friendly curve by design, since it has a small embedding degree $k = 6$. It is therefore possible to compute Weil pairings efficiently. However, in order to do so, we have to decide the groups \mathbb{G}_1 and \mathbb{G}_2 as explained in exercise 76.

check
reference

Since $BLS6_6$ has two non-trivial subgroups, it would be possible to use any of them as the n -torsion group. However, in cryptography, the only secure choice is to use the large prime-order subgroup, which in our case is $BLS6_6[13]$. We therefore decide to consider the 13-torsion and define $G_1[13]$ as the first argument for the Weil pairing function:

$$\mathbb{G}_1[13] = \{(13, 15) \rightarrow (33, 34) \rightarrow (38, 15) \rightarrow (35, 28) \rightarrow (26, 34) \rightarrow (27, 34) \rightarrow (27, 9) \rightarrow (26, 9) \rightarrow (35, 15) \rightarrow (38, 28) \rightarrow (33, 9) \rightarrow (13, 28) \rightarrow \mathcal{O}\}$$

In order to construct the domain for the second argument, we need to construct $\mathbb{G}_2[13]$, which, according to the general theory, should be defined by those elements P of the full 13-torsion group $BLS6_6[13]$ that are mapped to $43 \cdot P$ under the Frobenius endomorphism (equation 5.24).

check
reference

To compute $\mathbb{G}_2[13]$, we therefore have to find the full 13-torsion group first. To do so, we use the technique from XXX, which tells us that the full 13-torsion can be found in the curve extension over the extension field \mathbb{F}_{43^6} , since the embedding degree of $BLS6_6$ is 6:

$$BLS6_6 := \{(x, y) \mid y^2 = x^3 + 6 \text{ for all } x, y \in \mathbb{F}_{43^6}\} \quad (5.42)$$

Thus, we have to construct \mathbb{F}_{43^6} , a field that contains 6321363049 elements. In order to do so, we use the procedure of XXX and start by choosing a non-reducible polynomial of degree 6 from the ring of polynomials $\mathbb{F}_{43}[t]$. We choose $p(t) = t^6 + 6$. Using Sage, we get the following:

add refer-
ence

```
sage: F43 = GF(43) 619
sage: F43t.<t> = F43[] 620
sage: p = F43t(t^6+6) 621
sage: p.is_irreducible() 622
True 623
sage: F43_6.<v> = GF(43^6, name='v', modulus=p) 624
```

Recall from XXX that elements $x \in \mathbb{F}_{43^6}$ can be seen as polynomials $a_0 + a_1v + a_2v^2 + \dots + a_5v^5$ with the usual addition of polynomials and multiplication modulo $t^6 + 6$.

add refer-
ence

In order to compute $\mathbb{G}_2[13]$, we first have to extend $BLS6_6$ to \mathbb{F}_{43^6} , that is, we keep the defining equation, but expand the domain from \mathbb{F}_{43} to \mathbb{F}_{43^6} . After that, we have to find at least one element P from that curve that is not the point at infinity, is in the full 13-torsion and satisfies the identity $\pi(P) = [43]P$. We can then use this element as our generator of $\mathbb{G}_2[13]$ and construct all other elements by repeatedly adding the generator to itself.

Since $BLS6(\mathbb{F}_{43^6})$ contains 6321251664 elements, it's not a good strategy to simply loop through all elements. Fortunately, Sage has a way to loop through elements from the torsion group directly:

```
sage: BLS6 = EllipticCurve(F43_6, [0, 6]) # curve extension 625
sage: INF = BLS6(0) # point at infinity 626
```

```

4115 sage: for P in INF.division_points(13): # full 13-torsion           627
4116 .....: # PI(P) == [q]P                                           628
4117 .....:         if P.order() == 13: # exclude point at infinity    629
4118 .....:             PiP = BLS6([a.frobenius() for a in P])          630
4119 .....:             qP = 43*P                                         631
4120 .....:             if PiP == qP:                                     632
4121 .....:                 break                                         633
4122 sage: P.xy()                                                         634
4123 (7*v^2, 16*v^3)                                                     635

```

4124 We found an element from the full 13-torsion that is in the Eigenspace of the Eigenvalue 43,
 4125 which implies that it is an element of $\mathbb{G}_2[13]$. As $\mathbb{G}_2[13]$ is cyclic of prime order, this element
 4126 must be a generator:

$$g_{\mathbb{G}_2[13]} = (7v^2, 16v^3) \quad (5.43)$$

4127 We can use this generator to compute \mathbb{G}_2 in logarithmic order with respect to $g_{\mathbb{G}_2[13]}$. Using
 4128 Sage we get the following:

```

4129 sage: Q = BLS6(7*v^2, 16*v^3)                                       636
4130 sage: BLS6_13_2 = []                                                637
4131 sage: for x in range(0, 13):                                         638
4132 .....:     P = x*Q                                                  639
4133 .....:     BLS6_13_2.append(P)                                       640

```

$$\begin{aligned} \mathbb{G}_2 = \{ & (7v^2, 16v^3) \rightarrow (10v^2, 28v^3) \rightarrow (42v^2, 16v^3) \rightarrow (37v^2, 27v^3) \rightarrow \\ & (16v^2, 28v^3) \rightarrow (17v^2, 28v^3) \rightarrow (17v^2, 15v^3) \rightarrow (16v^2, 15v^3) \rightarrow \\ & (37v^2, 16v^3) \rightarrow (42v^2, 27v^3) \rightarrow (10v^2, 15v^3) \rightarrow (7v^2, 27v^3) \rightarrow \mathcal{O} \} \end{aligned}$$

Again, having a logarithmic description of $\mathbb{G}_2[13]$ is tremendously helpful in pen-and-paper computations, as it reduces complicated computation in the extended curves to modular 13 arithmetics, as in the following example:

$$\begin{aligned} (17v^2, 28v^3) \oplus (10v^2, 15v^3) &= [6](7v^2, 16v^3) \oplus [11](7v^2, 16v^3) \\ &= [6 + 11](7v^2, 16v^3) \\ &= [4](7v^2, 16v^3) \\ &= (37v^2, 27v^3) \end{aligned}$$

4134 So XXX is really all we need to do computations in $\mathbb{G}_2[13]$ in this book efficiently.

4135 To summarize the previous steps, we have found two subgroups, $\mathbb{G}_1[13]$ and $\mathbb{G}_2[13]$ suit-
 4136 able to do Weil pairings on $BLS6_6$ as explained in 5.28. Using the logarithmic order XXX
 4137 of $\mathbb{G}_1[13]$, the logarithmic order XXX of $\mathbb{G}_2[13]$ and the bilinearity in 5.44, we can do Weil
 4138 pairings on $BLS6_6$ in a pen-and-paper style:

$$e([k_1]g_{BLS6_6[13]}, [k_2]g_{\mathbb{G}_2[13]}) = e(g_{BLS6_6[13]}, g_{\mathbb{G}_2[13]})^{k_1 \cdot k_2} \quad (5.44)$$

4139 Observe that the Weil pairing between our two generators is given by the following identity:

$$e(g_{BLS6_6[13]}, g_{\mathbb{G}_2[13]}) = 5v^5 + 16v^4 + 16v^3 + 15v^2 + 3v + 41 \quad (5.45)$$

