
Operational notes

Document updated on **June 21, 2022**.

The following colors are **not** part of the final product, but serve as highlights in the editing/review process:










- text that needs attention from the Subject Matter Experts: Mirco, Anna,& Jan
- terms that have not yet been defined in the book
- things that need to be checked only at the very final typesetting stage (and it doesn't make sense to do them before)
- text that needs advice from the communications/marketing team: Aaron & Shane
- text that needs to be completed or otherwise edited (by Sylvia)

Todo list

12	■ Sylvia: This subsection should be in the introduction of the book as it applies to the	
13	entire math part	12
14	■ zero-knowledge proofs	12
15	■ add references	12
16	■ Sylvia: This subsection should be in the introduction of the book as it applies to the	
17	entire math part	13
18	■ Sylvia: We should put this table into the same introduction as 3.1.1	14
19	■ @jan @anna double check this definition. Is it clear enough? Proper definition re-	
20	quires the concept of equivalence or coprimeness first	14
21	■ Do we even need this quantum computing excursion?	16
22	■ @jan. You wrote: a and b are required to be non-zero in the definition above, so this	
23	can just be deleted. ... a can be zero and existence and uniqueness, non-zeroness	
24	are not obvious. Do you mean something else?	16
25	■ You wrote: if these should only satisfy the equation, why use definition symbols	
26	(:=) and not equality symbols (=)? But this is a definition the symbol a div b IS	
27	DEFINED to be the number b... Is that clear?	17
28	■ check algorithm floating	24
29	■ subtrahend	34
30	■ minuend	34
31	■ algorithm-floating	35
32	■ check algorithm floating	38
33	■ Sylvia: I would like to have a separate counter for definitions	41
34	■ check reference	47
35	■ runtime complexity	47
36	■ add reference	47
37	■ S: what does “efficiently” mean here?	47
38	■ computational hardness assumptions	48
39	■ check reference	48
40	■ check reference	48
41	■ explain last sentence more	49
42	■ “equation”?	50
43	■ check reference	50
44	■ what’s the difference between \mathbb{F}_p^* and \mathbb{Z}_p^* ?	50
45	■ Legendre symbol	50
46	■ Euler’s formular	50
47	■ These are only explained later in the text, ‘4.31’	50
48	■ are these going to be relevant later? yes, they are used in various snark proof systems	51
49	■ TODO: theorem: every factor of order defines a subgroup...	51
50	■ Is there a term for this property?	51

















































51	■ a few examples?	53
52	■ check reference	54
53	■ TODO: DOUBLE CHECK THIS REASONING.	54
54	■ Mirco: We can do better than this	56
55	■ check reference	57
56	■ add reference	58
57	■ pseudorandom	58
58	■ oracle	58
59	■ check reference	58
60	■ check reference	60
61	■ check reference	60
62	■ check reference	60
63	■ check reference	60
64	■ add more examples protocols of SNARK	60
65	■ check reference	60
66	■ add reference	61
67	■ Abelian groups	61
68	■ codomain	61
69	■ Check change of wording	62
70	■ add reference	62
71	■ Expand on this?	63
72	■ check reference	63
73	■ S: are we introducing elliptic curves in section 1 or 2?	64
74	■ check reference	65
75	■ check reference	65
76	■ add reference	65
77	■ check reference	65
78	■ write paragraph on exponentiation	66
79	■ add reference	66
80	■ check reference	66
81	■ add reference	66
82	■ group pairings	66
83	■ add reference	67
84	■ check reference	67
85	■ check reference	70
86	■ add reference	71
87	■ TODO: Elliptic Curve asymmetric cryptography examples. Private key, generator,	
88	public key.	73
89	■ add reference	73
90	■ maybe remove this sentence?	73
91	■ affine space	73
92	■ cusps	74
93	■ self-intersections	74
94	■ check reference	75
95	■ check reference	76
96	■ jubjub	76
97	■ check reference	76
98	■ affine plane	76

99	check reference	77
100	check reference	77
101	check reference	78
102	sign	78
103	more explanation of what the sign is	78
104	check reference	78
105	S: I don't follow this at all	79
106	check reference	79
107	add explanation of how this shows what we claim	79
108	should this def. be moved even earlier?	80
109	chord line	80
110	tangential	80
111	tangent line	80
112	remove Q ?	80
113	where?	81
114	check reference	81
115	check reference	81
116	check reference	81
117	check reference	82
118	check reference	82
119	check reference	83
120	check reference	83
121	check reference	84
122	add term	84
123	add term	84
124	add reference	84
125	cofactor clearing	84
126	add reference	84
127	check reference	84
128	check reference	85
129	add reference	85
130	add reference	85
131	check reference	85
132	check reference	85
133	check reference	86
134	check reference	86
135	check reference	86
136	Explain how	86
137	write example	87
138	check reference	87
139	add reference	87
140	check reference	87
141	add reference	88
142	check reference	88
143	add reference	88
144	check reference	88
145	add reference	88
146	check reference	88

147	 add reference	88
148	 add reference	88
149	 add reference	88
150	 check reference	88
151	 check reference	88
152	 Check if following Alg is floated too far	89
153	 add reference	89
154	 add reference	89
155	 write up this part	89
156	 is the label in L ^A T _E X correct here?	91
157	 check reference	91
158	 check reference	91
159	 check reference	91
160	 check reference	92
161	 check reference	92
162	 check reference	93
163	 check reference	93
164	 check reference	93
165	 check reference	93
166	 check reference	93
167	 add reference	93
168	 check reference	95
169	 check reference	95
170	 check reference	95
171	 check reference	95
172	 check reference	95
173	 check reference	97
174	 check reference	97
175	 check reference	98
176	 either expand on this or delete it	98
177	 add reference	98
178	 check reference	98
179	 check reference	98
180	 check reference	98
181	 check reference	98
182	 check reference	98
183	 check reference	99
184	 check reference	99
185	 check reference	100
186	 check reference	100
187	 add reference	100
188	 add reference	100
189	 This needs to be written (in Algebra)	100
190	 add reference	100
191	 add reference	100
192	 check reference	100
193	 towers of curve extensions	101
194	 check reference	101





































195	check reference	101
196	check reference	101
197	check reference	102
198	add reference	102
199	check reference	103
200	S: either add more explanation or move to a footnote	103
201	type 3 pairing-based cryptography	103
202	add references?	103
203	check reference	104
204	check reference	104
205	check floating of algorithm	105
206	add references	105
207	check reference	106
208	add reference	106
209	check reference	106
210	check reference	106
211	add reference	107
212	should all lines of all algorithms be numbered?	107
213	check reference	108
214	check reference	108
215	check reference	108
216	check if the algorithm is floated properly	108
217	check reference	108
218	again?	110
219	check reference	110
220	circuit	110
221	signature schemes	110
222	add reference	110
223	check reference	111
224	check reference	111
225	add references	111
226	add reference	111
227	reference text to be written in Algebra	111
228	check reference	111
229	check reference	111
230	check reference	112
231	add reference	112
232	algebraic closures	112
233	check reference	112
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235	check reference	113
236	check reference	113
237	check reference	114
238	disambiguate	114
239	add reference	114
240	unify terminology	114
241	check reference	115
242	actually make this a table?	115

243	exercise still to be written?	116
244	add reference	116
245	check reference	116
246	check reference	116
247	add reference	117
248	check reference	118
249	check reference	118
250	check reference	118
251	add reference	119
252	check reference	119
253	check reference	119
254	check reference	120
255	what does this mean? Maybe just delete it	121
256	write up this part	122
257	add reference	122
258	check reference	122
259	cyclotomic polynomial	122
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262	why? Because in this book elliptic curves are only defined for fields of chracteristic > 3	123
263	check reference	123
264	check reference	123
265	what does this mean?	123
266	add reference	123
267	add reference	123
268	check reference	123
269	check reference	124
270	add reference	125
271	add exercise	125
272	check reference	126
273	add reference	126
274	add reference	126
275	add reference	126
276	check reference	127
277	check reference	127
278	add reference	127
279	add reference	127
280	add reference	128
281	check reference	128
282	add reference	128
283	add reference	128
284	finish writing this up	129
285	add reference	129
286	correct computations	129
287	fill in missing parts	129
288	add reference	130
289	check equation	130
290	Chapter 1?	131

291		"rigorous"?	131
292		"proving"?	131
293		Add example	132
294		M: 1:1 correspondence might actually be wrong	132
295		binary tuples	132
296		add reference	133
297		add reference	133
298		check reference	133
299		check reference	133
300		Are we using w and x interchangeably or is there a difference between them?	134
301		check reference	134
302		jubjub	134
303		check reference	134
304		check reference	134
305		check wording	134
306		check reference	134
307		check references	135
308		add reference	135
309		add reference	135
310		check reference	136
311		add reference	136
312		check reference	137
313		check reference	137
314		add reference	137
315		add reference	138
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317		add reference	139
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319		check reference	140
320		add reference	141
321		check reference	142
322		check reference	142
323		check reference	142
324		check reference	142
325		check reference	142
326		add reference	143
327		add reference	144
328		check reference	144
329		check reference	144
330		check reference	145
331		check reference	145
332		add reference	146
333		check reference	148
334		add reference	148
335		check reference	149
336		check reference	149
337		check reference	149
338		Should we refer to R1CS satisfiability (p. 142 here?	150

339	check reference	151
340	add reference	151
341	check reference	151
342	check reference	152
343	check reference	152
344	check reference	153
345	check reference	155
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348	check reference	156
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354	clarify language	158
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373	add reference	171
374	check reference	172
375	check reference	172
376	check reference	172
377	check reference	172
378	add reference	172
379	check reference	173
380	check reference	174
381	"constraints" or "constrained"?	174
382	check reference	175
383	"constraints" or "constrained"?	175
384	add reference	175
385	"constraints" or "constrained"?	175
386	add reference	176

387	check references	176
388	check reference	176
389	add reference	177
390	can we rotate this by 90° ?	177
391	check reference	178
392	add reference	178
393	add reference	178
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402	add reference	187
403	add reference	187
404	check reference	188
405	add reference	188
406	add reference	188
407	add reference	190
408	check reference	191
409	check reference	192
410	common reference string	192
411	simulation trapdoor	192
412	check reference	192
413	check reference	192
414	add reference	193
415	check reference	193
416	check reference	193
417	check reference	193
418	"invariable"?	193
419	explain why	194
420	4 examples have the same title. Change it to be distinct	194
421	check reference	194
422	add reference	194
423	check reference	194
424	add reference	194
425	add reference	195
426	add reference	196
427	check reference	197
428	add reference	197
429	add reference	198
430	check reference	198
431	check reference	198
432	add reference	198
433	add reference	198
434	check reference	199

435	 add reference	199
436	 add reference	199
437	 add reference	199
438	 check reference	199
439	 add reference	199
440	 add reference	199
441	 add reference	199
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443	 add reference	200
444	 add reference	200
445	 add reference	200
446	 add reference	200
447	 check reference	202
448	 check reference	202
449	 add reference	202
450	 add reference	202
451	 add reference	202
452	 add reference	202
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456	 add reference	203
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458	 add reference	203
459	 check reference	204
460	 add reference	204
461	 add reference	204
462	 add reference	204
463	 add reference	205
464	 add reference	205
465	 add reference	205
466	 add reference	205
467	 add reference	205
468	 add reference	205
469	 add reference	205
470	 add reference	206

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MoonMath manual

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TechnoBob and the Least Scruples crew

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June 21, 2022

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Chapter 1

Introduction

This is dump from other papers as inspiration for the intro:

Zero knowledge proofs are a class of cryptographic protocols in which one can prove honest computation without revealing the inputs to that computation. A simple high-level example of a zero-knowledge proof is the ability to prove one is of legal voting age without revealing the respective age. In a typical zero knowledge proof system, there are two participants: a prover and a verifier. A prover will present a mathematical proof of computation to a verifier to prove honest computation. The verifier will then confirm whether the prover has performed honest computation based on predefined methods. Zero knowledge proofs are of particular interest to public blockchain activities as the verifier can be codified in smart contracts as opposed to trusted parties or third-party intermediaries.

Zero-knowledge proofs (ZKPs) are an important privacy-enhancing tool from cryptography. They allow proving the veracity of a statement, related to confidential data, without revealing any information beyond the validity of the statement. ZKPs were initially developed by the academic community in the 1980s, and have seen tremendous improvements since then. They are now of practical feasibility in multiple domains of interest to the industry, and to a large community of developers and researchers. ZKPs can have a positive impact in industries, agencies, and for personal use, by allowing privacy-preserving applications where designated private data can be made useful to third parties, despite not being disclosed to them.

ZKP systems involve at least two parties: a prover and a verifier. The goal of the prover is to convince the verifier that a statement is true, without revealing any additional information. For example, suppose the prover holds a birth certificate digitally signed by an authority. In order to access some service, the prover may have to prove being at least 18 years old, that is, that there exists a birth certificate, tied to the identity of the prover and digitally signed by a trusted certification authority, stating a birthdate consistent with the age claim. A ZKP allows this, without the prover having to reveal the birthdate.

1.1 Target audience

This book is accessible for both beginners and experienced developers alike. Concepts are gradually introduced in a logical and steady pace. Nonetheless, the chapters lend themselves rather well to being read in a different order. More experienced developers might get the most benefit by jumping to the chapters that interest them most. If you like to learn by example, then you should go straight to the chapter on Using Clarity.

It is assumed that you have a basic understanding of programming and the underlying logical concepts. The first chapter covers the general syntax of Clarity but it does not delve into what

programming itself is all about. If this is what you are looking for, then you might have a more difficult time working through this book unless you have an (undiscovered) natural affinity for such topics. Do not let that dissuade you though, find an introductory programming book and press on! The straightforward design of Clarity makes it a great first language to pick up.

1.2 The Zoo of Zero-Knowledge Proofs

First, a list of zero-knowledge proof systems:

1. Pinocchio (2013): Paper

– Notes: trusted setup

2. BCGTV (2013): Paper

– Notes: trusted setup, implementation

3. BCTV (2013): Paper

– Notes: trusted setup, implementation

4. Groth16 (2016): Paper

– Notes: trusted setup

– Other resources: Talk in 2019 by Georgios Konstantopoulos

5. GM17 (2017): Paper

– Notes: trusted setup

– Other resources: later Simulation extractability in ROM, 2018

6. Bulletproofs (2017): Paper

– Notes: no trusted setup

– Other resources: Polynomial Commitment Scheme on DL, 2016 and KZG10, Polynomial Commitment Scheme on Pairings, 2010

7. Ligero (2017): Paper

– Notes: no trusted setup

– Other resources:

8. Hyrax (2017): Paper

– Notes: no trusted setup

– Other resources:

9. STARKs (2018): Paper

– Notes: no trusted setup

– Other resources:

10. Aurora (2018): Paper

- Notes: transparent SNARK

- Other resources:

11. Sonic (2019): Paper

- Notes: SNORK - SNARK with universal and updateable trusted setup, PCS-based

- Other resources: Blog post by Mary Maller from 2019 and work on updateable and universal setup from 2018

12. Libra (2019): Paper

- Notes: trusted setup

- Other resources:

13. Spartan (2019): Paper

- Notes: transparent SNARK

- Other resources:

14. PLONK (2019): Paper

- Notes: SNORK, PCS-based

- Other resources: Discussion on Plonk systems and Awesome Plonk list

15. Halo (2019): Paper

- Notes: no trusted setup, PCS-based, recursive

- Other resources:

16. Marlin (2019): Paper

- Notes: SNORK, PCS-based

- Other resources: Rust Github

17. Fractal (2019): Paper

- Notes: Recursive, transparent SNARK

- Other resources:

18. SuperSonic (2019): Paper

- Notes: transparent SNARK, PCS-based

- Other resources: Attack on DARK compiler in 2021

19. Redshift (2019): Paper

- Notes: SNORK, PCS-based

- Other resources:

Other resources on the zoo: Awesome ZKP list on Github, ZKP community with the reference document

To Do List

- Make table for prover time, verifier time, and proof size
- Think of categories - *Achieved Goals*: Trusted setup or not, Post-quantum or not, ...
- Think of categories - *Mathematical background*: Polynomial commitment scheme, ...
- ... while we discuss the points above, we should also discuss a common notation/language for all these things. (E.g. transparent SNARK/no trusted setup/STARK)

Points to cover while writing

- Make a historical overview over the "discovery" of the different ZKP systems
- Make reader understand what paper is build on what result etc. - the tree of publications!
- Make reader understand the different terminology, e.g. SNARK/SNORK/STARK, PCS, R1CS, updateable, universal, ...
- Make reader understand the mathematical assumptions - and what this means for the zoo.
- Where will the development/evolution go? What are bottlenecks?

Other topics I fell into while compiling this list

- Vector commitments: <https://eprint.iacr.org/2020/527.pdf>
- Snark1: <http://ace.cs.ohio.edu/~gstewart/papers/snaark1.pdf>
- Virgo?: https://people.eecs.berkeley.edu/~kubitron/courses/cs262a-F19/projects/reports/project5_report_ver2.pdf

Chapter 2

Preliminaries

2.1 Preface and Acknowledgements

This book began as a set of lecture and notes accompanying the zk-Summit 0x and 0xx It arose from the desire to collect the scattered information of snarks [] and present them to an audience that does not have a strong background in cryptography []

2.2 Purpose of the book

The first version of this book is written by security auditors at Least Authority where we audited quite a few snark based systems. Its included "what we have learned" destilate of the time we spend on various audits.

We intend to let illustrative examples drive the discussion and present the key concepts of pairing computation with as little machinery as possible. For those that are fresh to pairing-based cryptography, it is our hope that this chapter might be particularly useful as a first read and prelude to more complete or advanced expositions (e.g. the related chapters in [Gal12]).

On the other hand, we also hope our beginner-friendly intentions do not leave any sophisticated readers dissatisfied by a lack of formality or generality, so in cases where our discussion does sacrifice completeness, we will at least endeavour to point to where a more thorough exposition can be found.

One advantage of writing a survey on pairing computation in 2012 is that, after more than a decade of intense and fast-paced research by mathematicians and cryptographers around the globe, the field is now racing towards full maturity. Therefore, an understanding of this text will equip the reader with most of what they need to know in order to tackle any of the vast literature in this remarkable field, at least for a while yet.

Since we are aiming the discussion at active readers, we have matched every example with a corresponding snippet of (hyperlinked) Magma [BCP97] code 1 , where we take inspiration from the helpful Magma pairing tutorial by Dominguez Perez et al. [DKS09].

Early in the book we will develop examples that we then later extend with most of the things we learn in each chapter. This way we incrementally build a few real world snarks but over full fledged cryptographic systems that are nevertheless simple enough to be computed by pen and paper to illustrate all steps in great detail.

2.3 How to read this book

Books and papers to read: XXXXXXXXXXXXX

Software to try: XXXXXXXXXXXXXXXXXXXXX

Correctly prescribing the best reading route for a beginner naturally requires individual diagnosis that depends on their prior knowledge and technical preparation.

2.4 Cryptological Systems

The science of information security is referred to as *cryptology*. In the broadest sense, it deals with encryption and decryption processes, with digital signatures, identification protocols, cryptographic hash functions, secrets sharing, electronic voting procedures and electronic money.
EXPAND

2.5 SNARKS

2.6 complexity theory

Before we deal with the mathematics behind zero knowledge proof systems, we must first clarify what is meant by the runtime of an algorithm or the time complexity of an entire mathematical problem. This is particularly important for us when we analyze the various snark systems...

For the reader who is interested in complexity theory, we recommend, or example or , as well as the references contained therein.

2.6.1 Runtime complexity

The runtime complexity of an algorithm describes, roughly speaking, the amount of elementary computation steps that this algorithm requires in order to solve a problem, depending on the size of the input data.

Of course, the exact amount of arithmetic operations required depends on many factors such as the implementation, the operating system used, the CPU and many more. However, such accuracy is seldom required and is mostly meaningful to consider only the asymptotic computational effort.

In computer science, the runtime of an algorithm is therefore not specified in individual calculation steps, but instead looks for an upper limit which approximates the runtime as soon as the input quantity becomes very large. This can be done using the so-called *Landau notation* (also called big- \mathcal{O} -notation) A precise definition would, however, go beyond the scope of this work and we therefore refer the reader to .

For us, only a rough understanding of transit times is important in order to be able to talk about the security of cryptographic systems. For example, $\mathcal{O}(n)$ means that the running time of the algorithm to be considered is linearly dependent on the size of the input set n , $\mathcal{O}(n^k)$ means that the running time is polynomial and $\mathcal{O}(2^n)$ stands for an exponential running time (chapter 2.4).

An algorithm which has a running time that is greater than a polynomial is often simply referred to as *slow*.

A generalization of the runtime complexity of an algorithm is the so-called *time complexity of a mathematical problem*, which is defined as the runtime of the fastest possible algorithm that can still solve this problem (chapter 3.1).

Since the time complexity of a mathematical problem is concerned with the runtime analysis of all possible (and thus possibly still undiscovered) algorithms, this is often a very difficult and deep-seated question .

For us, the time complexity of the so-called discrete logarithm problem will be important. This is a problem for which we only know slow algorithms on classical computers at the moment, but for which at the same time we cannot rule out that faster algorithms also exist.

STUFF ON CRYPTOGRAPHIC HASH FUNCTIOND

2.7 Software Used in This Book

2.7.1 Sagemath

It order to provide an interactive learning experience, and to allow getting hands-on with the concepts described in this book, we give examples for how to program them in the Sage programming language. Sage is a dialect of the learning-friendly programming language Python, which was extended and optimized for computing with, in and over algebraic objects. Therefore, we recommend installing Sage before diving into the following chapters.

The installation steps for various system configurations are described on the sage websit ¹. Note however that we use Sage version 9, so if you are using Linux and your package manager only contains version 8, you may need to choose a different installation path, such as using prebuilt binaries.

We recommend the interested reader, who is not familiar with sagemath to read on the many tutorial before starting this book. For example

¹<https://doc.sagemath.org/html/en/installation/index.html>

Chapter 3

arithmetic

3.1 Introduction

3.1.1 Aims and target audience

Sylvia: This subsection should be in the introduction of the book as it applies to the entire math part

How much mathematics do you need to understand **zero-knowledge proofs**? The answer, of course, depends on the level of understanding you aim for. It is possible to describe zero-knowledge proofs without using mathematics at all; however, to read a foundational paper like ? and to understand the internals of snark implementations, some knowledge of mathematics is needed to be able to follow the discussion.

The goal of this chapter is therefore to enable a reader who is starting out with nothing more than basic high school algebra to be able to solve simple tasks in the mathematics underlying most zero knowledge proofs without the need of a computer.

Without a solid grounding in mathematics, someone who is interested in learning the concepts of zero-knowledge proofs, but who has never seen or dealt with, say, a prime field 4.3.1, or an elliptic curve 5, may quickly become overwhelmed. This is not so much due to the complexity of the mathematics needed, but rather because of the vast amount of technical jargon, unknown terms, and obscure symbols that quickly makes a text unreadable, even though the concepts themselves are not actually that complicated. As a result, the reader might either lose interest, or pick up some incoherent bits and pieces of knowledge that, in the worst case scenario, result in immature and unsecure code.

This is why we dedicated this chapter to explaining the mathematical foundations needed to understand the basic concepts underlying snark development. We encourage the reader who is not familiar with basic number theory and elliptic curves to take the time and read this and the following chapters, until they are able to solve at least a few exercises in each chapter.

If, on the other hand, you are already skilled in elliptic curve cryptography, feel free to skip this chapter and only come back to it for reference and comparison. Maybe the most interesting parts are XXX .

We start our explanations at a very basic level, and only assume pre-existing knowledge of fundamental concepts like high school integer arithmetic. At the same time, we'll attempt to teach you to "think mathematically", and to show you that there are numbers and mathematical structures out there that appear to be very different from the things you learned about in high school, but on a deeper level, they are actually quite similar, as we will see in various examples below in this chapter.

Sylvia:
This subsection should be in the introduction of the book as it applies to the entire math part

zero-knowledge proofs

add references

To make it easier to memorize new concepts and symbols, we might frequently link to definitions in the beginning, but as too many links render a text unreadable, we will assume the reader will become familiar with definitions as the text proceeds at which point we will not link them anymore.

We want to stress, however that this introduction is informal, incomplete and optimized to enable the reader to understand zero-knowledge concepts as efficiently as possible. Our focus and design choices are to include as little theory as necessary, focusing on a wealth of numerical examples. We believe that such an informal, example-driven approach to learning mathematics may make it easier for beginners to digest the material in the initial stages.

For instance, as a beginner, you would probably find it more beneficial to first compute a simple toy **snark** with pen and paper all the way through, before actually developing real-world production-ready systems. In addition, it's useful to have a few simple examples in your head before getting started with reading actual academic papers.

However, in order to be able to derive these toy examples, some mathematical groundwork is needed. This chapter therefore will help the inexperienced reader to focus on what we believe is important, accompanied by exercises that you are encouraged to recompute yourself. Every section usually ends with a list of additional exercises in increasing order of difficulty, to help the reader memorize and apply the concepts.

3.1.2 The structure of this chapter

Sylvia: This subsection should be in the introduction of the book as it applies to the entire math part

We start with a brief recapitulation of basic integer arithmetic like long division, the greatest common divisor and Euclid's algorithm. After that, we introduce modular arithmetic as **the most important skill** to compute our pen-and-paper examples. We then introduce polynomials, compute their analogs to integer arithmetic and introduce the important concept of Lagrange interpolation.

After this practical warm up, we introduce some basic algebraic terms like groups and fields, because those terms are used very frequently in academic papers relating to zero-knowledge proofs. The beginner is advised to memorize those terms and think about them: Why are they defined the way they are? Can you come up with a better definition? How and why is every rule important and what happens if that rule is dropped?

We define these terms in the general abstract way of mathematics, hoping that the non mathematical trained reader will gradually learn to become comfortable with this style. We then give basic examples and do basic computations with these examples to get familiar with the concepts.

Sylvia: This subsection should be in the introduction of the book as it applies to the entire math part

3.2 Integer arithmetic

In a sense, integer arithmetic is at the heart of large parts of modern cryptography. Fortunately, most readers will probably remember integer arithmetic from school. It is, however, important that you can confidently apply those concepts to understand and execute computations in the many pen-and-paper examples that form an integral part of the MoonMath Manual. We will therefore recapitulate basic arithmetic concepts to refresh your memory and fill any knowledge gaps.

In what follows, we use many mathematical notations, which we summarized in the following table 3.2:

Symbol	Meaning of Symbol
$=$	equals
$:=$	defining the symbol on the r
\in	element from a set
\Leftrightarrow	logical equivalence
$\sum_{j=n}^k a_j$	summation

Sylvia: We should put this table into the same introduction as 3.1.1

In this book, we use the symbol \mathbb{Z} as a short description for the set of all **integers**, that is we write:

$$\mathbb{Z} := \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\} \quad (3.1)$$

Integers are also known as **whole numbers**, that is, numbers that can be written without fractional parts. Examples of numbers that are **not** integers are $\frac{2}{3}$, 1.2 and -1280.006 .

If $a \in \mathbb{Z}$ is an integer, then we write $|a|$ for the **absolute value** of a , that is, the non-negative value of a without regard to its sign:

$$|4| = 4 \quad (3.2)$$

$$|-4| = 4 \quad (3.3)$$

We use the symbol \mathbb{N} for the set of all positive integers, usually called the set of **natural numbers** and \mathbb{N}_0 for the set of all non negative integers. So whenever you see the symbol \mathbb{N} , think of the set of all positive integers excluding the number 0:

$$\mathbb{N} := \{1, 2, 3, \dots\} \quad \mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$$

In addition, we use the symbol \mathbb{Q} for the set of all **rational numbers**, which can be represented as the set of all fractions $\frac{n}{m}$, where $n \in \mathbb{Z}$ is an integer and $m \in \mathbb{N}$ is a natural number, such that there is no other fraction $\frac{n'}{m'}$ and natural number $k \in \mathbb{N}$ with $k \neq 1$ and

$$\frac{n}{m} = \frac{k \cdot n'}{k \cdot m'} \quad (3.4)$$

The sets \mathbb{N} , \mathbb{Z} and \mathbb{Q} have a notion of addition and multiplication defined on them. Most of us are probably able to do many integer computations in our head, but this gets more and more difficult as these increase in complexity. We will frequently invoke the SageMath system (2.7.1) for more complicated computations (We define rings and fields later in this book):

```
sage: ZZ # A sage notation for the integers
Integer Ring
sage: NN # A sage notation for the natural numbers
Non negative integer semiring
sage: QQ # A sage notation for the rational numbers
Rational Field
sage: ZZ(5) # Get an element from the integers
5
```

Sylvia:
We should
put this
table into
the same
introduc-
tion as
3.1.1

@jan
@anna
double
check this
definition.
Is it clear
enough?
Proper
definition
requires
the con-
cept of
equiv-
alence or
coprime-
ness first

```

935 sage: ZZ(5) + ZZ(3) 9
936 8 10
937 sage: ZZ(5) * NN(3) 11
938 15 12
939 sage: ZZ.random_element(10**50) 13
940 30673028501735198244901035439688857445645906776647 14
941 sage: ZZ(27713).str(2) # Binary string representation 15
942 110110001000001 16
943 sage: NN(27713).str(2) # Binary string representation 17
944 110110001000001 18
945 sage: ZZ(27713).str(16) # Hexadecimal string representation 19
946 6c41 20

```

One set of numbers that is of particular interest to us is the set of **prime numbers**, which are natural numbers $p \in \mathbb{N}$ with $p \geq 2$, which are only divisible by themselves and by 1. All prime numbers apart from the number 2 are called **odd** prime numbers. We write \mathbb{P} for the set of all prime numbers and $\mathbb{P}_{\geq 3}$ for the set of all odd prime numbers. The set of prime numbers \mathbb{P} is an infinite set and can be ordered according to size, which means that for any prime number $p \in \mathbb{P}$ one can always find another prime number $p' \in \mathbb{P}$ with $p < p'$. It follows that there is no largest prime number. Since prime numbers can be ordered by size, we can write them as follows:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, \dots \quad (3.5)$$

As the **fundamental theorem of arithmetic** tells us, prime numbers are, in a certain sense, the basic building blocks from which all other natural numbers are composed. To see that, let $n \in \mathbb{N}$ be any natural number with $n > 1$. Then there are always prime numbers $p_1, p_2, \dots, p_k \in \mathbb{P}$, such that

$$n = p_1 \cdot p_2 \cdot \dots \cdot p_k. \quad (3.6)$$

This representation is unique for each natural number (except for the order of the factors) and is called the **prime factorization** of n .

Example 1 (Prime Factorization). To see what we mean by prime factorization of a number, let's look at the number $504 \in \mathbb{N}$. To get its prime factors, we can successively divide it by all prime numbers in ascending order starting with 2:

$$504 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 7$$

We can double check our findings invoking Sage, which provides an algorithm to factor natural numbers:

```

962 sage: n = NN(504) 21
963 sage: factor(n) 22
964 2^3 * 3^2 * 7 23

```

The computation from the previous example reveals an important observation: Computing the factorization of an integer is computationally expensive, because we have to divide repeatedly by all prime numbers smaller than the number itself until all factors are prime numbers themselves. From this, an important question arises: How fast can we compute the prime factorization of a natural number? This question is the famous **integer factorization problem** and, as far as we know, there is currently no method known that can factor integers much faster than the naive approach that just divides the given number by all prime numbers in ascending order.

On the other hand computing the product of a given set of prime numbers, is fast (just multiply all factors) and this simple observation implies that the two processes "prime number multiplication" on the one side and its inverse process "natural number factorization" have very different computational costs. The factorization problem is therefore an example of a so-called **one-way function**: An invertible function that is easy to compute in one direction, but hard to compute in the other direction.

It should be pointed out, however, that the American mathematician Peter W. Shor developed an algorithm in ? which can calculate the prime factorization of a natural number in polynomial time on a quantum computer. The consequence of this is that cryptosystems, which are based on the prime factor problem, are unsafe as soon as practically usable quantum computers become available .

Exercise 1. What is the absolute value of the integers -123 , 27 and 0 ?

Exercise 2. Compute the factorization of 30030 and double check your results using Sage.

Exercise 3. Consider the following equation $4 \cdot x + 21 = 5$. Compute the set of all solutions for x under the following alternative assumptions:

1. The equation is defined over the natural numbers.

2. The equation is defined over the integers.

Exercise 4. Consider the following equation $2x^3 - x^2 - 2x = -1$. Compute the set of all solutions x under the following assumptions:

1. The equation is defined over the natural numbers.

2. The equation is defined over the integers.

3. The equation is defined over the rational numbers.

Euclidean Division As we know from high school mathematics, integers can be added, subtracted and multiplied and the result is guaranteed to always be an integer again. On the contrary division in the commonly understood sense is not defined for integers, as, for example, 7 divided by 3 will not be an integer again. However it is always possible to divide any two integers if we consider division with remainder. So for example 7 divided by 3 is equal to 2 with a remainder of 1 , since $7 = 2 \cdot 3 + 1$.

It is the content of this section to introduce division with remainder for integers which is usually called **Euclidean division**. It is an essential technique underlying many concepts in this book. The precise definition is as follows:

Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ be two integers with $b \neq 0$. Then there is always another integer $m \in \mathbb{Z}$ and a natural number $r \in \mathbb{N}$, with $0 \leq r < |b|$ such that

$$a = m \cdot b + r \quad (3.7)$$

This decomposition of a given b is called **Euclidean division**, where a is called the **dividend**, b is called the **divisor**, m is called the **quotient** and r is called the **remainder**. It can be shown that both the quotient and the remainder always exist and are unique, as long as the divisor is different from 0 .

Do we even need this quantum computing excursion?

@jan.
You wrote: a and b are required to be non-zero in the definition

1009 *Notation and Symbols* 1. Suppose that the numbers a, b, m and r satisfy equation (3.7). Then
 1010 we often write

$$a \operatorname{div} b := m, \quad a \bmod b := r \quad (3.8)$$

1011 to describe the quotient and the remainder of the Euclidean division. We also say that an integer
 1012 a is divisible by another integer b if $a \bmod b = 0$ holds. In this case we also write $b|a$ and call
 1013 the integer $a \operatorname{div} b$ the **cofactor** of b in a .

1014 So, in a nutshell Euclidean division is a process of dividing one integer by another in a way
 1015 that produces a quotient and a non-negative remainder, the latter of which is smaller than the
 1016 absolute value of the divisor.

1017 A special situation occurs whenever the remainder is zero, because in this case the dividend
 1018 is divisible by the divisor. Our notation $b|a$ reflects that.

Example 2. Applying Euclidean division and our previously defined notation 3.8 to the dividend
 -17 and the divisor 4 , we get

$$-17 \operatorname{div} 4 = -5, \quad -17 \bmod 4 = 3$$

1019 because $-17 = -5 \cdot 4 + 3$ is the Euclidean division of -17 and 4 (the remainder is, by definition,
 1020 a non-negative number). In this case 4 does not divide -17 , as the remainder is not zero. The
 1021 truth value of the expression $4|-17$ therefore is FALSE. On the other hand, the truth value of
 1022 $4|12$ is TRUE, since 4 divides 12 , as $12 \bmod 4 = 0$. We can invoke sage to do the computation
 1023 for us. We get the following:

```
1024 sage: ZZ(-17) // ZZ(4) # Integer quotient
1025 -5
1026 sage: ZZ(-17) % ZZ(4) # remainder
1027 3
1028 sage: ZZ(4).divides(ZZ(-17)) # self divides other
1029 False
1030 sage: ZZ(4).divides(ZZ(12))
1031 True
```

1032 *Remark 1.* In 3.8 we defined the notation of $a \operatorname{div} b$ and $a \bmod b$, in terms of Euclidean division.
 1033 It should be noted however that many programming languages like Python and Sage, implement
 1034 both the operator $(/)$ as well as the operator $(\%)$ differently. Programmers should be aware of
 1035 this, as the discrepancy between the mathematical notation and the implementation in program-
 1036 ming languages might become the source of subtle bugs in implementations of cryptographic
 1037 primitives.

To give an example consider the the dividend -17 and the divisor -4 . Note that in contrast
 to the previous example 2, we have a negative divisor. According to our definition we have

$$-17 \operatorname{div} -4 = 5, \quad -17 \bmod -4 = 3$$

1038 because $-17 = 5 \cdot (-4) + 3$ is the Euclidean division of -17 and -4 (the remainder is, by
 1039 definition, a non-negative number). However using the operators $(/)$ and $(\%)$ in Sage we get

```
1040 sage: ZZ(-17) // ZZ(-4) # Integer quotient
1041 4
1042 sage: ZZ(-17) % ZZ(-4) # remainder
1043 -1
```

24
 25 You
 26 wrote:
 27 If these
 28 should
 29 only sat-
 30 isfy the
 31 equation,
 why use
 definition
 symbols
 $(:=)$ and
 not equal-
 ity sym-
 bols $(=)$?
 But this
 is a defi-
 nition the
 symbol a
 $\operatorname{div} b$ IS
 DEFINED
 to be the
 number
 32... Is that
 33 clear?

```

1044 sage: ZZ(-17).quo_rem(ZZ(-4)) # not Euclidean division 36
1045 (4, -1) 37

```

1046 Methods to compute Euclidean division for integers are called **integer division algorithms**.
 1047 Probably the best known algorithm is the so-called **long division**, which most of us might have
 1048 learned in school.

1049 As long division is the standard method used for pen-and-paper division of multi-digit num-
 1050 bers expressed in decimal notation, the reader should become familiar with it as we use it
 1051 throughout this book when we do simple pen-and-paper computations. However, instead of
 1052 defining the algorithm formally, we rather give some examples that will hopefully make the
 1053 process clear.

1054 In a nutshell, the algorithm loops through the digits of the dividend from the left to right,
 1055 subtracting the largest possible multiple of the divisor (at the digit level) at each stage; the
 1056 multiples then become the digits of the quotient, and the remainder is the first digit of the
 1057 dividend.

1058 *Example 3 (Integer Long Division).* To give an example of integer long division algorithm, lets
 1059 divide the integer $a = 143785$ by the number $b = 17$. Our goal is therefore to find solutions
 1060 to equation 3.7, that is, we need to find the quotient $m \in \mathbb{Z}$ and the remainder $r \in \mathbb{N}$ such that
 1061 $143785 = m \cdot 17 + r$. Using a notation that is mostly used in Commonwealth countries, we
 1062 compute as follows:

$$\begin{array}{r}
 8457 \\
 17 \overline{) 143785} \\
 \underline{136} \\
 77 \\
 \underline{68} \\
 98 \\
 \underline{85} \\
 135 \\
 \underline{119} \\
 16
 \end{array} \tag{3.9}$$

1063 We therefore get $m = 8457$ as well as $r = 16$ and indeed we have $143785 = 8457 \cdot 17 + 16$,
 1064 which we can double check invoking Sage:

```

1065 sage: ZZ(143785).quo_rem(ZZ(17)) 38
1066 (8457, 16) 39
1067 sage: ZZ(143785) == ZZ(8457)*ZZ(17) + ZZ(16) # check 40
1068 True 41

```

1069 *Exercise 5 (Integer Long Division).* Find an $m \in \mathbb{Z}$ as well as an $r \in \mathbb{N}$ with $0 \leq r < |b|$ such that
 1070 $a = m \cdot b + r$ holds for the following pairs $(a, b) = (27, 5)$, $(a, b) = (27, -5)$, $(a, b) = (127, 0)$,
 1071 $(a, b) = (-1687, 11)$ and $(a, b) = (0, 7)$. In which cases are your solutions unique?

1072 *Exercise 6 (Long Division Algorithm).* Write an algorithm that computes integer long division
 1073 and handling all edge cases properly.

1074 **The Extended Euclidean Algorithm** One of the most critical parts in this book is the so
 1075 called modular arithmetic which we will define in 3.3 and its application in the computations of
 1076 **prime fields** as defined in 4.3.1. To be able to do computations in modular arithmetic, we have

to get familiar with the so-called **extended Euclidean algorithm**. We therefore introduce this algorithm here.

The **greatest common divisor** (GCD) of two non-zero integers a and b , is defined as the greatest non-zero natural number d such that d divides both a and b , that is, $d|a$ as well as $d|b$. We write $\gcd(a, b) := d$ for this number. Since the natural number 1 divides any other integer, 1 is always a common divisor of any two non-zero integers. However it must not be the greatest.

A common method to compute the greatest common divisor is the so called Euclidean algorithm. However since we don't need that algorithm in this book, we will introduce the Extended Euclidean algorithm which is a method to calculate the greatest common divisor of two natural numbers a and $b \in \mathbb{N}$, as well as two additional integers $s, t \in \mathbb{Z}$, such that the following equation holds:

$$\gcd(a, b) = s \cdot a + t \cdot b \quad (3.10)$$

The pseudocode in algorithm 1 shows in detail how to calculate the greatest common divisor and the numbers s and t with the extended Euclidean algorithm:

Algorithm 1 Extended Euclidean Algorithm

Require: $a, b \in \mathbb{N}$ with $a \geq b$

procedure EXT-EUCLID(a, b)

$r_0 \leftarrow a$

$r_1 \leftarrow b$

$s_0 \leftarrow 1$

$s_1 \leftarrow 0$

$k \leftarrow 1$

while $r_k \neq 0$ **do**

$q_k \leftarrow r_{k-1} \text{ div } r_k$

$r_{k+1} \leftarrow r_{k-1} - q_k \cdot r_k$

$s_{k+1} \leftarrow s_{k-1} - q_k \cdot s_k$

$k \leftarrow k + 1$

end while

return $\gcd(a, b) \leftarrow r_{k-1}$, $s \leftarrow s_{k-1}$ and $t := (r_{k-1} - s_{k-1} \cdot a) \text{ div } b$

end procedure

Ensure: $\gcd(a, b) = s \cdot a + t \cdot b$

The algorithm is simple enough to be done effectively in pen-and-paper examples, where it is common to write it as a table where the rows represent the while-loop and the columns represent the values of the the array r , s and t with index k . The following example provides a simple execution:

Example 4. To illustrate algorithm 1, we apply it to the numbers $a = 12$ and $b = 5$. Since $12, 5 \in \mathbb{N}$ as well as $12 \geq 5$ all requirements are met and we compute as follows:

k	r_k	s_k	$t_k = (r_k - s_k \cdot a) \text{ div } b$
0	12	1	0
1	5	0	1
2	2	1	-2
3	1	-2	5
4	0		

From this we can see that the greatest common divisor of 12 and 5 is $\gcd(12, 5) = 1$ and that the equation $1 = (-2) \cdot 12 + 5 \cdot 5$ holds. We can also invoke sage to double check our findings:


```

1099 sage: ZZ(12).xgcd(ZZ(5)) # (gcd(a,b), s, t)
1100 (1, -2, 5)

```

42

43

1101 *Exercise 7* (Extended Euclidean Algorithm). Find integers $s, t \in \mathbb{Z}$ such that $\gcd(a, b) = s \cdot a +$
 1102 $t \cdot b$ holds for the following pairs $(a, b) = (45, 10)$, $(a, b) = (13, 11)$, $(a, b) = (13, 12)$. What
 1103 pairs (a, b) are coprime?

1104 *Exercise 8* (Towards Prime fields). Let $n \in \mathbb{N}$ be a natural number and p a prime number, such
 1105 that $n < p$. What is the greatest common divisor $\gcd(p, n)$?

1106 *Exercise 9*. Find all numbers $k \in \mathbb{N}$ with $0 \leq k \leq 100$ such that $\gcd(100, k) = 5$.

1107 *Exercise 10*. Show that $\gcd(n, m) = \gcd(n + m, m)$ for all $n, m \in \mathbb{N}$.

1108 **Coprime Integers** Coprime integers are integers that do not have a common prime number
 1109 as a factor. As we will see in 3.3 those numbers are important for our purposes because in
 1110 modular arithmetic, computation that involve coprime numbers are substantially different from
 1111 computations on non-coprime numbers 3.3.

1112 The naive way to decide if two integers are coprime would be to divide both number suces-
 1113 sively by all prime numbers smaller then those numbers to see if they share a common prime
 1114 factor. However two integers are coprime if and only if their greatest common divisor is 1 and
 1115 hence computing the \gcd is the preferred method.

1116 *Example 5*. Consider example 4 again. As we have seen, the greatest common divisor of 12
 1117 and 5 is 1. This implies that the integers 12 and 5 are coprime, since they share no divisor other
 1118 then 1, which is not a prime number.

1119 *Exercise 11*. Consider exercise 7 again. Which pairs (a, b) from that exercise are coprime?

1120 3.3 Modular arithmetic

1121 **Modular arithmetic** is a system of integer arithmetic, where numbers “wrap around” when
 1122 reaching a certain value, much like calculations on a clock wrap around whenever the value
 1123 exceeds the number 12. For example, if the clock shows that it is 11 o’clock, then 20 hours
 1124 later it will be 7 o’clock, not 31 o’clock. The number 31 has no meaning on a normal clock that
 1125 shows hours.

1126 The number at which the wrap occurs is called the **modulus**. Modular arithmetic general-
 1127 izes the clock example to arbitrary moduli and studies equations and phenomena that arise in
 1128 this new kind of arithmetic. It is of central importance for understanding most modern crypto
 1129 systems, in large parts because modular arithmetic provides the computational infrastrutute for
 1130 algebraic types that have cryptographically useful examples of one-way functions.

1131 Although modular arithmetic appears very different from ordinary integer arithmetic that
 1132 we are all familiar with, we encourage the interested reader to work through the example and to
 1133 discover that, once they get used to the idea that this is a new kind of calculations, it will seem
 1134 much less daunting.

1135 **Congruence** In what follows, let $n \in \mathbb{N}$ with $n \geq 2$ be a fixed natural number that we will
 1136 call the **modulus** of our modular arithmetic system. With such an n given, we can then group
 1137 integers into classes, by saying that two integers are in the same class, whenever their Euclidean
 1138 division 3.2 by n will give the same remainder. We then say that two numbers are **congruent**
 1139 whenever they are in the same class.

Example 6. If we choose $n = 12$ as in our clock example, then the integers $-7, 5, 17$ and 29 are all congruent with respect to 12 , since all of them have the remainder 5 if we perform Euclidean division on them by 12 . In the picture of an analog 12-hour clock, starting at 5 o'clock, when we add 12 hours we are again at 5 o'clock, representing the number 17 . On the other hand, when we subtract 12 hours, we are at 5 o'clock again, representing the number -7 .

We can formalize this intuition of what congruence should be into a proper definition utilizing Euclidean division (as explained previously in 3.2): Let $a, b \in \mathbb{Z}$ be two integers and $n \in \mathbb{N}$ a natural number, such that $n \geq 2$. Then a and b are said to be **congruent with respect to the modulus n** , if and only if the following equation holds

$$a \bmod n = b \bmod n \quad (3.11)$$

If, on the other hand, two numbers are not congruent with respect to a given modulus n , we call them **incongruent** w.r.t. n .

A **congruence** is then nothing but an equation "up to congruence", which means that the equation only needs to hold if we take the modulus on both sides. In which case we write

$$a \equiv b \pmod{n} \quad (3.12)$$

Exercise 12. Which of the following pairs of numbers are congruent with respect to the modulus 13 : $(5, 19)$, $(13, 0)$, $(-4, 9)$, $(0, 0)$.

Exercise 13. Find all integers x , such that the congruence $x \equiv 4 \pmod{6}$ is satisfied.

Computational Rules Having defined the notion of a congruence as an equation "up to a modulus", a follow up question is if we can manipulate a congruence similar to an equation. Indeed we can almost apply the same substitution rules to a congruency then to an equation, with the main difference being that for some non-zero integer $k \in \mathbb{Z}$, the congruence $a \equiv b \pmod{n}$ is equivalent to the congruence $k \cdot a \equiv k \cdot b \pmod{n}$ only, if k and the modulus n are coprime 3.2. The following list gives a set of useful rules:

Suppose that integers $a_1, a_2, b_1, b_2, k \in \mathbb{Z}$ are given. Then the following arithmetic rules hold for congruencies:

- $a_1 \equiv b_1 \pmod{n} \Leftrightarrow a_1 + k \equiv b_1 + k \pmod{n}$ (compatibility with translation)
- $a_1 \equiv b_1 \pmod{n} \Rightarrow k \cdot a_1 \equiv k \cdot b_1 \pmod{n}$ (compatibility with scaling)
- $\gcd(k, n) = 1$ and $k \cdot a_1 \equiv k \cdot b_1 \pmod{n} \Rightarrow a_1 \equiv b_1 \pmod{n}$
- $k \cdot a_1 \equiv k \cdot b_1 \pmod{k \cdot n} \Rightarrow a_1 \equiv b_1 \pmod{n}$
- $a_1 \equiv b_1 \pmod{n}$ and $a_2 \equiv b_2 \pmod{n} \Rightarrow a_1 + a_2 \equiv b_1 + b_2 \pmod{n}$ (compatibility with addition)
- $a_1 \equiv b_1 \pmod{n}$ and $a_2 \equiv b_2 \pmod{n} \Rightarrow a_1 \cdot a_2 \equiv b_1 \cdot b_2 \pmod{n}$ (compatibility with multiplication)

Other rules, such as compatibility with subtraction, follow from the rules above. For example, compatibility with subtraction follows from compatibility with scaling by $k = -1$ and compatibility with addition.

Another property of congruencies, not known in the traditional arithmetic of integers is **Fermat's Little Theorem**. In simple words, it states that, in modular arithmetic, every number

1177 raised to the power of a prime number modulus is congruent to the number itself. Or, to be more
 1178 precise, if $p \in \mathbb{P}$ is a prime number and $k \in \mathbb{Z}$ is an integer, then:

$$k^p \equiv k \pmod{p}, \quad (3.13)$$

1179 If k is coprime to p , then we can divide both sides of this congruence by k and rewrite the
 1180 expression into the equivalent form

$$k^{p-1} \equiv 1 \pmod{p} \quad (3.14)$$

1181 The following sage code computes example effects of Fermat's little theorem and highlights the
 1182 effects of the exponent k being coprime and not coprime to p :

```

1183 sage: ZZ(137).gcd(ZZ(64))                                44
1184 1                                                            45
1185 sage: ZZ(64)^ZZ(137) % ZZ(137) == ZZ(64) % ZZ(137)      46
1186 True                                                         47
1187 sage: ZZ(64)^ZZ(137-1) % ZZ(137) == ZZ(1) % ZZ(137)      48
1188 True                                                         49
1189 sage: ZZ(1918).gcd(ZZ(137))                                50
1190 137                                                         51
1191 sage: ZZ(1918)^ZZ(137) % ZZ(137) == ZZ(1918) % ZZ(137)    52
1192 True                                                         53
1193 sage: ZZ(1918)^ZZ(137-1) % ZZ(137) == ZZ(1) % ZZ(137)    54
1194 False                                                         55

```

1195 Let's compute an example that contains most of the concepts described in this section:

Example 7. Assume that we consider the modulus 6 and that our task is to solve the following congruence for $x \in \mathbb{Z}$

$$7 \cdot (2x + 21) + 11 \equiv x - 102 \pmod{6}$$

As many rules for congruencies are more or less same as for integers, we can proceed in a similar way as we would if we had an equation to solve. Since both sides of a congruence contain ordinary integers, we can rewrite the left side as follows: $7 \cdot (2x + 21) + 11 = 14x + 147 + 11 = 14x + 158$. We can therefore rewrite the congruence into the equivalent form

$$14x + 158 \equiv x - 102 \pmod{6}$$

In the next step we want to shift all instances of x to left and every other term to the right. So we apply the “compatibility with translation” rules two times. In a first step we choose $k = -x$ and in a second step we choose $k = -158$. Since “compatibility with translation” transforms a congruence into an equivalent form, the solution set will not change and we get

$$\begin{aligned}
 14x + 158 &\equiv x - 102 \pmod{6} \Leftrightarrow \\
 14x - x + 158 - 158 &\equiv x - x - 102 - 158 \pmod{6} \Leftrightarrow \\
 13x &\equiv -260 \pmod{6}
 \end{aligned}$$

If our congruence would just be a normal integer equation, we would divide both sides by 13 to get $x = -20$ as our solution. However, in case of a congruence, we need to make sure that the modulus and the number we want to divide by are coprime first – only then will we get an equivalent expression (See rule XXX). So we need to find the greatest common divisor

$\gcd(13, 6)$. Since 13 is prime and 6 is not a multiple of 13, we know that $\gcd(13, 6) = 1$, so these numbers are indeed coprime. We therefore compute

$$13x \equiv -260 \pmod{6} \Leftrightarrow x \equiv -20 \pmod{6}$$

Our task is now to find all integers x , such that x is congruent to -20 with respect to the modulus 6. So we have to find all x such

$$x \bmod 6 = -20 \bmod 6$$

Since $-4 \cdot 6 + 4 = -20$ we know $-20 \bmod 6 = 4$ and hence we know that $x = 4$ is a solution to this congruence. However, 22 is another solution since $22 \bmod 6 = 4$ as well, and so is -20 . In fact, there are infinitely many solutions given by the set

$$\{\dots, -8, -2, 4, 10, 16, \dots\} = \{4 + k \cdot 6 \mid k \in \mathbb{Z}\}$$

Putting all this together, we have shown that the every x from the set $\{x = 4 + k \cdot 6 \mid k \in \mathbb{Z}\}$ is a solution to the congruence $7 \cdot (2x + 21) + 11 \equiv x - 102 \pmod{6}$. We double ckeck for, say, $x = 4$ as well as $x = 4 + 12 \cdot 6 = 76$ using sage:

```
1196 sage: (ZZ(7) * (ZZ(2) * ZZ(4) + ZZ(21)) + ZZ(11)) % ZZ(6) == (ZZ 56
1197         (4) - ZZ(102)) % ZZ(6)
1198
1199 True 57
1200
1201 sage: (ZZ(7) * (ZZ(2) * ZZ(76) + ZZ(21)) + ZZ(11)) % ZZ(6) == ( 58
1202         ZZ(76) - ZZ(102)) % ZZ(6)
1203
1204 True 59
```

Readers who had not been familiar with modular arithmetic until now and who might be discouraged by how complicated modular arithmetic seems at this point, should keep two things in mind. First, computing congruencies in modular arithmetic is not really more complicated than computations in more familiar number systems (e.g. rational numbers), it is just a matter of getting used to it. Second, once we introduce the idea of remainder class representations 3.3, computations become conceptually cleaner and more easy to handle.

Exercise 14. Consider the modulus 13 and find all solutions $x \in \mathbb{Z}$ to the following congruence $5x + 4 \equiv 28 + 2x \pmod{13}$

Exercise 15. Consider the modulus 23 and find all solutions $x \in \mathbb{Z}$ to the following congruence $69x \equiv 5 \pmod{23}$

Exercise 16. Consider the modulus 23 and find all solutions $x \in \mathbb{Z}$ to the following congruence $69x \equiv 46 \pmod{23}$

Exercise 17. Let a, b, k be integers, such that $a \equiv b \pmod{n}$ holds. Show $a^k \equiv b^k \pmod{n}$.

Exercise 18. Let a, n be integers, such that a and n are not coprime. For which $b \in \mathbb{Z}$ does the congruence $a \cdot x \equiv b \pmod{n}$ have a solution x and how does the solution set look in that case?

The Chinese Remainder Theorem We have seen how to solve congruencies in modular arithmetic. However, one question that remains is how to solve systems of congruencies with different moduli? The answer is given by the **Chinese reimainder theorem**, which states that

1224 for any $k \in \mathbb{N}$ and coprime natural numbers $n_1, \dots, n_k \in \mathbb{N}$ as well as integers $a_1, \dots, a_k \in \mathbb{Z}$, the
 1225 so-called **simultaneous congruences**

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2} \\ &\dots \\ x &\equiv a_k \pmod{n_k} \end{aligned} \tag{3.15}$$

1226 has a solution, and all possible solutions of this congruence system are congruent modulo the product $N = n_1 \cdot \dots \cdot n_k$.¹ In fact, the following algorithm computes the solution set:

check
algorithm
floating

Algorithm 2 Chinese Remainder Theorem

Require: $k \in \mathbb{Z}$, $j \in \mathbb{N}_0$ and $n_0, \dots, n_{k-1} \in \mathbb{N}$ coprime

procedure CONGRUENCE-SYSTEMS-SOLVER(a_0, \dots, a_{k-1})

$N \leftarrow n_0 \cdot \dots \cdot n_{k-1}$

while $j < k$ **do**

$N_j \leftarrow N/n_j$

$(_, s_j, t_j) \leftarrow \text{EXT-EUCLID}(N_j, n_j)$ $\triangleright 1 = s_j \cdot N_j + t_j \cdot n_j$

end while

$x' \leftarrow \sum_{j=0}^{k-1} a_j \cdot s_j \cdot N_j$

$x \leftarrow x' \bmod N$

return $\{x + m \cdot N \mid m \in \mathbb{Z}\}$

end procedure

Ensure: $\{x + m \cdot N \mid m \in \mathbb{Z}\}$ is the complete solution set to 3.15.

1227

Example 8. To illustrate how to solve simultaneous congruences using the Chinese remainder theorem, let's look at the following system of congruencies:

$$\begin{aligned} x &\equiv 4 \pmod{7} \\ x &\equiv 1 \pmod{3} \\ x &\equiv 3 \pmod{5} \\ x &\equiv 0 \pmod{11} \end{aligned}$$

Clearly all moduli are coprime and we have $N = 7 \cdot 3 \cdot 5 \cdot 11 = 1155$, as well as $N_1 = 165$, $N_2 = 385$, $N_3 = 231$ and $N_4 = 105$. From this we calculate with the extended Euclidean algorithm

$$\begin{aligned} 1 &= 2 \cdot 165 + (-47) \cdot 7 \\ 1 &= 1 \cdot 385 + (-128) \cdot 3 \\ 1 &= 1 \cdot 231 + (-46) \cdot 5 \\ 1 &= 2 \cdot 105 + (-19) \cdot 11 \end{aligned}$$

1228 so we have $x = 4 \cdot 2 \cdot 165 + 1 \cdot 1 \cdot 385 + 3 \cdot 1 \cdot 231 + 0 \cdot 2 \cdot 105 = 2398$ as one solution. Because
 1229 $2398 \bmod 1155 = 88$ the set of all solutions is $\{\dots, -2222, -1067, 88, 1243, 2398, \dots\}$. We
 1230 can invoke Sage's computation of the Chinese Remainder Theorem (CRT) to double check our
 1231 findings:

1232 **sage:** CRT_list([4, 1, 3, 0], [7, 3, 5, 11])

60

1233 **88**

61

¹This is the classical Chinese remainder theorem as it was already known in ancient China. Under certain circumstances, the theorem can be extended to non-coprime moduli n_1, \dots, n_k but this is beyond the scope of this book. Interested readers should consult XXX [add references](#)

Remainder Class Representation As we have seen in various examples before, computing congruencies can be cumbersome and solution sets are large in general. It is therefore advantageous to find some kind of simplification for modular arithmetic.

Fortunately, this is possible and relatively straightforward once we identify each set of numbers with equal remainder with that remainder itself and call it the **remainder class** or **residue class** representation in modulo n arithmetic.

It then follows from the properties of Euclidean division that there are exactly n different remainder classes for every modulus n and that integer addition and multiplication can be projected to a new kind of addition and multiplication on those classes.

Roughly speaking, the new rules for addition and multiplication are then computed by taking any element of the first remainder class and some element of the second, then add or multiply them in the usual way and see which remainder class the result is contained in. The following example makes this abstract description more concrete:

Example 9 (Arithmetic modulo 6). Choosing the modulus $n = 6$, we have six remainder classes of integers which are congruent modulo 6 (they have the same remainder when divided by 6) and when we identify each of those remainder classes with the remainder, we get the following identification:

$$\begin{aligned} 0 &:= \{\dots, -6, 0, 6, 12, \dots\} \\ 1 &:= \{\dots, -5, 1, 7, 13, \dots\} \\ 2 &:= \{\dots, -4, 2, 8, 14, \dots\} \\ 3 &:= \{\dots, -3, 3, 9, 15, \dots\} \\ 4 &:= \{\dots, -2, 4, 10, 16, \dots\} \\ 5 &:= \{\dots, -1, 5, 11, 17, \dots\} \end{aligned}$$

Now to compute the new addition law of those remainder class representatives, say $2 + 5$, one chooses arbitrary elements from both classes, say 14 and -1 , adds those numbers in the usual way and then looks at the remainder class of the result.

So we get $14 + (-1) = 13$, and 13 is in the remainder class (of) 1. Hence we find that $2 + 5 = 1$ in modular 6 arithmetic, which is a more readable way to write the congruence $2 + 5 \equiv 1 \pmod{6}$.

Applying the same reasoning to all remainder classes, addition and multiplication can be transferred to the representatives of the remainder classes. The results for modulus 6 arithmetic are summarized in the following addition and multiplication tables:

+	0	1	2	3	4	5	·	0	1	2	3	4	5
0	0	1	2	3	4	5	0	0	0	0	0	0	0
1	1	2	3	4	5	0	1	0	1	2	3	4	5
2	2	3	4	5	0	1	2	0	2	4	0	2	4
3	3	4	5	0	1	2	3	0	3	0	3	0	3
4	4	5	0	1	2	3	4	0	4	2	0	4	2
5	5	0	1	2	3	4	5	0	5	4	3	2	1

This way, we have defined a new arithmetic system that contains just 6 numbers and comes with its own definition of addition and multiplication. We call it **modular 6 arithmetic** and write the associated type as \mathbb{Z}_6 .

To see why such an identification of a remainder class with its remainder is useful and actually simplifies congruence computations a lot, let's go back to the congruence from example 7 again:

$$7 \cdot (2x + 21) + 11 \equiv x - 102 \pmod{6} \quad (3.16)$$

1263 As shown in example 7, the arithmetic of congruencies can deviate from ordinary arithmetic:
 1264 For example, division needs to check whether the modulus and the dividend are coprimes, and
 1265 solutions are not unique in general.

We can rewrite this congruence as an **equation** over our new arithmetic type \mathbb{Z}_6 by **projecting onto the remainder classes**. In particular, since $7 \bmod 6 = 1$, $21 \bmod 6 = 3$, $11 \bmod 6 = 5$ and $102 \bmod 6 = 0$ we have

$$7 \cdot (2x + 21) + 11 \equiv x - 102 \pmod{6} \text{ over } \mathbb{Z} \\ \Leftrightarrow 1 \cdot (2x + 3) + 5 = x \text{ over } \mathbb{Z}_6$$

1266 We can use the multiplication and addition table above to solves the equation on the right like
 1267 we would solve normal integer equations:

$$\begin{array}{ll} 1 \cdot (2x + 3) + 5 = x & \\ 2x + 3 + 5 = x & \# \text{ addition-table: } 3 + 5 = 2 \\ 2x + 2 = x & \# \text{ add 4 and } -x \text{ on both sides} \\ 2x + 2 + 4 - x = x + 4 - x & \# \text{ addition-table: } 2 + 4 = 0 \\ x = 4 & \end{array}$$

1268 As we can see, despite the somewhat unfamiliar rules of addition and multiplication, solving
 1269 congruencies this way is very similar to solving normal equations. And, indeed, the solution
 1270 set is identical to the solution set of the original congruence, since 4 is identified with the set
 1271 $\{4 + 6 \cdot k \mid k \in \mathbb{Z}\}$.

1272 We can invoke Sage to do computations in our modular 6 arithmetic type. This is particularly
 1273 useful to double-check our computations:

```
1274 sage: Z6 = Integers(6) 62
1275 sage: Z6(2) + Z6(5) 63
1276 1 64
1277 sage: Z6(7) * (Z6(2) * Z6(4) + Z6(21)) + Z6(11) == Z6(4) - Z6(102) 65
1278 True 66
```

1279 *Remark 2 (k-bit modulus).* In cryptographic papers, we sometimes read phrases like “[...] using
 1280 a 4096-bit modulus”. This means that the underlying modulus n of the modular arithmetic used
 1281 in the system has a binary representation with a length of 4096 bits. In contrast, the number 6
 1282 has the binary representation 110 and hence our example 9 describes a 3-bit modulus arithmetic
 1283 system.

1284 *Exercise 19.* Define \mathbb{Z}_{13} as the the arithmetic modulo 13 analog to example 9. Then consider
 1285 the congruence from exercise 14 and rewrite it into an equation in \mathbb{Z}_{13}

1286 **Modular Inverses** As we know, integers can be added, subtracted and multiplied so that the
 1287 result is also an integer, but this is not true for the division of integers in general: for example,
 1288 $3/2$ is not an integer anymore. To see why this is, from a more theoretical perspective, let us
 1289 consider the definition of a multiplicative inverse first. When we have a set that has some kind
 1290 of multiplication defined on it and we have a distinguished element of that set that behaves
 1291 neutrally with respect to that multiplication (doesn’t change anything when multiplied with any
 1292 other element), then we can define **multiplicative inverses** in the following way:

Let S be our set that has some notion $a \cdot b$ of multiplication and a **neutral element** $1 \in S$, such that $1 \cdot a = a$ for all elements $a \in S$. Then a **multiplicative inverse** a^{-1} of an element $a \in S$ is defined as follows:

$$a \cdot a^{-1} = 1 \quad (3.17)$$

Informally speaking, the definition of a multiplicative inverse means that it “cancels” the original element to give 1 when they are multiplied.

Numbers that have multiplicative inverses are of particular interest, because they immediately lead to the definition of division by those numbers. In fact, if a is number such that the multiplicative inverse a^{-1} exists, then we define **division** by a simply as multiplication by the inverse:

$$\frac{b}{a} := b \cdot a^{-1} \quad (3.18)$$

Example 10. Consider the set of rational numbers, also known as fractions, \mathbb{Q} . For this set, the neutral element of multiplication is 1, since $1 \cdot a = a$ for all rational numbers. For example, $1 \cdot 4 = 4$, $1 \cdot \frac{1}{4} = \frac{1}{4}$, or $1 \cdot 0 = 0$ and so on.

Every rational number $a \neq 0$ has a multiplicative inverse, given by $\frac{1}{a}$. For example, the multiplicative inverse of 3 is $\frac{1}{3}$, since $3 \cdot \frac{1}{3} = 1$, the multiplicative inverse of $\frac{5}{7}$ is $\frac{7}{5}$, since $\frac{5}{7} \cdot \frac{7}{5} = 1$, and so on.

Example 11. Looking at the set \mathbb{Z} of integers, we see that with respect to multiplication the neutral element is the number 1 and we notice that no integer other than 1 or -1 has a multiplicative inverse, since the equation $a \cdot x = 1$ has no integer solutions for $a \neq 1$ or $a \neq -1$.

The definition of multiplicative inverse works verbatim for addition as well where it is called the additive inverse. In the case of integers, the neutral element with respect to addition is 0, since $a + 0 = 0$ for all integers $a \in \mathbb{Z}$. The additive inverse always exist and is given by the negative number $-a$, since $a + (-a) = 0$.

Example 12. Looking at the set \mathbb{Z}_6 of residual classes modulo 6 from example 9, we can use the multiplication table to find multiplicative inverses. To do so, we look at the row of the element and then find the entry equal to 1. If such an entry exists, the element of that column is the multiplicative inverse. If, on the other hand, the row has no entry equal to 1, we know that the element has no multiplicative inverse.

For example in \mathbb{Z}_6 the multiplicative inverse of 5 is 5 itself, since $5 \cdot 5 = 1$. We can also see that 5 and 1 are the only elements that have multiplicative inverses in \mathbb{Z}_6 .

Now, since 5 has a multiplicative inverse in modulo 6 arithmetic, we can divide by 5 in \mathbb{Z}_6 , since we have a notation of multiplicative inverse and division is nothing but multiplication by the multiplicative inverse. For example

$$\frac{4}{5} = 4 \cdot 5^{-1} = 4 \cdot 5 = 2$$

From the last example, we can make the interesting observation that while 5 has no multiplicative inverse as an integer, it has a multiplicative inverse in modular 6 arithmetic.

This raises the question which numbers have multiplicative inverses in modular arithmetic. The answer is that, in modular n arithmetic, a number r has a multiplicative inverse, if and only if n and r are coprime. Since $\gcd(n, r) = 1$ in that case, we know from the extended Euclidean algorithm that there are numbers s and t , such that

$$1 = s \cdot n + t \cdot r \quad (3.19)$$

If we take the modulus n on both sides, the term $s \cdot n$ vanishes, which tells us that $t \bmod n$ is the multiplicative inverse of r in modular n arithmetic.

Example 13 (Multiplicative inverses in \mathbb{Z}_6). In the previous example, we looked up multiplicative inverses in \mathbb{Z}_6 from the lookup-table in Example 9. In real world examples, it is usually impossible to write down those lookup tables, as the modulus is way too large, and the sets occasionally contain more elements than there are atoms in the observable universe.

Now, trying to determine that $2 \in \mathbb{Z}_6$ has no multiplicative inverse in \mathbb{Z}_6 without using the lookup table, we immediately observe that 2 and 6 are not coprime, since their greatest common divisor is 2. It follows that equation 3.19 has no solutions s and t , which means that 2 has no multiplicative inverse in \mathbb{Z}_6 .

The same reasoning works for 3 and 4, as neither of these are coprime with 6. The case of 5 is different, since $\gcd(6, 5) = 1$. To compute the multiplicative inverse of 5, we use the extended Euclidean algorithm and compute the following:

k	r_k	s_k	$t_k = (r_k - s_k \cdot a) \div b$
0	6	1	0
1	5	0	1
2	1	1	-1
3	0	.	.

We get $s = 1$ as well as $t = -1$ and have $1 = 1 \cdot 6 - 1 \cdot 5$. From this, it follows that $-1 \bmod 6 = 5$ is the multiplicative inverse of 5 in modular 6 arithmetic. We can double check using Sage:

```
sage: ZZ(6).xgcd(ZZ(5))
(1, 1, -1)
```

67

68

At this point, the attentive reader might notice that the situation where the modulus is a prime number is of particular interest, because we know from exercise 8 that in these cases all remainder classes must have modular inverses, since $\gcd(r, n) = 1$ for prime n and any $r < n$. In fact, Fermat's little theorem provides a way to compute multiplicative inverses in this situation, since in case of a prime modulus p and $r < p$, we get the following:

$$\begin{aligned} r^p &\equiv r \pmod{p} \Leftrightarrow \\ r^{p-1} &\equiv 1 \pmod{p} \Leftrightarrow \\ r \cdot r^{p-2} &\equiv 1 \pmod{p} \end{aligned}$$

This tells us that the multiplicative inverse of a residue class r in modular p arithmetic is precisely r^{p-2} .

Example 14 (Modular 5 arithmetic). To see the unique properties of modular arithmetic when the modulus is a prime number, we will replicate our findings from example 9, but this time for the prime modulus 5. For $n = 5$ we have five equivalence classes of integers which are congruent modulo 5. We write this as follows:

$$\begin{aligned} 0 &:= \{\dots, -5, 0, 5, 10, \dots\} \\ 1 &:= \{\dots, -4, 1, 6, 11, \dots\} \\ 2 &:= \{\dots, -3, 2, 7, 12, \dots\} \\ 3 &:= \{\dots, -2, 3, 8, 13, \dots\} \\ 4 &:= \{\dots, -1, 4, 9, 14, \dots\} \end{aligned}$$

Addition and multiplication can be transferred to the equivalence classes, in a way exactly parallel to Example 9. This results in the following addition and multiplication tables:

+	0	1	2	3	4	·	0	1	2	3	4
0	0	1	2	3	4	0	0	0	0	0	0
1	1	2	3	4	0	1	0	1	2	3	4
2	2	3	4	0	1	2	0	2	4	1	3
3	3	4	0	1	2	3	0	3	1	4	2
4	4	0	1	2	3	4	0	4	3	2	1

1350

1351 Calling the set of remainder classes in modular 5 arithmetic with this addition and multiplication
 1352 \mathbb{Z}_5 , we see some subtle but important differences to the situation in \mathbb{Z}_6 . In particular, we see
 1353 that in the multiplication table, every remainder $r \neq 0$ has the entry 1 in its row and therefore
 1354 has a multiplicative inverse. In addition, there are no non-zero elements such that their product
 1355 is zero.

1356 To use Fermat's little theorem in \mathbb{Z}_5 for computing multiplicative inverses (instead of using
 1357 the multiplication table), let's consider $3 \in \mathbb{Z}_5$. We know that the multiplicative inverse is given
 1358 by the remainder class that contains $3^{5-2} = 3^3 = 3 \cdot 3 \cdot 3 = 4 \cdot 3 = 2$. And indeed $3^{-1} = 2$, since
 1359 $3 \cdot 2 = 1$ in \mathbb{Z}_5 .

1360 We can invoke Sage to do computations in our modular 5 arithmetic type to double-check
 1361 our computations:

```

1362 sage: Z5 = Integers(5)                                69
1363 sage: Z5(3) ** (5-2)                                    70
1364 2                                                         71
1365 sage: Z5(3) ** (-1)                                     72
1366 2                                                         73
1367 sage: Z5(3) ** (5-2) == Z5(3) ** (-1)                  74
1368 True                                                    75

```

Example 15. To understand one of the principal differences between prime number modular arithmetic and non-prime number modular arithmetic, consider the linear equation $a \cdot x + b = 0$ defined over both types \mathbb{Z}_5 and \mathbb{Z}_6 . Since in \mathbb{Z}_5 every non-zero element has a multiplicative inverse, we can always solve these equations in \mathbb{Z}_5 , which is not true in \mathbb{Z}_6 . To see that, consider the equation $3x + 3 = 0$. In \mathbb{Z}_5 we have the following:

$$\begin{array}{ll}
 3x + 3 = 0 & \# \text{ add 2 and on both sides} \\
 3x + 3 + 2 = 2 & \# \text{ addition-table: } 2 + 3 = 0 \\
 3x = 2 & \# \text{ divide by 3 (which equals multiplication by 2)} \\
 2 \cdot (3x) = 2 \cdot 2 & \# \text{ multiplication-table: } 2 \cdot 2 = 4 \\
 x = 4 &
 \end{array}$$

So in the case of our prime number modular arithmetic, we get the unique solution $x = 4$. Now consider \mathbb{Z}_6 :

$$\begin{array}{ll}
 3x + 3 = 0 & \# \text{ add 3 and on both sides} \\
 3x + 3 + 3 = 3 & \# \text{ addition-table: } 3 + 3 = 0 \\
 3x = 3 & \# \text{ division not possible (no multiplicative inverse of 3 exists)}
 \end{array}$$

1369 So, in this case, we cannot solve the equation for x by dividing by 3. And, indeed, when we look
 1370 at the multiplication table of \mathbb{Z}_6 (Example 9), we find that there are three solutions $x \in \{1, 3, 5\}$,
 1371 such that $3x + 3 = 0$ holds true for all of them.

1372 *Exercise 20.* Consider the modulus $n = 24$. Which of the integers 7, 1, 0, 805, -4255 have
 1373 multiplicative inverses in modular 24 arithmetic? Compute the inverses, in case they exist.

1374 *Exercise 21.* Find the set of all solutions to the congruence $17(2x + 5) - 4 \equiv 2x + 4 \pmod{5}$.
 1375 Then project the congruence into \mathbb{Z}_5 and solve the resulting equation in \mathbb{Z}_5 . Compare the results.

1376 *Exercise 22.* Find the set of all solutions to the congruence $17(2x + 5) - 4 \equiv 2x + 4 \pmod{6}$.
 1377 Then project the congruence into \mathbb{Z}_6 and try to solve the resulting equation in \mathbb{Z}_6 .

1378 3.4 Polynomial arithmetic

1379 A polynomial is an expression consisting of variables (also called indeterminates) and coeffi-
 1380 cients that involves only the operations of addition, subtraction and multiplication. All coeffi-
 1381 cients of a polynomial must have the same type, e.g. being integers or rational numbers etc. To
 1382 be more precise an *univariate polynomial* is an expression

$$P(x) := \sum_{j=0}^m a_j x^j = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0, \quad (3.20)$$

1383 where x is called the **indeterminate**, each a_j is called a **coefficient**. If R is the type of the
 1384 coefficients, then the set of all **univariate² polynomials with coefficients in R** is written as
 1385 $R[x]$. We often simply use **polynomial** instead of univariate polynomial, write $P(x) \in R[x]$ for a
 1386 polynomial and denote the constant term a_0 as $P(0)$.

1387 A polynomial is called the **zero polynomial** if all coefficients are zero and a polynomial is
 1388 called the **one polynomial** if the constant term is 1 and all other coefficients are zero.

1389 Given an univariate polynomial $P(x) = \sum_{j=0}^m a_j x^j$ that is not the zero polynomial, we call the
 1390 non-negative integer $\deg(P) := m$ the *degree* of P and define the degree of the zero polynomial
 1391 to be $-\infty$, where $-\infty$ (negative infinity) is a symbol with the properties that $-\infty + m = -\infty$ and
 1392 $-\infty < m$ for all non-negative integers $m \in \mathbb{N}_0$. In addition, we write

$$Lc(P) := a_m \quad (3.21)$$

1393 and call it the **leading coefficient** of the polynomial P . We can restrict the set $R[x]$ of **all**
 1394 polynomials with coefficients in R , to the set of all such polynomials that have a degree that
 1395 does not exceed a certain value. If m is the maximum degree allowed, we write $R_{\leq m}[x]$ for the
 1396 set of all polynomials with a degree less than or equal to m .

Example 16 (Integer Polynomials). The coefficients of a polynomial must all have the same type. The set of polynomials with integer coefficients is written as $\mathbb{Z}[x]$. Examples of such polynomials are:

$P_1(x) = 2x^2 - 4x + 17$	# with $\deg(P_1) = 2$ and $Lc(P_1) = 2$
$P_2(x) = x^{23}$	# with $\deg(P_2) = 23$ and $Lc(P_2) = 1$
$P_3(x) = x$	# with $\deg(P_3) = 1$ and $Lc(P_3) = 1$
$P_4(x) = 174$	# with $\deg(P_4) = 0$ and $Lc(P_4) = 174$
$P_5(x) = 1$	# with $\deg(P_5) = 0$ and $Lc(P_5) = 1$
$P_6(x) = 0$	# with $\deg(P_6) = -\infty$ and $Lc(P_6) = 0$
$P_7(x) = (x - 2)(x + 3)(x - 5)$	

²in our context the term univariate means that the polynomial contains a single variable only

In particular, every integer can be seen as an integer polynomial of degree zero. P_7 is a polynomial, because we can expand its definition into $P_7(x) = x^3 - 4x^2 - 11x + 30$, which is a polynomial of degree 3 and leading coefficient 1. The following expressions are not integer polynomials:

$$Q_1(x) = 2x^2 + 4 + 3x^{-2}$$

$$Q_2(x) = 0.5x^4 - 2x$$

$$Q_3(x) = 2^x$$

1397 In particular Q_1 is not an integer polynomial, because the expression x^{-2} has a negative expo-
 1398 nent, Q_2 is not an integer polynomial because the coefficient 0.5 is not an integer and Q_3 is not
 1399 an integer polynomial because the indeterminant appears in the exponent of of a coefficient.

1400 We can invoke Sage to do computations with polynomials. To do so, we have to specify the
 1401 symbol for the indeterminate and the type for the coefficients (For the definition of rings see
 1402 4.2). Note, however that Sage defines the degree of the zero polynomial to be -1 .

```

1403 sage: Zx = ZZ['x'] # integer polynomials with indeterminate x 76
1404 sage: Zt.<t> = ZZ[] # integer polynomials with indeterminate t 77
1405 sage: Zx 78
1406 Univariate Polynomial Ring in x over Integer Ring 79
1407 sage: Zt 80
1408 Univariate Polynomial Ring in t over Integer Ring 81
1409 sage: p1 = Zx([17,-4,2]) 82
1410 sage: p1 83
1411 2*x^2 - 4*x + 17 84
1412 sage: p1.degree() 85
1413 2 86
1414 sage: p1.leading_coefficient() 87
1415 2 88
1416 sage: p2 = Zt(t^23) 89
1417 sage: p2 90
1418 t^23 91
1419 sage: p6 = Zx([0]) 92
1420 sage: p6.degree() 93
1421 -1 94

```

Example 17 (Polynomials over \mathbb{Z}_6). Recall the definition of modular 6 arithmetics \mathbb{Z}_6 as defined in example 9. The set of all polynomials with indeterminate x and coefficients in \mathbb{Z}_6 is symbolized as $\mathbb{Z}_6[x]$. Example of polynomials from $\mathbb{Z}_6[x]$ are:

$P_1(x) = 2x^2 - 4x + 5$	# with $\deg(P_1) = 2$ and $Lc(P_1) = 2$
$P_2(x) = x^{23}$	# with $\deg(P_2) = 23$ and $Lc(P_2) = 1$
$P_3(x) = x$	# with $\deg(P_3) = 1$ and $Lc(P_3) = 1$
$P_4(x) = 3$	# with $\deg(P_4) = 0$ and $Lc(P_4) = 3$
$P_5(x) = 1$	# with $\deg(P_5) = 0$ and $Lc(P_5) = 1$
$P_6(x) = 0$	# with $\deg(P_5) = -\infty$ and $Lc(P_6) = 0$
$P_7(x) = (x-2)(x+3)(x-5)$	

Just like in the previous example, P_7 is a polynomial. However, since we are working with coefficients from \mathbb{Z}_6 now the expansion of P_7 is computed differently, as we have to invoke addition and multiplication in \mathbb{Z}_6 as defined in XXX. We get the following:

$$\begin{aligned}
 (x-2)(x+3)(x-5) &= (x+4)(x+3)(x+1) && \# \text{ additive inverses in } \mathbb{Z}_6 \\
 &= (x^2 + 4x + 3x + 3 \cdot 4)(x+1) && \# \text{ bracket expansion} \\
 &= (x^2 + 1x + 0)(x+1) && \# \text{ computation in } \mathbb{Z}_6 \\
 &= x^3 + x^2 + x^2 + x && \# \text{ bracket expansion} \\
 &= x^3 + 2x^2 + x
 \end{aligned}$$

1422 Again, we can use Sage to do computations with polynomials that have their coefficients in \mathbb{Z}_6
 1423 (For the definition of rings see 4.2). To do so, we have to specify the symbol for the indertemi-
 1424 nate and the type for the coefficients:

```

1425 sage: Z6 = Integers(6)                                     95
1426 sage: Z6x = Z6['x']                                       96
1427 sage: Z6x                                               97
1428 Univariate Polynomial Ring in x over Ring of integers modulo 6 98
1429 sage: p1 = Z6x([5,-4,2])                                  99
1430 sage: p1                                                100
1431 2*x^2 + 2*x + 5                                         101
1432 sage: p1 = Z6x([17,-4,2])                                102
1433 sage: p1                                                103
1434 2*x^2 + 2*x + 5                                         104
1435 sage: Z6x(x-2)*Z6x(x+3)*Z6x(x-5) == Z6x(x^3 + 2*x^2 + x) 105
1436 True                                                  106

```

1437 Given some element from the same type as the coefficients of a polynomial, the polyno-
 1438 mial can be evaluated at that element, which means that we insert the given element for every
 1439 occurrence of the indeterminate x in the polynomial expression.

1440 To be more precise, let $P \in R[x]$, with $P(x) = \sum_{j=0}^m a_j x^j$ be a polynomial with a coefficient
 1441 of type R and let $b \in R$ be an element of that type. Then the **evaluation** of P at b is given as
 1442 follows:

$$P(b) = \sum_{j=0}^m a_j b^j \quad (3.22)$$

Example 18. Consider the integer polynomials from example 16 again. To evaluate them at given points, we have to insert the point for all occurrences of x in the polynomial expression. Inserting arbitrary values from \mathbb{Z} , we get:

$$\begin{aligned}
 P_1(2) &= 2 \cdot 2^2 - 4 \cdot 2 + 17 = 17 \\
 P_2(3) &= 3^{23} = 94143178827 \\
 P_3(-4) &= -4 = -4 \\
 P_4(15) &= 174 \\
 P_5(0) &= 1 \\
 P_6(1274) &= 0 \\
 P_7(-6) &= (-6-2)(-6+3)(-6-5) = -264
 \end{aligned}$$

1443 Note, however, that it is not possible to evaluate any of those polynomial on values of different
 1444 type. For example, it is not strictly correct to write $P_1(0.5)$, since 0.5 is not an integer. We can
 1445 verify our computations using Sage:

```

1446 sage: Zx = ZZ['x']
1447 sage: p1 = Zx([17, -4, 2])
1448 sage: p7 = Zx(x-2)*Zx(x+3)*Zx(x-5)
1449 sage: p1(ZZ(2))
1450 17
1451 sage: p7(ZZ(-6)) == ZZ(-264)
1452 True

```

Example 19. Consider the polynomials with coefficients in \mathbb{Z}_6 from example again. To evaluate them at given values from \mathbb{Z}_6 , we have to insert the point for all occurrences of x in the polynomial expression. We get the following:

$$\begin{aligned}
 P_1(2) &= 2 \cdot 2^2 - 4 \cdot 2 + 5 = 2 - 2 + 5 = 5 \\
 P_2(3) &= 3^{23} = 3 \\
 P_3(-4) &= P_3(2) = 2 \\
 P_5(0) &= 1 \\
 P_6(4) &= 0
 \end{aligned}$$

```

1453
1454 sage: Z6 = Integers(6)
1455 sage: Z6x = Z6['x']
1456 sage: p1 = Z6x([5, -4, 2])
1457 sage: p1(Z6(2)) == Z6(5)
1458 True

```

1459 *Exercise 23.* Compare both expansions of P_7 from $\mathbb{Z}[x]$ and from $\mathbb{Z}_6[x]$ in example 16 and
 1460 example 19, and consider the definition of \mathbb{Z}_6 as given in example 9. Can you see how the
 1461 definition of P_7 over \mathbb{Z} projects to the definition over \mathbb{Z}_6 if you consider the residue classes of
 1462 \mathbb{Z}_6 ?

1463 **Polynomial arithmetic** Polynomials behave like integers in many ways. In particular, they
 1464 can be added, subtracted and multiplied. In addition, they have their own notion of Euclidean
 1465 division. Informally speaking, we can add two polynomials by simply adding the coefficients
 1466 of the same index, and we can multiply them by applying the distributive property, that is, by
 1467 multiplying every term of the left factor with every term of the right factor and adding the results
 1468 together.

1469 To be more precise let $\sum_{n=0}^{m_1} a_n x^n$ and $\sum_{n=0}^{m_2} b_n x^n$ be two polynomials from $R[x]$. Then the
 1470 **sum** and the **product** of these polynomials is defined as follows:

$$\sum_{n=0}^{m_1} a_n x^n + \sum_{n=0}^{m_2} b_n x^n = \sum_{n=0}^{\max\{m_1, m_2\}} (a_n + b_n) x^n \quad (3.23)$$

$$\left(\sum_{n=0}^{m_1} a_n x^n \right) \cdot \left(\sum_{n=0}^{m_2} b_n x^n \right) = \sum_{n=0}^{m_1+m_2} \sum_{i=0}^n a_i b_{n-i} x^n \quad (3.24)$$

1472 A rule for polynomial subtraction can be deduced from these two rules by first multiplying the
 1473 **subtrahend** with (the polynomial) -1 and then add the result to the **minuend**.

subtrahend

1474 Regarding the definition of the degree of a polynomial, we see that the degree of the sum is
 1475 always the maximum of the degrees of both summands, and the degree of the product is always
 1476 the degree of the sum of the factors, since we defined $-\infty + m = -\infty$ for every integer $m \in \mathbb{Z}$.

minuend

Example 20. To give an example of how polynomial arithmetic works, consider the following two integer polynomials $P, Q \in \mathbb{Z}[x]$ with $P(x) = 5x^2 - 4x + 2$ and $Q(x) = x^3 - 2x^2 + 5$. The sum of these two polynomials is computed by adding the coefficients of each term with equal exponent in x . This gives the following:

$$\begin{aligned}(P + Q)(x) &= (0 + 1)x^3 + (5 - 2)x^2 + (-4 + 0)x + (2 + 5) \\ &= x^3 + 3x^2 - 4x + 7\end{aligned}$$

The product of these two polynomials is computed by multiplication of each term in the first factor with each term in the second factor. We get the following:

$$\begin{aligned}(P \cdot Q)(x) &= (5x^2 - 4x + 2) \cdot (x^3 - 2x^2 + 5) \\ &= (5x^5 - 10x^4 + 25x^2) + (-4x^4 + 8x^3 - 20x) + (2x^3 - 4x^2 + 10) \\ &= 5x^5 - 14x^4 + 10x^3 + 21x^2 - 20x + 10\end{aligned}$$

1477

1478	sage: <code>Zx = ZZ['x']</code>	119
1479	sage: <code>P = Zx([2, -4, 5])</code>	120
1480	sage: <code>Q = Zx([5, 0, -2, 1])</code>	121
1481	sage: <code>P+Q == Zx(x^3 +3*x^2 -4*x +7)</code>	122
1482	True	123
1483	sage: <code>P*Q == Zx(5*x^5 -14*x^4 +10*x^3+21*x^2-20*x +10)</code>	124
1484	True	125

Example 21. Let us consider the polynomials of the previous example but interpreted in modular 6 arithmetic. So we consider $P, Q \in \mathbb{Z}_6[x]$ again with $P(x) = 5x^2 - 4x + 2$ and $Q(x) = x^3 - 2x^2 + 5$. This time we get the following:

$$\begin{aligned}(P + Q)(x) &= (0 + 1)x^3 + (5 - 2)x^2 + (-4 + 0)x + (2 + 5) \\ &= (0 + 1)x^3 + (5 + 4)x^2 + (2 + 0)x + (2 + 5) \\ &= x^3 + 3x^2 + 2x + 1\end{aligned}$$

$$\begin{aligned}(P \cdot Q)(x) &= (5x^2 - 4x + 2) \cdot (x^3 - 2x^2 + 5) \\ &= (5x^2 + 2x + 2) \cdot (x^3 + 4x^2 + 5) \\ &= (5x^5 + 2x^4 + 1x^2) + (2x^4 + 2x^3 + 4x) + (2x^3 + 2x^2 + 4) \\ &= 5x^5 + 4x^4 + 4x^3 + 3x^2 + 4x + 4\end{aligned}$$

1485

1486	sage: <code>Z6x = Integers(6)['x']</code>	126
1487	sage: <code>P = Z6x([2, -4, 5])</code>	127

```

1488 sage: Q = Z6x([5, 0, -2, 1]) 128
1489 sage: P+Q == Z6x(x^3 +3*x^2 +2*x +1) 129
1490 True 130
1491 sage: P*Q == Z6x(5*x^5 +4*x^4 +4*x^3+3*x^2+4*x +4) 131
1492 True 132

```

1493 *Exercise 24.* Compare the sum $P + Q$ and the product $P \cdot Q$ from the previous two examples
 1494 20 and 21 and consider the definition of \mathbb{Z}_6 as given in example 9. How can we derive the
 1495 computations in $\mathbb{Z}_6[x]$ from the computations in $\mathbb{Z}[x]$?

1496 **Euklidean Division** The arithmetic of polynomials share a lot of properties with the arith-
 1497 metic of integers and as a consequence the concept of Euclidean division and the algorithm of
 1498 long division is also defined for polynomials. Recalling the Euclidean division of integers 3.2,
 1499 we know that, given two integers a and $b \neq 0$, there is always another integer m and a natural
 1500 number r with $r < |b|$ such that $a = m \cdot b + r$ holds.

1501 We can generalize this to polynomials whenever the leading coefficient of the dividend
 1502 polynomial has a notion of multiplicative inverse. In fact, given two polynomials A and $B \neq 0$
 1503 from $R[x]$ such that $Lc(B)^{-1}$ exists in R , there exist two polynomials Q (the quotient) and P (the
 1504 remainder), such that the following equation holds:

$$A = Q \cdot B + P \quad (3.25)$$

1505 and $\deg(P) < \deg(B)$. Similarly to integer Euclidean division, both Q and P are uniquely
 1506 defined by these relations.

1507 *Notation and Symbols 2.* Suppose that the polynomials A, B, Q and P satisfy equation 3.25. We
 1508 often use the following notation to describe the quotient and the remainder polynomials of the
 1509 Euclidean division:

$$A \operatorname{div} B := Q, \quad A \operatorname{mod} B := P \quad (3.26)$$

1510 We also say that a polynomial A is divisible by another polynomial B if $A \operatorname{mod} B = 0$ holds. In
 1511 this case, we also write $B|A$ and call B a *factor* of A .

1512 Analogously to integers, methods to compute Euclidean division for polynomials are called
 1513 **polynomial division algorithms**. Probably the best known algorithm is the so called **polyno-**
 1514 **mial long division**.

1515 This algorithm works only when there is a notion of division by the leading coefficient of B .
 1516 It can be generalized, but we will only need this somewhat simpler method in what follows.

1517 *Example 22 (Polynomial Long Division).* To give an example of how the previous algorithm
 1518 works, let us divide the integer polynomial $A(x) = x^5 + 2x^3 - 9 \in \mathbb{Z}[x]$ by the integer polynomial
 1519 $B(x) = x^2 + 4x - 1 \in \mathbb{Z}[x]$. Since B is not the zero polynomial and the leading coefficient of B
 1520 is 1, which is invertible as an integer, we can apply algorithm 1. Our goal is to find solutions
 1521 to equation XXX, that is, we need to find the quotient polynomial $Q \in \mathbb{Z}[x]$ and the remainder
 1522 polynomial $P \in \mathbb{Z}[x]$ such that $x^5 + 2x^3 - 9 = Q(x) \cdot (x^2 + 4x - 1) + P(x)$. Using a notation that

algorithm-
floating

Algorithm 3 Polynomial Euclidean Algorithm**Require:** $A, B \in R[x]$ with $B \neq 0$, such that $Lc(B)^{-1}$ exists in R **procedure** POLY-LONG-DIVISION(A, B) $Q \leftarrow 0$ $P \leftarrow A$ $d \leftarrow \deg(B)$ $c \leftarrow Lc(B)$ **while** $\deg(P) \geq d$ **do** $S := Lc(P) \cdot c^{-1} \cdot x^{\deg(P)-d}$ $Q \leftarrow Q + S$ $P \leftarrow P - S \cdot B$ **end while****return** (Q, P) **end procedure****Ensure:** $A = Q \cdot B + P$

1523 is mostly used in anglophone countries, we compute as follows:

$$\begin{array}{r}
 X^3 - 4X^2 + 19X - 80 \\
 X^2 + 4X - 1 \overline{) \begin{array}{r} X^5 \\ - X^5 - 4X^4 \\ \hline -4X^4 + 3X^3 \\ 4X^4 + 16X^3 - 4X^2 \\ \hline 19X^3 - 4X^2 \\ - 19X^3 - 76X^2 + 19X \\ \hline -80X^2 + 19X - 9 \\ 80X^2 + 320X - 80 \\ \hline 339X - 89 \end{array}} \\
 \end{array} \tag{3.27}$$

1524 We therefore get $Q(x) = x^3 - 4x^2 + 19x - 80$ as well as $P(x) = 339x - 89$ and indeed we have
 1525 $x^5 + 2x^3 - 9 = (x^3 - 4x^2 + 19x - 80) \cdot (x^2 + 4x - 1) + (339x - 89)$, which we can double check
 1526 invoking Sage:

```

1527 sage: Zx = ZZ['x']                                     133
1528 sage: A = Zx([-9, 0, 0, 2, 0, 1])                       134
1529 sage: B = Zx([-1, 4, 1])                                135
1530 sage: Q = Zx([-80, 19, -4, 1])                          136
1531 sage: P = Zx([-89, 339])                                 137
1532 sage: A == Q*B + P                                       138
1533 True                                                    139

```

1534 *Example 23.* In the previous example, polynomial division gave a non-trivial (non-vanishing,
 1535 i.e non-zero) remainder. Of special interest are divisions that don't give a remainder. Such
 1536 divisors are called factors of the dividend.

1537 For example, consider the integer polynomial P_7 from example 16 again. As we have shown,
 1538 it can be written both as $x^3 - 4x^2 - 11x + 30$ and as $(x - 2)(x + 3)(x - 5)$. From this, we can
 1539 see that the polynomials $F_1(x) = (x - 2)$, $F_2(x) = (x + 3)$ and $F_3(x) = (x - 5)$ are all factors of
 1540 $x^3 - 4x^2 - 11x + 30$, since division of P_7 by any of these factors will result in a zero remainder.

1541 *Exercise 25.* Consider the polynomial expressions $A(x) := -3x^4 + 4x^3 + 2x^2 + 4$ and $B(x) =$
 1542 $x^2 - 4x + 2$. Compute the Euclidean division of A by B in the following types:

1543 1. $A, B \in \mathbb{Z}[x]$

1544 2. $A, B \in \mathbb{Z}_6[x]$

1545 3. $A, B \in \mathbb{Z}_5[x]$

1546 Now consider the result in $\mathbb{Z}[x]$ and in $\mathbb{Z}_6[x]$. How can we compute the result in $\mathbb{Z}_6[x]$ from the
 1547 result in $\mathbb{Z}[x]$?

1548 *Exercise 26.* Show that the polynomial $B(x) = 2x^4 - 3x + 4 \in \mathbb{Z}_5[x]$ is a factor of the polynomial
 1549 $A(x) = x^7 + 4x^6 + 4x^5 + x^3 + 2x^2 + 2x + 3 \in \mathbb{Z}_5[x]$ that is show $B|A$. What is $B \operatorname{div} A$?

1550 **Prime Factors** Recall that the fundamental theorem of arithmetic 3.6 tells us that every natu-
 1551 ral number is the product of prime numbers. In this chapter we will see that something similar
 1552 holds for univariate polynomials $R[x]$, too³.

1553 The polynomial analog to a prime number is a so called an **irreducible polynomial**, which
 1554 is defined as a polynomial that cannot be factored into the product of two non-constant poly-
 1555 nomials using Euclidean division. Irreducible polynomials are for polynomials what prime
 1556 numbers are for integer: They are the basic building blocks from which all other polynomials
 1557 can be constructed. To be more precise, let $P \in R[x]$ be any polynomial. Then there are always
 1558 irreducible polynomials $F_1, F_2, \dots, F_k \in R[x]$, such that the following holds:

$$P = F_1 \cdot F_2 \cdot \dots \cdot F_k. \quad (3.28)$$

1559 This representation is unique, except for permutations in the factors and is called the **prime**
 1560 **factorization** of P . Moreover each factor F_i is called a **prime factor** of P .

1561 *Example 24.* Consider the polynomial expression $P = x^2 - 3$. When we interpret P as an integer
 1562 polynomial $P \in \mathbb{Z}[x]$, we find that this polynomial is irreducible, since any factorization other
 1563 than $1 \cdot (x^2 - 3)$, must look like $(x - a)(x + a)$ for some integer a , but there is no integers a with
 1564 $a^2 = 3$.

1565	<code>sage: Zx = ZZ['x']</code>	140
1566	<code>sage: p = Zx(x^2-3)</code>	141
1567	<code>sage: p.factor()</code>	142
1568	<code>x^2 - 3</code>	143

1569 On the other hand interpreting P as a polynomial $P \in \mathbb{Z}_6[x]$ in modulo 6 arithmetic, we see that P
 1570 has two factors $F_1 = (x - 3)$ and $F_2 = (x + 3)$, since $(x - 3)(x + 3) = x^2 - 3x + 3x - 3 \cdot 3 = x^2 - 3$.

1571 Points where a polynomial evaluates to zero are called **roots** of the polynomial. To be more
 1572 precise, let $P \in R[x]$ be a polynomial. Then a root is a point $x_0 \in R$ with $P(x_0) = 0$ and the set
 1573 of all roots of P is defined as follows:

$$R_0(P) := \{x_0 \in R \mid P(x_0) = 0\} \quad (3.29)$$

1574 The roots of a polynomial are of special interest with respect to it's prime factorization, since it
 1575 can be shown that for any given root x_0 of P the polynomial $F(x) = (x - x_0)$ is a prime factor of
 1576 P .

³Strictly speaking this is not true for polynomials over arbitrary types R . However in this book we assume R to be a so called unique factorization domain for which the content of this section holds.

Finding the roots of a polynomial is sometimes called **solving the polynomial**. It is a hard problem and has been the subject of much research throughout history.

It can be shown that if m is the degree of a polynomial P , then P can not have more than m roots. However, in general, polynomials can have less than m roots.

Example 25. Consider the integer polynomial $P_7(x) = x^3 - 4x^2 - 11x + 30$ from example 16 again. We know that its set of roots is given by $R_0(P_7) = \{-3, 2, 5\}$.

On the other hand, we know from example 24 that the integer polynomial $x^2 - 3$ is irreducible. It follows that it has no roots, since every root defines a prime factor.

Example 26. To give another example, consider the integer polynomial $P = x^7 + 3x^6 + 3x^5 + x^4 - x^3 - 3x^2 - 3x - 1$. We can invoke Sage to compute the roots and prime factors of P :

```
sage: Zx = ZZ['x']
sage: p = Zx(x^7 + 3*x^6 + 3*x^5 + x^4 - x^3 - 3*x^2 - 3*x - 1)
sage: p.roots()
[(1, 1), (-1, 4)]
sage: p.factor()
(x - 1) * (x + 1)^4 * (x^2 + 1)
```

We see that P has the root 1 and that the associated prime factor $(x - 1)$ occurs once in P and that it has the root -1 , where the associated prime factor $(x + 1)$ occurs 4 times in P . This gives the following prime factorization:

$$P = (x - 1)(x + 1)^4(x^2 + 1)$$

Exercise 27. Show that if a polynomial $P \in R[x]$ of degree $\deg(P) = m$ has less than m roots, it must have a prime factor F of degree $\deg(F) > 1$.

Exercise 28. Consider the polynomial $P = x^7 + 3x^6 + 3x^5 + x^4 - x^3 - 3x^2 - 3x - 1 \in \mathbb{Z}_6[x]$. Compute the set of all roots of $R_0(P)$ and then compute the prime factorization of P .

Lagrange interpolation One particularly useful property of polynomials is that a polynomial of degree m is completely determined on $m + 1$ evaluation points, which implies that we can uniquely derive a polynomial of degree m from a set S :

$$S = \{(x_0, y_0), (x_1, y_1), \dots, (x_m, y_m) \mid x_i \neq x_j \text{ for all indices } i \text{ and } j\} \quad (3.30)$$

Polynomials therefore have the property that $m + 1$ pairs of points (x_i, y_i) for $x_i \neq x_j$ are enough to determine the set of pairs $(x, P(x))$ for all $x \in R$. This “few too many” property of polynomials is used in many places, like for example in erasure codes. It is also of importance in snarks and we therefore need to understand a method to actually compute a polynomial from a set of points.

If the coefficients of the polynomial we want to find have a notion of multiplicative inverse, it is always possible to find such a polynomial using a method called **Lagrange interpolation**, which works as follows: Given a set like 3.30, a polynomial P of degree m with $P(x_i) = y_i$ for all pairs (x_i, y_i) from S is given by the following algorithm:

Example 27. Let us consider the set $S = \{(0, 4), (-2, 1), (2, 3)\}$. Our task is to compute a polynomial of degree 2 in $\mathbb{Q}[x]$ with coefficients from the rational numbers \mathbb{Q} . Since \mathbb{Q} has multiplicative inverses, we can use the Lagrange interpolation algorithm from 4, to compute the

check
algorithm
floating

Algorithm 4 Lagrange Interpolation**Require:** R must have multiplicative inverses**Require:** $S = \{(x_0, y_0), (x_1, y_1), \dots, (x_m, y_m) \mid x_i, y_i \in R, x_i \neq x_j \text{ for all indices } i \text{ and } j\}$ **procedure** LAGRANGE-INTERPOLATION(S) **for** $j \in (0 \dots m)$ **do**

$$l_j(x) \leftarrow \prod_{i=0; i \neq j}^m \frac{x - x_i}{x_j - x_i} = \frac{(x - x_0)}{(x_j - x_0)} \cdots \frac{(x - x_{j-1})}{(x_j - x_{j-1})} \frac{(x - x_{j+1})}{(x_j - x_{j+1})} \cdots \frac{(x - x_m)}{(x_j - x_m)}$$

end for

$$P \leftarrow \sum_{j=0}^m y_j \cdot l_j$$

return P **end procedure****Ensure:** $P \in R[x]$ with $\deg(P) = m$ **Ensure:** $P(x_j) = y_j$ for all pairs $(x_j, y_j) \in S$

polynomial.

$$\begin{aligned} l_0(x) &= \frac{x - x_1}{x_0 - x_1} \cdot \frac{x - x_2}{x_0 - x_2} = \frac{x + 2}{0 + 2} \cdot \frac{x - 2}{0 - 2} = -\frac{(x + 2)(x - 2)}{4} \\ &= -\frac{1}{4}(x^2 - 4) \\ l_1(x) &= \frac{x - x_0}{x_1 - x_0} \cdot \frac{x - x_2}{x_1 - x_2} = \frac{x - 0}{-2 - 0} \cdot \frac{x - 2}{-2 - 2} = \frac{x(x - 2)}{8} \\ &= \frac{1}{8}(x^2 - 2x) \\ l_2(x) &= \frac{x - x_0}{x_2 - x_0} \cdot \frac{x - x_1}{x_2 - x_1} = \frac{x - 0}{2 - 0} \cdot \frac{x + 2}{2 + 2} = \frac{x(x + 2)}{8} \\ &= \frac{1}{8}(x^2 + 2x) \\ P(x) &= 4 \cdot \left(-\frac{1}{4}(x^2 - 4)\right) + 1 \cdot \frac{1}{8}(x^2 - 2x) + 3 \cdot \frac{1}{8}(x^2 + 2x) \\ &= -x^2 + 4 + \frac{1}{8}x^2 - \frac{1}{4}x + \frac{3}{8}x^2 + \frac{3}{4}x \\ &= -\frac{1}{2}x^2 + \frac{1}{2}x + 4 \end{aligned}$$

1609 And, indeed, evaluation of P on the x -values of S gives the correct points, since $P(0) = 4$,
 1610 $P(-2) = 1$ and $P(2) = 3$. Sage provides the following function:

```

1611 sage: Qx = QQ['x']                                     150
1612 sage: S=[(0,4), (-2,1), (2,3)]                         151
1613 sage: Qx.lagrange_polynomial(S)                       152
1614 -1/2*x^2 + 1/2*x + 4                                   153

```

Example 28. To give another example more relevant to the topics of this book, let us consider the same set $S = \{(0, 4), (-2, 1), (2, 3)\}$ as in the previous example. This time, the task is to compute a polynomial $P \in \mathbb{Z}_5[x]$ from this data. Since we know from example 14 that multiplicative inverses exist in \mathbb{Z}_5 , algorithm 4 applies and we can compute a unique polynomial of degree 2 in $\mathbb{Z}_5[x]$ from S . We can use the lookup tables from example 14 for computation in \mathbb{Z}_5

and get the following:

$$l_0(x) = \frac{x-x_1}{x_0-x_1} \cdot \frac{x-x_2}{x_0-x_2} = \frac{x+2}{0+2} \cdot \frac{x-2}{0-2} = \frac{(x+2)(x-2)}{-4} = \frac{(x+2)(x+3)}{1} \\ = x^2 + 1$$

$$l_1(x) = \frac{x-x_0}{x_1-x_0} \cdot \frac{x-x_2}{x_1-x_2} = \frac{x-0}{-2-0} \cdot \frac{x-2}{-2-2} = \frac{x}{3} \cdot \frac{x+3}{1} = 2(x^2 + 3x) \\ = 2x^2 + x$$

$$l_2(x) = \frac{x-x_0}{x_2-x_0} \cdot \frac{x-x_1}{x_2-x_1} = \frac{x-0}{2-0} \cdot \frac{x+2}{2+2} = \frac{x(x+2)}{3} = 2(x^2 + 2x) \\ = 2x^2 + 4x$$

$$P(x) = 4 \cdot (x^2 + 1) + 1 \cdot (2x^2 + x) + 3 \cdot (2x^2 + 4x) \\ = 4x^2 + 4 + 2x^2 + x + x^2 + 2x \\ = 2x^2 + 3x + 4$$

1615 And, indeed, evaluation of P on the x -values of S gives the correct points, since $P(0) = 4$,
1616 $P(-2) = 1$ and $P(2) = 3$. We can doublecheck our findings using Sage:

```
1617 sage: F5 = GF(5) 154
1618 sage: F5x = F5['x'] 155
1619 sage: S=[(0,4), (-2,1), (2,3)] 156
1620 sage: F5x.lagrange_polynomial(S) 157
1621 2*x^2 + 3*x + 4 158
```

1622 *Exercise 29.* Consider modular 5 arithmetic from example 14 and the set $S = \{(0,0), (1,1), (2,2), (3,2)\}$.
1623 Find a polynomial $P \in \mathbb{Z}_5[x]$ such that $P(x_i) = y_i$ for all $(x_i, y_i) \in S$.

1624 *Exercise 30.* Consider the set S from the previous example. Why is it not possible to apply
1625 algorithm 4 to construct a polynomial $P \in \mathbb{Z}_6[x]$, such that $P(x_i) = y_i$ for all $(x_i, y_i) \in S$?

Chapter 4

Algebra

In the previous chapter, we gave an introduction to the basic computational tools needed for a pen-and-paper approach to SNARKs and in this chapter we provide a more abstract clarification of relevant mathematical terminology such as **groups**, **rings** and **fields**.

Scientific literature on cryptography frequently contain such terms, and it is necessary to get at least some understanding of these terms to be able to follow the literature.

4.1 Commutative Groups

Commutative groups are abstractions that capture the essence of mathematical phenomena, like addition and subtraction, or multiplication and division.

To understand commutative groups, let us think back to when we learned about the addition and subtraction of integers in school. We have learned that, whenever we add two integers, the result is guaranteed to be an integer as well. We have also learned that adding zero to any integer means that “nothing happens”, that is, the result of the addition is the same integer we started with. Furthermore, we have learned that the order in which we add two (or more) integers does not matter, that brackets have no influence on the result of addition, and that, for every integer, there is always another integer (the negative) such that we get zero when we add them together.

These conditions are the defining properties of a commutative group, and mathematicians have realized that the exact same set of rules can be found in very different mathematical structures. It therefore makes sense to abstract from integers and to give a formal definition of what a group should be, detached from any concrete examples. This lets us handle entities of very different mathematical origins in a flexible way, while retaining essential structural aspects of many objects in abstract algebra and beyond.

Distilling these rules to the smallest independent list of properties and making them abstract, we arrive at the following definition of a commutative group:

Sylvia: I would like to have a separate counter for definitions

Definition 4.1.0.1. A **commutative group** (\mathbb{G}, \cdot) is a set \mathbb{G} , together with a **map** $\cdot : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G}$ called the **group law**, that combines two elements of the set \mathbb{G} into a third one such that the following properties hold:

- **Commutativity:** For all $g_1, g_2 \in \mathbb{G}$, the equation $g_1 \cdot g_2 = g_2 \cdot g_1$ holds.
- **Associativity:** For every $g_1, g_2, g_3 \in \mathbb{G}$ the equation $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$ holds.
- **Existence of a neutral element:** There is a $e \in \mathbb{G}$ for all $g \in \mathbb{G}$, such that $e \cdot g = g$.

Sylvia: I would like to have a separate counter for definitions

- **Existence of an inverse:** For every $g \in \mathbb{G}$ there is a $g^{-1} \in \mathbb{G}$, such that $g \cdot g^{-1} = e$.

If (\mathbb{G}, \cdot) is a group and $\mathbb{G}' \subset \mathbb{G}$ is a subset of \mathbb{G} , such that the restriction of the group law $\cdot : \mathbb{G}' \times \mathbb{G}' \rightarrow \mathbb{G}'$ is a group law on \mathbb{G}' , then (\mathbb{G}', \cdot) is called a **subgroup** of (\mathbb{G}, \cdot) .

Rephrasing the abstract definition in layman's terms, a group is something where we can do computations in a way that resembles the behavior of the addition of integers. Specifically, this means we can combine some element with another element into a new element in a way that is reversible and where the order of combining elements doesn't matter.

Notation and Symbols 3. Since we are exclusively concerned with commutative groups in this book, we often just call them groups, keeping the notation of commutativity implicit. A set \mathbb{G} with a map \cdot that satisfies all previously mentioned rules, except for the commutativity law is called a non-commutative group.

If there is no risk of ambiguity (about what the group law of that group is), we frequently drop the symbol \cdot and simply write \mathbb{G} as notation for the group, keeping the group law implicit. In this case we also say that \mathbb{G} is of group type, indicating that \mathbb{G} not simply a set but a set together with a group law.

For commutative groups (\mathbb{G}, \cdot) , we sometimes use the so-called **additive notation** $(\mathbb{G}, +)$, that is, we write $+$ instead of \cdot for the group law, 0 for the neutral element and $-g := g^{-1}$ for the inverse of an element $g \in \mathbb{G}$.

As we will see in the following chapters, groups are heavily used in cryptography and in SNARKs. But let us look at some more familiar examples first:

Example 29 (Integer Addition and Subtraction). The set $(\mathbb{Z}, +)$ of integers with integer addition is the archetypical example of a commutative group, where the group law is traditionally written in additive notation 3.

To compare integer addition against the abstract axioms of a commutative group, we first see that integer addition is commutative and associative, since $a + b = b + a$ as well as $(a + b) + c = a + (b + c)$ for all integers $a, b, c \in \mathbb{Z}$. The neutral element e is the number 0 , since $a + 0 = a$ for all integers $a \in \mathbb{Z}$. Furthermore, the inverse of a number is its negative counterpart, since $a + (-a) = 0$, for all $a \in \mathbb{Z}$. This implies that integers with addition are indeed a commutative group in the abstract sense.

To give an example of a subgroup for the group of integers, consider the set $\mathbb{Z}_{\text{even}} := \{\dots, -4, -2, 0, 2, 4, \dots\}$ of even numbers, including 0 . We can see that this set is a subgroup of $(\mathbb{Z}, +)$, since the sum of two even numbers is always an even number again, since the neutral element 0 is a member of \mathbb{Z}_{even} and since the negative of an even number is itself an even number.

Example 30 (The trivial group). The most basic example of a commutative group is the group with just one element $\{\bullet\}$ and the group law $\bullet \cdot \bullet = \bullet$. We call it the **trivial group**.

The trivial group is a subgroup of any group. To see that let (\mathbb{G}, \cdot) be a group with neutral element $e \in \mathbb{G}$. Then $e \cdot e = e$ as well as $e^{-1} = e$ and it follows that the set $\{e\}$ is a subgroup of \mathbb{G} . In particular $\{0\}$ is a subgroup of $(\mathbb{Z}, +)$, since $0 + 0 = 0$.

Example 31. Consider the addition in modulo 6 arithmetics $(\mathbb{Z}_6, +)$ as defined in in example 9. As we see, the remainder 0 is the neutral element in modulo 6 addition and the inverse of a remainder r is given by $6 - r$, because $r + (6 - r) = 6$, which is congruent to 0 , since $6 \bmod 6 = 0$. Moreover, $r_1 + r_2 = r_2 + r_1$ as well as $(r_1 + r_2) + r_3 = r_1 + (r_2 + r_3)$ are inherited from integer addition. We therefore see that $(\mathbb{Z}_6, +)$ is a group.

The previous example of a commutative group is a very important one for this book. Abstracting from this example and considering residue classes $(\mathbb{Z}_n, +)$ for arbitrary moduli n , it can be shown that $(\mathbb{Z}_n, +)$ is a commutative group with the neutral element 0 and the additive

inverse $n - r$ for any element $r \in \mathbb{Z}_n$. We call such a group the **remainder class group** of modulus n .

Exercise 31. Consider example 14 again and let \mathbb{Z}_5^* be the set of all remainder classes from \mathbb{Z}_5 without the class 0. Then $\mathbb{Z}_5^* = \{1, 2, 3, 4\}$. Show that (\mathbb{Z}_5^*, \cdot) is a commutative group.

Exercise 32. Generalizing the previous exercise, consider the general modulus n , and let \mathbb{Z}_n^* be the set of all remainder classes from \mathbb{Z}_n without the class 0. Then $\mathbb{Z}_n^* = \{1, 2, \dots, n - 1\}$. Provide a counter-example to show that (\mathbb{Z}_n^*, \cdot) is not a group in general.

Find a condition such that (\mathbb{Z}_n^*, \cdot) is a commutative group, compute the neutral element, give a closed form for the inverse of any element and prove the commutative group axioms.

Finite groups As we have seen in the previous examples, groups can either contain infinitely many elements (such as integers) or finitely many elements (as for example the remainder class groups $(\mathbb{Z}_n, +)$). To capture this distinction, a group is called a **finite group** if the underlying set of elements is finite. In that case, the number of elements of that group is called its **order**.

Notation and Symbols 4. Let \mathbb{G} be a finite group. We write $\text{ord}(\mathbb{G})$ or $|\mathbb{G}|$ for the order of \mathbb{G} .

Example 32. Consider the remainder class groups $(\mathbb{Z}_6, +)$ from example 9, the group $(\mathbb{Z}_5, +)$ from example 14, and the group (\mathbb{Z}_5^*, \cdot) from exercise 31. We can easily see that the order of $(\mathbb{Z}_6, +)$ is 6, the order of $(\mathbb{Z}_5, +)$ is 5 and the order of (\mathbb{Z}_5^*, \cdot) is 4.

Exercise 33. Let $n \in \mathbb{N}$ with $n \geq 2$ be some modulus. What is the order of the remainder class group $(\mathbb{Z}_n, +)$.

Generators The set of elements of a group can be complicated and it is not always obvious how to actually compute elements of a given group. From a practical point of view it is therefore desirable, if a group has a small subset of elements, such that all other elements can be generated by applying the group law repeatedly to the elements of that subset or their inverses only. Sets with these properties are called **generator sets**.

Of course, every group \mathbb{G} has a trivial set of generators, when we just consider every element of the group to be in the generator set. The more interesting question is to find smallest possible sets of generators for a given group. Of particular interest in this regard are groups that have a generator set that contains a single element only. In this case, there exists a (not necessarily unique) element $g \in \mathbb{G}$ such that every other element from \mathbb{G} can be computed by the repeated combination of g and its inverse g^{-1} only. Groups with single, not necessarily unique, generators are called **cyclic groups** and any element $g \in \mathbb{G}$ that is able to generate \mathbb{G} is called a **generator**.

Example 33. The most basic example of a cyclic group is the group of integers $(\mathbb{Z}, +)$ with integer addition. In this case, the number 1 is a generator of \mathbb{Z} , since every integer can be obtained by repeatedly adding either 1 or its inverse -1 to itself. For example -4 is generated by 1, since $-4 = -1 + (-1) + (-1) + (-1)$. Another generator of \mathbb{Z} is the number -1 .

Example 34. Consider the group (\mathbb{Z}_5^*, \cdot) from exercise 31. Since $2^1 = 2$, $2^2 = 4$, $2^3 = 3$ and $2^4 = 1$, the element 2 is a generator of (\mathbb{Z}_5^*, \cdot) . Moreover since $3^1 = 3$, $3^2 = 4$, $3^3 = 2$ and $3^4 = 1$, the element 3 is another generator of (\mathbb{Z}_5^*, \cdot) . Cyclic groups can therefore have more than one generator. However since $4^1 = 4$, $4^2 = 1$, $4^3 = 4$ and in general $4^k = 4$ for k odd and $4^k = 1$ for k even the element 4 is not a generator of (\mathbb{Z}_5^*, \cdot) . It follows that in general not every element of a finite cyclic group is a generator.

1746 *Example 35.* Consider a modulus n and the remainder class groups $(\mathbb{Z}_n, +)$ from exercise 33.
 1747 These groups are cyclic, with generator 1, since every other element of that group can be con-
 1748 structed by repeatedly adding the remainder class 1 to itself. Since \mathbb{Z}_n is also finite, we know
 1749 that $(\mathbb{Z}_n, +)$ is a finite cyclic group of order n .

1750 *Exercise 34.* Consider the group $(\mathbb{Z}_6, +)$ of modular 6 addition from example 9. Show that
 1751 $5 \in \mathbb{Z}_6$ is a generator and then show that $2 \in \mathbb{Z}_6$ is not a generator.