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```

4140 sage: g1 = BLS6([13,15]) 641
4141 sage: g2 = BLS6([7*v^2, 16*v^3]) 642
4142 sage: g1.weil_pairing(g2,13) 643
4143 5*v^5 + 16*v^4 + 16*v^3 + 15*v^2 + 3*v + 41 644

```

4144 **Hashing to pairing groups** We give various constructions to hash into \mathbb{G}_1 and \mathbb{G}_2 .

4145 We start with hashing to the scalar field... **TO APPEAR**

4146 None of these techniques work for hashing into \mathbb{G}_2 . We therefore implement Pederson's

4147 Hash for BLS6.

We start with \mathbb{G}_1 . Our goal is to define an 12-bit bounded hash function:

$$H_1 : \{0,1\}^{12} \rightarrow \mathbb{G}_1$$

Since $12 = 3 \cdot 4$ we “randomly” select 4 uniformly distributed generators $\{(38,15), (35,28), (27,34), (38,28)\}$ from \mathbb{G}_1 and use the pseudo-random function from XXX. Therefore, we have to choose a set of 4 randomly generated invertible elements from \mathbb{F}_{13} for every generator. We choose the following:

$$\begin{aligned}
 (38,15) &: \{2,7,5,9\} \\
 (35,28) &: \{11,4,7,7\} \\
 (27,34) &: \{5,3,7,12\} \\
 (38,28) &: \{6,5,1,8\}
 \end{aligned}$$

4148 Our hash function is then computed as follows:

$$\begin{aligned}
 H_1(x_{11}, x_1, \dots, x_0) = & [2 \cdot 7^{x_{11}} \cdot 5^{x_{10}} \cdot 9^{x_9}](38,15) + [11 \cdot 4^{x_8} \cdot 7^{x_7} \cdot 7^{x_6}](35,28) + \\
 & [5 \cdot 3^{x_5} \cdot 7^{x_4} \cdot 12^{x_3}](27,34) + [6 \cdot 5^{x_2} \cdot 1^{x_1} \cdot 8^{x_0}](38,28)
 \end{aligned}$$

4149 Note that $a^x = 1$ when $x = 0$. Hence, those terms can be omitted in the computation. In
 4150 particular, the hash of the 12-bit zero string is given as follows:

WRONG – ORDERING – REDO

$$\begin{aligned}
 H_1(0) = & [2](38,15) + [11](35,28) + [5](27,34) + [6](38,28) = \\
 & (27,34) + (26,34) + (35,28) + (26,9) = (33,9) + (13,28) = (38,28)
 \end{aligned}$$

The hash of 011010101100 is given as follows:

$$\begin{aligned}
 H_1(011010101100) = & \text{WRONG – ORDERING – REDO} \\
 & [2 \cdot 7^0 \cdot 5^1 \cdot 9^1](38,15) + [11 \cdot 4^0 \cdot 7^1 \cdot 7^0](35,28) + [5 \cdot 3^1 \cdot 7^0 \cdot 12^1](27,34) + [6 \cdot 5^1 \cdot 1^0 \cdot 8^0](38,28) = \\
 & [2 \cdot 5 \cdot 9](38,15) + [11 \cdot 7](35,28) + [5 \cdot 3 \cdot 12](27,34) + [6 \cdot 5](38,28) = \\
 & [12](38,15) + [12](35,28) + [11](27,34) + [4](38,28) =
 \end{aligned}$$

TO APPEAR

We can use the same technique to define a 12-bit bounded hash function in \mathbb{G}_2 :

$$H_2 : \{0,1\}^{12} \rightarrow \mathbb{G}_2$$

finish
writing
this up

add refer-
ence

correct
computa-
tions

fill in
missing
parts

Again, we “randomly” select 4 uniformly distributed generators $\{(7v^2, 16v^3), (42v^2, 16v^3), (17v^2, 15v^3), (10v^2, 15v^3)\}$ from \mathbb{G}_2 , and use the pseudo-random function from XXX. Therefore, we have to choose a set of 4 randomly generated invertible elements from \mathbb{F}_{13} for every generator:

add reference

$$\begin{aligned} (7v^2, 16v^3) &: \{8, 4, 5, 7\} \\ (42v^2, 16v^3) &: \{12, 1, 3, 8\} \\ (17v^2, 15v^3) &: \{2, 3, 9, 11\} \\ (10v^2, 15v^3) &: \{3, 6, 9, 10\} \end{aligned}$$

Our hash function is then computed like this:

$$H_1(x_{11}, x_{10}, \dots, x_0) = [8 \cdot 4^{x_{11}} \cdot 5^{x_{10}} \cdot 7^{x_9}](7v^2, 16v^3) + [12 \cdot 1^{x_8} \cdot 3^{x_7} \cdot 8^{x_6}](42v^2, 16v^3) + [2 \cdot 3^{x_5} \cdot 9^{x_4} \cdot 11^{x_3}](17v^2, 15v^3) + [3 \cdot 6^{x_2} \cdot 9^{x_1} \cdot 10^{x_0}](10v^2, 15v^3)$$

We extend this to a hash function that maps unbounded bitstrings to \mathbb{G}_2 by precomposing with an actual hash function like MD5, and feed the first 12 bits of its outcome into our previously defined hash function, with $TinyMD5_{\mathbb{G}_2}(s) = H_2(MD5(s)_{11}, \dots, MD5(s)_0)$:

$$TinyMD5_{\mathbb{G}_2} : \{0, 1\}^* \rightarrow \mathbb{G}_2$$

For example, since $MD5(“”) =$

$0xd41d8cd98f00b204e9800998ecf8427e$, and the binary representation of the hexadecimal number $0x27e$ is 001001111110 , we compute $TinyMD5_{\mathbb{G}_2}$ of the empty string as follows:

$$TinyMD5_{\mathbb{G}_2}(“”) = H_2(MD5(s)_{11}, \dots, MD5(s)_0) = H_2(001001111110) =$$

4151

check equation

Chapter 6

Statements

As we have seen in the informal introduction XXX, a SNARK is a short non-interactive argument of knowledge, where the knowledge-proof attests to the correctness of statements like “The prover knows the prime factorization of a given number” or “The prover knows the preimage to a given SHA2 digest value” and similar things. However, human-readable statements like these are imprecise and not very useful from a formal perspective.

Chapter 1?

In this chapter we therefore look more closely at ways to formalize statements in mathematically rigorous ways, useful for SNARK development. We start by introducing formal languages as a way to define statements properly (section 6.1). We will then look at algebraic circuits and rank-1 constraint systems as two particularly useful ways to define statements in certain formal languages (section 6.2). After that, we will have a look at fundamental building blocks of compilers that compile high-level languages to circuits and associated rank-1 constraint systems.

Proper statement design should be of high priority in the development of SNARKs, since unintended true statements can lead to potentially severe and almost undetectable security vulnerabilities in the applications of SNARKs.

6.1 Formal Languages

Formal languages provide the theoretical background in which statements can be formulated in a logically rigorous way and where proving the correctness of any given statement can be realized by computing words in that language.

"rigorous"?

One might argue that the understanding of formal languages is not very important in SNARK development and associated statement design, but terms from that field of research are standard jargon in many papers on zero-knowledge proofs. We therefore believe that at least some introduction to formal languages and how they fit into the picture of SNARK development is beneficial, mostly to give developers a better intuition about where all this is located in the bigger picture of the logic landscape. In addition, formal languages give a better understanding of what a formal proof for a statement actually is.

"proving"?

Roughly speaking, a formal language (or just language for short) is nothing but a set of words, w . Words, in turn, are strings of letters taken from some alphabet and formed according to some defining rules of the language.

To be more precise, let Σ be any set and Σ^* the set of all finite **tuples** (ordered lists) (x_1, \dots, x_n) of elements x_j from Σ including the empty tuple $() \in \Sigma^*$. Then, a **language** L , in its most general definition, is nothing but a subset of Σ^* . In this context, the set Σ is called the **alphabet** of the language L , elements from Σ are called letters and elements from L are called **words**. The rules that specify which tuples from Σ^* belong to the language and which don't,

are called the **grammar** of the language. S: I suggest adding an example based on English, e.g. “tea” and “eat” are words of English, but “aet” and “tae” are not

Add ex-ample

If L_1 and L_2 are two formal languages over the same alphabet, we call L_1 and L_2 **equivalent** if they generate the same set of words.

M: 1:1 correspondence might actually be wrong

Decision Functions Our previous definition of formal languages is very general and many subclasses of languages are known in the literature. However, in the context of SNARK development, languages are commonly defined as **decision problems** where a so-called **deciding relation** $R \subset \Sigma^*$ decides whether a given tuple $x \in \Sigma^*$ is a word in the language or not. If $x \in R$ then x is a word in the associated language L_R and if $x \notin R$ then not. The relation R therefore summarizes the grammar of language L_R .

Unfortunately, in some literature on proof systems, $x \in R$ is often written as $R(x)$, which is misleading since in general R is not a function but a relation in Σ^* . For the sake of this book, we therefore adopt a different point of view and work with what we might call a **decision function** instead:

$$R : \Sigma^* \rightarrow \{true, false\} \quad (6.1)$$

Decision functions decide if a tuple $x \in \Sigma^*$ is an element of a language or not. In case a decision function is given, the associated language itself can be written as the set of all tuples that are decided by R , i.e as the set:

$$L_R := \{x \in \Sigma^* \mid R(x) = true\} \quad (6.2)$$

In the context of formal languages and decision problems, a **statement** S is the claim that language L contains a word x , i.e a statement claims that there exist some $x \in L$. A constructive **proof** for statement S is given by some string $P \in \Sigma^*$ and such a proof is **verified** by checking $R(P) = true$. In this case, P is called an **instance** of the statement S .

While the term **language** might suggest a deeper relation to the well known **natural languages** like English, formal languages and natural languages differ in many ways. The following examples will provide some intuition about formal languages, highlighting the concepts of statements, proofs and instances:

Example 106 (Alternating Binary strings). To consider a very basic formal language with an almost trivial grammar, consider the set $\{0, 1\}$ of the two letters 0 and 1 as our alphabet Σ and imply the rule that a proper word must consist of alternating binary letters of arbitrary length.

Then, the associated language L_{alt} is the set of all finite binary tuples, where a 1 must follow a 0 and vice versa. So, for example, $(1, 0, 1, 0, 1, 0, 1, 0, 1) \in L_{alt}$ is a proper word in this languages as is $(0) \in L_{alt}$ or the empty word $() \in L_{alt}$. However, the binary tuple $(1, 0, 1, 0, 1, 0, 1, 1, 1) \in \{0, 1\}^*$ is not a proper word, as it violates the grammar of L_{alt} : the last3 letters are all 1. Furthermore, the tuple $(0, A, 0, A, 0, A, 0)$ is not a proper word, as not all its letters are not from the proper alphabet: we defined the alphabet Σ as the set $\{0, 1\}$, and A is not part of that set.

Attempting to write the grammar of this language in a more formal way, we can define the following decision function:

$$R : \{0, 1\}^* \rightarrow \{true, false\} ; (x_0, x_1, \dots, x_n) \mapsto \begin{cases} true & x_{j-1} \neq x_j \text{ for all } 1 \leq j \leq n \\ false & \text{else} \end{cases}$$

We can use this function to decide if arbitrary **binary tuples** are words in L_{alt} or not. Some examples are given below:

binary tuples

- $R(1, 0, 1) = true$,

- 4225 • $R(0) = \text{true}$,
- 4226 • $R() = \text{true}$,
- 4227 • but $R(1, 1) = \text{false}$.

4228 Inside our language L_{alt} , it makes sense to claim the following statement: “There exists an
 4229 alternating string.” One way to prove this statement constructively is by providing an actual
 4230 instance, that is, finding actual alternating string like $x = (1, 0, 1)$. Constructing string $(1, 0, 1)$
 4231 therefore proves the statement “There exists an alternating string.”, because it is easy to verify
 4232 that $R(1, 0, 1) = \text{true}$.

4233 *Example 107 (Programming Language).* Programming languages are a very important class of
 4234 formal languages. For these languages, the alphabet is usually (a subset) of the ASCII table,
 4235 and the grammar is defined by the rules of the programming language’s compiler. Words, then,
 4236 are nothing but properly written computer programs that the compiler accepts. The compiler
 4237 can therefore be interpreted as the decision function.

4238 To give an unusual example strange enough to highlight the point, consider the program-
 4239 ming language Malbolge as defined in XXX. This language was specifically designed to be
 4240 almost impossible to use and writing programs in this language is a difficult task. An inter-
 4241 esting claim is therefore the statement: “There exists a computer program in Malbolge”. As it
 4242 turned out, proving this statement constructively, that is, by providing an actual instance of such
 4243 a program, was not an easy task, as it took two years after the introduction of Malbolge to write
 4244 a program that its compiler accepts. So, for two years, no one was able to prove the statement
 4245 constructively.

To look at this high-level description more formally, we write $L_{Malbolge}$ for the language that
 uses the ASCII table as its alphabet and its words are tuples of ASCII letters that the Malbolge
 compiler accepts. Proving the statement “There exists a computer program in Malbolge” is then
 equivalent to the task of finding some word $x \in L_{Malbolge}$. The string

(=<#9] 6ZY327Uv4-QsqpMn&+Ij''E%e{Ab w=_:]Kw%o44Uqp0/Q?xNvL:'H%c#DD2^WV>gY;dts76qKJImZkj

4246 is an example of such a proof, as it is excepted by the Malbolge compiler and is compiled to
 4247 an executable binary that displays “Hello, World.” (See XXX). In this example, the Malbolge
 4248 compiler therefore serves as the verification process.

Example 108 (The Empty Language). To see that not every language has even one word, con-
 sider the alphabet $\Sigma = \mathbb{Z}_6$, where \mathbb{Z}_6 is the ring of modular 6 arithmetics as derived in example
 9 in chapter ??, together with the following decision function

$$R_\emptyset : \mathbb{Z}_6^* \rightarrow \{\text{true}, \text{false}\} ; (x_1, \dots, x_n) \mapsto \begin{cases} \text{true} & n = 4 \text{ and } x_1 \cdot x_1 = 2 \\ \text{true} & \text{else} \end{cases}$$

4249 We write L_\emptyset for the associated language. As we can see from the multiplication table of \mathbb{Z}_6 in
 4250 example 9 in chapter ??, the ring \mathbb{Z}_6 does not contain any element x such that $x^2 = 2$, which
 4251 implies $R_\emptyset(x_1, \dots, x_n) = \text{false}$ for all tuples $(x_1, \dots, x_n) \in \Sigma^*$. The language therefore does
 4252 not contain any words. Proving the statement “There exists a word in L_\emptyset ” constructively by
 4253 providing an instance is therefore impossible. The verification will never check any tuple.

4254 *Example 109 (3-Factorization).* We will use the following simple example repeatedly through-
 4255 out this book. The task is to develop a SNARK that proves knowledge of three factors of an
 4256 element from the finite field \mathbb{F}_{13} . There is nothing particularly useful about this example from

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an application point of view, however, in a sense, it is the most simple example that gives rise to a non trivial SNARK in some of the most common zero-knowledge proof systems.

Formalizing the high-level description, we use $\Sigma := \mathbb{F}_{13}$ as the underlying alphabet of this problem and define the language $L_{3.fac}$ to consists of those tuples of field elements from \mathbb{F}_{13} that contain exactly 4 letters w_1, w_2, w_3, w_4 which satisfy the equation $w_1 \cdot w_2 \cdot w_3 = w_4$.

So, for example, the tuple $(2, 12, 4, 5)$ is a word in $L_{3.fac}$, while neither $(2, 12, 11)$, nor $(2, 12, 4, 7)$ nor $(2, 12, 7, 168)$ are words in $L_{3.fac}$ as they don't satisfy the grammar or are not define over the proper alphabet.

We can describe the language $L_{3.fac}$ more formally by introducing a decision function (as described in equation 6.1):

$$R_{3.fac} : \mathbb{F}_{13}^* \rightarrow \{true, false\} ; (x_1, \dots, x_n) \mapsto \begin{cases} true & n = 4 \text{ and } x_1 \cdot x_2 \cdot x_3 = x_4 \\ false & \text{else} \end{cases}$$

Having defined the language $L_{3.fac}$, it then makes sense to claim the statement “There is a word in $L_{3.fac}$ ”. The way $L_{3.fac}$ is designed, this statement is equivalent to the statement “There are four elements w_1, w_2, w_3, w_4 from the finite field \mathbb{F}_{13} such that the equation $w_1 \cdot w_2 \cdot w_3 = w_4$ holds.”

Proving the correctness of this statement constructively means to actually find some concrete field elements like $x_1 = 2, x_2 = 12, x_3 = 4$ and $x_4 = 5$ that satisfy the relation $R_{3.fac}$. The tuple $(2, 12, 4, 5)$ is therefore a constructive proof for the statement and the computation $R_{3.fac}(2, 12, 4, 5) = true$ is a verification of that proof. In contrast, the tuple $(2, 12, 4, 7)$ is not a proof of the statement, since the check $R_{3.fac}(2, 12, 4, 7) = false$ does not verify the proof.

Example 110 (Tiny-jubjub Membership). In one of our main examples, we derive a SNARK that proves a pair (x, y) of field elements from \mathbb{F}_{13} to be a point on the tiny-jubjub curve in its Edwards form (see section 5.1.3).

In the first step, we define a language such that points on the tiny-jubjub curve are in 1:1 correspondence with words in that language.

Since the tiny-jubjub curve is an elliptic curve over the field \mathbb{F}_{13} , we choose the alphabet $\Sigma = \mathbb{F}_{13}$. In this case, the set \mathbb{F}_{13}^* consists of all finite strings of field elements from \mathbb{F}_{13} . To define the grammar, recall from 69 that a point on the tiny-jubjub curve is a pair (x, y) of field elements such that $3 \cdot x^2 + y^2 = 1 + 8 \cdot x^2 \cdot y^2$. We can use this equation to derive the following decision function:

$$R_{tiny.jj} : \mathbb{F}_{13}^* \rightarrow \{true, false\} ; (x_1, \dots, x_n) \mapsto \begin{cases} true & n = 2 \text{ and } 3 \cdot x_1^2 + x_2^2 = 1 + 8 \cdot x_1^2 \cdot x_2^2 \\ false & \text{else} \end{cases}$$

The associated language $L_{tiny.jj}$ is then given as the set of all strings from \mathbb{F}_{13}^* that are mapped onto *true* by $R_{tiny.jj}$. We get

$$L_{tiny.jj} = \{(x_1, \dots, x_n) \in \mathbb{F}_{13}^* \mid R_{tiny.jj}(x_1, \dots, x_n) = true\}$$

We can claim the statement “There is a word in $L_{tiny.jj}$ ” and because $L_{tiny.jj}$ is defined by $R_{tiny.jj}$, this statement is equivalent to the claim “The tiny-jubjub curve in its Edwards form has curve a point.”

A constructive proof for this statement is a pair (x, y) of field elements that satisfies the Edwards equation. Example 69 therefore implies that the tuple $(11, 6)$ is a constructive proof and the computation $R_{tiny.jj}(11, 6) = true$ is a proof verification. In contrast, the tuple $(1, 1)$ is not a proof of the statement, since the check $R_{tiny.jj}(1, 1) = false$ does not verify the proof.

Are we using w and x interchangeably or is there a difference between them?

check reference

jubjub

check reference

check reference

check wording

check reference

Exercise 54. Consider exercise XXX again. Define a decision function such that the associated language $L_{\text{Exercise}_{XX}}$ consist precisely of all solutions to the equation $5x + 4 = 28 + 2x$ over \mathbb{F}_{13} . Provide a constructive proof for the claim: “There exist a word in $L_{\text{Exercise}_{XX}}$ and verify the proof.

Exercise 55. Consider the modular 6 arithmetics \mathbb{Z}_6 from example 9 in chapter ??, the alphabet $\Sigma = \mathbb{Z}_6$ and the decision function

$$R_{\text{example}_9} : \Sigma^* \rightarrow \{true, false\} ; x \mapsto \begin{cases} true & x.\text{len}() = 1 \text{ and } 3 \cdot x + 3 = 0 \\ false & \text{else} \end{cases}$$

Compute all words in the associated language L_{example_9} , provide a constructive proof for the statement “There exist a word in L_{example_9} ” and verify the proof.

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Instance and Witness As we have seen in the previous paragraph, statements provide membership claims in formal languages, and instances serve as constructive proofs for those claims. However, in the context of **zero-knowledge** proof systems, our naive notion of constructive proofs is refined in such a way that its possible to hide parts of the proof instance and still be able to prove the statement. In this context, it is therefore necessary to split a proof into a **public part** called the **instance** and a private part called a **witness**.

To account for this separation of a proof instance into a public and a private part, our previous definition of formal languages needs a refinement in the context of zero-knowledge proof systems. Instead of a single alphabet, the refined definition considers two alphabets Σ_I and Σ_W , and a decision function defined as follows:

$$R : \Sigma_I^* \times \Sigma_W^* \rightarrow \{true, false\} ; (i; w) \mapsto R(i; w) \quad (6.3)$$

Words are therefore tuples $(i; w) \in \Sigma_I^* \times \Sigma_W^*$ with $R(i; w) = true$. The refined definition differentiates between public inputs $i \in \Sigma_I$ and private inputs $w \in \Sigma_W$. The public input i is called an **instance** and the private input w is called a **witness** of R .

If a decision function is given, the associated language is defined as the set of all tuples from the underlying alphabet that are verified by the decision function:

$$L_R := \{(i; w) \in \Sigma_I^* \times \Sigma_W^* \mid R(i; w) = true\} \quad (6.4)$$

In this refined context, a **statement** S is a claim that, given an instance $i \in \Sigma_I^*$, there is a witness $w \in \Sigma_W^*$ such that language L contains a word $(i; w)$. A constructive **proof** for statement S is given by some string $P = (i; w) \in \Sigma_I^* \times \Sigma_W^*$ and a proof is **verified** by checking $R(P) = true$.

It is worth understanding the difference between statements as defined in XXX and the refined notion of statements from this paragraph. While statements in the sense of the previous paragraph can be seen as membership claims, statements in the refined definition can be seen as knowledge-proofs, where a prover claims knowledge of a witness for a given instance. For a more detailed discussion on this topic see [XXX sec 1.4]

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Example 111 (SHA256 – Knowledge of Preimage). One of the most common examples in the context of zero-knowledge proof systems is the **knowledge-of-a-preimage proof** for some cryptographic hash function like *SHA256*, where a publicly known *SHA256* digest value is given, and the task is to prove knowledge of a preimage for that digest under the *SHA256* function, without revealing that preimage.

To understand this problem in detail, we have to introduce a language able to describe the knowledge-of-preimage problem in such a way that the claim “Given digest i , there is a

preimage w such that $SHA256(w) = i$ ” becomes a statement in that language. Since $SHA256$ is a function

$$SHA256 : \{0, 1\}^* \rightarrow \{0, 1\}^{256}$$

that maps binary strings of arbitrary length onto binary strings of length 256 and we want to prove knowledge of preimages, we have to consider binary strings of size 256 as instances and binary strings of arbitrary length as witnesses.

An appropriate alphabet Σ_I for the set of all instances and an appropriate alphabet Σ_W for the set of all witnesses is therefore given by the set $\{0, 1\}$ of the two binary letters and a proper decision function is given by:

$$R_{SHA256} : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \{true, false\};$$

$$(i; w) \mapsto \begin{cases} true & i.len() = 256, i = SHA256(w) \\ false & else \end{cases}$$

We write L_{SHA256} for the associated language and note that it consists of words, which are tuples $(i; w)$ such that the instance i is the $SHA256$ image of the witness w .

Given some instance $i \in \{0, 1\}^{256}$, a statement in L_{SHA256} is the claim “Given digest i , there is a preimage w such that $SHA256(w) = i$ ”, which is exactly what the knowledge-of-preimage problem is about. A constructive proof for this statement is therefore given by a preimage w to the digest i and proof verification is achieved by checking that $SHA256(w) = i$.

Example 112 (3-factorization). To give an intuition about the implication of refined languages, consider $L_{3, fac}$ from example 109 again. As we have seen, a constructive proof in $L_{3, fac}$ is given by 4 field elements x_1, x_2, x_3 and x_4 from \mathbb{F}_{13} such that the product in modular 13 arithmetics of the first three elements is equal to the 4'th element.

Splitting words from $L_{3, fac}$ into private and public parts, we can reformulate the problem and introduce different levels of privacy into the problem. For example, we could reformulate the membership statement of $L_{3, fac}$ into a statement where all factors x_1, x_2, x_3 of x_4 are private and only the product x_4 is public. A statement for this reformulation is then expressed by the claim: “Given a publicly known field element x_4 , there are three private factors of x_4 ”. Assuming some instance x_4 , a constructive proof for the associated knowledge claim is then provided by any tuple (x_1, x_2, x_3) such that $x_1 \cdot x_2 \cdot x_3 = x_4$.

At this point, it is important to note that, while constructive proofs in the refinement don't look very different from constructive proofs in the original language, we will see in XXX that there are proof systems able to prove the statement (at least with high probability) without revealing anything about the factors x_1, x_2 , or x_3 . This is why the importance of the refinement only becomes clear once more elaborate proofing methods beyond naive constructive proofs are provided.

We can formalize this new language, which we might call L_{3, fac_zk} , by defining the following decision function:

Definition 6.1.0.1.

$$R_{3, fac_zk} : \mathbb{F}_{13}^* \times \mathbb{F}_{13}^* \rightarrow \{true, false\};$$

$$((i_1, \dots, i_n); (w_1, \dots, w_m)) \mapsto \begin{cases} true & n = 1, m = 3, i_1 = w_1 \cdot w_2 \cdot w_3 \\ false & else \end{cases}$$

The associated language L_{3, fac_zk} is defined by all tuples from $\mathbb{F}_{13}^* \times \mathbb{F}_{13}^*$ that are mapped onto *true* under the decision function R_{3, fac_zk} .

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ence

Considering the distinction we made between the instance and the witness part in L_{3, fac_zk} , one might ask why we chose the factors x_1, x_2 and x_3 to be the witness and the product x_4 to be the instance and why we didn't choose another combination? This was an arbitrary choice in the example. Every other combination of private and public factors would be equally valid. For example, it would be possible to declare all variables as private or to declare all variables as public. Actual choices are determined by the application only.

Example 113 (The Tiny-Jubjub Curve). Consider the language $L_{tiny.jj}$ from example 110. As we have seen, a constructive proof in $L_{tiny.jj}$ is given by a pair (x_1, x_2) of field elements from \mathbb{F}_{13} such that the pair is a point of the tiny-jubjub curve in its Edwards representation.

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We look at a reasonable splitting of words from $L_{tiny.jj}$ into private and public parts. The two obvious choices are to either choose both coordinates x_1 as x_2 as public inputs, or to choose both coordinates x_1 as x_2 as private inputs.

In case both coordinates are public, we define the grammar of the associated language by introducing the following decision function:

$$R_{tiny.jj.1} : \mathbb{F}_{13}^* \times \mathbb{F}_{13}^* \rightarrow \{true, false\} ;$$

$$(I_1, \dots, I_n; W_1, \dots, W_m) \mapsto \begin{cases} true & n = 2, m = 0 \text{ and } 3 \cdot I_1^2 + I_2^2 = 1 + 8 \cdot I_1^2 \cdot I_2^2 \\ false & else \end{cases}$$

The language $L_{tiny.jj.1}$ is defined as the set of all strings from $\mathbb{F}_{13}^* \times \mathbb{F}_{13}^*$ that are mapped onto *true* by $R_{tiny.jj.1}$.

In case both coordinates are private, we define the grammar of the associated refined language by introducing the following decision function:

$$R_{tiny.jj.zk} : \mathbb{F}_{13}^* \times \mathbb{F}_{13}^* \rightarrow \{true, false\} ;$$

$$(I_1, \dots, I_n; W_1, \dots, W_m) \mapsto \begin{cases} true & n = 0, m = m \text{ and } 3 \cdot W_1^2 + W_2^2 = 1 + 8 \cdot W_1^2 \cdot W_2^2 \\ false & else \end{cases}$$

The language $L_{tiny.jj.zk}$ is defined as the set of all strings from $\mathbb{F}_{13}^* \times \mathbb{F}_{13}^*$ that are mapped onto *true* by $R_{tiny.jj.zk}$.

Exercise 56. Consider the modular 6 arithmetics \mathbb{Z}_6 from example 9 in chapter ?? as alphabets Σ_I and Σ_W and the following decision function