1752 *Exercise 35.* Let $p \in \mathbb{P}$ be prime number and (\mathbb{Z}_p^*, \cdot) the finite group from exercise 32. Show
 1753 that (\mathbb{Z}_p^*, \cdot) is cyclic.

1754 **The exponential map** Observe that, when \mathbb{G} is a cyclic group of order n and $g \in \mathbb{G}$ is a
 1755 generator of \mathbb{G} , then there is the following map with respect to the generator g :

$$g^{(\cdot)} : \mathbb{Z}_n \rightarrow \mathbb{G} \quad x \mapsto g^x \quad (4.1)$$

1756 In the map above, g^x means “multiply g x -times by itself” and $g^0 = e_{\mathbb{G}}$. This map, called
 1757 the **exponential map**, has the remarkable property that it maps the additive group law of the
 1758 remainder class group $(\mathbb{Z}_n, +)$ in a one-to-one correspondence to the group law of \mathbb{G} .

1759 To see this, first observe that, since $g^0 := e_{\mathbb{G}}$ by definition, the neutral element of \mathbb{Z}_n is
 1760 mapped to the neutral element of \mathbb{G} , and, since $g^{x+y} = g^x \cdot g^y$, the map respects the group law.

1761 Because the exponential map respects the group law, it doesn’t matter if we do our compu-
 1762 tation in \mathbb{Z}_n before we write the result into the exponent of g or afterwards: the result will be the
 1763 same in both cases. This is usually referred to as doing computations “in the exponent”. In cryp-
 1764 tography in general, and in SNARK development in particular, we often perform computations
 1765 “in the exponent” of a generator.

Example 36. Consider the multiplicative group (\mathbb{Z}_5^*, \cdot) from exercise 31. We know from 35 that
 \mathbb{Z}_5^* is a cyclic group of order 4, and that the element $3 \in \mathbb{Z}_5^*$ is a generator. We then know that
 the following map respects the group law of addition in \mathbb{Z}_4 and the group law of multiplication
 in \mathbb{Z}_5^* :

$$3^{(\cdot)} : \mathbb{Z}_4 \rightarrow \mathbb{Z}_5^*, \quad x \mapsto 3^x$$

To do an example computation “in the exponent” of 3, let’s perform the calculation $1 + 3 + 2$
 in the exponent of the generator 3:

$$\begin{aligned} 3^{1+3+2} &= 3^2 \\ &= 4 \end{aligned}$$

1766 What we did is, we first performed the computation $1 + 3 + 2 = 1$ in the remainder class group
 1767 $(\mathbb{Z}_4, +)$ and then applied the exponential map $3^{(\cdot)}$ to the result.

1768 However since the exponential map 4.1 “respects the group law” we also could map each
 1769 summand into (\mathbb{F}_5^*, \cdot) first and then apply the group law of (\mathbb{F}_5^*, \cdot) . The result is guaranteed to be
 1770 the same:

$$\begin{aligned} 3^1 \cdot 3^3 \cdot 3^2 &= 3 \cdot 2 \cdot 4 \\ &= 1 \cdot 4 \\ &= 4 \end{aligned}$$

Since the exponential map is a one-to-one correspondence that respects the group law, it can be shown that this map has an inverse with respect to the base g , called the **base g discrete logarithm map**:

$$\log_g(\cdot) : \mathbb{G} \rightarrow \mathbb{Z}_n \quad x \mapsto \log_g(x) \quad (4.2)$$

Discrete logarithms are highly important in cryptography, because there are finite cyclic groups where the exponential map and its inverse, the discrete logarithm map, are believed to be one-way functions, which informally means that computing the exponential map is fast, while computing the logarithm map is slow (We will look into a more precise definition in 4.1.1).

Example 37. Consider the exponential map $3^{(\cdot)}$ from example 36. Its inverse is the discrete logarithm to the base 3 and it is given by the map

$$\log_3(\cdot) : \mathbb{Z}_5^* \rightarrow \mathbb{Z}_4 \quad x \mapsto \log_3(x)$$

In contrast to the exponential map $3^{(\cdot)}$, we have no way to actually compute this map, other than by trying all elements of the group until we find the correct one. For example in order to compute $\log_3(4)$, we have to find some $x \in \mathbb{Z}_4$, such that $3^x = 4$ and all we can do is repeatedly insert elements x into the exponent of 3 until the result is 4. To do this let's write down all the images of $3^{(\cdot)}$:

$$3^0 = 1, \quad 3^1 = 3, \quad 3^2 = 4, \quad 3^3 = 2$$

Since the discrete logarithm $\log_3(\cdot)$ is defined as the inverse to this function, we can use those images to compute the discrete logarithm:

$$\log_3(1) = 0, \quad \log_3(2) = 3, \quad \log_3(3) = 1, \quad \log_3(4) = 2$$

Note that this computation was only possible, because we were able to write down all images of the exponential map. However, in real world applications the groups in consideration are too large to write down the images of the exponential map.

Factor Groups As we know from the fundamental theorem of arithmetics 3.6, every natural number n is a product of factors, the most basic of which are prime numbers. This reflects into subgroups of a finite cyclic group in an interesting way: If \mathbb{G} , is a finite cyclic group of order n , then every subgroup \mathbb{G}' of \mathbb{G} , is finite and cyclic and the order of \mathbb{G}' , is a factor of n . Moreover for each factor k of n , \mathbb{G} has exactly one subgroup of order k . This is known as the **fundamental theorem of finite cyclic groups**.

Notation and Symbols 5. If \mathbb{G} is a finite cyclic group of order n and k is a factor of n , then we write $\mathbb{G}[k]$ for the unique finite cyclic group, which is the order k subgroup of \mathbb{G} and call it a **factor group** of \mathbb{G} .

One particular interesting situation occurs if the order of a given finite cyclic group is a prime number. As we know from the fundamental theorem of arithmetics 3.6, prime numbers have only two factors, given by the number 1 and the prime number itself. It then follows from the fundamental theorem of finite cyclic groups 4.1, that those groups have no subgroups other than the trivial group 30 and the group itself.

Cryptographic protocols often assume the existence of finite cyclic groups of prime order but sometimes real world implementations of those protocols are not defined on prime order groups, but on groups where the order consist of a (usually large) prime number that has small cofactors 1. In this case a method called **cofactor clearing** has to be applied to ensure that the computations are not done in the group itself but in its (large) prime order subgroup.

To understand cofactor clearing in detail, let \mathbb{G} be a finite cyclic group of order n and let k be a factor of n with associated factor group $\mathbb{G}[k]$. We can project any element $g \in \mathbb{G}[k]$ onto the neutral element e of \mathbb{G} by multiplying g k -times with itself:

$$g^k = e \quad (4.3)$$

From this follows that if $c := n \operatorname{div} k$ is the cofactor 1 of k in n then any element from the full group $g \in \mathbb{G}$ can be projected into the factor group $\mathbb{G}[k]$ by multiplying g c -times with itself. This defines the following map, which is often called **cofactor clearing** in the cryptographic literature:

$$(\cdot)^c : \mathbb{G} \rightarrow \mathbb{G}[k] : g \mapsto g^c \quad (4.4)$$

Example 38. Consider the finite cyclic group (\mathbb{Z}_5^*, \cdot) from example 34. Since the order of \mathbb{Z}_5^* is 4 and 4 has the factors 1, 2 and 4, it follows from the fundamental theorem of finite cyclic groups, that \mathbb{Z}_5^* has 3 unique subgroups. In fact the unique subgroup $\mathbb{Z}_5^*[1]$ of order 1 is given by the trivial group $\{1\}$ that contains only the multiplicative neutral element 1 and the unique subgroup $\mathbb{Z}_5^*[4]$ of order 4 is \mathbb{Z}_5^* itself, since by definition every group is trivially a subgroup of itself. The unique subgroup $\mathbb{Z}_5^*[2]$ of order 2 is more interesting and is given by the set $\mathbb{Z}_5^*[2] = \{1, 4\}$.

Since \mathbb{Z}_5^* is not a prime order group and since the only prime factor of 4 is 2, the "large" prime order subgroup of \mathbb{Z}_5^* is $\mathbb{Z}_5^*[2]$. Moreover since the cofactor of 2 in 4 is also 2, we have the cofactor clearing map $(\cdot)^2 : \mathbb{Z}_5^* \rightarrow \mathbb{Z}_5^*[2]$ and indeed applying this map to all elements from \mathbb{Z}_5^* we see that it maps onto the elements of $\mathbb{Z}_5^*[2]$ only:

$$1^2 = 1, \quad 2^2 = 4, \quad 3^2 = 4, \quad 4^2 = 1$$

We can therefore use this map to "clear the cofactor" of any element from \mathbb{Z}_5^* which means that the element is projected into the "large" prime order subgroup $\mathbb{Z}_5^*[2]$.

Exercise 36. Consider the previous example 38 and show that $\mathbb{Z}_5^*[2]$ is a commutative group.

Exercise 37. Consider the finite cyclic group $(\mathbb{Z}_6, +)$ of modular 6 addition from example 34. Describe all subgroups of $(\mathbb{Z}_6, +)$. Identify the large prime order subgroup of \mathbb{Z}_6 , define its cofactor clearing map and apply that map to all elements of \mathbb{Z}_6 .

Exercise 38. Let (\mathbb{Z}_p^*, \cdot) be the cyclic group from exercise 35. Show that for $p \geq 5$, not every element $x \in \mathbb{F}_p^*$ is a generator of \mathbb{F}_p^* .

Pairings Of particular importance for the development of SNARKs are so-called pairing maps on commutative groups. To see the definition, let \mathbb{G}_1 , \mathbb{G}_2 and \mathbb{G}_3 be three commutative groups. Then a **pairing map** is a function

$$e(\cdot, \cdot) : \mathbb{G}_1 \times \mathbb{G}_2 \rightarrow \mathbb{G}_3 \quad (4.5)$$

This function takes pairs (g_1, g_2) of elements from \mathbb{G}_1 and \mathbb{G}_2 , and maps them to elements from \mathbb{G}_3 , such that the **bilinearity** property holds, which means that for all $g_1, g'_1 \in \mathbb{G}_1$ and $g_2, g'_2 \in \mathbb{G}_2$ the following two identities are satisfied:

$$e(g_1 \cdot g'_1, g_2) = e(g_1, g_2) \cdot e(g'_1, g_2) \quad \text{and} \quad e(g_1, g_2 \cdot g'_2) = e(g_1, g_2) \cdot e(g_1, g'_2) \quad (4.6)$$

Informally speaking, bilinearity means that it doesn't matter if we first execute the group law on one side and then apply the bilinear map, or if we first apply the bilinear map and then apply the group law in \mathbb{G}_3 .

A pairing map is called **non-degenerate** if, whenever the result of the pairing is the neutral element in \mathbb{G}_3 , one of the input values is the neutral element of \mathbb{G}_1 or \mathbb{G}_2 . To be more precise, $e(g_1, g_2) = e_{\mathbb{G}_3}$ implies $g_1 = e_{\mathbb{G}_1}$ or $g_2 = e_{\mathbb{G}_2}$.

Example 39. One of the most basic examples of a non-degenerate pairing involves $\mathbb{G}_1, \mathbb{G}_2$ and \mathbb{G}_3 all to be the group of integers with addition $(\mathbb{Z}, +)$. Then the following map defines a non-degenerate pairing:

$$e(\cdot, \cdot) : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \quad (a, b) \mapsto a \cdot b$$

Note that bilinearity follows from the distributive law of integers, since for $a, b, c \in \mathbb{Z}$, we have $e(a + b, c) = (a + b) \cdot c = a \cdot c + b \cdot c = e(a, c) + e(b, c)$ and the same reasoning is true for the second argument.

To see that $e(\cdot, \cdot)$ is non-degenerate, assume that $e(a, b) = 0$. Then $a \cdot b = 0$ implies that a or b must be zero.

Exercise 39 (Arithmetic laws for pairing maps). Let $\mathbb{G}_1, \mathbb{G}_2$ as well as \mathbb{G}_3 be finite cyclic groups of the same order n and let $e(\cdot, \cdot) : \mathbb{G}_1 \times \mathbb{G}_2 \rightarrow \mathbb{G}_3$ be a pairing map. Show that for given $g_1 \in \mathbb{G}_1$ and $g_2 \in \mathbb{G}_2$ and all $a, b \in \mathbb{Z}_n$ the following identity holds:

$$e(g_1^a, g_2^b) = e(g_1, g_2)^{a \cdot b} \quad (4.7)$$

Exercise 40. Consider the remainder class groups $(\mathbb{Z}_n, +)$ from example 33 for some modulus n . Show that the map

$$e(\cdot, \cdot) : \mathbb{Z}_n \times \mathbb{Z}_n \rightarrow \mathbb{Z}_n \quad (a, b) \mapsto a \cdot b$$

is a pairing map. Why is the pairing not non-degenerate in general and what condition must be imposed on n , such that the pairing will be non-degenerate?

4.1.1 Cryptographic Groups

In this section, we will look at families of groups that are believed to satisfy certain **computational hardness assumptions**, namely that a particular problem cannot be solved efficiently (where efficiently typically means “in polynomial time of a given security parameter”) in the groups under consideration.

Example 40. To highlight the concept of the computational hardness assumption, consider the group of integers \mathbb{Z} from example 3.5. One of the best known and most researched examples of computational hardness is the assumption that the factorization of integers into prime numbers cannot be solved by any algorithm in polynomial time with respect to the bit-length of the integer.

To be more precise, the computational hardness assumption of integer factorization assumes that, given any integer $z \in \mathbb{Z}$ with bit-length b , there is no integer k and no algorithm with the runtime complexity of $\mathcal{O}(b^k)$ that is able to find the prime numbers $p_1, p_2, \dots, p_j \in \mathbb{P}$, such that $z = p_1 \cdot p_2 \cdot \dots \cdot p_j$.

This hardness assumption was proven to be false, since Shor’s (?) algorithm shows that integer factorization is at least efficiently possible on a quantum computer, since the runtime complexity of this algorithm is $\mathcal{O}(b^3)$. However, no such algorithm is known on a classical computer.

In the realm of classical computers, however, we still have to call the non-existence of such an algorithm an “assumption” because, to date, there is no proof that it is actually impossible to find one. The problem is that it is hard to reason about algorithms that we don’t know.

So, despite the fact that there is currently no known algorithm that can factor integers efficiently on a classical computer, we cannot exclude that such an algorithm might exist in principle, and that someone eventually will discover it in the future.

However, what still makes the assumption plausible, despite the absence of any actual proof, is the fact that, after decades of extensive search, still no such algorithm has been found.

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reference

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S: what
does “effi-
ciently”
mean
here?

In what follows, we will describe a few **computational hardness assumptions** that arise in the context of groups in cryptography, because we will refer to them throughout the book.

computational
hardness
assump-
tions

The discrete logarithm assumption The so-called discrete logarithm problem is one of the most fundamental assumptions in cryptography. To define it, let \mathbb{G} be a finite cyclic group of order r and let g be a generator of \mathbb{G} . We know from 4.1 that there is an exponential map $g^{(\cdot)} : \mathbb{Z}_r \rightarrow \mathbb{G} : x \mapsto g^x$ that maps the residue classes from modulo r arithmetic onto the group in a 1 : 1 correspondence. The **discrete logarithm problem** is the task of finding inverses to this map, that is, to find a solution $x \in \mathbb{Z}_r$ to the following equation for some given $h \in \mathbb{G}$:

check
reference

$$h = g^x \quad (4.8)$$

In other words, the **discrete logarithm assumption (DL-A)** is the assumption that there exists no algorithm with polynomial running time in the security parameter $\log_2(r)$, that is able to compute some x if only h , g and g^x are given in \mathbb{G} . If this is the case for \mathbb{G} , we call \mathbb{G} a **DL-A group**.

Rephrasing the previous definition, DL-A groups are believed to have the property that it is infeasible to compute some number x that solves the equation $h = g^x$ for a given h and g , assuming that the size of the group r is large enough.

Example 41 (Public key cryptography). One the most basic examples of an application for DL-A groups is in public key cryptography, where the parties publicly agree on some pair (\mathbb{G}, g) such that \mathbb{G} is a finite cyclic group of sufficiently large order r , where \mathbb{G} is believed to be a DL-A group, and g is a generator of \mathbb{G} .

In this setting, a secret key is some number $sk \in \mathbb{Z}_r$ and the associated public key pk is the group element $pk = g^{sk}$. Since discrete logarithms are assumed to be hard, it is infeasible for an attacker to compute the secret key from the public key, since it is believed to be hard to find solutions x to the following equation:

$$pk = g^x \quad (4.9)$$

As the previous example shows, identifying DL-A groups is an important practical problem. Unfortunately, it is easy to see that it does not make sense to assume the hardness of the discrete logarithm problem in all finite cyclic groups: Counterexamples are common and easy to construct.

Example 42 (Modular arithmetics for Fermat's primes). It is widely believed that the discrete logarithm problem is hard in multiplicative groups \mathbb{Z}_p^* of prime number modular arithmetics. However, this is not true in general. To see that, consider any so-called Fermat's prime, which is a prime number $p \in \mathbb{P}$, such that $p = 2^n + 1$ for some number n .

We know from exercise 32 that in this case $\mathbb{Z}_p^* = \{1, 2, \dots, p-1\}$ is a group with respect to integer multiplication in modular p arithmetics and since $p = 2^n + 1$, the order of \mathbb{Z}_p^* is 2^n , which implies that the associated security parameter is given by $\log_2(2^n) = n$.

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reference

We show that, in this case, \mathbb{Z}_p^* is not a DL-A group, by constructing an algorithm, which is able to compute some $x \in \mathbb{Z}_{2^n}$ for any given generator g and arbitrary element h of \mathbb{F}_p^* , such that equation 4.10 holds, and the runtime complexity of the constructed algorithm is $\mathcal{O}(n^2)$, which is quadratic in the security parameter $n = \log_2(2^n)$.

$$h = g^x \quad (4.10)$$

To define such an algorithm, let us assume that the generator g is a public constant and that a group element h is given. Our task is to compute x efficiently.

The first thing to note is that, since x is a number in modular 2^n arithmetic, we can write the binary representation of x as in 4.11, with binary coefficients $c_j \in \{0, 1\}$. In particular, x is an n -bit number if interpreted as an integer.

$$x = c_0 \cdot 2^0 + c_1 \cdot 2^1 + \cdots + c_n \cdot 2^n \quad (4.11)$$

We then use this representation to construct an algorithm that computes the bits c_j one after another, starting at c_0 . To see how this can be achieved, observe that we can determine c_0 by raising the input h to the power of 2^{n-1} in \mathbb{F}_p^* . We use the exponential laws and compute as follows:

$$\begin{aligned} h^{2^{n-1}} &= (g^x)^{2^{n-1}} \\ &= \left(g^{c_0 \cdot 2^0 + c_1 \cdot 2^1 + \cdots + c_n \cdot 2^n} \right)^{2^{n-1}} \\ &= g^{c_0 \cdot 2^{n-1}} \cdot g^{c_1 \cdot 2^1 \cdot 2^{n-1}} \cdot g^{c_2 \cdot 2^2 \cdot 2^{n-1}} \cdots g^{c_n \cdot 2^n \cdot 2^{n-1}} \\ &= g^{c_0 2^{n-1}} \cdot g^{c_1 2^0 \cdot 2^n} \cdot g^{c_2 2^1 \cdot 2^n} \cdots g^{c_n 2^{n-1} \cdot 2^n} \end{aligned}$$

Now, since g is a generator and \mathbb{F}_p^* is cyclic of order 2^n , we know $g^{2^n} = 1$ and therefore $g^{k \cdot 2^n} = 1^k = 1$. From this, it follows that all but the first factor in the last expression are equal to 1 and we can simplify the expression into the following:

$$h^{2^{n-1}} = g^{c_0 2^{n-1}} \quad (4.12)$$

Now, in case $c_0 = 0$, we get $h^{2^{n-1}} = g^0 = 1$. In case $c_0 = 1$, we get $h^{2^{n-1}} = g^{2^{n-1}} \neq 1$ (To see that $g^{2^{n-1}} \neq 1$, recall that g is a generator of \mathbb{F}_p^* and hence, is \mathbb{F}_p^* a cyclic group of order 2^n , which implies $g^y \neq 1$ for all $y < 2^n$).

Raising h to the power of 2^{n-1} determines c_0 , and we can apply the same reasoning to the coefficient c_1 by raising $h \cdot g^{-c_0 \cdot 2^0}$ to the power of 2^{n-2} . This approach can then be repeated until all the coefficients c_j of x are found.

Assuming that exponentiation in \mathbb{F}_p^* can be done in logarithmic runtime complexity $\log(p)$, it follows that our algorithm has a runtime complexity of $\mathcal{O}(\log^2(p)) = \mathcal{O}(n^2)$, since we have to execute n exponentiations to determine the n binary coefficients of x .

From this, it follows that whenever p is a Fermat's prime, the discrete logarithm assumption does not hold in F_p^* .

The decisional Diffie–Hellman assumption Let \mathbb{G} be a finite cyclic group of order r and let g be a generator of \mathbb{G} . The decisional Diffie–Hellman assumption stipulates that there is no algorithm that has a polynomial runtime complexity in the security parameter $s = \log(r)$ that is able to distinguish the so-called DDH- triple (g^a, g^b, g^{ab}) from any triple (g^a, g^b, g^c) for randomly and independently chosen parameters $a, b, c \in \mathbb{Z}_r$. If this is the case for \mathbb{G} , we call \mathbb{G} a **DDH-A group**.

DDH-A is a stronger assumption than DL-A, in the sense that the discrete logarithm assumption is necessary for the decisional Diffie–Hellman assumption to hold, but not the other way around.

To see why this is the case, assume that the discrete logarithm assumption does not hold. In that case, given a generator g and a group element h , it is easy to compute some residue class $x \in \mathbb{Z}_p$ with $h = g^x$. Then the decisional Diffie–Hellman assumption cannot hold, since given

explain
last sen-
tence
more

some triple (g^a, g^b, z) , one could efficiently decide whether $z = g^{ab}$ is true by first computing the discrete logarithm b of g^b , then computing $g^{ab} = (g^a)^b$ and deciding whether or not $z = g^{ab}$.

On the other hand, the following example shows that there are groups where the discrete logarithm assumption holds but the decisional Diffie–Hellman assumption does not.

Example 43 (Efficiently computable pairings). Let \mathbb{G} be a finite, cyclic group of order r with generator g , such that the discrete logarithm assumption holds and there is a pairing map $e(\cdot, \cdot) : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G}_T$ for some target group \mathbb{G}_T that is computable in polynomial time of the parameter $\log(r)$.

In a setting like this, it is easy to show that DDH-A cannot hold, since given some triple (g^a, g^b, z) , it is possible to decide in polynomial time w.r.t $\log(r)$ whether $z = g^{ab}$ or not. To see that, check the following equation:

$$e(g^a, g^b) = e(g, z) \quad (4.13)$$

Since the bilinearity properties of $e(\cdot, \cdot)$ imply $e(g^a, g^b) = e(g, g)^{ab} = e(g, g^{ab})$, and $e(g, y) = e(g, y')$ implies $y = y'$ due to the non-degenerate property, the equality means $z = e^{ab}$.

It follows that DDH-A is indeed weaker than DL-A, and groups with efficient pairings cannot be DDH-A groups. The following example shows another important class of groups where DDH-A does not hold: multiplicative groups of prime number residue classes.

Example 44. Let p be a prime number and $\mathbb{Z}_p^* = \{1, 2, \dots, p-1\}$ the multiplicative group of modular p arithmetics as in exercise 32. As we have seen in XXX, this group is finite and cyclic of order $p-1$ and every element $g \neq 1$ is a generator.

To see that \mathbb{F}_p^* cannot be a DDH-A group, recall from XXX that the **Legendre symbol** $\left(\frac{x}{p}\right)$ of any $x \in \mathbb{F}_p^*$ is efficiently computable by **Euler’s formular**. But the Legendre symbol of g^a reveals whether a is even or odd. Given g^a, g^b and g^{ab} , one can thus efficiently compute and compare the least significant bit of a, b and ab , respectively, which provides a probabilistic method to distinguish g^{ab} from a random group element g^c .

The computational Diffie–Hellman assumption Let \mathbb{G} be a finite cyclic group of order r and let g be a generator of \mathbb{G} . The computational Diffie–Hellman assumption stipulates that, given randomly and independently chosen residue classes $a, b \in \mathbb{Z}_r$, it is not possible to compute g^{ab} if only g, g^a and g^b (but not a and b) are known. If this is the case for \mathbb{G} , we call \mathbb{G} a CDH-A group.

In general, we don’t know if CDH-A is a stronger assumption then DL-A, or if both assumptions are equivalent. We know that DL-A is necessary for CDH-A, but the other direction is currently not well understood. In particular, there are no groups known where DL-A holds but CDH-A does not hold [?].

To see why the discrete logarithm assumption is necessary, assume that it does not hold. So, given a generator g and a group element h , it is easy to compute some residue class $x \in \mathbb{Z}_p$ with $h = g^x$. In that case, the computational Diffie–Hellman assumption cannot hold, since, given g, g^a and g^b , one can efficiently compute b and hence is able to compute $g^{ab} = (g^a)^b$ from this data.

The computational Diffie–Hellman assumption is a weaker assumption than the decisional Diffie–Hellman assumption, which means that there are groups where CDH-A holds and DDH-A does not hold, while there cannot be groups such that DDH-A holds but CDH-A does not hold. To see that, assume that it is efficiently possible to compute g^{ab} from g, g^a and g^b . Then, given (g^a, g^b, z) it is easy to decide if $z = g^{ab}$ holds or not.

“equation”?

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reference

what’s the
difference
between
 \mathbb{F}_p^* and
 \mathbb{Z}_p^* ?

Legendre
symbol

Euler’s
formular

These
are only
explained
later in
the text,
‘4.31’

Several variations and special cases of the CDH-A exist. For example, the **square computational Diffie–Hellman assumption** assumes that, given g and g^x , it is computationally hard to compute g^{x^2} . The **inverse computational Diffie–Hellman assumption** assumes that, given g and g^x , it is computationally hard to compute $g^{x^{-1}}$.

Cofactor Clearing

4.1.2 Hashing to Groups

Hash functions Generally speaking, a hash function is any function that can be used to map data of arbitrary size to fixed-size values. Since binary strings of arbitrary length are a general way to represent arbitrary data, we can understand a general **hash function** as the following map where $\{0,1\}^*$ represents the set of all binary strings of arbitrary but finite length and $\{0,1\}^k$ represents the set of all binary strings that have a length of exactly k bits:

$$H : \{0,1\}^* \rightarrow \{0,1\}^k \quad (4.14)$$

In our definition, a hash function maps binary strings of arbitrary size onto binary strings of size exactly k . The **images** of H , that is, the values returned by the hash function H , are called **hash values**, **digests**, or simply **hashes**.

A hash function must be deterministic, that is, when we insert the same input x into H , the image $H(x)$ must always be the same. In addition, a hash function should be as uniform as possible, which means that it should map input values as evenly as possible over its output range. In mathematical terms, every string of length k from $\{0,1\}^k$ should be generated with roughly the same probability.

Example 45 (k -truncation hash). One of the most basic hash functions $H_k : \{0,1\}^* \rightarrow \{0,1\}^k$ is given by simply truncating every binary string s of size $s.len() > k$ to a string of size k and by filling any string s' of size $s'.len() < k$ with zeros. To make this hash function deterministic, we define that both truncation and filling should happen “on the left”.

For example, if $k = 3$, $x_1 = (0000101011101010011101010101)$ and $x_2 = 1$, then $H(x_1) = (101)$ and $H(x_2) = (001)$. It is easy to see that this hash function is deterministic and uniform.

Of particular interest are so-called **cryptographic** hash functions, which are hash functions that are also **one-way functions**, which essentially means that, given a string y from $\{0,1\}^k$ it is practically infeasible to find a string $x \in \{0,1\}^*$ such that $H(x) = y$ holds. This property is usually called **preimage-resistance**.

In addition, it should be infeasible to find two strings $x_1, x_2 \in \{0,1\}^*$, such that $H(x_1) = H(x_2)$, which is called **collision resistance**. It is important to note, though, that collisions always exist, since a function $H : \{0,1\}^* \rightarrow \{0,1\}^k$ inevitably maps infinitely many values onto the same hash. In fact, for any hash function with digests of length k , finding a preimage to a given digest can always be done using a brute force search in 2^k evaluation steps. It should just be practically impossible to compute those values, and statistically very unlikely to generate two of them by chance.

A third property of a cryptographic hash function is that small changes in the input string, like changing a single bit, should generate hash values that look completely different from each other.

Because cryptographically secure hash functions map tiny changes in input values onto large changes in the output, implementation errors that change the outcome are usually easy to spot by comparing them to expected output values. The definitions of cryptographically secure

are these going to be relevant later? yes, they are used in various snark proof systems

TODO: theorem: every factor of order defines a subgroup...

Is there a term for this property?

hash functions are therefore usually accompanied by some test vectors of common inputs and expected digests. Since the empty string $''$ is the only string of length 0, a common test vector is the expected digest of the empty string.

Example 46 (k -truncation hash). Consider the k -truncation hash from example 45. Since the empty string has length 0, it follows that the digest of the empty string is string of length k that only contains 0's:

$$H_k('') = (000 \dots 000) \quad (4.15)$$

It is pretty obvious from the definition of H_k that this simple hash function is not a cryptographic hash function. In particular, every digest is its own preimage, since $H_k(y) = y$ for every string of size exactly k . Finding preimages is therefore easy, so the property of preimage resistance does not hold.

In addition, it is easy to construct collisions as all strings of size $> k$ that share the same k -bits “on the right” are mapped to the same hash value, so this function is not collision resistant, either.

Finally, this hash function is not very chaotic, as changing bits that are not part of the k right-most bits don't change the digest at all.

Computing cryptographically secure hash functions in pen-and-paper style is possible but tedious. Fortunately, Sage can import the **hashlib** library, which is intended to provide a reliable and stable base for writing Python programs that require cryptographic functions. The following examples explain how to use hashlib in Sage.

Example 47. An example of a hash function that is generally believed to be a cryptographically secure hash function is the so-called **SHA256** hash, which, in our notation, is a function that maps binary strings of arbitrary length onto binary strings of length 256:

$$SHA256 : \{0, 1\}^* \rightarrow \{0, 1\}^{256} \quad (4.16)$$

To evaluate a proper implementation of the *SHA256* hash function, the digest of the empty string is supposed to be

$$SHA256('') = e3b0c44298fc1c149afbf4c8996fb92427ae41e4649b934ca495991b7852b855 \quad (4.17)$$

For better human readability, it is common practice to represent the digest of a string not in its binary form, but in a hexadecimal representation. We can use Sage to compute *SHA256* and freely transit between binary, hexadecimal and decimal representations. To do so, we import hashlib's implementation of *SHA256*:

```
sage: import hashlib 159
sage: test = 'e3b0c44298fc1c149afbf4c8996fb92427ae41e4649b934 160
ca495991b7852b855'
sage: hasher = hashlib.sha256(b'') 161
sage: str = hasher.hexdigest() 162
sage: type(str) 163
<class 'str'> 164
sage: d = ZZ('0x' + str) # conversion to integer type 165
sage: d.str(16) == str 166
True 167
sage: d.str(16) == test 168
True 169
```



```

2063 sage: d.str(16) 170
2064 e3b0c44298fc1c149afbf4c8996fb92427ae41e4649b934ca495991b7852b8 171
2065 55
2066 sage: d.str(2) 172
2067 11100011101100001100010001000010100110001111110000011100000101 173
2068 0010011010111110111110100110010001001100101101111101110010
2069 01001000010011110101110010000011110010001100100100110111001
2070 00110100110010100100100101011001100100011011011110000101001
2071 01011100001010101
2072 sage: d.str(10) 174
2073 10298733624955409702953521232258132278979990064819803499337939 175
2074 7001115665086549

```

Hashing to cyclic groups As we have seen in the previous paragraph, general hash functions map binary strings of arbitrary length onto binary strings of length k . However, it is desirable in various cryptographic primitives to not simply hash to binary strings of fixed length but to hash into algebraic structures like groups, while keeping (some of) the properties like preimage resistance or collision resistance.

Hash functions like this can be defined for various algebraic structures, but, in a sense, the most fundamental ones are hash functions that map into groups, because they can be easily extended to map into other structures like rings or fields.

To give a more precise definition, let \mathbb{G} be a group and $\{0, 1\}^*$ the set of all finite, binary strings, then a **hash-to-group** function is a deterministic map

$$H : \{0, 1\}^* \rightarrow \mathbb{G} \quad (4.18)$$

Common properties of hash functions, like uniformity, are desirable but not always realized in real-world instantiations of hash-to-group functions, so we skip those requirements for now and keep the definition very general.

As the following example shows, hashing to finite cyclic groups can be trivially achieved for the price of some undesirable properties of the hash function:

Example 48 (Naive cyclic group hash). Let \mathbb{G} be a finite cyclic group. If the task is to implement a hash-to-group function, one immediate approach can be based on the observation that binary strings of size k can be interpreted as integers $z \in \mathbb{Z}$ in the range $0 \leq z < 2^k$.

To be more precise, choose an ordinary hash function $H : \{0, 1\}^* \rightarrow \{0, 1\}^k$ for some parameter k and a generator g of \mathbb{G} . Then the expression below is a positive integer (where $H(s)_j$ means the bit at the j -th position of $H(s)$):

$$z_{H(s)} = H(s)_0 \cdot 2^0 + H(s)_1 \cdot 2^1 + \dots + H(s)_k \cdot 2^k \quad (4.19)$$

A hash-to-group function for the group \mathbb{G} can then be defined as a concatenation of the exponential map $g^{(\cdot)}$ of g with the interpretation of $H(s)$ as an integer:

$$H_g : \{0, 1\}^* \rightarrow \mathbb{G} : s \mapsto g^{z_{H(s)}} \quad (4.20)$$

Constructing a hash-to-group function like this is easy for cyclic groups, and it might be good enough in certain applications. It is, however, almost never adequate in cryptographic applications, as discrete log relations might be constructible between two given hash values $H_g(s)$ and $H_g(t)$.

a few examples?

To see that, assume that \mathbb{G} is of order r and that $z_{H(s)}$ has a multiplicative inverse in modular r arithmetics. In that case, we can compute $x = z_{H(t)} \cdot z_{H(s)}^{-1}$ in \mathbb{Z}_r and find a discrete log relation between the group hash values, that is, find some x with $H_g(t) = (H_g(s))^x$:

$$\begin{aligned} H_g(t) &= (H_g(s))^x && \Leftrightarrow \\ g^{z_{H(t)}} &= g^{z_{H(s)} \cdot x} && \Leftrightarrow \\ g^{z_{H(t)}} &= g^{z_{H(t)}} \end{aligned}$$

Therefore applications where discrete log relations between hash values are undesirable need different approaches. Many of these approaches start with a way to hash into the set \mathbb{Z}_r of modular r arithmetics.

Hashing to modular arithmetics One of the most widely used applications of hash-into-group functions are hash functions that map into the set \mathbb{Z}_r of modular r arithmetics for some modulus r . Different approaches to construct such a function are known, but probably the most widely used ones are based on the insight that the images of arbitrary hash functions can be interpreted as binary representations of integers, as explained in example 48.

It follows from this interpretation that one simple method of hashing into \mathbb{Z}_r is constructed by observing that if r is a modulus with a bit-length of $k = r.\text{nbits}()$, then every binary string $(b_0, b_1, \dots, b_{k-2})$ of length $k - 1$ defines an integer z in the range $0 \leq z < 2^{k-1} \leq r$, by defining z :

$$z = b_0 \cdot 2^0 + b_1 \cdot 2^1 + \dots + b_{k-2} \cdot 2^{k-2} \quad (4.21)$$

Now, since $z < r$, we know that z is guaranteed to be in the set $\{0, 1, \dots, r-1\}$, and hence it can be interpreted as an element of \mathbb{Z}_r . From this it follows that if $H : \{0, 1\}^* \rightarrow \{0, 1\}^{k-1}$ is a hash function, then a hash-to-group function can be constructed as follows (where $H(s)_j$ means the j -th bit of the image binary string $H(s)$ of the original binary hash function):

$$H_{r.\text{nbits}()-1} : \{0, 1\}^* \rightarrow \mathbb{Z}_r : s \mapsto H(s)_0 \cdot 2^0 + H(s)_1 \cdot 2^1 + \dots + H(s)_{k-2} \cdot 2^{k-2} \quad (4.22)$$

A drawback of this hash function is that the distribution of the hash values in \mathbb{Z}_r is not necessarily uniform. In fact, if $r - 2^{k-1} \neq 0$, then by design $H_{r.\text{nbits}()-1}$ will never hash onto values $z \geq 2^{k-1}$. Good moduli r are therefore as close to 2^{k-1} as possible, while less good moduli are closer to 2^k . In the worst case, when $r = 2^k - 1$, it misses $2^{k-1} - 1$, that is, almost half of all elements, from \mathbb{Z}_r .

An advantage of this approach is that properties like preimage resistance or collision resistance of the original hash function $H(\cdot)$ are preserved.

Example 49. To give an implementation of the $H_{r.\text{nbits}()-1}$ hash function, we use a 5-bit truncation of the *SHA256* hash from example 47 and define a hash into \mathbb{Z}_{16} as follows:

$$H_{16.\text{nbits}()-1} : \{0, 1\}^* \rightarrow \mathbb{Z}_{16} : s \mapsto \text{SHA256}(s)_0 \cdot 2^0 + \text{SHA256}(s)_1 \cdot 2^1 + \dots + \text{SHA256}(s)_4 \cdot 2^4$$

Since $k = 16.\text{nbits}() = 5$ and $16 - 2^{k-1} = 0$, this hash maps uniformly onto \mathbb{Z}_{16} . We can invoke Sage to implement it:

```

2126 sage: import hashlib                                176
2127 sage: def Hash5(x):                                177
2128     ....:     hasher = hashlib.sha256(x)            178
2129     ....:     digest = hasher.hexdigest()           179
2130     ....:     d = ZZ(digest, base=16)               180
2131     ....:     d = d.str(2)[-4:]                     181
2132     ....:     return ZZ(d, base=2)                  182

```

check
reference

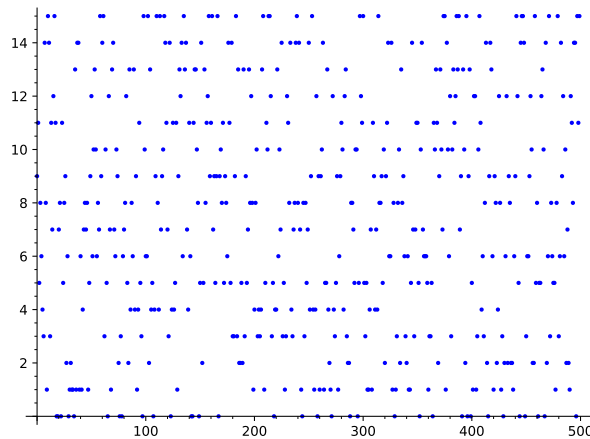
TODO:
DOUBLE
CHECK
THIS
REA-
SONING.

2133 `sage: Hash5(b'')`
 2134 `5`

183

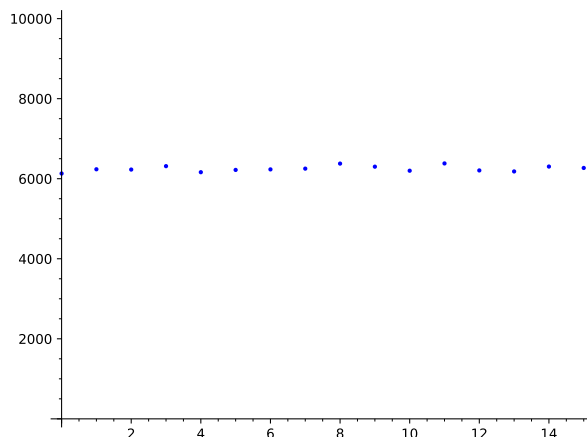
184

2135 We can then use Sage to apply this function to a large set of input values in order to plot a
 2136 visualization of the distribution over the set $\{0, \dots, 15\}$. Executing over 500 input values gives
 2137 the following plot:



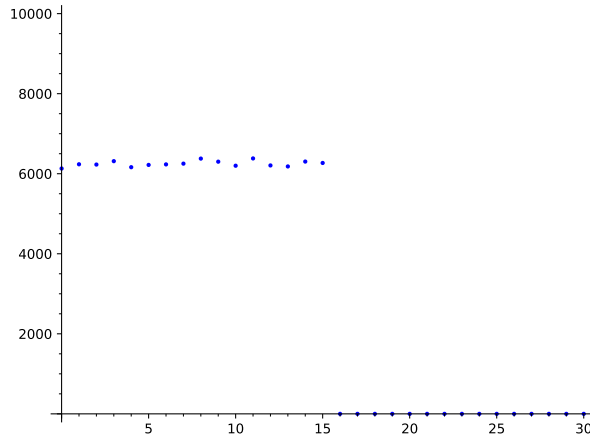
2138

2139 To get an intuition of uniformity, we can count the number of times the hash function $H_{16.nb\text{its}()-1}$
 2140 maps onto each number in the set $\{0, 1, \dots, 15\}$ in a loop of 100000 hashes, and compare that
 2141 to the ideal uniform distribution, which would map exactly 6250 samples to each element. This
 2142 gives the following result:



2143

2144 The uniformity of distribution problem becomes apparent if we want to construct a similar hash
 2145 function for \mathbb{Z}_r for any r in the range $17 \leq r \leq 31$. In this case, the definition of the hash
 2146 function is exactly the same as for \mathbb{Z}_{16} , and hence, the images will not exceed the value 16.
 2147 So, for example, even in the case of hashing to \mathbb{Z}_{31} , the hash function never maps to any value
 2148 larger than 16, leaving almost half of all numbers out of the image range.



Mirco:
We can
do better
than this

The second widely used method of hashing into \mathbb{Z}_r is constructed by observing the following: If r is a modulus with a bit-length of $r.bits() = k_1$ and $H : \{0, 1\}^* \rightarrow \{0, 1\}^{k_2}$ is a hash function that produces digests of size k_2 , with $k_2 \geq k_1$, then a hash-to-group function can be constructed by interpreting the image of H as a binary representation of an integer and then taking the modulus by r to map into \mathbb{Z}_r . This is formalized in the equation below, where $H(s)_j$ means the j 'th bit of the image binary string $H(s)$ of the original binary hash function.

$$H'_{mod_r} : \{0, 1\}^* \rightarrow \mathbb{Z}_r : s \mapsto \left(H(s)_0 \cdot 2^0 + H(s)_1 \cdot 2^1 + \dots + H(s)_{k_2} \cdot 2^{k_2} \right) \bmod n \quad (4.23)$$

A drawback of this hash function is that computing the modulus requires some computational effort. In addition, the distribution of the hash values in \mathbb{Z}_r might not be even, depending on the difference $2^{k_2+1} - r$. An advantage of it is that potential properties of the original hash function $H(\cdot)$ (like preimage resistance or collision resistance) are preserved, and the distribution can be made almost uniform, with only negligible bias depending on what modulus r and images size k_2 are chosen.

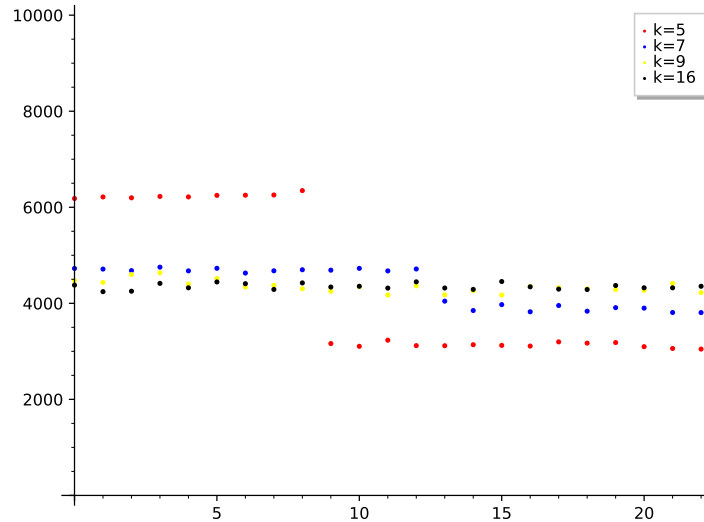
Example 50. To give an implementation of the H_{mod_r} hash function, we use k_2 -bit truncation of the *SHA256* hash from example 47, and define a hash into \mathbb{Z}_{23} as follows:

$$H_{mod_{23}, k_2} : \{0, 1\}^* \rightarrow \mathbb{Z}_{23} : \\ s \mapsto \left(SHA256(s)_0 \cdot 2^0 + SHAH256(s)_1 \cdot 2^1 + \dots + SHA256(s)_{k_2} \cdot 2^{k_2} \right) \bmod 23$$

We want to use various instantiations of k_2 to visualize the impact of truncation length on the distribution of the hashes in \mathbb{Z}_{23} . We can invoke Sage to implement it as follows:

```
sage: import hashlib
sage: Z23 = Integers(23)
sage: def Hash_mod23(x, k2):
.....:     hasher = hashlib.sha256(x.encode('utf-8'))
.....:     digest = hasher.hexdigest()
.....:     d = ZZ(digest, base=16)
.....:     d = d.str(2)[-k2:]
.....:     d = ZZ(d, base=2)
.....:     return Z23(d)
```

We can then use Sage to apply this function to a large set of input values in order to plot visualizations of the distribution over the set $\{0, \dots, 22\}$ for various values of k_2 , by counting the number of times it maps onto each number in a loop of 100000 hashes. We get the following plot:



A third method that can sometimes be found in implementations is the so-called “**try-and-increment**” method. To understand this method, we define an integer $z \in \mathbb{Z}$ from any hash value $H(s)$ as we did in the previous methods, that is, we define $z = H(s)_0 \cdot 2^0 + H(s)_1 \cdot 2^1 + \dots + H(s)_{k-1} \cdot 2^k$.

Hashing into \mathbb{Z}_r is then achievable by first computing z , and then trying to see if $z \in \mathbb{Z}_r$. If it is, then the hash is done; if not, the string s is modified in a deterministic way and the process is repeated until a suitable number z is found. A suitable, deterministic modification could be to concatenate the original string by some bit counter. A “try-and-increment” algorithm would then work like in algorithm 5.

check
reference

Algorithm 5 Hash-to- \mathbb{Z}_n

Require: $r \in \mathbb{Z}$ with $r.\text{nbits}() = k$ and $s \in \{0, 1\}^*$

procedure TRY-AND-INCREMENT(r, k, s)

$c \leftarrow 0$

repeat

$s' \leftarrow s || c_bits()$

$z \leftarrow H(s')_0 \cdot 2^0 + H(s')_1 \cdot 2^1 + \dots + H(s')_k \cdot 2^k$

$c \leftarrow c + 1$

until $z < r$

return x

end procedure

Ensure: $z \in \mathbb{Z}_r$

Depending on the parameters, this method can be very efficient. In fact, if k is sufficiently large and r is close to 2^{k+1} , the probability for $z < r$ is very high and the repeat loop will almost always be executed a single time only. A drawback is, however, that the probability of having to execute the loop multiple times is not zero.

Once some hash function into modular arithmetics is found, it can often be combined with additional techniques to hash into more general finite cyclic groups. The following paragraphs describe a few of those methods widely adopted in SNARK development.

Pedersen Hashes The so-called **Pedersen hash function** [?] provides a way to map binary inputs of fixed size k onto elements of finite cyclic groups that avoids discrete log relations

between the images as they occur in the naive approach XXX. Combining it with a classical hash function provides a hash function that maps strings of arbitrary length onto group elements.

To be more precise, let j be an integer, \mathbb{G} a finite cyclic group of order r and $\{g_1, \dots, g_j\} \subset \mathbb{G}$ a uniform randomly generated set of generators of \mathbb{G} . Then **Pedersen's hash function** is defined as follows:

$$H_{Ped} : (\mathbb{Z}_r)^j \rightarrow \mathbb{G} : (x_1, \dots, x_j) \mapsto \prod_{i=1}^j g_i^{x_i} \quad (4.24)$$

It can be shown that Pedersen's hash function is collision-resistant under the assumption that \mathbb{G} is a DL-A group. However, it is important to note that Pedersen hashes cannot be assumed to be **pseudorandom** and should therefore not be used where a hash function serves as an approximation of a random **oracle**. *will these be explained in the initial chapters?*

From an implementation perspective, it is important to derive the set of generators $\{g_1, \dots, g_j\}$ in such a way that they are as uniform and random as possible. In particular, any known discrete log relation between two generators, that is, any known $x \in \mathbb{Z}_r$ with $g_h = (g_i)^x$ must be avoided.

To see how Pedersen hashes can be used to define an actual hash-to-group function according to our definition, we can use any of the hash-to- \mathbb{Z}_r functions as we have derived them in equation 4.22.

MimC Hashes [?]

Pseudorandom Functions in DDH-A groups As noted in above, Pederson's hash function does not have the properties a random function and should therefore not be instantiated as such. To see an example of a random oracle function in groups where the decisional Diffie–Hellman construction is assumed to hold true, let \mathbb{G} be a DDH-A group of order r with generator g and $\{a_0, a_1, \dots, a_k\} \subset \mathbb{Z}_r^*$ a uniform randomly generated set of numbers invertible in modular r arithmetics. Then a pseudo-random function is given by the as follows:

$$F_{rand} : \{0, 1\}^{k+1} \rightarrow \mathbb{G} : (b_0, \dots, b_k) \mapsto g^{b_0 \cdot \prod_{i=1}^k a_i^{b_i}} \quad (4.25)$$

Of course, if $H : \{0, 1\}^* \rightarrow \{0, 1\}^{k+1}$ is a random oracle, then the concatenation of F_{rand} and H also defines a random oracle

$$H_{rand, \mathbb{G}} : \{0, 1\}^* \rightarrow \mathbb{G} : s \mapsto F_{rand}(H(s)) \quad (4.26)$$

4.2 Commutative Rings

Thinking back to operations on integers, we know that there are two of these: addition and multiplication. As we have seen, addition defines a group structure on the set of integers. However, multiplication does not define a group structure, given that integers generally don't have multiplicative inverses.

Configurations like these constitute so-called **commutative rings with unit**. To be more precise, a commutative ring with unit $(R, +, \cdot, 1)$ is a set R provided with two maps $+: R \cdot R \rightarrow R$ and $\cdot: R \cdot R \rightarrow R$, called **addition** and **multiplication**, such that the following conditions hold:

Definition 4.2.0.1. Commutative ring with unit

- $(R, +)$ is a commutative group, where the neutral element is denoted with 0. **Commutativity of multiplication:** $r_1 \cdot r_2 = r_2 \cdot r_1$ for all $r_1, r_2 \in R$.
- **Existence of a unit:** There is an element $1 \in R$, such that $1 \cdot g$ holds for all $g \in R$,

• **Associativity:** For every $g_1, g_2, g_3 \in \mathbb{G}$ the equation $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$ holds.

• **Distributivity:** For all $g_1, g_2, g_3 \in R$ the distributive laws $g_1 \cdot (g_2 + g_3) = g_1 \cdot g_2 + g_1 \cdot g_3$ holds.

Example 51 (The ring of integers). The set \mathbb{Z} of integers with the usual addition and multiplication is the archetypical example of a commutative ring with unit 1.

Example 52 (Underlying commutative group of a ring). Every commutative ring with unit $(R, +, \cdot, 1)$ gives rise to a group, if we disregard multiplication.

The following example is somewhat unusual, but we encourage you to think through it because it helps to detach the mind from familiar styles of computation and concentrate on the abstract algebraic explanation.

Example 53. Let $S := \{\bullet, \star, \odot, \otimes\}$ be a set that contains four elements, and let addition and multiplication on S be defined as follows:

\cup	\bullet	\star	\odot	\otimes
\bullet	\bullet	\star	\odot	\otimes
\star	\star	\odot	\otimes	\bullet
\odot	\odot	\otimes	\bullet	\star
\otimes	\otimes	\bullet	\star	\odot

\circ	\bullet	\star	\odot	\otimes
\bullet	\bullet	\bullet	\bullet	\bullet
\star	\bullet	\star	\odot	\otimes
\odot	\bullet	\odot	\bullet	\odot
\otimes	\bullet	\otimes	\odot	\star

Then (S, \cup, \circ) is a ring with unit \star and zero \bullet . It therefore makes sense to ask for solutions to equations like this one: Find $x \in S$ such that

$$\otimes \circ (x \cup \odot) = \star$$

To see how such a “moonmath equation” can be solved, we have to keep in mind that rings behaves mostly like normal numbers when it comes to bracketing and computation rules. The only differences are the symbols, and the actual way to add and multiply them. With this in mind, we solve the equation for x in the “usual way”:

$$\begin{aligned}
 \otimes \circ (x \cup \odot) &= \star && \# \text{ apply the distributive law} \\
 \otimes \circ x \cup \otimes \circ \odot &= \star && \# \otimes \circ \odot = \odot \\
 \otimes \circ x \cup \odot &= \star && \# \text{ concatenate the } \cup \text{ inverse of } \odot \text{ to both sides} \\
 \otimes \circ x \cup \odot \cup -\odot &= \star \cup -\odot && \# \odot \cup -\odot = \bullet \\
 \otimes \circ x \cup \bullet &= \star \cup -\odot && \# \bullet \text{ is the } \cup \text{ neutral element} \\
 \otimes \circ x &= \star \cup -\odot && \# \text{ for } \cup \text{ we have } -\odot = \odot \\
 \otimes \circ x &= \star \cup \odot && \# \star \cup \odot = \otimes \\
 \otimes \circ x &= \otimes && \# \text{ concatenate the } \circ \text{ inverse of } \otimes \text{ to both sides} \\
 (\otimes)^{-1} \circ \otimes \circ x &= (\otimes)^{-1} \circ \otimes && \# \text{ multiply with the multiplicative inverse} \\
 \star \circ x &= \star \\
 x &= \star
 \end{aligned}$$

So, even though this equation looked really alien at first glance, we could solve it basically exactly the way we solve “normal” equations containing numbers.

Note, however, that whenever a multiplicative inverse would be needed to solve an equation in the usual way in a ring, things can be very different than most of us are used to. For example,

the following equation cannot be solved for x in the usual way, since there is no multiplicative inverse for \odot in our ring.

$$\odot \circ x = \otimes \quad (4.27)$$

We can confirm this by looking at the multiplication table to see that no such x exists. As another example, the following equation does not have a single solution but two: $x \in \{\star, \otimes\}$.

$$\odot \circ x = \odot \quad (4.28)$$

Having no solution or two solutions is certainly not something to expect from types like \mathbb{Q} (rational numbers). [can we use another set as an example? we hardly talked about \$\mathbb{Q}\$ so far](#)

Example 54. Considering polynomials again, we note from their definition that what we have called the type R of the coefficients must in fact be a commutative ring with a unit, since we need addition, multiplication, commutativity and the existence of a unit for $R[x]$ to have the properties we expect.

Considering R to be a ring with addition and multiplication of polynomials as defined in 4.2.0.1 [actually makes \$R\[x\]\$ into a commutative ring with a unit, too, where the polynomial 1 is the multiplicative unit.](#)

check
reference

Example 55. Let n be a modulus and $(\mathbb{Z}_n, +, \cdot)$ the set of all remainder classes of integers modulo n , with the projection of integer addition and multiplication as defined in 4.2.0.1. It can be shown that $(\mathbb{Z}_n, +, \cdot)$ is a commutative ring with unit 1.

check
reference

Considering the exponential map from page 44 [again, let \$\mathbb{G}\$ be a finite cyclic group of order \$n\$ with generator \$g \in \mathbb{G}\$. Then the ring structure of \$\(\mathbb{Z}_n, +, \cdot\)\$ is mapped onto the group structure of \$\mathbb{G}\$ in the following way:](#)

check
reference

$$\begin{aligned} g^{x+y} &= g^x \cdot g^y & \text{for all } x, y \in \mathbb{Z}_n \\ g^{x \cdot y} &= (g^x)^y & \text{for all } x, y \in \mathbb{Z}_n \end{aligned}$$

This is of particular interest in cryptography and SNARKs, as it allows for the evaluation of polynomials with coefficients in \mathbb{Z}_n to be evaluated “in the exponent”. To be more precise, let $p \in \mathbb{Z}_n[x]$ be a polynomial with $p(x) = a_m \cdot x^m + a_{m-1}x^{m-1} + \dots + a_1x + a_0$. Then the previously defined exponential laws (example 36) [imply the following:](#)

check
reference

$$\begin{aligned} g^{p(x)} &= g^{a_m \cdot x^m + a_{m-1}x^{m-1} + \dots + a_1x + a_0} \\ &= (g^{x^m})^{a_m} \cdot (g^{x^{m-1}})^{a_{m-1}} \cdot \dots \cdot (g^x)^{a_1} \cdot g^{a_0} \end{aligned}$$

Hence, to evaluate p at some point s in the exponent, we can insert s into the right-hand side of the last equation and evaluate the product.

As we will see, this is a key insight to understanding many SNARK protocols like e.g. Groth16 [?] or XXX.

add more
examples
proto-
cols of
SNARK

Example 56. To give an example of the evaluation of a polynomial in the exponent of a finite cyclic group, consider the exponential map from example 36:

$$3^{(\cdot)} : \mathbb{Z}_4 \rightarrow \mathbb{Z}_5^* \quad x \mapsto 3^x$$

check
reference

Choosing the polynomial $p(x) = 2x^2 + 3x + 1$ from $\mathbb{Z}_4[x]$, we can evaluate the polynomial at say $x = 2$ in the exponent of 3 in two different ways. On the one hand, we can evaluate p at 2 and then write the result into the exponent as follows:

$$\begin{aligned} 3^{p(2)} &= 3^{2 \cdot 2^2 + 3 \cdot 2 + 1} \\ &= 3^{2 \cdot 0 + 2 + 1} \\ &= 3^3 \\ &= 2 \end{aligned}$$

On the other hand, we can use the right-hand side of the equation to evaluate p at 2 in the exponent of 3 as follows:

$$\begin{aligned} 3^{p(2)} &= (3^{2^2})^2 \cdot (3^2)^3 \cdot 3^1 \\ &= (3^0)^2 \cdot 3^3 \cdot 3 \\ &= 1^2 \cdot 2 \cdot 3 \\ &= 2 \cdot 3 \\ &= 2 \end{aligned}$$

2271 **Hashing to Commutative Rings** As we have seen in XXX, various constructions for hashing-
 2272 to-groups are known and used in applications. As commutative rings are **Abelian groups**, when
 2273 we disregard the multiplicative structure, hash-to-group constructions can be applied for hash-
 2274 ing into commutative rings, too. This is possible in general, as the **codomain** of a general hash
 2275 function $\{0, 1\}^*$ is just the set of binary strings of arbitrary but finite length, which has no
 2276 algebraic structure that the hash function must respect.

add refer-
ence

Abelian
groups

codomain

2277 4.3 Fields

2278 We started this chapter with the definition of a group (section 4.1), which we then expanded into
 2279 the definition of a commutative ring with a unit (section 4.2). Such rings generalize the behavior
 2280 of integers. In this section, we will look at those special cases of commutative rings where every
 2281 element other than the neutral element of addition has a multiplicative inverse. Those structures
 2282 behave very much like the rational numbers \mathbb{Q} . Rational numbers are, in a sense, an extension
 2283 of the ring of integers, that is, they are constructed by including newly defined multiplicative
 2284 inverses (fractions) to the integers.

2285 Now, considering the definition of a ring (4.2.0.1) again, we define a **field** $(\mathbb{F}, +, \cdot)$ to be a
 2286 set \mathbb{F} , together with two maps $+: \mathbb{F} \cdot \mathbb{F} \rightarrow \mathbb{F}$ and $\cdot: \mathbb{F} \cdot \mathbb{F} \rightarrow \mathbb{F}$, called *addition* and *multiplication*,
 2287 such that the following conditions hold:

2288 **Definition 4.3.0.1. Field**

- 2289 • $(\mathbb{F}, +)$ is a commutative group, where the neutral element is denoted by 0.
- 2290 • $(\mathbb{F} \setminus \{0\}, \cdot)$ is a commutative group, where the neutral element is denoted by 1.
- 2291 • (Distributivity) For all $g_1, g_2, g_3 \in \mathbb{F}$ the distributive law $g_1 \cdot (g_2 + g_3) = g_1 \cdot g_2 + g_1 \cdot g_3$
 2292 holds.