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$$R_{linear} : \Sigma^* \times \Sigma^* \rightarrow \{true, false\} ;$$

$$(i; w) \mapsto \begin{cases} true & i.len() = 3 \text{ and } w.len() = 1 \text{ and } i_1 \cdot w_1 + i_2 = i_3 \\ false & else \end{cases}$$

Which of the following instances (i_1, i_2, i_3) has a proof of knowledge in L_{linear} ?

• (3, 3, 0)

• (2, 1, 0)

• (4, 4, 2)

Exercise 57 (Edwards Addition on Tiny-Jubjub). Consider the tiny-jubjub curve together with its Edwards addition law from example XXX. Define an instance alphabet Σ_I , a witness alphabet Σ_W and a decision function R_{add} with associated language L_{add} such that a string $(i; w) \in \Sigma_I^* \times$

add refer-
ence

4374 Σ_W^* is a word in L_{add} if and only if i is a pair of curve points on the tiny-jubjub curve in Edwards
 4375 form and w is the Edwards sum of those curve points.

4376 Choose some instance $i \in \Sigma_I^*$, provide a constructive proof for the statement “There is a
 4377 witness $w \in \Sigma_W^*$ such that $(i;w)$ is a word in L_{add} ” and verify that proof. Then find some
 4378 instance $i \in \Sigma_I^*$ such that i has no knowledge proof in L_{add} .

4379 **Modularity** From a developers perspective, it is often useful to construct complex statements
 4380 and their representing languages from simple ones. In the context of zero-knowledge proof
 4381 systems, those simple building blocks are often called **gadgets**, and gadget libraries usually
 4382 contain representations of atomic types like booleans, integers, various hash functions, elliptic
 4383 curve cryptography and many more. In order to synthesize statements, developers then combine
 4384 predefined gadgets into complex logic. We call the ability to combine statements into more
 4385 complex statements **modularity**.

4386 To understand the concept of modularity on the level of formal languages defined by deci-
 4387 sion functions, we need to look at the **intersection** of two languages, which exists whenever
 4388 both languages are defined over the same alphabet. In this case, the intersection is a language
 4389 that consists of strings which are words in both languages.

4390 To be more precise, let L_1 and L_2 be two languages defined over the same instance and
 4391 witness alphabets Σ_I and Σ_W . Then the intersection $L_1 \cap L_2$ of L_1 and L_2 is defined as

$$L_1 \cap L_2 := \{x \mid x \in L_1 \text{ and } x \in L_2\} \quad (6.5)$$

4392 If both languages are defined by decision functions R_1 and R_2 , the following function is a
 4393 decision function for the intersection language $L_1 \cap L_2$:

$$R_{L_1 \cap L_2} : \Sigma_I^* \times \Sigma_W^* \rightarrow \{true, false\}; (i, w) \mapsto R_1(i, w) \text{ and } R_2(i, w) \quad (6.6)$$

4394 Thus, the intersection of two decision-function-based languages is a also decision-function-
 4395 based language. This is important from an implementations point of view: It allows us to
 4396 construct complex decision functions, their languages and associated statements from simple
 4397 building blocks. Given a publicly known instance $i \in \Sigma_I^*$ a statement in an intersection language
 4398 then claims knowledge of a witness that satisfies all relations simultaneously.

4399 6.2 Statement Representations

4400 As we have seen in the previous section, formal languages and their definitions by decision
 4401 functions are a powerful tool to describe statements in a formally rigorous manner.

4402 However, from the perspective of existing zero-knowledge proof systems, not all ways to
 4403 actually represent decision functions are equally useful. Depending on the proof system, some
 4404 are more suitable than others. In this section, will describe two of the most common ways to
 4405 represent decision functions and their statements.

4406 6.2.1 Rank-1 Quadratic Constraint Systems

4407 Although decision functions are expressible in various ways, many contemporary proofing sys-
 4408 tems require the deciding relation to be expressed in terms of a system of quadratic equations
 4409 over a finite field. This is true in particular for pairing-based proofing systems like XXX,
 4410 roughly because it is possible to check solutions to those equations “in the exponent” of pairing-
 4411 friendly cryptographic groups.

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ence

In this section, we will therefore have a closer look at a particular type of quadratic equation called **rank-1 quadratic constraints systems**, which are a common standard in zero-knowledge proof systems. We will start with a general introduction to those systems and then look at their relation to formal languages. We will look into a common way to compute solutions to those systems, and then show how a simple compiler might derive rank-1 constraint systems from more high-level programming code.

R1CS representation To understand what **rank-1 (quadratic) constraint systems** are in detail, let \mathbb{F} be a field, n, m and $k \in \mathbb{N}$ three numbers and a_j^i, b_j^i and $c_j^i \in \mathbb{F}$ constants from \mathbb{F} for every index $0 \leq j \leq n+m$ and $1 \leq i \leq k$. Then a rank-1 constraint system (R1CS) is defined as follows:

Definition 6.2.1.1. R1CS representation

$$\begin{aligned} (a_0^1 + \sum_{j=1}^n a_j^1 \cdot I_j + \sum_{j=1}^m a_{n+j}^1 \cdot W_j) \cdot (b_0^1 + \sum_{j=1}^n b_j^1 \cdot I_j + \sum_{j=1}^m b_{n+j}^1 \cdot W_j) &= c_0^1 + \sum_{j=1}^n c_j^1 \cdot I_j + \sum_{j=1}^m c_{n+j}^1 \cdot W_j \\ &\vdots \\ (a_0^k + \sum_{j=1}^n a_j^k \cdot I_j + \sum_{j=1}^m a_{n+j}^k \cdot W_j) \cdot (b_0^k + \sum_{j=1}^n b_j^k \cdot I_j + \sum_{j=1}^m b_{n+j}^k \cdot W_j) &= c_0^k + \sum_{j=1}^n c_j^k \cdot I_j + \sum_{j=1}^m c_{n+j}^k \cdot W_j \end{aligned}$$

If a rank-1 constraint system is given, the parameter k is called the **number of constraints**. If a tuple $(I_1, \dots, I_n; W_1, \dots, W_m)$ of field elements satisfies these equations, (I_1, \dots, I_n) is called an **instance** and (W_1, \dots, W_m) is called an associated **witness** of the system.

Remark 5 (Matrix notation). The presentation of rank-1 constraint systems can be simplified using the notation of vectors and matrices, which abstracts over the indices. In fact if $x = (1, I, W) \in \mathbb{F}^{1+n+m}$ is a $(n+m+1)$ -dimensional vector, A, B, C are $(n+m+1) \times k$ -dimensional matrices and \odot is the **Schur/Hadamard product**, then a R1CS can be written as

$$Ax \odot Bx = Cx$$

However, since we did not introduced matrix calculus in the book, we use XXX as the defining equation for rank-1 constraints systems. We only highlighted the matrix notation, because it is sometimes used in the literature.

Generally speaking, the idea of a rank-1 constraint system is to keep track of all the values that any variable can assume during a computation and to bind the relationships among all those variables that are implied by the computation itself. Enforcing relations between all the steps of a computer program, the execution is then constrained to be computed in exactly the expected way without any opportunity for deviations. In this sense, solutions to rank-1 constraint systems are proofs of proper program execution.

Example 114 (3-Factorization). To provide a better intuition of rank-1 constraint systems, consider the language L_{3, fac_zk} from example 109 again. As we have seen, L_{3, fac_zk} consists of words $(I_1; W_1, W_2, W_3)$ over the alphabet \mathbb{F}_{13} such that $I_1 = W_1 \cdot W_2 \cdot W_3$. We show how to rewrite the decision function as a rank-1 constraint system.

Since R1CS are systems of quadratic equations, expressions like $W_1 \cdot W_2 \cdot W_3$ which contain products of more than two factors (which are therefore not quadratic) have to be rewritten in a process often called **flattening**. To flatten the defining equation $I_1 = W_1 \cdot W_2 \cdot W_3$ of L_{3, fac_zk} we introduce a new variable W_4 , which captures two of the three multiplications in $W_1 \cdot W_2 \cdot W_3$. We get the following two constraints

$$\begin{aligned} W_1 \cdot W_2 &= W_4 && \text{constraint 1} \\ W_4 \cdot W_3 &= I_1 && \text{constraint 2} \end{aligned}$$

Given some instance I_1 , any solution (W_1, W_2, W_3, W_4) to this system of equations provides a solution to the original equation $I_1 = W_1 \cdot W_2 \cdot W_3$ and vice versa. Both equations are therefore equivalent in the sense that solutions are in a 1:1 correspondence.

Looking at both equations, we see how each constraint enforces a step in the computation. In fact, the first constraint forces any computation to multiply the witness W_1 and W_2 first. Otherwise it would not be possible to compute the witness W_4 , which is needed to solve the second constraint. Witness W_4 therefore expresses the constraining of an intermediate computational state.

At this point, one might ask why equation 1 constrains the system to compute $W_1 \cdot W_2$ first, since computing $W_2 \cdot W_3$, or $W_1 \cdot W_3$ in the beginning and then multiplying with the remaining factor gives the exact same result. The reason is that the way we designed the R1CS prohibits any of these alternative computations, which shows that R1CS are in general **not unique** descriptions of a language: many different R1CS are able to describe the same problem.

To see that the two quadratic equations qualify as a rank-1 constraint system, choose the parameter $n = 1, m = 4$ and $k = 2$ as well as

$$\begin{aligned} a_0^1 &= 0 & a_1^1 &= 0 & a_2^1 &= 1 & a_3^1 &= 0 & a_4^1 &= 0 & a_5^1 &= 0 \\ a_0^2 &= 0 & a_1^2 &= 0 & a_2^2 &= 0 & a_3^2 &= 0 & a_4^2 &= 0 & a_5^2 &= 1 \\ b_0^1 &= 0 & b_1^1 &= 0 & b_2^1 &= 0 & b_3^1 &= 1 & b_4^1 &= 0 & b_5^1 &= 0 \\ b_0^2 &= 0 & b_1^2 &= 0 & b_2^2 &= 0 & b_3^2 &= 0 & b_4^2 &= 1 & b_5^2 &= 0 \\ c_0^1 &= 0 & c_1^1 &= 0 & c_2^1 &= 0 & c_3^1 &= 0 & c_4^1 &= 0 & c_5^1 &= 1 \\ c_0^2 &= 0 & c_1^2 &= 1 & c_2^2 &= 0 & c_3^2 &= 0 & c_4^2 &= 0 & c_5^2 &= 0 \end{aligned}$$

With this choice, the rank-1 constraint system of our 3-factorization problem can be written in its most general form as follows:

$$\begin{aligned} (a_0^1 + a_1^1 I_1 + a_2^1 W_1 + a_3^1 W_2 + a_4^1 W_3 + a_5^1 W_4) \cdot (b_0^1 + b_1^1 I_1 + b_2^1 W_1 + b_3^1 W_2 + b_4^1 W_3 + b_5^1 W_4) &= (c_0^1 + c_1^1 I_1 + c_2^1 W_1 + c_3^1 W_2 + c_4^1 W_3 + c_5^1 W_4) \\ (a_0^2 + a_1^2 I_1 + a_2^2 W_2 + a_3^2 W_2 + a_4^2 W_3 + a_5^2 W_4) \cdot (b_0^2 + b_1^2 I_1 + b_2^2 W_2 + b_3^2 W_2 + b_4^2 W_3 + b_5^2 W_4) &= (c_0^2 + c_1^2 I_1 + c_2^2 W_2 + c_3^2 W_2 + c_4^2 W_3 + c_5^2 W_4) \end{aligned}$$

Example 115 (The Tiny-Jubjub curve). Consider the languages $L_{\text{tiny.jj.1}}$ from example 110, which consist of words (I_1, I_2) over the alphabet \mathbb{F}_{13} such that $3 \cdot I_1^2 + I_2^2 = 1 + 8 \cdot I_1^2 \cdot I_2^2$.

We derive a rank-1 constraint system such that its associated language is equivalent to $L_{\text{tiny.jj.1}}$. To achieve this, we first rewrite the defining equation:

$$\begin{aligned} 3 \cdot I_1^2 + I_2^2 &= 1 + 8 \cdot I_1^2 \cdot I_2^2 && \Leftrightarrow \\ 0 &= 1 + 8 \cdot I_1^2 \cdot I_2^2 - 3 \cdot I_1^2 - I_2^2 && \Leftrightarrow \\ 0 &= 1 + 8 \cdot I_1^2 \cdot I_2^2 + 10 \cdot I_1^2 + 12 \cdot I_2^2 \end{aligned}$$

Since R1CSs are systems of quadratic equations, we have to reformulate this expression into a system of quadratic equations. To do so, we have to introduce new variables that constrain intermediate steps in the computation and we have to decide if those variables should be public or private. We decide to declare all new variables as private and get the following constraints

$$\begin{aligned} I_1 \cdot I_1 &= W_1 && \text{constraint 1} \\ I_2 \cdot I_2 &= W_2 && \text{constraint 2} \\ (8 \cdot W_1) \cdot W_2 &= W_3 && \text{constraint 3} \\ (12 \cdot W_2 + W_3 + 10 \cdot W_1 + 1) \cdot 1 &= 0 && \text{constraint 4} \end{aligned}$$

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To see that these four quadratic equations qualify as a rank-1 constraint system according to definition XXX, choose the parameter $n = 2$, $m = 3$ and $k = 4$:

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$$\begin{array}{llllll} a_0^1 = 0 & a_1^1 = 1 & a_2^1 = 0 & a_3^1 = 0 & a_4^1 = 0 & a_5^1 = 0 \\ a_0^2 = 0 & a_1^2 = 0 & a_2^2 = 1 & a_3^2 = 0 & a_4^2 = 0 & a_5^2 = 0 \\ a_0^3 = 0 & a_1^3 = 0 & a_2^3 = 0 & a_3^3 = 8 & a_4^3 = 0 & a_5^3 = 0 \\ a_0^4 = 1 & a_1^4 = 0 & a_2^4 = 0 & a_3^4 = 10 & a_4^4 = 12 & a_5^4 = 1 \end{array}$$

$$\begin{array}{llllll} b_0^1 = 0 & b_1^1 = 1 & b_2^1 = 0 & b_3^1 = 0 & b_4^1 = 0 & b_5^1 = 0 \\ b_0^2 = 0 & b_1^2 = 0 & b_2^2 = 1 & b_3^2 = 0 & b_4^2 = 0 & b_5^2 = 0 \\ b_0^3 = 0 & b_1^3 = 0 & b_2^3 = 0 & b_3^3 = 0 & b_4^3 = 1 & b_5^3 = 0 \\ b_0^4 = 1 & b_1^4 = 0 & b_2^4 = 0 & b_3^4 = 0 & b_4^4 = 0 & b_5^4 = 0 \end{array}$$

$$\begin{array}{llllll} c_0^1 = 0 & c_1^1 = 0 & c_2^1 = 0 & c_3^1 = 1 & c_4^1 = 0 & c_5^1 = 0 \\ c_0^2 = 0 & c_1^2 = 0 & c_2^2 = 0 & c_3^2 = 0 & c_4^2 = 1 & c_5^2 = 0 \\ c_0^3 = 0 & c_1^3 = 0 & c_2^3 = 0 & c_3^3 = 0 & c_4^3 = 0 & c_5^3 = 1 \\ c_0^4 = 0 & c_1^4 = 0 & c_2^4 = 0 & c_3^4 = 0 & c_4^4 = 0 & c_5^4 = 0 \end{array}$$

With this choice, the rank-1 constraint system of our tiny-jubjub curve point problem can be written in its most general form as follows:

$$\begin{aligned} (a_0^1 + a_1^1 I_1 + a_2^1 I_2 + a_3^1 W_1 + a_4^1 W_2 + a_5^1 W_3) \cdot (b_0^1 + b_1^1 I_1 + b_2^1 I_2 + b_3^1 W_1 + b_4^1 W_2 + b_5^1 W_3) &= (c_0^1 + c_1^1 I_1 + c_2^1 I_2 + c_3^1 W_1 + c_4^1 W_2 + c_5^1 W_3) \\ (a_0^2 + a_1^2 I_1 + a_2^2 I_2 + a_3^2 W_1 + a_4^2 W_2 + a_5^2 W_3) \cdot (b_0^2 + b_1^2 I_1 + b_2^2 I_2 + b_3^2 W_1 + b_4^2 W_2 + b_5^2 W_3) &= (c_0^2 + c_1^2 I_1 + c_2^2 I_2 + c_3^2 W_1 + c_4^2 W_2 + c_5^2 W_3) \\ (a_0^3 + a_1^3 I_1 + a_2^3 I_2 + a_3^3 W_1 + a_4^3 W_2 + a_5^3 W_3) \cdot (b_0^3 + b_1^3 I_1 + b_2^3 I_2 + b_3^3 W_1 + b_4^3 W_2 + b_5^3 W_3) &= (c_0^3 + c_1^3 I_1 + c_2^3 I_2 + c_3^3 W_1 + c_4^3 W_2 + c_5^3 W_3) \\ (a_0^4 + a_1^4 I_1 + a_2^4 I_2 + a_3^4 W_1 + a_4^4 W_2 + a_5^4 W_3) \cdot (b_0^4 + b_1^4 I_1 + b_2^4 I_2 + b_3^4 W_1 + b_4^4 W_2 + b_5^4 W_3) &= (c_0^4 + c_1^4 I_1 + c_2^4 I_2 + c_3^4 W_1 + c_4^4 W_2 + c_5^4 W_3) \end{aligned}$$

4453 In what follows, we write L_{jubjub} for the associated language that consists of solutions to the
4454 R1CS.

4455 To see that L_{jubjub} is equivalent to $L_{tiny.jj.1}$, let $(I_1, I_2; W_1, W_2, W_3)$ be a word in L_{jubjub} , then
4456 (I_1, I_2) is a word in $L_{tiny.jj.1}$, since the defining R1CS of L_{jubjub} implies that I_1 and I_2 satisfy the
4457 Edwards equation of the tiny-jubjub curve. On the other hand, let (I_1, I_2) be a word in $L_{tiny.jj.1}$.
4458 Then $(I_1, I_2; I_1^2, I_2^2, 8 \cdot I_1^2 \cdot I_2^2)$ is a word in L_{jubjub} and both maps are inverses of each other.

4459 **Exercise 58.** Consider the language $L_{tiny.jj_zk}$ and define a rank-1 constraint relation with a
4460 decision function such that the associated language is equivalent to $L_{tiny.jj_zk}$.

4461 **R1CS Satisfiability** To understand how rank-1 constraint systems define formal languages,
4462 observe that every R1CS over a field \mathbb{F} defines a decision function over the alphabet $\Sigma_I \times \Sigma_W =$
4463 $\mathbb{F} \times \mathbb{F}$ in the following way:

$$R_{R1CS} : \mathbb{F}^* \times \mathbb{F}^* \rightarrow \{true, false\} ; (I; W) \mapsto \begin{cases} true & (I; W) \text{ satisfies R1CS} \\ false & \text{else} \end{cases} \quad (6.7)$$

4464 Every R1CS therefore defines a formal language. The grammar of this language is encoded
4465 in the constraints, words are solutions to the equations and a **statement** is a knowledge claim
4466 “Given instance I , there is a witness W such that $(I; W)$ is a solution to the rank-1 constraint
4467 system”. A constructive proof to this claim is therefore an assignment of a field element to every
4468 witness variable, which is verified whenever the set of all instance and witness variables solves
4469 the R1CS.

Remark 6 (R1CS satisfiability). It should be noted that in our definition, every R1CS defines its own language. However, in more theoretical approaches, another language usually called **R1CS satisfiability** is often considered, which is useful when it comes to more abstract problems like expressiveness or the computational complexity of the class of **all** R1CS. From our perspective, the R1CS satisfiability language is obtained by the union of all R1CS languages that are in our definition. To be more precise, let the alphabet $\Sigma = \mathbb{F}$ be a field. Then

$$L_{R1CS_SAT}(\mathbb{F}) = \{(i; w) \in \Sigma^* \times \Sigma^* \mid \text{there is a R1CS } R \text{ such that } R(i; w) = \text{true}\}$$

4470 *Example 116 (3-Factorization).* Consider the language $L_{3.\text{fac_zk}}$ from example 109 and the
 4471 R1CS defined in example ex:3-factorization-r1cs. As we have seen in ex:3-factorization-r1cs,
 4472 solutions to the R1CS are in 1:1 correspondence with solutions to the decision function of
 4473 $L_{3.\text{fac_zk}}$. Both languages are therefore equivalent in the sense that there is a 1:1 correspon-
 4474 dence between words in both languages.

To give an intuition of what constructive proofs in $L_{3.\text{fac_zk}}$ look like, consider the instance $I_1 = 11$. To prove the statement “There exists a witness W such that $(I_1; W)$ is a word in $L_{3.\text{fac_zk}}$ ” constructively, a proof has to provide assignments to all witness variables W_1, W_2, W_3 and W_4 . Since the alphabet is \mathbb{F}_{13} , an example assignment is given by $W = (2, 3, 4, 6)$ since $(I_1; W)$ satisfies the R1CS

$$\begin{array}{ll} W_1 \cdot W_2 = W_4 & \# 2 \cdot 3 = 6 \\ W_4 \cdot W_3 = I_1 & \# 6 \cdot 4 = 11 \end{array}$$

4475 A proper constructive proof is therefore given by $P = (2, 3, 4, 6)$. Of course, P is not the only
 4476 possible proof for this statement. Since factorization is not unique in a field in general, another
 4477 constructive proof is given by $P' = (3, 5, 12, 2)$.

Example 117 (The tiny-jubjub curve). Consider the language L_{jubjub} from example 110 and
 its associated R1CS. To see how constructive proofs in L_{jubjub} look like, consider the instance
 $(I_1, I_2) = (11, 6)$. To prove the statement “There exists a witness W such that $(I_1, I_2; W)$ is a
 word in L_{jubjub} ” constructively, a proof has to provide assignments to all witness variables $W_1,$
 W_2 and W_3 . Since the alphabet is \mathbb{F}_{13} , an example assignment is given by $W = (4, 10, 8)$ since
 $(I_1, I_2; W)$ satisfies the R1CS

$$\begin{array}{ll} I_1 \cdot I_1 = W_1 & 11 \cdot 11 = 4 \\ I_2 \cdot I_2 = W_2 & 6 \cdot 6 = 10 \\ (8 \cdot W_1) \cdot W_2 = W_3 & (8 \cdot 4) \cdot 10 = 8 \\ (12 \cdot W_2 + W_3 + 10 \cdot W_1 + 1) \cdot 1 = 0 & 12 \cdot 10 + 8 + 10 \cdot 4 + 1 = 0 \end{array}$$

4478 A proper constructive proof is therefore given by $P = (4, 10, 8)$, which shows that the instance
 4479 $(11, 6)$ is a point on the tiny-jubjub curve.

4480 **Modularity** As we discussed on page 139 XXX, it is often useful to construct complex state-
 4481 ments and their representing languages from simple ones. Rank-1 constraint systems are par-
 4482 ticularly useful for this, as the intersection of two R1CS over the same alphabet results in a new
 4483 R1CS over that same alphabet.

4484 To be more precise, let S_1 and S_2 be two R1CS over \mathbb{F} , then a new R1CS S_3 is obtained
 4485 by the intersection $S_3 = S_1 \cap S_2$ of S_1 and S_2 . In this context, intersection means that both the
 4486 equations of S_1 **and** the equations of S_2 have to be satisfied in order to provide a solution for the
 4487 system S_3 .

As a consequence, developers are able to construct complex R1CS from simple ones and this modularity provides the theoretical foundation for many R1CS compilers, as we will see in XXX.

add reference

6.2.2 Algebraic Circuits

As we have seen in the previous paragraphs, rank-1 constraint systems are quadratic equations such that solutions are knowledge proofs for the existence of words in associated languages. From the perspective of a proofer, it is therefore important to solve those equations efficiently.

However, in contrast to systems of linear equation, no general methods are known that solve systems of quadratic equations efficiently. Rank-1 constraint systems are therefore impractical from a proofers perspective and auxiliary information is needed that helps to compute solutions efficiently.

Methods which compute R1CS solutions are sometimes called **witness generator functions**. To provide a common example, we introduce another class of decision functions called **algebraic circuits**. As we will see, every algebraic circuit defines an associated R1CS and also provides an efficient way to compute solutions for that R1CS.

It can be shown that every space- and time-bounded computation is expressible as an algebraic circuit. Transforming high-level computer programs into those circuits is a process often called **flattening**.

To understand this in more detail, we will introduce our model for algebraic circuits and look at the concept of circuit execution and valid assignments. After that, we will show how to derive rank-1 constraint systems from circuits and how circuits are useful to compute solutions to their R1CS efficiently.

Algebraic circuit representation To see what algebraic circuits are, let \mathbb{F} be a field. An algebraic circuit is then a directed acyclic (multi)graph that computes a polynomial function over \mathbb{F} . Nodes with only outgoing edges (source nodes) represent the variables and constants of the function and nodes with only incoming edges (sink nodes) represent the outcome of the function. All other nodes have exactly two incoming edges and represent the defining field operations **addition** as well as **multiplication**. Graph edges represent the flow of the computation along the nodes.

To be more precise, we call a directed acyclic multi-graph $C(\mathbb{F})$ an **algebraic circuit** over \mathbb{F} in this book if the following conditions hold:

Definition 6.2.2.1. Algebraic circuit

- The set of edges has a total order.
- Every source node has a label that represents either a variable or a constant from the field \mathbb{F} .
- Every sink node has exactly one incoming edge and a label that represents either a variable or a constant from the field \mathbb{F} .
- Every node that is neither a source nor a sink has exactly two incoming edges and a label from the set $\{+, *\}$ that represents either addition or multiplication in \mathbb{F} .
- All outgoing edges from a node have the same label.
- Outgoing edges from a node with a label that represents a variable have a label.

- Outgoing edges from a node with a label that represents multiplication have a label, if there is at least one labeled edge in both input path.
- All incoming edges to sink nodes have a label.
- If an edge has two labels S_i and S_j it gets a new label $S_i = S_j$.
- No other edge has a label.
- Incoming edges to sink nodes that are labeled with a constant $c \in \mathbb{F}$ are labeled with the same constant. Every other edge label is taken from the set $\{W, I\}$ and indexed compatible with the order of the edge set.

It should be noted that the details in the definitions of algebraic circuits vary between different sources. We use this definition as it is conceptually straightforward and well-suited for pen-and-paper computations.

To get a better intuition of our definition, let $C(\mathbb{F})$ be an algebraic circuit. Source nodes are the inputs to the circuit and either represent variables or constants. In a similar way, sink nodes represent termination points of the circuit and are either output variables or constants. Constant sink nodes enforce computational outputs to take on certain values.

Nodes that are neither source nodes nor sink nodes are called **arithmetic gates**. Arithmetic gates that are decorated with the “+”-label are called **addition-gates** and arithmetic gates that are decorated with the “·”-label are called **multiplication-gates**. Every arithmetic gate has exactly two inputs, represented by the two incoming edges.

Since the set of edges is ordered, we can write it as $\{E_1, E_2, \dots, E_n\}$ for some $n \in \mathbb{N}$ and we use those indices to index the edge labels, too. Edge labels are therefore either constants or symbols like I_j , W_j or S_j , where j is an index compatible with the edge order. Labels I_j represent instance variables, labels W_j witness variables. Labels on the outgoing edges of input variables constrain the associated variable to that edge. Every other edge defines a constraining equation in the associated R1CS. We will explain this in more detail in XXX.

Notation and Symbols 13. In synthesizing algebraic circuits, assigning instance I_j or witness W_j labels to appropriate edges is often the final step. It is therefore convenient to not distinguish these two types of edges in previous steps. To account for that, we often simply write S_j for an edge label, indicating that the private/public property of the label is unspecified and it might represent an instance or a witness label.

Example 118 (Generalized factorization SNARK). To give a simple example of an algebraic circuit, consider our 3-factorization problem from example 109 again. To express the problem in the algebraic circuit model, consider the following function

$$f_{3.fac} : \mathbb{F}_{13} \times \mathbb{F}_{13} \times \mathbb{F}_{13} \rightarrow \mathbb{F}_{13}; (x_1, x_2, x_3) \mapsto x_1 \cdot x_2 \cdot x_3$$

Using this function, we can describe the zero-knowledge 3-factorization problem from 109 in the following way: Given instance $I_1 \in \mathbb{F}_{13}$, a valid witness is a preimage of $f_{3.fac}$ at the point I_1 , i.e., a valid witness consists of three values W_1 , W_2 and W_3 from \mathbb{F}_{13} such that $f_{3.fac}(W_1, W_2, W_3) = I_1$.

To see how this function can be transformed into an algebraic circuit over \mathbb{F}_{13} , it is a common first step to introduce brackets into the function’s definition and then write the operations as binary operators, in order to highlight how exactly every field operation acts on its two inputs.

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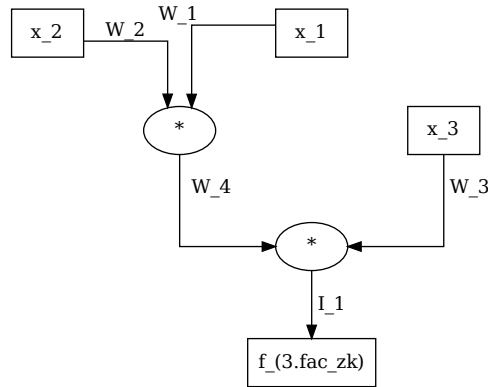
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Due to the associativity laws in a field, we have several choices. We choose

$$\begin{aligned}
 f_{3.fac}(x_1, x_2, x_3) &= x_1 \cdot x_2 \cdot x_3 && \# \text{ bracket choice} \\
 &= (x_1 \cdot x_2) \cdot x_3 && \# \text{ operator notation} \\
 &= MUL(MUL(x_1, x_2), x_3)
 \end{aligned}$$

4566 Using this expression, we can write an associated algebraic circuit by first constraining the
 4567 variables to edge labels $W_1 = x_1$, $W_2 = x_2$ and $W_3 = x_3$ as well as $I_1 = f_{3.fac}(x_1, x_2, x_3)$, taking
 4568 the distinction between private and public inputs into account. We then rewrite the operator
 4569 representation of $f_{3.fac}$ into circuit nodes and get the following:



4570

4571 In this case, the directed acyclic multi-graph is a binary tree with three leaves (the source
 4572 nodes) labeled by x_1 , x_2 and x_3 , one root (the single sink node) labeled by $f(x_1, x_2, x_3)$ and two
 4573 internal nodes, which are labeled as multiplication gates.

4574 The order we use to label the edges is chosen to make the edge labeling consistent with
 4575 the choice of W_4 as defined in definition 6.2.2.1. This order can be obtained by a depth-first
 4576 right-to-left-first traversal algorithm.

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Example 119. To give a more realistic example of an algebraic circuit, look at the defining
 equation of the tiny-jubjub curve (69) again. A pair of field elements $(x, y) \in \mathbb{F}_{13}^2$ is a curve
 point, precisely if the following equation holds:

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$$3 \cdot x^2 + y^2 = 1 + 8 \cdot x^2 \cdot y^2$$

To understand how one might transform this identity into an algebraic circuit, we first rewrite
 this equation by shifting all terms to the right. We get the following:

$$\begin{aligned}
 3 \cdot x^2 + y^2 &= 1 + 8 \cdot x^2 \cdot y^2 && \Leftrightarrow \\
 0 &= 1 + 8 \cdot x^2 \cdot y^2 - 3 \cdot x^2 - y^2 && \Leftrightarrow \\
 0 &= 1 + 8 \cdot x^2 \cdot y^2 + 10 \cdot x^2 + 12 \cdot y^2
 \end{aligned}$$

Then we use this expression to define a function such that all points of the tiny-jubjub curve are
 characterized as the function preimages at 0.

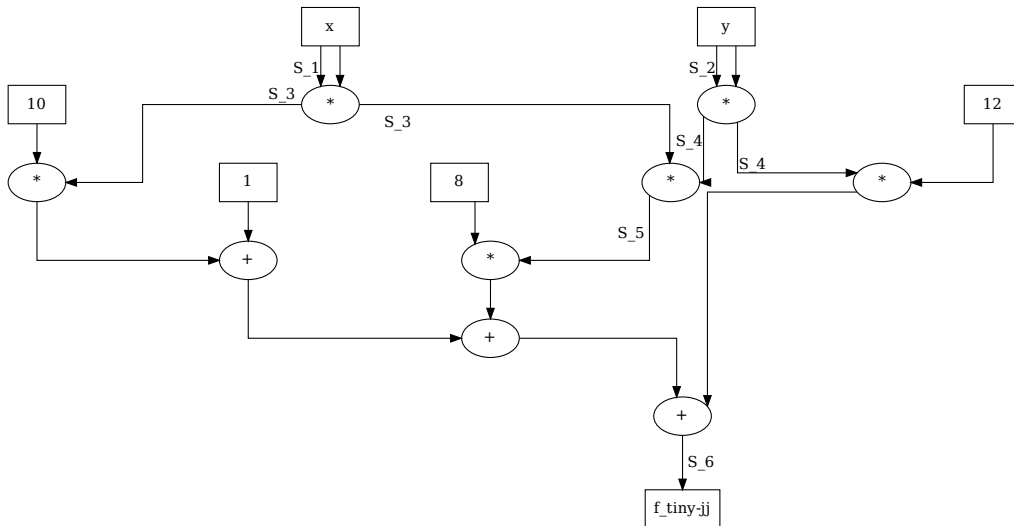
$$f_{tiny-jj} : \mathbb{F}_{13} \times \mathbb{F}_{13} \rightarrow \mathbb{F}_{13} ; (x, y) \mapsto 1 + 8 \cdot x^2 \cdot y^2 + 10 \cdot x^2 + 12 \cdot y^2$$

4577 Every pair of points $(x, y) \in \mathbb{F}_{13}^2$ with $f_{\text{tiny-jj}}(x, y) = 0$ is a point on the tiny-jubjub curve, and
 4578 there are no other curve points. The preimage $f_{\text{tiny-jj}}^{-1}(0)$ is therefore a complete description of
 4579 the tiny-jubjub curve.

We can transform this function into an algebraic circuit over \mathbb{F}_{13} . We first introduce brackets into potentially ambiguous expressions and then rewrite the function in terms of binary operators. We get the following:

$$\begin{aligned}
 f_{\text{tiny-jj}}(x, y) &= 1 + 8 \cdot x^2 \cdot y^2 + 10 \cdot x^2 + 12y^2 && \Leftrightarrow \\
 &= ((8 \cdot ((x \cdot x) \cdot (y \cdot y))) + (1 + 10 \cdot (x \cdot x))) + (12 \cdot (y \cdot y)) && \Leftrightarrow \\
 &= \text{ADD}(\text{ADD}(\text{MUL}(8, \text{MUL}(\text{MUL}(x, x), \text{MUL}(y, y))), \text{ADD}(1, \text{MUL}(10, \text{MUL}(x, x)))), \text{MUL}(12, \text{MUL}(y, y)))
 \end{aligned}$$

4580 Since we haven't decided which part of the computation should be public and which part should
 4581 be private, we use the unspecified symbol S to represent edge labels. Constraining all vari-
 4582 ables to edge labels $S_1 = x$, $S_2 = y$ and $S_6 = f_{\text{tiny-jj}}$, we get the following circuit, representing
 4583 the function $f_{\text{tiny-jj}}$, by inductively replacing binary operators with their associated arithmetic
 4584 gates:



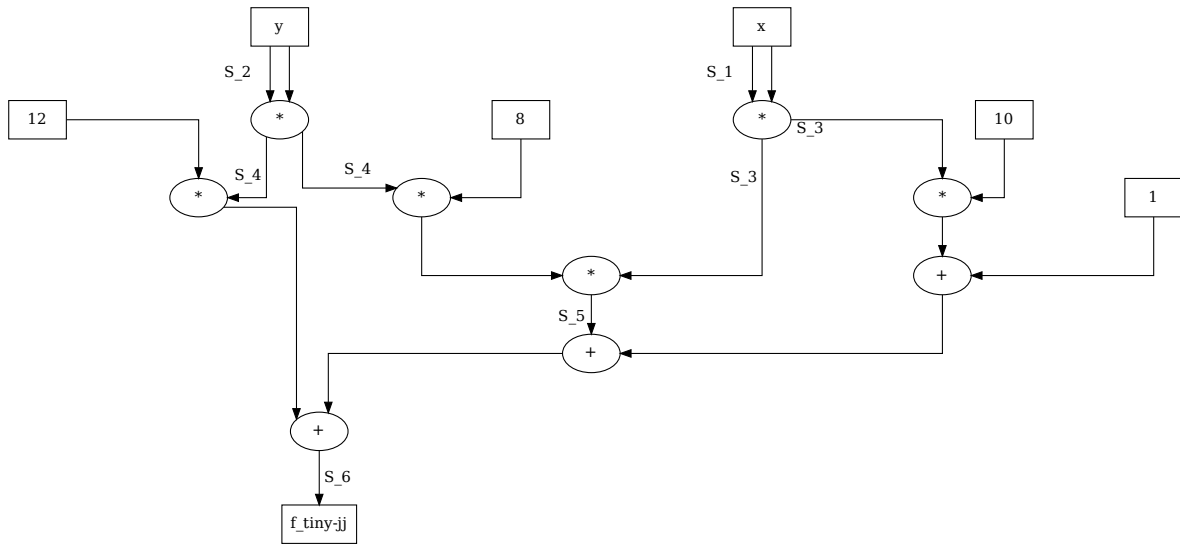
4585

4586 This circuit is not a graph, but a multigraph, since there is more than one edge between some of
 4587 the nodes.

4588 In the process of designing of circuits from functions, it should be noted that circuit rep-
 4589 resentations are not unique in general. In case of the function $f_{\text{tiny-jj}}$, the circuit shape is
 4590 dependent on our choice of bracketing in XXX. An alternative design i,s for example, given by
 4591 the following circuit, which occurs when the bracketed expression $8 \cdot ((x \cdot x) \cdot (y \cdot y))$ is replaced
 4592 by the expression $(x \cdot x) \cdot (8 \cdot (y \cdot y))$.

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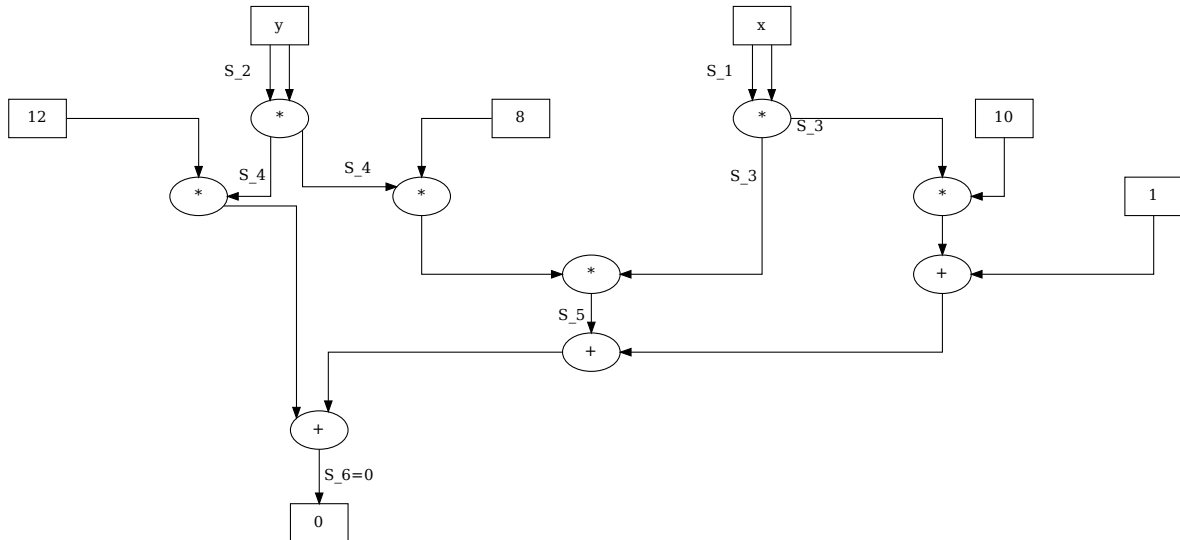
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4595

4596 Of course, both circuits represent the same function, due to the associativity and commutativity
 4597 laws that hold true in any field.

4598 With a circuit that represents the function $f_{\text{tiny-jj}}$, we can now proceed to derive a circuit
 4599 that constrains arbitrary pairs (x, y) of field elements to be points on the tiny-jubjub curve. To do
 4600 so, we have to constrain the output to be zero, that is, we have to constrain $S_6 = 0$. To indicate
 4601 this in the circuit, we replace the output variable by the constant 0 and constrain the related edge
 4602 label accordingly. We get the following:

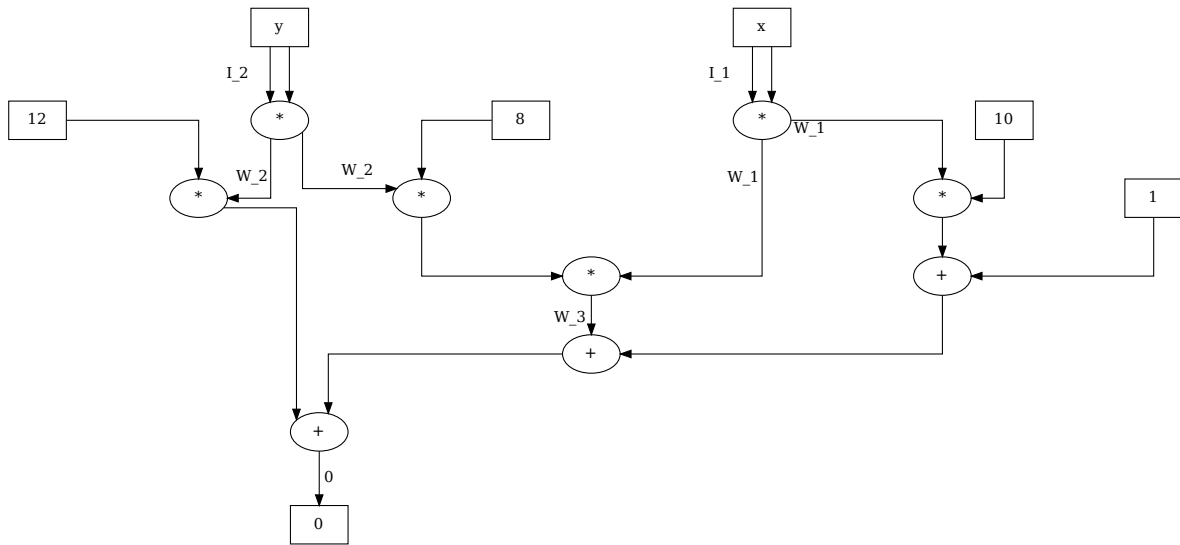
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4605

4606 The previous circuit enforces input values assigned to the labels S_1 and S_2 to be points on the
 4607 tiny-jubjub curve. However, it does not specify which labels are considered public and which
 4608 are considered private. The following circuit defines the inputs to be public, while all other
 4609 labels are private:



It can be shown that every space- and time-bounded computation can be transformed into an algebraic circuit. We call any process that transforms a bounded computation into a circuit **flattening**.

Circuit Execution Algebraic circuits are directed, acyclic multi-graphs, where nodes represent variables, constants, or addition and multiplication gates. In particular, every algebraic circuit with n input nodes decorated with variable symbols and m output nodes decorated with variables can be seen as a function that transforms an input tuple (x_1, \dots, x_n) from \mathbb{F}^n into an output tuple (f_1, \dots, f_m) from \mathbb{F}^m . The transformation is done by sending values associated to nodes along their outgoing edges to other nodes. If those nodes are gates, then the values are transformed according to the gate label and the process is repeated along all edges until a sink node is reached. We call this computation **circuit execution**.

When executing a circuit, it is possible to not only compute the output values of the circuit but to derive field elements for all edges, and, in particular, for all edge labels in the circuit. The result is a tuple (S_1, S_2, \dots, S_n) of field elements associated to all labeled edges, which we call a **valid assignment** to the circuit. In contrast, any assignment $(S'_1, S'_2, \dots, S'_n)$ of field elements to edge labels that can not arise from circuit execution is called an **invalid assignment**.

Valid assignments can be interpreted as **proofs for proper circuit execution** because they keep a record of the computational result as well as intermediate computational steps.

Example 120 (3-factorization). Consider the 3-factorization problem from example 109 and its representation as an algebraic circuit from XXX. We know that the set of edge labels is given by $S := \{I_1; W_1, W_2, W_3, W_4\}$.

To understand how this circuit is executed, consider the variables $x_1 = 2$, $x_2 = 3$ as well as $x_3 = 4$. Following all edges in the graph, we get the assignments $W_1 = 2$, $W_2 = 3$ and $W_3 = 4$. Then the assignments of W_1 and W_2 enter a multiplication gate and the output of the gate is $2 \cdot 3 = 6$, which we assign to W_4 , i.e. $W_4 = 6$. The values W_4 and W_3 then enter the second multiplication gate and the output of the gate is $6 \cdot 4 = 24$, which we assign to I_1 , i.e. $I_1 = 24$.

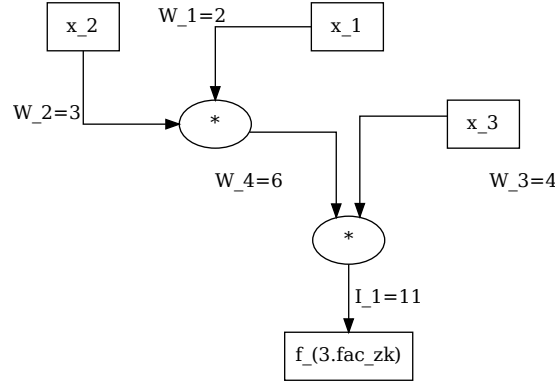
A valid assignment to the 3-factorization circuit $C_{3, fac}(\mathbb{F}_{13})$ is therefore given by the following set:

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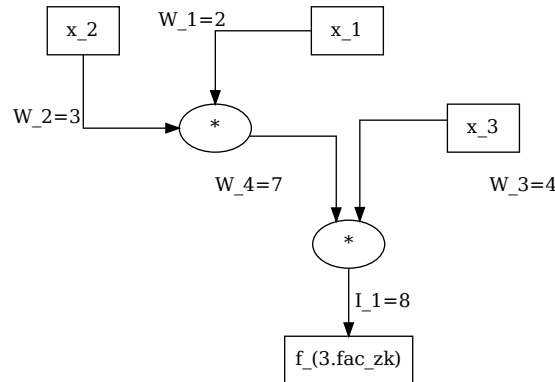
$$S_{\text{valid}} := \{11; 2, 3, 4, 6\} \quad (6.8)$$

4641 We can visualise this assignment in the circuit as follows:



4642

4643 To see what an invalid assignment looks like, consider the assignment $S_{\text{err}} := \{8; 2, 3, 4, 7\}$. In
 4644 this assignment, the input values are the same as in the previous case. The associated circuit is:



4645

4646 This assignment is invalid, as the assignments of I_1 and W_4 cannot be obtained by executing the
 4647 circuit.

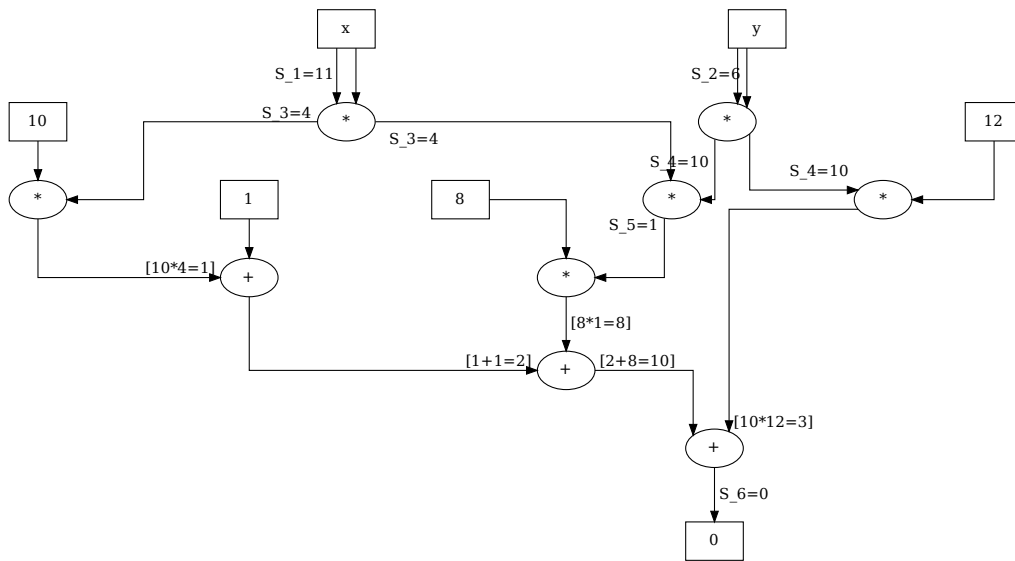
4648 *Example 121.* To compute a more realistic algebraic circuit execution, consider the defining
 4649 circuit $C_{\text{tiny-jj}}(\mathbb{F}_{13})$ from example 117 again. We already know from the way this circuit is
 4650 constructed that any valid assignment with $S_1 = x$, $S_2 = y$ and $S_6 = 0$ will ensure that the pair
 4651 (x, y) is a point on the tiny-jubjub curve in its Edwards representation (equation 5.20).

4652 From example 117, we know that the pair (11, 6) is a proper point on the tiny-jubjub curve
 4653 and we use this point as input to a circuit execution. We get the following:

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4654

Executing the circuit, we indeed compute $S_6 = 0$ as expected, which proves that $(11, 6)$ is a point on the tiny-jubjub curve in its Edwards representation. A valid assignment of $C_{\text{tiny-jj}}(\mathbb{F}_{13})$ is therefore given by the following equation:

$$S_{\text{tiny-jj}} = \{S_1, S_2, S_3, S_4, S_5, S_6\} = \{11, 6, 4, 10, 1, 0\}$$

4655 **Circuit Satisfiability** To understand how algebraic circuits give rise to formal languages, ob-
 4656 serve that every algebraic circuit $C(\mathbb{F})$ over a fields \mathbb{F} defines a decision function over the al-
 4657 phabet $\Sigma_I \times \Sigma_W = \mathbb{F} \times \mathbb{F}$ in the following way:

$$R_{C(\mathbb{F})} : \mathbb{F}^* \times \mathbb{F}^* \rightarrow \{true, false\} ; (I; W) \mapsto \begin{cases} true & (I; W) \text{ is valid assignment to } C(\mathbb{F}) \\ false & \text{else} \end{cases} \quad (6.9)$$

4658 Every algebraic circuit therefore defines a formal language. The grammar of this language is
 4659 encoded in the shape of the circuit, words are assignments to edge labels that are derived from
 4660 circuit execution, and **statements** are knowledge claims “Given instance I , there is a witness
 4661 W such that $(I; W)$ is a valid assignment to the circuit”. A constructive proof to this claim
 4662 is therefore an assignment of a field element to every witness variable, which is verified by
 4663 executing the circuit to see if the assignment of the execution meets the assignment of the
 4664 proof.

4665 In the context of zero-knowledge proof systems, executing circuits is also often called **wit-**
 4666 **ness generation**, since in applications the instance part is usually public, while its the task of a
 4667 prover to compute the witness part.

Remark 7 (Circuit satisfiability). It should be noted that, in our definition, every circuit defines its own language. However, in more theoretical approaches another language usually called **circuit satisfiability** is often considered, which is useful when it comes to more abstract problems like expressiveness, or computational complexity of the class of **all** algebraic circuits over a given field. From our perspective the circuit satisfiability language is obtained by union of all circuit languages that are in our definition. To be more precise, let the alphabet $\Sigma = \mathbb{F}$ be a field. Then

$$L_{\text{CIRCUIT_SAT}(\mathbb{F})} = \{(i; w) \in \Sigma^* \times \Sigma^* \mid \text{there is a circuit } C(\mathbb{F}) \text{ such that } (i; w) \text{ is valid assignment}\}$$

Should we refer to RICS satisfiability (p. 143 here?)

4668 *Example 122 (3-Factorization).* Consider the circuit $C_{3, fac}$ from equation 6.8 again. We call the
 4669 associated language L_{3, fac_circ} .

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reference

4670 To understand how a constructive proof of a statement in L_{3, fac_circ} looks like, consider the
 4671 instance $I_1 = 11$. To provide a proof for the statement “There exist a witness W such that $(I_1; W)$
 4672 is a word in L_{3, fac_circ} ” a proof therefore has to consists of proper values for the variables W_1 ,
 4673 W_2 , W_3 and W_4 . Any proofer therefore has to find input values for W_1 , W_2 and W_3 and then
 4674 execute the circuit to compute W_4 under the assumption $I_1 = 11$.

4675 Example XXX implies that $(2, 3, 4, 6)$ is a proper constructive proof and in order to verify
 4676 the proof a verifier needs to execute the circuit with instance $I_1 = 11$ and inputs $W_1 = 2$, $W_2 = 3$
 4677 and $W_3 = 4$ to decide whether the proof is a valid assignment or not.

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4678 **Associated Constraint Systems** As we have seen in 6.2.1.1, rank-1 constraint systems define
 4679 a way to represent statements in terms of a system of quadratic equations over finite fields,
 4680 suitable for pairing-based zero-knowledge proof systems. However, those equations provide no
 4681 practical way for a proofer to actually compute a solution. On the other hand, algebraic circuits
 4682 can be executed in order to derive valid assignments efficiently.

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4683 In this paragraph, we show how to transform any algebraic circuit into a rank-1 constraint
 4684 system such that valid circuit assignments are in 1:1 correspondence with solutions to the asso-
 4685 ciated R1CS.

4686 To see this, let $C(\mathbb{F})$ be an algebraic circuit over a finite field \mathbb{F} , with a set of edge labels
 4687 $\{S_1, S_2, \dots, S_n\}$. Then one of the following steps is executed for every edge label S_j from that
 4688 set:

4689 *Definition 6.2.2.2.* • If the edge label S_j is an outgoing edge of a multiplication gate, the
 4690 R1CS gets a new quadratic constraint

$$(\text{left input}) \cdot (\text{right input}) = S_j \quad (6.10)$$

4691 where (left input) respectively (right input) is the output from the symbolic execution of
 4692 the subgraph that consists of the left respectively right input edge of this gate, and all
 4693 edges and nodes that have this edge in their path, starting with constant inputs or labeled
 4694 outgoing edges of other nodes.

4695 • If the edge label S_j is an outgoing edge of an addition gate, the R1CS gets a new quadratic
 4696 constraint

$$(\text{left input} + \text{right input}) \cdot 1 = S_j \quad (6.11)$$

4697 where (left input) respectively (right input) is the output from the symbolic execution of
 4698 the subgraph that consists of the left respectively right input edge of this gate and all
 4699 edges and nodes that have this edge in their path, starting with constant inputs or labeled
 4700 outgoing edges of other nodes.

4701 • No other edge label adds a constraint to the system.

4702 The result of this method is a rank-1 constraint system, and, in this sense, every algebraic
 4703 circuit $C(\mathbb{F})$ generates a R1CS R , which we call the **associated R1CS** of the circuit. It can be
 4704 shown that a tuple of field elements (S_1, S_2, \dots, S_n) is a valid assignment to a circuit if and only
 4705 if the same tuple is a solution to the associated R1CS. Circuit executions therefore compute
 4706 solutions to rank-1 constraints systems efficiently.

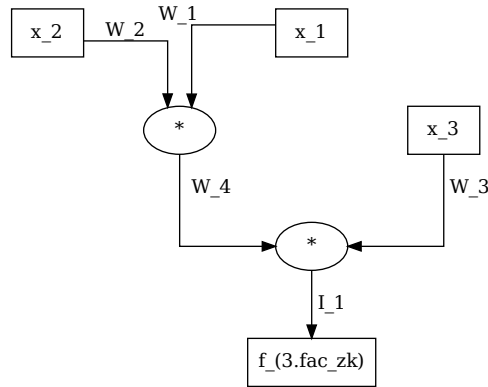
4707 To understand the contribution of algebraic gates to the number of constraints, note that, by
 4708 definition, multiplication gates have labels on their outgoing edges if and only if there is at least

one labeled edge in both input paths, or if the outgoing edge is an input to a sink node. This implies that multiplication with a constant is essentially free in the sense that it doesn't add a new constraint to the system, as long as that multiplication gate is not an input to an output node.

Moreover, addition gates have labels on their outgoing edges if and only if they are inputs to sink nodes. This implies that addition is essentially free in the sense that it doesn't add a new constraint to the system, as long as that addition gate is not an input to an output node.

Example 123 (3-factorization). Consider our 3-factorization problem from equation 6.8 and the associated circuit $C_{3.fac}(\mathbb{F}_{13})$. Our task is to transform this circuit into an equivalent rank-1 constraint system.

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We start with an empty R1CS, and, in order to generate all constraints, we have to iterate over the set of edge labels $\{I_1; W_1, W_2, W_3, W_4\}$.

Starting with the edge label I_1 , we see that it is an outgoing edge of a multiplication gate, and, since both input edges are labeled, we have to add the following constraint to the system:

$$\begin{aligned} (\text{left input}) \cdot (\text{right input}) &= I_1 \\ W_4 \cdot W_3 &= I_1 \end{aligned} \quad \Leftrightarrow$$

Next, we consider the edge label W_1 and, since, it's not an outgoing edge of a multiplication or addition label, we don't add a constraint to the system. The same holds true for the labels W_2 and W_3 .

For edge label W_4 , we see that it is an outgoing edge of a multiplication gate, and, since both input edges are labeled, we have to add the following constraint to the system:

$$\begin{aligned} (\text{left input}) \cdot (\text{right input}) &= W_4 \\ W_2 \cdot W_1 &= W_4 \end{aligned} \quad \Leftrightarrow$$

Since there are no more labeled edges, all constraints are generated, and we have to combine them to get the associated R1CS of $C_{3.fac}(\mathbb{F}_{13})$:

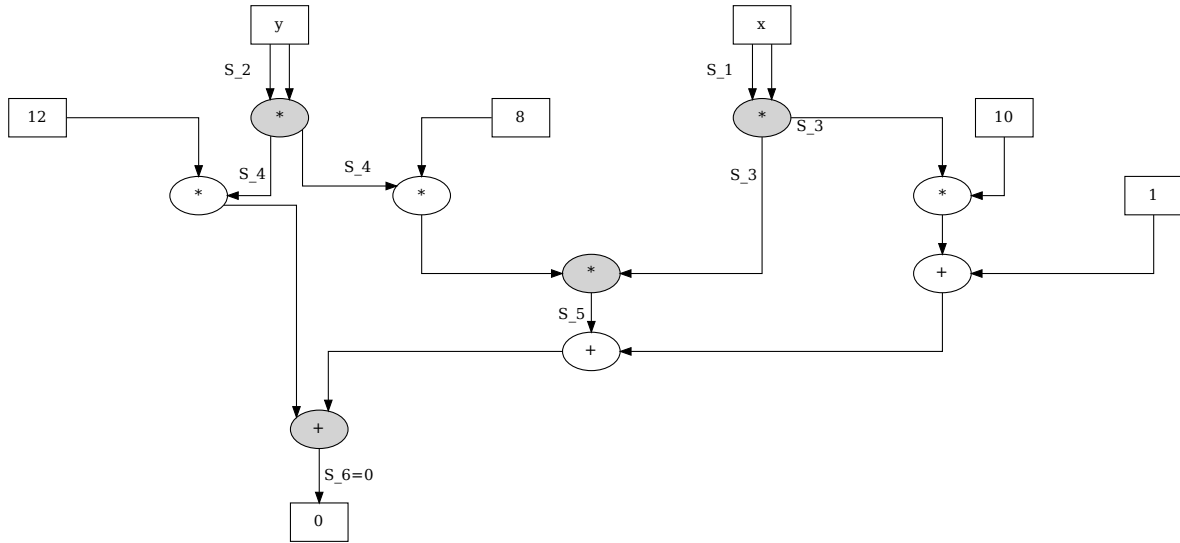
$$\begin{aligned} W_4 \cdot W_3 &= I_1 \\ W_2 \cdot W_1 &= W_4 \end{aligned}$$

This system is equivalent to the R1CS we derived in example 114. The languages $L_{3.fac_zk}$ and $L_{3.fac_circ}$ are therefore equivalent and both the circuit as well as the R1CS are just two different ways of expressing the same language.

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Example 124. To consider a more general transformation, we consider the tiny-jubjub circuit from example 117 again. A proper circuit is given by

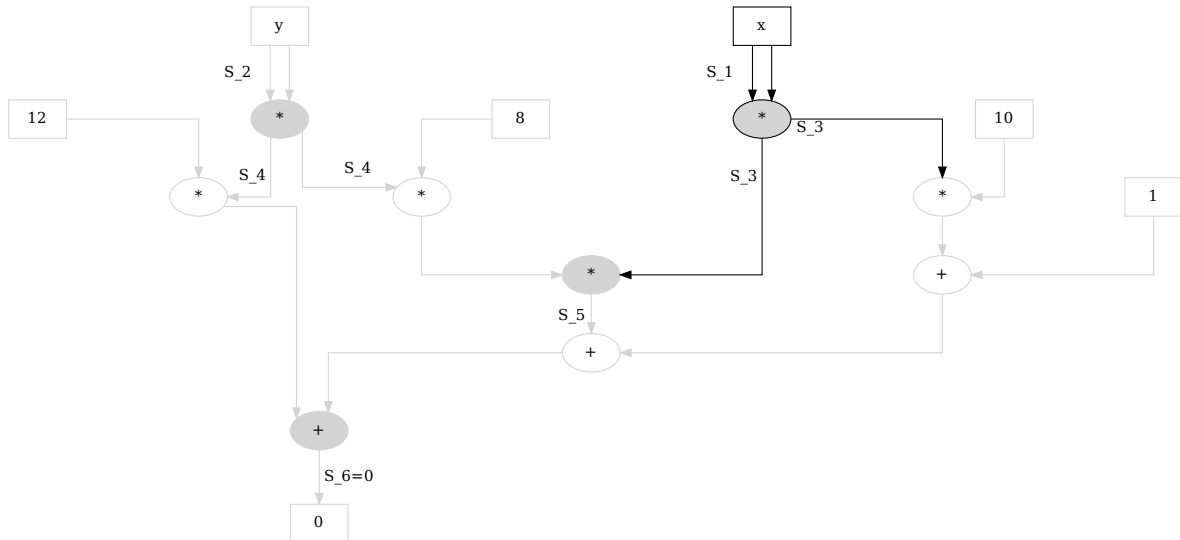
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To compute the number of constraints, observe that we have 3 multiplication gates that have labels on their outgoing edges and 1 addition gate that has a label on its outgoing edge. We therefore have to compute 4 quadratic constraints.

In order to derive the associated R1CS, we have start with an empty R1CS and then iterate over the set $\{S_1, S_2, S_3, S_4, S_5, S_6 = 0\}$ of all edge labels, in order to generate the constraints.

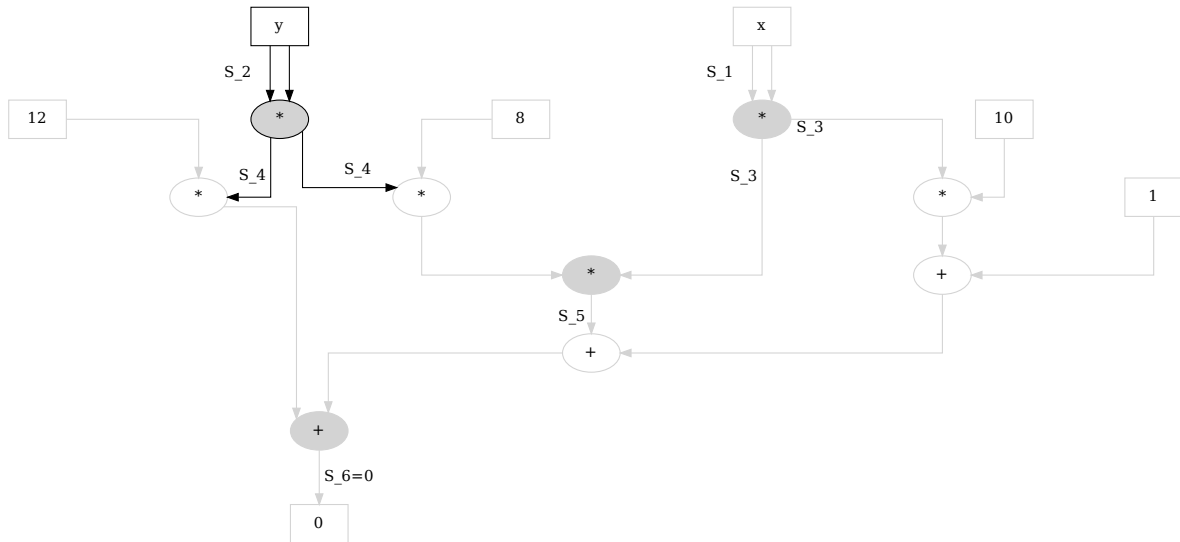
Considering edge label S_1 , we see that the associated edges are not outgoing edges of any algebraic gate, and we therefore have to add no new constraint to the system. The same holds true for edge label S_2 . Looking at edge label S_3 , we see that the associated edges are outgoing edges of a multiplication gate and that the associated subgraph is given by:



Both the left and the right input to this multiplication gate are labeled by S_1 . We therefore have to add the following constraint to the system:

$$S_1 \cdot S_1 = S_3$$

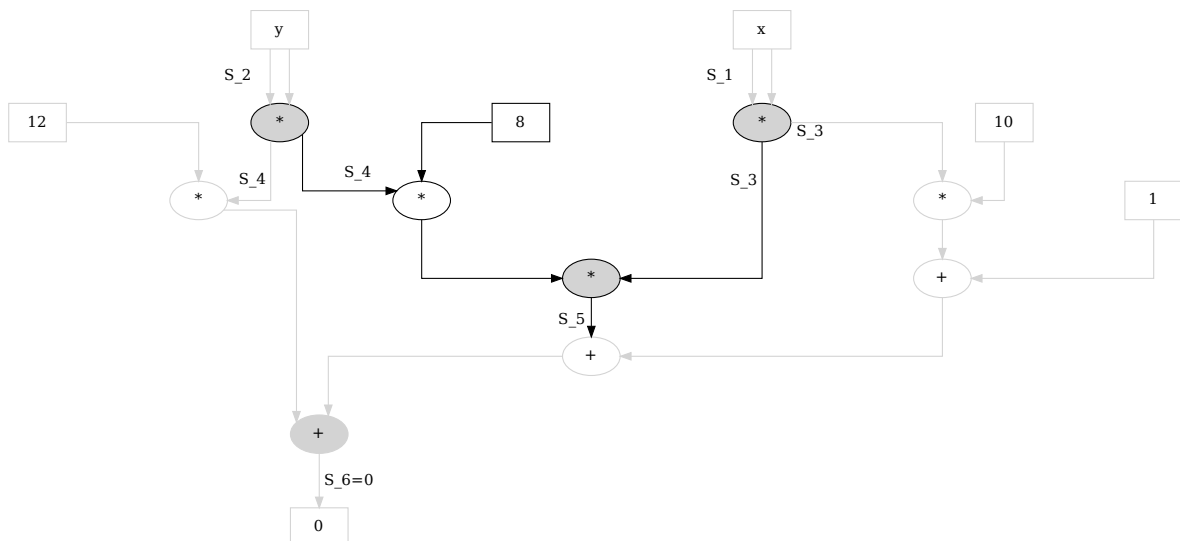
Looking at edge label S_4 , we see that the associated edges are outgoing edges of a multiplication gate and that the associated subgraph is given by:



Both the left and the right input to this multiplication gate are labeled by S_2 and we therefore have to add the following constraint to the system:

$$S_2 \cdot S_2 = S_4$$

Edge label S_5 is more interesting. To see if it implies a constraint, we have to construct the associated subgraph first, which consists of all edges and all nodes in all path starting either at a constant input or a labeled edge. We get



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The right input to the associated multiplication gate is given by the labeled edge S_3 . However, the left input is not a labeled edge, but has a labeled edge in one of its path. This implies that we have to add a constraint to the system. To compute the left factor of that constraint, we have to compute the output of subgraph associated to the left edge, which is $8 \cdot W_2$. This gives the constraint

$$(S_4 \cdot 8) \cdot S_3 = S_5$$

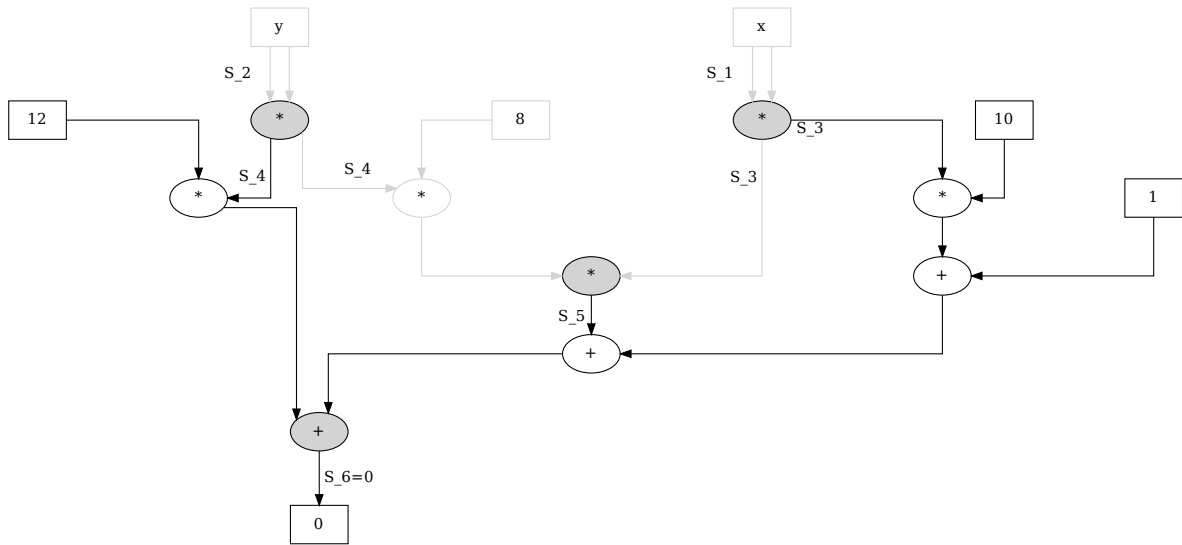
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The last edge label is the constant $S_6 = 0$. To see if it implies a constraint, we have to construct the associated subgraph, which consists of all edges and all nodes in all path starting either at a constant input or a labeled edge. We get

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Both the left and the right input are unlabeled, but have a labeled edges in their path. This implies that we have to add a constraint to the system. Since the gate is an addition gate, the right factor in the quadratic constraint is always 1 and the left factor is computed by symbolically executing all inputs to all gates in sub-circuit. We get

$$(12 \cdot S_4 + S_5 + 10 \cdot S_3 + 1) \cdot 1 = 0$$

Since there are no more labeled outgoing edges, we are done deriving the constraints. Combining all constraints together, we get the following R1CS:

$$S_1 \cdot S_1 = S_3$$

$$S_2 \cdot S_2 = S_4$$

$$(S_4 \cdot 8) \cdot S_3 = S_5$$

$$(12 \cdot S_4 + S_5 + 10 \cdot S_3 + 1) \cdot 1 = 0$$

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which is equivalent to the R1CS we derived in example 117. The languages L_{3, fac_zk} and L_{3, fac_circ} are therefore equivalent and both the circuit as well as the R1CS are just two different ways to express the same language.

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6.2.3 Quadratic Arithmetic Programs

We have introduced algebraic circuits and their associated rank-1 constraints systems as two particular models able to represent space- and time-bounded computation. Both models define formal languages, and associated membership as well as knowledge claims can be constructively proved by executing the circuit in order to compute solutions to its associated R1CS.

One reason why those systems are useful in the context of succinct zero-knowledge proof systems is because any R1CS can be transformed into another computational model called **quadratic arithmetic programs** (QAP), which serve as the basis for some of the most efficient succinct non-interactive zero-knowledge proof generators that currently exist.

As we will see, proving statements for languages that have checking relations defined by quadratic arithmetic programs can be achieved by providing certain polynomials, and those proofs can be verified by checking a particular divisibility property.

QAP representation To understand what quadratic arithmetic programs are in detail, let \mathbb{F} be a field and R a rank-1 constraints system over \mathbb{F} such that the number of non-zero elements in \mathbb{F} is strictly larger than the number k of constraints in R . Moreover, let a_j^i, b_j^i and $c_j^i \in \mathbb{F}$ for every index $0 \leq j \leq n + m$ and $1 \leq i \leq k$, be the defining constants of the R1CS and m_1, \dots, m_k be arbitrary, invertible and distinct elements from \mathbb{F} .

Then a **quadratic arithmetic program** [QAP] of the R1CS is the following set of polynomials over \mathbb{F} :

$$QAP(R) = \left\{ T \in \mathbb{F}[x], \{A_j, B_j, C_j \in \mathbb{F}[x]\}_{h=0}^{n+m} \right\} \quad (6.12)$$

In the equation above, $T(x) := \prod_{l=1}^k (x - m_l)$ is a polynomial of degree k , called the **target polynomial** of the QAP and A_j, B_j as well as C_j are the unique degree $k - 1$ polynomials defined by the following equation:

$$A_j(m_i) = a_j^i \quad B_j(m_i) = b_j^i \quad C_j(m_i) = c_j^i \quad j = 1, \dots, n + m + 1, i = 1, \dots, k \quad (6.13)$$

Given some rank-1 constraint system, an associated quadratic arithmetic program is therefore nothing but a set of polynomials, computed from the constants in the R1CS. To see that the polynomials A_j, B_j and C_j are uniquely defined by the equations in XXX, recall that a polynomial of degree $k - 1$ is completely determined on k evaluation points and the equation 4 precisely determines those k evaluation points.

Since we only consider polynomials over fields, Lagrange's interpolation method from 3.31 in chapter ?? can be used to derive the polynomials A_j, B_j and C_j from their defining equations XXX. A practical method to compute a QAP from a given R1CS therefore consists of two steps. If the R1CS consists of k constraints, first choose k invertible and mutually different points from the underlying field. Every choice defines a different QAP for the same R1CS. Then use Lagrange's method and equation XXX to compute the polynomials A_j, B_j and C_j for every $1 \leq j \leq k$.

Example 125 (Generalized factorization SNARK). To provide a better intuition of quadratic arithmetic programs and how they are computed from their associated rank-1 constraint systems, consider the language L_{3, fac_zk} from example 109 and its associated R1CS:

$$\begin{array}{ll} W_1 \cdot W_2 = W_4 & \text{constraint 1} \\ W_4 \cdot W_3 = I_1 & \text{constraint 2} \end{array}$$

In this example we want to transform this R1CS into an associated QAP. In a first step, we have to compute the defining constants a_j^i, b_j^i and c_j^i of the R1CS. According to XXX, we have

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$$\begin{array}{cccccc} a_0^1 = 0 & a_1^1 = 0 & a_2^1 = 1 & a_3^1 = 0 & a_4^1 = 0 & a_5^1 = 0 \\ a_0^2 = 0 & a_1^2 = 0 & a_2^2 = 0 & a_3^2 = 0 & a_4^2 = 0 & a_5^2 = 1 \end{array}$$

$$\begin{array}{cccccc} b_0^1 = 0 & b_1^1 = 0 & b_2^1 = 0 & b_3^1 = 1 & b_4^1 = 0 & b_5^1 = 0 \\ b_0^2 = 0 & b_1^2 = 0 & b_2^2 = 0 & b_3^2 = 0 & b_4^2 = 1 & b_5^2 = 0 \end{array}$$

$$\begin{array}{cccccc} c_0^1 = 0 & c_1^1 = 0 & c_2^1 = 0 & c_3^1 = 0 & c_4^1 = 0 & c_5^1 = 1 \\ c_0^2 = 0 & c_1^2 = 1 & c_2^2 = 0 & c_3^2 = 0 & c_4^2 = 0 & c_5^2 = 0 \end{array}$$

Since the R1CS is defined over the field \mathbb{F}_{13} and has two constraining equations, we need to choose two arbitrary but distinct elements m_1 and m_2 from \mathbb{F}_{13} . We choose $m_1 = 5$, and $m_2 = 7$ and with this choice we get the target polynomial

$$\begin{aligned} T(x) &= (x - m_1)(x - m_2) && \# \text{ Definition of } T \\ &= (x - 5)(x - 7) && \# \text{ Insert our choice} \\ &= (x + 8)(x + 6) && \# \text{ Negatives in } \mathbb{F}_{13} \\ &= x^2 + x + 9 && \# \text{ expand} \end{aligned}$$

4799 Then we have to compute the polynomials A_j , B_j and C_j by their defining equation from the
4800 R1CS coefficients. Since the R1CS has two constraining equations, those polynomials are of
4801 degree 1 and they are defined by their evaluation at the point $m_1 = 5$ and the point $m_2 = 7$.

At point m_1 , each polynomial A_j is defined to be a_j^1 and at point m_2 , each polynomial A_j is defined to be a_j^2 . The same holds true for the polynomials B_j as well as C_j . Writing all these equations now, we get:

$$\begin{array}{l} A_0(5) = 0, \quad A_1(5) = 0, \quad A_2(5) = 1, \quad A_3(5) = 0, \quad A_4(5) = 0, \quad A_5(5) = 0 \\ A_0(7) = 0, \quad A_1(7) = 0, \quad A_2(7) = 0, \quad A_3(7) = 0, \quad A_4(7) = 0, \quad A_5(7) = 1 \end{array}$$

$$\begin{array}{l} B_0(5) = 0, \quad B_1(5) = 0, \quad B_2(5) = 0, \quad B_3(5) = 1, \quad B_4(5) = 0, \quad B_5(5) = 0 \\ B_0(7) = 0, \quad B_1(7) = 0, \quad B_2(7) = 0, \quad B_3(7) = 0, \quad B_4(7) = 1, \quad B_5(7) = 0 \end{array}$$

$$\begin{array}{l} C_0(5) = 0, \quad C_1(5) = 0, \quad C_2(5) = 0, \quad C_3(5) = 0, \quad C_4(5) = 0, \quad C_5(5) = 1 \\ C_0(7) = 0, \quad C_1(7) = 1, \quad C_2(7) = 0, \quad C_3(7) = 0, \quad C_4(7) = 0, \quad C_5(7) = 0 \end{array}$$

4802 Lagrange's interpolation implies that a polynomial of degree k , that is, that zero on $k + 1$ points
4803 has to be the zero polynomial. Since our polynomials are of degree 1 and determined on 2
4804 points, we therefore know that the only non-zero polynomials in our QAP are A_2 , A_5 , B_3 , B_4 ,
4805 C_1 and C_5 , and that we can use Lagrange's interpolation to compute them.

To compute A_2 we note that the set S in our version of Lagrange's method is given by $S = \{(x_0, y_0), (x_1, y_1)\} = \{(5, 1), (7, 0)\}$. Using this set we get:

$$\begin{aligned} A_2(x) &= y_0 \cdot l_0 + y_1 \cdot l_1 \\ &= y_0 \cdot \left(\frac{x - x_1}{x_0 - x_1} \right) + y_1 \cdot \left(\frac{x - x_0}{x_1 - x_0} \right) = 1 \cdot \left(\frac{x - 7}{5 - 7} \right) + 0 \cdot \left(\frac{x - 5}{7 - 5} \right) \\ &= \frac{x - 7}{-2} = \frac{x - 7}{11} && \# 11^{-1} = 6 \\ &= 6(x - 7) = 6x + 10 && \# -7 = 6 \text{ and } 6 \cdot 6 = 10 \end{aligned}$$

To compute A_5 , we note that the set S in our version of Lagrange's method is given by $S = \{(x_0, y_0), (x_1, y_1)\} = \{(5, 0), (7, 1)\}$. Using this set we get:

$$\begin{aligned}
 A_5(x) &= y_0 \cdot l_0 + y_1 \cdot l_1 \\
 &= y_0 \cdot \left(\frac{x - x_1}{x_0 - x_1}\right) + y_1 \cdot \left(\frac{x - x_0}{x_1 - x_0}\right) = 0 \cdot \left(\frac{x - 7}{5 - 7}\right) + 1 \cdot \left(\frac{x - 5}{7 - 5}\right) \\
 &= \frac{x - 5}{2} \quad \# 2^{-1} = 7 \\
 &= 7(x - 5) = 7x + 4 \quad \# -5 = 8 \text{ and } 7 \cdot 8 = 4
 \end{aligned}$$

4806 Using Lagrange's interpolation, we can deduce that $A_2 = B_3 = C_5$ as well as $A_5 = B_4 = C_1$,
 4807 since they are polynomials of degree 1 that evaluate to same values on 2 points. Using this, we
 4808 get the following set of polynomials

$A_0(x) = 0$	$B_0(x) = 0$	$C_0(x) = 0$
$A_1(x) = 0$	$B_1(x) = 0$	$C_1(x) = 7x + 4$
$A_2(x) = 6x + 10$	$B_2(x) = 0$	$C_2(x) = 0$
$A_3(x) = 0$	$B_3(x) = 6x + 10$	$C_3(x) = 0$
$A_4(x) = 0$	$B_4(x) = 7x + 4$	$C_4(x) = 0$
$A_5(x) = 7x + 4$	$B_5(x) = 0$	$C_5(x) = 6x + 10$

4810 We can invoke Sage to verify our computation. In sage every polynomial ring has a function
 4811 `lagrange_polynomial` that takes the defining points as inputs and the associated Lagrange
 4812 polynomial as output.