If a field is given and the definition of its addition and multiplication is not ambiguous, we will often simply write \mathbb{F} instead of $(\mathbb{F}, +, \cdot)$ to denote the field. Moreover, we use \mathbb{F}^* to describe the multiplicative group of the field, that is, the set of elements with multiplication as the group law, excluding the neutral element of addition.

The **characteristic** of a field \mathbb{F} , represented as $\text{char}(\mathbb{F})$, is the smallest natural number $n \geq 1$ for which the n -fold sum of 1 equals zero, i.e. for which $\sum_{i=1}^n 1 = 0$. If such an $n > 0$ exists, the field is also said to have a **finite characteristic**. If, on the other hand, every finite sum of 1 is such that it is not equal to zero, then the field is defined to have characteristic 0. S: Tried to disambiguate the scope of negation between 1. “It is true of every finite sum of 1 that it is not equal to 0” and 2. “It is not true of every finite sum of 1 that it is equal to 0” From the example below, it looks like 1. is the intended meaning here, correct?

Check
change of
wording

Example 57 (Field of rational numbers). Probably the best known example of a field is the set of rational numbers \mathbb{Q} together with the usual definition of addition, subtraction, multiplication and division. Since there is no natural number $n \in \mathbb{N}$, such that $\sum_{j=0}^n 1 = 0$ in the set of rational numbers, the characteristic $\text{char}(\mathbb{Q})$ of the field \mathbb{Q} is zero. In Sage, rational numbers are called as follows:

```
sage: QQ                                     194
Rational Field                               195
sage: QQ(1/5) # Get an element from the field of rational 196
      numbers
1/5                                           197
sage: QQ(1/5) / QQ(3) # Division             198
1/15                                         199
```

Example 58 (Field with two elements). It can be shown that, in any field, the neutral element 0 of addition must be different from the neutral element 1 of multiplication, that is, $0 \neq 1$ always holds in a field. From this, it follows that the smallest field must contain at least two elements. As the following addition and multiplication tables show, there is indeed a field with two elements, which is usually called \mathbb{F}_2 :

Let $\mathbb{F}_2 := \{0, 1\}$ be a set that contains two elements and let addition and multiplication on \mathbb{F}_2 be defined as follows:

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

Since $1 + 1 = 0$ in the field \mathbb{F}_2 , we know that the characteristic of \mathbb{F}_2 is there, that is, we have $\text{char}(\mathbb{F}_2) = 2$.

For reasons we will understand better in XXX, Sage defines this field as a so-called Galois field with 2 elements. You can call it in Sage as follows:

add refer-
ence

```
sage: F2 = GF(2)                             200
sage: F2(1) # Get an element from GF(2)       201
1                                               202
sage: F2(1) + F2(1) # Addition                 203
0                                               204
sage: F2(1) / F2(1) # Division                 205
1                                               206
```

2335 *Example 59.* Both the real numbers \mathbb{R} as well as the complex numbers \mathbb{C} are well known ex-
 2336 amples of fields.

Expand
on this?

2337 *Exercise 41.* Consider our remainder class ring $(\mathbb{Z}_5, +, \cdot)$ and show that it is a field. What is the
 2338 characteristic of \mathbb{Z}_5 ?

2339 4.3.1 Prime fields

2340 As we have seen in the various examples of the previous sections, modular arithmetics behaves
 2341 similarly to the ordinary arithmetics of integers in many ways. This is due to the fact that
 2342 remainder class sets \mathbb{Z}_n are commutative rings with units.

2343 However, we have also seen in 35 that, whenever the modulus is a prime number, every
 2344 remainder class other than the zero class has a modular multiplicative inverse. This is an im-
 2345 portant observation, since it immediately implies that, in case of a prime number, the remainder
 2346 class set \mathbb{Z}_n is not just a ring but actually a **field**. Moreover, since $\sum_{j=0}^n 1 = 0$ in \mathbb{Z}_n , we know
 2347 that those fields have the finite characteristic n .

check
reference

2348 To distinguish this important case from arbitrary remainder class rings, we write $(\mathbb{F}_p, +, \cdot)$
 2349 for the field of all remainder classes for a prime number modulus $p \in \mathbb{P}$ and call it the **prime**
 2350 **field** of characteristic p .

2351 Prime fields are the foundation for many of the contemporary algebra-based cryptographic
 2352 systems, as they have many desirable properties. One of them is that, since these sets are finite
 2353 and a prime field of characteristic, p can be represented on a computer in roughly $\log_2(p)$
 2354 amount of space without precision problems that are unavoidable for computer representations
 2355 of infinite sets such as rational numbers or integers.

2356 Since prime fields are special cases of remainder class rings, all computations remain the
 2357 same. Addition and multiplication can be computed by first doing normal integer addition
 2358 and multiplication, and then taking the remainder modulus p . Subtraction and division can be
 2359 computed by adding or multiplying with the additive or the multiplicative inverse, respectively.
 2360 The additive inverse $-x$ of a field element $x \in \mathbb{F}_p$ is given by $p - x$, and the multiplicative inverse
 2361 of $x \neq 0$ is given by x^{p-2} , or can be computed using the Extended Euclidean Algorithm.

2362 Note that these computations might not be the fastest to implement on a computer. They
 2363 are, however, useful in this book, as they are easy to compute for small prime numbers.

2364 *Example 60.* The smallest field is the field \mathbb{F}_2 of characteristic 2 as we have seen in example
 2365 58. It is the prime field of the prime number 2.

Example 61. To summarize the basic aspects of computation in prime fields, let us consider the
 prime field \mathbb{F}_5 and simplify the following expression:

$$\left(\frac{2}{3} - 2\right) \cdot 2$$

A first thing to note is that since \mathbb{F}_5 is a field, all rules are identical to the rules we learned in
 school when we were dealing with rational, real or complex numbers. This means we can use

e.g. bracketing (distributivity) or addition as usual:

$$\begin{aligned}
 \left(\frac{2}{3} - 2\right) \cdot 2 &= \frac{2}{3} \cdot 2 - 2 \cdot 2 && \# \text{ distributive law} \\
 &= \frac{2 \cdot 2}{3} - 2 \cdot 2 && 4 \bmod 5 = 4 \\
 &= \frac{4}{3} - 4 && \# \text{ multiplicative inverse of 3 is } 3^{5-2} \bmod 5 = 2 \\
 &= 4 \cdot 2 - 4 && \# \text{ additive inverse of 4 is } 5 - 4 = 1 \\
 &= 4 \cdot 2 + 1 && 8 \bmod 5 = 3 \\
 &= 3 + 1 && 4 \bmod 5 = 4 \\
 &= 4
 \end{aligned}$$

2366 In this computation, we computed the multiplicative inverse of 3 using the identity $x^{-1} = x^{p-2}$
 2367 in a prime field. This is impractical for large prime numbers. Recall that another way of
 2368 computing the multiplicative inverse is the Extended Euclidean Algorithm (see 3.10 on page
 2369 19). To refresh our memory, the task is to compute $x^{-1} \cdot 3 + t \cdot 5 = 1$, but t is actually irrelevant.
 2370 We get

k	r_k	x_k^{-1}	$t_k = (r_k - s_k \cdot a) \operatorname{div} b$
0	3	1	.
1	5	0	.
2	3	1	.
3	2	-1	.
4	1	2	.

2372 So the multiplicative inverse of 3 in \mathbb{Z}_5 is 2, and, indeed, if compute $3 \cdot 2$, we get 1 in \mathbb{F}_5 .

2373 **Square Roots** In this part, we deal with square numbers, also called **quadratic residues** and
 2374 **square roots** in prime fields. This is of particular importance in our studies on elliptic curves,
 2375 as only square numbers can actually be points on an elliptic curve.

2376 To make the intuition of quadratic residues and roots precise, let $p \in \mathbb{P}$ be a prime number
 2377 and \mathbb{F}_p its associated prime field. Then a number $x \in \mathbb{F}_p$ is called a **square root** of another
 2378 number $y \in \mathbb{F}_p$, if x is a solution to the following equation:

$$x^2 = y \quad (4.29)$$

2379 In this case, y is called a **quadratic residue**. On the other hand, if y is given and the quadratic
 2380 equation has no solution x , we call y a **quadratic non-residue**. For any $y \in \mathbb{F}_p$, we denote the
 2381 set of all square roots of y in the prime field \mathbb{F}_n as follows:

$$\sqrt{y} := \{x \in \mathbb{F}_p \mid x^2 = y\} \quad (4.30)$$

2382 If y is a quadratic non-residue, then $\sqrt{y} = \emptyset$ (an empty set), and if $y = 0$, then $\sqrt{y} = \{0\}$.

2383 Informally speaking, quadratic residues are numbers such that we can take the square root
 2384 of them, while quadratic non-residues are numbers that don't have square roots. The situation
 2385 therefore parallels the familiar case of integers, where some integers like 4 or 9 have square
 2386 roots and others like 2 or 3 don't (as integers).

2387 It can be shown that, in any prime field, every non zero element has either no square root or
 2388 two of them. We adopt the convention to call the smaller one (when interpreted as an integer)

S: are we introducing elliptic curves in section 1 or 2?

as the **positive** square root and the larger one as the **negative**. This makes sense, as the larger one can always be computed as the modulus minus the smaller one, which is the definition of the negative in prime fields.

Example 62 (Quadratic (Non)-Residues and roots in \mathbb{F}_5). Let us consider our example prime field \mathbb{F}_5 again. All square numbers can be found on the main diagonal of the multiplication table in example 14 on page 28. As you can see, in \mathbb{F}_5 only the numbers 0, 1 and 4 have square roots and we get $\sqrt{0} = \{0\}$, $\sqrt{1} = \{1, 4\}$, $\sqrt{2} = \emptyset$, $\sqrt{3} = \emptyset$ and $\sqrt{4} = \{2, 3\}$. The numbers 0, 1 and 4 are therefore quadratic residues, while the numbers 2 and 3 are quadratic non-residues.

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In order to describe whether an element of a prime field is a square number or not, the so-called **Legendre symbol** can sometimes be found in the literature, which is why we will summarize it here:

Let $p \in \mathbb{P}$ be a prime number and $y \in \mathbb{F}_p$ an element from the associated prime field. Then the *Legendre symbol* of y is defined as follows:

$$\left(\frac{y}{p}\right) := \begin{cases} 1 & \text{if } y \text{ has square roots} \\ -1 & \text{if } y \text{ has no square roots} \\ 0 & \text{if } y = 0 \end{cases} \quad (4.31)$$

Example 63. Looking at the quadratic residues and non residues in \mathbb{F}_5 from example 14 again, we can deduce the following Legendre symbols, from example XXX.

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$$\left(\frac{0}{5}\right) = 0, \quad \left(\frac{1}{5}\right) = 1, \quad \left(\frac{2}{5}\right) = -1, \quad \left(\frac{3}{5}\right) = -1, \quad \left(\frac{4}{5}\right) = 1.$$

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The Legendre symbol provides a criterion to decide whether or not an element from a prime field has a quadratic root or not. This, however, is not just of theoretical use: The so-called **Euler criterion** provides a compact way to actually compute the Legendre symbol. To see that, let $p \in \mathbb{P}_{\geq 3}$ be an odd prime number and $y \in \mathbb{F}_p$. Then the Legendre symbol can be computed as follows:

$$\left(\frac{y}{p}\right) = y^{\frac{p-1}{2}}. \quad (4.32)$$

Example 64. Looking at the quadratic residues and non residues in \mathbb{F}_5 from example 14 again, we can compute the following Legendre symbols using the Euler criterion:

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$$\begin{aligned} \left(\frac{0}{5}\right) &= 0^{\frac{5-1}{2}} = 0^2 = 0 \\ \left(\frac{1}{5}\right) &= 1^{\frac{5-1}{2}} = 1^2 = 1 \\ \left(\frac{2}{5}\right) &= 2^{\frac{5-1}{2}} = 2^2 = 4 = -1 \\ \left(\frac{3}{5}\right) &= 3^{\frac{5-1}{2}} = 3^2 = 4 = -1 \\ \left(\frac{4}{5}\right) &= 4^{\frac{5-1}{2}} = 4^2 = 1 \end{aligned}$$

Exercise 42. Consider the prime field \mathbb{F}_{13} . Find the set of all pairs $(x, y) \in \mathbb{F}_{13} \times \mathbb{F}_{13}$ that satisfy the equation

$$x^2 + y^2 = 1 + 7 \cdot x^2 \cdot y^2$$

2409 **Exponentiation** TO APPEAR...

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paragraph
on expo-
nentiation

2410 **Hashing into prime fields**

2411 An important problem in SNARK development is the ability to hash to (various subsets) of
2412 elliptic curves. As we will see in XXX, those curves are often defined over prime fields, and
2413 hashing to a curve might start with hashing to the prime field. It is therefore important to
2414 understand how to hash into prime fields.

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2415 On pages 54–57, we looked at a few methods of hashing into the residue class rings \mathbb{Z}_n for
2416 arbitrary $n > 1$. As prime fields are just special instances of those rings, all methods for hashing
2417 into \mathbb{Z}_n functions can be used for hashing into prime fields, too.

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2418 **MiMC Hash functions** As we will see in XXX,

2419 To define a MiMC hash function, let \mathbb{F}_p be a prime field with prime modulus $p \in \mathbb{P}$, let n be
2420 the smallest natural number, such that $\gcd(n, p-1) = 1$. Let r be the smallest integer greater
2421 than or equal to $\frac{\log(p)}{\log_2(3)}$ and let $C = \{c_i \in \mathbb{F}_p \mid 0 \leq i \leq r\}$ be a set of randomly generated field
2422 elements. The definition of the MiMC hash function then starts with an invertible map

$$E(\cdot) : \mathbb{F}_p \rightarrow \mathbb{F}_p \quad x \mapsto F_{r-1}(\cdots F_1(F_0(x)) \cdots) \quad (4.33)$$

2423 where $F_i(x) = (x + c_i)^n$

2424 4.3.2 Extension Fields

2425 Prime fields, defined in the previous section, are the basic building blocks for cryptography in
2426 general and SNARKs in particular.

2427 However, as we will see in XXX so-called **pairing-based** SNARK systems are crucially
2428 dependent on **group pairings** XXX defined over the group of rational points of elliptic curves.
2429 For those pairings to be non-trivial, the elliptic curve must not only be defined over a prime
2430 field, but over a so-called **extension field** of a given prime field.

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group
pairings

2431 We therefore have to understand field extensions. First note that the field \mathbb{F}' is called an
2432 **extension** of a field \mathbb{F} if \mathbb{F} is a subfield of \mathbb{F}' , that is, \mathbb{F} is a subset of \mathbb{F}' and restricting the
2433 addition and multiplication laws of \mathbb{F}' to the subset \mathbb{F} recovers the appropriate laws of \mathbb{F} .

2434 Now it can be shown that whenever $p \in \mathbb{P}$ is a prime and $m \in \mathbb{N}$ a natural number, then there
2435 is a field \mathbb{F}_{p^m} with characteristic p and p^m elements such that \mathbb{F}_{p^m} is an extension field of the
2436 prime field \mathbb{F}_p .

2437 Similarly to the way prime fields \mathbb{F}_p are generated by starting with the ring of integers
2438 and then dividing by a prime number p and keeping the remainder, prime field extensions \mathbb{F}_{p^m}
2439 are generated by starting with the ring $\mathbb{F}_p[x]$ of polynomials and then dividing them by an
2440 irreducible polynomial of degree m and keeping the remainder.

2441 To be more precise, let $P \in \mathbb{F}_p[x]$ be an irreducible polynomial of degree m with coefficients
2442 from the given prime field \mathbb{F}_p . Then the underlying set \mathbb{F}_{p^m} of the extension field is given by
2443 the set of all polynomials with a degree less than m :

$$\mathbb{F}_{p^m} := \{a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \cdots + a_1x + a_0 \mid a_i \in \mathbb{F}_p\} \quad (4.34)$$

2444 This can be shown to be the set of all remainders when dividing any polynomial $Q \in \mathbb{F}_p[x]$ by
2445 P , consequently, elements of the extension field are polynomials of degree less than m . This is
2446 analogous to how \mathbb{F}_p is the set of all remainders when dividing integers by p .

2447 Addition is inherited from $\mathbb{F}_p[x]$, which means that addition on \mathbb{F}_{p^m} is defined like normal
 2448 addition of polynomials:

$$+ : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \rightarrow \mathbb{F}_{p^m}, \left(\sum_{j=0}^m a_j x^j, \sum_{j=0}^m b_j x^j \right) \mapsto \sum_{j=0}^m (a_j + b_j) x^j \quad (4.35)$$

2449 We can see that the neutral element is (the polynomial) 0, and that the additive inverse is given
 2450 by the polynomial with all negative coefficients.

2451 Multiplication is inherited from $\mathbb{F}_p[x]$, too, but we have to divide the result by our modulus
 2452 polynomial P whenever the degree of the resulting polynomial is equal or greater to m :

$$\cdot : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \rightarrow \mathbb{F}_{p^m}, \left(\sum_{j=0}^m a_j x^j, \sum_{j=0}^m b_j x^j \right) \mapsto \left(\sum_{n=0}^{2m} \sum_{i=0}^n a_i b_{n-i} x^n \right) \bmod P \quad (4.36)$$

2453 We can see that the neutral element is (the polynomial) 1. It is, however, not obvious from this
 2454 definition how the multiplicative inverse looks.

2455 We can easily see from the definition of \mathbb{F}_{p^m} that the field is of characteristic p , since the
 2456 multiplicative neutral element 1 is equivalent to the multiplicative element 1 from the underlying
 2457 prime field, and hence $\sum_{j=0}^p 1 = 0$. Moreover, \mathbb{F}_{p^m} is finite and contains p^m many elements,
 2458 since elements are polynomials of degree $< m$, and every coefficient a_j can have a p number
 2459 of different values. In addition, we see that the prime field \mathbb{F}_p is a subfield of \mathbb{F}_{p^m} that occurs
 2460 when we restrict the elements of \mathbb{F}_{p^m} to polynomials of degree zero.

2461 One key point is that the construction of \mathbb{F}_{p^m} depends on the choice of an irreducible polyno-
 2462 mial, and, in fact, different choices will give different multiplication tables, since the remainders
 2463 from dividing a product by P will be different.

2464 It can, however, be shown that the fields for different choices of P are **isomorphic**, which
 2465 means that there is a one-to-one correspondence between all of them. Consequently, from
 2466 an abstract point of view, they are the same thing. From an implementations point of view,
 2467 however, some choices are preferable to others because they allow for faster computations.

2468 To summarize, we have seen that when a prime field \mathbb{F}_p is given, any field \mathbb{F}_{p^m} constructed in
 2469 the above manner is a field extension of \mathbb{F}_p . To be more general, a field $\mathbb{F}_{p^{m_2}}$ is a field extension
 2470 of a field $\mathbb{F}_{p^{m_1}}$, if and only if m_1 divides m_2 . From this, we can deduce that, for any given fixed
 2471 prime number, there are nested sequences of fields whenever the power m_j divides the power
 2472 m_{j+1} , such that $\mathbb{F}_{p^{m_j}}$ is a subfield of $\mathbb{F}_{p^{m_{j+1}}}$:

$$\mathbb{F}_p \subset \mathbb{F}_{p^{m_1}} \subset \cdots \subset \mathbb{F}_{p^{m_k}} \quad (4.37)$$

2473 To get a more intuitive picture of this, we construct an extension field of the prime field \mathbb{F}_3
 2474 in the following example, and we can see how \mathbb{F}_3 sits inside that extension field.

Example 65 (The Extension field \mathbb{F}_{3^2}). In (XXX) we have constructed the prime field \mathbb{F}_3 . In
 this example, we apply the definition of a field extension (page 66) to construct \mathbb{F}_{3^2} . We start
 by choosing an irreducible polynomial of degree 2 with coefficients in \mathbb{F}_3 . We try $P(t) = t^2 + 1$.
 Possibly the fastest way to show that P is indeed irreducible is to just insert all elements from
 \mathbb{F}_3 to see if the result is ever zero. We compute as follows:

$$\begin{aligned} P(0) &= 0^2 + 1 = 1 \\ P(1) &= 1^2 + 1 = 2 \\ P(2) &= 2^2 + 1 = 1 + 1 = 2 \end{aligned}$$

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This implies that P is irreducible. The set \mathbb{F}_{3^2} contains all polynomials of degrees lower than two, with coefficients in \mathbb{F}_3 , which are precisely as listed below:

$$\mathbb{F}_{3^2} = \{0, 1, 2, t, t+1, t+2, 2t, 2t+1, 2t+2\}$$

As expected, our extension field contains 9 elements. Addition is defined as addition of polynomials; for example $(t+2) + (2t+2) = (1+2)t + (2+2) = 1$. Doing this computation for all elements gives the following addition table

+	0	1	2	t	t+1	t+2	2t	2t+1	2t+2
0	0	1	2	t	t+1	t+2	2t	2t+1	2t+2
1	1	2	0	t+1	t+2	t	2t+1	2t+2	2t
2	2	0	1	t+2	t	t+1	2t+2	2t	2t+1
t	t	t+1	t+2	2t	2t+1	2t+2	0	1	2
t+1	t+1	t+2	t	2t+1	2t+2	2t	1	2	0
t+2	t+2	t	t+1	2t+2	2t	2t+1	2	0	1
2t	2t	2t+1	2t+2	0	1	2	t	t+1	t+2
2t+1	2t+1	2t+2	2t	1	2	0	t+1	t+2	t
2t+2	2t+2	2t	2t+1	2	0	1	t+2	t	t+1

As we can see, the group $(\mathbb{F}_3, +)$ is a subgroup of the group $(\mathbb{F}_{3^2}, +)$, obtained by only considering the first three rows and columns of this table.

As it was the case in previous examples, we can use the table to deduce the negative of any element from \mathbb{F}_{3^2} . For example, in \mathbb{F}_{3^2} we have $-(2t+1) = t+2$, since $(2t+1) + (t+2) = 0$

Multiplication needs a bit more computation, as we first have to multiply the polynomials, and whenever the result has a degree ≥ 2 , we have to divide it by P and keep the remainder. To see how this works, let us compute the product of $t+2$ and $2t+2$ in \mathbb{F}_{3^2} :

$$\begin{aligned}
 (t+2) \cdot (2t+2) &= (2t^2 + 2t + t + 1) \bmod (t^2 + 1) \\
 &= (2t^2 + 1) \bmod (t^2 + 1) & \# \ 2t^2 + 1 : t^2 + 1 &= 2 + \frac{2}{t^2 + 1} \\
 &= 2
 \end{aligned}$$

This means that the product of $t+2$ and $2t+2$ in \mathbb{F}_{3^2} is 2. Performing this computation for all elements gives the following multiplication table:

·	0	1	2	t	t+1	t+2	2t	2t+1	2t+2
0	0	0	0	0	0	0	0	0	0
1	0	1	2	t	t+1	t+2	2t	2t+1	2t+2
2	0	2	1	2t	2t+2	2t+1	t	t+2	t+1
t	0	t	2t	2	t+2	2t+2	1	t+1	2t+1
t+1	0	t+1	2t+2	t+2	2t	1	2t+1	2	t
t+2	0	t+2	2t+1	2t+2	1	t	t+1	2t	2
2t	0	2t	t	1	2t+1	t+1	2	2t+2	t+2
2t+1	0	2t+1	t+2	t+1	2	2t	2t+2	t	1
2t+2	0	2t+2	t+1	2t+1	t	2	t+2	1	2t

As it was the case in previous examples, we can use the table to deduce the multiplicative inverse of any non-zero element from \mathbb{F}_{3^2} . For example, in \mathbb{F}_{3^2} we have $(2t+1)^{-1} = 2t+2$, since $(2t+1) \cdot (2t+2) = 1$.

2489 From the multiplication table, we can also see that the only quadratic residues in \mathbb{F}_{3^2} are
 2490 from the set $\{0, 1, 2, t, 2t\}$, with $\sqrt{0} = \{0\}$, $\sqrt{1} = \{1, 2\}$, $\sqrt{2} = \{t, 2t\}$, $\sqrt{t} = \{t+2, 2t+1\}$ and
 2491 $\sqrt{2t} = \{t+1, 2t+2\}$.

Since \mathbb{F}_{3^2} is a field, we can solve equations as we would for other fields, (such as rational numbers). To see that, let us find all $x \in \mathbb{F}_{3^2}$ that solve the quadratic equation $(t+1)(x^2 + (2t+2)) = 2$. We compute as follows:

$$\begin{aligned}
 (t+1)(x^2 + (2t+2)) &= 2 && \# 2 \text{ distributive law} \\
 (t+1)x^2 + (t+1)(2t+2) &= 2 \\
 (t+1)x^2 + (t) &= 2 && \# 2 \text{ add the additive inverse of } t \\
 (t+1)x^2 + (t) + (2t) &= (2) + (2t) \\
 (t+1)x^2 &= 2t+2 && \# \text{ multiply with the multiplicative invers of } t+1 \\
 (t+2)(t+1)x^2 &= (t+2)(2t+2) && \# \text{ multiply with the multiplicative invers of } t+1 \\
 x^2 &= 2 && \# 2 \text{ is quadratic residue. Take the roots.} \\
 x &\in \{t, 2t\}
 \end{aligned}$$

2492 Computations in extension fields are arguably on the edge of what can reasonably be done with
 2493 pen and paper. Fortunately, Sage provides us with a simple way to do the computations.

```

2494 sage: Z3 = GF(3) # prime field 207
2495 sage: Z3t.<t> = Z3[] # polynomials over Z3 208
2496 sage: P = Z3t(t^2+1) 209
2497 sage: P.is_irreducible() 210
2498 True 211
2499 sage: F3_2.<t> = GF(3^2, name='t', modulus=P) 212
2500 sage: F3_2 213
2501 Finite Field in t of size 3^2 214
2502 sage: F3_2(t+2)*F3_2(2*t+2) == F3_2(2) 215
2503 True 216
2504 sage: F3_2(2*t+2)^(-1) # multiplicative inverse 217
2505 2*t + 1 218
2506 sage: # verify our solution to (t+1)(x^2 + (2t+2)) = 2 219
2507 sage: F3_2(t+1)*(F3_2(t)**2 + F3_2(2*t+2)) == F3_2(2) 220
2508 True 221
2509 sage: F3_2(t+1)*(F3_2(2*t)**2 + F3_2(2*t+2)) == F3_2(2) 222
2510 True 223

```

2511 *Exercise 43.* Consider the extension field \mathbb{F}_{3^2} from the previous example and find all pairs of
 2512 elements $(x, y) \in \mathbb{F}_{3^2}$, for which the following equation holds:

$$y^2 = x^3 + 4 \quad (4.38)$$

2513 *Exercise 44.* Show that the polynomial $P = x^3 + x + 1$ from $\mathbb{F}_5[x]$ is irreducible. Then consider
 2514 the extension field \mathbb{F}_{5^3} defined relative to P . Compute the multiplicative inverse of $(2t^2 + 4) \in$
 2515 \mathbb{F}_{5^3} using the extended Euclidean algorithm. Then find all $x \in \mathbb{F}_{5^3}$ that solve the following
 2516 equation:

$$(2t^2 + 4)(x - (t^2 + 4t + 2)) = (2t + 3) \quad (4.39)$$

2517 **Hashing into extension fields** On page 66, we have seen how to hash into prime fields. As
 2518 elements of extension fields can be seen as polynomials over prime fields, hashing into extension
 2519 fields is therefore possible if every coefficient of the polynomial is hashed independently.

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2520 4.4 Projective Planes

2521 Projective planes are particular geometric constructs defined over a given field. In a sense,
 2522 projective planes extend the concept of the ordinary Euclidean plane by including “points at
 2523 infinity.”

2524 Such an inclusion of infinity points makes projective planes particularly useful in the de-
 2525 scription of elliptic curves, as the description of such a curve in an ordinary plane needs an
 2526 additional symbol “the point at infinity” to give the set of points on the curve the structure of
 2527 a group. Translating the curve into projective geometry includes this “point at infinity” more
 2528 naturally into the set of all points on a projective plane.

2529 To understand the idea of constructing of projective planes, note that in an ordinary Eu-
 2530 clidean plane, two lines either intersect in a single point or are parallel. In the latter case, both
 2531 lines are either the same, that is, they intersect in all points, or do not intersect at all. A projec-
 2532 tive plane can then be thought of as an ordinary plane, but equipped with additional “point at
 2533 infinity” such that two different lines always intersect in a single point. Parallel lines intersect
 2534 “at infinity”.

2535 To be more precise, let \mathbb{F} be a field, $\mathbb{F}^3 := \mathbb{F} \times \mathbb{F} \times \mathbb{F}$ the set of all three tuples over \mathbb{F} and
 2536 $x \in \mathbb{F}^3$ with $x = (X, Y, Z)$. Then there is exactly one *line* in \mathbb{F}^3 that intersects both $(0, 0, 0)$ and
 2537 x . This line is given as follows:

$$[X : Y : Z] := \{(k \cdot X, k \cdot Y, k \cdot Z) \mid k \in \mathbb{F}\} \quad (4.40)$$

2538 A **point** in the **projective plane** over \mathbb{F} is defined as such a **line**, and the projective plane is the
 2539 set of all such points:

$$\mathbb{FP}^2 := \{[X : Y : Z] \mid (X, Y, Z) \in \mathbb{F}^3 \text{ with } (X, Y, Z) \neq (0, 0, 0)\} \quad (4.41)$$

2540 It can be shown that a projective plane over a finite field \mathbb{F}_{p^m} contains $p^{2m} + p^m + 1$ number of
 2541 elements.

2542 To understand why $[X : Y : Z]$ is called a line, consider the situation where the underlying
 2543 field \mathbb{F} is the set of real numbers \mathbb{R} . In this case, \mathbb{R}^3 can be seen as the three-dimensional space,
 2544 and $[X : Y : Z]$ is an ordinary line in this 3-dimensional space that intersects zero and the point
 2545 with coordinates X, Y and Z .

2546 The key observation here is that points in the projective plane are lines in the 3-dimensional
 2547 space \mathbb{F}^3 . Additionally, for finite fields, the terms **space** and **line** share very little visual simi-
 2548 larity with their counterparts over the set of real numbers.

2549 It follows from this that points $[X : Y : Z] \in \mathbb{FP}^2$ are not simply described by fixed co-
 2550 ordinates (X, Y, Z) , but by **sets of coordinates**, where two different coordinates (X_1, Y_1, Z_1)
 2551 and (X_2, Y_2, Z_2) describe the same point if and only if there is some field element k such that
 2552 $(X_1, Y_1, Z_1) = (k \cdot X_2, k \cdot Y_2, k \cdot Z_2)$. Points $[X : Y : Z]$ are called **projective coordinates**.

2553 *Notation and Symbols 6* (Projective coordinates). Projective coordinates of the form $[X : Y : 1]$
 2554 are descriptions of so-called **affine points**. Projective coordinates of the form $[X : Y : 0]$ are
 2555 descriptions of so-called **points at infinity**. In particular, the projective coordinate $[1 : 0 : 0]$
 2556 describes the so-called **line at infinity**.

2557 *Example 66.* Consider the field \mathbb{F}_3 from example XXX. As this field only contains three ele-
 2558 ments, it does not take too much effort to construct its associated projective plane $\mathbb{F}_3\mathbb{P}^2$, as we
 2559 know that it only contains 13 elements.

add refer-
ence

To find $\mathbb{F}_3\mathbb{P}^2$, we have to compute the set of all lines in $\mathbb{F}_3 \times \mathbb{F}_3 \times \mathbb{F}_3$ that intersect $(0,0,0)$. Since those lines are parameterized by tuples (x_1, x_2, x_3) , we compute as follows:

$$\begin{aligned}
 [0 : 0 : 1] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(0, 0, 1), (0, 0, 2)\} \\
 [0 : 0 : 2] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(0, 0, 2), (0, 0, 1)\} = [0 : 0 : 1] \\
 [0 : 1 : 0] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(0, 1, 0), (0, 2, 0)\} \\
 [0 : 1 : 1] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(0, 1, 1), (0, 2, 2)\} \\
 [0 : 1 : 2] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(0, 1, 2), (0, 2, 1)\} \\
 [0 : 2 : 0] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(0, 2, 0), (0, 1, 0)\} = [0 : 1 : 0] \\
 [0 : 2 : 1] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(0, 2, 1), (0, 1, 2)\} = [0 : 1 : 2] \\
 [0 : 2 : 2] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(0, 2, 2), (0, 1, 1)\} = [0 : 1 : 1] \\
 [1 : 0 : 0] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(1, 0, 0), (2, 0, 0)\} \\
 [1 : 0 : 1] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(1, 0, 1), (2, 0, 2)\} \\
 [1 : 0 : 2] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(1, 0, 2), (2, 0, 1)\} \\
 [1 : 1 : 0] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(1, 1, 0), (2, 2, 0)\} \\
 [1 : 1 : 1] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(1, 1, 1), (2, 2, 2)\} \\
 [1 : 1 : 2] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(1, 1, 2), (2, 2, 1)\} \\
 [1 : 2 : 0] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(1, 2, 0), (2, 1, 0)\} \\
 [1 : 2 : 1] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(1, 2, 1), (2, 1, 2)\} \\
 [1 : 2 : 2] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(1, 2, 2), (2, 1, 1)\} \\
 [2 : 0 : 0] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(2, 0, 0), (1, 0, 0)\} = [1 : 0 : 0] \\
 [2 : 0 : 1] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(2, 0, 1), (1, 0, 2)\} = [1 : 0 : 2] \\
 [2 : 0 : 2] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(2, 0, 2), (1, 0, 1)\} = [1 : 0 : 1] \\
 [2 : 1 : 0] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(2, 1, 0), (1, 2, 0)\} = [1 : 2 : 0] \\
 [2 : 1 : 1] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(2, 1, 1), (1, 2, 2)\} = [1 : 2 : 2] \\
 [2 : 1 : 2] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(2, 1, 2), (1, 2, 1)\} = [1 : 2 : 1] \\
 [2 : 2 : 0] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(2, 2, 0), (1, 1, 0)\} = [1 : 1 : 0] \\
 [2 : 2 : 1] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(2, 2, 1), (1, 1, 2)\} = [1 : 1 : 2] \\
 [2 : 2 : 2] &= \{(k \cdot x_1, k \cdot x_2, k \cdot x_3) \mid k \in \mathbb{F}_3\} = \{(2, 2, 2), (1, 1, 1)\} = [1 : 1 : 1]
 \end{aligned}$$

These lines define the 13 points in the projective plane $\mathbb{F}_3\mathbb{P}$:

$$\begin{aligned}
 \mathbb{F}_3\mathbb{P} = \{ & [0 : 0 : 1], [0 : 1 : 0], [0 : 1 : 1], [0 : 1 : 2], [1 : 0 : 0], [1 : 0 : 1], \\
 & [1 : 0 : 2], [1 : 1 : 0], [1 : 1 : 1], [1 : 1 : 2], [1 : 2 : 0], [1 : 2 : 1], [1 : 2 : 2] \}
 \end{aligned}$$

2560 This projective plane contains 9 affine points, three points at infinity and one line at infinity.

2561 To understand the ambiguity in projective coordinates a bit better, let us consider the point
 2562 $[1 : 2 : 2]$. As this point in the projective plane is a line in \mathbb{F}_3^3 , it has the projective coordinates
 2563 $(1, 2, 2)$ as well as $(2, 1, 1)$, since the former coordinate gives the latter when multiplied in \mathbb{F}_3
 2564 by the factor 2. In addition, note that, for the same reasons, the points $[1 : 2 : 2]$ and $[2 : 1 : 1]$
 2565 are the same, since their underlying sets are equal.

2566 *Exercise 45.* Construct the so-called **Fano plane**, that is, the projective plane over the finite
2567 field \mathbb{F}_2 .

Chapter 5

Elliptic Curves

Generally speaking, elliptic curves are “curves” defined in geometric planes like the Euclidean or the projective plane over some given field. One of the key features of elliptic curves over finite fields from the point of view of cryptography is that their set of points has a group law such that the resulting group is finite and cyclic, and it is believed that the discrete logarithm problem on these groups is hard.

A special class of elliptic curves are so-called **pairing-friendly curves**, which have a notation of a group pairing as defined in XXX. This pairing has cryptographically advantageous properties. Those curve are useful in the development of SNARKs, since they allow to compute so-called RICS-satisfiability “in the exponent”

In this chapter, we introduce elliptic curves as they are used in pairing-based approaches to the construction of SNARKs. The elliptic curves we consider are all defined over prime fields or prime field extensions and the reader should be familiar with the content of the previous section on those fields.

In its most generality elliptic curves are defined as a smooth projective curve of genus 1 defined over some field \mathbb{F} with a distinguished \mathbb{F} -rational point, but this definition is not very useful for the introductory character of this book. We will therefore look at 3 more practical definitions in the following sections, by introducing Weierstraß, Montgomery and Edwards curves. All of them are widely used in cryptography, and understanding them is crucial to being able to follow the rest of this book.

5.1 Elliptic Curve Arithmetics

5.1.1 Short Weierstraß Curves

In this section, we introduce **short Weierstraß** curves, which are the most general types of curves over finite fields of characteristic greater than 3.

We start with their representation in **affine space**. This representation has the advantage that affine points correspond to pairs of numbers, which makes it more accessible for beginners. However, it has the disadvantage that a special “point at infinity”, that is not a point on the curve, is necessary to describe the group structure. We introduce the elliptic curve group law and describe elliptic curve scalar multiplication, which is an instantiation of the exponential map from general cyclic groups.

Then we look at the projective representation of short Weierstraß curves. This has the advantage that no special symbol is necessary to represent the point at infinity but comes with

TODO:
Elliptic
Curve
asymmet-
ric cryp-
tography
examples.
Private
key, gen-
erator,
public
key.

add refer-
ence

maybe re-
move this
sentence?

affine
space

the drawback that projective points are classes of numbers, which might be a bit unusual for a beginner.

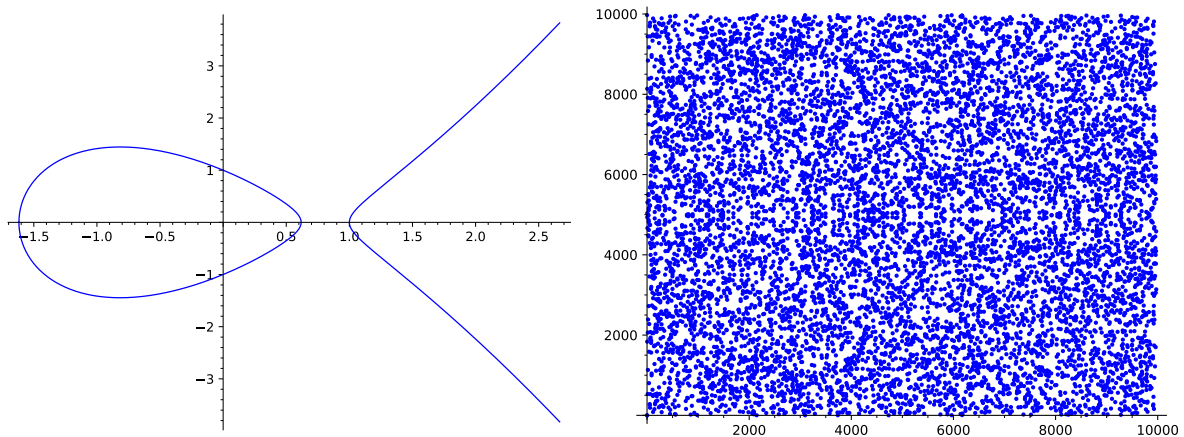
We finish this section with an explicit equivalence that transforms affine representations into projective ones and vice versa.

Affine short Weierstraß form Probably the least abstract and most straight-forward way to introduce elliptic curves for non-mathematicians and beginners is the so-called affine representation of a short Weierstraß curve. To see what this is, let \mathbb{F} be a finite field of order q and $a, b \in \mathbb{F}$ two field elements such that $4a^3 + 27b^2 \bmod q \neq 0$. Then a **short Weierstraß elliptic curve** $E(\mathbb{F})$ over \mathbb{F} in its affine representation is the set of all pairs of field elements $(x, y) \in \mathbb{F} \times \mathbb{F}$ that satisfy the short Weierstraß cubic equation $y^2 = x^3 + a \cdot x + b$, together with a distinguished symbol \mathcal{O} , called the **point at infinity**:

$$E(\mathbb{F}) = \{(x, y) \in \mathbb{F} \times \mathbb{F} \mid y^2 = x^3 + a \cdot x + b\} \cup \{\mathcal{O}\} \quad (5.1)$$

Notation and Symbols 7. In the literature, the set $E(\mathbb{F})$, which includes the symbol \mathcal{O} , is often called the set of **rational points** of the elliptic curve, in which case the curve itself is usually written as E/\mathbb{F} . However, in what follows, we will frequently identify an elliptic curve with its set of rational points and therefore use the notation $E(\mathbb{F})$ instead. This is possible in our case, since we only the group structure of the curve in consideration is relevant for us.

The term “curve” is used here because, in the ordinary 2 dimensional plane \mathbb{R}^2 , the set of all points (x, y) that satisfy $y^2 = x^3 + a \cdot x + b$ looks like a curve. We should note however that visualizing elliptic curves over finite fields as “curves” has its limitations, and we will therefore not stress the geometric picture too much, but focus on the computational properties instead. To understand the visual difference, consider the following two elliptic curves:



Both elliptic curves are defined by the same short Weierstraß equation $y^2 = x^3 - 2x + 1$, but the first curve is defined in the real affine plane \mathbb{R}^2 , that is, the pair (x, y) contains real numbers, while the second one is defined in the affine plane \mathbb{F}_{9973}^2 , which means that both x and y are from the prime field \mathbb{F}_{9973} . Every blue dot represents a pair (x, y) , that is a solution to $y^2 = x^3 - 2x + 1$. As we can see, the second curve hardly looks like a geometric structure one would naturally call a curve. This shows that our geometric intuitions from \mathbb{R}^2 are obfuscated in curves over finite fields.

The identity $6 \cdot (4a^3 + 27b^2) \bmod q \neq 0$ ensures that the curve is non-singular, which basically means that the curve has no **cusps** or **self-intersections**.

cusps

self-intersections

Throughout this book, the reader is advised to do as many computations in a pen-and-paper fashion as possible, as this helps getting a deeper understanding of the details. However, when dealing with elliptic curves, computations can quickly become cumbersome and tedious, and one might get lost in the details. Fortunately, Sage is very helpful in dealing with elliptic curves. This book introduces the reader to the great elliptic curve capabilities of Sage. The following snippet shows a way to define elliptic curves and work with them in Sage:

```

sage: F5 = GF(5) # define the base field
sage: a = F5(2) # parameter a
sage: b = F5(4) # parameter b
sage: # check non-singularity
sage: F5(6)*(F5(4)*a^3+F5(27)*b^2) != F5(0)
True
sage: # short Weierstrass curve
sage: E = EllipticCurve(F5,[a,b]) # y^2 == x^3 + ax +b
sage: P = E(0,2) # 2^2 == 0^3 + 2*0 + 4
sage: P.xy() # affine coordinates
(0, 2)
sage: INF = E(0) # point at infinity
sage: try: # point at infinity has no affine coordinates
.....:     INF.xy()
.....: except ZeroDivisionError:
.....:     pass
sage: P = E.plot() # create a plotted version

```

The following three examples give a more practical understanding of what an elliptic curve is and how we can compute it. The reader is advised to read them carefully, and ideally, to also carry out the computation themselves. We will repeatedly build on these examples in this chapter, and use the second example throughout this book.

Example 67. To provide the reader with an example of a small elliptic curve where all computation can be done with pen and paper, consider the prime field \mathbb{F}_5 from example 61 (page 63), which should be quite familiar to readers who had worked through the examples and exercises in the previous chapter.

To define an elliptic curve over \mathbb{F}_5 , we have to choose two numbers a and b from that field. Assuming we choose $a = 1$ and $b = 1$ then $4a^3 + 27b^2 \equiv 1 \pmod{5}$ from which follows that the corresponding elliptic curve $E_1(\mathbb{F}_5)$ is given by the set of all pairs (x, y) from \mathbb{F}_5 that satisfy the equation $y^2 = x^3 + x + 1$, together with the special symbol \mathcal{O} , which represents the “point at infinity”.

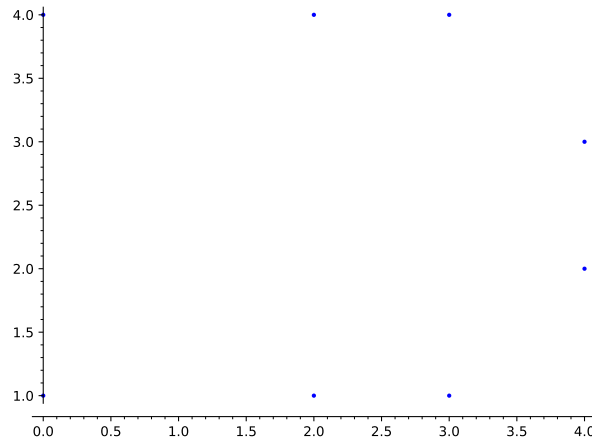
To get a better understanding of that curve, observe that if we choose arbitrarily the pair $(x, y) = (1, 1)$, we see that $1^2 \neq 1^3 + 1 + 1$ and hence $(1, 1)$ is not an element of the curve $E_1(\mathbb{F}_5)$. On the other hand choosing for example $(x, y) = (2, 1)$ gives $1^2 = 2^3 + 2 + 1$ and hence the pair $(2, 1)$ is an element of $E_1(\mathbb{F}_5)$ (Remember that all computations are done in modulo 5 arithmetics).

Now since the set $\mathbb{F}_5 \times \mathbb{F}_5$ of all pairs (x, y) from \mathbb{F}_5 contains only $5 \cdot 5 = 25$ pairs, we can compute the curve, by just inserting every possible pair (x, y) into the short Weierstrass equation $y^2 = x^3 + x + 1$. If the equation holds, the pair is a curve point, if not that means that the point is not on the curve. Combining the result of this computation with the point at infinity gives the curve as follows:

$$E_1(\mathbb{F}_5) = \{\mathcal{O}, (0, 1), (2, 1), (3, 1), (4, 2), (4, 3), (0, 4), (2, 4), (3, 4)\}$$

check
reference

2674 This means that our elliptic curve is a set of 9 elements, 8 of which are pairs of numbers and
 2675 one special symbol \mathcal{O} . Visualizing $E1$ gives the following plot:



2676

2677 In the development of SNARKs, it is sometimes necessary to do elliptic curve cryptography
 2678 “in a circuit”, which basically means that the elliptic curves need to be implemented in a certain
 2679 SNARK-friendly way. We will look at what this means in chapter 7. [To be able to do this](#)
 2680 efficiently, it is desirable to have curves with special properties. The following example is a
 2681 pen-and-paper version of such a curve, called **Baby-jubjub**, which resembles cryptographically
 2682 secure curves extensively used in real-world SNARKs. The interested reader is advised to study
 2683 this example carefully, as we will use it and build on it in various places throughout the book. S:
 2684 I feel like a lot of people won’t get the Lewis Carroll reference unless we make it more explicit.
 2685 M: The term Baby-JubJub is actually the name of a curve used in zCash and Ethereum a lot.
 2686 **IDK why they choosed that name.**

check
reference

jubjub

2687 *Example 68 (Tiny-Jubjub).* Consider the prime field \mathbb{F}_{13} from exercise 4.3.1 (page 65. [If we](#)
 2688 choose $a = 8$ and $b = 8$, then $4a^3 + 27b^2 \equiv 6 \pmod{13}$ and the corresponding elliptic curve
 2689 is given by all pairs (x, y) from \mathbb{F}_{13} such that $y^2 = x^3 + 8x + 8$ holds. We call this curve the
 2690 **Tiny-jubjub** curve, or TJJ_13 for short.

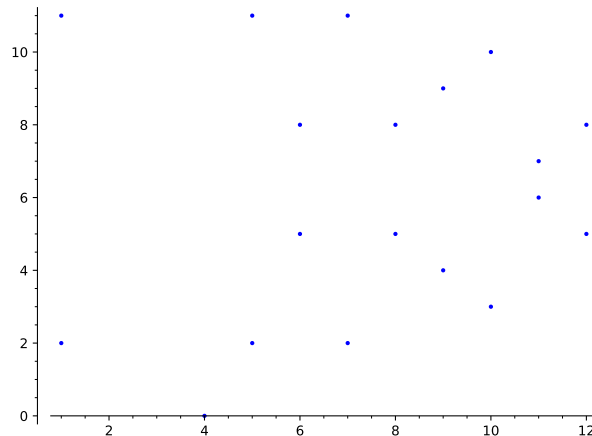
check
reference

2691 Now, since the set $\mathbb{F}_{13} \times \mathbb{F}_{13}$ of all pairs (x, y) from \mathbb{F}_{13} contains only $13 \cdot 13 = 169$ pairs,
 2692 we can compute the curve by just inserting every possible pair (x, y) into the short Weierstraß
 2693 equation $y^2 = x^3 + 8x + 8$. We get the following result:

$$TJJ_13 = \{\mathcal{O}, (1, 2), (1, 11), (4, 0), (5, 2), (5, 11), (6, 5), (6, 8), (7, 2), (7, 11), (8, 5), (8, 8), \\ (9, 4), (9, 9), (10, 3), (10, 10), (11, 6), (11, 7), (12, 5), (12, 8)\} \quad (5.2)$$

2694 As we can see, the curve consists of 20 points; 19 points from the [affine plane](#) and the point at
 2695 infinity. To get a visual impression of the TJJ_13 curve, we might plot all of its points (except
 2696 the point at infinity) in the $\mathbb{F}_{13} \times \mathbb{F}_{13}$ affine plane. We get the following plot:

affine
plane



2697

2698 As we will see in what follows, this curve is rather special, as it is possible to represent it in two
 2699 alternative forms called the **Montgomery** and the **twisted Edwards form** (See sections 5.1.2
 2700 and 5.1.3, respectively).

check
reference

2701 Now that we have seen two pen-and-paper friendly elliptic curves, let us look at a curve, that
 2702 is used in actual cryptography. Cryptographically secure elliptic curves are not **qualitatively**
 2703 different from the curves we looked at so far, but the prime number modulus of their prime field
 2704 is much larger. Typical examples use prime numbers that have binary representations in the
 2705 magnitude of more than double the size of the desired security level. If, for example, a security
 2706 of 128 bits is desired, a prime modulus of binary size ≥ 256 is chosen. The following example
 2707 provides such a curve.

check
reference

Example 69 (Bitcoin's Secp256k1 curve). To give an example of a real-world, cryptographically secure curve, let us look at curve Secp256k1, which is famous for being used in the public key cryptography of Bitcoin. The prime field \mathbb{F}_p of Secp256k1 is defined by the following prime number:

$$p = 115792089237316195423570985008687907853269984665640564039457584007908834671663$$

2708 The binary representation of this number needs 256 bits, which implies that the prime field
 2709 \mathbb{F}_p contains approximately 2^{256} many elements, which is considered quite large. To get a better
 2710 impression of how large the base field is, consider that the number 2^{256} is approximately in the
 2711 same order of magnitude as the estimated number of atoms in the observable universe.

The curve Secp256k1 is defined by the parameters $a, b \in \mathbb{F}_p$ with $a = 0$ and $b = 7$. Since $4 \cdot 0^3 + 27 \cdot 7^2 \bmod p = 1323$, those parameters indeed define an elliptic curve given as follows:

$$\text{Secp256k1} = \{(x, y) \in \mathbb{F}_p \times \mathbb{F}_p \mid y^2 = x^3 + 7\}$$

Clearly, the Secp256k1 curve is too large to do computations by hand, since it can be shown that the number of its elements is a prime number r that also has a binary representation of 256 bits:

$$r = 115792089237316195423570985008687907852837564279074904382605163141518161494337$$

2712 Cryptographically secure elliptic curves are therefore not useful in pen-and-paper computations.
 2713 Fortunately, Sage handles large curves efficiently:

2714 **sage:** `p = 1157920892373161954235709850086879078532699846656405` 241
 2715 `64039457584007908834671663`

```

2716 sage: # Hexadecimal representation
2717 sage: p.str(16)
2718 ffffffffffffffffffffffffffffffffffffffffffffffffffefffffc
2719 2f
2720 sage: p.is_prime()
2721 True
2722 sage: p.nbits()
2723 256
2724 sage: Fp = GF(p)
2725 sage: Secp256k1 = EllipticCurve(Fp, [0, 7])
2726 sage: r = Secp256k1.order() # number of elements
2727 sage: r.str(16)
2728 fffffffffffffffffffffffffffffffebaaedce6af48a03bbfd25e8cd03641
2729 41
2730 sage: r.is_prime()
2731 True
2732 sage: r.nbits()
2733 256

```

2734 *Exercise 46.* Look up the definition of curve BLS12-381, implement it in Sage and compute its
 2735 order.

2736 **Affine compressed representation** As we have seen in example 69, cryptographically secure
 2737 elliptic curves are defined over large prime fields, where elements of those fields typically need
 2738 more than 255 bits of storage on a computer. Since elliptic curve points consist of pairs of those
 2739 field elements, they need double that amount of storage.

2740 However, we can reduce the amount of space needed to represent a curve point by using
 2741 a technique called **point compression**. Note that, up to a sign, the y coordinate of a curve
 2742 point can be computed from the x coordinate by simply inserting x into the Weierstraß equation
 2743 and then computing the roots of the result. This gives two results, and it means that we can
 2744 represent a curve point in **compressed form** by simply storing the x coordinate together with
 2745 a single sign bit only, the latter of which deterministically decides which of the two roots to
 2746 choose. One convention could be to always choose the root closer to 0 when the sign bit is 0,
 2747 and the root closer to the order of \mathbb{F} when the sign bit is 1. In case the y coordinate is zero, both
 2748 sign bits give the same result.

Example 70 (Tiny-jubjub). To understand the concept of compressed curve points a bit better,
 consider the *TJJ_13* curve from example 68 again. Since this curve is defined over the prime
 field \mathbb{F}_{13} , and numbers between 0 and 13 need approximately 4 bits to be represented, each
TJJ_13 point on this curve needs 8 bits of storage in uncompressed form. The following set
 represents the uncompressed form of the points on this curve:

$$TJJ_13 = \{\mathcal{O}, (1, 2), (1, 11), (4, 0), (5, 2), (5, 11), (6, 5), (6, 8), (7, 2), (7, 11), \\ (8, 5), (8, 8), (9, 4), (9, 9), (10, 3), (10, 10), (11, 6), (11, 7), (12, 5), (12, 8)\}$$

Using the technique of point compression, we can reduce the bits needed to represent the points
 on this curve to 5 per point. To achieve this, we can replace the y coordinate in each (x, y) pair
 by a sign bit indicating whether or not y is closer to 0 or to 13. As a result y values in the range
 $[0, \dots, 6]$ will have the sign bit 0, while y -values in the range $[7, \dots, 12]$ will have the sign bit 1.

check
reference

sign

more ex-
planation
of what
the sign is

check
reference

Applying this to the points in *TJJ_13* gives the compressed representation as follows:

$$TJJ_{13} = \{\mathcal{O}, (1,0), (1,1), (4,0), (5,0), (5,1), (6,0), (6,1), (7,0), (7,1), (8,0), (8,1), (9,0), (9,1), (10,0), (10,1), (11,0), (11,1), (12,0), (12,1)\}$$

Note that the numbers $7, \dots, 12$ are the negatives (additive inverses) of the numbers $1, \dots, 6$ in modular 13 arithmetics and that $-0 = 0$. Calling the compression bit a “sign bit” therefore makes sense.

To recover the uncompressed counterpart of, say, the compressed point $(5, 1)$, we insert the x coordinate 5 into the Weierstraß equation and get $y^2 = 5^3 + 8 \cdot 5 + 8 = 4$. As expected, 4 is a quadratic residue in \mathbb{F}_{13} with roots $\sqrt{4} = \{2, 11\}$. Since the sign bit of the point is 1, we have to choose the root closer to the modulus 13, which is 11. The uncompressed point is therefore $(5, 11)$.

Looking at the previous examples, the compression rate does not look very impressive. However, looking at the real-life example of the Secp256k1 curve shows that compression is has significant practical advantages.

Example 71. Consider the Secp256k1 curve from example 69 again. The following code invokes Sage to generate a random affine curve point, then applies our compression method to it:

```
sage: P = Secp256k1.random_point().xy()
sage: P
(1947671157535255271568804626855091201525278356885973944134897
 5889194772189583, 67910864198023178457214489206019381199414
 819273749289220137871356982528892902)
sage: # uncompressed affine point size
sage: ZZ(P[0]).nbits()+ZZ(P[1]).nbits()
510
sage: # compute the compression
sage: if P[1] > Fp(-1)/Fp(2):
.....:     PARITY = 1
.....: else:
.....:     PARITY = 0
sage: PCOMPRESSED = [P[0], PARITY]
sage: PCOMPRESSED
[1947671157535255271568804626855091201525278356885973944134897
 5889194772189583, 1]
sage: # compressed affine point size
sage: ZZ(PCOMPRESSED[0]).nbits()+ZZ(PCOMPRESSED[1]).nbits()
255
```

Affine group law One of the key properties of an elliptic curve is that it is possible to define a group law on the set of its rational points such that the point at infinity serves as the neutral element and inverses are reflections on the x -axis.

The origin of this law can be understood in a geometric picture and is known as the **chord-and-tangent rule**. In the affine representation of a short Weierstraß curve, the rule can be described in the following way:

S: I don't follow this at all

check reference

add explanation of how this shows what we claim

Definition 5.1.1.1. Chord-and-tangent rule

- (Point at infinity) We define the point at infinity \mathcal{O} as the neutral element of addition, that is, we define $P + \mathcal{O} = P$ for all points $P \in E(\mathbb{F})$.

should
this def.
be moved
even ear-
lier?

- (Point addition) Let $P, Q \in E(\mathbb{F}) \setminus \{\mathcal{O}\}$ with $P \neq Q$ be two distinct points on an elliptic curve, neither of them the point at infinity. The sum of P and Q is defined as follows: Consider the line l which intersects the curve in P and Q . If l intersects the elliptic curve at a third point R' , define the sum $R = P \oplus Q$ of P and Q as the reflection of R' at the x -axis. If the line l does not intersect the curve at a third point, define the sum to be the point at infinity \mathcal{O} . It can be shown that no such **chord line** will intersect the curve in more than three points, so addition is not ambiguous.

chord line

- (Point doubling) Let $P \in E(\mathbb{F}) \setminus \{\mathcal{O}\}$ be a point on an elliptic curve, that is not the point at infinity. The sum of P with itself (the doubling of P) is defined as follows:

Consider the line which is **tangential** to the elliptic curve at P . If this line intersects the elliptic curve at a second point R' , the sum $2P = P + P$ is the reflection of R' at the x -axis.

tangential

If it does not intersect the curve at a third, point define the sum to be the point at infinity \mathcal{O} . It can be shown that no such **tangent line** will intersect the curve in more than two points, so addition is not ambiguous.

tangent
line

It can be shown that the points of an elliptic curve form a commutative group with respect to the tangent-and-chord rule such that \mathcal{O} acts the neutral element, and the inverse of any element $P \in E(\mathbb{F})$ is the reflection of P on the x -axis.

To translate the geometric description into algebraic equations, first observe that, for any two given curve points $(x_1, y_1), (x_2, y_2) \in E(\mathbb{F})$, it can be shown that the identity $x_1 = x_2$ implies $y_2 = \pm y_1$, which shows that the following rules are a complete description of the affine addition law.