```

4813 sage: F13 = GF(13)
4814 sage: F13t.<t> = F13[]
4815 sage: T = F13t((t-5)*(t-7))
4816 sage: A2 = F13t.lagrange_polynomial([(5, 1), (7, 0)])
4817 sage: A5 = F13t.lagrange_polynomial([(5, 0), (7, 1)])
4818 sage: T == F13t(t^2 + t + 9)
4819 True
4820 sage: A2 == F13t(6*t + 10)
4821 True
4822 sage: A5 == F13t(7*t + 4)
4823 True

```

4824 Combining this computation with the target polynomial we derived earlier, a quadratic arith-
 4825 metic program associated to the rank-1 constraint system $R_{3.fac_zk}$ is given as follows:

$$\begin{aligned}
 QAP(R_{3.fac_zk}) &= \{x^2 + x + 9, \\
 \{0, 0, 6x + 10, 0, 0, 7x + 4\}, \{0, 0, 0, 6x + 10, 7x + 4, 0\}, \{0, 7x + 4, 0, 0, 0, 6x + 10\}\}
 \end{aligned} \quad (6.14)$$

4826 **QAP Satisfiability** One of the major points of quadratic arithmetic programs in proofing sys-
 4827 tems is that solutions of their associated rank-1 constraints systems are in 1:1 correspondence
 4828 with certain polynomials P such that P is divisible by the target polynomial T of the QAP if and
 4829 only if the solution is a solution. Verifying solutions to the R1CS and hence, checking proper
 4830 circuit execution is then achievable by polynomial division of P by T .

4831 To be more specific, let R be some rank-1 constraints system with associated assignment
 4832 variables $(I_1, \dots, I_n; W_1, \dots, W_m)$ and let $QAP(R)$ be a quadratic arithmetic program of R . Then

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the tuple $(I_1, \dots, I_n; W_1, \dots, W_m)$ is a solution to the R1CS if and only if the following polynomial is divisible by the target polynomial T :

$$P_{(I;W)} = (A_0 + \sum_j^n I_j \cdot A_j + \sum_j^m W_j \cdot A_{n+j}) \cdot (B_0 + \sum_j^n I_j \cdot B_j + \sum_j^m W_j \cdot B_{n+j}) - (C_0 + \sum_j^n I_j \cdot C_j + \sum_j^m W_j \cdot C_{n+j}) \quad (6.15)$$

Every tuple $(I; W)$ defines a polynomial $P_{(I;W)}$, and, since each polynomial A_j , B_j and C_j is of degree $k-1$, $P_{(I;W)}$ is of degree $(k-1) \cdot (k-1) = k^2 - 2k + 1$.

To understand how quadratic arithmetic programs define formal languages, observe that every QAP over a field \mathbb{F} defines a decision function over the alphabet $\Sigma_I \times \Sigma_W = \mathbb{F} \times \mathbb{F}$ in the following way:

$$R_{QAP} : \mathbb{F}^* \times \mathbb{F}^* \rightarrow \{true, false\} ; (I; W) \mapsto \begin{cases} true & P_{(I;W)} \text{ is divisible by } T \\ false & \text{else} \end{cases} \quad (6.16)$$

This means that every QAP defines a formal language, and, if the QAP is associated to an R1CS, it can be shown that the two languages are equivalent. A **statement** is a membership claim “There is a word $(I; W)$ in L_{QAP} ”. A proof to this claim is therefore a polynomial $P_{(I;W)}$, which is verified by dividing $P_{(I;W)}$ by T .

Note the structural similarity to the definition of an R1CS in 6.2.1.1 and the different ways of computing proofs in both systems. For circuits and their associated rank-1 constraints systems, a constructive proof consists of a valid assignment of field elements to the edges of the circuit, or the variables in the R1CS. However, in the case of QAPs, a valid proof consists of a polynomial $P_{(I;W)}$.

To compute a proof for a statement in L_{QAP} given some instance I , a proofer first needs to compute a constructive proof W , e.g. by executing the circuit. With $(I; W)$ at hand, the proofer can then compute the polynomial $P_{(I;W)}$ and publish it as proof.

Verifying a constructive proof in the case of a circuit is achieved by executing the circuit, comparing the result to the given proof, and verifying the same proof in the R1CS picture means checking if the elements of the proof satisfy all equation.

In contrast, verifying a proof in the case of a QAP is done by polynomial division of the proof P by the target polynomial T of the QAP. The proof checks out if and only if P is divisible by T .

Example 126. Consider the quadratic arithmetic program $QAP(R_{3, fac_zk})$ from example XXX, and its associated R1CS from equation 6.1.0.1. To give an intuition of how proofs in the language $L_{QAP(R_{3, fac_zk})}$ let's consider the instance $I_1 = 11$. As we know from example XXX, $(W_1, W_2, W_3, W_5) = (2, 3, 4, 6)$ is a proper witness, since $(I_1; W_1, W_2, W_3, W_5) = (11; 2, 3, 4, 6)$ is a valid circuit assignment and hence, a solution to R_{3, fac_zk} and a constructive proof for language $L_{R_{3, fac_zk}}$.

In order to transform this constructive proof into a membership proof in language $L_{QAP(R_{3, fac_zk})}$ a proofer has to use the elements of the constructive proof, to compute the polynomial $P_{(I;W)}$.

In the case of $(I_1; W_1, W_2, W_3, W_5) = (11; 2, 3, 4, 6)$, the associated proof is computed as fol-

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lows:

$$\begin{aligned}
P_{(I;W)} &= (A_0 + \sum_j^n I_j \cdot A_j + \sum_j^m W_j \cdot A_{n+j}) \cdot (B_0 + \sum_j^n I_j \cdot B_j + \sum_j^m W_j \cdot B_{n+j}) - (C_0 + \sum_j^n I_j \cdot C_j + \sum_j^m W_j \cdot C_{n+j}) \\
&= (2(6x + 10) + 6(7x + 4)) \cdot (3(6x + 10) + 4(7x + 4)) - (11(7x + 4) + 6(6x + 10)) \\
&= ((12x + 7) + (3x + 11)) \cdot ((5x + 4) + (2x + 3)) - ((12x + 5) + (10x + 8)) \\
&= (2x + 5) \cdot (7x + 7) - (9x) \\
&= (x^2 + 2 \cdot 7x + 5 \cdot 7x + 5 \cdot 7) - (9x) \\
&= (x^2 + x + 9x + 9) - (9x) \\
&= x^2 + x + 9
\end{aligned}$$

4866 Given instance $I_1 = 11$ a proofer therefore provides the polynomial $x^2 + x + 9$ as proof. To verify
4867 this proof, any verifier can then look up the target polynomial T from the QAP and divide $P_{(I;W)}$
4868 by T . In this particular example, $P_{(I;W)}$ is equal to the target polynomial T , and hence, it is
4869 divisible by T with $P/T = 1$. The verification therefore checks the proof.

```

4870 sage: F13 = GF(13) 656
4871 sage: F13t.<t> = F13[] 657
4872 sage: T = F13t(t^2 + t + 9) 658
4873 sage: P = F13t((2*(6*t+10)+6*(7*t+4))*(3*(6*t+10)+4*(7*t+4)) 659
4874             -(11*(7*t+4)+6*(6*t+10)))
4875 sage: P == T 660
4876 True 661
4877 sage: P % T # remainder 662
4878 0 663

```

To give an example of a false proof, consider the tuple $(I_1; W_1, W_2, W_3, W_4) = (11, 2, 3, 4, 8)$. Executing the circuit, we can see that this is not a valid assignment and not a solution to the R1CS, and hence, not a constructive knowledge proof in $L_{3.fac_zk}$. However, a proofer might use these values to construct a false proof $P_{(I;W)}$:

$$\begin{aligned}
P'_{(I;W)} &= (A_0 + \sum_j^n I_j \cdot A_j + \sum_j^m W_j \cdot A_{n+j}) \cdot (B_0 + \sum_j^n I_j \cdot B_j + \sum_j^m W_j \cdot B_{n+j}) - (C_0 + \sum_j^n I_j \cdot C_j + \sum_j^m W_j \cdot C_{n+j}) \\
&= (2(6x + 10) + 8(7x + 4)) \cdot (3(6x + 10) + 4(7x + 4)) - (8(6x + 10) + 11(7x + 4)) \\
&= 8x^2 + 6
\end{aligned}$$

Given instance $I_1 = 11$, a proofer therefore provides the polynomial $8x^2 + 6$ as proof. To verify this proof, any verifier can look up the target polynomial T from the QAP and divide $P_{(I;W)}$ by T . However, polynomial division has the following remainder:

$$(8x^2 + 6)/(x^2 + x + 9) = 8 + \frac{5x + 12}{x^2 + x + 9}$$

4879 This implies that $P_{(I;W)}$ is not divisible by T , and hence, the verification does not check the
4880 proof. Any verifier can therefore show that the proof is false.

```

4881 sage: F13 = GF(13) 664
4882 sage: F13t.<t> = F13[] 665
4883 sage: T = F13t(t^2 + t + 9) 666
4884 sage: P = F13t((2*(6*t+10)+8*(7*t+4))*(3*(6*t+10)+4*(7*t+4))-( 667
4885             8*(6*t+10)+11*(7*t+4)))

```

4886	sage: $P == F13t(8*t^2 + 6)$	668
4887	True	669
4888	sage: $P \% T$ # remainder	670
4889	$5*t + 12$	671

Chapter 7

Circuit Compilers

As we have seen in the previous chapter, statements can be formalized as membership or knowledge claims in formal language, and algebraic circuits as well as rank-1 constraint systems are two practically important ways to define those languages.

However, both algebraic circuits and rank-1 constraint systems are not ideal from a developers point of view, because they deviate substantially from common programming paradigms. Writing real-world applications as circuits and the associated verification in terms of rank-1 constraint systems is at least as troublesome as writing any other low-level language like assembler code. To allow for complex statement design, it is therefore necessary to have some kind of compiler framework, capable of transforming high-level languages into arithmetic circuits and associated rank-1 constraint systems.

As we have seen in chapter 6 and in 6.2.1.1., both arithmetic circuits and rank-1 constraint systems have a modularity property by which it is possible to synthesize complex circuits from simple ones. A basic approach taken by many circuit/R1CS compilers is therefore to provide a library of atomic and simple circuits and then define a way to combine those basic building blocks into arbitrary complex systems.

In this chapter, we provide an introduction to basic concepts of so-called **circuit compilers** and derive a toy language which we can “compile” in a pen-and-paper approach into algebraic circuits and their associated rank-1 constraint systems.

We start with a general introduction to our language, and then introduce atomic types like booleans and unsigned integers. Then we define the fundamental control flow primitives like the if-then-else conditional and the bounded loop. We will look at basic functionality primitives like elliptic curve cryptography. Primitives like these are often called **gadgets** in the literature.

7.1 A Pen-and-Paper Language

To explain basic concepts of circuit compilers and their associated high-level languages, we derive an informal toy language and associated “brain-compiler” which we name PAPER (**Pen-And-Paper Execution Rules**). PAPER allows programmers to define statements in Rust-like pseudo-code. The language is inspired by ZOKRATES and `circom`.

7.1.1 The Grammar

In PAPER, any statement is defined as an ordered list of functions, where any function has to be declared in the list before it is called in another function of that list. The last entry in a statement has to be a special function, called `main`. Functions take a list of typed parameters as inputs

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and compute a tuple of typed variables as output, where types are special functions that define how to transform that type into another type, ultimately transforming any type into elements of the base field where the circuit is defined over.

Any statement is parameterized over the field that the circuit will be defined on, and has additional optional parameters of unsigned type, needed to define the size of array or the counter of bounded loops. The following definition makes the grammar of a statement precise using a command line language like description:

```
statement <Name> {F:<Field> [ , <N_1: unsigned>, ... ] } {
  [fn <Name>([[pub]<Arg>:<Type>, ...]) -> (<Type>, ...)] {
    [let [pub] <Var>:<Type> ; ... ]
    [let const <Const>:<Type>=<Value> ; ... ]
    Var<==>(fn ([<Arg>|<Const>|<Var>, ...]) | (<Arg>|<Const>|<Var>)) ;
    return (<Var>, ...) ;
  } ; ...]
  fn main([[pub]<Arg>:<Type>, ...]) -> (<Type>, ...) {
    [let [pub] <Var>:<Type> ; ... ]
    [let const <Const>:<Type>=<Value> ; ... ]
    Var<==>(fn ([<Arg>|<Const>|<Var>, ...]) | (<Arg>|<Const>|<Var>)) ;
    return (<Var>, ...) ;
  } ;
}
```

Function arguments and variables are private by default, but can be declared as public by the `pub` specifier. Declaring arguments and variables as public always overwrites any previous or conflicting private declarations. Every argument, constant or variable has a type, and every type is defined as a function that transforms that type into another type:

```
type <TYPE>( t1 : <TYPE_1>) -> TYPE_2{
  let t2: TYPE_2 <== fn(TYPE_1)
  return t2
}
```

Many real-world circuit languages are based on a similar, but of course more sophisticated approach than PAPER. The purpose of PAPER is to show basic principles of circuit compilers and their associated high-level languages.

Example 127. To get a better understanding of the grammar of PAPER, the following constitutes proper high-level code that follows the grammar of the PAPER language, assuming that all types in that code have been defined elsewhere.

```
statement MOCK_CODE {F: F_43, N_1 = 1024, N_2 = 8} {
  fn foo(in_1 : F, pub in_2 : TYPE_2) -> F {
    let const c_1 : F = 0 ;
    let const c_2 : TYPE_2 = SOME_VALUE ;
    let pub out_1 : F ;
    out_1<== c_1 ;
    return out_1 ;
  } ;

  fn bar(pub in_1 : F) -> F {
    let out_1 : F ;
    out_1<==foo(in_1);
    return out_1 ;
  } ;
}
```



```

4971 } ;
4972
4973 fn main(in_1 : TYPE_1) -> (F, TYPE_2) {
4974   let const c_1 : TYPE_1 = SOME_VALUE ;
4975   let const c_2 : F = 2 ;
4976   let const c_3 : TYPE_2 = SOME_VALUE ;
4977   let pub out_1 : F ;
4978   let out_2 : TYPE_2 ;
4979   c_1 <== in_1 ;
4980   out_1 <== foo(c_2) ;
4981   out_2 <== TYPE_2 ;
4982   return (out_1, out_2) ;
4983 } ;
4984 }

```

7.1.2 The Execution Phases

In contrast to normal executable programs, programs for circuit compilers have two modes of execution. The first mode, usually called the **setup phase**, is executed in order to generate the circuit and its associated rank-1 constraint system, the latter of which is then usually used as input to some zero-knowledge proof system.

The second mode of execution is usually called the **prover phase**. In this phase, a prover usually computes a valid assignment to the circuit. Depending on the use case, this valid assignment is then either directly used as constructive proof for proper circuit execution or is transferred as input to the proof generation algorithm of some zero-knowledge proof system, where the full-sized, non hiding constructive proof is processed into a succinct proof with various levels of zero knowledge.

Modern circuit languages and their associated compilers abstract over those two phases and provide a unified **interphase** to the developer, who then writes a single program that can be used in both phases.

To give the reader a clear, conceptual distinction between the two phases, PAPER keeps them separated. Code can be “brain-compiled” during the **setup-phase** in a pen-and-paper approach into visual circuits. Once a circuit is derived, it can be executed in a **prover phase** to generate a valid assignment. The valid assignment is then interpreted as a constructive proof for a knowledge claim in the associated language.

The Setup Phase In PAPER, the task of the setup phase is to compile code in the PAPER language into a visual representation of an algebraic circuit. Deriving the circuit from the code in a pen-and-paper style is what we call **brain-compiling**.

Given some statement description that adheres to the correct grammar, we start circuit development with an empty circuit, compile the main function first and then inductively compile all other functions as they are called during the process.

For every function we compile, we draw a box-node for every argument, every variable and every constant of that function. If the node represents a variable, we label it with that variable’s name, and if it represents a constant, we label it with that constant’s value. We group arguments into a subgraph labeled “inputs” and return values into a subgraph labeled “outputs”. We then group everything into a subgraph and label that subgraph with the function’s name.

After this is done, we have to do a consistency and type check for every occurrence of the

assignment operator `<==`. We have to ensure that the expression on the right side of the operator is well defined and that the types of both side match.

Then we compile the right side of every occurrence of the assignment operator `<==`. If the right side is a constant or variable defined in this function, we draw a dotted line from the box-node that represents the left side of `<==` to the box node that represents the right side of the same operator. If the right side represents an argument of that function we draw a line from the box-node that represents the left side of `<==` to the box node that represents the right side of the same operator.

If the right side of the `<==` operator is a function, we look into our database, find its associated circuit and draw it. If no circuit is associated to that function yet, we repeat the compilation process for that function, drawing edges from the function's argument to its input nodes and from the functions output nodes to the nodes on the right side of `<==`.

During that process, edge labels are drawn according to the rules from 6.2.2.1. If the associated variable represents a private value, we use the *W* label to indicate a witness, and if it represents a public value, we use the *I* label to indicate an instance.

check
reference

Once this is done, we compile all occurring types in a function, by compiling the function of each type. We do this inductively until we reach the type of the base field. Circuits have no notion of types, only of field elements; hence, every type needs to be compiled to the field type in a sequence of compilation steps.

The compilation stops once we have inductively replaced all functions by their circuits. The result is a circuit that contains many unnecessary box nodes. In a final optimization step, all box nodes that are directly linked to each other are collapsed into a single node, and all box nodes that represent the same constants are collapsed into a single node.

Of course, PAPER's brain-compiler is not properly defined in any formal manner. Its purpose is to highlight important steps that real-world compilers undergo in their setup phases.

Example 128 (A trivial Circuit). To give an intuition of how to write and compile circuits in the PAPER language, consider the following statement description:

```
statement trivial_circuit {F:F_13} {
  fn main{F}(in1 : F, pub in2 : F) -> (F,F) {
    let const outc1 : F = 0 ;
    let const incl : F = 7 ;
    let out1 : F ;
    let out2 : F ;
    out1 <== incl;
    out2 <== in1;
    outc1 <== in2;
    return (out1, out2) ;
  }
}
```

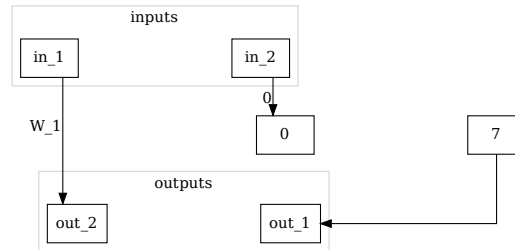
To brain-compile this statement into an algebraic circuit with PAPER, we start with an empty circuit and evaluate function `main`, which is the only function in this statement.

We draw box-nodes for every argument, every constant and every variable of the function and label them with their names or values, respectively. Then we do a consistency and type check for every `<==` operator in the function. Since the circuit only wires inputs to outputs and all elements have the same type, the check is valid.

Then we evaluate the right side of the assignment operators. Since, in our case, the right side of each operator is not a function, we draw edges from the box-nodes on the right side to the associated box node on the left side. To label those edges, we use the general rules of

algebraic circuits as defined in 6.2.2.1. According to those rules, every incoming edge of a sink node has a label and every outgoing edge of a source node has a label, if the node is labeled with a variable. Since nodes that represent constants are implicitly assumed to be private, and since the public specifier determines if an edge is labeled with W or I , we get the following circuit:

check
reference



The Prover Phase In PAPER, a so-called **prover phase** can be executed once the setup phase has generated a circuit image from its associated high-level code. This is done by executing the circuit while assigning proper values to all input nodes of the circuit. However, in contrast to most real-world compilers, PAPER does not tell the prover how to find proper input values to a given circuit. Real-world programming languages usually provide this data by computations that are done outside of the circuit.

Example 129. Consider the circuit from example 128. Valid assignments to this circuit are constructive proofs that the pair of inputs (S_1, S_2) is a point on the tiny-jubjub curve. However, the circuit does not provide a way to actually compute proper values for S_1 and S_2 . Any real-world system therefore needs an auxiliary computation that provides those values.

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reference

7.2 Common Programing concepts

In this section, we cover concepts that appear in almost every programming language, and see how they can be implemented in circuit compilers.

7.2.1 Primitive Types

Primitive data types like booleans, (unsigned) integers, or strings are the most basic building blocks one can expect to find in every general high-level programming language. In order to write statements as computer programs that compile into circuits, it is therefore necessary to implement primitive types as constraint systems, and define their associated operations as circuits.

In this section, we look at some common ways to achieve this. After a recapitulation of the atomic type of prime field elements, we start with an implementation of the boolean type and its associated boolean algebra as circuits. After that, we define unsigned integers based on the boolean type, and leave the implementation of signed integers as an exercise to the reader.

It should be noted, however, that while primitive data types in common programming languages (like C, Go, or Rust) have a one-to-one correspondence with objects in the computer's memory, this is not the case for most languages that compile into algebraic circuits. As we will see in the following paragraphs, common primitives like booleans or unsigned integers require many constraints and memory. Primitives different from the underlying field elements can be expensive.

The base-field type

Since both algebraic circuits and their associated rank-1 constraint systems are defined over a finite field, elements from that field are the atomic informational units in those models. In this sense, field elements $x \in \mathbb{F}$ are for algebraic circuits what bits are for computers.

In PAPER, we write `F` for this type and specify the actual field instance for every statement in curly brackets after the name of that statement. Two functions are associated to this type, which are induced by the **addition** and **multiplication** law in the field `F`. We write

$$\text{MUL} : F \times F \rightarrow F ; (x, y) \mapsto \text{MUL}(x, y) \quad (7.1)$$

$$\text{ADD} : F \times F \rightarrow F ; (x, y) \mapsto \text{ADD}(x, y) \quad (7.2)$$

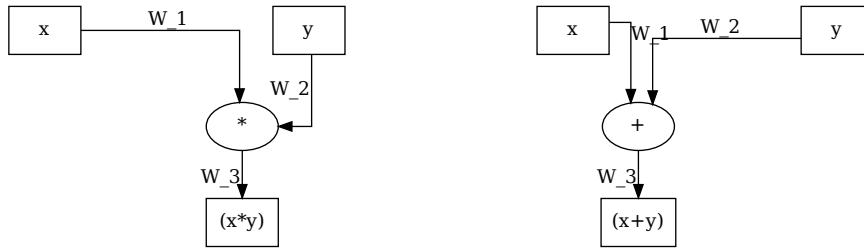
Circuit compilers have to compile these functions into algebraic gates, as explained in 6.2.2.2. Every other function has to be expressed in terms of them and proper wiring.

To represent addition and multiplication in the PAPER language, we define the following two functions:

```
fn MUL(x : F, y : F) -> (MUL(x, y) : F) {}
```

```
fn ADD(x : F, y : F) -> (ADD(x, y) : F) {}
```

The compiler then compiles every occurrence of the `MUL` or the `ADD` function into the following circuits:



Example 130 (Basic gates). To give an intuition of how a real-world compiler might transform addition and multiplication in algebraic expressions into a circuit, consider the following PAPER statement:

```
statement basic_ops {F:F_13} {
  fn main(in_1 : F, pub in_2 : F) -> (out_1:F, out_2:F) {
    out_1 <== MUL(in_1, in_2) ;
    out_2 <== ADD(in_1, in_2) ;
  }
}
```

To compile this into an algebraic circuit, we start with an empty circuit and evaluate the function `main`, which is the only function in this statement.

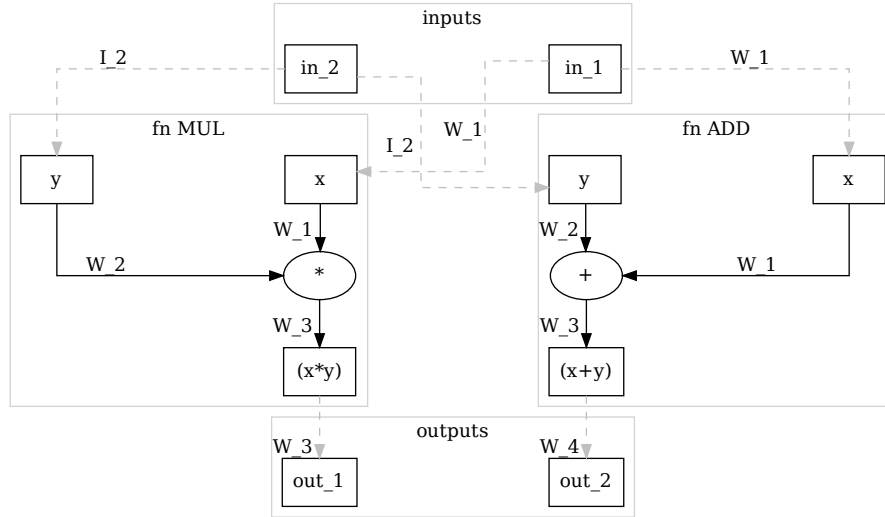
We draw an inputs subgraph containing box-nodes for every argument of the function, and an outputs subgraph containing box-nodes for every factor in the return value. Since all of these nodes represent variables of the `field` type, we don't have to add any type constraints to the circuit.

We check the validity of every expression on the right side of every `<==` operator including a type check. In our case, every variable is of the `field` type and hence the types match the types of the `MUL` as well as the `ADD` function and the type of the left sides of `<==` operators.

check
reference

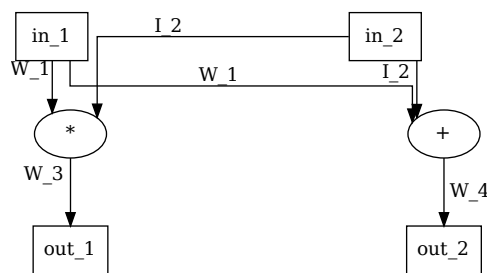
We evaluate the expressions on the right side of every `<==` operator inductively, replacing every occurrence of a function with a subgraph that represents its associated circuit.

According to PAPER, every occurrence of the `public` specifier overwrites the associate `private` default value. Using the appropriate edge labels we get:



Any real-world compiler might process its associated high-level language in a similar way, replacing functions, or gadgets by predefined associated circuits. This process is often followed by various optimization steps that try to reduce the number of constraints as much as possible.

In PAPER, we optimize this circuit by collapsing all box nodes that are directly connected to other box nodes, adhering to the rule that a variable's `public` specifier overwrites any `private` specifier. Reindexing edge labels, we get the following circuit as our pen and pencil compiler output:



Example 131 (3-factorization). Consider our 3-factorization problem from example 109 and the associated circuit $C_{3.\text{fac_zk}}(\mathbb{F}_{13})$ we provided in equation 6.8. To understand the process of replacing high-level functions by their associated circuits inductively, we want define a PAPER statement that we brain-compile into an algebraic circuit equivalent to $C_{3.\text{fac_zk}}(\mathbb{F}_{13})$:

```
statement 3_fac_zk {F:F_13} {
  fn main(x_1 : F, x_2 : F, x_3 : F) -> (pub 3_fac_zk : F) {
    f_3.fac_zk <== MUL( MUL( x_1 , x_2 ) , x_3 ) ;
  }
}
```

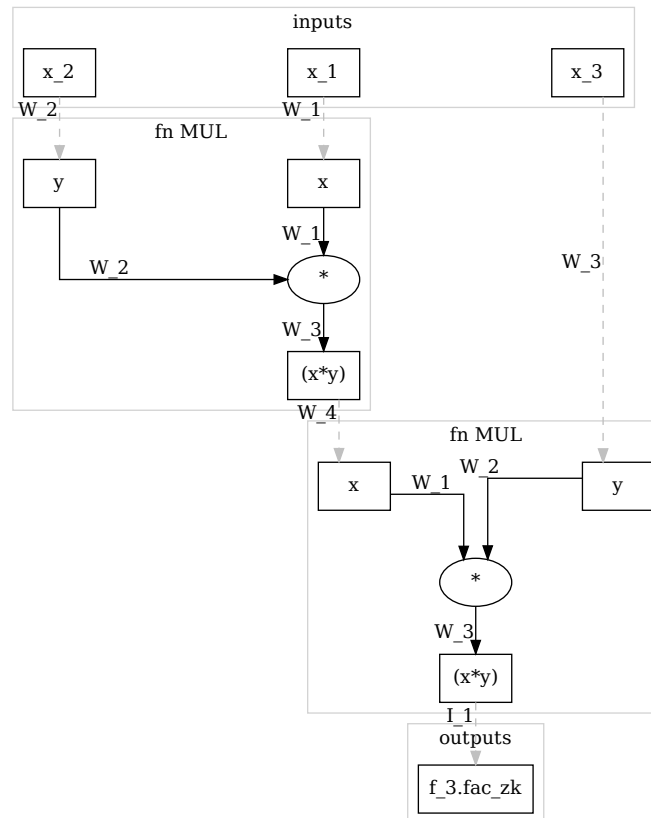
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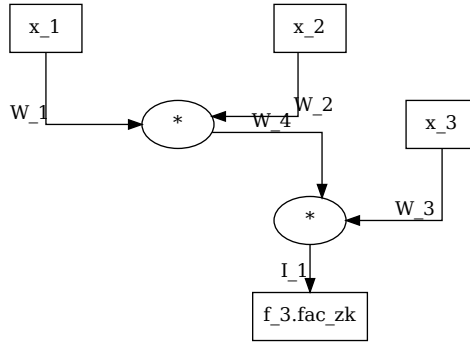
Using PAPER, we start with an empty circuit and then add 3 input nodes to the input subgraph as well as 1 output node to the output subgraph. All these nodes are decorated with the associated variable names. Since all of these nodes represent variables of the `field` type, we don't have to add any type constraints to the circuit.

We check the validity of every expression on the right side of the single `<==` operator including a type check.

We evaluate the expressions on the right side of every `<==` operator inductively. We have two nested multiplication functions and we replace them by the associated multiplication circuits, starting with the most outer function. We get:



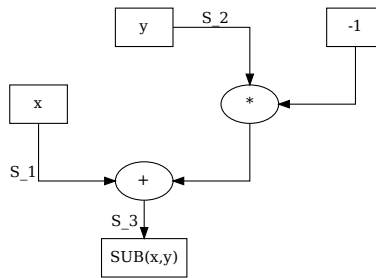
In a final optimization step, we collapse all box nodes directly connected to other box nodes, adhering to the rule that a variable's `public` specifier overwrites any `private` specifier. Reindexing edge labels we get the following circuit:



5167

5168 **The Subtraction Constraint System** By definition, algebraic circuits only contain addition
 5169 and multiplication gates, and it follows that there is no single gate for field subtraction, despite
 5170 the fact that subtraction is a native operation in every field.

5171 High-level languages and their associated circuit compilers, therefore, need another way to
 5172 deal with subtraction. To see how this can be achieved, recall that subtraction is defined by addi-
 5173 tion with the additive inverse, and that the inverse can be computed efficiently by multiplication
 5174 with -1 . A circuit for field subtraction is therefore given by



5175

5176 Using the general method from 6.2.1.1, the circuits associated rank-1 constraint system is given
 5177 by:

$$(S_1 + (-1) \cdot S_2) \cdot 1 = S_3 \quad (7.3)$$

5178 Any valid assignment $\{S_1, S_2, S_3\}$ to this circuit therefore enforces the value S_3 to be the differ-
 5179 ence $S_1 - S_2$.

5180 Real-world compilers usually provide a gadget or a function to abstract over this circuit
 5181 such that programers can use subtraction as if it were native to circuits. In PAPER, we define
 5182 the following subtraction function that compiles to the previous circuit:

```
5183 fn SUB(x : F, y : F) -> (SUB(x,y) : F) {
5184   constant c : F = -1 ;
5185   SUB <== ADD(x , MUL( y , c ) );
5186 }
```

5187 In the setup phase of a statement, we compile every occurrence of the SUB function into an
 5188 instance of its associated subtraction circuit, and edge labels are generated according to the
 5189 rules from 6.2.2.1.

check
reference

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reference

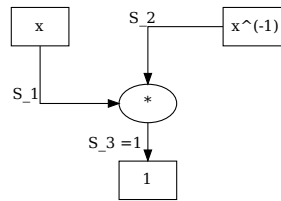
The Inversion Constraint System By definition, algebraic circuits only contain addition and multiplication gates, and it follows that there is no single gate for field inversion, despite the fact that inversion is a native operation in every field.

If the underlying field is a prime field, one approach would be to use Fermat’s little theorem 3.3 to compute the multiplicative inverse inside the circuit. To see how this works, let \mathbb{F}_p be the prime field. The multiplicative inverse x^{-1} of a field element $x \in \mathbb{F}$ with $x \neq 0$ is then given by $x^{-1} = x^{p-2}$, and computing x^{p-2} in the circuit therefore computes the multiplicative inverse.

Unfortunately, real-world primes p are large and computing x^{p-2} by repeated multiplication of x with itself is infeasible. A “double and multiply” approach (as described in XXX) is faster, as it computes the power in roughly $\log_2(p)$ steps, but still adds a lot of constraints to the circuit.

Computing inverses in the circuit makes no use of the fact that inversion is a native operation in any field. A more constraints friendly approach is therefore to compute the multiplicative inverse outside of the circuit and then only enforce correctness of the computation in the circuit.

To understand how this can be achieved, observe that a field element $y \in \mathbb{F}$ is the multiplicative inverse of a field element $x \in \mathbb{F}$ if and only if $x \cdot y = 1$ in \mathbb{F} . We can use this, and define a circuit that has two inputs, x and y , and enforces $x \cdot y = 1$. It is then guaranteed that y is the multiplicative inverse of x . The price we pay is that we can not compute y by circuit execution, but auxiliary data is needed to tell any prover which value of y is needed for a valid circuit assignment. The following circuit defines the constraint



Using the general method from 6.2.1.1, the circuit is transformed into the following rank-1 constraint system:

$$S_1 \cdot S_2 = 1 \quad (7.4)$$

Any valid assignment $\{S_1, S_2\}$ to this circuit enforces that S_2 is the multiplicative inverse of S_1 , and, since there is no field element S_2 such that $0 \cdot S_2 = 1$, it also handles the fact that the multiplicative inverse of 0 is not defined in any field.

Real-world compilers usually provide a gadget or a function to abstract over this circuit, and those functions compute the inverse x^{-1} as part of their witness generation process. Programmers then don’t have to care about providing the inverse as auxiliary data to the circuit. In PAPER, we define the following inversion function that compiles to the previous circuit:

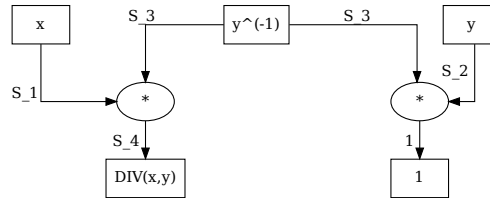
```
fn INV(x : F, y : F) -> (x_inv : F) {
  constant c : F = 1 ;
  c <== MUL( x , y ) ;
  x_inv <== y ;
}
```

As we see, this functions takes two inputs, the field value and its inverse. It therefore does not handle the computation of the inverse by itself. This is to keep PAPER as simple as possible.

In the setup phase, we compile every occurrence of the INV function into an instance of the inversion circuit XXX, and edge labels are generated according to the rules from XXX.

The Division Constraint System By definition, algebraic circuits only contain addition and multiplication gates, and it follows that there is no single gate for field division, despite the fact that division is a native operation in every field.

Implementing division as a circuit, we use the fact that division is multiplication with the multiplicative inverse. We therefore define division as a circuit using the inversion circuit and constraint system from the previous paragraph. Expensive inversion is computed outside of the circuit and then provided as circuit input. We get



Using the method from 6.2.1.1, we transform this circuit into the following rank-1 constraint system:

$$\begin{aligned} S_2 \cdot S_3 &= 1 \\ S_1 \cdot S_3 &= S_4 \end{aligned}$$

Any valid assignment $\{S_1, S_2, S_3, S_4\}$ to this circuit enforces S_4 to be the field division of S_1 by S_2 . It handles the fact that division by 0 is not defined, since there is no valid assignment in case $S_2 = 0$.

In PAPER, we define the following division function that compiles to the previous circuit:

```
fn DIV(x : F, y : F, y_inv : F) -> (DIV : F) {
  DIV <== MUL( x , INV( y, y_inv ) ) ;
}
```

In the setup phase, we compile every occurrence of the binary INV operator into an instance of the inversion circuit.

Exercise 59. Let F be the field \mathbb{F}_5 of modular 5 arithmetics from example 14. Brain-compile the following PAPER statement into an algebraic circuit:

```
statement STUPID_CIRC {F: F_5} {
  fn foo(in_1 : F, in_2 : F)->(out_1 : F, out_2 : F){
    constant c_1 : F = 3 ;
    out_1<== ADD( MUL( c_1 , in_1 ) , in_1 ) ;
    out_2<== INV( c_1 , in_2 ) ;
  } ;

  fn main(in_1 : F, in_2 : F)->(out_1 : F, out_2 : TYPE_2){
    constant (c_1,c_2) : (F,F) = (3,2) ;
    (out_1,out_2) <== foo(in_1, in_2) ;
  } ;
}
```

Exercise 60. Consider the tiny-jubjub curve from example 69 and its associated circuit 128. Write a statement in PAPER that brain-compiles the statement into a circuit equivalent to the one derived in XXX, assuming that curve points are instances and every other assignment is a witness.

5263 *Exercise 61.* Let $F = \mathbb{F}_{13}$ be the modular 13 prime field and $x \in F$ some field element. Define a
 5264 statement in PAPER such that given instance x a field element $y \in F$ is a witness for the statement
 5265 if and only if y is the square root of x .

5266 Brain-compile the statement into a circuit and derive its associated rank-1 constraint system.
 5267 Consider the instance $x = 9$ and compute a constructive proof for the statement.

5268 The boolean Type

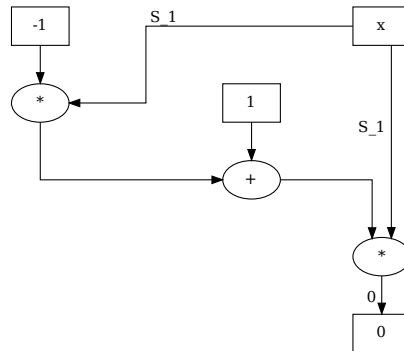
5269 Booleans are a classical primitive type, implemented by virtually every higher programing lan-
 5270 guage. It is therefore important to implement booleans in circuits. One of the most common
 5271 ways to do this is by interpreting the additive and multiplicative neutral element $\{0, 1\} \subset \mathbb{F}$ as
 5272 the two boolean values such that 0 represents *false* and 1 represents *true*. boolean operators
 5273 like *and*, *or*, or *xor* are then expressible as algebraic computations inside \mathbb{F} .

5274 Representing booleans this way is convenient, because the elements 0 and 1 are defined in
 5275 any field. The representation is therefore independent of the actual field in consideration.

5276 To fix boolean algebra notation, we write 0 to represent *false* and 1 to represent *true*, and
 5277 we write \wedge to represent the boolean AND as well as \vee to represent the boolean OR operator.
 5278 The boolean NOT operator is written as \neg .

5279 **The boolean Constraint System** To represent booleans by the additive and multiplicative
 5280 neutral elements of a field, a constraint is required to actually enforce variables of boolean type
 5281 to be either 1 or 0. In fact, many of the following circuits that represent boolean functions are
 5282 only correct under the assumption that their input variables are constrained to be either 0 or 1.
 5283 Not constraining boolean variables is a common problem in circuit design.

5284 In order to constrain an arbitrary field element $x \in \mathbb{F}$ to be 1 or 0, the key observation is
 5285 that the equation $x \cdot (1 - x) = 0$ has only two solutions 0 and 1 in any field. Implementing this
 5286 equation as a circuit therefore generates the correct constraint:



5287

Using the method from 6.2.1.1, we transform this circuit into the following rank-1 constraint system:

$$S_1 \cdot (1 - S_1) = 0$$

check
reference

5288 Any valid assignment $\{S_1\}$ to this circuit enforces S_1 to be either 0 or 1.

5289 Some real-world circuit compilers (like ZOKRATES or BELLMAN) are typed, while others
 5290 (like circom) are not. However, all of them have their way of dealing with the binary con-
 5291 straint. In PAPER, we define the following boolean type that compiles to the previous circuit:

```

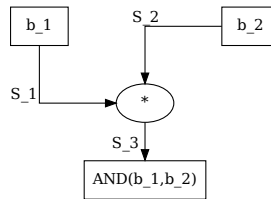
5292 type BOOL(b : BOOL) -> (x : F) {
5293     constant c1 : F = 0 ;
5294     constant c2 : F = 1 ;
5295     constant c3 : F = -1 ;
5296     c1 <== MUL( x , ADD( c2 , MUL( x , c3 ) ) ) ;
5297     x <== b ;
5298 }

```

5299 In the setup phase of a statement, we compile every occurrence of a variable of boolean type
5300 into an instance of its associated boolean circuit.

5301 **The AND operator constraint system** Given two field elements b_1 and b_2 from \mathbb{F} that are
5302 constrained to represent boolean variables, we want to find a circuit that computes the logical
5303 **and** operator $AND(b_1, b_2)$ as well as its associated R1CS that enforces $b_1, b_2, AND(b_1, b_2)$ to
5304 satisfy the constraint system if and only if $b_1 \wedge b_2 = AND(b_1, b_2)$ holds true.

5305 The key insight here is that, given three boolean constraint variables b_1, b_2 and b_3 , the
5306 equation $b_1 \cdot b_2 = b_3$ is satisfied in \mathbb{F} if and only if the equation $b_1 \wedge b_2 = b_3$ is satisfied in
5307 boolean algebra. The logical operator \wedge is therefore implementable in \mathbb{F} by field multiplication
5308 of its arguments and the following circuit computes the \wedge operator in \mathbb{F} , assuming all inputs are
5309 restricted to be 0 or 1:



5310

5311 The associated rank-1 constraint system can be deduced from the general process 6.2.1.1 and
5312 consists of the following constraint:

$$S_1 \cdot S_2 = S_3 \quad (7.5)$$

check
reference

5313 Common circuit languages typically provide a gadget or a function to abstract over this circuit
5314 such that programers can use the \wedge operator without caring about the associated circuit. In
5315 PAPER, we define the following function that compiles to the \wedge -operator's circuit:

```

5316 fn AND(b_1 : BOOL, b_2 : BOOL) -> AND(b_1, b_2) : BOOL{
5317     AND(b_1, b_2) <== MUL( b_1 , b_2 ) ;
5318 }

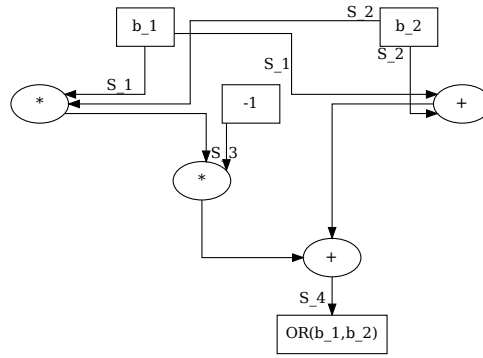
```

5319 In the setup phase of a statement, we compile every occurrence of the AND function into an
5320 instance of its associated \wedge -operator's circuit.

5321 **The OR operator constraint system** Given two field elements b_1 and b_2 from \mathbb{F} that are
5322 constrained to represent boolean variables, we want to find a circuit that computes the logical
5323 **or** operator $OR(b_1, b_2)$ as well as its associated R1CS that enforces $b_1, b_2, OR(b_1, b_2)$ to satisfy
5324 the constraint system if and only if $b_1 \vee b_2 = OR(b_1, b_2)$ holds true.

5325 Assuming that three variables b_1, b_2 and b_3 are boolean constraint, the equation $b_1 + b_2 - b_1 \cdot$
5326 $b_2 = b_3$ is satisfied in \mathbb{F} if and only if the equation $b_1 \vee b_2 = b_3$ is satisfied in boolean algebra.
5327 The logical operator \vee is therefore implementable in \mathbb{F} by the following circuit, assuming all
5328 inputs are restricted to be 0 or 1:

"constraints"
or "con-
strained"?



5329

The associated rank-1 constraint system can be deduced from the general process 6.2.1.1 and consists of the following constraints:

check
reference

$$\begin{aligned} S_1 \cdot S_2 &= S_3 \\ (S_1 + S_2 - S_3) \cdot 1 &= S_4 \end{aligned}$$

Common circuit languages typically provide a gadget or a function to abstract over this circuit such that programers can use the \vee operator without caring about the associated circuit. In PAPER, we define the following function that compiles to the \vee -operator's circuit:

```
5333 fn OR(b_1 : BOOL, b_2 : BOOL) -> OR(b_1,b_2) : BOOL{
5334   constant c1 : F = -1 ;
5335   OR(b_1,b_2) <== ADD(ADD(b_1,b_2),MUL(c1,MUL(b_1,b_2))) ;
5336 }
```

In the setup phase of a statement, we compile every occurrence of the OR function into an instance of its associated \vee -operator's circuit.

Exercise 62. Let \mathbb{F} be a finite field and let b_1 as well as b_2 two boolean constraint variables from \mathbb{F} . Show that the equation $OR(b_1, b_2) = 1 - (1 - b_1) \cdot (1 - b_2)$ holds true.

Use this equation to derive an algebraic circuit with ingoing variables b_1 and b_2 and outgoing variable $OR(b_1, b_2)$ such that b_1 and b_2 are boolean **constraint** and the circuit has a valid assignment, if and only if $OR(b_1, b_2) = b_1 \vee b_2$.

"constraints"
or "con-
strained"?

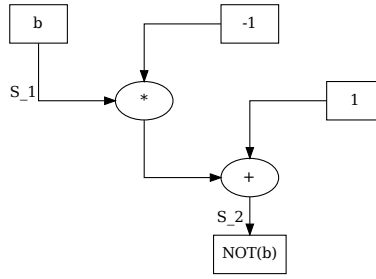
Use the technique from XXX to transform this circuit into a rank-1 constraint system and find its full solution set. Define a PAPER function that brain-compiles into the circuit.

add refer-
ence

The NOT operator constraint system Given a field element b from \mathbb{F} that is constrained to represent a boolean variable, we want to find a circuit that computes the logical **NOT** operator $NOT(b)$ as well as its associated RICS that enforces $b, NOT(b)$ to satisfy the constraint system if and only if $\neg b = NOT(b)$ holds true.

Assuming that two variables b_1 and b_2 are boolean **constraint**, the equation $(1 - b_1) = b_2$ is satisfied in \mathbb{F} if and only if the equation $\neg b_1 = b_2$ is satisfied in boolean algebra. The logical operator \neg is therefore implementable in \mathbb{F} by the following circuit, assuming all inputs are restricted to be 0 or 1:

"constraints"
or "con-
strained"?



5354

The associated rank-1 constraint system can be deduced from the general process XXX and consists of the following constraints

add reference

$$(1 - S_1) \cdot 1 = S_2$$

5355 Common circuit languages typically provide a gadget or a function to abstract over this circuit
 5356 such that programers can use the \neg operator without caring about the associated circuit. In
 5357 PAPER, we define the following function that compiles to the \neg -operator's circuit:

```

5358 fn NOT(b : BOOL -> NOT(b) : BOOL{
5359   constant c1 = 1 ;
5360   constant c2 = -1 ;
5361   NOT(b_1) <== ADD( c1 , MUL( c2 , b ) ) ;
5362 }
```

5363 In the setup phase of a statement, we compile every occurrence of the NOT function into an
 5364 instance of its associated \neg -operator's circuit.

5365 *Exercise 63.* Let \mathbb{F} be a finite field. Derive the algebraic circuit and associated rank-1 constraint
 5366 system for the following operators: NOR, XOR, NAND, EQU.

5367 **Modularity** As we have seen in chapter 6,, both algebraic circuits and R1CS have a modular-
 5368 ity property, and as we have seen in this section, all basic boolean functions are expressible in
 5369 circuits. Combining those two properties, show that it is possible to express arbitrary boolean
 5370 functions as algebraic circuits.

check references

5371 This shows that the expressiveness of algebraic circuits and therefore rank-1 constraint sys-
 5372 tems is as general as the expressiveness of boolean circuits. An important implication is that
 5373 the languages $L_{R1CS-SAT}$ and $L_{Circuit-SAT}$ as defined in 6, are as general as the famous language
 5374 L_{3-SAT} , which is known to be \mathcal{NP} -complete.

check reference

5375 *Example 132.* To give an example of how a compiler might construct complex boolean expres-
 5376 sions in algebraic circuits from simple ones and how we derive their associated rank-1 constraint
 5377 systems, let's look at the following PAPER statement:

```

5378 statement BOOLEAN_STAT {F: F_p} {
5379   fn main(b_1:BOOL,b_2:BOOL,b_3:BOOL,b_4:BOOL )-> pub b_5:BOOL {
5380     b_5 <== AND( OR( b_1 , b_2 ) , AND( b_3 , NOT( b_4 ) ) ) ;
5381   } ;
5382 }
```

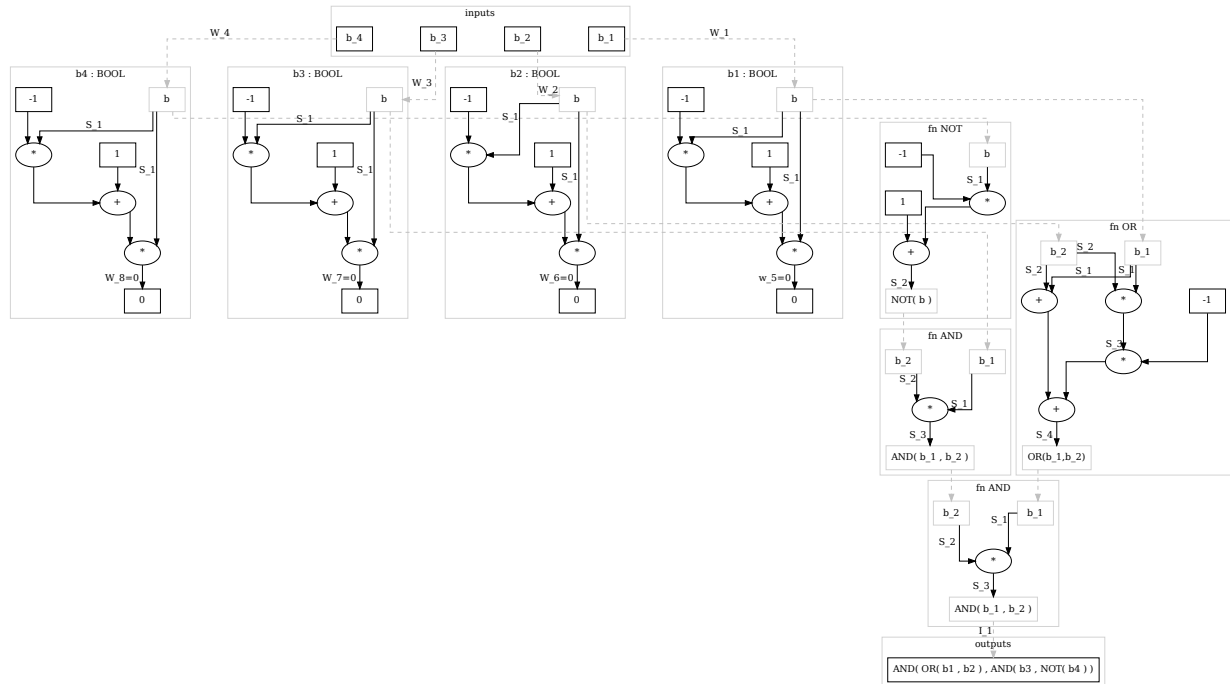
The code describes a circuit that takes four private inputs b_1, b_2, b_3 and b_4 of boolean type and computes a public output b_5 such that the following boolean expression holds true:

$$(b_1 \vee b_2) \wedge (b_3 \wedge \neg b_4) = b_5$$

During a setup-phase, a circuit compiler transforms this high-level language statement into a circuit and associated rank-1 constraint systems and hence defines a language $L_{BOOLEAN_STAT}$.

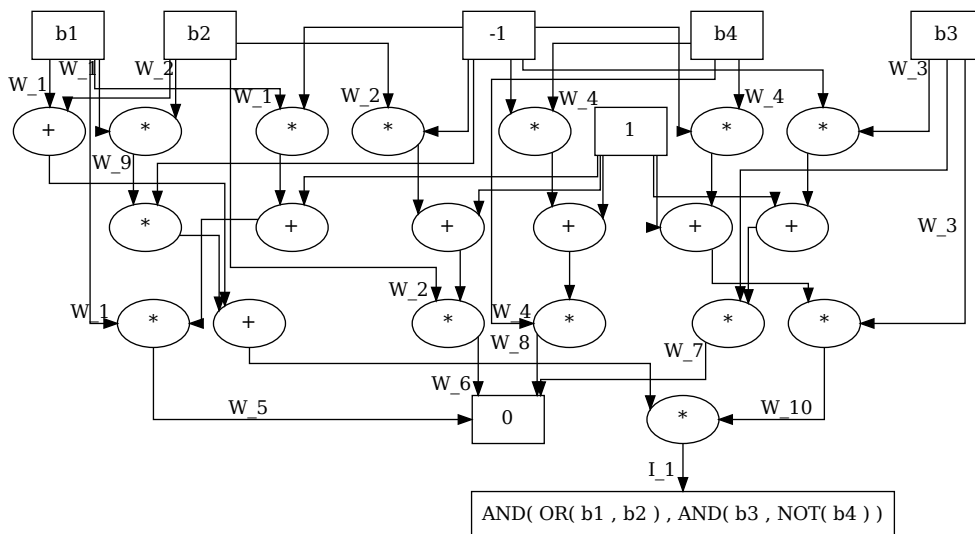
To see how this might be achieved, we use PAPER as an example to execute the setup-phase and compile `BOOLEAN_STAT` into a circuit. Taking the definition of the boolean constraint `XXX` as well as the definitions of the appropriate boolean operators into account, we get the following circuit:

add reference



Simple optimization then collapses all box-nodes that are directly linked and all box nodes that represent the same constants. After relabeling the edges, the following circuit represents the circuit associated to the `BOOLEAN_STAT` statement:

can we rotate this by 90°?



5397 Given some public input I_1 from \mathbb{F}_{13} , a valid assignment to this circuit consists of private inputs
 5398 W_1, W_2, W_3, W_4 from \mathbb{F}_{13} such that the equation $I_1 = (W_1 \vee W_2) \wedge (W_3 \wedge \neg W_4)$ holds true. In
 5399 addition, a valid assignment also has to contain private inputs W_5, W_6, W_7, W_8, W_9 and W_{10} ,
 5400 which can be derived from circuit execution. The inputs W_5, \dots, W_8 ensure that the first four
 5401 private inputs are either 0 or 1 but not any other field element, and the others enforce the boolean
 5402 operations in the expression.

To compute the associated R1CS, we can use the general method from 6.2.1.1 and look at every labeled outgoing edge not coming from a source node. We declare the edges coming from input nodes as well as the edge going to the single output node as public, and every other edge as private input. In this case we get:

check
reference

$$\begin{array}{ll}
 W_5 : W_1 \cdot (1 - W_1) = 0 & \text{boolean constraints} \\
 W_6 : W_2 \cdot (1 - W_2) = 0 & \\
 W_7 : W_3 \cdot (1 - W_3) = 0 & \\
 W_8 : W_4 \cdot (1 - W_4) = 0 & \\
 W_9 : W_1 \cdot W_2 = W_9 & \text{first OR-operator constraint} \\
 W_{10} : W_3 \cdot (1 - W_4) = W_{10} & \text{AND(.,NOT(.))-operator constraints} \\
 I_1 : (W_1 + W_2 - W_9) \cdot W_{10} = I_1 & \text{AND-operator constraints}
 \end{array}$$

5403 The reason why this R1CS only contains a single constraint for the multiplication gate in the
 5404 OR-circuit, while the general definition XXX requires two constraints, is that the second con-
 5405 straint in XXX only appears because the final addition gate is connected to an output node. In
 5406 this case, however, the final addition gate from the OR-circuit is enforced in the left factor of
 5407 the I_1 constraint. Something similar holds true for the negation circuit.

add refer-
ence

add refer-
ence

During a prover-phase, some public instance I_5 must be given. To compute a constructive proof for the statement of the associated languages with respect to instance I_5 , a prover has to find four boolean values W_1, W_2, W_3 and W_4 such that

$$(W_1 \vee W_2) \wedge (W_3 \wedge \neg W_4) = I_5$$

5408 holds true. In our case neither the circuit nor the PAPER statement specifies how to find those
 5409 values, and it is a problem that any prover has to solve outside of the circuit. This might or
 5410 might not be true for other problems, too. In any case, once the prover found those values, they
 5411 can execute the circuit to find a valid assignment.

To give a concrete example, let $I_1 = 1$ and assume $W_1 = 1, W_2 = 0, W_3 = 1$ and $W_4 = 0$. Since $(1 \vee 0) \wedge (1 \wedge \neg 0) = 1$, those values satisfy the problem and we can use them to execute the circuit. We get

$$\begin{array}{l}
 W_5 = W_1 \cdot (1 - W_1) = 0 \\
 W_6 = W_2 \cdot (1 - W_2) = 0 \\
 W_7 = W_3 \cdot (1 - W_3) = 0 \\
 W_8 = W_4 \cdot (1 - W_4) = 0 \\
 W_9 = W_1 \cdot W_2 = 0 \\
 W_{10} = W_3 \cdot (1 - W_4) = 1 \\
 I_1 = (W_1 + W_2 - W_9) \cdot W_{10} = 1
 \end{array}$$

5412 A constructive proof of knowledge of a witness, for instance, $I_1 = 1$, is therefore given by the
 5413 tuple $P = (W_5, W_6, W_7, W_8, W_9, W_{10}) = (0, 0, 0, 0, 0, 1)$.

Arrays

The `array` type represents a fixed-size collection of elements of equal type, each selectable by one or more indices that can be computed at run time during program execution.

Arrays are a classical type, implemented by many higher programming languages that compile to circuits or rank-1 constraint systems. However, most high-level circuit languages support **static** arrays, i.e., arrays whose length is known at compile time only.

The most common way to compile arrays to circuits is to transform any array of a given type τ and size N into N circuit variables of type τ . Arrays are therefore **syntactic sugar**, that is, parts of the formal language that makes the code easier for humans to read, which the compiler transforms into input nodes, much like any other variable. In PAPER, we define the following array type:

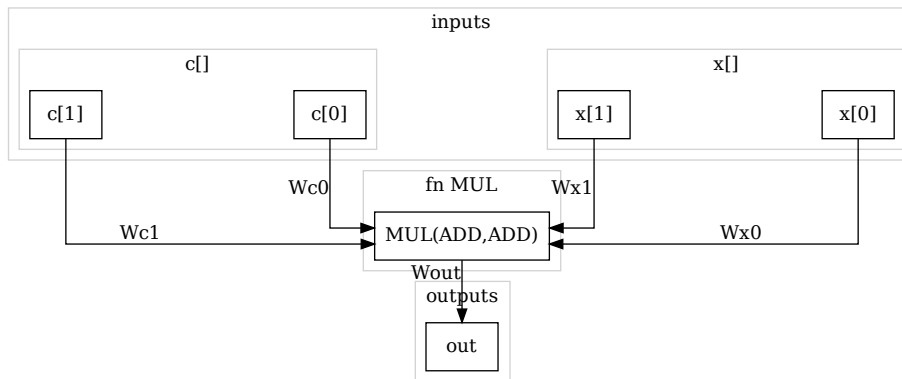
```
type <Name>: <Type>[N : unsigned] -> (Type,...) {
  return (<Name>[0],...)
}
```

In the setup phase of a statement, we compile every occurrence of an array of size N that contains elements of type `Type` into N variables of type `Type`.

Example 133. To give an intuition of how a real-world compiler might transform arrays into circuit variables, consider the following PAPER statement:

```
statement ARRAY_TYPE {F: F_5} {
  fn main(x: F[2]) -> F {
    let constant c: F[2] = [2,4] ;
    let out:F <== MUL(ADD(x[1],c[0]),ADD(x[0],c[1])) ;
    return out ;
  } ;
}
```

During a setup phase, a circuit compiler might then replace any occurrence of the array type by a tuple of variables of the underlying type, and then use those variables in the circuit synthesis process instead. To see how this can be achieved, we use PAPER as an example. Abstracting over the sub-circuit of the computation, we get the following circuit:



The Unsigned Integer Type

Unsigned integers of size N , where N is usually a power of two, represent non-negative integers in the range $0 \dots 2^N - 1$. They have a notion of addition, subtraction and multiplication, defined

5447 by modular 2^N arithmetics. If some N is given, we write uN for the associated type.

5448 **The uN Constraint System** Many high-level circuit languages define the the various uN types
 5449 as arrays of size N , where each element is of boolean type. This is similar to their representa-
 5450 tion on common computer hardware and allows for efficient and straightforward definition of
 5451 common operators, like the various **shift**, or logical operators.

shift

5452 If some unsigned integer N is known at compile time in PAPER, we define the following uN
 5453 type:

```
5454 type uN -> BOOL[N] {
5455   let base2 : BOOL[N] <== BASE_2 (uN) ;
5456   return base2 ;
5457 }
```

To enforce an N -tuple of field elements (b_0, \dots, b_{N-1}) to represent an element of type uN we therefore need N boolean constraints

$$\begin{aligned} S_0 \cdot (1 - S_0) &= 0 \\ S_1 \cdot (1 - S_1) &= 0 \\ &\dots \\ S_{N-1} \cdot (1 - S_{N-1}) &= 0 \end{aligned}$$

5458 In the setup phase of a statement, we compile every occurrence of the uN type by a size N array
 5459 of boolean type. During a=the prover phase, actual elements of the uN type are first transformed
 5460 into binary representation and then this binary representation is assigned to the boolean array
 5461 that represents the uN type.

5462 *Remark 8.* Representing the uN type as boolean arrays is conceptually clean and works over
 5463 generic base fields. However, representing unsigned integers in this way requires a lot of space
 5464 as every bit is represented as a field element and if the base field is large, those field elements
 5465 require considerable space in hardware.

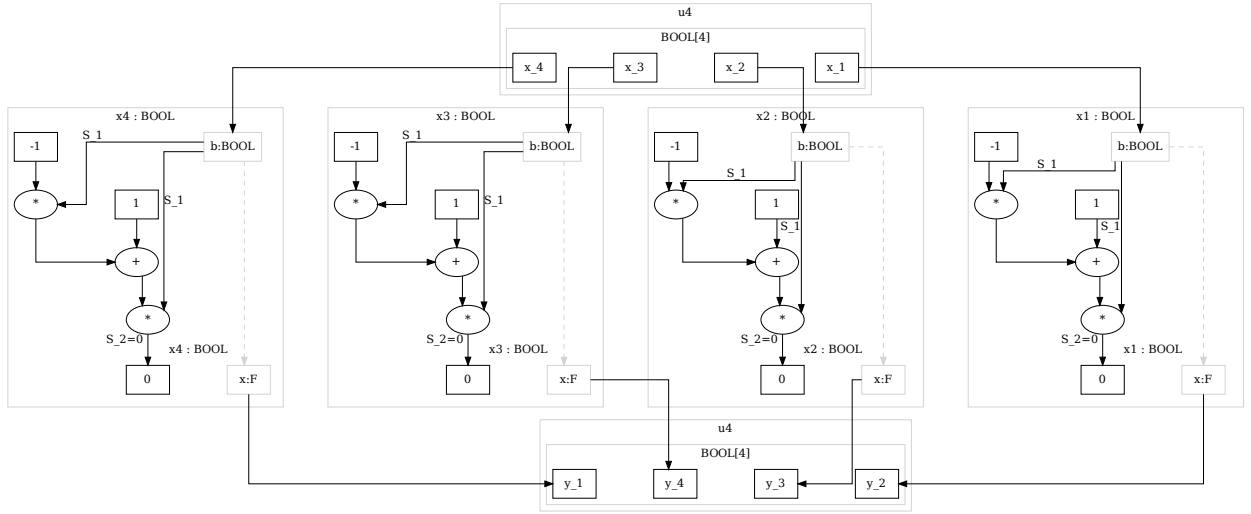
5466 It should be noted that, in some cases, there is another, more space- and constraint-efficient
 5467 approach for representing unsigned integers that can be used whenever the underlying base field
 5468 is sufficiently large. To understand this, recall that addition and multiplication in a prime field
 5469 \mathbb{F}_p is equal to addition and multiplication of integers, as long as the sum or the product does
 5470 not exceed the modulus p . It is therefore possible to represent the uN type inside the base-field
 5471 type whenever N is small enough. In this case, however, care has to be taken to never overflow
 5472 the modulus. It is also important to make sure that, in the case of subtraction, the subtrahend is
 5473 never larger than the minuend.

5474 *Example 134.* To give an intuition of how a real-world compiler might transform unsigned
 5475 integers into circuit variables, consider the following PAPER statement:

```
5476 statement RING_SHIFT{F: F_p, N=4} {
5477   fn main(x: uN) -> uN {
5478     let y:uN <== [x[1], x[2], x[3], x[0]] ;
5479     return y ;
5480   } ;
5481 }
```

5482 During the setup-phase, a circuit compiler might then replace any occurrence of the uN type
 5483 by N variables of `boolean` type. Using the definition of booleans, each of these variables is

then transformed into the `field` type and a boolean constraint system. To see how this can be achieved, we use PAPER as an example and get the following circuit:



During the prover phase, the function `main` is called with an actual input of `u4` type, say $x=14$. The high-level language then has to transform the decimal value 14 into its 4-bit binary representation $14_2 = (0, 1, 1, 1)$ outside of the circuit. Then the array of field values $x[4] = [0, 1, 1, 1]$ is used as an input to the circuit. Since all 4 field elements are either 0 or 1, the four boolean constraints are satisfiable and the output is an array of the four field elements $[1, 1, 1, 0]$, which represents the `u4` element 7.

The Unigned Integer Operators Since elements of `uN` type are represented as boolean arrays, shift operators are implemented in circuits simply by rewiring the boolean input variables to the output variables accordingly.

Logical operators, like AND, OR, or NOT are defined on the `uN` type by invoking the appropriate boolean operators bitwise to every bit in the boolean array that represents the `uN` element.

Addition and multiplication can be represented similarly to how machines represent those operations. Addition can be implemented by first defining the **full adder** circuit and then combining N of these circuits into a circuit that adds two elements from the `uN` type.

Exercise 64. Let $F = \mathbb{F}_{13}$ and $N=4$ be fixed. Define circuits and associated R1CS for the left and right **bishift** operators $x \ll 2$ as well as $x \gg 2$ that operate on the `uN` type. Execute the associated circuit for $x : u4 = 11$.

Exercise 65. Let $F = \mathbb{F}_{13}$ and $N=2$ be fixed. Define a circuit and associated R1CS for the addition operator $\text{ADD} : F \times F \rightarrow F$. Execute the associated circuit to compute $\text{ADD}(2, 7)$.

Exercise 66. Brain-compile the following PAPER code into a circuit and derive the associated R1CS.

```
statement MASK_MERGE {F:F_5, N=4} {
  fn main(pub a : uN, pub b : uN) -> F {
    let constant mask : uN = 10 ;
    let r : uN <== XOR(a, AND(XOR(a, b), mask)) ;
    return r ;
```

```
5515     }
5516 }
```

5517 Let L_{mask_merge} be the language defined by the circuit. Provide a constructive knowledge proof
5518 in L_{mask_merge} for the instance $I = (I_a, I_b) = (14, 7)$.

5519 7.2.2 Control Flow

5520 Most programming languages of the imperative or functional style have some notion of basic
5521 control structures to direct the order in which instructions are evaluated. Contemporary circuit
5522 compilers usually provide a single thread of execution and provide basic flow constructs that
5523 implement control flow in circuits.

5524 The Conditional Assignment

5525 Writing high-level code that compiles to circuits, it is often necessary to have a way for condi-
5526 tional assignment of values or computational output to variables.

5527 One way to realize this in many programming languages is in terms of the conditional
5528 ternary assignment operator $?:$ that branches the control flow of a program according to some
5529 condition and then assigns the output of the computed branch to some variable.

```
5530 variable = condition ? value_if_true : value_if_false
```

5531 In this description, `condition` is a boolean expression and `value_if_true` as well as
5532 `value_if_false` are expressions that evaluate to the same type as `variable`.

5533 In programming languages like Rust, another way to write the conditional assignment oper-
5534 ator that is more familiar to many programmers is given by

```
5535 variable = if condition then {
5536     value_if_true
5537 } else {
5538     value_if_false
5539 }
```

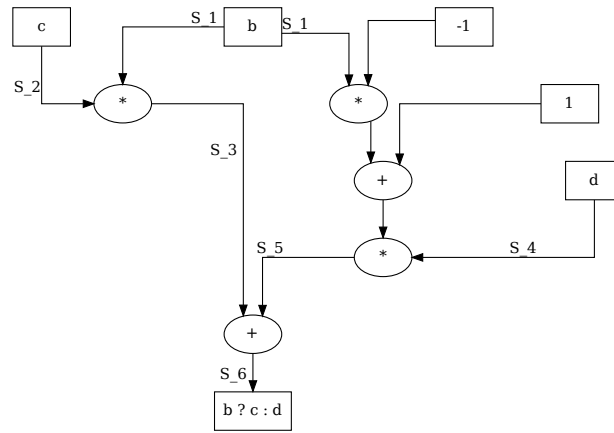
5540 In most programming languages, it is a key property of the ternary assignment operator that
5541 the expression `value_if_true` is only evaluated if `condition` evaluates to true and the
5542 expression `value_if_false` is only evaluated if `condition` evaluates to false. In fact,
5543 computer programs would turn out to be very inefficient if the ternary operator would evaluate
5544 both expressions regardless of the value of `condition`.

5545 A simple way to implement conditional assignment operator as a circuit can be achieved
5546 if the requirement that only one branch of the conditional operator is executed is dropped. To
5547 see that, let b , c and d be field elements such that b is a boolean constraint. In this case, the
5548 following equation enforces a field element x to be the result of the conditional assignment
5549 operator:

$$x = b \cdot c + (1 - b) \cdot d \quad (7.6)$$

5550 Expressing this equation in terms of the addition and multiplication operators from XXX, we
5551 can flatten it into the following algebraic circuit:

add refer-
ence



5552

5553 Note that, in order to compute a valid assignment to this circuit, both S_2 as well as S_4 are
 5554 necessary. If the inputs to the nodes c and d are circuits themselves, both circuits need valid
 5555 assignments and therefore have to be executed. As a consequence, this implementation of
 5556 the conditional assignment operator has to execute all branches of all circuits, which is very
 5557 different from the execution of common computer programs and contributes to the increased
 5558 computational effort any prover has to invest, in contrast to the execution in other programming
 5559 models.

We can use the general technique from 6.2.1.1 to derive the associated rank-1 constraint system of the conditional assignment operator. We get the following:

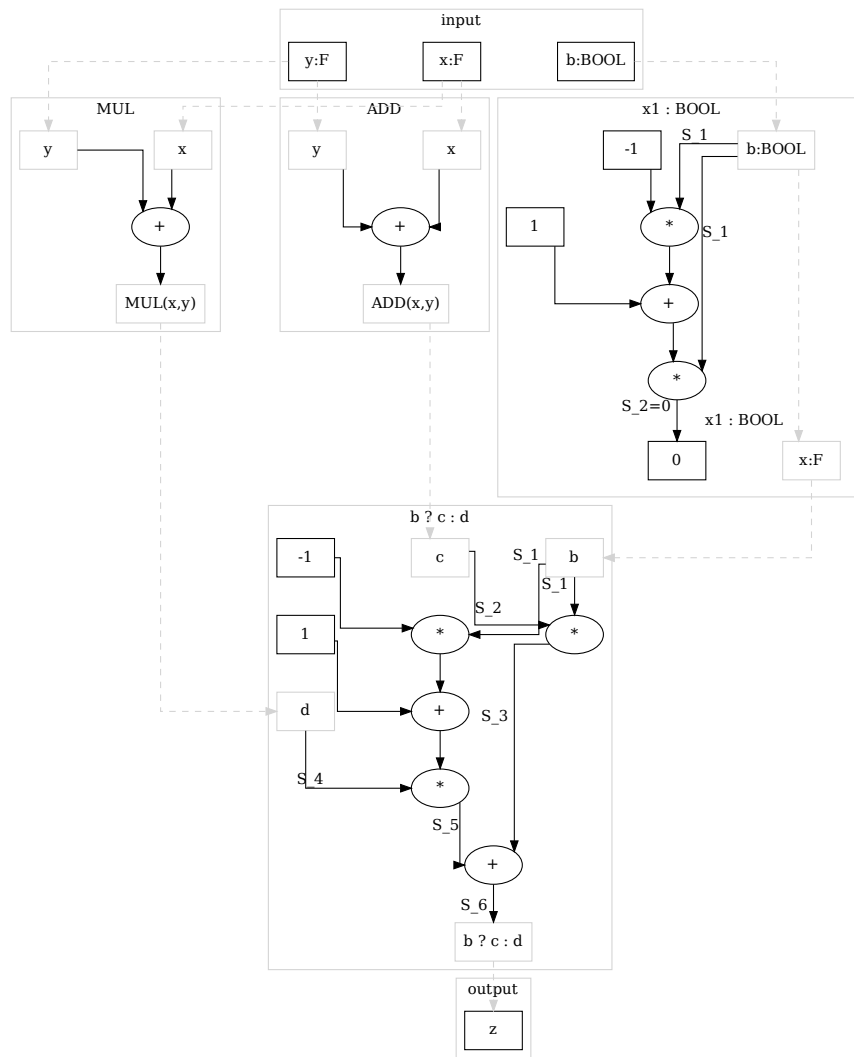
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reference

$$\begin{aligned} S_1 \cdot S_2 &= S_3 \\ (1 - S_1) \cdot S_4 &= S_5 \\ (S_3 + S_5) \cdot 1 &= S_6 \end{aligned}$$

5560 *Example 135.* To give an intuition of how a real-world circuit compiler might transform any
 5561 high-level description of the conditional assignment operator into a circuit, consider the follow-
 5562 ing PAPER code:

```
5563 statement CONDITIONAL_OP {F:F_p} {
5564   fn main(x : F, y : F, b : BOOL) -> F {
5565     let z : F <== if b then {
5566       ADD(x,y)
5567     } else {
5568       MUL(x,y)
5569     } ;
5570     return z ;
5571   }
5572 }
```

5573 Brain-compiling this code into a circuit, we first draw box nodes for all input and output vari-
 5574 ables, and then transform the boolean type into the field type together with its associated con-
 5575 straint. Then we evaluate the assignments to the output variables. Since the conditional assign-
 5576 ment operator is the top level function, we draw its circuit and then draw the circuits for both
 5577 conditional expressions. We get the following:



5578

5579 **Loops**

5580 In many programming languages, various loop control structures are defined that allow devel-
 5581 opers to execute expressions with a specified number of repetitions or arguments. In particular,
 5582 it is often possible to implement unbounded loops like the loop structure give below, where the
 5583 number of executions depends on execution inputs and is therefore unknown at compile time:

5584 **while** true do {
 5585

5586 **M:** Add another example where the loop actually depends on the input.

Add ex-
ample

5587 In contrast to this, it should be noted that algebraic circuits and rank-1 constraint systems
 5588 are not general enough to express arbitrary computation, but bounded computation only. As a
 5589 consequence, it is not possible to implement unbounded loops, or loops with bounds that are
 5590 unknown at compile time in those models. This can be easily seen since circuits are acyclic
 5591 by definition, and implementing an unbounded loop as an acyclic graph requires a circuits of
 5592 unbounded size.

5593 However, circuits are general enough to express bounded loops, where the upper bound on
 5594 its execution is known at compile time. Those loop can be implemented in circuits by enrolling
 5595 the loop.

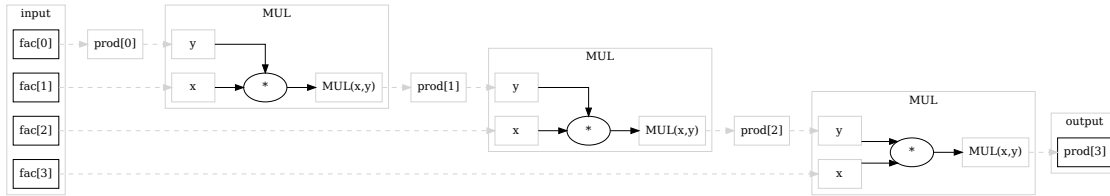
As a consequence, any programming language that compiles to algebraic circuits can only provide loop structures where the bound is a constant known at compile time. This implies that loops cannot depend on execution inputs, but on compile time parameters only.

Example 136. To give an intuition of how a real-world circuit compiler might transform any high-level description of a bounded `for` loop into a circuit, consider the following PAPER code:

```
statement FOR_LOOP {F:F_p, N: unsigned = 4} {
  fn main(fac : F[N]) -> F {
    let prod[N] : F ;
    prod[0] <== fac[0] ;
    for unsigned i in 1..N do [{
      prod[i] <== MUL(fac[i], prod[i-1]) ;
    }]
    return prod[N] ;
  }
}
```

Note that, in a program like this, the loop counter `i` has no expression in the derived circuit. It is pure syntactic sugar, telling the compiler how to unroll the loop.

Brain-compiling this code into a circuit, we first draw box nodes for all input and output variables, noting that the loop counter is not represented in the circuit. Since all variables are of `field` type, we don't have to compile any type constraints. Then we evaluate the assignments to the output variables by unrolling the loop into 3 individual assignment operators. We get:



7.2.3 Binary Field Representations

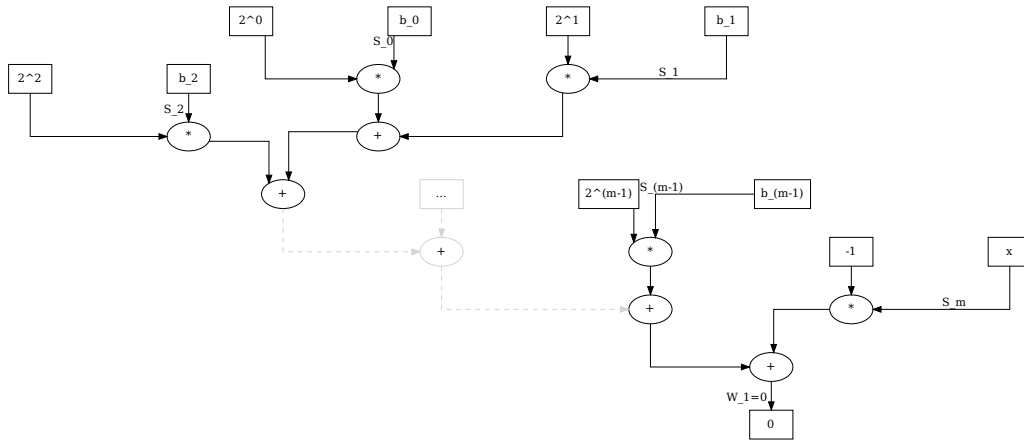
In applications, it is often necessary to enforce a binary representation of elements from the `field` type. To derive an appropriate circuit over a prime field \mathbb{F}_p , let $m = \lceil \log_2 p \rceil$ be the smallest number of bits necessary to represent the prime modulus p . Then a bitstring $(b_0, \dots, b_{m-1}) \in \{0, 1\}^m$ is a binary representation of a field element $x \in \mathbb{F}_p$, if and only if

$$x = b_0 \cdot 2^0 + b_1 \cdot 2^1 + \dots + b_{m-1} \cdot 2^{m-1}$$

In this expression, addition and exponentiation is considered to be executed in \mathbb{F}_p , which is well defined since all terms 2^j for $0 \leq j < m$ are elements of \mathbb{F}_p . Note, however, that in contrast to the binary representation of unsigned integers $n \in \mathbb{N}$, this representation is not unique in general, since the modular p equivalence class might contain more than one binary representative.

Considering that the underlying prime field is fixed and the most significant bit of the prime modulus is m , the following circuit flattens equation XXX, assuming all inputs b_1, \dots, b_m are of boolean type.

add reference



5627

Applying the general transformation rule to compute the associated rank-1 constraint systems, we see that we actually only need a single constraint to enforce some binary representation of any field element. We get

$$(S_0 \cdot 2^0 + S_1 \cdot 2^1 + \dots + S_{m-1} \cdot 2^{m-1} - S_m) \cdot 1 = 0$$

5628 Given an array `BOOL[N]` of N boolean constraint field elements and another field element x ,
 5629 the circuit enforces `BOOL[N]` to be one of the binary representations of x . If `BOOL[N]` is not
 5630 a binary representation of x , no valid assignment and hence no solution to the associated R1CS
 5631 can exist.

5632 *Example 137.* Consider the prime field \mathbb{F}_{13} . To compute binary representations of elements
 5633 from that field, we start with the binary representation of the prime modulus 13, which is $|13|_2 =$
 5634 $(1, 0, 1, 1)$ since $13 = 1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3$. So $m = 4$ and we need up to 4 bits to represent
 5635 any element $x \in \mathbb{F}_{13}$.

To see that binary representations are not unique in general, consider the element $2 \in \mathbb{F}_{13}$. It has the binary representations $|2|_2 = (0, 1, 0, 0)$ and $|2|_2 = (1, 1, 1, 1)$, since in \mathbb{F}_{13} we have

$$2 = \begin{cases} 0 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 0 \cdot 2^3 \\ 1 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3 \end{cases}$$

5636 This is because the unsigned integers 2 and 15 are both in the modular 13 remainder class of 2
 5637 and hence are both representatives of 2 in \mathbb{F}_{13} .

To see how circuit XXX works, we want to enforce the binary representation of $7 \in \mathbb{F}_{13}$. Since $m = 4$ we have to enforce a 4-bit representation for 7, which is $(1, 1, 1, 0)$, since $7 = 1 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2 + 0 \cdot 2^3$. A valid circuit assignment is therefore given by $(S_0, S_1, S_2, S_3, S_4) = (1, 1, 1, 0, 7)$ and, indeed, the assignment satisfies the required 5 constraints including the 4 boolean constraints for S_0, \dots, S_3 :

$$\begin{aligned} 1 \cdot (1 - 1) &= 0 & // \text{boolean constraints} \\ 1 \cdot (1 - 1) &= 0 \\ 1 \cdot (1 - 1) &= 0 \\ 0 \cdot (1 - 0) &= 0 \\ (1 + 2 + 4 + 0 - 7) \cdot 1 &= 0 & // \text{binary rep. constraint} \end{aligned}$$

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ence

7.2.4 Cryptographic Primitives

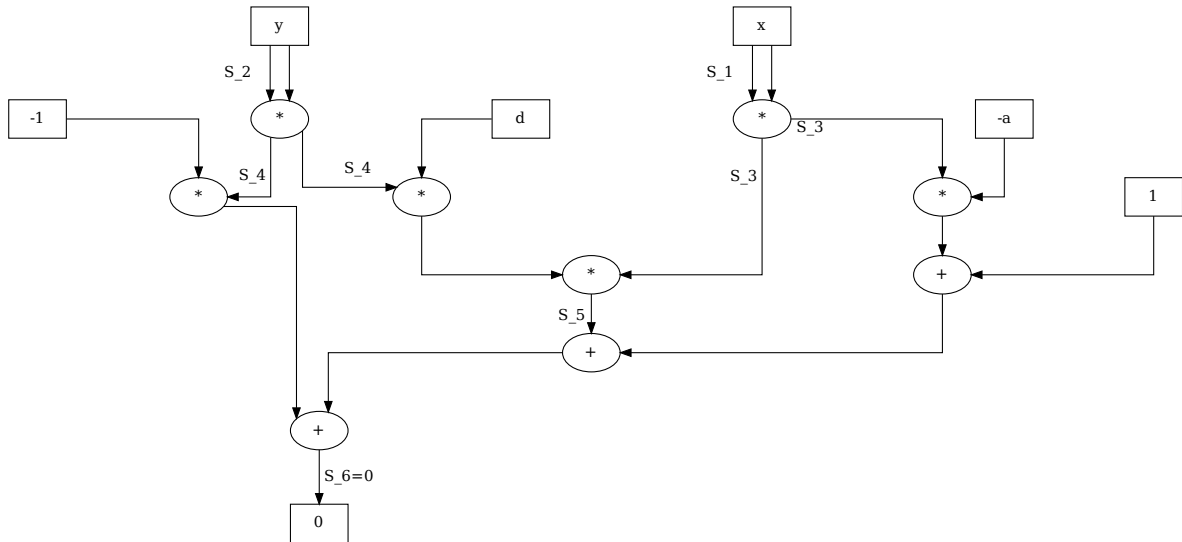
In applications, it is often required to do cryptography in a circuit. To do this, basic cryptographic primitives like hash functions or elliptic curve cryptography needs to be implemented as circuits. In this section, we give a few basic examples of how to implement such primitives.

Twisted Edwards curves

Implementing elliptic curve cryptography in circuits means to implement the field equation as well as the algebraic operations of an elliptic curve as circuits. To do this efficiently, the curve must be defined over the same base field as the field that is used in the circuit.

For efficiency reasons, it is advantageous to choose an elliptic curve such that that all required constraints and operations can be implement with as few gates as possible. Twisted Edwards curves are particularly useful for that matter, since their addition law is particularly simple and the same equation can be used for all curve points including the point at infinity. This simplifies the circuit a lot.

Twisted Edwards curve constraints As we have seen in section 5.1.3, a twisted Edwards curve over a finite field F is defined as the set of all pairs of points $(x, y) \in \mathbb{F} \times \mathbb{F}$ such that x and y satisfy the equation $a \cdot x^2 + y^2 = 1 + d \cdot x^2 y^2$. As we have seen in example XXX, we can transform this equation into the following circuit:



The circuit enforces the two inputs of `field` type to satisfy the twisted Edwards curve equation and, as we know from example XXX, the associated rank-1 constraint system is given by:

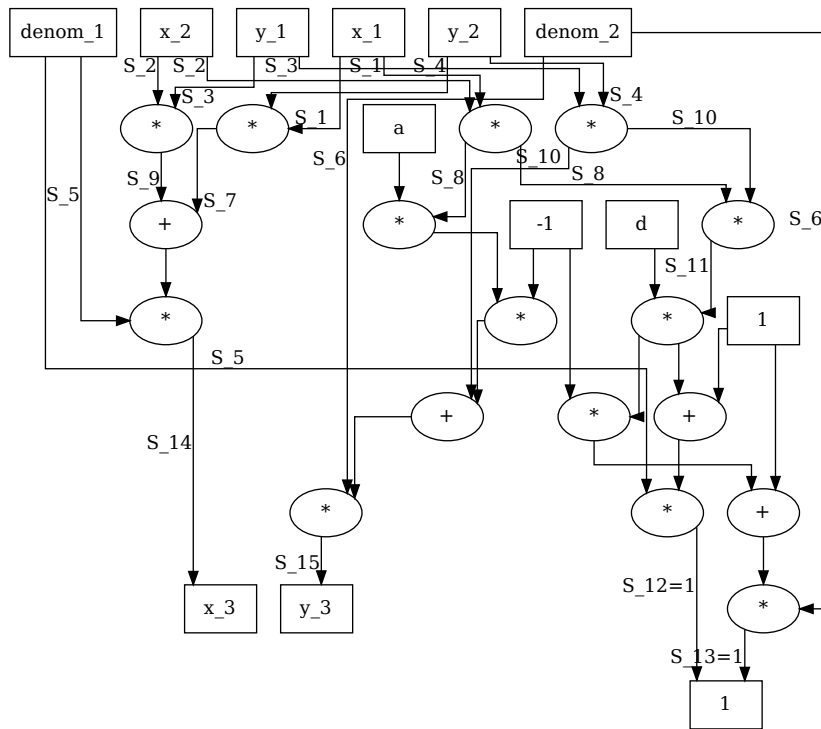
$$\begin{aligned}
 S_1 \cdot S_1 &= S_3 \\
 S_2 \cdot S_2 &= S_4 \\
 (S_4 \cdot 8) \cdot S_3 &= S_5 \\
 (12 \cdot S_4 + S_5 + 10 \cdot S_3 + 1) \cdot 1 &= 0
 \end{aligned}$$

Exercise 67. Write the circuit and associated rank-1 constraint system for a Weierstraß curve of a given field \mathbb{F} .

Twisted Edwards curve addition As we have seen in 5.1.3, a major advantage of twisted Edwards curves is the existence of an addition law that contains no branching and is valid for all curve points. Moreover, the neutral element is not “at infinity” but the actual curve point $(0, 1)$. In fact, given two points (x_1, y_1) and (x_2, y_2) on a twisted Edwards curve, their sum is defined as

$$(x_3, y_3) = \left(\frac{x_1 y_2 + y_1 x_2}{1 + d \cdot x_1 x_2 y_1 y_2}, \frac{y_1 y_2 - a \cdot x_1 x_2}{1 - d \cdot x_1 x_2 y_1 y_2} \right)$$

We can use the division circuit from XXX to flatten this equation into an algebraic circuit. Inputs to the circuit are then not only the two curve points (x_1, y_1) and (x_2, y_2) , but also the two denominators $denum_1 = 1 + d \cdot x_1 x_2 y_1 y_2$ as well as $denum_2 = 1 - d \cdot x_1 x_2 y_1 y_2$, which any prover needs to compute outside of the circuit. We get



Using the general technique from XXX to derive the associated rank-1 constraint system, we get the following result:

$$\begin{aligned} S_1 \cdot S_4 &= S_7 \\ S_1 \cdot S_2 &= S_8 \\ S_2 \cdot S_3 &= S_9 \\ S_3 \cdot S_4 &= S_{10} \\ S_8 \cdot S_{10} &= S_{11} \\ S_5 \cdot (1 + d \cdot S_{11}) &= 1 \\ S_6 \cdot (1 - d \cdot S_{11}) &= 1 \\ S_5 \cdot (S_9 + S_7) &= S_{14} \\ S_6 \cdot (S_{10} - a \cdot S_8) &= S_{15} \end{aligned}$$

5665 *Exercise 68.* Let \mathbb{F} be a field. Define a circuit that enforces field inversion for a point of a
5666 twisted Edwards curve over \mathbb{F} .

Chapter 8

Zero Knowledge Protocols

A so-called **zero-knowledge protocol** is a set of mathematical rules by which one party (usually called **the prover**) can convince another party (usually called **the verifier**) that a given statement is true, while not revealing any additional information apart from the truth of the statement.

As we have seen in chapter 6, given some language L and instance I , the knowledge claim “there is a witness W such that $(I; W)$ is a word in L ” is constructively provable by providing W to the verifier. However, the challenge for a zero-knowledge protocol is to prove knowledge of a witness without revealing any information beyond its bare existence.

In this chapter, we look at various systems that exist to solve this task. We start with an introduction to the basic concepts and terminology in zero-knowledge proving systems and then introduce the so-called Groth_16 protocol as one of the most efficient systems. We plan to update this chapter with new inventions in future versions of this book.

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8.1 Proof Systems

From an abstract point of view, a proof system is a set of rules which models the generation and exchange of messages between two parties: a prover and a verifier. Its task is to ascertain whether a given string belongs to a formal language or not.

Proof systems are often classified by certain trust assumptions and the computational capabilities of both parties. In its most general form, the prover usually possesses unlimited computational resources but cannot be trusted, while the verifier has bounded computation power but is assumed to be honest.

Proving the membership statement for some string is executed by the generation of certain messages that are sent between prover and verifier, until the verifier is convinced that the string is an element of the language in consideration.

To be more specific, let Σ be an alphabet, and let L be a formal language defined over Σ . Then a **proof system** for language L is a pair of probabilistic interactive algorithms (P, V) , where P is called the **prover** and V is called the **verifier**.

Both algorithms are able to send messages to one another, and each algorithm has its own state, some shared initial state and access to the messages. The verifier is bounded to a number of steps which is polynomial in the size of the shared initial state, after which it stops and outputs either `accept` or `reject` indicating that it accepts or rejects a given string to be in L . In contrast, there are bounds on the computational power of the prover.

When the execution of the verifier algorithm stops the following conditions are required to hold:

- (Completeness) If the tuple $x \in \Sigma^*$ is a word in language L and both prover and verifier follow the protocol, the verifier outputs `accept`.
- (Soundness) If the tuple $x \in \Sigma^*$ is not a word in language L and the verifier follows the protocol, the verifier outputs `reject`, except with some small probability.

In addition, a proof system is called **zero-knowledge** if the verifier learns nothing about x other than $x \in L$.

The previous definition of proof systems is very general, and many subclasses of proving systems are known in the field. The type of languages that a proof system can support crucially depends on the abilities of the verifier (for example, whether it can make random choices) or on the nature and number of the messages that can be exchanged. If the system only requires to send a single message from the prover to the verifier, the proof system is called **non-interactive**, because no interaction other than sending the actual proof is required. In contrast, any other proof system is called **interactive**.

A proof system is usually called **succinct** if the size of the proof is shorter than the witness necessary to generate the proof. Moreover, a proof system is called **computationally sound** if soundness only holds under the assumption that the computational capabilities of the prover are polynomially bound. To distinguish general proofs from computationally sound proofs, the latter are often called **arguments**. Zero-knowledge, succinct, non-interactive arguments of knowledge claims are often abbreviated **zk-SNARKs**.

Example 138 (Constructive Proofs for Algebraic Circuits). To formalize our previous notion of constructive proofs for algebraic circuits, let \mathbb{F} be a finite field, and let $C(\mathbb{F})$ be an algebraic circuit over \mathbb{F} with associated language $L_{C(\mathbb{F})}$. A non-interactive proof system for $L_{C(\mathbb{F})}$ is given by the following two algorithms:

Given some instance I , the prover algorithm P uses its unlimited computational power to compute a witness W such that the pair $(I; W)$ is a valid assignment to $C(\mathbb{F})$ whenever the circuit is satisfiable for I . The prover then sends the constructive proof $(I; W)$ to the verifier.

On receiving a message $(I; W)$, the verifier algorithm V assigns the constructive proof $(I; W)$ to circuit $C(\mathbb{F})$, and decides whether the assignment is valid by executing all gates in the circuit. The runtime is polynomial in the number of gates. If the assignment is valid, the verifier returns `accepts`, if not, it returns `reject`.

To see that this proof system has the completeness and soundness properties, let $C(\mathbb{F})$ be a circuit of the field \mathbb{F} , and let I be an instance. The circuit may or may not have a witness W such that $(I; W)$ is a valid assignment to $C(\mathbb{F})$.

If no W exists, I is not part of any word in $L_{C(\mathbb{F})}$, and there is no way for P to generate a valid assignment. It follows that the verifier will not accept any claimed proof sent by P , which implies that the system has **soundness**.

If, on the other hand, W exists and P is honest, P can use its unlimited computational power to compute W and send $(I; W)$ to V , which V will accept in polynomial time. This implies that the system has **completeness**.

The system is non-interactive because the prover only sends a single message to the verifier, which contains the proof itself. Because in this simple system the witness itself is the proof, the proof system is **not** succinct.

8.2 The “Groth16” Protocol

In chapter 6, we have introduced algebraic circuits, their associated rank-1 constraint systems and their induced quadratic arithmetic programs. These models define formal languages, and

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associated memberships and knowledge claims can be constructively proofed by executing the circuit to compute a solution to its associated R1CS. The solution can then be transformed into a polynomial such that the polynomial is divisible by another polynomial if and only if the solution is correct.

In [\[1\]](#), Jens Groth provides a method that can transform those proofs into zero-knowledge succinct non-interactive arguments of knowledge. Assuming that the pairing groups $(\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T, b)$ are given, the arguments are of constant size and consist of 2 elements from \mathbb{G}_1 and a single element from \mathbb{G}_2 , regardless of the size of the witness. They are zero-knowledge in the sense that the verifier learns nothing about the witness beside the fact that the instance-witness pair is a proper word in the language of the problem.

Verification is non-interactive and needs to compute a number of exponentiations proportional to the size of the instance, together with 3 group pairings in order to check a single equation.

The generated argument is perfectly zero-knowledge has perfect completeness and soundness in the generic bilinear group model, assuming the existence of a trusted third party that executes a preprocessing phase to generate a **common reference string** and a **simulation trapdoor**. This party must be trusted to delete the simulation trapdoor, since everyone in possession of it can simulate proofs.

To be more precise, let R be a rank-1 constraint system defined over some finite field \mathbb{F}_r . Then **Groth_16 parameters** for R are given by the following set:

$$\text{Groth_16} - \text{Param}(R) = (r, \mathbb{G}_1, \mathbb{G}_2, e(\cdot, \cdot), g_1, g_2) \quad (8.1)$$

In the equation above, \mathbb{G}_1 and \mathbb{G}_2 are finite cyclic groups of order r , g_1 is a generator of \mathbb{G}_1 , g_2 is a generator of \mathbb{G}_2 and $e : \mathbb{G}_1 \times \mathbb{G}_2 \rightarrow \mathbb{G}_T$ is a non-degenerate, bilinear pairing for some target group \mathbb{G}_T . In real-world applications, the parameter set is usually agreed on in advance.

Given some Groth_16 parameters, a **Groth_16 protocol** is then a quadruple of probabilistic polynomial algorithms (SETUP, PROVE, VFY, SIM) such that the following conditions hold:

- (Setup-Phase): $(CRS, \tau) \leftarrow \text{Setup}(R)$: Algorithm Setup takes the R1CS R as input and computes a common reference string CRS and a simulation trapdoor τ .
- (Prover-Phase): $\pi \leftarrow \text{Prove}(R, CRS, I, W)$: Given a constructive proof (I, W) for R , algorithm Prove takes the R1CS R , the common reference string CRS and the constructive proof (I, W) as input and computes an zk-SNARK π .
- Verify: $\{\text{accept}, \text{reject}\} \leftarrow \text{Vfy}(R, CRS, I, \pi)$: Algorithm Vfy takes the R1CS R , the common reference string CRS , the instance I and the zk-SNARK π as input and returns `reject` or `accept`.
- $\pi \leftarrow \text{Sim}(R, \tau, CRS, I)$: Algorithm Sim takes the R1CS R , the common reference string CRS , the simulation trapdoor τ and the instance I as input and returns a zk-SNARK π .

We will explain these algorithms together with detailed examples in the remainder of this section.

Assuming a trusted third party for the setup, the protocol is able to compute a zk-SNARK from a constructive proof for R , provided that r is sufficiently large, and, in particular, larger than the number of constraints in the associated R1CS.

Example 139 (The 3-Factorization Problem). Consider the 3-factorization problem from 109 and its associated algebraic circuit and rank-1 constraint system from 6.8. In this example,

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common
reference
string

simulation
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we want to agree on a parameter set $(R, r, \mathbb{G}_1, \mathbb{G}_2, e(\cdot, \cdot), g_1, g_2)$ in order to use the Groth_16 protocol for our 3-factorization problem.

To find proper parameters, first observe that the circuit XXX, as well as its associated R1CS R_{3, fac_zk} ex:3-factorization-r1cs and the derived QAP 6.14, are defined over the field \mathbb{F}_{13} . We therefore have $r = 13$ and need pairing groups \mathbb{G}_1 and \mathbb{G}_2 of order 13.

We know from 5.4 that the moon-math curve BLS6_6 has two subgroups $\mathbb{G}_1[13]$ and $\mathbb{G}_2[13]$, which are both of order 13. The associated Weil pairing b (5.45) is a proper bilinear map. We therefore choose those groups and the Weil pairing together with the generators $g_1 = (13, 15)$ and $g_2 = (7v^2, 16v^3)$ of $\mathbb{G}_1[13]$ and $\mathbb{G}_2[13]$, as a parameter:

$$\text{Groth_16} - \text{Param}(R_{3, fac_zk}) = (r, \mathbb{G}_1[13], \mathbb{G}_2[13], e(\cdot, \cdot), (13, 15), (7v^2, 16v^3))$$

It should be noted that our choice is not unique. Every pair of finite cyclic groups of order 13 that has a proper bilinear pairing qualifies as a Groth_16 parameter set. The situation is similar to real-world applications, where SNARKs with equivalent behavior are defined over different curves, used in different applications.

The Setup Phase To generate zk-SNARKs from constructive knowledge proofs in the Groth16 protocol, a preprocessing phase is required. This has to be executed a single time for every rank-1 constraint system and any associated quadratic arithmetic program. The outcome of this phase is a common reference string that prover and verifier need in order to generate and verify the zk-SNARK. In addition, a simulation trapdoor is produced that can be used to simulate proofs.

To be more precise, let L be a language defined by some rank-1 constraint system R such that a constructive proof of knowledge for an instance (I_1, \dots, I_n) in L consists of a witness (W_1, \dots, W_m) . Let $QAP(R) = \{T \in \mathbb{F}[x], \{A_j, B_j, C_j \in \mathbb{F}[x]\}_{j=0}^{n+m}\}$ be a quadratic arithmetic program associated to R , and let $\{\mathbb{G}_1, \mathbb{G}_2, e(\cdot, \cdot), g_1, g_2, \mathbb{F}_r\}$ be the set of Groth_16 parameters.

The setup phase then samples 5 random, **inverible** elements $\alpha, \beta, \gamma, \delta$ and s from the scalar field \mathbb{F}_r of the protocol and outputs the **simulation trapdoor** τ :

$$\tau = (\alpha, \beta, \gamma, \delta, s) \quad (8.2)$$

In addition, the setup phase uses those 5 random elements together with the two generators g_1 and g_2 and the quadratic arithmetic program to generate a **common reference string** $CRS_{QAP} = (CRS_{\mathbb{G}_1}, CRS_{\mathbb{G}_2})$ of language L :

$$CRS_{\mathbb{G}_1} = \left\{ \begin{array}{l} g_1^\alpha, g_1^\beta, g_1^\delta, \left(g_1^{s^j}, \dots \right)_{j=0}^{deg(T)-1}, \left(g_1^{\frac{\beta \cdot A_j(s) + \alpha \cdot B_j(s) + C_j(s)}{\gamma}}, \dots \right)_{j=0}^n \\ \left(g_1^{\frac{\beta \cdot A_{j+n}(s) + \alpha \cdot B_{j+n}(s) + C_{j+n}(s)}{\delta}}, \dots \right)_{j=1}^m, \left(g_1^{\frac{s^j \cdot T(s)}{\delta}}, \dots \right)_{j=0}^{deg(T)-2} \end{array} \right\}$$

$$CRS_{\mathbb{G}_2} = \left\{ g_2^\beta, g_2^\gamma, g_2^\delta, \left(g_2^{s^j}, \dots \right)_{j=0}^{deg(T)-1} \right\}$$

Common reference strings depend on the simulation trapdoor, and are therefore not unique to the problem. Any language can have more than one common reference string. The size of a common reference string is linear in the size of the instance and the size of the witness.

If a simulation trapdoor $\tau = (\alpha, \beta, \gamma, \delta, s)$ is given, we call the element s a **secret evaluation point** of the protocol, because if \mathbb{F}_r is the scalar field of the finite cyclic groups \mathbb{G}_1 and \mathbb{G}_2 , then

5813 a key feature of any common reference string is that it provides data to compute the evaluation
 5814 of any polynomial $P \in \mathbb{F}_r[x]$ of degree $\deg(P) < \deg(T)$ at the point s in the exponent of the
 5815 generator g_1 or g_2 , without knowing s .

To be more precise, let s be the secret evaluation point and let $P(x) = a_0 \cdot x^0 + a_1 \cdot x^1 + \dots + a_k \cdot x^k$ be a polynomial of degree $k < \deg(T)$ with coefficients in \mathbb{F}_r . Then we can compute $g_1^{P(s)}$ without knowing what the actual value of s is:

$$\begin{aligned} g_1^{P(s)} &= g_1^{a_0 \cdot s^0 + a_1 \cdot s^1 + \dots + a_k \cdot s^k} \\ &= g_1^{a_0 \cdot s^0} \cdot g_1^{a_1 \cdot s^1} \cdot \dots \cdot g_1^{a_k \cdot s^k} \\ &= \left(g_1^{s^0}\right)^{a_0} \cdot \left(g_1^{s^1}\right)^{a_1} \cdot \dots \cdot \left(g_1^{s^k}\right)^{a_k} \end{aligned}$$

5816 In this expression, all group points $g_1^{s^j}$ are part of the common reference string, hence, they can
 5817 be used to compute the result. The same holds true for the evaluation of $g_2^{P(s)}$, since the \mathbb{G}_2 part
 5818 of the common reference string contains the points $g_2^{s^j}$.

5819 In real-world applications, the simulation trapdoor is often called the **toxic waste** of the
 5820 setup-phase, while a common reference string is also-called the pair of **prover and verifier**
 5821 **key**.

explain
why

5822 In order to make the protocol secure, the setup needs to be executed in a way that guarantees
 5823 that the simulation trapdoor is deleted. Anyone in possession of it can generate arguments
 5824 without knowledge of a constructive proof. The most simple approach to achieving deletion of
 5825 the toxic waste is by a so-called **trusted third party**, where the trust assumption is that that
 5826 the party generates the common reference string precisely as defined and deletes the simulation
 5827 backdoor afterwards.

5828 However, as trusted third parties are not easy to find, more sophisticated protocols exist
 5829 in real-world applications. They execute the setup phase as a multi party computation, where
 5830 the proper execution can be publicly verified and the simulation trapdoor is deleted if at least
 5831 one participant deletes their individual contribution to the randomness. Each participant only
 5832 possesses a fraction of the simulation trapdoor, so it can only be recovered if all participants
 5833 collude and share their fraction.

Example 140 (The 3-factorization Problem). To see how the setup phase of a Groth_16 zk-SNARK can be computed, consider the 3-factorization problem from 109 and the parameters from page 194. As we have seen in 6.14, an associated quadratic arithmetic program is given as follows:

$$\begin{aligned} QAP(R_{3, fac_zk}) &= \{x^2 + x + 9, \\ &\quad \{0, 0, 6x + 10, 0, 0, 7x + 4\}, \{0, 0, 0, 6x + 10, 7x + 4, 0\}, \{0, 7x + 4, 0, 0, 0, 6x + 10\}\} \end{aligned}$$

To transform this QAP into a common reference string, we choose the field elements $\alpha = 6$, $\beta = 5$, $\gamma = 4$, $\delta = 3$, $s = 2$ from \mathbb{F}_{13} . In real-world applications, it is important to sample those values randomly from the scalar field, but in our approach, we choose those non-random values to make them more memorable, which helps in pen-and-paper computations. Our simulation trapdoor is then given as follows:

$$\tau = (6, 5, 4, 3, 2)$$

5834 We keep this secret in order to simulate proofs later on, but we are careful though to hide τ
 5835 from anyone who hasn't read this book. Then we instantiate the common reference string XXX

4 ex-
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Change it
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5836 from those values. Since our groups are subgroups of the BLS_{6_6} elliptic curve, we use scalar
 5837 product notation instead of exponentiation.

To compute the \mathbb{G}_1 part of the common reference string, we use the logarithmic order of the group \mathbb{G}_1 XXX, the generator $g_1 = (13, 15)$, as well as the values from the simulation backdoor. Since $\deg(T) = 2$, we get the following:

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$$\begin{aligned} [\alpha]_{g_1} &= [6](13, 15) = (27, 34) \\ [\beta]_{g_1} &= [5](13, 15) = (26, 34) \\ [\delta]_{g_1} &= [3](13, 15) = (38, 15) \end{aligned}$$

To compute the rest of the \mathbb{G}_1 part of the common reference string, we expand the indexed tuples and insert the secret random elements from the simulation backdoor. We get the following:

$$\begin{aligned} \left([s^j]_{g_1}, \dots \right)_{j=0}^1 &= \left([2^0](13, 15), [2^1](13, 15) \right) \\ &= \left((13, 15), (33, 34) \right) \\ \left(\left[\frac{\beta A_j(s) + \alpha B_j(s) + C_j(s)}{\gamma} \right]_{g_1}, \dots \right)_{j=0}^1 &= \left(\left[\frac{5A_0(2) + 6B_0(2) + C_0(2)}{4} \right](13, 15), \right. \\ &\quad \left. \left[\frac{5A_1(2) + 6B_1(2) + C_1(2)}{4} \right](13, 15) \right) \\ \left(\left[\frac{\beta A_{j+n}(s) + \alpha B_{j+n}(s) + C_{j+n}(s)}{\delta} \right]_{g_1}, \dots \right)_{j=1}^4 &= \left(\left[\frac{5A_2(2) + 6B_2(2) + C_2(2)}{3} \right](13, 15), \right. \\ &\quad \left[\frac{5A_3(2) + 6B_3(2) + C_3(2)}{3} \right](13, 15), \\ &\quad \left[\frac{5A_4(2) + 6B_4(2) + C_4(2)}{3} \right](13, 15), \\ &\quad \left. \left[\frac{5A_5(2) + 6B_5(2) + C_6(2)}{3} \right](13, 15) \right) \\ \left(\left[\frac{s^j \cdot T(s)}{\delta} \right]_{g_1} \right)_{j=0}^0 &= \left(\left[\frac{2^0 \cdot T(2)}{3} \right](13, 15) \right) \end{aligned}$$

To compute the curve points on the right side of these expressions, we need the polynomials from the associated quadratic arithmetic program and evaluate them on the secret point $s = 2$.

Since $4^{-1} = 10$ and $3^{-1} = 9$ in \mathbb{F}_{13} , we get the following:

$$\left\lfloor \frac{5A_0(2) + 6B_0(2) + C_0(2)}{4} \right\rfloor(13, 15) = [(5 \cdot 0 + 6 \cdot 0 + 0) \cdot 10](13, 15) = [0](13, 15) =$$

\emptyset

$$\left\lfloor \frac{5A_1(2) + 6B_1(2) + C_1(2)}{4} \right\rfloor(13, 15) = [(5 \cdot 0 + 6 \cdot 0 + (7 \cdot 2 + 4)) \cdot 10](13, 15) = [11](13, 15) =$$

(33, 9)

$$\left\lfloor \frac{5A_2(2) + 6B_2(2) + C_2(2)}{3} \right\rfloor(13, 15) = [(5 \cdot (6 \cdot 2 + 10) + 6 \cdot 0 + 0) \cdot 9](13, 15) = [2](13, 15) =$$

(33, 34)

$$\left\lfloor \frac{5A_3(2) + 6B_3(2) + C_3(2)}{3} \right\rfloor(13, 15) = [(5 \cdot 0 + 6 \cdot (6 \cdot 2 + 10) + 0) \cdot 9](13, 15) = [5](13, 15) =$$

(26, 34)

$$\left\lfloor \frac{5A_4(2) + 6B_4(2) + C_4(2)}{3} \right\rfloor(13, 15) = [(5 \cdot 0 + 6 \cdot (7 \cdot 2 + 4) + 0) \cdot 9](13, 15) = [10](13, 15) =$$

(38, 28)

$$\left\lfloor \frac{5A_5(2) + 6B_5(2) + C_5(2)}{3} \right\rfloor(13, 15) = [(5 \cdot (7 \cdot 2 + 4) + 6 \cdot 0 + 0) \cdot 9](13, 15) = [4](13, 15) =$$

(35, 28)

$$\left\lfloor \frac{2^0 \cdot T(2)}{3} \right\rfloor(13, 15) = [1 \cdot (2^2 + 2 + 9) \cdot 9](13, 15) = [5](13, 15) =$$

(26, 34)

Putting all those values together, we see that the \mathbb{G}_1 part of the common reference string is given by the following set of 12 points from the BLS6_6 13-torsion group \mathbb{G}_1 :

$$CRS_{\mathbb{G}_1} = \left\{ \begin{array}{l} (27, 34), (26, 34), (38, 15), \left((13, 15), (33, 34) \right), \left(\emptyset, (33, 9) \right) \\ \left((33, 34), (26, 34), (38, 28), (35, 28) \right), \left((26, 34) \right) \end{array} \right\}$$

To compute the \mathbb{G}_2 part of the common reference string, we use the logarithmic order of the group \mathbb{G}_2 XXX, the generator $g_2 = (7v^2, 16v^3)$, as well as the values from the simulation backdoor. Since $\deg(T) = 2$, we get the following:

$$[\beta]g_2 = [5](7v^2, 16v^3) = (16v^2, 28v^3)$$

$$[\gamma]g_2 = [4](7v^2, 16v^3) = (37v^2, 27v^3)$$

$$[\delta]g_2 = [3](7v^2, 16v^3) = (42v^2, 16v^3)$$

To compute the rest of the \mathbb{G}_2 part of the common reference string, we expand the indexed tuple and insert the secret random elements from the simulation backdoor. We get the following:

$$\begin{aligned} \left([s^j]g_2, \dots \right)_{j=0}^1 &= ([2^0](7v^2, 16v^3), [2^1](7v^2, 16v^3)) \\ &= ((7v^2, 16v^3), (10v^2, 28v^3)) \end{aligned}$$

Putting all these values together, we see that the \mathbb{G}_2 part of the common reference string is given by the following set of 5 points from the BLS6_6 13-torsion group \mathbb{G}_2 :

$$CRS_{\mathbb{G}_2} = \left\{ (16v^2, 28v^3), (37v^2, 27v^3), (42v^2, 16v^3), (7v^2, 16v^3), (10v^2, 28v^3) \right\}$$

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Given the simulation trapdoor τ and the quadratic arithmetic program 6.14, the associated common reference string of the 3-factorization problem is as follows:

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$$\begin{aligned} CRS_{\mathbb{G}_1} &= \left\{ (27, 34), (26, 34), (38, 15), \left((13, 15), (33, 34) \right), \left(\mathcal{O}, (33, 9) \right) \right\} \\ &\quad \left\{ \left((33, 34), (26, 34), (38, 28), (35, 28) \right), \left((26, 34) \right) \right\} \\ CRS_{\mathbb{G}_2} &= \left\{ (16v^2, 28v^3), (37v^2, 27v^3), (42v^2, 16v^3), \left(7v^2, 16v^3 \right), (10v^2, 28v^3) \right\} \end{aligned}$$

We then publish this data to everyone who wants to participate in the generation of a zk-SNARK or its verification in the 3-factorization problem.

To understand how this common reference string can be used to evaluate polynomials at the secret evaluation point in the exponent of a generator, let's assume that we have deleted the simulation trapdoor. In that case, we have no way to know the secret evaluation point anymore, hence, we cannot evaluate polynomials at that point. However, we can evaluate polynomials of smaller degree than the degree of the target polynomial in the exponent of both generators at that point.

To see that, consider e.g. the polynomials $A_2(x) = 6x + 10$ and $A_5(x) = 7x + 4$ from the QAP of this problem. To evaluate these polynomials in the exponent of g_1 and g_2 at the secret point s without knowing the value of s (which is 2), we can use the common reference string and equation XXX. Using the scalar product notation instead of exponentiation, we get the following:

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$$\begin{aligned} [A_2(s)]g_1 &= [6 \cdot s^1 + 10 \cdot s^0]g_1 \\ &= [6](33, 34) + [10](13, 15) & \# [s^0]g_1 = (13, 15), [s^1]g_1 = (33, 34) \\ &= [6 \cdot 2](13, 15) + [10](13, 15) = [9](13, 15) & \# \text{logarithmic order on } \mathbb{G}_1 \\ &= (35, 15) \\ [A_5(s)]g_1 &= [7 \cdot s^1 + 4 \cdot s^0]g_1 \\ &= [7](33, 34) + [4](13, 15) \\ &= [7 \cdot 2](13, 15) + [4](13, 15) = [5](13, 15) \\ &= (26, 34) \end{aligned}$$

Indeed, we are able to evaluate the polynomials in the exponent at a secret evaluation point, because that point is encrypted in the curve point $(33, 34)$ and its secrecy is protected by the discrete logarithm assumption. Of course, in our computation, we recovered the secret point $s = 2$, but that was only possible because we have a group of logarithmic order in order to simplify our pen-and-paper computations. Such an order is infeasible for computing in cryptographically secure curves. We can do the same computation on \mathbb{G}_2 and get the following:

$$\begin{aligned} [A_2(s)]g_2 &= [6 \cdot s^1 + 10 \cdot s^0]g_2 \\ &= [6](10v^2, 28v^3) + [10](7v^2, 16v^3) \\ &= [6 \cdot 2](7v^2, 16v^3) + [10](7v^2, 16v^3) = [9](7v^2, 16v^3) \\ &= (37v^2, 16v^3) \\ [A_5(s)]g_2 &= [7 \cdot s^1 + 4 \cdot s^0]g_2 \\ &= [7](10v^2, 28v^3) + [4](7v^2, 16v^3) \\ &= [7 \cdot 2](7v^2, 16v^3) + [4](7v^2, 16v^3) = [5](7v^2, 16v^3) \\ &= (16v^2, 28v^3) \end{aligned}$$

Apart from the target polynomial T , all other polynomials of the quadratic arithmetic program can be evaluated in the exponent this way.

The Prover Phase Given some rank-1 constraint system R and instance $I = (I_1, \dots, I_n)$, the task of the prover phase is to convince any verifier that a prover knows a witness W to instance I such that $(I; W)$ is a word in the language L_R of the system, without revealing anything about W .

To achieve this in the Groth_16 protocol, we assume that any prover has access to the rank-1 constraint system of the problem, in addition to some algorithm that tells the prover how to compute constructive proofs for the R1CS. In addition, the prover has access to a common reference string and its associated quadratic arithmetic program.

In order to generate a zk-SNARK for this instance, the prover first computes a valid constructive proof as explained in XXX, that is, the prover generates a proper witness $W = (W_1, \dots, W_m)$ such that $(I_1, \dots, I_n; W_1, \dots, W_m)$ is a solution to the rank-1 constraint system R .

The prover then uses the quadratic arithmetic program and computes the polynomial $P_{(I;W)}$, as explained in 6.15. They then divide $P_{(I;W)}$ by the target polynomial T of the quadratic arithmetic program. Since $P_{(I;W)}$ is constructed from a valid solution to the R1CS, we know from 6.15 that it is divisible by T . This implies that polynomial division of P by T generates another polynomial $H := P/T$, with $\deg(H) < \deg(T)$.

The prover then evaluates the polynomial $(H \cdot T)\delta^{-1}$ in the exponent of the generator g_1 at the secret point s , as explained in XXX. To see how this can be achieved, let $H(x)$ be the quotient polynomial P/T :

$$H(x) = H_0 \cdot x^0 + H_1 \cdot x^1 + \dots + H_k \cdot x^k \quad (8.3)$$

To evaluate $H \cdot T$ at s in the exponent of g_1 , the prover uses the common reference string and computes as follows:

$$g_1^{\frac{H(s) \cdot T(s)}{\delta}} = \left(g_1^{\frac{s^0 \cdot T(s)}{\delta}}\right)^{H_0} \cdot \left(g_1^{\frac{s^1 \cdot T(s)}{\delta}}\right)^{H_1} \dots \left(g_1^{\frac{s^k \cdot T(s)}{\delta}}\right)^{H_k}$$

After this has been done, the prover samples two random field elements $r, t \in \mathbb{F}_r$, and uses the common reference string, the instance variables I_1, \dots, I_n and the witness variables W_1, \dots, W_m to compute the following curve points:

$$\begin{aligned} g_1^W &= \left(g_1^{\frac{\beta \cdot A_{1+n}(s) + \alpha \cdot B_{1+n}(s) + C_{1+n}(s)}{\delta}}\right)^{W_1} \dots \left(g_1^{\frac{\beta \cdot A_{m+n}(s) + \alpha \cdot B_{m+n}(s) + C_{m+n}(s)}{\delta}}\right)^{W_m} \\ g_1^A &= g_1^\alpha \cdot g_1^{A_0(s)} \cdot \left(g_1^{A_1(s)}\right)^{I_1} \dots \left(g_1^{A_n(s)}\right)^{I_n} \cdot \left(g_1^{A_{n+1}(s)}\right)^{W_1} \dots \left(g_1^{A_{n+m}(s)}\right)^{W_m} \cdot \left(g_1^\delta\right)^r \\ g_1^B &= g_1^\beta \cdot g_1^{B_0(s)} \cdot \left(g_1^{B_1(s)}\right)^{I_1} \dots \left(g_1^{B_n(s)}\right)^{I_n} \cdot \left(g_1^{B_{n+1}(s)}\right)^{W_1} \dots \left(g_1^{B_{n+m}(s)}\right)^{W_m} \cdot \left(g_1^\delta\right)^t \\ g_2^B &= g_2^\beta \cdot g_2^{B_0(s)} \cdot \left(g_2^{B_1(s)}\right)^{I_1} \dots \left(g_2^{B_n(s)}\right)^{I_n} \cdot \left(g_2^{B_{n+1}(s)}\right)^{W_1} \dots \left(g_2^{B_{n+m}(s)}\right)^{W_m} \cdot \left(g_2^\delta\right)^t \\ g_1^C &= g_1^W \cdot g_1^{\frac{H(s) \cdot T(s)}{\delta}} \cdot \left(g_1^A\right)^r \cdot \left(g_1^B\right)^t \cdot \left(g_1^\delta\right)^{-r \cdot t} \end{aligned}$$

In this computation, the group elements $g_1^{A_j(s)}$, $g_1^{B_j(s)}$ and $g_2^{B_j(s)}$ can be derived from the common reference string and the quadratic arithmetic program of the problem, as we have seen in XXX. In fact, those points only have to be computed once, and can be published and reused

for multiple proof generations because they are the same for all instances and witnesses. All other group elements are part of the common reference string.

After all these computations have been done, a valid zero-knowledge succinct non-interactive argument of knowledge π in the Groth_16 protocol is given by the following three curve points:

$$\pi = (g_1^A, g_1^C, g_2^B) \quad (8.4)$$

As we can see, a Groth_16 zk-SNARK consists of 3 curve points, two points from \mathbb{G}_1 and 1 point from \mathbb{G}_2 . The argument is specifically designed this way because, in typical applications, \mathbb{G}_1 is a torsion group of an elliptic curve over some prime field, while \mathbb{G}_2 is a subgroup of a torsion group over an extension field. Elements from \mathbb{G}_1 therefore need less space to be stored, and computations in \mathbb{G}_1 are typically faster than in \mathbb{G}_2 .

Since the witness is encoded in the exponent of a generator of a cryptographically secure elliptic curve, it is hidden from anyone but the prover. Moreover, since any proof is randomized by the occurrence of the random field elements r and t , proofs are not unique to any given witness. This is an important feature because, if all proofs for the same witness would be the same, knowledge of a witness would destroy the zero-knowledge property of those proofs.

Example 141 (The 3-factorization Problem). To see how a prover might compute a zk-SNARK, consider the 3-factorization problem from 109, our protocol parameters from XXX as well as the common reference string from XXX.

Our task is to compute a zk-SNARK for the instance $I_1 = 11$ and its constructive proof $(W_1, W_2, W_3, W_4) = (2, 3, 4, 6)$ as computed in XXX. As we know from 6.15, the associated polynomial $P_{(I;W)}$ of the quadratic arithmetic program from XXX is given by the following equation:

$$P_{(I;W)} = x^2 + x + 9$$

Since $P_{(I;W)}$ is identical to the target polynomial $T(x) = x^2 + x + 9$ in this example, we know from XXX that the quotient polynomial $H = P/T$ is the constant degree 0 polynomial:

$$H(x) = H_0 \cdot x^0 = 1 \cdot x^0$$

We therefore use $[\frac{s^0 \cdot T(s)}{\delta}]_{g_1} = (26, 34)$ from our common reference string XXX of the 3-factorization problem and compute as follows:

$$\begin{aligned} [\frac{H(s) \cdot T(s)}{\delta}]_{g_1} &= [H_0](26, 34) = [1](26, 34) \\ &= (26, 34) \end{aligned}$$

In the next step, we have to compute all group elements required for a proper Groth16 zk-SNARK. We start with g_1^W . Using scalar products instead of the exponential notation, and \oplus for the group law on the BLS6_6 curve, we have to compute the point $[W]g_1$:

$$\begin{aligned} [W]g_1 &= [W_1]g_1 \frac{\beta \cdot A_2(s) + \alpha \cdot B_2(s) + C_2(s)}{\delta} \oplus [W_2]g_1 \frac{\beta \cdot A_3(s) + \alpha \cdot B_3(s) + C_3(s)}{\delta} \oplus [W_3]g_1 \frac{\beta \cdot A_4(s) + \alpha \cdot B_4(s) + C_4(s)}{\delta} \\ &\quad \oplus [W_4]g_1 \frac{\beta \cdot A_5(s) + \alpha \cdot B_5(s) + C_5(s)}{\delta} \end{aligned}$$

To compute this point, we have to remember that a prover should not be in possession of the simulation trapdoor, hence, they do not know what α , β , δ and s are. In order to compute this group element, the prover therefore needs the common reference string. Using the logarithmic order from XXX and the witness, we get the following:

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$$\begin{aligned}
[W]g_1 &= [2](33, 34) \oplus [3](26, 34) \oplus [4](38, 28) \oplus [6](35, 28) \\
&= [2 \cdot 2](13, 15) \oplus [3 \cdot 5](13, 15) \oplus [4 \cdot 10](13, 15) \oplus [6 \cdot 4](13, 15) \\
&= [2 \cdot 2 + 3 \cdot 5 + 4 \cdot 10 + 6 \cdot 4](13, 15) = [5](13, 15) \\
&= (26, 34)
\end{aligned}$$

5891 In a next step, we compute g_1^A . We sample the random point $r = 11$ from \mathbb{F}_{13} , using scalar
5892 products instead of the exponential notation, and \oplus for the group law on the BLS6_6 curve.
5893 We then have to compute the following expression:

$$\begin{aligned}
[A]g_1 &= [\alpha]g_1 \oplus [A_0(s)]g_1 \oplus [I_1][A_1(s)]g_1 \oplus [W_1][A_2(s)]g_1 \oplus [W_2][A_3(s)]g_1 \\
&\quad \oplus [W_3][A_4(s)]g_1 \oplus [W_4][A_5(s)]g_1 \oplus [r][\delta]g_1
\end{aligned}$$

Since we don't know what α , δ and s are, we look up $[\alpha]g_1$ and $[\delta]g_1$ from the common reference string. Recall from XXX that we can evaluate $[A_j(s)]g_1$ without knowing the secret evaluation point s . According to XXX, we have $[A_2(s)]g_1 = (35, 15)$, $[A_5(s)]g_1 = (26, 34)$ and $[A_j(s)]g_1 = \mathcal{O}$ for all other indices $0 \leq j \leq 5$. Since \mathcal{O} is the neutral element on \mathbb{G}_1 , we get the following:

$$\begin{aligned}
[A]g_1 &= (27, 34) \oplus \mathcal{O} \oplus [11]\mathcal{O} \oplus [2](35, 15) \oplus [3]\mathcal{O} \oplus [4]\mathcal{O} \oplus [6](26, 34) \oplus [11](38, 15) \\
&= (27, 34) \oplus [2](35, 15) \oplus [6](26, 34) \oplus [11](38, 15) \\
&= [6](13, 15) \oplus [2 \cdot 9](13, 15) \oplus [6 \cdot 5](13, 15) \oplus [11 \cdot 3](13, 15) \\
&= [6 + 2 \cdot 9 + 6 \cdot 5 + 11 \cdot 3](13, 15) = [9](13, 15) \\
&= (35, 15)
\end{aligned}$$

5894 In order to compute the two curve points $[B]g_1$ and $[B]g_2$, we sample another random element
5895 $t = 4$ from \mathbb{F}_{13} . Using the scalar product instead of the exponential notation, and \oplus for the group
5896 law on the BLS6_6 curve, we have to compute the following expressions:

$$\begin{aligned}
[B]g_1 &= [\beta]g_1 \oplus [B_0(s)]g_1 \oplus [I_1][B_1(s)]g_1 \oplus [W_1][B_2(s)]g_1 \oplus [W_2][B_3(s)]g_1 \\
&\quad \oplus [W_3][B_4(s)]g_1 \oplus [W_4][B_5(s)]g_1 \oplus [t][\delta]g_1 \\
[B]g_2 &= [\beta]g_2 \oplus [B_0(s)]g_2 \oplus [I_1][B_1(s)]g_2 \oplus [W_1][B_2(s)]g_2 \oplus [W_2][B_3(s)]g_2 \\
&\quad \oplus [W_3][B_4(s)]g_2 \oplus [W_4][B_5(s)]g_2 \oplus [t][\delta]g_2
\end{aligned}$$

Since we don't know what β , δ and s are, we look up the associated group elements from the common reference string. Recall from XXX that we can evaluate $[B_j(s)]g_1$ without knowing the secret evaluation point s . Since $B_3 = A_2$ and $B_4 = A_5$, we have $[B_3(s)]g_1 = (35, 15)$, $[B_4(s)]g_1 = (26, 34)$ according to XXX, and $[B_j(s)]g_1 = \mathcal{O}$ for all other indices $0 < j < 5$. Since \mathcal{O} is the neutral element on \mathbb{G}_1 , we get the following:

$$\begin{aligned}
[B]g_1 &= (26, 34) \oplus \mathcal{O} \oplus [11]\mathcal{O} \oplus [2]\mathcal{O} \oplus [3](35, 15) \oplus [4](26, 34) \oplus [6]\mathcal{O} \oplus [4](38, 15) \\
&= (26, 34) \oplus [3](35, 15) \oplus [4](26, 34) \oplus [4](38, 15) \\
&= [5](13, 15) \oplus [3 \cdot 9](13, 15) \oplus [4 \cdot 5](13, 15) \oplus [4 \cdot 3](13, 15) \\
&= [5 + 3 \cdot 9 + 4 \cdot 5 + 4 \cdot 3](13, 15) = [12](13, 15) \\
&= (13, 28)
\end{aligned}$$

$$\begin{aligned}
 [B]g_2 &= (16v^2, 28v^3) \oplus \mathcal{O} \oplus [11]\mathcal{O} \oplus [2]\mathcal{O} \oplus [3](37v^2, 16v^3) \oplus [4](16v^2, 28v^3) \oplus [6]\mathcal{O} \oplus [4](42v^2, 16v^3) \\
 &= (16v^2, 28v^3) \oplus [3](37v^2, 16v^3) \oplus [4](16v^2, 28v^3) \oplus [4](42v^2, 16v^3) \\
 &= [5](7v^2, 16v^3) \oplus [3 \cdot 9](7v^2, 16v^3) \oplus [4 \cdot 5](7v^2, 16v^3) \oplus [4 \cdot 3](7v^2, 16v^3) \\
 &= [5 + 3 \cdot 9 + 4 \cdot 5 + 4 \cdot 3](7v^2, 16v^3) = [12](7v^2 + 16v^3) \\
 &= (7v^2, 27v^3)
 \end{aligned}$$

In a last step, we combine the previous computations to compute the point $[C]g_1$ in the group \mathbb{G}_1 as follows:

$$\begin{aligned}
 [C]g_1 &= [W]g_1 \oplus \left[\frac{H(s) \cdot T(s)}{\delta} \right]g_1 \oplus [t][A]g_1 \oplus [r][B]g_1 \oplus [-r \cdot t][\delta]g_1 \\
 &= (26, 34) \oplus (26, 34) \oplus [4](35, 15) \oplus [11](13, 28) \oplus [-11 \cdot 4](38, 15) \\
 &= [5](13, 15) \oplus [5](13, 15) \oplus [4 \cdot 9](13, 15) \oplus [11 \cdot 12](13, 15) \oplus [-11 \cdot 4 \cdot 3](13, 15) \\
 &= [5 + 5 + 4 \cdot 9 + 11 \cdot 12 - 11 \cdot 4 \cdot 3](13, 15) = [7](13, 15) \\
 &= (27, 9)
 \end{aligned}$$

Given the instance $I_1 = 11$, we can now combine these computations and see that the following 3 curve points are a zk-SNARK for the witness $(W_1, W_2, W_3, W_4) = (2, 3, 4, 6)$:

$$\pi = ((35, 15), (27, 9), (7v^2, 27v^3))$$

5897 We can now publish this zk-SNARK, or send it to a designated verifier. Note that, if we had
 5898 sampled different values for r and t , we would have computed a different SNARK for the same
 5899 witness. The SNARK, therefore, hides the witness perfectly, which means that it is impossible
 5900 to reconstruct the witness from the SNARK.

5901 **The Verification Phase** Given some rank-1 constraint system R , instance $I = (I_1, \dots, I_n)$ and
 5902 zk-SNARK π , the task of the verification phase is to check that π is indeed an argument for
 5903 a constructive proof. Assuming that the simulation trapdoor does not exist anymore and the
 5904 verification checks the proof, the verifier can be convinced that someone knows a witness $W =$
 5905 (W_1, \dots, W_m) such that $(I; W)$ is a word in the language of R .

To achieve this in the Groth16 protocol, we assume that any verifier is able to compute the pairing map $e(\cdot, \cdot)$ efficiently, and has access to the common reference string used to produce the SNARK π . In order to verify the SNARK with respect to the instance (I_1, \dots, I_n) , the verifier computes the following curve point:

$$g_1^I = \left(g_1^{\frac{\beta \cdot A_0(s) + \alpha \cdot B_0(s) + C_0(s)}{\gamma}} \right) \cdot \left(g_1^{\frac{\beta \cdot A_1(s) + \alpha \cdot B_1(s) + C_1(s)}{\gamma}} \right)^{I_1} \cdots \left(g_1^{\frac{\beta \cdot A_n(s) + \alpha \cdot B_n(s) + C_n(s)}{\gamma}} \right)^{I_n}$$

5906 With this group element, the verifier is able to verify the SNARK $\pi = (g_1^A, g_1^C, g_2^B)$ by checking
 5907 the following equation using the pairing map:

$$e(g_1^A, g_2^B) = e(g_1^\alpha, g_2^\beta) \cdot e(g_1^I, g_2^\gamma) \cdot e(g_1^C, g_2^\delta) \quad (8.5)$$

5908 If the equation holds true, the SNARK is accepted. If the equation does not hold, the
 5909 SNARK is rejected.

Remark 9. We know from chapter 5 that computing pairings in cryptographically secure pairing groups is computationally expensive. As we can see, in the Groth16 protocol, 3 pairings are required to verify the SNARK, because the pairing $e(g_1^\alpha, g_2^\beta)$ is independent of the proof, meaning that can be computed once and then stored as an amendment to the verifier key.

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In [?], the author showed that 2 is the minimal amount of pairings that any protocol with similar properties has to use. This protocol is therefore close to the theoretical minimum. In the same paper, the author outlined an adaptation that only uses 2 pairings. However, that reduction comes with the price of much more overhead computation. Having 3 pairings is therefore a compromise that gives the overall best performance. To date, the Groth16 protocol is the most efficient in its class.

Example 142 (The 3-factorization Problem). To see how a verifier might check a zk-SNARK for some given instance I , consider the 3-factorization problem from 109, our protocol parameters from XXX, the common reference string from XXX as well as the zk-SNARK $\pi = ((35, 15), (27, 9), (7v^2, 27v^3))$, which claims to be an argument of knowledge for a witness for the instance $I_1 = 11$.

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In order to verify the zk-SNARK for that instance, we first compute the curve point g_1^I . Using scalar products instead of the exponential notation, and \oplus for the group law on the BLS6_6 curve, we have to compute the point $[I]g_1$ as follows:

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$$[I]g_1 = \left[\frac{\beta \cdot A_0(s) + \alpha \cdot B_0(s) + C_0(s)}{\gamma} \right]_{g_1} \oplus [I_1] \left[\frac{\beta \cdot A_1(s) + \alpha \cdot B_1(s) + C_1(s)}{\gamma} \right]_{g_1}$$

To compute this point, we have to remember that a verifier should not be in possession of the simulation trapdoor, which means that they do not know what α , β , γ and s are. In order to compute this group element, the verifier therefore needs the common reference string. Using the logarithmic order from XXX and instance I_1 , we get the following:

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$$\begin{aligned} [I]g_1 &= \left[\frac{\beta \cdot A_0(s) + \alpha \cdot B_0(s) + C_0(s)}{\gamma} \right]_{g_1} \oplus [I_1] \left[\frac{\beta \cdot A_1(s) + \alpha \cdot B_1(s) + C_1(s)}{\gamma} \right]_{g_1} \\ &= \mathcal{O} \oplus [11](33, 9) \\ &= [11 \cdot 11](13, 15) = [4](13, 15) \\ &= (35, 28) \end{aligned}$$

In the next step, we have to compute all the pairings involved in equation XXX. Using the logarithmic order on \mathbb{G}_1 and \mathbb{G}_2 as well as the bilinearity property of the pairing map we get the following:

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$$\begin{aligned}
e([A]g_1, [B]g_2) &= e((35, 15), (7v^2, 27v^3)) = e([9](13, 15), [12](7v^2, 16v^3)) \\
&= e((13, 15), (7v^2, 16v^3))^{9 \cdot 12} \\
&= e((13, 15), (7v^2, 16v^3))^{108} \\
e([\alpha]g_1, [\beta]g_2) &= e((27, 34), (16v^2, 28v^3)) = e([6](13, 15), [5](7v^2, 16v^3)) \\
&= e((13, 15), (7v^2, 16v^3))^{6 \cdot 5} \\
&= e((13, 15), (7v^2, 16v^3))^{30} \\
e([I]g_1, [\gamma]g_2) &= e((35, 28), (37v^2, 27v^3)) = e([4](13, 15), [4](7v^2, 16v^3)) \\
&= e((13, 15), (7v^2, 16v^3))^{4 \cdot 4} \\
&= e((13, 15), (7v^2, 16v^3))^{16} \\
e([C]g_1, [\delta]g_2) &= e((27, 9), (42v^2, 16v^3)) = e([7](13, 15), [3](7v^2, 16v^3)) \\
&= e((13, 15), (7v^2, 16v^3))^{7 \cdot 3} \\
&= e((13, 15), (7v^2, 16v^3))^{21}
\end{aligned}$$

5935 In order to check equation XXX, observe that the target group \mathbb{G}_T of the Weil pairing is
 5936 a finite cyclic group of order 13. Exponentiation is therefore done in modular 13 arithmetics.
 5937 Accordingly, since $108 \bmod 13 = 4$, we evaluate the left side of equation XXX as follows:

$$e([A]g_1, [B]g_2) = e((13, 15), (7v^2, 16v^3))^{108} = e((13, 15), (7v^2, 16v^3))^4$$

5938 Similarly, we evaluate the right side of equation XXX using modular 13 arithmetics and the
 5939 exponential law $a^x \cdot a^y = a^{x+y}$:

$$\begin{aligned}
&e([\alpha]g_1, [\beta]g_2) \cdot e([I]g_1, [\gamma]g_2) \cdot e([C]g_1, [\delta]g_2) = \\
&e((13, 15), (7v^2, 16v^3))^{30} \cdot e((13, 15), (7v^2, 16v^3))^{16} \cdot e((13, 15), (7v^2, 16v^3))^{21} = \\
&e((13, 15), (7v^2, 16v^3))^4 \cdot e((13, 15), (7v^2, 16v^3))^3 \cdot e((13, 15), (7v^2, 16v^3))^8 = \\
&e((13, 15), (7v^2, 16v^3))^{4+3+8} = \\
&e((13, 15), (7v^2, 16v^3))^2
\end{aligned}$$

5940 As we can see, both the left and the right side of equation XXX are identical, which implies
 5941 that the verification process accepts the simulated proof.

5942 **NOTE: UNFORTUNATELY NOT! :-((HENCE THERE IS AN ERROR SOMEWHERE ...**
 5943 **NEED TO FIX IT AFTER VACATION**

5944 **Proof Simulation** During the execution of a setup phase, a common reference string is gen-
 5945 erated, along with a simulation trapdoor, the latter of which must be deleted at the end of the
 5946 setup-phase. As an alternative, a more complicated multi-party protocol like [XXX] can be
 5947 used to split the knowledge of the simulation trapdoor among many different parties.

5948 In this paragraph, we will show why knowledge of the simulation trapdoor is problematic,
 5949 and how it can be used to generate zk-SNARKs for a given instance without any knowledge or
 5950 the existence of associated witness.

5951 To be more precise, let I be an instance for some R1CS language L_R . We call a zk-SNARK
 5952 for L_R **forged** or **simulated** if it passes a verification but its generation does not require the
 5953 existence of a witness W such that $(I; W)$ is a word in L_R .

To see how simulated zk-SNARKs can be computed, assume that a forger has knowledge of proper Groth_16 parameters, a quadratic arithmetic program of the problem, a common reference string and its associated simulation trapdoor τ :

$$\tau = (\alpha, \beta, \gamma, \delta, s) \quad (8.6)$$

Given some instance I , the forger’s task is to generate a zk-SNARK for this instance that passes the verification process, without having access to any other zk-SNARKs for this instance and without knowledge of a valid witness W .

To achieve this in the Groth_16 protocol, the forger can use the simulation trapdoor in combination with the QAP and two arbitrary field elements A and B from the scalar field \mathbb{F}_r of the pairing groups to g_1^C compute for the instance (I_1, \dots, I_n) as follows:

$$g_1^C = g_1^{\frac{A \cdot B}{\delta}} \cdot g_1^{-\frac{\alpha \cdot \beta}{\delta}} \cdot g_1^{-\frac{\beta A_0(s) + \alpha B_0(s) + C_0(s)}{\delta}} \cdot \left(g_1^{-\frac{\beta A_1(s) + \alpha B_1(s) + C_1(s)}{\delta}} \right)^{I_1} \cdots \left(g_1^{-\frac{\beta A_n(s) + \alpha B_n(s) + C_n(s)}{\delta}} \right)^{I_n}$$

The forger then publishes the zk-SNARK $\pi_{\text{forged}} = (g_1^A, g_1^C, g_2^B)$, which will pass the verification process and is computable without the existence of a witness (W_1, \dots, W_m) .

To see that the simulation trapdoor is necessary and sufficient to compute the simulated proof π_{forged} , first observe that both generators g_1 and g_2 are known to the forger, as they are part of the common reference string, encoded as $g_1^{s^0}$ and $g_2^{s^0}$. The forger is therefore able to compute $g_1^{A \cdot B}$. Moreover, since the forger knows α, β, δ and s from the trapdoor, they are able to compute all factors in the computation of g_1^C .

If, on the other hand, the simulation trapdoor is unknown, it is not possible to compute g_1^C , since, for example, the computational Diffie-Hellman assumption makes the derivation of $g_1^{\alpha \cdot \beta}$ from g_1^α and g_1^β infeasible.

Example 143 (The 3-factorization Problem). To see how a forger might simulate a zk-SNARK for some given instance I , consider the 3-factorization problem from 109, our protocol parameters from XXX, the common reference string from XXX and the simulation trapdoor $\tau = (6, 5, 4, 3, 2)$ of that CRS.

In order to forge a zk-SNARK for instance $I_1 = 11$, we don’t need a constructive proof for the associated rank-1 constraint system, which implies that we don’t have to execute the circuit $C_{3.\text{fac}}(\mathbb{F}_{13})$. Instead, we have to choose 2 arbitrary elements A and B from \mathbb{F}_{13} , and compute g_1^A, g_2^B and g_1^C as defined in XXX. We choose $A = 9$ and $B = 3$, and, since $\delta^{-1} = 3$, we compute as follows:

$$\begin{aligned} [A]g_1 &= [9](13, 15) = (35, 15) \\ [B]g_2 &= [3](7v^2, 16v^3) = (42v^2, 16v^3) \\ [C]g_1 &= \left[\frac{A \cdot B}{\delta} \right]g_1 \oplus \left[-\frac{\alpha \cdot \beta}{\delta} \right]g_1 \oplus \left[-\frac{\beta A_0(s) + \alpha B_0(s) + C_0(s)}{\delta} \right]g_1 \oplus \\ &\quad [I_1] \left[-\frac{\beta A_1(s) + \alpha B_1(s) + C_1(s)}{\delta} \right]g_1 \\ &= [(9 \cdot 3) \cdot 9](13, 15) \oplus [-(6 \cdot 5) \cdot 9](13, 15) \oplus [0](13, 15) \oplus [11][-(7 \cdot 2 + 4) \cdot 9](13, 15) \\ &= [9](13, 15) \oplus [3](13, 15) \oplus [12](13, 15) = [11](13, 15) \\ &= (33, 9) \end{aligned}$$

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This is all we need to generate our forged proof for the 3-factorization problem. We publish the simulated zk-SNARK:

$$\pi_{fake} = ((35, 15), (33, 9), (42v^2, 16v^3))$$

Despite the fact that this zk-SNARK was generated without knowledge of a proper witness, it is indistinguishable from a zk-SNARK that proves knowledge of a proper witness.

To see that, we show that our forged SNARK passes the verification process. In order to verify π_{fake} , we proceed as in XXX, and compute the curve point g_1^I for the instance $I_1 = 11$. Since the instance is the same as in example XXX, we can parallel the computation from XXX:

$$\begin{aligned} [I]g_1 &= \left[\frac{\beta \cdot A_0(s) + \alpha \cdot B_0(s) + C_0(s)}{\gamma} \right]_{g_1} \oplus [I_1] \left[\frac{\beta \cdot A_1(s) + \alpha \cdot B_1(s) + C_1(s)}{\gamma} \right]_{g_1} \\ &= (35, 28) \end{aligned}$$

In a next step we have to compute all the pairings involved in equation XXX. Using the logarithmic order on \mathbb{G}_1 and \mathbb{G}_2 as well as the bilinearity property of the pairing map we get

$$\begin{aligned} e([A]g_1, [B]g_2) &= e((35, 15), (42v^2, 16v^3)) = e([9](13, 15), [3](7v^2, 16v^3)) \\ &= e((13, 15), (7v^2, 16v^3))^{9 \cdot 3} \\ &= e((13, 15), (7v^2, 16v^3))^{27} \\ e([\alpha]g_1, [\beta]g_2) &= e((27, 34), (16v^2, 28v^3)) = e([6](13, 15), [5](7v^2, 16v^3)) \\ &= e((13, 15), (7v^2, 16v^3))^{6 \cdot 5} \\ &= e((13, 15), (7v^2, 16v^3))^{30} \\ e([I]g_1, [\gamma]g_2) &= e((35, 28), (37v^2, 27v^3)) = e([4](13, 15), [4](7v^2, 16v^3)) \\ &= e((13, 15), (7v^2, 16v^3))^{4 \cdot 4} \\ &= e((13, 15), (7v^2, 16v^3))^{16} \\ e([C]g_1, [\delta]g_2) &= e((33, 9), (42v^2, 16v^3)) = e([11](13, 15), [3](7v^2, 16v^3)) \\ &= e((13, 15), (7v^2, 16v^3))^{11 \cdot 3} \\ &= e((13, 15), (7v^2, 16v^3))^{33} \end{aligned}$$

In order to check equation XXX, observe that the target group \mathbb{G}_T of the Weil pairing is a finite cyclic group of order 13. Exponentiation is therefore done in modular 13 arithmetics. Using this, we evaluate the left side of equation XXX as follows:

$$e([A]g_1, [B]g_2) = e((13, 15), (7v^2, 16v^3))^{27} = e((13, 15), (7v^2, 16v^3))^1$$

since $27 \bmod 13 = 1$. Similarly, we evaluate the right side of equation XXX using modular 13 arithmetics and the exponential law $a^x \cdot a^y = a^{x+y}$. We get

$$\begin{aligned} e([\alpha]g_1, [\beta]g_2) \cdot e([I]g_1, [\gamma]g_2) \cdot e([C]g_1, [\delta]g_2) &= \\ e((13, 15), (7v^2, 16v^3))^{30} \cdot e((13, 15), (7v^2, 16v^3))^{16} \cdot e((13, 15), (7v^2, 16v^3))^{33} &= \\ e((13, 15), (7v^2, 16v^3))^4 \cdot e((13, 15), (7v^2, 16v^3))^3 \cdot e((13, 15), (7v^2, 16v^3))^7 &= \\ e((13, 15), (7v^2, 16v^3))^{4+3+7} &= \\ e((13, 15), (7v^2, 16v^3))^1 & \end{aligned}$$

5987 As we can see, both the left and the right side of equation XXX are identical, which implies
5988 that the verification process accepts the simulated proof. π_{fake} therefore convinces the veri-
5989 fier that a witness to the 3-factorization problem exists. However, no such witness was really
5990 necessary to generate the proof.

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ence

5991 **Chapter 9**

5992 **Exercises and Solutions**

5993 TODO: All exercises we provided should have a solution, which we give here in all detail.