Definition 5.1.1.2. Chord-and-tangent rule: algebraic equations

- (Neutral element) The point at infinity \mathcal{O} is the neutral element.
- (Additive inverse) The additive inverse of \mathcal{O} is \mathcal{O} . For any other curve point $(x, y) \in E(\mathbb{F}) \setminus \{\mathcal{O}\}$, the additive inverse is given by $(x, -y)$.
- (Addition rule) For any two curve points $P, Q \in E(\mathbb{F})$, addition is defined by one of the following three cases:

1. (Adding the neutral element) If $Q = \mathcal{O}$, then the sum is defined as $P \oplus Q = P$.
2. (Adding inverse elements) If $P = (x, y)$ and $Q = (x, -y)$, the sum is defined as $P \oplus Q = \mathcal{O}$.
3. (Adding non-self-inverse equal points) If $P = (x, y)$ and $Q = (x, y)$ with $y \neq 0$, the sum $2P = (x', y')$ is defined as follows: **We only referred to P in the definition of point doubling above so Q seems a bit confusing here even though it's defined as equal to P**

remove
 Q ?

$$x' = \left(\frac{3x^2 + a}{2y} \right)^2 - 2x \quad , \quad y' = \left(\frac{3x^2 + a}{2y} \right)^2 (x - x') - y$$

4. (Adding non-inverse different points) If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ such that $x_1 \neq x_2$, the sum $R = P + Q$ with $R = (x_3, y_3)$ is defined as follows:

$$x_3 = \left(\frac{y_2 - y_1}{x_2 - x_1} \right)^2 - x_1 - x_2 \quad , \quad y_3 = \left(\frac{y_2 - y_1}{x_2 - x_1} \right) (x_1 - x_3) - y_1$$

2823 Note that short Weierstraß curve points P with $P = (x, 0)$ are inverses of themselves, which
 2824 implies $2P = \mathcal{O}$ in this case.

2825 *Notation and Symbols 8.* Let \mathbb{F} be a field and $E(\mathbb{F})$ be an elliptic curve over \mathbb{F} . We write \oplus for
 2826 the group law on $E(\mathbb{F})$ and $(E(\mathbb{F}), \oplus)$ for the group of rational points.

2827 As we can see, it is very efficient to compute inverses on elliptic curves. However, com-
 2828 puting the addition of elliptic curve points in the affine representation needs to consider many
 2829 cases and involves extensive finite field divisions. As we will see in the next paragraph, this can
 2830 be simplified in projective coordinates. where?

2831 To get some practical impression of how the group law on an elliptic curve is computed,
 2832 let's look at some actual cases:

Example 72. Consider the elliptic curve $E_1(\mathbb{F}_5)$ from example 67 again. As we have seen, the
 set of rational points contains 9 elements: check
reference

$$E_1(\mathbb{F}_5) = \{\mathcal{O}, (0, 1), (2, 1), (3, 1), (4, 2), (4, 3), (0, 4), (2, 4), (3, 4)\}$$

2833 We know that this set defines a group, so we can add any two elements from $E_1(\mathbb{F}_5)$ to get a
 2834 third element.

To give an example, consider the elements $(0, 1)$ and $(4, 2)$. Neither of these elements is
 the neutral element \mathcal{O} , and since, the x coordinate of $(0, 1)$ is different from the x coordinate of
 $(4, 2)$, we know that we have to use the chord rule, that is, rule number 4 from definition 5.1.1.2,
 to compute the sum $(0, 1) \oplus (4, 2)$: check
reference

$$\begin{aligned} x_3 &= \left(\frac{y_2 - y_1}{x_2 - x_1} \right)^2 - x_1 - x_2 && \# \text{ insert points} \\ &= \left(\frac{2 - 1}{4 - 0} \right)^2 - 0 - 4 && \# \text{ simplify in } \mathbb{F}_5 \\ &= \left(\frac{1}{4} \right)^2 + 1 = 4^2 + 1 = 1 + 1 = 2 \end{aligned}$$

$$\begin{aligned} y_3 &= \left(\frac{y_2 - y_1}{x_2 - x_1} \right) (x_1 - x_3) - y_1 && \# \text{ insert points} \\ &= \left(\frac{2 - 1}{4 - 0} \right) (0 - 2) - 1 && \# \text{ simplify in } \mathbb{F}_5 \\ &= \left(\frac{1}{4} \right) \cdot 3 + 4 = 4 \cdot 3 + 4 = 2 + 4 = 1 \end{aligned}$$

So, in our elliptic curve $E_1(\mathbb{F}_5)$ we get $(0, 1) \oplus (4, 2) = (2, 1)$, and, indeed, the pair $(2, 1)$ is an
 element of $E_1(\mathbb{F}_5)$ as expected. On the other hand, $(0, 1) \oplus (0, 4) = \mathcal{O}$, since both points have
 equal x coordinates and inverse y coordinates, rendering them inverses of each other. Adding
 the point $(4, 2)$ to itself, we have to use the tangent rule, that is, rule 3 from definition 5.1.1.2: check
reference

$$\begin{aligned}
 x' &= \left(\frac{3x^2 + a}{2y} \right)^2 - 2x && \# \text{ insert points} \\
 &= \left(\frac{3 \cdot 4^2 + 1}{2 \cdot 2} \right)^2 - 2 \cdot 4 && \# \text{ simplify in } \mathbb{F}_5 \\
 &= \left(\frac{3 \cdot 1 + 1}{4} \right)^2 + 3 \cdot 4 = \left(\frac{4}{4} \right)^2 + 2 = 1 + 2 = 3
 \end{aligned}$$

$$\begin{aligned}
 y' &= \left(\frac{3x^2 + a}{2y} \right)^2 (x - x') - y && \# \text{ insert points} \\
 &= \left(\frac{3 \cdot 4^2 + 1}{2 \cdot 2} \right)^2 (4 - 3) - 2 && \# \text{ simplify in } \mathbb{F}_5 \\
 &= 1 \cdot 1 + 3 = 4
 \end{aligned}$$

2835 So, in our elliptic curve $E_1(\mathbb{F}_5)$, we get the doubling of $(4, 2)$, that is, $(4, 2) \oplus (4, 2) = (3, 4)$,
 2836 and, indeed the pair $(3, 4)$ is an element of $E_1(\mathbb{F}_5)$ as expected. The group $E_1(\mathbb{F}_5)$ has no self-
 2837 inverse points other than the neutral element \mathcal{O} , since no point has 0 as its y coordinate. We can
 2838 invoke Sage to double-check the computations.

```

2839 sage: F5 = GF(5)                                275
2840 sage: E1 = EllipticCurve(F5, [1, 1])             276
2841 sage: INF = E1(0) # point at infinity             277
2842 sage: P1 = E1(0, 1)                               278
2843 sage: P2 = E1(4, 2)                               279
2844 sage: P3 = E1(0, 4)                               280
2845 sage: R1 = E1(2, 1)                               281
2846 sage: R2 = E1(3, 4)                               282
2847 sage: R1 == P1+P2                                 283
2848 True                                              284
2849 sage: INF == P1+P3                                285
2850 True                                              286
2851 sage: R2 == P2+P2                                 287
2852 True                                              288
2853 sage: R2 == 2*P2                                  289
2854 True                                              290
2855 sage: P3 == P3 + INF                              291
2856 True                                              292

```

Example 73 (Tiny-jubjub). Consider the *TJJ_13*-curve from example 68 again and recall that its group of rational points is given as follows:

check
reference

$$\begin{aligned}
 TJJ_{13} = \{ &\mathcal{O}, (1, 2), (1, 11), (4, 0), (5, 2), (5, 11), (6, 5), (6, 8), (7, 2), (7, 11), \\
 &(8, 5), (8, 8), (9, 4), (9, 9), (10, 3), (10, 10), (11, 6), (11, 7), (12, 5), (12, 8) \}
 \end{aligned}$$

2857 In contrast to the group from the previous example, this group contains a self-inverse point,
 2858 which is different from the neutral element, defined by $(4, 0)$. To see what this means, observe
 2859 that we cannot add $(4, 0)$ to itself using the tangent rule 3 from definition 5.1.1.2, as the y
 2860 coordinate is zero. Instead, we have to use rule 2, since $0 = -0$. We therefore get $(4, 0) \oplus$

check
reference

2861 $(4,0) = \mathcal{O}$ in *TJJ_13*. The point $(4,0)$ is therefore the inverse of itself, as adding it to itself
 2862 results in the neutral element.

```

2863 sage: F13 = GF(13)                                     293
2864 sage: MJJ = EllipticCurve(F13, [8, 8])                  294
2865 sage: P = MJJ(4, 0)                                     295
2866 sage: INF = MJJ(0) # Point at infinity                  296
2867 sage: INF == P+P                                         297
2868 True                                                    298
2869 sage: INF == 2*P                                         299
2870 True                                                  300

```

2871 *Example 74.* Consider the Secp256k1 curve from example 69 again. The following code in-
 2872 vokes Sage to generate a random affine curve point, then applies our compression method:

check
reference

```

2873 sage: P = Secp256k1.random_point()                      301
2874 sage: Q = Secp256k1.random_point()                      302
2875 sage: INF = Secp256k1(0)                                 303
2876 sage: R1 = -P                                           304
2877 sage: R2 = P + Q                                         305
2878 sage: R3 = Secp256k1.order()*P                          306
2879 sage: P.xy()                                             307
2880 (5944888894207008764967348038870772122893488962940891291048219 308
2881      303875099472805, 308996414314973423805019033631509654839443
2882      0850451394476647400543468838362981)
2883 sage: Q.xy()                                             309
2884 (5183203912423152646279683763991696131184870317454767143468243 310
2885      1790964472162005, 11511873981572006937607059249499333905161
2886      3963290296124946223535431070983287487)
2887 sage: (ZZ(R1[0]).str(16), ZZ(R1[1]).str(16))            311
2888 ('d24b01883f52c695ea4aea779574f319709dd40a67e5a611c5e012992cbb 312
2889      3a5', 'f92b246bf2c0c931afbc9d3175ec265fa9a2413571a6f6b8f645
2890      97a4558d5cca')
2891 sage: R2.xy()                                             313
2892 (4946893428164461316043416558644979381298505518379497238771765 314
2893      0109491560467507, 52218831593492946206796837678800981103608
2894      676764722765094852517648201408061949)
2895 sage: R3 == INF                                           315
2896 True                                                    316
2897 sage: P[1]+R1[1] == Fp(0) # -(x,y) = (x,-y)           317
2898 True                                                  318

```

2899 *Exercise 47.* Consider the *TJJ_13*-curve from example 68.

check
reference

- 2900 1. Compute the inverse of $(10,10)$, \mathcal{O} , $(4,0)$ and $(1,2)$.
- 2901 2. Compute the expression $3 \cdot (1,11) - (9,9)$.
- 2902 3. Solve the equation $x + 2(9,4) = (5,2)$ for some $x \in TJJ_{13}$
- 2903 4. Solve the equation $x \cdot (7,11) = (8,5)$ for $x \in \mathbb{Z}$

Scalar multiplication As we have seen in the previous section, elliptic curves $E(\mathbb{F})$ have the structure of a commutative group associated to them. Moreover, It can moreover be shown that this group is finite and cyclic whenever the field is finite.

To understand elliptic curve scalar multiplication, recall from page 43 that every finite cyclic group of order q has a generator g and an associated exponential map $g^{(\cdot)} : \mathbb{Z}_q \rightarrow \mathbb{G}$, where g^n is the n -fold product of g with itself.

check
reference

Elliptic curve scalar multiplication is the exponential map written in additive notation. To be more precise, let \mathbb{F} be a finite field, $E(\mathbb{F})$ an elliptic curve of order r , and P a generator of $E(\mathbb{F})$. Then the **elliptic curve scalar multiplication** with base P is defined as follows (where $[0]P = \mathcal{O}$ and $[m]P = P + P + \dots + P$ is the m -fold sum of P with itself):

$$[\cdot]P : \mathbb{Z}_r \rightarrow E(\mathbb{F}); m \mapsto [m]P$$

therefore, elliptic curve scalar multiplication is an instantiation of the general exponential map using additive instead of multiplicative notation. This map is a homomorph of groups, which means that $[n + m]P = [n]P \oplus [m]P$.

As with all finite, cyclic groups, the inverse of the exponential map exists and is usually called the **elliptic curve discrete logarithm map**. However, elliptic curves are believed to be XXX-groups, which means that we don't know of any efficient way to actually compute this map.

add term

Scalar multiplication and its inverse, the elliptic curve discrete logarithm, define the **elliptic curve discrete logarithm problem**, which consists of finding solutions $m \in \mathbb{Z}_r$ such that the following equation holds:

$$P = [m]Q \tag{5.3}$$

Any solution m is usually called a **discrete logarithm relation** between P and Q . If Q is a generator of the curve, then there is a discrete logarithm relation between Q and any other point, since Q generates the group by repeatedly adding Q to itself. Therefore, we know that some discrete logarithm relation exists for generator Q and point P . However, since elliptic curves are believed to be XXX-groups, finding actual relations m is computationally hard, with runtimes being approximately the size of the order of the group. In practice, we often need the assumption that a discrete logarithm relation exists, while the relation itself is not known.

add term

One useful property of the exponential map in regard to the examples in this book is that it can be used to greatly simplify pen-and-paper computations. As we have seen in example XXX, computing the elliptic curve addition law takes quite a bit of effort when done without a computer. However, when g is a generator of a small pen-and-paper elliptic curve group of order r , we can use the exponential map to write the group using cofactor clearing, which implies that $[r]g = \mathcal{O}$:

add refer-
encecofactor
clearing

$$\mathbb{G} = \{[1]g \rightarrow [2]g \rightarrow [3]g \rightarrow \dots \rightarrow [r-1]g \rightarrow \mathcal{O}\} \tag{5.4}$$

“Logarithmic ordering” like this greatly simplifies complicated elliptic curve addition to the much simpler case of modular r addition. In order to add two curve points P and Q , we only have to look up their discrete log relations with the generator, say $P = [n]g$ and $Q = [m]g$, and compute the sum as $P \oplus Q = [n + m]g$. This is, of course, only possible for small groups where we can keep a clear overview, such as XXX.

add refer-
ence

In the following example, we will look at some implications of the fact that elliptic curves are finite cyclic groups. We will apply the fundamental theorem of finite cyclic groups and look how it reflects on the curves in consideration.

Example 75. Consider the elliptic curve group $E_1(\mathbb{F}_5)$ from example 67. Since it is a finite cyclic group of order 9, and the prime factorization of 9 is $3 \cdot 3$, we can use the fundamental

check
reference

theorem of finite cyclic groups to reason about all its subgroups. In fact, since the only prime factor of 9 is 3, we know that $E_1(\mathbb{F}_5)$ has the following subgroups:

- $\mathbb{G}_1 = E_1(\mathbb{F}_5)$ is a subgroup of order 9. By definition, any group is a subgroup of itself.
- $\mathbb{G}_2 = \{(2, 1), (2, 4), \mathcal{O}\}$ is a subgroup of order 3. This is the subgroup associated to the prime factor 3.
- $\mathbb{G}_3 = \{\mathcal{O}\}$ is a subgroup of order 1. This is the trivial subgroup.

Moreover, since $E_1(\mathbb{F}_5)$ and all its subgroups are cyclic, we know from page 43 that they must have generators. For example, the curve point $(2, 1)$ is a generator of the order 3 subgroup \mathbb{G}_2 , since every element of \mathbb{G}_2 can be generated by repeatedly adding $(2, 1)$ to itself:

check
reference

$$\begin{aligned}[1](2, 1) &= (2, 1) \\ [2](2, 1) &= (2, 4) \\ [3](2, 1) &= \mathcal{O}\end{aligned}$$

Since $(2, 1)$ is a generator, we know from XXX that it gives rise to an exponential map from the finite field \mathbb{F}_3 onto \mathbb{G}_2 defined by scalar multiplication:

add refer-
ence

$$[\cdot](2, 1) : \mathbb{F}_3 \rightarrow \mathbb{G}_2 : x \mapsto [x](2, 1)$$

To give an example of a generator that generates the entire group $E_1(\mathbb{F}_5)$, consider the point $(0, 1)$. Applying the tangent rule repeatedly, we compute as follows:

$$\begin{array}{ll} [0](0, 1) = \mathcal{O} & [1](0, 1) = (0, 1) \\ [2](0, 1) = (4, 2) & [3](0, 1) = (2, 1) \\ [4](0, 1) = (3, 4) & [5](0, 1) = (3, 1) \\ [6](0, 1) = (2, 4) & [7](0, 1) = (4, 3) \\ [8](0, 1) = (0, 4) & [9](0, 1) = \mathcal{O} \end{array}$$

Again, since $(2, 1)$ is a generator, we know from XXX that it gives rise to an exponential map. However, since the group order is not a prime number, the exponential map does not map from any field, but from the residue class ring \mathbb{Z}_9 only:

add refer-
ence

$$[\cdot](0, 1) : \mathbb{Z}_9 \rightarrow \mathbb{G}_1 : x \mapsto [x](0, 1)$$

Using the generator $(0, 1)$ and its associated exponential map, we can write $E(\mathbb{F}_1)$ i logarithmic order with respect to $(0, 1)$ as explained in equation 5.4. We get the following:

check
reference

$$E_1(\mathbb{F}_5) = \{(0, 1) \rightarrow (4, 2) \rightarrow (2, 1) \rightarrow (3, 4) \rightarrow (3, 1) \rightarrow (2, 4) \rightarrow (4, 3) \rightarrow (0, 4) \rightarrow \mathcal{O}\}$$

This indicates that the first element is a generator, and the n -th element is the scalar product of n and the generator. To see how logarithmic orders like this simplify the computations in small elliptic curve groups, consider example 72 again. In that example, we use the chord-and-tangent rule to compute $(0, 1) \oplus (4, 2)$. Now, in the logarithmic order of $E_1(\mathbb{F})$, we can compute that sum much easier, since we can directly see that $(0, 1) = [1](0, 1)$ and $(4, 2) = [2](0, 1)$. We can then deduce $(0, 1) \oplus (4, 2) = (2, 1)$ immediately, since $[1](0, 1) \oplus [2](0, 1) = [3](0, 1) = (2, 1)$.

check
reference

To give another example, we can immediately see that $(3, 4) \oplus (4, 3) = (4, 2)$, without doing any expensive elliptic curve addition, since we know $(3, 4) = [4](0, 1)$ as well as $(4, 3) =$

2957 $[7](0, 1)$ from the logarithmic representation of $E_1(\mathbb{F}_5)$. Since $4 + 7 = 2$ in \mathbb{Z}_9 , the result must
 2958 be $[2](0, 1) = (4, 2)$.

2959 Finally we can use $E_1(\mathbb{F}_5)$ as an example to understand the concept of cofactor clearing from
 2960 5.4. Since the order of $E_1(\mathbb{F}_5)$ is 9, we only have a single factor, which happen to be the cofactor
 2961 as well. Cofactor clearing then implies that we can map any element from $E_1(\mathbb{F}_5)$ onto its prime
 2962 factor group \mathbb{G}_2 by scalar multiplication with 3. For example, taking the element $(3, 4)$, which
 2963 is not in \mathbb{G}_2 , and multiplying it with 3, we get $[3](3, 4) = (2, 1)$, which is an element of \mathbb{G}_2 as
 2964 expected.

check
reference

2965 In the following example, we will look at the subgroups of our tiny-jubjub curve, define
 2966 generators, and compute the logarithmic order for pen-and-paper computations. Then we take
 2967 another look at the principle of cofactor clearing.

2968 *Example 76.* Consider the tiny-jubjub curve TJJ_13 from example 68 again. Since the order of
 2969 TJJ_13 is 20, and the prime factorization of 20 is $2^2 \cdot 5$, we know that the TJJ_13 contains a
 2970 “large” prime-order subgroup of size 5 and a small prime order subgroup of size 2.

check
reference

2971 To compute those groups, we can apply the technique of cofactor clearing in a try-and-repeat
 2972 loop. We start the loop by arbitrarily choosing an element $P \in TJJ_13$, then multiplying that
 2973 element with the cofactor of the group that we want to compute. If the result is \mathcal{O} , we try a
 2974 different element and repeat the process until the result is different from the point at infinity \mathcal{O} .

2975 To compute a generator for the small prime-order subgroup $(TJJ_13)_2$, first observe that the
 2976 cofactor is 10, since $20 = 2 \cdot 10$. We then arbitrarily choose the curve point $(5, 11) \in TJJ_13$
 2977 and compute $[10](5, 11) = \mathcal{O}$. Since the result is the point at infinity, we have to try another
 2978 curve point, say $(9, 4)$. We get $[10](9, 4) = (4, 0)$ and we can deduce that $(4, 0)$ is a generator
 2979 of $(TJJ_13)_2$. Logarithmic order then gives $(TJJ_13)_2 = \{(4, 0) \rightarrow \mathcal{O}\}$ as expected, since we
 2980 know from example 73 that $(4, 0)$ is self-inverse, with $(4, 0) \oplus (4, 0) = \mathcal{O}$. We double check the
 2981 computations using Sage:

check
reference

2982	sage: <code>F13 = GF(13)</code>	319
2983	sage: <code>TJJ = EllipticCurve(F13, [8, 8])</code>	320
2984	sage: <code>P = TJJ(5, 11)</code>	321
2985	sage: <code>INF = TJJ(0)</code>	322
2986	sage: <code>10*P == INF</code>	323
2987	True	324
2988	sage: <code>Q = TJJ(9, 4)</code>	325
2989	sage: <code>R = TJJ(4, 0)</code>	326
2990	sage: <code>10*Q == R</code>	327
2991	True	328

We can apply the same reasoning to the “large” prime-order subgroup $(TJJ_13)_5$, which contains 5 elements. To compute a generator for this group, first observe that the associated cofactor is 4, since $20 = 5 \cdot 4$. We choose the curve point $(9, 4) \in TJJ_13$ again, and compute $[4](9, 4) = (7, 11)$. We can deduce that $(7, 11)$ is a generator of $(TJJ_13)_5$. Using the generator $(7, 11)$, we compute the exponential map $[\cdot](7, 11) : \mathbb{F}_5 \rightarrow TJJ_13$ and get the following:

Explain
how

$$\begin{aligned}
 [0](7, 11) &= \mathcal{O} \\
 [1](7, 11) &= (7, 11) \\
 [2](7, 11) &= (8, 5) \\
 [3](7, 11) &= (8, 8) \\
 [4](7, 11) &= (7, 2)
 \end{aligned}$$

We can use this computation to write the large-order prime group $(TJJ_13)_5$ of the tiny-jubjub curve in logarithmic order, which we will use quite frequently in what follows. We get the following:

$$(TJJ_13)_5 = \{(7, 11) \rightarrow (8, 5) \rightarrow (8, 8) \rightarrow (7, 2) \rightarrow \mathcal{O}\} \quad (5.5)$$

From this, we can immediately see, for example that $(8, 8) \oplus (7, 2) = (8, 5)$, since $3 + 4 = 2$ in \mathbb{F}_5 .

From the previous two examples, the reader might get the impression that elliptic curve computation can be largely replaced by modular arithmetics. This however, is not true in general, but only an artifact of small groups, where it is possible to write the entire group in a logarithmic order. The following example gives some understanding of why this is not possible in cryptographically secure groups.

Example 77. SEKTP BICOIN. DISCRETE LOG HARDNESS PROHIBITS ADDITION IN THE FIELD...

write example

Projective short Weierstraß form As we have seen in the previous section, describing elliptic curves as pairs of points that satisfy a certain equation is relatively straight-forward. However, in order to define a group structure on the set of points, we had to add a special point at infinity to act as the neutral element.

Recalling from the definition of projective planes (section 4.4), we know that points at infinity are handled as ordinary points in projective geometry. Therefore, it makes sense to look at the definition of a short Weierstraß curve in projective geometry.

check reference

To see what a short Weierstraß curve in projective coordinates is, let \mathbb{F} be a finite field of order q and characteristic > 3 , let $a, b \in \mathbb{F}$ be two field elements such that $4a^3 + 27b^2 \bmod q \neq 0$ and let \mathbb{FP}^2 be the projective plane over \mathbb{F} . Then a **short Weierstraß elliptic curve** over \mathbb{F} in its projective representation is the set of all points $[X : Y : Z] \in \mathbb{FP}^2$ from the projective plane that satisfy the **homogenous** cubic equation $Y^2 \cdot Z = X^3 + a \cdot X \cdot Z^2 + b \cdot Z^3$:

$$E(\mathbb{FP}^2) = \{[X : Y : Z] \in \mathbb{FP}^2 \mid Y^2 \cdot Z = X^3 + a \cdot X \cdot Z^2 + b \cdot Z^3\} \quad (5.6)$$

To understand how the point at infinity is unified in this definition, recall from XXX that, in projective geometry, points at infinity are given by homogeneous coordinates $[X : Y : 0]$. Inserting representatives $(x_1, y_1, 0) \in [X : Y : 0]$ from those classes into the defining homogenous cubic equations gives the following:

add reference

$$\begin{aligned} y_1^2 \cdot 0 &= x_1^3 + a \cdot x_1 \cdot 0^2 + b \cdot 0^3 \\ 0 &= x_1^3 \end{aligned} \quad \Leftrightarrow$$

This shows that the only point at infinity, that is also a point on a projective short Weierstraß curve is the class $[0, 1, 0] = \{(0, y, 0) \mid y \in \mathbb{F}\}$.

This point is the projective representation of \mathcal{O} . The projective representation of a short Weierstraß curve, therefore, has the advantage that it does not need a special symbol to represent the point at infinity \mathcal{O} from the affine definition.

Example 78. To get an intuition of how an elliptic curve in projective geometry looks, consider curve $E_1(\mathbb{F}_5)$ from example (67). We know that, in its affine representation, the set of rational points is given as follows:

check reference

$$E_1(\mathbb{F}_5) = \{\mathcal{O}, (0, 1), (2, 1), (3, 1), (4, 2), (4, 3), (0, 4), (2, 4), (3, 4)\} \quad (5.7)$$

3024 This is defined as the set of all pairs $(x, y) \in \mathbb{F}_5 \times \mathbb{F}_5$ such that the affine short Weierstraß
 3025 equation $y^2 = x^3 + ax + b$ with $a = 1$ and $b = 1$ is satisfied.

3026 To find the projective representation of a short Weierstraß curve with the same parameters
 3027 $a = 1$ and $b = 1$, we have to compute the set of projective points $[X : Y : Z]$ from the projec-
 3028 tive plane $\mathbb{F}_5\mathbb{P}^2$ that satisfy the following homogenous cubic equation for any representative
 3029 $(x_1, y_1, z_1) \in [X : Y : Z]$:

$$y_1^2 z_1 = x_1^3 + 1 \cdot x_1 z_1^2 + 1 \cdot z_1^3 \quad (5.8)$$

3030 We know from XXX that the projective plane $\mathbb{F}_5\mathbb{P}^2$ contains $5^2 + 5 + 1 = 31$ elements, so we
 3031 can take the effort and insert all elements into equation 5.8 and see if both sides match.

For example, consider the projective point $[0 : 4 : 1]$. We know from XXX that this point in
 the projective plane represents the following line in the three-dimensional space \mathbb{F}^3 :

$$[0 : 4 : 1] = \{(0, 0, 0), (0, 4, 1), (0, 3, 2), (0, 2, 3), (0, 1, 4)\}$$

To check whether or not $[0 : 4 : 1]$ satisfies 5.8, we can insert any representative, in other words,
 any element from XXX. Each element satisfies the equation if and only if all other elements
 satisfy the equation. So, we insert $(0, 4, 1)$ and get the following result:

$$1^2 \cdot 1 = 0^3 + 1 \cdot 0 \cdot 1^2 + 1 \cdot 1^3$$

This tells us that the affine point $[0 : 4 : 1]$ is indeed a solution to the equation 5.8, but we
 could just as well have inserted any other representative. For example, inserting $(0, 3, 2)$ also
 satisfies 5.8:

$$3^2 \cdot 2 = 0^3 + 1 \cdot 0 \cdot 2^2 + 1 \cdot 2^3$$

3032 To find the projective representation of E_1 , we first observe that the projective line at infinity
 3033 $[1 : 0 : 0]$ is not a curve point on any projective short Weierstraß curve, since it cannot satisfy
 3034 XXX for any parameter a and b . Therefore, we can exclude it from our consideration.

3035 Moreover, a point at infinity $[X : Y : 0]$ can only satisfy equation XXX for any a and b , if
 3036 $X = 0$, which implies that the only point at infinity relevant for short Weierstraß elliptic curves
 3037 is $[0 : 1 : 0]$, since $[0 : k : 0] = [0 : 1 : 0]$ for all k from the finite field. Therefore, we can exclude
 3038 all points at infinity except the point $[0 : 1 : 0]$.

3039 All points that remain are the affine points $[X : Y : 1]$. Inserting all of them into XXX, we
 3040 get the set of all projective curve points as follows:

$$E_1(\mathbb{F}_5\mathbb{P}^2) = \{[0 : 1 : 0], [0 : 1 : 1], [2 : 1 : 1], [3 : 1 : 1], \\ [4 : 2 : 1], [4 : 3 : 1], [0 : 4 : 1], [2 : 4 : 1], [3 : 4 : 1]\}$$

3041 If we compare this with the affine representation, we see that there is a 1:1 correspondence
 3042 between the points in the affine representation in 5.7 and the affine points in projective geometry,
 3043 and that the point $[0 : 1 : 0]$ represents the additional point \mathcal{O} in the projective representation.

3044 *Exercise 48.* Compute the projective representation of the tiny-jubjub curve and the logarithmic
 3045 order of its large prime-order subgroup with respect to the generator $(7, 11)$.

3046 **Projective Group law** As we have seen on page 73, one of the key properties of an elliptic
 3047 curve is that it comes with a definition of a group law on the set of its rational points, described
 3048 geometrically by the chord-and-tangent rule (definition 5.1.1.1). This rule was kind of intuitive,

with the exception of the distinguished point at infinity, which appeared whenever the chord or the tangent did not have a third intersection point with the curve.

One of the key features of projective coordinates is that, in projective space, it is guaranteed that any chord will always intersect the curve in three points, and any tangent will intersect it in two points including the tangent point. So, the geometric picture simplifies, as we don't need to consider external symbols and associated cases.

Again, it can be shown that the points of an elliptic curve in projective space form a commutative group with respect to the tangent-and-chord rule such that the projective point $[0 : 1 : 0]$ is the neutral element, and the additive inverse of a point $[X : Y : Z]$ is given by $[X : -Y : Z]$. The addition law is usually described by the following algorithm, minimizing the number of necessary additions and multiplications in the base field.

Exercise 49. Compare the affine addition law for short Weierstraß curves with the projective addition rule. Which branch in the projective rule corresponds to which case in the affine law?

Check if following Alg is floated too far

Coordinate Transformations As we have seen in example XXX, there was a close relation between the affine and the projective representation of a short Weierstraß curve. This was not a coincidence. In fact, from a mathematical point of view, projective and affine short Weierstraß curves describe the same thing, as there is a one-to-one correspondence (an isomorphism) between both representations for any arbitrary parameters a and b .

add reference

To specify the isomorphism, let $E(\mathbb{F})$ and $E(\mathbb{FP}^2)$ be an affine and a projective short Weierstraß curve defined for the same parameters a and b . Then the map in 5.9 maps points from the affine representation to points from the projective representation of a short Weierstraß curve. In other words, if the pair of points (x, y) satisfies the affine equation $y^2 = x^3 + ax + b$, then all homogeneous coordinates $(x_1, y_1, z_1) \in [x : y : 1]$ satisfy the projective equation $y_1^2 \cdot z_1 = x_1^3 + ay_1 \cdot z_1^2 + b \cdot z_1^3$.

$$\Phi : E(\mathbb{F}) \rightarrow E(\mathbb{FP}^2) : \begin{array}{ll} (x, y) & \mapsto [x : y : 1] \\ \mathcal{O} & \mapsto [0 : 1 : 0] \end{array} \quad (5.9)$$

The inverse is given by the following map:

$$\Phi^{-1} : E(\mathbb{FP}^2) \rightarrow E(\mathbb{F}) : [X : Y : Z] \mapsto \begin{cases} (\frac{X}{Z}, \frac{Y}{Z}) & \text{if } Z \neq 0 \\ \mathcal{O} & \text{if } Z = 0 \end{cases} \quad (5.10)$$

Note that the only projective point $[X : Y : Z]$ with $Z \neq 0$ that satisfies XXX is given by the class $[0 : 1 : 0]$.

add reference

One key feature of Φ and its inverse is that it respects the group structure, which means that $\Phi((x_1, y_1) \oplus (x_2, y_2))$ is equal to $\Phi(x_1, y_1) \oplus \Phi(x_2, y_2)$. The same holds true for the inverse map Φ^{-1} .

Maps with these properties are called **group isomorphisms**, and, from a mathematical point of view, the existence of Φ implies that these two definitions are equivalent, and implementations can choose freely between these representations.

5.1.2 Montgomery Curves

History and use of them (optimized scalar multiplication)

write up this part

Algorithm 6 Projective Weierstraß Addition Law

Require: $[X_1 : Y_1 : Z_1], [X_2 : Y_2 : Z_2] \in E(\mathbb{F}_{\mathbb{P}^2})$

procedure ADD-RULE($[X_1 : Y_1 : Z_1], [X_2 : Y_2 : Z_2]$)

if $[X_1 : Y_1 : Z_1] == [0 : 1 : 0]$ **then**

$[X_3 : Y_3 : Z_3] \leftarrow [X_2 : Y_2 : Z_2]$

else if $[X_2 : Y_2 : Z_2] == [0 : 1 : 0]$ **then**

$[X_3 : Y_3 : Z_3] \leftarrow [X_1 : Y_1 : Z_1]$

else

$U_1 \leftarrow Y_2 \cdot Z_1$

$U_2 \leftarrow Y_1 \cdot Z_2$

$V_1 \leftarrow X_2 \cdot Z_1$

$V_2 \leftarrow X_1 \cdot Z_2$

if $V_1 == V_2$ **then**

if $U_1 \neq U_2$ **then** $[X_3 : Y_3 : Z_3] \leftarrow [0 : 1 : 0]$

else

if $Y_1 == 0$ **then** $[X_3 : Y_3 : Z_3] \leftarrow [0 : 1 : 0]$

else

$W \leftarrow a \cdot Z_1^2 + 3 \cdot X_1^2$

$S \leftarrow Y_1 \cdot Z_1$

$B \leftarrow X_1 \cdot Y_1 \cdot S$

$H \leftarrow W^2 - 8 \cdot B$

$X' \leftarrow 2 \cdot H \cdot S$

$Y' \leftarrow W \cdot (4 \cdot B - H) - 8 \cdot Y_1^2 \cdot S^2$

$Z' \leftarrow 8 \cdot S^3$

$[X_3 : Y_3 : Z_3] \leftarrow [X' : Y' : Z']$

end if

end if

else

$U = U_1 - U_2$

$V = V_1 - V_2$

$W = Z_1 \cdot Z_2$

$A = U^2 \cdot W - V^3 - 2 \cdot V^2 \cdot V_2$

$X' = V \cdot A$

$Y' = U \cdot (V^2 \cdot V_2 - A) - V^3 \cdot U_2$

$Z' = V^3 \cdot W$

$[X_3 : Y_3 : Z_3] \leftarrow [X' : Y' : Z']$

end if

end if

return $[X_3 : Y_3 : Z_3]$

end procedure

Ensure: $[X_3 : Y_3 : Z_3] == [X_1 : Y_1 : Z_1] \oplus [X_2 : Y_2 : Z_2]$

Affine Montgomery Form To see what a Montgomery curve in affine coordinates is, let \mathbb{F} be a finite field of characteristic > 2 , and let $A, B \in \mathbb{F}$ be two field elements such that $B \neq 0$ and $A^2 \neq 4$. A **Montgomery elliptic curve** $M(\mathbb{F})$ over \mathbb{F} in its affine representation is the set of all pairs of field elements $(x, y) \in \mathbb{F} \times \mathbb{F}$ that satisfy the Montgomery cubic equation $B \cdot y^2 = x^3 + A \cdot x^2 + x$, together with a distinguished symbol \mathcal{O} , called the **point at infinity**.

$$M(\mathbb{F}) = \{(x, y) \in \mathbb{F} \times \mathbb{F} \mid B \cdot y^2 = x^3 + A \cdot x^2 + x\} \cup \{\mathcal{O}\} \quad (5.11)$$

Despite the fact that Montgomery curves look different from short Weierstraß curves, they are just a special way to describe certain short Weierstraß curves. In fact, every curve in affine Montgomery form can be transformed into an elliptic curve in Weierstraß form. To see that, assume that a curve is given in Montgomery form $By^2 = x^3 + Ax^2 + x$. The associated Weierstraß form is then as follows:

$$y^2 = x^3 + \frac{3 - A^2}{3B^2} \cdot x + \frac{2A^3 - 9A}{27B^3} \quad (5.12)$$

On the other hand, an elliptic curve $E(\mathbb{F})$ over base field \mathbb{F} in Weierstraß form $y^2 = x^3 + ax + b$ can be converted to Montgomery form if and only if the following conditions hold:

Definition 5.1.2.1. Requirements for Montgomery curves

- The number of points on $E(F)$ is divisible by 4
- The polynomial $z^3 + az + b \in \mathbb{F}[z]$ has at least one root $z_0 \in \mathbb{F}$
- $3z_0^2 + a$ is a quadratic residue in \mathbb{F} .

When these conditions are satisfied, then for $s = (\sqrt{3z_0^2 + a})^{-1}$, the equivalent Montgomery curve is defined by the following equation:

$$sy^2 = x^3 + (3z_0s)x^2 + x \quad (5.13)$$

In the following example we will look at our tiny-jubjub curve again, and show that it is actually a Montgomery curve.

Example 79. Consider the prime field \mathbb{F}_{13} and the tiny-jubjub curve TJJ_13 from example 68. To see that it is a Montgomery curve, we have to check the requirements from 5.1.2.1:

Since the order of TJJ_13 is 20, which is divisible by 4, the first requirement is met.

Next, since $a = 8$ and $b = 8$, we have to check if the polynomial $P(z) = z^3 + 8z + 8$ has a root in \mathbb{F}_{13} . We simply evaluate P at all numbers $z \in \mathbb{F}_{13}$, and find that $P(4) = 0$, so a root is given by $z_0 = 4$.

In the last step, we have to check that $3 \cdot z_0^2 + a$ has a root in \mathbb{F}_{13} . We compute as follows:

$$\begin{aligned} 3z_0^2 + a &= 3 \cdot 4^2 + 8 \\ &= 3 \cdot 3 + 8 \\ &= 9 + 8 \\ &= 4 \end{aligned}$$

To see if 4 is a quadratic residue, we can use Euler's criterion (4.32) to compute the Legendre symbol of 4. We get the following:

is the label in L^AT_EX correct here?

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$$\left(\frac{4}{13}\right) = 4^{\frac{13-1}{2}} = 4^6 = 1$$

3113 This means that 4 does have a root in \mathbb{F}_{13} . In fact, computing a root of 4 in \mathbb{F}_{13} is easy, since
 3114 the integer root 2 of 4 is also one of its roots in \mathbb{F}_{13} . The other root is given by $13 - 4 = 9$.

Since all requirements are met, we have now shown that *TJJ_13* is indeed a Montgomery curve, and we can use 5.13 to compute its associated Montgomery form. We compute as follows:

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$$\begin{aligned} s &= \left(\sqrt{3 \cdot z_0^2 + 8} \right)^{-1} \\ &= 2^{-1} && \# \text{ Fermat's little theorem} \\ &= 2^{13-2} && \# 2048 \bmod 13 = 7 \\ &= 7 \end{aligned}$$

The defining equation for the Montgomery form of our tiny-jubjub curve is then given by the following equation:

$$\begin{aligned} sy^2 &= x^3 + (3z_0s)x^2 + x && \Rightarrow \\ 7 \cdot y^2 &= x^3 + (3 \cdot 4 \cdot 7)x^2 + x && \Leftrightarrow \\ 7 \cdot y^2 &= x^3 + 6x^2 + x \end{aligned}$$

3115 So, we get the defining parameters as $B = 7$ and $A = 6$, and we can write the tiny-jubjub curve
 3116 in its affine Montgomery representation as follows:

$$TJJ_13 = \{(x, y) \in \mathbb{F}_{13} \times \mathbb{F}_{13} \mid 7 \cdot y^2 = x^3 + 6x^2 + x\} \cup \{\mathcal{O}\} \quad (5.14)$$

Now that we have the abstract definition of our tiny-jubjub curve in Montgomery form, we can compute the set of points by inserting all pairs $(x, y) \in \mathbb{F}_{13} \times \mathbb{F}_{13}$ similarly to how we computed the curve points in its Weierstraß representation. We get the following:

$$TJJ_13 = \{\mathcal{O}, (0, 0), (1, 4), (1, 9), (2, 4), (2, 9), (3, 5), (3, 8), (4, 4), (4, 9), (5, 1), (5, 12), (7, 1), (7, 12), (8, 1), (8, 12), (9, 2), (9, 11), (10, 3), (10, 10)\}$$

3117

```

3118 sage: F13 = GF(13)                                     329
3119 sage: L_MTJJ = []                                       330
3120 ....: for x in F13:                                     331
3121 ....:     for y in F13:                                  332
3122 ....:         if F13(7)*y^2 == x^3 + F13(6)*x^2 + x:    333
3123 ....:             L_MTJJ.append((x, y))                 334
3124 sage: MTJJ = Set(L_MTJJ)                                335
3125 sage: # does not compute the point at infinity          336

```

3126 **Affine Montgomery coordinate transformation** Comparing the Montgomery representa-
 3127 tion of the previous example (equation 5.14) with the Weierstraß representation of the same
 3128 curve (equation 5.2), we see that there is a 1:1 correspondence between the curve points in both

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examples. This is no accident. In fact, if $M_{A,B}$ is a Montgomery curve, and $E_{a,b}$ a Weierstraß curve with $a = \frac{3-A^2}{3B^2}$ and $b = \frac{2A^2-9A}{27B^3}$ then the following function maps all points in Montgomery representation onto the points in Weierstraß representation:

$$\Phi : M_{A,B} \rightarrow E_{a,b} : (x,y) \mapsto \left(\frac{3x+A}{3B}, \frac{y}{B} \right) \quad (5.15)$$

This map is a 1:1 correspondence (an isomorphism), and its inverse map is given by the following equation (where z_0 is a root of the polynomial $z^3 + az + b \in \mathbb{F}[z]$ and $s = (\sqrt{3z_0^2 + a})^{-1}$).

$$\Phi^{-1} : E_{a,b} \rightarrow M_{A,B} : (x,y) \mapsto (s \cdot (x - z_0), s \cdot y) \quad (5.16)$$

Using this map, it is therefore possible for implementations of Montgomery curves to freely transit between the Weierstraß and the Montgomery representation. However, as we saw in definition 5.1.2.1, not every Weierstraß curve is a Montgomery curve, as all criteria in 5.1.2.1 have to be satisfied. This means that the map Φ^{-1} does not always exist.

Example 80. Consider our tiny-jubjub curve again. In equation 5.2 we derived its Weierstraß representation and in example 5.14, we derived its Montgomery representation.

To see how coordinate transformation Φ works in this example, let's map points from the Montgomery representation onto points from the Weierstraß representation. Inserting, for example, the point $(0,0)$ from the Montgomery representation 5.14 into Φ gives the following:

$$\begin{aligned} \Phi(0,0) &= \left(\frac{3 \cdot 0 + A}{3B}, \frac{0}{B} \right) \\ &= \left(\frac{3 \cdot 0 + 6}{3 \cdot 7}, \frac{0}{7} \right) \\ &= \left(\frac{6}{8}, 0 \right) \\ &= (4,0) \end{aligned}$$

As we can see, the Montgomery point $(0,0)$ maps to the self-inverse point $(4,0)$ of the Weierstraß representation. On the other hand, we can use our computations of $s = 7$ and $z_0 = 4$ from XXX to compute the inverse map Φ^{-1} , which maps points on the Weierstraß representation to points on the Montgomery form. Inserting, for example, $(4,0)$ we get the following:

$$\begin{aligned} \Phi^{-1}(4,0) &= (s \cdot (4 - z_0), s \cdot 0) \\ &= (7 \cdot (4 - 4), 0) \\ &= (0,0) \end{aligned}$$

As expected, the inverse map maps the Weierstraß point back to where it originated in the Montgomery form. We can invoke Sage to check that our computation of Φ is correct:

```

3142 sage: # Compute PHI of Montgomery form: 337
3143 sage: L_PHI_MTJJ = [] 338
3144 sage: for (x,y) in L_MTJJ: # LMJJ as defined previously 339
3145     . . . . . v = (F13(3)*x + F13(6)) / (F13(3)*F13(7)) 340
3146     . . . . . w = y/F13(7) 341
3147     . . . . . L_PHI_MTJJ.append((v,w)) 342

```

```

3148 sage: PHI_MTJJ = Set(L_PHI_MTJJ) 343
3149 sage: # Computation Weierstrass form 344
3150 sage: C_WTJJ = EllipticCurve(F13, [8, 8]) 345
3151 sage: L_WTJJ = [P.xy() for P in C_WTJJ.points() if P.order() > 346
3152 1]
3153 sage: WTJJ = Set(L_WTJJ) 347
3154 sage: # check PHI(Montgomery) == Weierstrass 348
3155 sage: WTJJ == PHI_MTJJ 349
3156 True 350
3157 sage: # check the inverse map PHI^(-1) 351
3158 sage: L_PHIINV_WTJJ = [] 352
3159 sage: for (v,w) in L_WTJJ: 353
3160 .....:     x = F13(7)*(v-F13(4)) 354
3161 .....:     y = F13(7)*w 355
3162 .....:     L_PHIINV_WTJJ.append((x,y)) 356
3163 sage: PHIINV_WTJJ = Set(L_PHIINV_WTJJ) 357
3164 sage: MTJJ == PHIINV_WTJJ 358
3165 True 359

```

3166 **Montgomery group law** We have seen that Montgomery curves special cases of short Weier-
3167 straß curves. As such, they have a group structure defined on the set of their points, which can
3168 also be derived from the chord-and-tangent rule. In accordance with short Weierstraß curves, it
3169 can be shown that the identity $x_1 = x_2$ implies $y_2 = \pm y_1$, meaning that the following rules are a
3170 complete description of the affine addition law.

3171 *Definition 5.1.2.2. Montgomery group law*

- 3172 • (Neutral element) Point at infinity \mathcal{O} is the neutral element.
- 3173 • (Additive inverse) The additive inverse of \mathcal{O} is \mathcal{O} . For any other curve point $(x,y) \in$
3174 $M(\mathbb{F}_q) \setminus \{\mathcal{O}\}$, the additive inverse is given by $(x, -y)$.
- 3175 • (Addition rule) For any two curve points $P, Q \in M(\mathbb{F}_q)$, addition is defined by one of the
3176 following cases:
 - 3177 1. (Adding the neutral element) If $Q = \mathcal{O}$, then the sum is defined as $P + Q = P$.
 - 3178 2. (Adding inverse elements) If $P = (x,y)$ and $Q = (x, -y)$, the sum is defined as $P +$
3179 $Q = \mathcal{O}$.
 3. (Adding non-self-inverse equal points) If $P = (x,y)$ and $Q = (x,y)$ with $y \neq 0$, the
sum $2P = (x', y')$ is defined as follows:

$$x' = \left(\frac{3x_1^2 + 2Ax_1 + 1}{2By_1} \right)^2 \cdot B - (x_1 + x_2) - A, \quad y' = \frac{3x_1^2 + 2Ax_1 + 1}{2By_1} (x_1 - x') - y_1$$

4. (Adding non-inverse different points) If $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ such that $x_1 \neq$
 x_2 , the sum $R = P + Q$ with $R = (x_3, y_3)$ is defined as follows:

$$x' = \left(\frac{y_2 - y_1}{x_2 - x_1} \right)^2 B - (x_1 + x_2) - A, \quad y' = \frac{y_2 - y_1}{x_2 - x_1} (x_1 - x') - y_1$$

5.1.3 Twisted Edwards Curves

As we have seen in 5.1.2.2 both Weierstraß and Montgomery curves have somewhat complicated addition and doubling laws, as many cases have to be distinguished. Those various cases translate to branches in computer programs.

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In the context of SNARK development, two computational models for bounded computations are used, called **circuits** and **rank-1 constraint systems**. Program branches are undesirably costly when implemented in those models. It is therefore advantageous to look for curves with an addition/doubling rule that requires no branches and as few field operations as possible.

Twisted Edwards curves are particularly useful here, as a subclass of these curves has a compact and easily implementable addition law that works for all points including the point at infinity. Implementing this law needs no branching.

Twisted Edwards Form To see what an affine **twisted Edwards curve** looks like, let \mathbb{F} be a finite field of characteristic > 2 , and let $a, d \in \mathbb{F} \setminus \{0\}$ be two non-zero field elements with $a \neq d$. A **twisted Edwards elliptic curve** in its affine representation is the set of all pairs (x, y) from $\mathbb{F} \times \mathbb{F}$ that satisfy the twisted Edwards equation $a \cdot x^2 + y^2 = 1 + d \cdot x^2 y^2$, given below:

$$E(\mathbb{F}) = \{(x, y) \in \mathbb{F} \times \mathbb{F} \mid a \cdot x^2 + y^2 = 1 + d \cdot x^2 y^2\} \quad (5.17)$$

A twisted Edwards curve is called an **Edwards curve (non-twisted)**, if the parameter a is equal to 1, and it is called a **SNARK-friendly twisted Edwards curve** if the parameter a is a quadratic residue and the parameter d is a quadratic non-residue.

As we can see from the definition, affine twisted Edwards curves look somewhat different from Weierstraß curves, as their affine representation does not need a special symbol to represent the point at infinity. In fact, we will see that the pair $(0, 1)$ is always a point on any twisted Edwards curve, and that it takes the role of the point at infinity.

Despite their different appearances however, twisted Edwards curves are equivalent to Montgomery curves in the sense that, for every twisted Edwards curve, there is a Montgomery curve, and a way to map the points of one curve in a 1:1 correspondence onto the other and vice versa. To see that, assume that a curve in twisted Edwards form $a \cdot x^2 + y^2 = 1 + d \cdot x^2 y^2$ is given. The associated Montgomery curve is then defined by the Montgomery equation:

$$\frac{4}{a-d} y^2 = x^3 + \frac{2(a+d)}{a-d} \cdot x^2 + x \quad (5.18)$$

On the other hand, a Montgomery curve $By^2 = x^3 + Ax^2 + x$ with $B \neq 0$ and $A^2 \neq 4$ can give rise to a twisted Edwards curve defined by the following equation:

$$\left(\frac{A+2}{B}\right)x^2 + y^2 = 1 + \left(\frac{A-2}{B}\right)x^2 y^2 \quad (5.19)$$

As we have seen in equation 5.12 and the following discussion, Montgomery curves are just a special class of Weierstraß curves. Furthermore we now know that twisted Edwards curves are special Weierstraß curves too. This means that the more general way to describe elliptic curves is as Weierstraß curves.

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reference

Example 81. Consider the tiny-jubjub curve from example 68 again. We know from example 79 that it is a Montgomery curve, and, since Montgomery curves are equivalent to twisted Edwards curves, we want to write this curve in twisted Edwards form. We use equation 5.19,

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and compute the parameters a and d as follows:

$$\begin{aligned}
 a &= \frac{A+2}{B} && \# \text{ insert } A=6 \text{ and } B=7 \\
 &= \frac{8}{7} = 3 && \# 7^{-1} = 2 \\
 \\
 d &= \frac{A-2}{B} \\
 &= \frac{4}{7} = 8
 \end{aligned}$$

Thus, we get the defining parameters as $a = 3$ and $d = 8$. Since our goal is to use this curve later on in implementations of pen-and-paper SNARKs, let us show that tiny-jubjub is also a **SNARK-friendly** twisted Edwards curve. To see that, we have to show that a is a quadratic residue and d is a quadratic non-residue. We therefore compute the Legendre symbols of a and d using Euler's criterion. We get the following:

$$\begin{aligned}
 \left(\frac{3}{13} \right) &= 3^{\frac{13-1}{2}} \\
 &= 3^6 = 1
 \end{aligned}$$

$$\begin{aligned}
 \left(\frac{8}{13} \right) &= 8^{\frac{13-1}{2}} \\
 &= 8^6 = 12 = -1
 \end{aligned}$$

3213 This proves that tiny-jubjub is SNARK-friendly. We can write the tiny-jubjub curve in its
3214 affine twisted Edwards representation as follows:

$$TJJ_{13} = \{(x, y) \in \mathbb{F}_{13} \times \mathbb{F}_{13} \mid 3 \cdot x^2 + y^2 = 1 + 8 \cdot x^2 \cdot y^2\} \quad (5.20)$$

3215 Now that we have the abstract definition of our tiny-jubjub curve in twisted Edwards form,
3216 we can compute the set of points by inserting all pairs $(x, y) \in \mathbb{F}_{13} \times \mathbb{F}_{13}$, similarly to how we
3217 computed the curve points in its Weierstraß or Edwards representation. We get the following:

$$\begin{aligned}
 TJJ_{13} = \{ &(0, 1), (0, 12), (1, 2), (1, 11), (2, 6), (2, 7), (3, 0), (5, 5), (5, 8), (6, 4), \\
 &(6, 9), (7, 4), (7, 9), (8, 5), (8, 8), (10, 0), (11, 6), (11, 7), (12, 2), (12, 11) \}
 \end{aligned} \quad (5.21)$$

3218

```

3219 sage: F13 = GF(13)                                     360
3220 sage: L_ETJJ = []                                       361
3221 .....: for x in F13:                                    362
3222 .....:     for y in F13:                                363
3223 .....:         if F13(3)*x^2 + y^2 == 1+ F13(8)*x^2*y^2: 364
3224 .....:             L_ETJJ.append((x, y))                365
3225 sage: ETJJ = Set(L_ETJJ)                               366

```

Twisted Edwards group law As we have seen, twisted Edwards curves are equivalent to Montgomery curves, and, as such, also have a group law. However, in contrast to Montgomery and Weierstraß curves, the group law of SNARK-friendly twisted Edwards curves can be described by a single computation that works in all cases, no matter if we add the neutral element, the inverse, or if we have to double a point. To see what the group law looks like, first observe that the point $(0, 1)$ is a solution to $a \cdot x^2 + y^2 = 1 + d \cdot x^2 \cdot y^2$ for any curve. The sum of any two points $(x_1, y_1), (x_2, y_2)$ on an Edwards curve $E(\mathbb{F})$ is then given by the following equation:

$$(x_1, y_1) \oplus (x_2, y_2) = \left(\frac{x_1 y_2 + y_1 x_2}{1 + d x_1 x_2 y_1 y_2}, \frac{y_1 y_2 - a x_1 x_2}{1 - d x_1 x_2 y_1 y_2} \right) \quad (5.22)$$

and it can be shown that the point $(0, 1)$ serves as the neutral element and the inverse of a point (x_1, y_1) is given by $(-x_1, y_1)$.

Example 82. Lets look at the tiny-jubjub curve in Edwards form from example 5.20 again. As we have seen, this curve is given by

$$TJJ_13 = \{(0, 1), (0, 12), (1, 2), (1, 11), (2, 6), (2, 7), (3, 0), (5, 5), (5, 8), (6, 4), (6, 9), (7, 4), (7, 9), (8, 5), (8, 8), (10, 0), (11, 6), (11, 7), (12, 2), (12, 11)\}$$

To get an understanding of the twisted Edwards addition law, let's first add the neutral element $(0, 1)$ to itself. We apply the group law 5.22 and get the following:

$$\begin{aligned} (0, 1) \oplus (0, 1) &= \left(\frac{0 \cdot 1 + 1 \cdot 0}{1 + 8 \cdot 0 \cdot 0 \cdot 1 \cdot 1}, \frac{1 \cdot 1 - 3 \cdot 0 \cdot 0}{1 - 8 \cdot 0 \cdot 0 \cdot 1 \cdot 1} \right) \\ &= (0, 1) \end{aligned}$$

So, as expected, the neutral element added to itself gives the neutral element again. Now let's add the neutral element to some other curve point. We get the following:

$$\begin{aligned} (0, 1) \oplus (8, 5) &= \left(\frac{0 \cdot 5 + 1 \cdot 8}{1 + 8 \cdot 0 \cdot 8 \cdot 1 \cdot 5}, \frac{1 \cdot 5 - 3 \cdot 0 \cdot 8}{1 - 8 \cdot 0 \cdot 8 \cdot 1 \cdot 5} \right) \\ &= (8, 5) \end{aligned}$$

Again, as expected, adding the neutral element to any element will result in that element again. Given any curve point (x, y) , we know that its inverse is given by $(-x, y)$. To see how the addition of a point to its inverse works, we compute as follows:

$$\begin{aligned} (5, 5) \oplus (8, 5) &= \left(\frac{5 \cdot 5 + 5 \cdot 8}{1 + 8 \cdot 5 \cdot 8 \cdot 5 \cdot 5}, \frac{5 \cdot 5 - 3 \cdot 5 \cdot 8}{1 - 8 \cdot 5 \cdot 8 \cdot 5 \cdot 5} \right) \\ &= \left(\frac{12 + 1}{1 + 5}, \frac{12 - 3}{1 - 5} \right) \\ &= \left(\frac{0}{6}, \frac{12 + 10}{1 + 8} \right) \\ &= \left(0, \frac{9}{9} \right) \\ &= (0, 1) \end{aligned}$$

Adding a curve point to its inverse gives the neutral element, as expected. As we have seen from these examples, the twisted Edwards addition law handles edge cases particularly well and in a unified way.

5.2 Elliptic Curve Pairings

As we have seen in equation 4.5, some groups come with the notation of a so-called pairing map, which is a non-degenerate bilinear map from two groups into another group.

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In this section, we discuss **pairings on elliptic curves**, which form the basis of several zk-SNARKs and other zero-knowledge proof schemes. The SNARKs derived from pairings have the advantage of constant proof sizes, which is crucial to blockchains.

We start out by defining elliptic curve pairings and discussing a simple application which bears some resemblance to more advanced SNARKs. We then introduce the pairings arising from elliptic curves and describe Miller's algorithm, which makes these pairings practical rather than just theoretically interesting.

Elliptic curves have a few structures, like the Weil or the Tate map that qualifies as pairing.

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Embedding Degrees As we will see in what follows, every elliptic curve gives rise to a pairing map. However, we will also see in example XXX that not every such pairing can be efficiently computed. In order to distinguish curves with efficiently computable pairings from the rest, we need to start with an introduction to the so-called **embedding degree** of a curve.

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ence

Definition 5.2.0.1. Embedding degree

Let \mathbb{F} be a finite field, let $E(\mathbb{F})$ be an elliptic curve over \mathbb{F} , and let n be a prime number that divides the order of $E(\mathbb{F})$. The embedding degree of $E(\mathbb{F})$ with respect to n is then the smallest integer k such that n divides $q^k - 1$.

Fermat's little theorem (page 21 ff.) implies that every curve has at least **some** embedding degree k , since at least $k = n - 1$ is always a solution to the congruency $q^k \equiv 1 \pmod{n}$. This implies that the remainder of the integer division of $q^k - 1$ by n is 0.

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Example 83. To get a better intuition of the embedding degree, let's consider the elliptic curve $E_1(\mathbb{F}_5)$ from example 67. We know from 67 that the order of $E_1(\mathbb{F}_5)$ is 9, and, since the only prime factor of 9 is 3, we compute the embedding degree of $E_1(\mathbb{F}_5)$ with respect to 3.

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To find the embedding degree, we have to find the smallest integer k such that 3 divides $q^k - 1 = 5^k - 1$. We try and increment until we find a proper k .

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$$k = 1: 5^1 - 1 = 4$$

not divisible by 3

$$k = 2: 5^2 - 1 = 24$$

divisible by 3

Now we know that the embedding degree of $E_1(\mathbb{F}_5)$ is 2 relative to the prime factor 3.

Example 84. Let us consider the tiny-jubjub curve TJJ_13 from example 68. We know from 68 that the order of TJJ_13 is 20, and that the order therefore has two prime factors. A "large" prime factor 5 and a small prime factor 2.

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We start by computing the embedding degree of TJJ_13 with respect to the large prime factor 5. To find that embedding degree, we have to find the smallest integer k such that 5 divides $q^k - 1 = 13^k - 1$. We try and increment until we find a proper k .

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$$k = 1: 13^1 - 1 = 12$$

not divisible by 5

$$k = 2: 13^2 - 1 = 168$$

not divisible by 5

$$k = 3: 13^3 - 1 = 2196$$

not divisible by 5

$$k = 4: 13^4 - 1 = 28560$$

divisible by 5

Now we know that the embedding degree of TJJ_13 is 4 relative to the prime factor 5.

In real-world applications, like on pairing-friendly elliptic curves such as BLS_12-381, usually only the embedding degree of the large prime factor is relevant, which in the case of our tiny-jubjub curve is represented by 5. It should be noted, however that every prime factor of a curve's order has its own notation of embedding degree despite the fact that this is mostly irrelevant in applications.

To find the embedding degree of the small prime factor 2, we have to find the smallest integer k such that 2 divides $q^k - 1 = 13^k - 1$. We try and increment until we find a proper k .

$$k = 1: 13^1 - 1 = 12 \quad \text{divisible by 2}$$

Now we know that the embedding degree of TJJ_13 is 1 relative to the prime factor 2. As we have seen, different prime factors can have different embedding degrees in general.

```

sage: p = 13
sage: # large prime factor
sage: n = 5
sage: for k in range(1,5): # Fermat's little theorem
.....:     if (p^k-1)%n == 0:
.....:         break
sage: k
4
sage: # small prime factor
sage: n = 2
sage: for k in range(1,2): # Fermat's little theorem
.....:     if (p^k-1)%n == 0:
.....:         break
sage: k
1

```

Example 85. To give an example of a cryptographically secure real-world elliptic curve that does not have a small embedding degree, let's look at curve Secp256k1 again. We know from 69 that the order of this curve is a prime number, so we only have a single embedding degree.

To test potential embedding degrees k , say, in the range $1 \dots 1000$, we can invoke Sage and compute as follows:

```

sage: p = 1157920892373161954235709850086879078532699846656405
      64039457584007908834671663
sage: n = 1157920892373161954235709850086879078528375642790749
      04382605163141518161494337
sage: for k in range(1,1000):
.....:     if (p^k-1)%n == 0:
.....:         break
sage: k
999

```

We see that Secp256k1 has at least no embedding degree $k < 1000$, which renders Secp256k1 a curve that has no small embedding degree. This property will be of importance later on.

Elliptic Curves over extension fields Suppose that p is a prime number, and \mathbb{F}_p its associated prime field. We know from equation 4.34 that the fields \mathbb{F}_{p^m} are extensions of \mathbb{F}_p in the sense

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that \mathbb{F}_p is a subfield of \mathbb{F}_{p^m} . This implies that we can extend the affine plane that an elliptic curve is defined on by changing the base field to any extension field. To be more precise, let $E(\mathbb{F}) = \{(x, y) \in \mathbb{F} \times \mathbb{F} \mid y^2 = x^3 + a \cdot x + b\}$ be an affine short Weierstraß curve, with parameters a and b taken from \mathbb{F} . If \mathbb{F}' is an extension field of \mathbb{F} , then we extend the domain of the curve by defining $E(\mathbb{F}')$ as follows:

$$E(\mathbb{F}') = \{(x, y) \in \mathbb{F}' \times \mathbb{F}' \mid y^2 = x^3 + a \cdot x + b\} \quad (5.23)$$

While we did not change the defining parameters, we consider curve points from the affine plane over the extension field now. Since $\mathbb{F} \subset \mathbb{F}'$, it can be shown that the original elliptic curve $E(\mathbb{F})$ is a sub-curve of the extension curve $E(\mathbb{F}')$.

Example 86. Consider the prime field \mathbb{F}_5 from example 61 and the elliptic curve $E_1(\mathbb{F}_5)$ from example 67. Since we know from XXX that \mathbb{F}_{5^2} is an extension field of \mathbb{F}_5 , we can extend the definition of $E_1(\mathbb{F}_5)$ to define a curve over \mathbb{F}_{5^2} :

$$E_1(\mathbb{F}_{5^2}) = \{(x, y) \in \mathbb{F} \times \mathbb{F} \mid y^2 = x^3 + x + 1\}$$

Since \mathbb{F}_{5^2} contains 25 points, in order to compute the set $E_1(\mathbb{F}_{5^2})$, we have to try $25 \cdot 25 = 625$ pairs, which is probably a bit too much for the average motivated reader. Instead, we invoke Sage to compute the curve for us. To do, we so choose the representation of \mathbb{F}_{5^2} from XXX. We get:

```

3322 sage: F5= GF(5)                                     389
3323 sage: F5t.<t> = F5[]                                 390
3324 sage: P = F5t(t^2+2)                                 391
3325 sage: P.is_irreducible()                             392
3326 True                                                393
3327 sage: F5_2.<t> = GF(5^2, name='t', modulus=P)        394
3328 sage: E1F5_2 = EllipticCurve(F5_2, [1, 1])          395
3329 sage: E1F5_2.order()                                 396
3330 27                                                  397

```

The curve $E_1(\mathbb{F}_{5^2})$ consist of 27 points, in contrast to curve $E_1(\mathbb{F}_5)$, which consists of 9 points. Printing the points gives the following:

$$\begin{aligned}
 E_1(\mathbb{F}_{5^2}) = \{ & \mathcal{O}, (0, 4), (0, 1), (3, 4), (3, 1), (4, 3), (4, 2), (2, 4), (2, 1), \\
 & (4t + 3, 3t + 4), (4t + 3, 2t + 1), (3t + 2, t), (3t + 2, 4t), \\
 & (2t + 2, t), (2t + 2, 4t), (2t + 1, 4t + 4), (2t + 1, t + 1), \\
 & (2t + 3, 3), (2t + 3, 2), (t + 3, 2t + 4), (t + 3, 3t + 1), \\
 & (3t + 1, t + 4), (3t + 1, 4t + 1), (3t + 3, 3), (3t + 3, 2), (1, 4t) \}
 \end{aligned}$$

As we can see, curve $E_1(\mathbb{F}_5)$ sits inside curve $E(\mathbb{F}_{5^2})$, which is implied by \mathbb{F}_5 being a subfield of \mathbb{F}_{5^2} .

Full torsion groups The fundamental theorem of finite cyclic groups XXX implies that every prime factor n of a cyclic group's order defines a subgroup of the size of the prime factor. Such a subgroup is called an n -torsion group. We have seen many of those subgroups in the examples XXX and XXX.

When we consider elliptic curve extensions as defined in 5.23, we could ask what happens to the n -torsion groups in the extension. One might intuitively think that their extension just

parallels the extension of the curve. For example, when $E(\mathbb{F}_p)$ is a curve over prime field \mathbb{F}_p , with some n -torsion group \mathbb{G} and when we extend the curve to $E(\mathbb{F}_{p^m})$, then there is a bigger n -torsion group such that \mathbb{G} is a subgroup. This might make intuitive sense, as $E(\mathbb{F}_p)$ is a sub-curve of $E(\mathbb{F}_{p^m})$.

However, the actual situation is a bit more surprising than that. To see that, let \mathbb{F}_p be a prime field and let $E(\mathbb{F}_p)$ be an elliptic curve of order r , with embedding degree k and n -torsion group $E(\mathbb{F}_p)[n]$ for the same prime factor n of r . Then it can be shown that the n -torsion group $E(\mathbb{F}_{p^m})[n]$ of a curve extension is equal to $E(\mathbb{F}_p)[n]$, as long as the power m is less than the embedding degree k of $E(\mathbb{F}_p)$.

However, for the prime power p^m , for any $m \geq k$, $E(\mathbb{F}_{p^m})[n]$ is strictly larger than $E(\mathbb{F}_p)[n]$ and contains $E(\mathbb{F}_p)[n]$ as a subgroup. We call the n -torsion group $E(\mathbb{F}_{p^k})[n]$ of the extension of E over \mathbb{F}_{p^k} the **full n -torsion group** of that elliptic curve. It can be shown that it contains n^2 many elements and consists of $n + 1$ subgroups, one of which is $E(\mathbb{F}_p)[n]$.

So, roughly speaking, when we consider **towers of curve extensions** $E(\mathbb{F}_{p^m})$ ordered by the prime power m , then the n -torsion group stays constant for every level m , that is smaller than the embedding degree, while it suddenly blossoms into a larger group on level k with $n + 1$ subgroups, and then stays like that for any level m larger than k . In other words, once the extension field is big enough to find one more point of order n (that is not defined over the base field), then we actually find all of the points in the full torsion group.

Example 87. Consider curve $E_1(\mathbb{F}_5)$ again. We know that it contains a 3-torsion group and that the embedding degree of 3 is 2. From this we can deduce that we can find the full 3-torsion group $E_1[3]$ in the curve extension $E_1(\mathbb{F}_{5^2})$, the latter of which we computed in example 86.

Since that curve is small, in order to find the full 3-torsion, we can loop through all elements of $E_1(\mathbb{F}_{5^2})$ and check the defining equation $[3]P = \mathcal{O}$. Invoking Sage, we compute as follows:

```
sage: INF = E1F5_2(0) # Point at infinity          398
sage: L_E1_3 = []                                     399
sage: for p in E1F5_2:                               400
.....:     if 3*p == INF:                           401
.....:         L_E1_3.append(p)                       402
sage: E1_3 = Set(L_E1_3) # Full 3-torsion set        403
```

We get the following result:

$$E_1[3] = \{\mathcal{O}, (1, t), (1, 4t), (2, 1), (2, 4), (2t + 1, t + 1), (2t + 1, 4t + 4), (3t + 1, t + 4), (3t + 1, 4t + 1)\}$$

Example 88. Consider the tiny-jubjub curve from example 68. We know from example 84 that it contains a 5-torsion group and that the embedding degree of 5 is 4. This implies that we can find the full 5-torsion group $TJJ_13[5]$ in the curve extension $TJJ_13(\mathbb{F}_{13^4})$.

To compute the full torsion, first observe that, since \mathbb{F}_{13^4} contains 28561 elements, computing $TJJ_13(\mathbb{F}_{13^4})$ means checking $28561^2 = 815730721$ elements. From each of these curve points P , we then have to check the equation $[5]P = \mathcal{O}$. Doing this for 815730721 is a bit too slow even on a computer.

Fortunately, Sage has a way to loop through points of a given order efficiently. The following Sage code provides a way to compute the full torsion group:

```
sage: # define the extension field                    404
sage: F13= GF(13) # prime field                      405
sage: F13t.<t> = F13[] # polynomials over t          406
```

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```

3382 sage: P = F13t(t^4+2) # irreducible polynomial of degree 4 407
3383 sage: P.is_irreducible() 408
3384 True 409
3385 sage: F13_4.<t> = GF(13^4, name='t', modulus=P) # F_{13^4} 410
3386 sage: TJJF13_4 = EllipticCurve(F13_4,[8,8]) # tiny-jubjub 411
3387     extension
3388 sage: # compute the full 5-torsion 412
3389 sage: L_TJJF13_4_5 = [] 413
3390 sage: INF = TJJF13_4(0) 414
3391 sage: for P in INF.division_points(5): # [5]P == INF 415
3392     ....:     L_TJJF13_4_5.append(P) 416
3393 sage: len(L_TJJF13_4_5) 417
3394 25 418
3395 sage: TJJF13_4_5 = Set(L_TJJF13_4_5) 419

```

As expected, we get a group that contains $5^2 = 25$ elements. As it's rather tedious to write this group down, and as we don't need it in what follows, we forgo doing this. To see that the embedding degree 4 is actually the smallest prime power to find the full 5-torsion group, let's compute the 5-torsion group over of the tiny-jubjub curve of the extension field \mathbb{F}_{13^3} . We get the following:

```

3401 sage: # define the extension field 420
3402 sage: P = F13t(t^3+2) # irreducible polynomial of degree 3 421
3403 sage: P.is_irreducible() 422
3404 True 423
3405 sage: F13_3.<t> = GF(13^3, name='t', modulus=P) # F_{13^3} 424
3406 sage: TJJF13_3 = EllipticCurve(F13_3,[8,8]) # tiny-jubjub 425
3407     extension
3408 sage: # compute the 5-torsion 426
3409 sage: L_TJJF13_3_5 = [] 427
3410 sage: INF = TJJF13_3(0) 428
3411 sage: for P in INF.division_points(5): # [5]P == INF 429
3412     ....:     L_TJJF13_3_5.append(P) 430
3413 sage: len(L_TJJF13_3_5) 431
3414 5 432
3415 sage: TJJF13_3_5 = Set(L_TJJF13_3_5) # full 5-torsion 433

```

As we can see, the 5-torsion group of tiny-jubjub over \mathbb{F}_{13^3} is equal to the 5-torsion group of tiny-jubjub over \mathbb{F}_{13} itself.

Example 89. Let's look at the curve Secp256k1. We know from example 69 that the curve is of some prime order r . Because of this, the only n -torsion group to consider is the curve itself, so the curve group is the r -torsion.

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However, in order to find the full r -torsion of Secp256k1, we need to compute the embedding degree k . And as we have seen in XXX it is at least not small. However, we know from Fermat's little theorem (page 21 ff.) that a finite embedding degree must exist. It can be shown that it is given by the following 256-bit number:

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$$k = 192986815395526992372618308347813175472927379845817397100860523586360249056$$

This means that the embedding degree is **very large**, which implies that the field extension \mathbb{F}_{p^k} is very large too. To understand how big \mathbb{F}_{p^k} is, recall that an element of \mathbb{F}_{p^m} can be represented as

3423 a string $[x_0, \dots, x_m]$ of m elements, each containing a number from the prime field \mathbb{F}_p . Now, in
 3424 the case of Secp256k1, such a representation has k -many entries, each of them 256 bits in size.
 3425 So, without any optimizations, representing such an element would need $k \cdot 256$ bits, which is
 3426 too much to be represented in the observable universe.

3427 **Torsion subgroups** As we have stated above, any full n -torsion group contains $n + 1$ cyclic
 3428 subgroups, two of which are of particular interest in pairing-based elliptic curve cryptography.
 3429 To characterize these groups, we need to consider the so-called **Frobenius endomorphism** of
 3430 an elliptic curve $E(\mathbb{F})$ over some finite field \mathbb{F} of characteristic p :

$$\pi : E(\mathbb{F}) \rightarrow E(\mathbb{F}) : \begin{array}{ccc} (x, y) & \mapsto & (x^p, y^p) \\ \mathcal{O} & \mapsto & \mathcal{O} \end{array} \quad (5.24)$$

3431 It can be shown that π maps curve points to curve points. The first thing to note is that, in case
 3432 \mathbb{F} is a prime field, the Frobenius endomorphism acts trivially, since $(x^p, y^p) = (x, y)$ on prime
 3433 fields due to Fermat's little theorem (page 21 ff.). This means that the Frobenius map is more
 3434 interesting over prime field extensions.

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3435 With the Frobenius map at hand, we can characterize two important subgroups of the full
 3436 n -torsion. The first subgroup is the n -torsion group that already exists in the curve over the
 3437 base field. In pairing-based cryptography, this group is usually written as \mathbb{G}_1 , assuming that the
 3438 prime factor n in the definition is implicitly given. Since we know that the Frobenius map acts
 3439 trivially on curves over the prime field, we can define \mathbb{G}_1 as follows:

$$\mathbb{G}_1[n] := \{(x, y) \in E[n] \mid \pi(x, y) = (x, y)\} \quad (5.25)$$

3440 In more mathematical terms, this definition means that \mathbb{G}_1 is the **Eigenspace** of the Frobenius
 3441 map with respect to the **Eigenvalue** 1.

3442 It can be shown that there is another subgroup of the full n -torsion group that can be char-
 3443 acterized by the Frobenius map. In the context of so-called **type 3 pairing-based cryptography**,
 3444 this subgroup is usually called \mathbb{G}_2 and it is defined as follows:

$$\mathbb{G}_2[n] := \{(x, y) \in E[n] \mid \pi(x, y) = [p](x, y)\} \quad (5.26)$$

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ation or
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footnote

3445 In mathematical terms, \mathbb{G}_2 is the **Eigenspace** of the Frobenius map with respect to the
 3446 **Eigenvalue** p .

type 3
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based
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phy

3447 *Notation and Symbols* 9. If the prime factor n of a curve's order is clear from the context, we
 3448 sometimes simply write \mathbb{G}_1 and \mathbb{G}_2 to mean $\mathbb{G}_1[n]$ and $\mathbb{G}_2[n]$, respectively.

3449 It should be noted, however that other definitions of \mathbb{G}_2 also exists in the literature. However,
 3450 in the context of pairing-based cryptography, this is the most common one. It is particularly
 3451 useful because we can define hash functions that map into \mathbb{G}_2 , which is not possible for all
 3452 subgroups of the full n -torsion.

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ences?

3453 *Example 90.* Consider the curve $E_1(\mathbb{F}_5)$ from example 67 again. As we have seen, this curve
 3454 has the embedding degree $k = 2$, and a full 3-torsion group is given as follows:

$$\begin{aligned} E_1[3] = \{ & \mathcal{O}, (2, 1), (2, 4), (1, t), (1, 4t), (2t + 1, t + 1), \\ & (2t + 1, 4t + 4), (3t + 1, t + 4), (3t + 1, 4t + 1) \} \end{aligned} \quad (5.27)$$

3455 According to the general theory, $E_1[3]$ contains 4 subgroups, and we can characterize the
 3456 subgroups \mathbb{G}_1 and \mathbb{G}_2 using the Frobenius endomorphism. Unfortunately, at the time of writing,

Sage does not have a predefined Frobenius endomorphism for elliptic curves, so we have to use the Frobenius endomorphism of the underlying field as a temporary workaround. We compute as follows:

```

3460 sage: L_G1 = []
3461 sage: for P in E1_3:
3462     ....:     PiP = E1F5_2([a.frobenius() for a in P]) # pi(P)
3463     ....:     if P == PiP:
3464     ....:         L_G1.append(P)
3465 sage: G1 = Set(L_G1)

```

As expected, the group $\mathbb{G}_1 = \{\mathcal{O}, (2, 4), (2, 1)\}$ is identical to the 3-torsion group of the (unextended) curve over the prime field $E_1(\mathbb{F}_5)$. We can use almost the same algorithm to compute the group \mathbb{G}_2 and get the following:

```

3469 sage: L_G2 = []
3470 sage: for P in E1_3:
3471     ....:     PiP = E1F5_2([a.frobenius() for a in P]) # pi(P)
3472     ....:     pP = 5*P # [5]P
3473     ....:     if pP == PiP:
3474     ....:         L_G2.append(P)
3475 sage: G2 = Set(L_G2)

```

Thus, we have computed the the second subgroup of the full 3-torsion group of curve E_1 as the set $\mathbb{G}_2 = \{\mathcal{O}, (1, t), (1, 4t)\}$.

Example 91. Consider the tiny-jubjub curve *TJJ_13* from example 68. In example 88, we computed its full 5 torsion, which is a group that has 6 subgroups. We compute G_1 using Sage as follows:

```

3481 sage: L_TJJ_G1 = []
3482 sage: for P in TJJF13_4_5:
3483     ....:     PiP = TJJF13_4([a.frobenius() for a in P]) # pi(P)
3484     ....:     if P == PiP:
3485     ....:         L_TJJ_G1.append(P)
3486 sage: TJJ_G1 = Set(L_TJJ_G1)

```

We get $\mathbb{G}_1 = \{\mathcal{O}, (7, 2), (8, 8), (8, 5), (7, 11)\}$

```

3488 sage: L_TJJ_G1 = []
3489 sage: for P in TJJF13_4_5:
3490     ....:     PiP = TJJF13_4([a.frobenius() for a in P]) # pi(P)
3491     ....:     pP = 13*P # [5]P
3492     ....:     if pP == PiP:
3493     ....:         L_TJJ_G1.append(P)
3494 sage: TJJ_G1 = Set(L_TJJ_G1)

```

$\mathbb{G}_2 = \{\mathcal{O}, (9t^2 + 7, t^3 + 11t), (9t^2 + 7, 12t^3 + 2t), (4t^2 + 7, 5t^3 + 10t), (4t^2 + 7, 8t^3 + 3t)\}$

Example 92. Consider Bitcoin's curve Secp256k1 again. Since the group \mathbb{G}_1 is identical to the torsion group of the unextended curve, and since Secp256k1 has prime order, we know that, in this case, \mathbb{G}_1 is identical to Secp256k1. It is however, infeasible not to compute not only \mathbb{G}_2 itself, but to even compute an average element of \mathbb{G}_2 , as elements need too much storage to be

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representable in this universe.

The Weil pairing In this part, we consider a pairing function defined on the subgroups $\mathbb{G}_1[r]$ and $\mathbb{G}_2[r]$ of the full r -torsion $E[r]$ of a short Weierstraß elliptic curve. To be more precise, let $E(\mathbb{F}_p)$ be an elliptic curve of embedding degree k such that r is a prime factor of its order. Then the **Weil pairing** is a bilinear, non-degenerate map:

$$e(\cdot, \cdot) : \mathbb{G}_1[r] \times \mathbb{G}_2[r] \rightarrow \mathbb{F}_{p^k} ; (P, Q) \mapsto (-1)^r \cdot \frac{f_{r,P}(Q)}{f_{r,Q}(P)} \quad (5.28)$$

The extension field elements $f_{r,P}(Q), f_{r,Q}(P) \in \mathbb{F}_{p^k}$ are computed by **Miller's algorithm**:

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Algorithm 7 Miller's algorithm for short Weierstraß curves $y^2 = x^3 + ax + b$

Require: $r > 3, P \in E[r], Q \in E[r]$ and

$b_0, \dots, b_t \in \{0, 1\}$ with $r = b_0 \cdot 2^0 + b_1 \cdot 2^1 + \dots + b_t \cdot 2^t$ and $b_t = 1$

procedure MILLER'S ALGORITHM(P, Q)

if $P = \mathcal{O}$ or $Q = \mathcal{O}$ or $P = Q$ **then**

return $f_{r,P}(Q) \leftarrow (-1)^r$

end if

$(x_T, y_T) \leftarrow (x_P, y_P)$

$f_1 \leftarrow 1$

$f_2 \leftarrow 1$

for $j \leftarrow t - 1, \dots, 0$ **do**

$m \leftarrow \frac{3 \cdot x_T^2 + a}{2 \cdot y_T}$

$f_1 \leftarrow f_1^2 \cdot (y_Q - y_T - m \cdot (x_Q - x_T))$

$f_2 \leftarrow f_2^2 \cdot (x_Q + 2x_T - m^2)$

$x_{2T} \leftarrow m^2 - 2x_T$

$y_{2T} \leftarrow -y_T - m \cdot (x_{2T} - x_T)$

$(x_T, y_T) \leftarrow (x_{2T}, y_{2T})$

if $b_j = 1$ **then**

$m \leftarrow \frac{y_T - y_P}{x_T - x_P}$

$f_1 \leftarrow f_1 \cdot (y_Q - y_T - m \cdot (x_Q - x_T))$

$f_2 \leftarrow f_2 \cdot (x_Q + (x_P + x_T) - m^2)$

$x_{T+P} \leftarrow m^2 - x_T - x_P$

$y_{T+P} \leftarrow -y_T - m \cdot (x_{T+P} - x_T)$

$(x_T, y_T) \leftarrow (x_{T+P}, y_{T+P})$

end if

end for

$f_1 \leftarrow f_1 \cdot (x_Q - x_T)$

return $f_{r,P}(Q) \leftarrow \frac{f_1}{f_2}$

end procedure

Understanding how the algorithm works in detail requires the concept of **divisors**, which is outside of the scope this book. The interested reader might look at XXX.

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ences

In real-world applications of pairing-friendly elliptic curves, the embedding degree is usually a small number like 2, 4, 6 or 12, and the number r is the largest prime factor of the curve's order.

3511 *Example 93.* Consider curve $E_1(\mathbb{F}_5)$ from example 67. Since the only prime factor of the
 3512 group's order is 3, we cannot compute the Weil pairing on this group using our definition of
 3513 Miller's algorithm. In fact, since \mathbb{G}_1 is of order 3, executing the if statement on line XXX will
 3514 lead to a "division by zero" error in the computation of the slope m .

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Example 94. Consider the tiny-jubjub curve $TJJ_13(\mathbb{F}_{13})$ from example 68 again. We want to
 instantiate the general definition of the Weil pairing for this example. To do so, recall that, as we
 have see in example 84, its embedding degree is 4, and that we have the following type-3 pairing
 groups (where \mathbb{G}_1 and \mathbb{G}_2 are subgroups of the full 5-torsion found in the curve $TJJ_13(\mathbb{F}_{13^4})$):

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$$\mathbb{G}_1 = \{\mathcal{O}, (7, 2), (8, 8), (8, 5), (7, 11)\}$$

$$\mathbb{G}_2 = \{\mathcal{O}, (9t^2 + 7, t^3 + 11t), (9t^2 + 7, 12t^3 + 2t), (4t^2 + 7, 5t^3 + 10t), (4t^2 + 7, 8t^3 + 3t)\}$$

3515 The type-3 Weil pairing is a map $e(\cdot, \cdot) : \mathbb{G}_1 \times \mathbb{G}_2 \rightarrow \mathbb{F}_{13^4}$. From the first if-statement in
 3516 Miller's algorithm, we can deduce that $e(\mathcal{O}, Q) = 1$ as well as $e(P, \mathcal{O}) = 1$ for all arguments
 3517 $P \in \mathbb{G}_1$ and $Q \in \mathbb{G}_2$. In order to compute a non-trivial Weil pairing, we choose the arguments
 3518 $P = (7, 2)$ and $Q = (9t^2 + 7, 12t^3 + 2t)$.

3519 To compute the pairing $e((7, 2), (9t^2 + 7, 12t^3 + 2t))$, we have to compute the extension field
 3520 elements $f_{5,P}(Q)$ and $f_{5,Q}(P)$ by applying Miller's algorithm. Do do so, observe that we have
 3521 $5 = 1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2$, so we get $t = 2$ as well as $b_0 = 1, b_1 = 0$ and $b_2 = 1$. The loop therefore
 3522 needs to be executed two times.

Computing $f_{5,P}(Q)$, we initiate $(x_T, y_T) = (7, 2)$ as well as $f_1 = 1$ and $f_2 = 1$. Then we
 proceed as follows:

j	b_j	m	f_1	f_2	x_{2T}	y_{2T}	x_{T+P}	y_{T+P}
1	.							

$$\begin{aligned}
m &= \frac{3 \cdot x_T^2 + a}{2 \cdot y_T} \\
&= \frac{3 \cdot 2^2 + 1}{2 \cdot 4} = \frac{3}{3} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
f_1 &= f_1^2 \cdot (y_Q - y_T - m \cdot (x_Q - x_T)) \\
&= 1^2 \cdot (t - 4 - 1 \cdot (1 - 2)) = t - 4 + 1 \\
&= t + 2
\end{aligned}$$

$$\begin{aligned}
f_2 &= f_2^2 \cdot (x_Q + 2x_T - m^2) \\
&= 1^2 \cdot (1 + 2 \cdot 2 - 1^2) = (1 + 4 - 1) \\
&= 4
\end{aligned}$$

$$\begin{aligned}
x_{2T} &= m^2 - 2x_T \\
&= 1^2 - 2 \cdot 2 = -3 \\
&= 2
\end{aligned}$$

$$\begin{aligned}
y_{2T} &= -y_T - m \cdot (x_{2T} - x_T) \\
&= -4 - 1 \cdot (2 - 2) = -4 \\
&= 1
\end{aligned}$$

We update $(x_T, y_T) = (2, 1)$ and, since $b_0 = 1$, we have to execute the if statement on line XXX in the **for** loop. However, since we only loop a single time, we don't need to compute y_{T+P} , since we only need the updated x_T in the final step. We get:

$$\begin{aligned}
m &= \frac{y_T - y_P}{x_T - x_P} \\
&= \frac{1 - 4}{2 - x_P}
\end{aligned}$$

$$f_1 = f_1 \cdot (y_Q - y_T - m \cdot (x_Q - x_T))$$

$$f_2 = f_2 \cdot (x_Q + (x_P + x_T) - m^2)$$

$$x_{T+P} = m^2 - x_T - x_P$$

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should all lines of all algorithms be numbered?

3523 5.3 Hashing to Curves

3524 Elliptic curve cryptography frequently requires the ability to hash data onto elliptic curves. If
3525 the order of the curve is not a prime number, hashing to prime number subgroups is also of

importance. In the context of pairing-friendly curves, it is also sometimes necessary to hash specifically onto the group \mathbb{G}_1 or \mathbb{G}_2 .

As we have seen in section 4.1.2, many general methods are known for hashing into groups in general, and finite cyclic groups in particular. As elliptic groups are cyclic, those methods can be utilized in this case, too. However, in what follows we want to describe some methods specific to elliptic curves that are frequently used in real-world applications.

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Try-and-increment hash functions One of the most straight-forward ways of hashing a bit-string onto an elliptic curve point in a secure way is to use a cryptographic hash function together with one of the methods we described in section 4.1.2 to hash to the modular arithmetic base field of the curve. Ideally, the hash function generates an image that is at least one bit longer than the bit representation of the base field modulus.

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The image in the base field can then be interpreted as the x coordinate of the curve point, and the two possible y coordinates are derived from the curve equation, while one of the bits that exceeded the modulus determines which of the two y coordinates to choose.

Such an approach would be deterministic and easy to implement, and it would conserve the cryptographic properties of the original hash function. However, not all x coordinates generated in such a way will result in quadratic residues when inserted into the defining equation. It follows that not all field elements give rise to actual curve points. In fact, on a prime field, only half of the field elements are quadratic residues. Hence, assuming an even distribution of the hash values in the field, this method would fail to generate a curve point in about half of the attempts.

One way to account for this problem is the so-called **try-and-increment** method. Its basic assumption is that, when hashing different values, the result will eventually lead to a valid curve point.

Therefore, instead of simply hashing a string s to the field, we hash the concatenation of s with additional bytes to the field instead. In other words, we use a try-and-increment hash as described in 5. If the first try of hashing to the field does not result in a valid curve point, the counter is incremented, and the hashing is repeated again. This is done until a valid curve point is found.

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This method has a number of advantages: It is relatively easy to implement in code, and it maintains the cryptographic properties of the original hash function. However, it is not guaranteed to find a valid curve point, as there is a chance that all possible values in the chosen size of the counter will fail to generate a quadratic residue. Fortunately, it is possible to make the probability for this arbitrarily small by choosing large enough counters and relying on the (approximate) uniformity of the hash-to-field function.

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If the curve is not of prime order, the result will be a general curve point that might not be in the “large” prime-order subgroup. In this case, a **cofactor clearing** step is then necessary to project the curve point onto the subgroup. This is done by scalar multiplication with the cofactor of prime order with respect to the curves order.

Example 95. Consider the tiny-jubjub curve from example 68. We want to construct a try-and-increment hash function that hashes a binary string s of arbitrary length onto the large prime-order subgroup of size 5.

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Since the curve, as well as our targeted subgroup, is defined over the field \mathbb{F}_{13} , and the binary representation of 13 is $13.bits() = 1101$, we apply SHA256 from Sage’s hashlib library on the concatenation $s||c$ for some binary counter string, and use the first 4 bits of the image to try to hash into \mathbb{F}_{13} . In case we are able to hash to a value z such that $z^3 + 8 \cdot z + 8$ is a quadratic residue in \mathbb{F}_{13} , we use the 5-th bit to decide which of the two possible roots of $z^3 + 8 \cdot z + 8$ we

Algorithm 8 Hash-to- $E(\mathbb{F}_r)$ **Require:** $r \in \mathbb{Z}$ with $r.\text{nbits}() = k$ and $s \in \{0, 1\}^*$ **Require:** Curve equation $y^2 = x^3 + ax + b$ over \mathbb{F}_r **procedure** TRY-AND-INCREMENT(r, k, s) $c \leftarrow 0$ **repeat** $s' \leftarrow s || c.\text{bits}()$ $z \leftarrow H(s')_0 \cdot 2^0 + H(s')_1 \cdot 2^1 + \dots + H(s')_k \cdot 2^k$ $x \leftarrow z^3 + a \cdot z + b$ $c \leftarrow c + 1$ **until** $z < r$ and $x^{\frac{r-1}{2}} \bmod r = 1$ **if** $H(s')_{k+1} == 0$ **then** $y \leftarrow \sqrt{x}$ # (root in \mathbb{F}_r)**else** $y \leftarrow r - \sqrt{x}$ # (root in \mathbb{F}_r)**end if****return** (x, y) **end procedure****Ensure:** $(x, y) \in E(\mathbb{F}_r)$

will choose as the y coordinate. The result is a curve point different from the point at infinity. To project it to a point of \mathbb{G}_1 , we multiply it with the cofactor 4. If the result is still not the point at infinity, it is the result of the hash.

To make this concrete, let $s = '10011001111010110100000111'$ be our binary string that we want to hash onto \mathbb{G}_1 . We use a 4-bit binary counter starting at zero, that is, we choose $c = 0000$. Invoking Sage, we define the try-hash function as follows:

```

sage: import hashlib
sage: def try_hash(s, c):
.....:     s_1 = s+c
.....:     hasher = hashlib.sha256(s_1.encode('utf-8'))
.....:     digest = hasher.hexdigest()
.....:     d = Integer(digest, base=16)
.....:     sign = d.str(2)[-5:-4]
.....:     d = d.str(2)[-4:]
.....:     z = Integer(d, base=2)
.....:     return (z, sign)
sage: try_hash('10011001111010110100000111', '0000')
(15, '1')
```

As we can see, our first attempt to hash into \mathbb{F}_{13} was not successful, as 15 is not a number in \mathbb{F}_{13} , so we increment the binary counter by 1 and try again:

```

sage: try_hash('10011001111010110100000111', '0001')
(3, '0')
```

With this try, we found a hash into \mathbb{F}_{13} . However, this point is not guaranteed to define a curve point. To see that, we insert $z = 3$ into the right side of the Weierstraß equation of the tiny-jubjub curve, and compute $3^3 + 8 * 3 + 8 = 7$. However, 7 is not a quadratic residue in

3598 \mathbb{F}_{13} , since $7^{\frac{13-1}{2}} = 7^6 = 12 = -1$. This means that 3 is not a suitable point, and we have to
 3599 increment the counter two more times:

```
3600 sage: try_hash('10011001111010110100000111', '0010') 474
3601 (3, '0') 475
3602 sage: try_hash('10011001111010110100000111', '0011') 476
3603 (6, '1') 477
```

Since $6^3 + 8 \cdot 6 + 8 = 12$, and we have $\sqrt{12} \in \{5, 8\}$, we finally found the valid x coordinate $x = 6$ for the curve point hash. Now, since the sign bit of this hash is 1, we choose the larger root $y = 8$ as the y coordinate and get the following hash which is a valid curve point point on the tiny-jubjub curve:

$$H('10011001111010110100000111') = (6, 8)$$

In order to project this onto the “large” prime-order subgroup, we have to do cofactor clearing, that is, we have to multiply the point with the cofactor 4. We get the following:

$$[4](6, 8) = \mathcal{O}$$

3604 This means that the hash value is still not right. We therefore have to increment the counter
 3605 two more times again, until we finally find a correct hash to \mathbb{G}_1 :

```
3606 sage: try_hash('10011001111010110100000111', '0100') 478
3607 (0, '1') 479
3608 sage: try_hash('10011001111010110100000111', '0101') 480
3609 (12, '0') 481
```

Since $12^3 + 8 \cdot 12 + 8 = 12$, and we have $\sqrt{12} \in \{5, 8\}$, we found another valid x coordinate $x = 12$ for the curve point hash. Since the sign bit of this hash is 0, we choose the smaller root $y = 5$ as the y coordinate, and get the following hash, which is a valid curve point point on the tiny-jubjub curve:

$$H('10011001111010110100000111') = (12, 5)$$

In order to project this onto the “large” prime-order subgroup we have to do cofactor clearing, again? that is, we have to multiply the point with the cofactor 4. We get the following:

$$[4](12, 5) = (8, 5)$$

3610 So, hashing the binary string '10011001111010110100000111' onto \mathbb{G}_1 gives the hash value
 3611 (8, 5) as a result.

3612 5.4 Constructing elliptic curves

3613 Cryptographically secure elliptic curves like Secp256k1 from example 69 have been known for
 3614 quite some time. Given the latest advancements of cryptography, however, it is often necessary
 3615 to design and instantiate elliptic curves from scratch that satisfy certain very specific properties.

3616 For example, in the context of SNARK development, it was necessary to design a curve that
 3617 can be efficiently implemented inside of a so-called **circuit** in order to enable primitives like
 3618 elliptic curve **signature schemes** in a zero-knowledge proof. Such a curve is given by the Baby-
 3619 jubjub curve in XXX, and we have paralleled its definition by introducing the tiny-jubjub curve

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from example 68. As we have seen, those curves are instances of so-called twisted Edwards curves, and as such have easy to implement addition laws that work without branching. However, we introduced the tiny-jubjub curve out of thin air, as we just gave the curve parameters without explaining how we came up with them.

Another requirement in the context of many so-called **pairing-based zero-knowledge proofing systems** is the existence of a suitable, pairing-friendly curve with a specified security level and a low embedding degree as defined in 5.2.0.1. Famous examples are the BLS_12 and the NMT curves.

The major goal of this section is to explain the most important method of designing elliptic curves with predefined properties from scratch, called the **complex multiplication method**. We will apply this method in section XXX to synthesize a particular BLS_6 curve, which is one of the most insecure curves, but it will serve as the main curve to build our pen-and-paper SNARKs on. As we will see, this curve has a “large” prime factor subgroup of order 13, which implies that we can use our tiny-jubjub curve to implement certain elliptic curve cryptographic primitives in circuits over that BLS_6 curve.

Before we introduce the complex multiplication method, we have to explain a few properties of elliptic curves that are of key importance in understanding the complex multiplication method.

The Trace of Frobenius To understand the complex multiplication method of elliptic curves, we have to define the so-called **trace** of an elliptic curve first.

We know from XXX that elliptic curves over finite fields are products of cyclic groups of finite order. Therefore, an interesting question is whether it is possible to estimate the number of elements that this curve contains. Since an affine short Weierstraß curve consists of pairs (x, y) of elements from a finite field \mathbb{F}_q plus the point at infinity, and the field \mathbb{F}_q contains q elements, the number of curve points cannot be arbitrarily large, since it can contain at most $q^2 + 1$ many elements.

There is however, a more precise estimation, usually called the **Hasse bound**. To understand it, let $E(\mathbb{F}_q)$ be an affine short Weierstraß curve over a finite field \mathbb{F}_q of order q , and let $|E(\mathbb{F}_q)|$ be the order of the curve. Then there is an integer $t \in \mathbb{Z}$, called the **trace of Frobenius** of the curve, such that $|t| \leq 2\sqrt{q}$ and the following equation holds:

$$|E(\mathbb{F}_q)| = q + 1 - t \quad (5.29)$$

A positive trace, therefore, implies that the curve contains less points than the underlying field, whereas a negative trace means that the curve contains more points. However, the estimation $|t| \leq 2\sqrt{q}$ implies that the difference is not very large in either direction, and the number of elements in an elliptic curve is always approximately in the same order of magnitude as the size of the curve’s base field.

Example 96. Consider the elliptic curve $E_1(\mathbb{F}_5)$ from example 67. We know that it contains 9 curve points. Since the order of \mathbb{F}_5 is 5, we compute the trace of $E_1(\mathbb{F})$ to be $t = -3$, since the Hasse bound is given by the following equation:

$$9 = 5 + 1 - (-3)$$

Indeed, we have $|t| \leq 2\sqrt{q}$, since $\sqrt{5} > 2.23$ and $|-3| = 3 \leq 4.46 = 2 \cdot 2.23 < 2 \cdot \sqrt{5}$.

Example 97. To compute the trace of the tiny-jubjub curve, recall from example 76 that the order of TJJ_{13} is 20. Since the order of \mathbb{F}_{13} is 13, we can therefore use the Hasse bound and compute the trace as $t = -6$:

$$20 = 13 + 1 - (-6) \quad (5.30)$$

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3659 Again, we have $|t| \leq 2\sqrt{q}$, since $\sqrt{13} > 3.60$ and $|-6| = 6 \leq 7.20 = 2 \cdot 3.60 < 2 \cdot \sqrt{13}$.

Example 98. To compute the trace of Secp256k1, recall from example 69 [that this curve is](#) defined over a prime field with p elements, and that the order of that group is given by r :

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```
p = 115792089237316195423570985008687907853269984665640564039457584007908834671663
r = 115792089237316195423570985008687907852837564279074904382605163141518161494337
```

Using the Hesse bound $r = p + 1 - t$, we therefore compute $t = p + 1 - r$, which gives the trace of curve Secp256k1 as follows:

```
t = 432420386565659656852420866390673177327
```

3660 As we can see, Secp256k1 contains less elements than its underlying field. However, the
3661 difference is tiny, since the order of Secp256k1 is in the same order of magnitude as the order
3662 of the underlying field. Compared to p and r , t is tiny.

```
3663 sage: p = 1157920892373161954235709850086879078532699846656405 482
3664         64039457584007908834671663
3665 sage: r = 1157920892373161954235709850086879078528375642790749 483
3666         04382605163141518161494337
3667 sage: t = p + 1 - r 484
3668 sage: t.nbits() 485
3669 129 486
3670 sage: abs(RR(t)) <= 2*sqrt(RR(p)) 487
3671 True 488
```

3672 **The j -invariant** As we have seen in XXX, two elliptic curves $E_1(\mathbb{F})$ defined by $y^2 = x^3 + ax +$
3673 b and $E_2(\mathbb{F})$ defined by $y^2 + a'x + b'$ are strictly isomorphic if and only if there is a quadratic
3674 residue $d \in \mathbb{F}$ such that $a' = ad^2$ and $b' = bd^3$.

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3675 There is, however, a more general way to classify elliptic curves over finite fields \mathbb{F}_q , based
3676 on the so-called **j -invariant** of an elliptic curve with $j(E(\mathbb{F}_q)) \in \mathbb{F}_q$, as defined below:

$$j(E(\mathbb{F}_q)) = (1728 \bmod q) \frac{4 \cdot a^3}{4 \cdot a^3 + (27 \bmod q) \cdot b^2} \quad (5.31)$$

3677 A detailed description of the j -invariant is beyond the scope of this book. For our present
3678 purposes, it is sufficient to note that two elliptic curves $E_1(\mathbb{F})$ and $E_2(\mathbb{F}')$ are isomorphic over
3679 the **algebraic closures** of \mathbb{F} and \mathbb{F}' , if and only if $\overline{\mathbb{F}} = \overline{\mathbb{F}'}$ and $j(E_1) = j(E_2)$.

algebraic
closures

3680 So, the j -invariant is an important tool to classify elliptic curves and it is needed in the com-
3681 plex multiplication method to decide on an actual curve instantiation that implements abstractly
3682 chosen properties.

Example 99. Consider the elliptic curve $E_1(\mathbb{F}_5)$ from example 67. [We compute its \$j\$ -invariant](#) as follows:

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$$\begin{aligned} j(E_1(\mathbb{F}_5)) &= (1728 \bmod 5) \frac{4 \cdot 1^3}{4 \cdot 1^3 + (27 \bmod 5) \cdot 1^2} \\ &= 3 \frac{4}{4 + 2} \\ &= 3 \cdot 4 = 2 \end{aligned}$$

Example 100. Consider the elliptic curve *TJJ_13* from example 68. We compute its *j*-invariant as follows:

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$$\begin{aligned}
 j(E_1(\mathbb{F}_5)) &= (1728 \bmod 13) \frac{4 \cdot 8^3}{4 \cdot 8^3 + (27 \bmod 13) \cdot 8^2} \\
 &= 12 \cdot \frac{4 \cdot 5}{4 \cdot 5 + 1 \cdot 12} \\
 &= 12 \cdot \frac{7}{7 + 12} \\
 &= 12 \cdot 7 \cdot 6^{-1} \\
 &= 12 \cdot 7 \cdot 11 \\
 &01
 \end{aligned}$$

Example 101. Consider *Secp256k1* from example *Secp256k1*. We compute its *j*-invariant using Sage:

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```

3683 sage: p = 1157920892373161954235709850086879078532699846656405 489
3684      64039457584007908834671663
3685
3686 sage: F = GF(p) 490
3687
3688 sage: j = F(1728) * ( (F(4) * F(0) ^3) / (F(4) * F(0) ^3 + F(27) * F(7) ^2) ) 491
3689
3690 sage: j == F(0) 492
3691 True 493

```

The Complex Multiplication Method As we have seen in the previous sections, elliptic curves have various defining properties, like their order, their prime factors, the embedding degree, or the cardinality (number of elements) of the base field. The **complex multiplication** (CM) method provides a practical way of constructing elliptic curves with pre-defined restrictions on the order and the base field.

The method usually starts by choosing a base field \mathbb{F}_q of the curve $E(\mathbb{F}_q)$ we want to construct such that $q = p^m$ for some prime number p , and “ $m \in \mathbb{N}$ with $m \geq 1$. We assume $p > 3$ to simplify things in what follows.

Next, the trace of Frobenius $t \in \mathbb{Z}$ of the curve is chosen such that p and t are coprime, that is, $\gcd(p, t) = 1$ holds true. The choice of t also defines the curve’s order r , since $r = p + 1 - t$ by the Hasse bound (equation 5.29), so choosing t will define the large order subgroup as well as all small cofactors. r has to be defined in such a way that the elliptic curve meets the security requirements of the application it is designed for.

check
reference

Note that the choice of p and t also determines the embedding degree k of any prime-order subgroup of the curve, since k is defined as the smallest number such that the prime order n divides the number $q^k - 1$.

$$\begin{aligned}
 D &< 0 \\
 D \bmod 4 &= 0 \text{ or } D \bmod 4 = 1 \\
 4q &= t^2 + |D|v^2
 \end{aligned} \tag{5.32}$$

In order for the complex multiplication method to work, neither q nor t can be arbitrary, but must be chosen in such a way that two additional integers $D \in \mathbb{Z}$ and $v \in \mathbb{Z}$ exist and the following conditions hold:

If such numbers exist, we call D the **CM-discriminant**, and we know that we can construct a curve $E(\mathbb{F}_q)$ over a finite field \mathbb{F}_q such that the order of the curve is $|E(\mathbb{F}_q)| = q + 1 - t$.

It is the content of the complex multiplication method to actually construct such a curve, that is finding the parameters a and b from \mathbb{F}_q in the defining Weierstraß equation such that the curve has the desired order r .

Finding solutions to equation 5.29, can be achieved in different ways, but we will forego
the fine detail here. In general, it can be said that there are well-known constraints for elliptic curve families (e.g. the BLS (ECT) families) that provides families of solutions. In what follows, we will look at one type curve in the BLS-family, which gives an entire range of solutions. Are we looking at a subtype of BLS or is BLS the specific type we're referring to?

check
reference

disambiguate

Assuming that the proper parameters q , t , D and v are found, we have to compute the so-called **Hilbert class polynomial** $H_D \in \mathbb{Z}[x]$ of the CM-discriminant D , which is a polynomial with integer coefficients. To do so, we first have to compute the following set:

$$ICG(D) = \{(A, B, C) \mid A, B, C \in \mathbb{Z}, D = B^2 - 4AC, \gcd(A, B, C) = 1,$$

$$|B| \leq A \leq \sqrt{\frac{|D|}{3}}, A \leq C, \text{ if } B < 0 \text{ then } |B| < A < C\}$$

One way to compute this set is to first compute the integer $A_{max} = \text{Floor}(\sqrt{\frac{|D|}{3}})$, then loop through all the integers A in the range $[0, \dots, A_{max}]$, as well as through all the integers B in the range $[-A_{max}, \dots, A_{max}]$, then see if there is an integer C that satisfies $D = B^2 - 4AC$ and the rest of the requirements in XXX.

add refer-
ence

3724 To compute the Hilbert class polynomial, the so-called ***j*-function** (or *j*-invariant) is needed,
3725 which is a complex function defined on the upper half \mathbb{H} of the complex plane \mathbb{C} , usually written
3726 as follows:
3727 **is this the same as equation eq:j-invariant1** No they just sound very similar. But they
are very different?

unify terminology

$$j: \mathbb{H} \rightarrow \mathbb{C} \quad (5.33)$$

3728 Roughly speaking, what this means is that the j -functions takes complex numbers $(x + i \cdot y)$
3729 with a positive imaginary part $y > 0$ as inputs and returns a complex number $j(x + i \cdot y)$ as a
3730 result.

For the purposes of this book, it is not important to understand the j -function in detail, and we can use Sage to compute it in a similar way that we would use Sage to compute any other well-known function. It should be noted, however, that the computation of the j -function in Sage is sometimes prone to precision errors. For example, the j -function has a root in $\frac{-1+i\sqrt{3}}{2}$, which Sage only approximates. Therefore, when using Sage to compute the j -function, we need to take precision loss into account and possibly round to the nearest integer.

```

3737 sage: z = ComplexField(100)(0,1) 494
3738 sage: z # (0+1i) 495
3739 1.00000000000000000000000000000000*I 496
3740 sage: elliptic_j(z) 497
3741 1728.000000000000000000000000000000 498
3742 sage: # j-function only defined for positive imaginary 499
3743 arguments
3744 sage: z = ComplexField(100)(1,-1) 500
3745 sage: try: 501
3746 ....:     elliptic_j(z) 502
3747 ....: except PariError: 503

```



```

3748     ....:         pass
3749 sage: # root at (-1+i sqrt(3))/2
3750 sage: z = ComplexField(100)(-1, sqrt(3))/2
3751 sage: elliptic_j(z)
3752 -2.6445453750358706361219364880e-88
3753 sage: elliptic_j(z).imag().round()
3754 0
3755 sage: elliptic_j(z).real().round()
3756 0

```

3757 With a way to compute the j -function and the precomputed set $ICG(D)$ at hand, we can now
 3758 compute the Hilbert class polynomial as follows:

$$H_D(x) = \prod_{(A,B,C) \in ICG(D)} \left(x - j \left(\frac{-B + \sqrt{D}}{2A} \right) \right) \quad (5.34)$$

3759 In other words, we loop over all elements (A, B, C) from the set $ICG(D)$ and compute the
 3760 j -function at the point $\frac{-B + \sqrt{D}}{2A}$, where D is the CM-discriminant that we chose in a previous
 3761 step. The result defines a factor of the Hilbert class polynomial and all factors are multiplied
 3762 together.

3763 It can be shown that the Hilbert class polynomial is an integer polynomial, but actual com-
 3764 putations need high-precision arithmetics to avoid approximation errors that usually occur in
 3765 computer approximations of the j -function (as shown above). So, in case the calculated Hilbert
 3766 class polynomial does not have integer coefficients, we need to round the result to the nearest
 3767 integer. Given that the precision we used was high enough, the result will be correct.

In the next step, we use the Hilbert class polynomial $H_D \in \mathbb{Z}[x]$, and project it to a poly-
 nomial $H_{D,q} \in \mathbb{F}_q[x]$ with coefficients in the base field \mathbb{F}_q as chosen in the first step. We do
 this by simply computing the new coefficients as the old coefficients modulus p , that is, if
 $H_D(x) = a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0$, we compute the q -modulus of each coefficient
 $\tilde{a}_j = a_j \bmod p$, which defines the **projected Hilbert class polynomial** as follows:

$$H_{D,p}(x) = \tilde{a}_m x^m + \tilde{a}_{m-1} x^{m-1} + \dots + \tilde{a}_1 x + \tilde{a}_0$$

3768 We then search for roots of $H_{D,p}$, since every root j_0 of $H_{D,p}$ defines a family of elliptic curves
 3769 over \mathbb{F}_q , which all have a j -invariant 5.31 or 5.33 equal to j_0 . We can pick any root, since all of
 3770 them will lead to proper curves eventually.

check
reference

3771 However, some of the curves with the correct j -invariant might have an order different from
 3772 the one we initially decided on. Therefore, we need a way to decide on a curve with the correct
 3773 order.

3774 To compute such a curve, we have to distinguish a few different cases based on our choice
 3775 of the root j_0 and of the CM-discriminant D . If $j_0 \neq 0$ or $j_0 \neq 1728 \bmod q$, we compute
 3776 $c_1 = \frac{j_0}{(1728 \bmod q) - j_0}$, then we chose some arbitrary quadratic non-residue $c_2 \in \mathbb{F}_q$, and some
 3777 arbitrary cubic non-residue $c_3 \in \mathbb{F}_q$.

3778 The following table is guaranteed to define a curve with the correct order $r = q + 1 - t$ for
 3779 the trace of Frobenius t we initially decided on:

actually
make this
a table?

3780 **Definition 5.4.0.1.** • Case $j_0 \neq 0$ and $j_0 \neq 1728 \bmod q$. A curve with the correct order is
 3781 defined by one of the following equations:

$$y^2 = x^3 + 3c_1 x + 2c_1 \quad \text{or} \quad y^2 = x^3 + 3c_1 c_2^2 x + 2c_1 c_2^3 \quad (5.35)$$

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- Case $j_0 = 0$ and $D \neq -3$. A curve with the correct order is defined by one of the following equations:

$$y^2 = x^3 + 1 \quad \text{or} \quad y^2 = x^3 + c_2^3 \quad (5.36)$$

- Case $j_0 = 0$ and $D = -3$. A curve with the correct order is defined by one of the following equations:

$$\begin{aligned} y^2 &= x^3 + 1 \quad \text{or} \quad y^2 = x^3 + c_2^3 \quad \text{or} \\ y^2 &= x^3 + c_3^2 \quad \text{or} \quad y^2 = c_3^2 c_2^3 \quad \text{or} \\ y^2 &= x^3 + c_3^{-2} \quad \text{or} \quad y^2 = x^3 + c_3^{-2} c_2^3 \end{aligned}$$

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- Case $j_0 = 1728 \bmod q$ and $D \neq -4$. A curve with the correct order is defined by one of the following equations:

$$y^2 = x^3 + x \quad \text{or} \quad y^2 = x^3 + c_2^2 x \quad (5.37)$$

- Case $j_0 = 1728 \bmod q$ and $D = -4$. A curve with the correct order is defined by one of the following equations:

$$\begin{aligned} y^2 &= x^3 + x \quad \text{or} \quad y^2 = x^3 + c_2 x \quad \text{or} \\ y^2 &= x^3 + c_2^2 x \quad \text{or} \quad y^2 = x^3 + c_2^3 x \end{aligned}$$

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To decide the proper defining Weierstraß equation, we therefore have to compute the order of any of the potential curves above, and then choose the one that fits our initial requirements. Since it can be shown that the Hilbert class polynomials for the CM-discriminants $D = -3$ and $D = -4$ are given by $H_{-3,q}(x) = x$ and $H_{-4,q}(x) = x - (1728 \bmod q)$ (EXERCISE), the previous cases are exhaustive.

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To summarize, using the complex multiplication method, it is possible to synthesize elliptic curves with predefined order over predefined base fields from scratch. However, the curves that are constructed this way are just some representatives of a larger class of curves, all of which have the same order. Therefore, in real-world applications, it is sometimes more advantageous to choose a different representative from that class. To do so recall from XXX that any curve defined by the Weierstraß equation $y^2 = x^3 + ax + b$ is isomorphic to a curve of the form $y^2 = x^3 + ad^2x + bd^3$ for some quadratic residue $d \in \mathbb{F}_q$.

In order to find a suitable representative (e.g. with small parameters a and b) in the last step, the curve designer might choose a quadratic residue d such that the transformed curve has the properties they wanted.

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Example 102. Consider curve $E_1(\mathbb{F}_5)$ from example 67. We want to use the complex multiplication method to derive that curve from scratch. Since $E_1(\mathbb{F}_5)$ is a curve of order $r = 9$ over the prime field of order $q = 5$, we know from example 96 that its trace of Frobenius is $t = -3$, which also implies that q and $|t|$ are coprime.

We then have to find parameters $D, v \in \mathbb{Z}$ such that the criteria in 5.32 hold. We get the following:

$$\begin{aligned} 4q &= t^2 + |D|v^2 && \Rightarrow \\ 20 &= (-3)^2 + |D|v^2 && \Leftrightarrow \\ 11 &= |D|v^2 \end{aligned}$$

exercise
still to be
written?

add refer-
ence

check
reference

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reference

Now, since 11 is a prime number, the only solution is $|D| = 11$ and $v = 1$ here. With $D = -11$ and the Euclidean division of -11 by 4 being $-11 = -3 \cdot 4 + 1$, we have $-11 \bmod 4 = 1$, which shows that $D = -11$ is a proper choice.

In the next step, we have to compute the Hilbert class polynomial H_{-11} . To do so, we first have to find the set $ICG(D)$. To compute that set, observe that, since $\sqrt{\frac{|D|}{3}} \approx 1.915 < 2$, we know from $A \leq \sqrt{\frac{|D|}{3}}$ and $A \in \mathbb{Z}$ that A must be either 0 or 1.

For $A = 0$, we know $B = 0$ from the constraint $|B| \leq A$. However, in this case, there could be no C satisfying $-11 = B^2 - 4AC$. So we try $A = 1$ and deduce $B \in \{-1, 0, 1\}$ from the constraint $|B| \leq A$. The case $B = -1$ can be excluded, since then $B < 0$ has to imply $|B| < A$. The case $B = 0$ can also be excluded, as there cannot be an integer C with $-11 = -4C$, since 11 is a prime number.

This leaves the case $B = 1$, and we compute $C = 3$ from the equation $-11 = 1^2 - 4C$, which gives the solution $(A, B, C) = (1, 1, 3)$:

$$ICG(D) = \{(1, 1, 3)\}$$

With the set $ICG(D)$ at hand, we can compute the Hilbert class polynomial of $D = -11$. To do so, we have to insert the term $\frac{-1+\sqrt{-11}}{2}$ into the j -function. To do so, first observe that $\sqrt{-11} = i\sqrt{11}$, where i is the imaginary unit, defined by $i^2 = -1$. Using this, we can invoke Sage to compute the j -invariant and get the following:

$$H_{-11}(x) = x - j\left(\frac{-1+i\sqrt{11}}{2}\right) = x + 32768$$

As we can see, in this particular case, the Hilbert class polynomial is a linear function with a single integer coefficient. In the next step, we have to project it onto a polynomial from $\mathbb{F}_5[x]$ by computing the modular 5 remainder of the coefficients 1 and 32768. We get $32768 \bmod 5 = 3$, from which it follows that the projected Hilbert class polynomial is considered a polynomial from $\mathbb{F}_5[x]$:

$$H_{-11,5}(x) = x + 3$$

As we can see, the only root of this polynomial is $j = 2$, since $H_{-11,5}(2) = 2 + 3 = 0$. We therefore have a situation with $j \neq 0$ and $j \neq 1728$, which tells us that we have to compute the parameter c_1 in modular 5 arithmetics:

$$c_1 = \frac{2}{1728 - 2}$$

Since $1728 \bmod 5 = 3$, we get $c_1 = 2$.

Next, we have to check if the curve $E(\mathbb{F}_5)$ defined by the Weierstraß equation $y^2 = x^3 + 3 \cdot 2x + 2 \cdot 2$ has the correct order. We invoke Sage, and find that the order is indeed 9, so it is a curve with the required parameters. Thus, we have successfully constructed the curve with the desired properties.

Note, however, that in real-world applications, it might be useful to choose parameters a and b that have certain properties, e.g. to be as small as possible. As we know from XXX, choosing any quadratic residue $d \in \mathbb{F}_5$ gives a curve of the same order defined by $y^2 = x^2 + ak^2x + bk^3$. Since 4 is a quadratic residue in \mathbb{F}_4 , we can transform the curve defined by $y^2 = x^3 + x + 4$ into the curve $y^2 = x^3 + 4^2 + 4 \cdot 4^3$ which gives the following:

$$y^2 = x^3 + x + 1$$

add reference

3821 This is the curve $E_1(\mathbb{F}_5)$ that we used extensively throughout this book. Thus, using the
 3822 complex multiplication method, we were able to derive a curve with specific properties from
 3823 scratch.

3824 *Example 103.* Consider the tiny-jubjub curve TJJ_13 from example 68. We want to use the
 3825 complex multiplication method to derive that curve from scratch. Since TJJ_13 is a curve of
 3826 order $r = 20$ over the prime field of order $q = 13$, we know from example 97 that its trace of
 3827 Frobenius is $t = -6$, which also implies that q and $|t|$ are coprime.

check
referencecheck
reference

We then have to find parameters $D, v \in \mathbb{Z}$ such that 5.32 holds. We get the following:

$$\begin{aligned} 4q &= t^2 + |D|v^2 && \Rightarrow \\ 4 \cdot 13 &= (-6)^2 + |D|v^2 && \Rightarrow \\ 52 &= 36 + |D|v^2 && \Leftrightarrow \\ 16 &= |D|v^2 \end{aligned}$$

3828 This equation has two solutions for (D, v) , namely $(-4, \pm 2)$ and $(-16, \pm 1)$. Looking at the
 3829 first solution, we know that $D = -4$ implies $j = 1728$, and the constructed curve is defined by
 3830 a Weierstraß equation 5.1 that has a vanishing parameter $b = 0$. We can therefore conclude that
 3831 choosing $D = -4$ will not help us reconstructing TJJ_13 . It will produce curves with order 20,
 3832 just not the one we are looking for.

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reference

So we choose the second solution $D = -16$. In the next step, we have to compute the Hilbert class polynomial H_{-16} . To do so, we first have to find the set $ICG(D)$. To compute that set, observe that since $\sqrt{\frac{|-16|}{3}} \approx 2.31 < 3$, we know from $A \leq \sqrt{\frac{|-16|}{3}}$ and $A \in \mathbb{Z}$ that A must be in the range $0..2$. So we loop through all possible values of A and through all possible values of B under the constraints $|B| \leq A$, and if $B < 0$ then $|B| < A$. Then we compute potential C 's from $-16 = B^2 - 4AC$. We get the following two solutions for $ICG(D)$: we get

$$ICG(D) = \{(1, 0, 4), (2, 0, 2)\}$$

With the set $ICG(D)$ at hand, we can compute the Hilbert class polynomial of $D = -16$. We can invoke Sage to compute the j -invariant and get the following:

$$\begin{aligned} H_{-16}(x) &= \left(x - j \left(\frac{i\sqrt{16}}{2} \right) \right) \left(x - j \left(\frac{i\sqrt{16}}{4} \right) \right) \\ &= (x - 287496)(x - 1728) \end{aligned}$$

As we can see, in this particular case, the Hilbert class polynomial is a quadratic function with two integer coefficients. In the next step, we have to project it onto a polynomial from $\mathbb{F}_5[x]$ by computing the modular 5 remainder of the coefficients 1, 287496 and 1728. We get $287496 \bmod 13 = 1$ and $1728 \bmod 13 = 2$, which means that the projected Hilbert class polynomial is as follows:

$$H_{-11,5}(x) = (x - 1)(x - 12) = (x + 12)(x + 1)$$

3833 This is considered a polynomial from $\mathbb{F}_5[x]$. Thus, we have two roots, namely $j = 1$ and $j =$
 3834 12. We already know that $j = 12$ is the wrong root to construct the tiny-jubjub curve, since
 3835 $1728 \bmod 13 = 2$, and that case is not compatible with a curve with $b \neq 0$. So we choose $j = 1$.

Another way to decide the proper root is to compute the j -invariant of the tiny-jubjub curve. We get the following:

$$\begin{aligned}
 j(TJJ_13) &= 12 \frac{4 \cdot 8^3}{4 \cdot 8^3 + 1 \cdot 8^2} \\
 &= 12 \frac{4 \cdot 5}{4 \cdot 5 + 12} \\
 &= 12 \frac{7}{7 + 12} \\
 &= 12 \frac{7}{7 + 12} \\
 &= 1
 \end{aligned}$$

This is equal to the root $j = 1$ of the Hilbert class polynomial $H_{-16,13}$ as expected. We therefore have a situation with $j \neq 0$ and $j \neq 1728$, which tells us that we have to compute the parameter c_1 in modular 5 arithmetics:

$$c_1 = \frac{1}{12 - 1} = 6$$

Since $1728 \bmod 13 = 12$, we get $c_1 = 6$. Then we have to check if the curve $E(\mathbb{F}_5)$ defined by the Weierstraß equation $y^2 = x^3 + 3 \cdot 6x + 2 \cdot 6$, which is equivalent to $y^2 = x^3 + 5x + 12$, has the correct order. We invoke Sage and find that the order is 8, which implies that the trace of this curve is 6, not -6 as required. So we have to consider the second possibility, and choose some quadratic non-residue $c_2 \in \mathbb{F}_{13}$. We choose $c_2 = 5$ and compute the Weierstraß equation $y^2 = x^3 + 5c_2^2 + 12c_2^3$ as follows:

$$y^2 = x^3 + 8x + 5$$

We invoke Sage and find that the order is 20, which is indeed the correct one. As we know from XXX, choosing any quadratic residue $d \in \mathbb{F}_5$ gives a curve of the same order defined by $y^2 = x^2 + ad^2x + bd^3$. Since 12 is a quadratic residue in \mathbb{F}_{13} , we can transform the curve defined by $y^2 = x^3 + 8x + 5$ into the curve $y^2 = x^3 + 12^2 \cdot 8 + 5 \cdot 12^3$ which gives the following:

$$y^2 = x^3 + 8x + 8$$

add reference

3836 This is the tiny-jubjub curve that we used extensively throughout this book. So using the
3837 complex multiplication method, we were able to derive a curve with specific properties from
3838 scratch.

Example 104. To consider a real-world example, we want to use the complex multiplication method in combination with Sage to compute Secp256k1 from scratch. So based on example 69, we decided to compute an elliptic curve over a prime field \mathbb{F}_p of order r for the following security parameters:

check reference

$$\begin{aligned}
 p &= 115792089237316195423570985008687907853269984665640564039457584007908834671663 \\
 r &= 115792089237316195423570985008687907852837564279074904382605163141518161494337
 \end{aligned}$$

According to example 98, this gives the following trace of Frobenius:

check reference

$$t = 43242038656559659652420866390673177327$$

3839 We also decided that we want a curve of the form $y^2 = x^3 + b$, that is, we want the parameter
3840 a to be zero. This implies that the j -invariant of our curve must be zero.

In a first step, we have to find a CM-discriminant D and some integer v such that the equation $4p = t^2 + |D|v^2$ is satisfied. Since we aim for a vanishing j -invariant, the first thing to try is $D = -3$. In this case, we can compute $v^2 = (4p - t^2)$, and if v^2 happens to be an integer that has a square root v , we are done. Invoking Sage we compute as follows:

```

3845 sage: D = -3 513
3846 sage: p = 1157920892373161954235709850086879078532699846656405 514
3847         64039457584007908834671663
3848 sage: r = 1157920892373161954235709850086879078528375642790749 515
3849         04382605163141518161494337
3850 sage: t = p+1-r 516
3851 sage: v_sqr = (4*p - t^2)/abs(D) 517
3852 sage: v_sqr.is_integer() 518
3853 True 519
3854 sage: v = sqrt(v_sqr) 520
3855 sage: v.is_integer() 521
3856 True 522
3857 sage: 4*p == t^2 + abs(D)*v^2 523
3858 True 524
3859 sage: v 525
3860 303414439467246543595250775667605759171 526

```

The pair $(D, v) = (-3, 303414439467246543595250775667605759171)$ does indeed solve the equation, which tells us that there is a curve of order r over a prime field of order p , defined by a Weierstraß equation $y^2 = x^3 + b$ for some $b \in \mathbb{F}_p$. Now we need to compute b .

For $D = -3$, we already know that the associated Hilbert class polynomial is given by $H_{-3}(x) = x$, which gives the projected Hilbert class polynomial as $H_{-3,p} = x$ and the j -invariant of our curve is guaranteed to be $j = 0$. Now, looking at 5.4.0.1, we see that there are 6 possible cases to construct a curve with the correct order r . In order to construct the curves in question, we have to choose some arbitrary quadratic and cubic non-residue. So we loop through \mathbb{F}_p to find them, invoking Sage:

```

3870 sage: F = GF(p) 527
3871 sage: for c2 in F: 528
3872     ....:     try: # quadratic residue 529
3873     ....:         _ = c2.nth_root(2) 530
3874     ....:     except ValueError: # quadratic non-residue 531
3875     ....:         break 532
3876 sage: c2 533
3877 3 534
3878 sage: for c3 in F: 535
3879     ....:     try: 536
3880     ....:         _ = c3.nth_root(3) 537
3881     ....:     except ValueError: 538
3882     ....:         break 539
3883 sage: c3 540
3884 2 541

```

We found the quadratic non-residue $c_2 = 3$ and the cubic non-residue $c_3 = 2$. Using those numbers, we check the six cases against the the expected order r of the curve we want to

check
reference

3887 synthesize:

```

3888 sage: C1 = EllipticCurve(F, [0, 1])           542
3889 sage: C1.order() == r                         543
3890 False                                         544
3891 sage: C2 = EllipticCurve(F, [0, c2^3])        545
3892 sage: C2.order() == r                         546
3893 False                                         547
3894 sage: C3 = EllipticCurve(F, [0, c3^2])        548
3895 sage: C3.order() == r                         549
3896 False                                         550
3897 sage: C4 = EllipticCurve(F, [0, c3^2*c2^3])   551
3898 sage: C4.order() == r                         552
3899 False                                         553
3900 sage: C5 = EllipticCurve(F, [0, c3^(-2)])     554
3901 sage: C5.order() == r                         555
3902 False                                         556
3903 sage: C6 = EllipticCurve(F, [0, c3^(-2)*c2^3]) 557
3904 sage: C6.order() == r                         558
3905 True                                          559

```

As expected, we found an elliptic curve of the correct order r over a prime field of size p . In principle, we are done, as we have found a curve with the same basic properties as Secp256k1. However, the curve is defined by the following equation, which uses a very large parameter b_1 , and so it might perform too slowly in certain algorithms.

$$y^2 = x^3 + 86844066927987146567678238756515930889952488499230423029593188005931626003754$$

It is also not very elegant to be written down by hand. It might therefore be advantageous to find an isomorphic curve with the smallest possible parameter b_2 . In order to find such a b_2 , we have to choose a quadratic residue d such that $b_2 = b_1 \cdot d^3$ is as small as possible. To do so, we rewrite the last equation into the following form:

$$d = \sqrt[3]{\frac{b_2}{b_1}}$$

what
does this
mean?
Maybe
just delete
it

3906 Then we invoke Sage to loop through values $b_2 \in \mathbb{F}_p$ until it finds some number such that
3907 the quotient $\frac{b_2}{b_1}$ has a cube root d and this cube root itself is a quadratic residue.

```

3908 sage: b1=86844066927987146567678238756515930889952488499230423  560
3909      029593188005931626003754
3910 sage: for b2 in F:                                           561
3911     ....:     try:                                           562
3912     ....:         d = (b2/b1).nth_root(3)                   563
3913     ....:         try:                                       564
3914     ....:             _ = d.nth_root(2)                     565
3915     ....:             if d != 0:                             566
3916     ....:                 break                               567
3917     ....:         except ValueError:                         568
3918     ....:             pass                                    569
3919     ....:     except ValueError:                             570
3920     ....:         pass                                        571

```

3921 **sage: b2**
 3922 **7**

572
 573

3923 Indeed, the smallest possible value is $b_2 = 7$ and the defining Weierstraß equation of a curve
 3924 over \mathbb{F}_p with prime order r is $y^2 = x^3 + 7$, which we might call Secp256k1. As we have just
 3925 seen, the complex multiplication method is powerful enough to derive cryptographically secure
 3926 curves like Secp256k1 from scratch.

3927 **The BLS6_6 pen-and-paper curve** In this paragraph, we summarize our understanding of
 3928 elliptic curves to derive our main pen-and-paper example for the rest of the book. To do so, we
 3929 want to use the complex multiplication method to derive a pairing-friendly elliptic curve that
 3930 has similar properties to curves that are used in actual cryptographic protocols. However, we
 3931 design the curve specifically to be useful in pen-and-paper examples, which mostly means that
 3932 the curve should contain only a few points so that we are able to derive exhaustive addition and
 3933 pairing tables.

3934 A well-understood family of pairing-friendly curves is the the group of BLS curves (STUFF
 3935 ABOUT THE HISTORY AND THE NAMING CONVENTION), which are derived in [XXX].
 3936 BLS curves are particularly useful in our case if the embedding degree k satisfies $k \equiv 6 \pmod{0}$.
 3937 Of course, the smallest embedding degree k that satisfies this congruency is $k = 6$ and we there-
 3938 fore aim for a BLS6 curve as our main pen-and-paper example.

write up
this part

add refer-
ence

3939 To apply the complex multiplication method from page 113 ff., recall that this method starts
 3940 with a definition of the base field \mathbb{F}_{p^m} , as well as the trace of Frobenius t and the order of the
 3941 curve. If the order $p^m + 1 - t$ is not a prime number, then the order r of the largest prime factor
 3942 group needs to be controlled.

check
reference

3943 In the case of BLS_6 curves, the parameter m is chosen to be 1, which means that the
 3944 curves are defined over prime fields. All relevant parameters p , t and r are then themselves
 3945 parameterized by the following three polynomials:

$$\begin{aligned} r(x) &= \Phi_6(x) \\ t(x) &= x + 1 \\ q(x) &= \frac{1}{3}(x-1)^2(x^2 - x + 1) + x \end{aligned} \tag{5.38}$$

3946 In the equations above, Φ_6 is the 6-th cyclotomic polynomial and $x \in \mathbb{N}$ is a parameter
 3947 that the designer has to choose in such a way that the evaluation of p , t and r at the point x
 3948 gives integers that have the proper size to meet the security requirements of the curve that they
 3949 want to design. It is then guaranteed that the complex multiplication method can be used in
 3950 combination with those parameters to define an elliptic curve with CM-discriminant $D = -3$,
 3951 embedding degree $k = 6$, and curve equation $y^2 = x^3 + b$ for some $b \in \mathbb{F}_p$.

cyclotomic
polyno-
mial

3952 For example, if the curve should target the 128-bit security level, due to the Pholaard-rho
 3953 attack (TODO) the parameter r should be prime number of at least 256 bits.

Pholaard-
rho attack

3954 In order to design the smallest BLS_6 curve, we therefore have to find a parameter x such
 3955 that $r(x)$, $t(x)$ and $q(x)$ are the smallest natural numbers that satisfy $q(x) > 3$ and $r(x) > 3$.¹

todo

We therefore initiate the design process of our BLS6 curve by looking up the 6-th cyclo-
 tomic polynomial, which is $\Phi_6 = x^2 - x + 1$, and then insert small values for x into the defining

¹The smallest BLS curve will also be the most insecure BLS curve. However, since our goal with this curve is ease of pen-and-paper computation rather than security, it fits the purposes of this book.

polynomials r, t, q . We get the following results:

$$\begin{array}{lll} x = 1 & (r(x), t(x), q(x)) & (1, 2, 1) \\ x = 2 & (r(x), t(x), q(x)) & (3, 3, 3) \\ x = 3 & (r(x), t(x), q(x)) & (7, 4, \frac{37}{3}) \\ x = 4 & (r(x), t(x), q(x)) & (13, 5, 43) \end{array}$$

Since $q(1) = 1$ is not a prime number, the first x that gives a proper curve is $x = 2$. However, such a curve would be defined over a base field of characteristic 3, and we would rather like to avoid that. We therefore find $x = 4$, which defines a curve over the prime field of characteristic 43 that has a trace of Frobenius $t = 5$ and a larger order prime group of size $r = 13$.

Since the prime field \mathbb{F}_{43} has 43 elements and 43's binary representation is $43_2 = 101011$, which consists of 6 digits, the name of our pen-and-paper curve should be *BLS6_6*, since its is common to name a BLS curve by its embedding degree and the bit-length of the modulus in the base field. We call *BLS6_6* the **moon-math-curve**.

Based on 5.29, we know that the Hasse bound implies that *BLS6_6* will contain exactly 39 elements. Since the prime factorization of 39 is $39 = 3 \cdot 13$, we have a “large” prime factor group of size 13, as expected, and a small cofactor group of size 3. Fortunately, a subgroup of order 13 is well suited for our purposes, as 13 elements can be easily handled in the associated addition, scalar multiplication and pairing tables in a pen-and-paper style.

We can check that the embedding degree is indeed 6 as expected, since $k = 6$ is the smallest number k such that $r = 13$ divides $43^k - 1$.

```
sage: for k in range(1, 42): # Fermat's little theorem
      ....:     if (43^k - 1) % 13 == 0:
      ....:         break
sage: k
6
```

In order to compute the defining equation $y^2 = x^3 + ax + b$ of *BLS6_6*, we use the complex multiplication method as described in 5.4. The goal is to find $a, b \in \mathbb{F}_{43}$ representations that are particularly nice to work with. The authors of XXX showed that the CM-discriminant of every BLS curve is $D = -3$ and, indeed, the following equation has the four solutions $(D, v) \in \{(-3, -7), (-3, 7), (-49, -1), (-49, 1)\}$ if D is required to be negative, as expected:

$$\begin{aligned} 4p &= t^2 + |D|v^2 && \Rightarrow \\ 4 \cdot 43 &= 5^2 + |D|v^2 && \Rightarrow \\ 172 &= 25 + |D|v^2 && \Leftrightarrow \\ 49 &= |D|v^2 \end{aligned}$$

This means that $D = -3$ is indeed a proper CM-discriminant, and we can deduce that the parameter a has to be 0, and that the Hilbert class polynomial is given by $H_{-3, 43}(x) = x$.

This implies that the j -invariant of *BLS6_6* is given by $j(\text{BLS6}_6) = 0$. We therefore have to look at case XXX in table 5.4.0.1 to derive a parameter b . To decide the proper case for $j_0 = 0$ and $D = -3$, we therefore have to choose some arbitrary quadratic non-residue c_2 and cubic non-residue c_3 in \mathbb{F}_{43} . We choose $c_2 = 5$ and $c_3 = 36$. We check these with Sage:

```
sage: F43 = GF(43)
```

why? Because in this book elliptic curves are only defined for fields of characteristic > 3

check reference

574

575

576

577

578

check reference

what does this mean?

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check reference

```

3988 sage: c2 = F43(5) 580
3989 ..... try: # quadratic residue 581
3990 ..... c2.nth_root(2) 582
3991 ..... except ValueError: # quadratic non-residue 583
3992 ..... c2 584
3993 sage: c3 = F43(36) 585
3994 ..... try: 586
3995 ..... c3.nth_root(3) 587
3996 ..... except ValueError: 588
3997 ..... c3 589

```

3998 Using those numbers we check the six possible cases from 5.4.0.1 against the the expected
3999 order 39 of the curve we want to synthesize:

check
reference

```

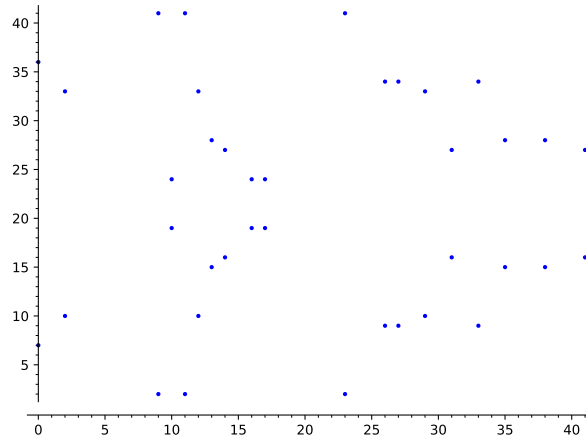
4000 sage: BLS61 = EllipticCurve(F43, [0, 1]) 590
4001 sage: BLS61.order() == 39 591
4002 False 592
4003 sage: BLS62 = EllipticCurve(F43, [0, c2^3]) 593
4004 sage: BLS62.order() == 39 594
4005 False 595
4006 sage: BLS63 = EllipticCurve(F43, [0, c3^2]) 596
4007 sage: BLS63.order() == 39 597
4008 True 598
4009 sage: BLS64 = EllipticCurve(F43, [0, c3^2*c2^3]) 599
4010 sage: BLS64.order() == 39 600
4011 False 601
4012 sage: BLS65 = EllipticCurve(F43, [0, c3^(-2)]) 602
4013 sage: BLS65.order() == 39 603
4014 False 604
4015 sage: BLS66 = EllipticCurve(F43, [0, c3^(-2)*c2^3]) 605
4016 sage: BLS66.order() == 39 606
4017 False 607
4018 sage: BLS6 = BLS63 # our BLS6 curve in the book 608

```

4019 As expected, we found an elliptic curve of the correct order 39 over a prime field of size 43,
4020 defined by the following equation:

$$BLS6_6 := \{(x, y) \mid y^2 = x^3 + 6 \text{ for all } x, y \in \mathbb{F}_{43}\} \quad (5.39)$$

4021 There are other choices for b , such as $b = 10$ or $b = 23$, but all these curves are isomorphic,
4022 and hence represent the same curve in a different way. Since BLS6-6 only contains 39 points, it
4023 is possible to give a visual impression of the curve:



As we can see, our curve has some desirable properties: it does not contain self-inverse points, that is, points with $y = 0$. It follows that the addition law can be optimized, since the branch for those cases can be eliminated.

Summarizing the previous procedure, we have used the method of Barreto, Lynn and Scott to construct a pairing-friendly elliptic curve of embedding degree 6. However, in order to do elliptic curve cryptography on this curve, note that, since the order of $BLS6_6$ is 39, its group of rational points is not a finite cyclic group of prime order. We therefore have to find a suitable subgroup as our main target. Since $39 = 13 \cdot 3$, we know that the curve must contain a “large” prime-order group of size 13 and a small cofactor group of order 3.

The following step is to construct this group. One way to do so is to find a generator. We can achieve this by choosing an arbitrary element of the group that is not the point at infinity, and then multiply that point with the cofactor of the group’s order. If the result is not the point at infinity, the result will be a generator. If it is the point at infinity we have to choose a different element.

In order to find a generator for the large order subgroup of size 13, we first notice that the cofactor of 13 is 3, since $39 = 3 \cdot 13$. We then need to construct an arbitrary element from $BLS6_6$. To do so in a pen-and-paper style, we can choose some *arbitrary* $x \in \mathbb{F}_{43}$ and see if there is some solution $y \in \mathbb{F}_{43}$ that satisfies the defining Weierstraß equation $y^2 = x^3 + 6$. We choose $x = 9$, and check that $y = 2$ is a proper solution:

$$\begin{aligned} y^2 &= x^3 + 6 && \Rightarrow \\ 2^2 &= 9^3 + 6 && \Leftrightarrow \\ 4 &= 4 \end{aligned}$$

This implies that $P = (9, 2)$ is therefore a point on $BLS6_6$. To see if we can project this point onto a generator of the large order prime group $BLS6_6[13]$, we have to multiply P with the cofactor, that is, we have to compute $[3](9, 2)$. After some computation (EXERCISE) we get $[3](9, 2) = (13, 15)$. Since this is not the point at infinity, we know that $(13, 15)$ must be a generator of $BLS6_6[13]$. The generator $g_{BLS6_6[13]}$, which we will use in pairing computations in the remainder of this book, is given as follows:

$$g_{BLS6_6[13]} = (13, 15) \tag{5.40}$$

Since $g_{BLS6_6[13]}$ is a generator, we can use it to construct the subgroup $BLS6_6[13]$ by repeatedly adding the generator to itself. Using Sage, we get the following:

```
sage: P = BLS6(9, 2)
```

add reference

add exercise

```

4048 sage: Q = 3*P                                     610
4049 sage: Q.xy()                                       611
4050 (13, 15)                                           612
4051 sage: BLS6_13 = []                                 613
4052 sage: for x in range(0,13): # cyclic of order 13    614
4053     ....:     P = x*Q                               615
4054     ....:     BLS6_13.append(P)                     616

```

Repeatedly adding a generator to itself, as we just did, will generate small groups in logarithmic order with respect to the generator as, explained on page 43 ff. We therefore get the following description of the large prime-order subgroup of $BLS6_6$:

check
reference

$$\begin{aligned}
 BLS6_6[13] = \\
 \{ (13, 15) \rightarrow (33, 34) \rightarrow (38, 15) \rightarrow (35, 28) \rightarrow (26, 34) \rightarrow (27, 34) \rightarrow \\
 (27, 9) \rightarrow (26, 9) \rightarrow (35, 15) \rightarrow (38, 28) \rightarrow (33, 9) \rightarrow (13, 28) \rightarrow \mathcal{O} \} \quad (5.41)
 \end{aligned}$$

Having a logarithmic description of this group is tremendously helpful in pen-and-paper computations. To see that, observe that we know from XXX that there is an exponential map from the scalar field \mathbb{F}_{13} to $BLS6_6[13]$ with respect to our generator, which generates the group in logarithmic order:

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$$[\cdot]_{(13,15)} : \mathbb{F}_{13} \rightarrow BLS6_6[13] ; x \mapsto [x](13, 15)$$

So, for example, we have $[1]_{(13,15)} = (13, 15)$, $[7]_{(13,15)} = (27, 9)$ and $[0]_{(13,15)} = \mathcal{O}$ and so on. The relevant point here is that we can use this representation to do computations in $BLS6_6[13]$ efficiently in our head using XXX, as in the following example:

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$$\begin{aligned}
 (27, 34) \oplus (33, 9) &= [6](13, 15) \oplus [11](13, 15) \\
 &= [6 + 11](13, 15) \\
 &= [4](13, 15) \\
 &= (35, 28)
 \end{aligned}$$

So XXX is really all we need to do computations in $BLS6_6[13]$ in this book efficiently. However, out of convenience, the following picture lists the entire addition table of that group, as it might be useful in pen-and-paper computations:

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ence

\oplus	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)	(35, 28)	(26, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)	(13, 28)
\mathcal{O}	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)	(35, 28)	(26, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)	(13, 28)
(13, 15)	(13, 15)	(33, 34)	(38, 15)	(35, 28)	(26, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)	(13, 28)	\mathcal{O}
(33, 34)	(33, 34)	(38, 15)	(35, 28)	(26, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)	(13, 28)	\mathcal{O}	(13, 15)
(38, 15)	(38, 15)	(35, 28)	(26, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)	(13, 28)	\mathcal{O}	(13, 15)	(33, 34)
(35, 28)	(35, 28)	(26, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)	(13, 28)	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)
(26, 34)	(26, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)	(13, 28)	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)	(35, 28)
(27, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)	(13, 28)	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)	(35, 28)	(26, 34)
(27, 9)	(27, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)	(13, 28)	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)	(35, 28)	(26, 34)	(27, 34)
(26, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)	(13, 28)	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)	(35, 28)	(26, 34)	(27, 34)	(27, 9)
(35, 15)	(35, 15)	(38, 28)	(33, 9)	(13, 28)	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)	(35, 28)	(26, 34)	(27, 34)	(27, 9)	(26, 9)
(38, 28)	(38, 28)	(33, 9)	(13, 28)	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)	(35, 28)	(26, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)
(33, 9)	(33, 9)	(13, 28)	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)	(35, 28)	(26, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)	(38, 28)
(13, 28)	(13, 28)	\mathcal{O}	(13, 15)	(33, 34)	(38, 15)	(35, 28)	(26, 34)	(27, 34)	(27, 9)	(26, 9)	(35, 15)	(38, 28)	(33, 9)

Now that we have constructed a “large” cyclic prime-order subgroup of $BLS6_6$ suitable for many pen-and-paper computations in elliptic curve cryptography, we have to look at how to do pairings in this context. We know that $BLS6_6$ is a pairing-friendly curve by design, since it has a small embedding degree $k = 6$. It is therefore possible to compute Weil pairings efficiently. However, in order to do so, we have to decide the groups \mathbb{G}_1 and \mathbb{G}_2 as explained in exercise 75.

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reference

Since $BLS6_6$ has two non-trivial subgroups, it would be possible to use any of them as the n -torsion group. However, in cryptography, the only secure choice is to use the large prime-order subgroup, which in our case is $BLS6_6[13]$. We therefore decide to consider the 13-torsion and define $G_1[13]$ as the first argument for the Weil pairing function:

$$\mathbb{G}_1[13] = \{(13, 15) \rightarrow (33, 34) \rightarrow (38, 15) \rightarrow (35, 28) \rightarrow (26, 34) \rightarrow (27, 34) \rightarrow (27, 9) \rightarrow (26, 9) \rightarrow (35, 15) \rightarrow (38, 28) \rightarrow (33, 9) \rightarrow (13, 28) \rightarrow \mathcal{O}\}$$

In order to construct the domain for the second argument, we need to construct $\mathbb{G}_2[13]$, which, according to the general theory, should be defined by those elements P of the full 13-torsion group $BLS6_6[13]$ that are mapped to $43 \cdot P$ under the Frobenius endomorphism (equation 5.24).

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reference

To compute $\mathbb{G}_2[13]$, we therefore have to find the full 13-torsion group first. To do so, we use the technique from XXX, which tells us that the full 13-torsion can be found in the curve extension over the extension field \mathbb{F}_{43^6} , since the embedding degree of $BLS6_6$ is 6:

$$BLS6_6 := \{(x, y) \mid y^2 = x^3 + 6 \text{ for all } x, y \in \mathbb{F}_{43^6}\} \quad (5.42)$$

Thus, we have to construct \mathbb{F}_{43^6} , a field that contains 6321363049 elements. In order to do so, we use the procedure of XXX and start by choosing a non-reducible polynomial of degree 6 from the ring of polynomials $\mathbb{F}_{43}[t]$. We choose $p(t) = t^6 + 6$. Using Sage, we get the following:

add refer-
ence

```
sage: F43 = GF(43) 617
sage: F43t.<t> = F43[] 618
sage: p = F43t(t^6+6) 619
sage: p.is_irreducible() 620
True 621
sage: F43_6.<v> = GF(43^6, name='v', modulus=p) 622
```

Recall from XXX that elements $x \in \mathbb{F}_{43^6}$ can be seen as polynomials $a_0 + a_1v + a_2v^2 + \dots + a_5v^5$ with the usual addition of polynomials and multiplication modulo $t^6 + 6$.

add refer-
ence

In order to compute $\mathbb{G}_2[13]$, we first have to extend $BLS6_6$ to \mathbb{F}_{43^6} , that is, we keep the defining equation, but expand the domain from \mathbb{F}_{43} to \mathbb{F}_{43^6} . After that, we have to find at least one element P from that curve that is not the point at infinity, is in the full 13-torsion and satisfies the identity $\pi(P) = [43]P$. We can then use this element as our generator of $\mathbb{G}_2[13]$ and construct all other elements by repeatedly adding the generator to itself.

Since $BLS6(\mathbb{F}_{43^6})$ contains 6321251664 elements, it's not a good strategy to simply loop through all elements. Fortunately, Sage has a way to loop through elements from the torsion group directly:

```
sage: BLS6 = EllipticCurve(F43_6, [0, 6]) # curve extension 623
sage: INF = BLS6(0) # point at infinity 624
```

```

4097 sage: for P in INF.division_points(13): # full 13-torsion           625
4098 .....: # PI(P) == [q]P                                           626
4099 .....:         if P.order() == 13: # exclude point at infinity    627
4100 .....:             PiP = BLS6([a.frobenius() for a in P])          628
4101 .....:             qP = 43*P                                         629
4102 .....:             if PiP == qP:                                     630
4103 .....:                 break                                         631
4104 sage: P.xy()                                                         632
4105 (7*v^2, 16*v^3)                                                     633

```

4106 We found an element from the full 13-torsion that is in the Eigenspace of the Eigenvalue 43,
4107 which implies that it is an element of $\mathbb{G}_2[13]$. As $\mathbb{G}_2[13]$ is cyclic of prime order, this element
4108 must be a generator:

$$g_{\mathbb{G}_2[13]} = (7v^2, 16v^3) \quad (5.43)$$

4109 We can use this generator to compute \mathbb{G}_2 in logarithmic order with respect to $g_{\mathbb{G}_2[13]}$. Using
4110 Sage we get the following:

```

4111 sage: Q = BLS6(7*v^2, 16*v^3)                                       634
4112 sage: BLS6_13_2 = []                                                635
4113 sage: for x in range(0, 13):                                         636
4114 .....:     P = x*Q                                                  637
4115 .....:     BLS6_13_2.append(P)                                       638

```

$$\begin{aligned} \mathbb{G}_2 = \{ & (7v^2, 16v^3) \rightarrow (10v^2, 28v^3) \rightarrow (42v^2, 16v^3) \rightarrow (37v^2, 27v^3) \rightarrow \\ & (16v^2, 28v^3) \rightarrow (17v^2, 28v^3) \rightarrow (17v^2, 15v^3) \rightarrow (16v^2, 15v^3) \rightarrow \\ & (37v^2, 16v^3) \rightarrow (42v^2, 27v^3) \rightarrow (10v^2, 15v^3) \rightarrow (7v^2, 27v^3) \rightarrow \mathcal{O} \} \end{aligned}$$

Again, having a logarithmic description of $\mathbb{G}_2[13]$ is tremendously helpful in pen-and-paper computations, as it reduces complicated computation in the extended curves to modular 13 arithmetics, as in the following example:

$$\begin{aligned} (17v^2, 28v^3) \oplus (10v^2, 15v^3) &= [6](7v^2, 16v^3) \oplus [11](7v^2, 16v^3) \\ &= [6 + 11](7v^2, 16v^3) \\ &= [4](7v^2, 16v^3) \\ &= (37v^2, 27v^3) \end{aligned}$$

4116 So XXX is really all we need to do computations in $\mathbb{G}_2[13]$ in this book efficiently.

4117 To summarize the previous steps, we have found two subgroups, $\mathbb{G}_1[13]$ and $\mathbb{G}_2[13]$ suit-
4118 able to do Weil pairings on $BLS6_6$ as explained in 5.28. Using the logarithmic order XXX
4119 of $\mathbb{G}_1[13]$, the logarithmic order XXX of $\mathbb{G}_2[13]$ and the bilinearity in 5.44, we can do Weil
4120 pairings on $BLS6_6$ in a pen-and-paper style:

$$e([k_1]g_{BLS6_6[13]}, [k_2]g_{\mathbb{G}_2[13]}) = e(g_{BLS6_6[13]}, g_{\mathbb{G}_2[13]})^{k_1 \cdot k_2} \quad (5.44)$$

4121 Observe that the Weil pairing between our two generators is given by the following identity:

$$e(g_{BLS6_6[13]}, g_{\mathbb{G}_2[13]}) = 5v^5 + 16v^4 + 16v^3 + 15v^2 + 3v + 41 \quad (5.45)$$

add refer-
ence

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```

4122 sage: g1 = BLS6([13,15]) 639
4123 sage: g2 = BLS6([7*v^2, 16*v^3]) 640
4124 sage: g1.weil_pairing(g2,13) 641
4125 5*v^5 + 16*v^4 + 16*v^3 + 15*v^2 + 3*v + 41 642

```

4126 **Hashing to pairing groups** We give various constructions to hash into \mathbb{G}_1 and \mathbb{G}_2 .

4127 We start with hashing to the scalar field... **TO APPEAR**

4128 None of these techniques work for hashing into \mathbb{G}_2 . We therefore implement Pederson's
 4129 Hash for BLS6.

We start with \mathbb{G}_1 . Our goal is to define an 12-bit bounded hash function:

$$H_1 : \{0,1\}^{12} \rightarrow \mathbb{G}_1$$

Since $12 = 3 \cdot 4$ we “randomly” select 4 uniformly distributed generators $\{(38,15), (35,28), (27,34), (38,28)\}$ from \mathbb{G}_1 and use the pseudo-random function from XXX. Therefore, we have to choose a set of 4 randomly generated invertible elements from \mathbb{F}_{13} for every generator. We choose the following:

$$\begin{aligned}
 (38,15) &: \{2,7,5,9\} \\
 (35,28) &: \{11,4,7,7\} \\
 (27,34) &: \{5,3,7,12\} \\
 (38,28) &: \{6,5,1,8\}
 \end{aligned}$$

4130 Our hash function is then computed as follows:

$$\begin{aligned}
 H_1(x_{11}, x_1, \dots, x_0) = & [2 \cdot 7^{x_{11}} \cdot 5^{x_{10}} \cdot 9^{x_9}](38,15) + [11 \cdot 4^{x_8} \cdot 7^{x_7} \cdot 7^{x_6}](35,28) + \\
 & [5 \cdot 3^{x_5} \cdot 7^{x_4} \cdot 12^{x_3}](27,34) + [6 \cdot 5^{x_2} \cdot 1^{x_1} \cdot 8^{x_0}](38,28)
 \end{aligned}$$

4131 Note that $a^x = 1$ when $x = 0$. Hence, those terms can be omitted in the computation. In
 4132 particular, the hash of the 12-bit zero string is given as follows:

WRONG – ORDERING – REDO

$$\begin{aligned}
 H_1(0) = & [2](38,15) + [11](35,28) + [5](27,34) + [6](38,28) = \\
 & (27,34) + (26,34) + (35,28) + (26,9) = (33,9) + (13,28) = (38,28)
 \end{aligned}$$

The hash of 011010101100 is given as follows:

$$\begin{aligned}
 H_1(011010101100) = & \text{WRONG – ORDERING – REDO} \\
 & [2 \cdot 7^0 \cdot 5^1 \cdot 9^1](38,15) + [11 \cdot 4^0 \cdot 7^1 \cdot 7^0](35,28) + [5 \cdot 3^1 \cdot 7^0 \cdot 12^1](27,34) + [6 \cdot 5^1 \cdot 1^0 \cdot 8^0](38,28) = \\
 & [2 \cdot 5 \cdot 9](38,15) + [11 \cdot 7](35,28) + [5 \cdot 3 \cdot 12](27,34) + [6 \cdot 5](38,28) = \\
 & [12](38,15) + [12](35,28) + [11](27,34) + [4](38,28) =
 \end{aligned}$$

TO APPEAR

We can use the same technique to define a 12-bit bounded hash function in \mathbb{G}_2 :

$$H_2 : \{0,1\}^{12} \rightarrow \mathbb{G}_2$$

Again, we “randomly” select 4 uniformly distributed generators $\{(7v^2, 16v^3), (42v^2, 16v^3), (17v^2, 15v^3), (10v^2, 15v^3)\}$ from \mathbb{G}_2 , and use the pseudo-random function from XXX. Therefore, we have to choose a set of 4 randomly generated invertible elements from \mathbb{F}_{13} for every generator:

add reference

$$\begin{aligned} (7v^2, 16v^3) &: \{8, 4, 5, 7\} \\ (42v^2, 16v^3) &: \{12, 1, 3, 8\} \\ (17v^2, 15v^3) &: \{2, 3, 9, 11\} \\ (10v^2, 15v^3) &: \{3, 6, 9, 10\} \end{aligned}$$

Our hash function is then computed like this:

$$H_1(x_{11}, x_{10}, \dots, x_0) = [8 \cdot 4^{x_{11}} \cdot 5^{x_{10}} \cdot 7^{x_9}](7v^2, 16v^3) + [12 \cdot 1^{x_8} \cdot 3^{x_7} \cdot 8^{x_6}](42v^2, 16v^3) + [2 \cdot 3^{x_5} \cdot 9^{x_4} \cdot 11^{x_3}](17v^2, 15v^3) + [3 \cdot 6^{x_2} \cdot 9^{x_1} \cdot 10^{x_0}](10v^2, 15v^3)$$

We extend this to a hash function that maps unbounded bitstrings to \mathbb{G}_2 by precomposing with an actual hash function like MD5, and feed the first 12 bits of its outcome into our previously defined hash function, with $TinyMD5_{\mathbb{G}_2}(s) = H_2(MD5(s)_{11}, \dots, MD5(s)_0)$:

$$TinyMD5_{\mathbb{G}_2} : \{0, 1\}^* \rightarrow \mathbb{G}_2$$

For example, since $MD5(“”) =$

$0xd41d8cd98f00b204e9800998ecf8427e$, and the binary representation of the hexadecimal number $0x27e$ is 001001111110 , we compute $TinyMD5_{\mathbb{G}_2}$ of the empty string as follows:

$$TinyMD5_{\mathbb{G}_2}(“”) = H_2(MD5(s)_{11}, \dots, MD5(s)_0) = H_2(001001111110) =$$

4133

check equation

Chapter 6

Statements

As we have seen in the informal introduction XXX, a SNARK is a short non-interactive argument of knowledge, where the knowledge-proof attests to the correctness of statements like “The prover knows the prime factorization of a given number” or “The prover knows the preimage to a given SHA2 digest value” and similar things. However, human-readable statements like these are imprecise and not very useful from a formal perspective.

In this chapter we therefore look more closely at ways to formalize statements in mathematically rigorous ways, useful for SNARK development. We start by introducing formal languages as a way to define statements properly (section 6.1). We will then look at algebraic circuits and rank-1 constraint systems as two particularly useful ways to define statements in certain formal languages (section 6.2). After that, we will have a look at fundamental building blocks of compilers that compile high-level languages to circuits and associated rank-1 constraint systems.

Proper statement design should be of high priority in the development of SNARKs, since unintended true statements can lead to potentially severe and almost undetectable security vulnerabilities in the applications of SNARKs.

Chapter 1?

6.1 Formal Languages

Formal languages provide the theoretical background in which statements can be formulated in a logically rigorous way and where proving the correctness of any given statement can be realized by computing words in that language.

"rigorous"?

One might argue that the understanding of formal languages is not very important in SNARK development and associated statement design, but terms from that field of research are standard jargon in many papers on zero-knowledge proofs. We therefore believe that at least some introduction to formal languages and how they fit into the picture of SNARK development is beneficial, mostly to give developers a better intuition about where all this is located in the bigger picture of the logic landscape. In addition, formal languages give a better understanding of what a formal proof for a statement actually is.

"proving"?

Roughly speaking, a formal language (or just language for short) is nothing but a set of words, \mathcal{L} . Words, in turn, are strings of letters taken from some alphabet and formed according to some defining rules of the language.

To be more precise, let Σ be any set and Σ^* the set of all finite **tuples** (ordered lists) (x_1, \dots, x_n) of elements x_j from Σ including the empty tuple $() \in \Sigma^*$. Then, a **language** L , in its most general definition, is nothing but a subset of Σ^* . In this context, the set Σ is called the **alphabet** of the language L , elements from Σ are called letters and elements from L are called **words**. The rules that specify which tuples from Σ^* belong to the language and which don't,

are called the **grammar** of the language. S: I suggest adding an example based on English, e.g. “tea” and “eat” are words of English, but “aet” and “tae” are not

Add ex-ample

If L_1 and L_2 are two formal languages over the same alphabet, we call L_1 and L_2 **equivalent** if they generate the same set of words.

M: 1:1 correspondence might actually be wrong

Decision Functions Our previous definition of formal languages is very general and many subclasses of languages are known in the literature. However, in the context of SNARK development, languages are commonly defined as **decision problems** where a so-called **deciding relation** $R \subset \Sigma^*$ decides whether a given tuple $x \in \Sigma^*$ is a word in the language or not. If $x \in R$ then x is a word in the associated language L_R and if $x \notin R$ then not. The relation R therefore summarizes the grammar of language L_R .

Unfortunately, in some literature on proof systems, $x \in R$ is often written as $R(x)$, which is misleading since in general R is not a function but a relation in Σ^* . For the sake of this book, we therefore adopt a different point of view and work with what we might call a **decision function** instead:

$$R : \Sigma^* \rightarrow \{true, false\} \quad (6.1)$$

Decision functions decide if a tuple $x \in \Sigma^*$ is an element of a language or not. In case a decision function is given, the associated language itself can be written as the set of all tuples that are decided by R , i.e as the set:

$$L_R := \{x \in \Sigma^* \mid R(x) = true\} \quad (6.2)$$

In the context of formal languages and decision problems, a **statement** S is the claim that language L contains a word x , i.e a statement claims that there exist some $x \in L$. A constructive **proof** for statement S is given by some string $P \in \Sigma^*$ and such a proof is **verified** by checking $R(P) = true$. In this case, P is called an **instance** of the statement S .

While the term **language** might suggest a deeper relation to the well known **natural languages** like English, formal languages and natural languages differ in many ways. The following examples will provide some intuition about formal languages, highlighting the concepts of statements, proofs and instances:

Example 105 (Alternating Binary strings). To consider a very basic formal language with an almost trivial grammar, consider the set $\{0, 1\}$ of the two letters 0 and 1 as our alphabet Σ and imply the rule that a proper word must consist of alternating binary letters of arbitrary length.

Then, the associated language L_{alt} is the set of all finite binary tuples, where a 1 must follow a 0 and vice versa. So, for example, $(1, 0, 1, 0, 1, 0, 1, 0, 1) \in L_{alt}$ is a proper word in this languages as is $(0) \in L_{alt}$ or the empty word $() \in L_{alt}$. However, the binary tuple $(1, 0, 1, 0, 1, 0, 1, 1, 1) \in \{0, 1\}^*$ is not a proper word, as it violates the grammar of L_{alt} : the last3 letters are all 1. Furthermore, the tuple $(0, A, 0, A, 0, A, 0)$ is not a proper word, as not all its letters are not from the proper alphabet: we defined the alphabet Σ as the set $\{0, 1\}$, and A is not part of that set.

Attempting to write the grammar of this language in a more formal way, we can define the following decision function:

$$R : \{0, 1\}^* \rightarrow \{true, false\} ; (x_0, x_1, \dots, x_n) \mapsto \begin{cases} true & x_{j-1} \neq x_j \text{ for all } 1 \leq j \leq n \\ false & \text{else} \end{cases}$$

We can use this function to decide if arbitrary **binary tuples** are words in L_{alt} or not. Some examples are given below:

binary tuples

- $R(1, 0, 1) = true$,

- $R(0) = \text{true}$,
- $R() = \text{true}$,
- but $R(1, 1) = \text{false}$.

Inside our language L_{alt} , it makes sense to claim the following statement: “There exists an alternating string.” One way to prove this statement constructively is by providing an actual instance, that is, finding actual alternating string like $x = (1, 0, 1)$. Constructing string $(1, 0, 1)$ therefore proves the statement “There exists an alternating string.”, because it is easy to verify that $R(1, 0, 1) = \text{true}$.

Example 106 (Programming Language). Programming languages are a very important class of formal languages. For these languages, the alphabet is usually (a subset) of the ASCII table, and the grammar is defined by the rules of the programming language’s compiler. Words, then, are nothing but properly written computer programs that the compiler accepts. The compiler can therefore be interpreted as the decision function.

To give an unusual example strange enough to highlight the point, consider the programming language Malbolge as defined in XXX. This language was specifically designed to be almost impossible to use and writing programs in this language is a difficult task. An interesting claim is therefore the statement: “There exists a computer program in Malbolge”. As it turned out, proving this statement constructively, that is, by providing an actual instance of such a program, was not an easy task, as it took two years after the introduction of Malbolge to write a program that its compiler accepts. So, for two years, no one was able to prove the statement constructively.

To look at this high-level description more formally, we write $L_{Malbolge}$ for the language that uses the ASCII table as its alphabet and its words are tuples of ASCII letters that the Malbolge compiler accepts. Proving the statement “There exists a computer program in Malbolge” is then equivalent to the task of finding some word $x \in L_{Malbolge}$. The string

(=<#9] 6ZY327Uv4-QsqpMn&+Ij''E%e{Ab w=_:]Kw%o44Uqp0/Q?xNvL:'H%c#DD2^WV>gY;dts76qKJImZk j

is an example of such a proof, as it is excepted by the Malbolge compiler and is compiled to an executable binary that displays “Hello, World.” (See XXX). In this example, the Malbolge compiler therefore serves as the verification process.

Example 107 (The Empty Language). To see that not every language has even one word, consider the alphabet $\Sigma = \mathbb{Z}_6$, where \mathbb{Z}_6 is the ring of modular 6 arithmetics as derived in example 9 in chapter ??, together with the following decision function

$$R_\emptyset : \mathbb{Z}_6^* \rightarrow \{\text{true}, \text{false}\} ; (x_1, \dots, x_n) \mapsto \begin{cases} \text{true} & n = 4 \text{ and } x_1 \cdot x_1 = 2 \\ \text{true} & \text{else} \end{cases}$$

We write L_\emptyset for the associated language. As we can see from the multiplication table of \mathbb{Z}_6 in example 9 in chapter ??, the ring \mathbb{Z}_6 does not contain any element x such that $x^2 = 2$, which implies $R_\emptyset(x_1, \dots, x_n) = \text{false}$ for all tuples $(x_1, \dots, x_n) \in \Sigma^*$. The language therefore does not contain any words. Proving the statement “There exists a word in L_\emptyset ” constructively by providing an instance is therefore impossible. The verification will never check any tuple.

Example 108 (3-Factorization). We will use the following simple example repeatedly throughout this book. The task is to develop a SNARK that proves knowledge of three factors of an element from the finite field \mathbb{F}_{13} . There is nothing particularly useful about this example from

an application point of view, however, in a sense, it is the most simple example that gives rise to a non trivial SNARK in some of the most common zero-knowledge proof systems.

Formalizing the high-level description, we use $\Sigma := \mathbb{F}_{13}$ as the underlying alphabet of this problem and define the language $L_{3.fac}$ to consists of those tuples of field elements from \mathbb{F}_{13} that contain exactly 4 letters w_1, w_2, w_3, w_4 which satisfy the equation $w_1 \cdot w_2 \cdot w_3 = w_4$.

So, for example, the tuple $(2, 12, 4, 5)$ is a word in $L_{3.fac}$, while neither $(2, 12, 11)$, nor $(2, 12, 4, 7)$ nor $(2, 12, 7, 168)$ are words in $L_{3.fac}$ as they don't satisfy the grammar or are not define over the proper alphabet.

We can describe the language $L_{3.fac}$ more formally by introducing a decision function (as described in equation 6.1):

$$R_{3.fac} : \mathbb{F}_{13}^* \rightarrow \{true, false\} ; (x_1, \dots, x_n) \mapsto \begin{cases} true & n = 4 \text{ and } x_1 \cdot x_2 \cdot x_3 = x_4 \\ false & else \end{cases}$$

Having defined the language $L_{3.fac}$, it then makes sense to claim the statement “There is a word in $L_{3.fac}$ ”. The way $L_{3.fac}$ is designed, this statement is equivalent to the statement “There are four elements w_1, w_2, w_3, w_4 from the finite field \mathbb{F}_{13} such that the equation $w_1 \cdot w_2 \cdot w_3 = w_4$ holds.”

Proving the correctness of this statement constructively means to actually find some concrete field elements like $x_1 = 2, x_2 = 12, x_3 = 4$ and $x_4 = 5$ that satisfy the relation $R_{3.fac}$. The tuple $(2, 12, 4, 5)$ is therefore a constructive proof for the statement and the computation $R_{3.fac}(2, 12, 4, 5) = true$ is a verification of that proof. In contrast, the tuple $(2, 12, 4, 7)$ is not a proof of the statement, since the check $R_{3.fac}(2, 12, 4, 7) = false$ does not verify the proof.

Example 109 (Tiny-jubjub Membership). In one of our main examples, we derive a SNARK that proves a pair (x, y) of field elements from \mathbb{F}_{13} to be a point on the tiny-jubjub curve in its Edwards form (see section 5.1.3).

In the first step, we define a language such that points on the tiny-jubjub curve are in 1:1 correspondence with words in that language.

Since the tiny-jubjub curve is an elliptic curve over the field \mathbb{F}_{13} , we choose the alphabet $\Sigma = \mathbb{F}_{13}$. In this case, the set \mathbb{F}_{13}^* consists of all finite strings of field elements from \mathbb{F}_{13} . To define the grammar, recall from 68 that a point on the tiny-jubjub curve is a pair (x, y) of field elements such that $3 \cdot x^2 + y^2 = 1 + 8 \cdot x^2 \cdot y^2$. We can use this equation to derive the following decision function:

$$R_{tiny.jj} : \mathbb{F}_{13}^* \rightarrow \{true, false\} ; (x_1, \dots, x_n) \mapsto \begin{cases} true & n = 2 \text{ and } 3 \cdot x_1^2 + x_2^2 = 1 + 8 \cdot x_1^2 \cdot x_2^2 \\ false & else \end{cases}$$

The associated language $L_{tiny.jj}$ is then given as the set of all strings from \mathbb{F}_{13}^* that are mapped onto *true* by $R_{tiny.jj}$. We get

$$L_{tiny.jj} = \{(x_1, \dots, x_n) \in \mathbb{F}_{13}^* \mid R_{tiny.jj}(x_1, \dots, x_n) = true\}$$

We can claim the statement “There is a word in $L_{tiny.jj}$ ” and because $L_{tiny.jj}$ is defined by $R_{tiny.jj}$, this statement is equivalent to the claim “The tiny-jubjub curve in its Edwards form has curve a point.”

A constructive proof for this statement is a pair (x, y) of field elements that satisfies the Edwards equation. Example 68 therefore implies that the tuple $(11, 6)$ is a constructive proof and the computation $R_{tiny.jj}(11, 6) = true$ is a proof verification. In contrast, the tuple $(1, 1)$ is not a proof of the statement, since the check $R_{tiny.jj}(1, 1) = false$ does not verify the proof.

Are we using w and x interchangeably or is there a difference between them?

check reference

jubjub

check reference

check reference

check wording

check reference

Exercise 50. Consider exercise XXX again. Define a decision function such that the associated language $L_{Exercise_{XXX}}$ consist precisely of all solutions to the equation $5x + 4 = 28 + 2x$ over \mathbb{F}_{13} . Provide a constructive proof for the claim: “There exist a word in $L_{Exercise_{XXX}}$ and verify the proof.

Exercise 51. Consider the modular 6 arithmetics \mathbb{Z}_6 from example 9 in chapter ??, the alphabet $\Sigma = \mathbb{Z}_6$ and the decision function

$$R_{example_9} : \Sigma^* \rightarrow \{true, false\} ; x \mapsto \begin{cases} true & x.len() = 1 \text{ and } 3 \cdot x + 3 = 0 \\ false & \text{else} \end{cases}$$

Compute all words in the associated language $L_{example_9}$, provide a constructive proof for the statement “There exist a word in $L_{example_example_9}$ ” and verify the proof.

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references

Instance and Witness As we have seen in the previous paragraph, statements provide membership claims in formal languages, and instances serve as constructive proofs for those claims. However, in the context of **zero-knowledge** proof systems, our naive notion of constructive proofs is refined in such a way that its possible to hide parts of the proof instance and still be able to prove the statement. In this context, it is therefore necessary to split a proof into a **public part** called the **instance** and a private part called a **witness**.

To account for this separation of a proof instance into a public and a private part, our previous definition of formal languages needs a refinement in the context of zero-knowledge proof systems. Instead of a single alphabet, the refined definition considers two alphabets Σ_I and Σ_W , and a decision function defined as follows:

$$R : \Sigma_I^* \times \Sigma_W^* \rightarrow \{true, false\} ; (i; w) \mapsto R(i; w) \quad (6.3)$$

Words are therefore tuples $(i; w) \in \Sigma_I^* \times \Sigma_W^*$ with $R(i; w) = true$. The refined definition differentiates between public inputs $i \in \Sigma_I$ and private inputs $w \in \Sigma_W$. The public input i is called an **instance** and the private input w is called a **witness** of R .

If a decision function is given, the associated language is defined as the set of all tuples from the underlying alphabet that are verified by the decision function:

$$L_R := \{(i; w) \in \Sigma_I^* \times \Sigma_W^* \mid R(i; w) = true\} \quad (6.4)$$

In this refined context, a **statement** S is a claim that, given an instance $i \in \Sigma_I^*$, there is a witness $w \in \Sigma_W^*$ such that language L contains a word $(i; w)$. A constructive **proof** for statement S is given by some string $P = (i; w) \in \Sigma_I^* \times \Sigma_W^*$ and a proof is **verified** by checking $R(P) = true$.

It is worth understanding the difference between statements as defined in XXX and the refined notion of statements from this paragraph. While statements in the sense of the previous paragraph can be seen as membership claims, statements in the refined definition can be seen as knowledge-proofs, where a prover claims knowledge of a witness for a given instance. For a more detailed discussion on this topic see [XXX sec 1.4]

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Example 110 (SHA256 – Knowledge of Preimage). One of the most common examples in the context of zero-knowledge proof systems is the **knowledge-of-a-preimage proof** for some cryptographic hash function like *SHA256*, where a publicly known *SHA256* digest value is given, and the task is to prove knowledge of a preimage for that digest under the *SHA256* function, without revealing that preimage.

To understand this problem in detail, we have to introduce a language able to describe the knowledge-of-preimage problem in such a way that the claim “Given digest i , there is a

preimage w such that $SHA256(w) = i$ ” becomes a statement in that language. Since $SHA256$ is a function

$$SHA256 : \{0, 1\}^* \rightarrow \{0, 1\}^{256}$$

that maps binary strings of arbitrary length onto binary strings of length 256 and we want to prove knowledge of preimages, we have to consider binary strings of size 256 as instances and binary strings of arbitrary length as witnesses.

An appropriate alphabet Σ_I for the set of all instances and an appropriate alphabet Σ_W for the set of all witnesses is therefore given by the set $\{0, 1\}$ of the two binary letters and a proper decision function is given by:

$$R_{SHA256} : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \{true, false\};$$

$$(i; w) \mapsto \begin{cases} true & i.len() = 256, i = SHA256(w) \\ false & else \end{cases}$$

We write L_{SHA256} for the associated language and note that it consists of words, which are tuples $(i; w)$ such that the instance i is the $SHA256$ image of the witness w .

Given some instance $i \in \{0, 1\}^{256}$, a statement in L_{SHA256} is the claim “Given digest i , there is a preimage w such that $SHA256(w) = i$ ”, which is exactly what the knowledge-of-preimage problem is about. A constructive proof for this statement is therefore given by a preimage w to the digest i and proof verification is achieved by checking that $SHA256(w) = i$.

Example 111 (3-factorization). To give an intuition about the implication of refined languages, consider $L_{3, fac}$ from example 108 again. As we have seen, a constructive proof in $L_{3, fac}$ is given by 4 field elements x_1, x_2, x_3 and x_4 from \mathbb{F}_{13} such that the product in modular 13 arithmetics of the first three elements is equal to the 4'th element.

Splitting words from $L_{3, fac}$ into private and public parts, we can reformulate the problem and introduce different levels of privacy into the problem. For example, we could reformulate the membership statement of $L_{3, fac}$ into a statement where all factors x_1, x_2, x_3 of x_4 are private and only the product x_4 is public. A statement for this reformulation is then expressed by the claim: “Given a publicly known field element x_4 , there are three private factors of x_4 ”. Assuming some instance x_4 , a constructive proof for the associated knowledge claim is then provided by any tuple (x_1, x_2, x_3) such that $x_1 \cdot x_2 \cdot x_3 = x_4$.

At this point, it is important to note that, while constructive proofs in the refinement don't look very different from constructive proofs in the original language, we will see in XXX that there are proof systems able to prove the statement (at least with high probability) without revealing anything about the factors x_1, x_2 , or x_3 . This is why the importance of the refinement only becomes clear once more elaborate proofing methods beyond naive constructive proofs are provided.

We can formalize this new language, which we might call L_{3, fac_zk} , by defining the following decision function:

Definition 6.1.0.1.

$$R_{3, fac_zk} : \mathbb{F}_{13}^* \times \mathbb{F}_{13}^* \rightarrow \{true, false\};$$

$$((i_1, \dots, i_n); (w_1, \dots, w_m)) \mapsto \begin{cases} true & n = 1, m = 3, i_1 = w_1 \cdot w_2 \cdot w_3 \\ false & else \end{cases}$$

The associated language L_{3, fac_zk} is defined by all tuples from $\mathbb{F}_{13}^* \times \mathbb{F}_{13}^*$ that are mapped onto *true* under the decision function R_{3, fac_zk} .

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ence

Considering the distinction we made between the instance and the witness part in L_{3, fac_zk} , one might ask why we chose the factors x_1, x_2 and x_3 to be the witness and the product x_4 to be the instance and why we didn't choose another combination? This was an arbitrary choice in the example. Every other combination of private and public factors would be equally valid. For example, it would be possible to declare all variables as private or to declare all variables as public. Actual choices are determined by the application only.

Example 112 (The Tiny-Jubjub Curve). Consider the language $L_{tiny.jj}$ from example 109. As we have seen, a constructive proof in $L_{tiny.jj}$ is given by a pair (x_1, x_2) of field elements from \mathbb{F}_{13} such that the pair is a point of the tiny-jubjub curve in its Edwards representation.

check
reference

We look at a reasonable splitting of words from $L_{tiny.jj}$ into private and public parts. The two obvious choices are to either choose both coordinates x_1 as x_2 as public inputs, or to choose both coordinates x_1 as x_2 as private inputs.

In case both coordinates are public, we define the grammar of the associated language by introducing the following decision function:

$$R_{tiny.jj.1} : \mathbb{F}_{13}^* \times \mathbb{F}_{13}^* \rightarrow \{true, false\};$$

$$(I_1, \dots, I_n; W_1, \dots, W_m) \mapsto \begin{cases} true & n = 2, m = 0 \text{ and } 3 \cdot I_1^2 + I_2^2 = 1 + 8 \cdot I_1^2 \cdot I_2^2 \\ false & else \end{cases}$$

The language $L_{tiny.jj.1}$ is defined as the set of all strings from $\mathbb{F}_{13}^* \times \mathbb{F}_{13}^*$ that are mapped onto *true* by $R_{tiny.jj.1}$.

In case both coordinates are private, we define the grammar of the associated refined language by introducing the following decision function:

$$R_{tiny.jj.zk} : \mathbb{F}_{13}^* \times \mathbb{F}_{13}^* \rightarrow \{true, false\};$$

$$(I_1, \dots, I_n; W_1, \dots, W_m) \mapsto \begin{cases} true & n = 0, m = m \text{ and } 3 \cdot W_1^2 + W_2^2 = 1 + 8 \cdot W_1^2 \cdot W_2^2 \\ false & else \end{cases}$$

The language $L_{tiny.jj.zk}$ is defined as the set of all strings from $\mathbb{F}_{13}^* \times \mathbb{F}_{13}^*$ that are mapped onto *true* by $R_{tiny.jj.zk}$.

Exercise 52. Consider the modular 6 arithmetics \mathbb{Z}_6 from example 9 in chapter ?? as alphabets Σ_I and Σ_W and the following decision function

check
reference

$$R_{linear} : \Sigma^* \times \Sigma^* \rightarrow \{true, false\};$$

$$(i; w) \mapsto \begin{cases} true & i.len() = 3 \text{ and } w.len() = 1 \text{ and } i_1 \cdot w_1 + i_2 = i_3 \\ false & else \end{cases}$$

Which of the following instances (i_1, i_2, i_3) has a proof of knowledge in L_{linear} ?

• (3, 3, 0)

• (2, 1, 0)

• (4, 4, 2)

Exercise 53 (Edwards Addition on Tiny-Jubjub). Consider the tiny-jubjub curve together with its Edwards addition law from example XXX. Define an instance alphabet Σ_I , a witness alphabet Σ_W and a decision function R_{add} with associated language L_{add} such that a string $(i; w) \in \Sigma_I^* \times$

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ence

4356 Σ_W^* is a word in L_{add} if and only if i is a pair of curve points on the tiny-jubjub curve in Edwards
 4357 form and w is the Edwards sum of those curve points.

4358 Choose some instance $i \in \Sigma_I^*$, provide a constructive proof for the statement “There is a
 4359 witness $w \in \Sigma_W^*$ such that $(i;w)$ is a word in L_{add} ” and verify that proof. Then find some
 4360 instance $i \in \Sigma_I^*$ such that i has no knowledge proof in L_{add} .

4361 **Modularity** From a developers perspective, it is often useful to construct complex statements
 4362 and their representing languages from simple ones. In the context of zero-knowledge proof
 4363 systems, those simple building blocks are often called **gadgets**, and gadget libraries usually
 4364 contain representations of atomic types like booleans, integers, various hash functions, elliptic
 4365 curve cryptography and many more. In order to synthesize statements, developers then combine
 4366 predefined gadgets into complex logic. We call the ability to combine statements into more
 4367 complex statements **modularity**.

4368 To understand the concept of modularity on the level of formal languages defined by deci-
 4369 sion functions, we need to look at the **intersection** of two languages, which exists whenever
 4370 both languages are defined over the same alphabet. In this case, the intersection is a language
 4371 that consists of strings which are words in both languages.

4372 To be more precise, let L_1 and L_2 be two languages defined over the same instance and
 4373 witness alphabets Σ_I and Σ_W . Then the intersection $L_1 \cap L_2$ of L_1 and L_2 is defined as

$$L_1 \cap L_2 := \{x \mid x \in L_1 \text{ and } x \in L_2\} \quad (6.5)$$

4374 If both languages are defined by decision functions R_1 and R_2 , the following function is a
 4375 decision function for the intersection language $L_1 \cap L_2$:

$$R_{L_1 \cap L_2} : \Sigma_I^* \times \Sigma_W^* \rightarrow \{true, false\}; (i, w) \mapsto R_1(i, w) \text{ and } R_2(i, w) \quad (6.6)$$

4376 Thus, the intersection of two decision-function-based languages is a also decision-function-
 4377 based language. This is important from an implementations point of view: It allows us to
 4378 construct complex decision functions, their languages and associated statements from simple
 4379 building blocks. Given a publicly known instance $i \in \Sigma_I^*$ a statement in an intersection language
 4380 then claims knowledge of a witness that satisfies all relations simultaneously.

4381 6.2 Statement Representations

4382 As we have seen in the previous section, formal languages and their definitions by decision
 4383 functions are a powerful tool to describe statements in a formally rigorous manner.

4384 However, from the perspective of existing zero-knowledge proof systems, not all ways to
 4385 actually represent decision functions are equally useful. Depending on the proof system, some
 4386 are more suitable than others. In this section, will describe two of the most common ways to
 4387 represent decision functions and their statements.

4388 6.2.1 Rank-1 Quadratic Constraint Systems

4389 Although decision functions are expressible in various ways, many contemporary proofing sys-
 4390 tems require the deciding relation to be expressed in terms of a system of quadratic equations
 4391 over a finite field. This is true in particular for pairing-based proofing systems like XXX,
 4392 roughly because it is possible to check solutions to those equations “in the exponent” of pairing-
 4393 friendly cryptographic groups.

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ence

In this section, we will therefore have a closer look at a particular type of quadratic equation called **rank-1 quadratic constraints systems**, which are a common standard in zero-knowledge proof systems. We will start with a general introduction to those systems and then look at their relation to formal languages. We will look into a common way to compute solutions to those systems, and then show how a simple compiler might derive rank-1 constraint systems from more high-level programming code.

R1CS representation To understand what **rank-1 (quadratic) constraint systems** are in detail, let \mathbb{F} be a field, n, m and $k \in \mathbb{N}$ three numbers and a_j^i, b_j^i and $c_j^i \in \mathbb{F}$ constants from \mathbb{F} for every index $0 \leq j \leq n+m$ and $1 \leq i \leq k$. Then a rank-1 constraint system (R1CS) is defined as follows:

Definition 6.2.1.1. R1CS representation

$$\begin{aligned} (a_0^1 + \sum_{j=1}^n a_j^1 \cdot I_j + \sum_{j=1}^m a_{n+j}^1 \cdot W_j) \cdot (b_0^1 + \sum_{j=1}^n b_j^1 \cdot I_j + \sum_{j=1}^m b_{n+j}^1 \cdot W_j) &= c_0^1 + \sum_{j=1}^n c_j^1 \cdot I_j + \sum_{j=1}^m c_{n+j}^1 \cdot W_j \\ &\vdots \\ (a_0^k + \sum_{j=1}^n a_j^k \cdot I_j + \sum_{j=1}^m a_{n+j}^k \cdot W_j) \cdot (b_0^k + \sum_{j=1}^n b_j^k \cdot I_j + \sum_{j=1}^m b_{n+j}^k \cdot W_j) &= c_0^k + \sum_{j=1}^n c_j^k \cdot I_j + \sum_{j=1}^m c_{n+j}^k \cdot W_j \end{aligned}$$

If a rank-1 constraint system is given, the parameter k is called the **number of constraints**. If a tuple $(I_1, \dots, I_n; W_1, \dots, W_m)$ of field elements satisfies these equations, (I_1, \dots, I_n) is called an **instance** and (W_1, \dots, W_m) is called an associated **witness** of the system.

Remark 3 (Matrix notation). The presentation of rank-1 constraint systems can be simplified using the notation of vectors and matrices, which abstracts over the indices. In fact if $x = (1, I, W) \in \mathbb{F}^{1+n+m}$ is a $(n+m+1)$ -dimensional vector, A, B, C are $(n+m+1) \times k$ -dimensional matrices and \odot is the **Schur/Hadamard product**, then a R1CS can be written as

$$Ax \odot Bx = Cx$$

However, since we did not introduced matrix calculus in the book, we use XXX as the defining equation for rank-1 constraints systems. We only highlighted the matrix notation, because it is sometimes used in the literature.

Generally speaking, the idea of a rank-1 constraint system is to keep track of all the values that any variable can assume during a computation and to bind the relationships among all those variables that are implied by the computation itself. Enforcing relations between all the steps of a computer program, the execution is then constrained to be computed in exactly the expected way without any opportunity for deviations. In this sense, solutions to rank-1 constraint systems are proofs of proper program execution.

Example 113 (3-Factorization). To provide a better intuition of rank-1 constraint systems, consider the language L_{3, fac_zk} from example 108 again. As we have seen, L_{3, fac_zk} consists of words $(I_1; W_1, W_2, W_3)$ over the alphabet \mathbb{F}_{13} such that $I_1 = W_1 \cdot W_2 \cdot W_3$. We show how to rewrite the decision function as a rank-1 constraint system.

Since R1CS are systems of quadratic equations, expressions like $W_1 \cdot W_2 \cdot W_3$ which contain products of more than two factors (which are therefore not quadratic) have to be rewritten in a process often called **flattening**. To flatten the defining equation $I_1 = W_1 \cdot W_2 \cdot W_3$ of L_{3, fac_zk} we introduce a new variable W_4 , which captures two of the three multiplications in $W_1 \cdot W_2 \cdot W_3$. We get the following two constraints

$$\begin{aligned} W_1 \cdot W_2 &= W_4 && \text{constraint 1} \\ W_4 \cdot W_3 &= I_1 && \text{constraint 2} \end{aligned}$$

Schur/Hadamard product

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Given some instance I_1 , any solution (W_1, W_2, W_3, W_4) to this system of equations provides a solution to the original equation $I_1 = W_1 \cdot W_2 \cdot W_3$ and vice versa. Both equations are therefore equivalent in the sense that solutions are in a 1:1 correspondence.

Looking at both equations, we see how each constraint enforces a step in the computation. In fact, the first constraint forces any computation to multiply the witness W_1 and W_2 first. Otherwise it would not be possible to compute the witness W_4 , which is needed to solve the second constraint. Witness W_4 therefore expresses the constraining of an intermediate computational state.

At this point, one might ask why equation 1 constrains the system to compute $W_1 \cdot W_2$ first, since computing $W_2 \cdot W_3$, or $W_1 \cdot W_3$ in the beginning and then multiplying with the remaining factor gives the exact same result. The reason is that the way we designed the R1CS prohibits any of these alternative computations, which shows that R1CS are in general **not unique** descriptions of a language: many different R1CS are able to describe the same problem.

To see that the two quadratic equations qualify as a rank-1 constraint system, choose the parameter $n = 1, m = 4$ and $k = 2$ as well as

$$\begin{array}{llllll} a_0^1 = 0 & a_1^1 = 0 & a_2^1 = 1 & a_3^1 = 0 & a_4^1 = 0 & a_5^1 = 0 \\ a_0^2 = 0 & a_1^2 = 0 & a_2^2 = 0 & a_3^2 = 0 & a_4^2 = 0 & a_5^2 = 1 \\ \\ b_0^1 = 0 & b_1^1 = 0 & b_2^1 = 0 & b_3^1 = 1 & b_4^1 = 0 & b_5^1 = 0 \\ b_0^2 = 0 & b_1^2 = 0 & b_2^2 = 0 & b_3^2 = 0 & b_4^2 = 1 & b_5^2 = 0 \\ \\ c_0^1 = 0 & c_1^1 = 0 & c_2^1 = 0 & c_3^1 = 0 & c_4^1 = 0 & c_5^1 = 1 \\ c_0^2 = 0 & c_1^2 = 1 & c_2^2 = 0 & c_3^2 = 0 & c_4^2 = 0 & c_5^2 = 0 \end{array}$$

With this choice, the rank-1 constraint system of our 3-factorization problem can be written in its most general form as follows:

$$\begin{aligned} (a_0^1 + a_1^1 I_1 + a_2^1 W_1 + a_3^1 W_2 + a_4^1 W_3 + a_5^1 W_4) \cdot (b_0^1 + b_1^1 I_1 + b_2^1 W_1 + b_3^1 W_2 + b_4^1 W_3 + b_5^1 W_4) &= (c_0^1 + c_1^1 I_1 + c_2^1 W_1 + c_3^1 W_2 + c_4^1 W_3 + c_5^1 W_4) \\ (a_0^2 + a_1^2 I_1 + a_2^2 W_2 + a_3^2 W_2 + a_4^2 W_3 + a_5^2 W_4) \cdot (b_0^2 + b_1^2 I_1 + b_2^2 W_2 + b_3^2 W_2 + b_4^2 W_3 + b_5^2 W_4) &= (c_0^2 + c_1^2 I_1 + c_2^2 W_2 + c_3^2 W_2 + c_4^2 W_3 + c_5^2 W_4) \end{aligned}$$

Example 114 (The Tiny-Jubjub curve). Consider the languages $L_{\text{tiny.jj.1}}$ from example 109, which consist of words (I_1, I_2) over the alphabet \mathbb{F}_{13} such that $3 \cdot I_1^2 + I_2^2 = 1 + 8 \cdot I_1^2 \cdot I_2^2$.

We derive a rank-1 constraint system such that its associated language is equivalent to $L_{\text{tiny.jj.1}}$. To achieve this, we first rewrite the defining equation:

$$\begin{aligned} 3 \cdot I_1^2 + I_2^2 &= 1 + 8 \cdot I_1^2 \cdot I_2^2 && \Leftrightarrow \\ 0 &= 1 + 8 \cdot I_1^2 \cdot I_2^2 - 3 \cdot I_1^2 - I_2^2 && \Leftrightarrow \\ 0 &= 1 + 8 \cdot I_1^2 \cdot I_2^2 + 10 \cdot I_1^2 + 12 \cdot I_2^2 \end{aligned}$$

Since R1CSs are systems of quadratic equations, we have to reformulate this expression into a system of quadratic equations. To do so, we have to introduce new variables that constrain intermediate steps in the computation and we have to decide if those variables should be public or private. We decide to declare all new variables as private and get the following constraints

$$\begin{array}{ll} I_1 \cdot I_1 = W_1 & \text{constraint 1} \\ I_2 \cdot I_2 = W_2 & \text{constraint 2} \\ (8 \cdot W_1) \cdot W_2 = W_3 & \text{constraint 3} \\ (12 \cdot W_2 + W_3 + 10 \cdot W_1 + 1) \cdot 1 = 0 & \text{constraint 4} \end{array}$$

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To see that these four quadratic equations qualify as a rank-1 constraint system according to definition XXX, choose the parameter $n = 2$, $m = 3$ and $k = 4$:

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$$\begin{array}{llllll} a_0^1 = 0 & a_1^1 = 1 & a_2^1 = 0 & a_3^1 = 0 & a_4^1 = 0 & a_5^1 = 0 \\ a_0^2 = 0 & a_1^2 = 0 & a_2^2 = 1 & a_3^2 = 0 & a_4^2 = 0 & a_5^2 = 0 \\ a_0^3 = 0 & a_1^3 = 0 & a_2^3 = 0 & a_3^3 = 8 & a_4^3 = 0 & a_5^3 = 0 \\ a_0^4 = 1 & a_1^4 = 0 & a_2^4 = 0 & a_3^4 = 10 & a_4^4 = 12 & a_5^4 = 1 \end{array}$$

$$\begin{array}{llllll} b_0^1 = 0 & b_1^1 = 1 & b_2^1 = 0 & b_3^1 = 0 & b_4^1 = 0 & b_5^1 = 0 \\ b_0^2 = 0 & b_1^2 = 0 & b_2^2 = 1 & b_3^2 = 0 & b_4^2 = 0 & b_5^2 = 0 \\ b_0^3 = 0 & b_1^3 = 0 & b_2^3 = 0 & b_3^3 = 0 & b_4^3 = 1 & b_5^3 = 0 \\ b_0^4 = 1 & b_1^4 = 0 & b_2^4 = 0 & b_3^4 = 0 & b_4^4 = 0 & b_5^4 = 0 \end{array}$$

$$\begin{array}{llllll} c_0^1 = 0 & c_1^1 = 0 & c_2^1 = 0 & c_3^1 = 1 & c_4^1 = 0 & c_5^1 = 0 \\ c_0^2 = 0 & c_1^2 = 0 & c_2^2 = 0 & c_3^2 = 0 & c_4^2 = 1 & c_5^2 = 0 \\ c_0^3 = 0 & c_1^3 = 0 & c_2^3 = 0 & c_3^3 = 0 & c_4^3 = 0 & c_5^3 = 1 \\ c_0^4 = 0 & c_1^4 = 0 & c_2^4 = 0 & c_3^4 = 0 & c_4^4 = 0 & c_5^4 = 0 \end{array}$$

With this choice, the rank-1 constraint system of our tiny-jubjub curve point problem can be written in its most general form as follows:

$$\begin{aligned} (a_0^1 + a_1^1 I_1 + a_2^1 I_2 + a_3^1 W_1 + a_4^1 W_2 + a_5^1 W_3) \cdot (b_0^1 + b_1^1 I_1 + b_2^1 I_2 + b_3^1 W_1 + b_4^1 W_2 + b_5^1 W_3) &= (c_0^1 + c_1^1 I_1 + c_2^1 I_2 + c_3^1 W_1 + c_4^1 W_2 + c_5^1 W_3) \\ (a_0^2 + a_1^2 I_1 + a_2^2 I_2 + a_3^2 W_1 + a_4^2 W_2 + a_5^2 W_3) \cdot (b_0^2 + b_1^2 I_1 + b_2^2 I_2 + b_3^2 W_1 + b_4^2 W_2 + b_5^2 W_3) &= (c_0^2 + c_1^2 I_1 + c_2^2 I_2 + c_3^2 W_1 + c_4^2 W_2 + c_5^2 W_3) \\ (a_0^3 + a_1^3 I_1 + a_2^3 I_2 + a_3^3 W_1 + a_4^3 W_2 + a_5^3 W_3) \cdot (b_0^3 + b_1^3 I_1 + b_2^3 I_2 + b_3^3 W_1 + b_4^3 W_2 + b_5^3 W_3) &= (c_0^3 + c_1^3 I_1 + c_2^3 I_2 + c_3^3 W_1 + c_4^3 W_2 + c_5^3 W_3) \\ (a_0^4 + a_1^4 I_1 + a_2^4 I_2 + a_3^4 W_1 + a_4^4 W_2 + a_5^4 W_3) \cdot (b_0^4 + b_1^4 I_1 + b_2^4 I_2 + b_3^4 W_1 + b_4^4 W_2 + b_5^4 W_3) &= (c_0^4 + c_1^4 I_1 + c_2^4 I_2 + c_3^4 W_1 + c_4^4 W_2 + c_5^4 W_3) \end{aligned}$$

4435 In what follows, we write L_{jubjub} for the associated language that consists of solutions to the
4436 R1CS.

4437 To see that L_{jubjub} is equivalent to $L_{tiny.jj.1}$, let $(I_1, I_2; W_1, W_2, W_3)$ be a word in L_{jubjub} , then
4438 (I_1, I_2) is a word in $L_{tiny.jj.1}$, since the defining R1CS of L_{jubjub} implies that I_1 and I_2 satisfy the
4439 Edwards equation of the tiny-jubjub curve. On the other hand, let (I_1, I_2) be a word in $L_{tiny.jj.1}$.
4440 Then $(I_1, I_2; I_1^2, I_2^2, 8 \cdot I_1^2 \cdot I_2^2)$ is a word in L_{jubjub} and both maps are inverses of each other.

4441 **Exercise 54.** Consider the language $L_{tiny.jj_zk}$ and define a rank-1 constraint relation with a
4442 decision function such that the associated language is equivalent to $L_{tiny.jj_zk}$.

4443 **R1CS Satisfiability** To understand how rank-1 constraint systems define formal languages,
4444 observe that every R1CS over a field \mathbb{F} defines a decision function over the alphabet $\Sigma_I \times \Sigma_W =$
4445 $\mathbb{F} \times \mathbb{F}$ in the following way:

$$R_{R1CS} : \mathbb{F}^* \times \mathbb{F}^* \rightarrow \{true, false\} ; (I; W) \mapsto \begin{cases} true & (I; W) \text{ satisfies R1CS} \\ false & \text{else} \end{cases} \quad (6.7)$$

4446 Every R1CS therefore defines a formal language. The grammar of this language is encoded
4447 in the constraints, words are solutions to the equations and a **statement** is a knowledge claim
4448 “Given instance I , there is a witness W such that $(I; W)$ is a solution to the rank-1 constraint
4449 system”. A constructive proof to this claim is therefore an assignment of a field element to every
4450 witness variable, which is verified whenever the set of all instance and witness variables solves
4451 the R1CS.

Remark 4 (R1CS satisfiability). It should be noted that in our definition, every R1CS defines its own language. However, in more theoretical approaches, another language usually called **R1CS satisfiability** is often considered, which is useful when it comes to more abstract problems like expressiveness or the computational complexity of the class of **all** R1CS. From our perspective, the R1CS satisfiability language is obtained by the union of all R1CS languages that are in our definition. To be more precise, let the alphabet $\Sigma = \mathbb{F}$ be a field. Then

$$L_{R1CS_SAT}(\mathbb{F}) = \{(i; w) \in \Sigma^* \times \Sigma^* \mid \text{there is a R1CS } R \text{ such that } R(i; w) = \text{true}\}$$

4452 *Example 115 (3-Factorization).* Consider the language $L_{3.fac_zk}$ from example 108 and the
 4453 R1CS defined in example ex:3-factorization-r1cs. As we have seen in ex:3-factorization-r1cs,
 4454 solutions to the R1CS are in 1:1 correspondence with solutions to the decision function of
 4455 $L_{3.fac_zk}$. Both languages are therefore equivalent in the sense that there is a 1:1 correspon-
 4456 dence between words in both languages.

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To give an intuition of what constructive proofs in $L_{3.fac_zk}$ look like, consider the instance $I_1 = 11$. To prove the statement “There exists a witness W such that $(I_1; W)$ is a word in $L_{3.fac_zk}$ ” constructively, a proof has to provide assignments to all witness variables W_1, W_2, W_3 and W_4 . Since the alphabet is \mathbb{F}_{13} , an example assignment is given by $W = (2, 3, 4, 6)$ since $(I_1; W)$ satisfies the R1CS

$$\begin{array}{ll} W_1 \cdot W_2 = W_4 & \# 2 \cdot 3 = 6 \\ W_4 \cdot W_3 = I_1 & \# 6 \cdot 4 = 11 \end{array}$$

4457 A proper constructive proof is therefore given by $P = (2, 3, 4, 6)$. Of course, P is not the only
 4458 possible proof for this statement. Since factorization is not unique in a field in general, another
 4459 constructive proof is given by $P' = (3, 5, 12, 2)$.

Example 116 (The tiny-jubjub curve). Consider the language L_{jubjub} from example 109 and
 its associated R1CS. To see how constructive proofs in L_{jubjub} look like, consider the instance
 $(I_1, I_2) = (11, 6)$. To prove the statement “There exists a witness W such that $(I_1, I_2; W)$ is a
 word in L_{jubjub} ” constructively, a proof has to provide assignments to all witness variables $W_1,$
 W_2 and W_3 . Since the alphabet is \mathbb{F}_{13} , an example assignment is given by $W = (4, 10, 8)$ since
 $(I_1, I_2; W)$ satisfies the R1CS

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$$\begin{array}{ll} I_1 \cdot I_1 = W_1 & 11 \cdot 11 = 4 \\ I_2 \cdot I_2 = W_2 & 6 \cdot 6 = 10 \\ (8 \cdot W_1) \cdot W_2 = W_3 & (8 \cdot 4) \cdot 10 = 8 \\ (12 \cdot W_2 + W_3 + 10 \cdot W_1 + 1) \cdot 1 = 0 & 12 \cdot 10 + 8 + 10 \cdot 4 + 1 = 0 \end{array}$$

4460 A proper constructive proof is therefore given by $P = (4, 10, 8)$, which shows that the instance
 4461 $(11, 6)$ is a point on the tiny-jubjub curve.

4462 **Modularity** As we discussed on page 138 XXX, it is often useful to construct complex state-
 4463 ments and their representing languages from simple ones. Rank-1 constraint systems are par-
 4464 ticularly useful for this, as the intersection of two R1CS over the same alphabet results in a new
 4465 R1CS over that same alphabet.

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4466 To be more precise, let S_1 and S_2 be two R1CS over \mathbb{F} , then a new R1CS S_3 is obtained
 4467 by the intersection $S_3 = S_1 \cap S_2$ of S_1 and S_2 . In this context, intersection means that both the
 4468 equations of S_1 **and** the equations of S_2 have to be satisfied in order to provide a solution for the
 4469 system S_3 .

As a consequence, developers are able to construct complex R1CS from simple ones and this modularity provides the theoretical foundation for many R1CS compilers, as we will see in XXX.

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6.2.2 Algebraic Circuits

As we have seen in the previous paragraphs, rank-1 constraint systems are quadratic equations such that solutions are knowledge proofs for the existence of words in associated languages. From the perspective of a proofer, it is therefore important to solve those equations efficiently.

However, in contrast to systems of linear equation, no general methods are known that solve systems of quadratic equations efficiently. Rank-1 constraint systems are therefore impractical from a proofers perspective and auxiliary information is needed that helps to compute solutions efficiently.

Methods which compute R1CS solutions are sometimes called **witness generator functions**. To provide a common example, we introduce another class of decision functions called **algebraic circuits**. As we will see, every algebraic circuit defines an associated R1CS and also provides an efficient way to compute solutions for that R1CS.

It can be shown that every space- and time-bounded computation is expressible as an algebraic circuit. Transforming high-level computer programs into those circuits is a process often called **flattening**.

To understand this in more detail, we will introduce our model for algebraic circuits and look at the concept of circuit execution and valid assignments. After that, we will show how to derive rank-1 constraint systems from circuits and how circuits are useful to compute solutions to their R1CS efficiently.

Algebraic circuit representation To see what algebraic circuits are, let \mathbb{F} be a field. An algebraic circuit is then a directed acyclic (multi)graph that computes a polynomial function over \mathbb{F} . Nodes with only outgoing edges (source nodes) represent the variables and constants of the function and nodes with only incoming edges (sink nodes) represent the outcome of the function. All other nodes have exactly two incoming edges and represent the defining field operations **addition** as well as **multiplication**. Graph edges represent the flow of the computation along the nodes.

To be more precise, we call a directed acyclic multi-graph $C(\mathbb{F})$ an **algebraic circuit** over \mathbb{F} in this book if the following conditions hold:

Definition 6.2.2.1. Algebraic circuit

- The set of edges has a total order.
- Every source node has a label that represents either a variable or a constant from the field \mathbb{F} .
- Every sink node has exactly one incoming edge and a label that represents either a variable or a constant from the field \mathbb{F} .
- Every node that is neither a source nor a sink has exactly two incoming edges and a label from the set $\{+, *\}$ that represents either addition or multiplication in \mathbb{F} .
- All outgoing edges from a node have the same label.
- Outgoing edges from a node with a label that represents a variable have a label.

- Outgoing edges from a node with a label that represents multiplication have a label, if there is at least one labeled edge in both input path.
- All incoming edges to sink nodes have a label.
- If an edge has two labels S_i and S_j it gets a new label $S_i = S_j$.
- No other edge has a label.
- Incoming edges to sink nodes that are labeled with a constant $c \in \mathbb{F}$ are labeled with the same constant. Every other edge label is taken from the set $\{W, I\}$ and indexed compatible with the order of the edge set.

It should be noted that the details in the definitions of algebraic circuits vary between different sources. We use this definition as it is conceptually straightforward and well-suited for pen-and-paper computations.

To get a better intuition of our definition, let $C(\mathbb{F})$ be an algebraic circuit. Source nodes are the inputs to the circuit and either represent variables or constants. In a similar way, sink nodes represent termination points of the circuit and are either output variables or constants. Constant sink nodes enforce computational outputs to take on certain values.

Nodes that are neither source nodes nor sink nodes are called **arithmetic gates**. Arithmetic gates that are decorated with the “+”-label are called **addition-gates** and arithmetic gates that are decorated with the “·”-label are called **multiplication-gates**. Every arithmetic gate has exactly two inputs, represented by the two incoming edges.

Since the set of edges is ordered, we can write it as $\{E_1, E_2, \dots, E_n\}$ for some $n \in \mathbb{N}$ and we use those indices to index the edge labels, too. Edge labels are therefore either constants or symbols like I_j , W_j or S_j , where j is an index compatible with the edge order. Labels I_j represent instance variables, labels W_j witness variables. Labels on the outgoing edges of input variables constrain the associated variable to that edge. Every other edge defines a constraining equation in the associated R1CS. We will explain this in more detail in XXX.

Notation and Symbols 10. In synthesizing algebraic circuits, assigning instance I_j or witness W_j labels to appropriate edges is often the final step. It is therefore convenient to not distinguish these two types of edges in previous steps. To account for that, we often simply write S_j for an edge label, indicating that the private/public property of the label is unspecified and it might represent an instance or a witness label.

Example 117 (Generalized factorization SNARK). To give a simple example of an algebraic circuit, consider our 3-factorization problem from example 108 again. To express the problem in the algebraic circuit model, consider the following function

$$f_{3.fac} : \mathbb{F}_{13} \times \mathbb{F}_{13} \times \mathbb{F}_{13} \rightarrow \mathbb{F}_{13}; (x_1, x_2, x_3) \mapsto x_1 \cdot x_2 \cdot x_3$$

Using this function, we can describe the zero-knowledge 3-factorization problem from 108, in the following way: Given instance $I_1 \in \mathbb{F}_{13}$, a valid witness is a preimage of $f_{3.fac}$ at the point I_1 , i.e., a valid witness consists of three values W_1 , W_2 and W_3 from \mathbb{F}_{13} such that $f_{3.fac}(W_1, W_2, W_3) = I_1$.

To see how this function can be transformed into an algebraic circuit over \mathbb{F}_{13} , it is a common first step to introduce brackets into the function’s definition and then write the operations as binary operators, in order to highlight how exactly every field operation acts on its two inputs.

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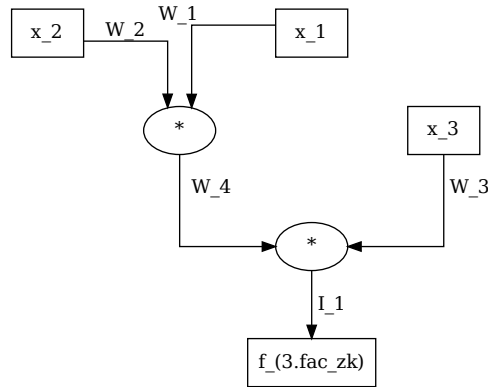
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Due to the associativity laws in a field, we have several choices. We choose

$$\begin{aligned}
 f_{3.fac}(x_1, x_2, x_3) &= x_1 \cdot x_2 \cdot x_3 && \# \text{ bracket choice} \\
 &= (x_1 \cdot x_2) \cdot x_3 && \# \text{ operator notation} \\
 &= MUL(MUL(x_1, x_2), x_3)
 \end{aligned}$$

4548 Using this expression, we can write an associated algebraic circuit by first constraining the
 4549 variables to edge labels $W_1 = x_1$, $W_2 = x_2$ and $W_3 = x_3$ as well as $I_1 = f_{3.fac}(x_1, x_2, x_3)$, taking
 4550 the distinction between private and public inputs into account. We then rewrite the operator
 4551 representation of $f_{3.fac}$ into circuit nodes and get the following:



4552

4553 In this case, the directed acyclic multi-graph is a binary tree with three leaves (the source
 4554 nodes) labeled by x_1 , x_2 and x_3 , one root (the single sink node) labeled by $f(x_1, x_2, x_3)$ and two
 4555 internal nodes, which are labeled as multiplication gates.

4556 The order we use to label the edges is chosen to make the edge labeling consistent with
 4557 the choice of W_4 as defined in definition 6.2.2.1. This order can be obtained by a depth-first
 4558 right-to-left-first traversal algorithm.

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Example 118. To give a more realistic example of an algebraic circuit, look at the defining
 equation of the tiny-jubjub curve (68) again. A pair of field elements $(x, y) \in \mathbb{F}_{13}^2$ is a curve
 point, precisely if the following equation holds:

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$$3 \cdot x^2 + y^2 = 1 + 8 \cdot x^2 \cdot y^2$$

To understand how one might transform this identity into an algebraic circuit, we first rewrite
 this equation by shifting all terms to the right. We get the following:

$$\begin{aligned}
 3 \cdot x^2 + y^2 &= 1 + 8 \cdot x^2 \cdot y^2 && \Leftrightarrow \\
 0 &= 1 + 8 \cdot x^2 \cdot y^2 - 3 \cdot x^2 - y^2 && \Leftrightarrow \\
 0 &= 1 + 8 \cdot x^2 \cdot y^2 + 10 \cdot x^2 + 12 \cdot y^2
 \end{aligned}$$

Then we use this expression to define a function such that all points of the tiny-jubjub curve are
 characterized as the function preimages at 0.

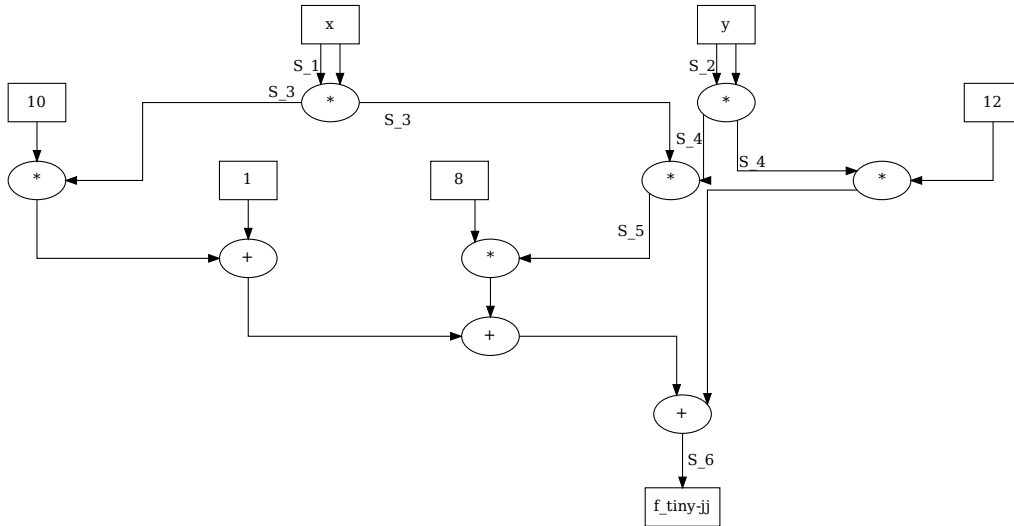
$$f_{tiny-jj} : \mathbb{F}_{13} \times \mathbb{F}_{13} \rightarrow \mathbb{F}_{13} ; (x, y) \mapsto 1 + 8 \cdot x^2 \cdot y^2 + 10 \cdot x^2 + 12 \cdot y^2$$

4559 Every pair of points $(x, y) \in \mathbb{F}_{13}^2$ with $f_{\text{tiny-jj}}(x, y) = 0$ is a point on the tiny-jubjub curve, and
 4560 there are no other curve points. The preimage $f_{\text{tiny-jj}}^{-1}(0)$ is therefore a complete description of
 4561 the tiny-jubjub curve.

We can transform this function into an algebraic circuit over \mathbb{F}_{13} . We first introduce brackets into potentially ambiguous expressions and then rewrite the function in terms of binary operators. We get the following:

$$\begin{aligned}
 f_{\text{tiny-jj}}(x, y) &= 1 + 8 \cdot x^2 \cdot y^2 + 10 \cdot x^2 + 12y^2 && \Leftrightarrow \\
 &= ((8 \cdot ((x \cdot x) \cdot (y \cdot y))) + (1 + 10 \cdot (x \cdot x))) + (12 \cdot (y \cdot y)) && \Leftrightarrow \\
 &= \text{ADD}(\text{ADD}(\text{MUL}(8, \text{MUL}(\text{MUL}(x, x), \text{MUL}(y, y))), \text{ADD}(1, \text{MUL}(10, \text{MUL}(x, x)))), \text{MUL}(12, \text{MUL}(y, y)))
 \end{aligned}$$

4562 Since we haven't decided which part of the computation should be public and which part should
 4563 be private, we use the unspecified symbol S to represent edge labels. Constraining all vari-
 4564 ables to edge labels $S_1 = x$, $S_2 = y$ and $S_6 = f_{\text{tiny-jj}}$, we get the following circuit, representing
 4565 the function $f_{\text{tiny-jj}}$, by inductively replacing binary operators with their associated arithmetic
 4566 gates:



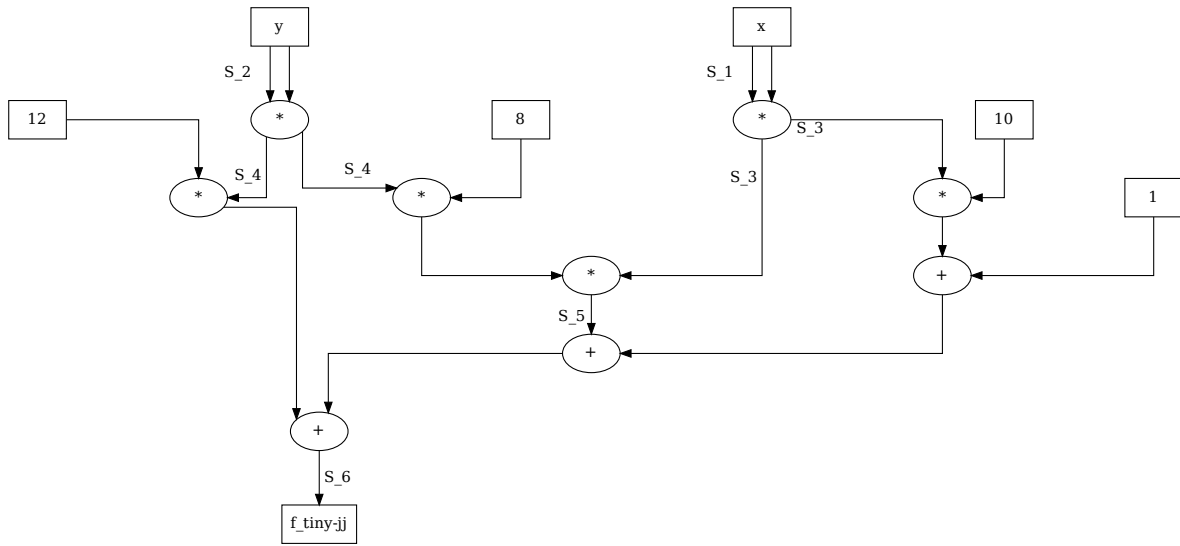
4567

4568 This circuit is not a graph, but a multigraph, since there is more than one edge between some of
 4569 the nodes.

4570 In the process of designing of circuits from functions, it should be noted that circuit rep-
 4571 resentations are not unique in general. In case of the function $f_{\text{tiny-jj}}$, the circuit shape is
 4572 dependent on our choice of bracketing in XXX. An alternative design i,s for example, given by
 4573 the following circuit, which occurs when the bracketed expression $8 \cdot ((x \cdot x) \cdot (y \cdot y))$ is replaced
 4574 by the expression $(x \cdot x) \cdot (8 \cdot (y \cdot y))$.

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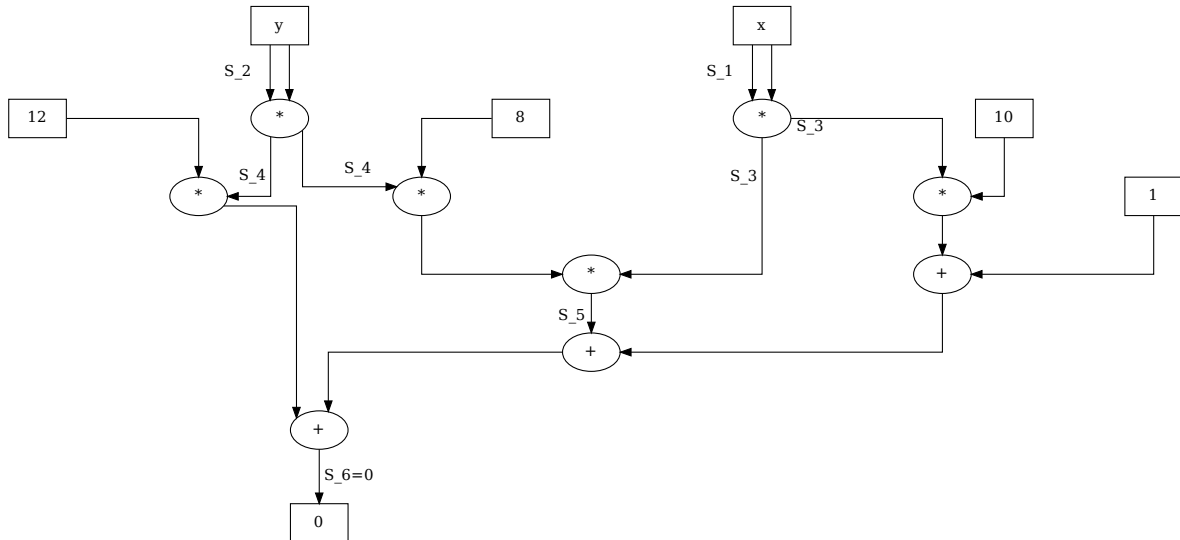
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4577

4578 Of course, both circuits represent the same function, due to the associativity and commutativity
 4579 laws that hold true in any field.

4580 With a circuit that represents the function $f_{\text{tiny-jj}}$, we can now proceed to derive a circuit
 4581 that constrains arbitrary pairs (x, y) of field elements to be points on the tiny-jubjub curve. To do
 4582 so, we have to constrain the output to be zero, that is, we have to constrain $S_6 = 0$. To indicate
 4583 this in the circuit, we replace the output variable by the constant 0 and constrain the related edge
 4584 label accordingly. We get the following:

4585

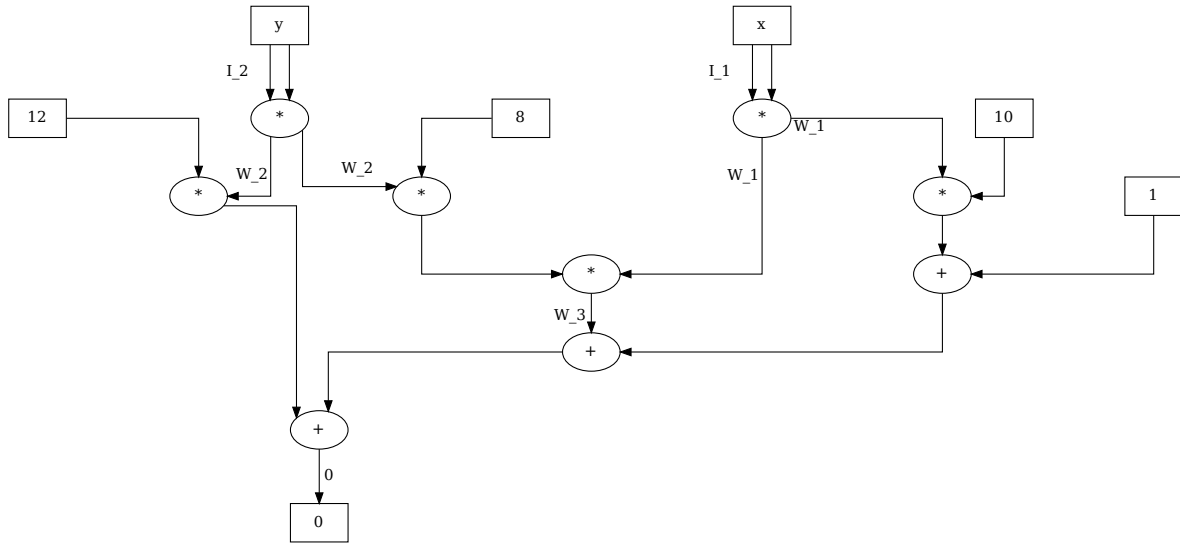


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4587

4588 The previous circuit enforces input values assigned to the labels S_1 and S_2 to be points on the
 4589 tiny-jubjub curve. However, it does not specify which labels are considered public and which
 4590 are considered private. The following circuit defines the inputs to be public, while all other
 4591 labels are private:

4592



4593

4594

4595 It can be shown that every space- and time-bounded computation can be transformed into
 4596 an algebraic circuit. We call any process that transforms a bounded computation into a circuit
 4597 **flattening**.

4598 **Circuit Execution** Algebraic circuits are directed, acyclic multi-graphs, where nodes repre-
 4599 sent variables, constants, or addition and multiplication gates. In particular, every algebraic
 4600 circuit with n input nodes decorated with variable symbols and m output nodes decorated with
 4601 variables can be seen a function that transforms an input tuple (x_1, \dots, x_n) from \mathbb{F}^n into an out-
 4602 put tuple (f_1, \dots, f_m) from \mathbb{F}^m . The transformation is done by sending values associated to
 4603 nodes along their outgoing edges to other nodes. If those nodes are gates, then the values are
 4604 transformed according to the gate label and the process is repeated along all edges until a sink
 4605 node is reached. We call this computation **circuit execution**.

4606 When executing a circuit, it is possible to not only compute the output values of the circuit
 4607 but to derive field elements for all edges, and, in particular, for all edge labels in the circuit. The
 4608 result is a tuple (S_1, S_2, \dots, S_n) of field elements associated to all labeled edges, which we call a
 4609 **valid assignment** to the circuit. In contrast, any assignment $(S'_1, S'_2, \dots, S'_n)$ of field elements to
 4610 edge labels that can not arise from circuit execution is called an **invalid assignment**.

4611 Valid assignments can be interpreted as **proofs for proper circuit execution** because they
 4612 keep a record of the computational result as well as intermediate computational steps.

4613 *Example 119 (3-factorization).* Consider the 3-factorization problem from example 108 and its
 4614 representation as an algebraic circuit from XXX. We know that the set of edge labels is given
 4615 by $S := \{I_1; W_1, W_2, W_3, W_4\}$.

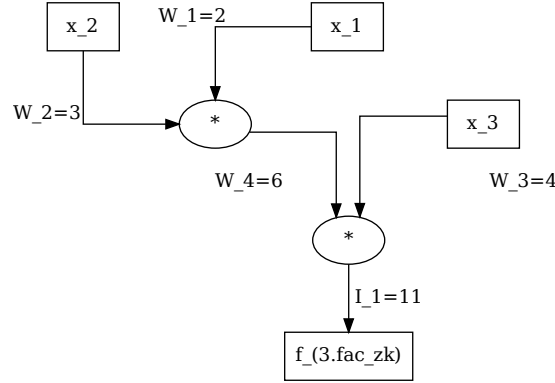
4616 To understand how this circuit is executed, consider the variables $x_1 = 2$, $x_2 = 3$ as well as
 4617 $x_3 = 4$. Following all edges in the graph, we get the assignments $W_1 = 2$, $W_2 = 3$ and $W_3 = 4$.
 4618 Then the assignments of W_1 and W_2 enter a multiplication gate and the output of the gate is
 4619 $2 \cdot 3 = 6$, which we assign to W_4 , i.e. $W_4 = 6$. The values W_4 and W_3 then enter the second
 4620 multiplication gate and the output of the gate is $6 \cdot 4 = 11$, which we assign to I_1 , i.e. $I_1 = 11$.

4621 A valid assignment to the 3-factorization circuit $C_{3, fac}(\mathbb{F}_{13})$ is therefore given by the fol-
 4622 lowing set:

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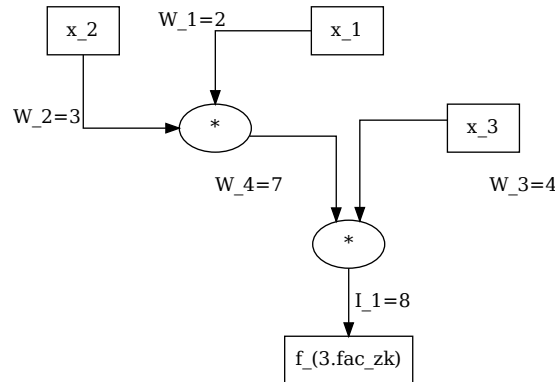
$$S_{\text{valid}} := \{11; 2, 3, 4, 6\} \quad (6.8)$$

4623 We can visualise this assignment in the circuit as follows:



4624

4625 To see what an invalid assignment looks like, consider the assignment $S_{\text{err}} := \{8; 2, 3, 4, 7\}$. In
 4626 this assignment, the input values are the same as in the previous case. The associated circuit is:



4627

4628 This assignment is invalid, as the assignments of I_1 and W_4 cannot be obtained by executing the
 4629 circuit.

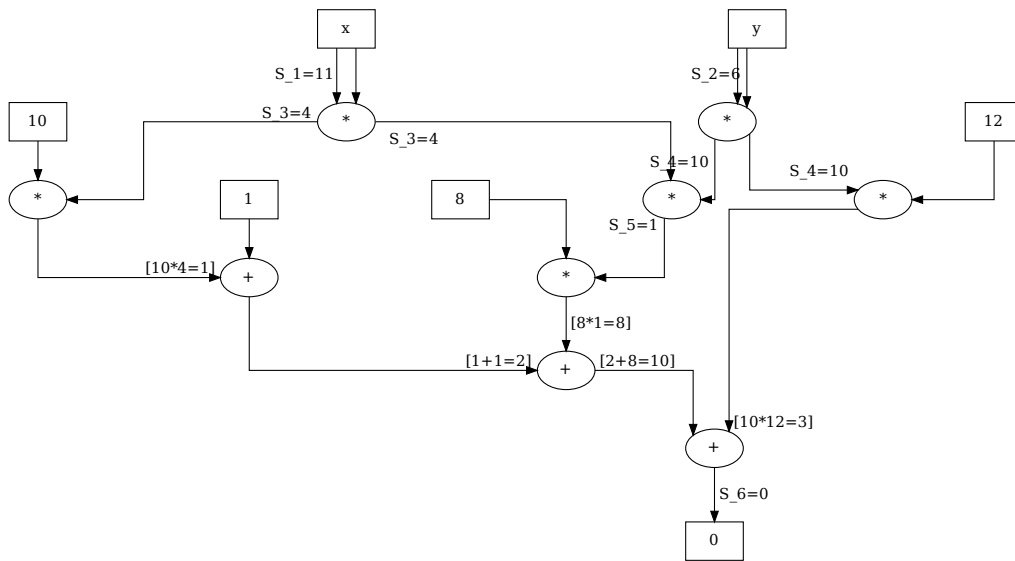
4630 *Example 120.* To compute a more realistic algebraic circuit execution, consider the defining
 4631 circuit $C_{\text{tiny-jj}}(\mathbb{F}_{13})$ from example 116 again. We already know from the way this circuit is
 4632 constructed that any valid assignment with $S_1 = x$, $S_2 = y$ and $S_6 = 0$ will ensure that the pair
 4633 (x, y) is a point on the tiny-jubjub curve in its Edwards representation (equation 5.20).

4634 From example 116, we know that the pair $(11, 6)$ is a proper point on the tiny-jubjub curve
 4635 and we use this point as input to a circuit execution. We get the following:

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4636

Executing the circuit, we indeed compute $S_6 = 0$ as expected, which proves that $(11, 6)$ is a point on the tiny-jubjub curve in its Edwards representation. A valid assignment of $C_{\text{tiny-jj}}(\mathbb{F}_{13})$ is therefore given by the following equation:

$$S_{\text{tiny-jj}} = \{S_1, S_2, S_3, S_4, S_5, S_6\} = \{11, 6, 4, 10, 1, 0\}$$

4637 **Circuit Satisfiability** To understand how algebraic circuits give rise to formal languages, ob-
 4638 serve that every algebraic circuit $C(\mathbb{F})$ over a fields \mathbb{F} defines a decision function over the al-
 4639 phabet $\Sigma_I \times \Sigma_W = \mathbb{F} \times \mathbb{F}$ in the following way:

$$R_{C(\mathbb{F})} : \mathbb{F}^* \times \mathbb{F}^* \rightarrow \{true, false\} ; (I; W) \mapsto \begin{cases} true & (I; W) \text{ is valid assignment to } C(\mathbb{F}) \\ false & \text{else} \end{cases} \quad (6.9)$$

4640 Every algebraic circuit therefore defines a formal language. The grammar of this language is
 4641 encoded in the shape of the circuit, words are assignments to edge labels that are derived from
 4642 circuit execution, and **statements** are knowledge claims “Given instance I , there is a witness
 4643 W such that $(I; W)$ is a valid assignment to the circuit”. A constructive proof to this claim
 4644 is therefore an assignment of a field element to every witness variable, which is verified by
 4645 executing the circuit to see if the assignment of the execution meets the assignment of the
 4646 proof.

4647 In the context of zero-knowledge proof systems, executing circuits is also often called **wit-**
 4648 **ness generation**, since in applications the instance part is usually public, while its the task of a
 4649 prover to compute the witness part.

Remark 5 (Circuit satisfiability). It should be noted that, in our definition, every circuit defines its own language. However, in more theoretical approaches another language usually called **circuit satisfiability** is often considered, which is useful when it comes to more abstract problems like expressiveness, or computational complexity of the class of **all** algebraic circuits over a given field. From our perspective the circuit satisfiability language is obtained by union of all circuit languages that are in our definition. To be more precise, let the alphabet $\Sigma = \mathbb{F}$ be a field. Then

Should we refer to RICS satisfiability (p. 142 here?)

$$L_{\text{CIRCUIT_SAT}(\mathbb{F})} = \{(i; w) \in \Sigma^* \times \Sigma^* \mid \text{there is a circuit } C(\mathbb{F}) \text{ such that } (i; w) \text{ is valid assignment}\}$$

4650 *Example 121 (3-Factorization).* Consider the circuit $C_{3, fac}$ from equation 6.8 again. We call the
 4651 associated language L_{3, fac_circ} .

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4652 To understand how a constructive proof of a statement in L_{3, fac_circ} looks like, consider the
 4653 instance $I_1 = 11$. To provide a proof for the statement “There exist a witness W such that $(I_1; W)$
 4654 is a word in L_{3, fac_circ} ” a proof therefore has to consists of proper values for the variables W_1 ,
 4655 W_2 , W_3 and W_4 . Any proofer therefore has to find input values for W_1 , W_2 and W_3 and then
 4656 execute the circuit to compute W_4 under the assumption $I_1 = 11$.

4657 Example XXX implies that $(2, 3, 4, 6)$ is a proper constructive proof and in order to verify
 4658 the proof a verifier needs to execute the circuit with instance $I_1 = 11$ and inputs $W_1 = 2$, $W_2 = 3$
 4659 and $W_3 = 4$ to decide whether the proof is a valid assignment or not.

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4660 **Associated Constraint Systems** As we have seen in 6.2.1.1, rank-1 constraint systems define
 4661 a way to represent statements in terms of a system of quadratic equations over finite fields,
 4662 suitable for pairing-based zero-knowledge proof systems. However, those equations provide no
 4663 practical way for a proofer to actually compute a solution. On the other hand, algebraic circuits
 4664 can be executed in order to derive valid assignments efficiently.

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4665 In this paragraph, we show how to transform any algebraic circuit into a rank-1 constraint
 4666 system such that valid circuit assignments are in 1:1 correspondence with solutions to the asso-
 4667 ciated R1CS.

4668 To see this, let $C(\mathbb{F})$ be an algebraic circuit over a finite field \mathbb{F} , with a set of edge labels
 4669 $\{S_1, S_2, \dots, S_n\}$. Then one of the following steps is executed for every edge label S_j from that
 4670 set:

4671 *Definition 6.2.2.2.* • If the edge label S_j is an outgoing edge of a multiplication gate, the
 4672 R1CS gets a new quadratic constraint

$$(\text{left input}) \cdot (\text{right input}) = S_j \quad (6.10)$$

4673 where (left input) respectively (right input) is the output from the symbolic execution of
 4674 the subgraph that consists of the left respectively right input edge of this gate, and all
 4675 edges and nodes that have this edge in their path, starting with constant inputs or labeled
 4676 outgoing edges of other nodes.

4677 • If the edge label S_j is an outgoing edge of an addition gate, the R1CS gets a new quadratic
 4678 constraint

$$(\text{left input} + \text{right input}) \cdot 1 = S_j \quad (6.11)$$

4679 where (left input) respectively (right input) is the output from the symbolic execution of
 4680 the subgraph that consists of the left respectively right input edge of this gate and all
 4681 edges and nodes that have this edge in their path, starting with constant inputs or labeled
 4682 outgoing edges of other nodes.

4683 • No other edge label adds a constraint to the system.

4684 The result of this method is a rank-1 constraint system, and, in this sense, every algebraic
 4685 circuit $C(\mathbb{F})$ generates a R1CS R , which we call the **associated R1CS** of the circuit. It can be
 4686 shown that a tuple of field elements (S_1, S_2, \dots, S_n) is a valid assignment to a circuit if and only
 4687 if the same tuple is a solution to the associated R1CS. Circuit executions therefore compute
 4688 solutions to rank-1 constraints systems efficiently.

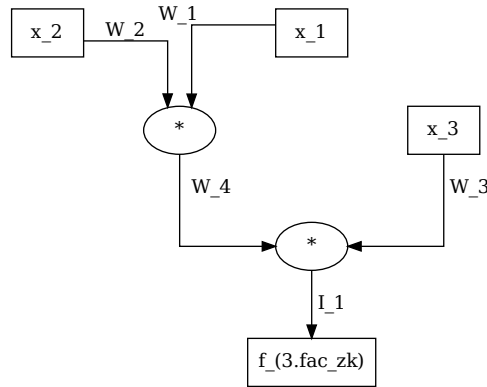
4689 To understand the contribution of algebraic gates to the number of constraints, note that, by
 4690 definition, multiplication gates have labels on their outgoing edges if and only if there is at least

one labeled edge in both input paths, or if the outgoing edge is an input to a sink node. This implies that multiplication with a constant is essentially free in the sense that it doesn't add a new constraint to the system, as long as that multiplication gate is not an input to an output node.

Moreover, addition gates have labels on their outgoing edges if and only if they are inputs to sink nodes. This implies that addition is essentially free in the sense that it doesn't add a new constraint to the system, as long as that addition gate is not an input to an output node.

Example 122 (3-factorization). Consider our 3-factorization problem from equation 6.8 and the associated circuit $C_{3.fac}(\mathbb{F}_{13})$. Our task is to transform this circuit into an equivalent rank-1 constraint system.

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We start with an empty R1CS, and, in order to generate all constraints, we have to iterate over the set of edge labels $\{I_1; W_1, W_2, W_3, W_4\}$.

Starting with the edge label I_1 , we see that it is an outgoing edge of a multiplication gate, and, since both input edges are labeled, we have to add the following constraint to the system:

$$\begin{aligned} (\text{left input}) \cdot (\text{right input}) &= I_1 \\ W_4 \cdot W_3 &= I_1 \end{aligned} \quad \Leftrightarrow$$

Next, we consider the edge label W_1 and, since, it's not an outgoing edge of a multiplication or addition label, we don't add a constraint to the system. The same holds true for the labels W_2 and W_3 .

For edge label W_4 , we see that it is an outgoing edge of a multiplication gate, and, since both input edges are labeled, we have to add the following constraint to the system:

$$\begin{aligned} (\text{left input}) \cdot (\text{right input}) &= W_4 \\ W_2 \cdot W_1 &= W_4 \end{aligned} \quad \Leftrightarrow$$

Since there are no more labeled edges, all constraints are generated, and we have to combine them to get the associated R1CS of $C_{3.fac}(\mathbb{F}_{13})$:

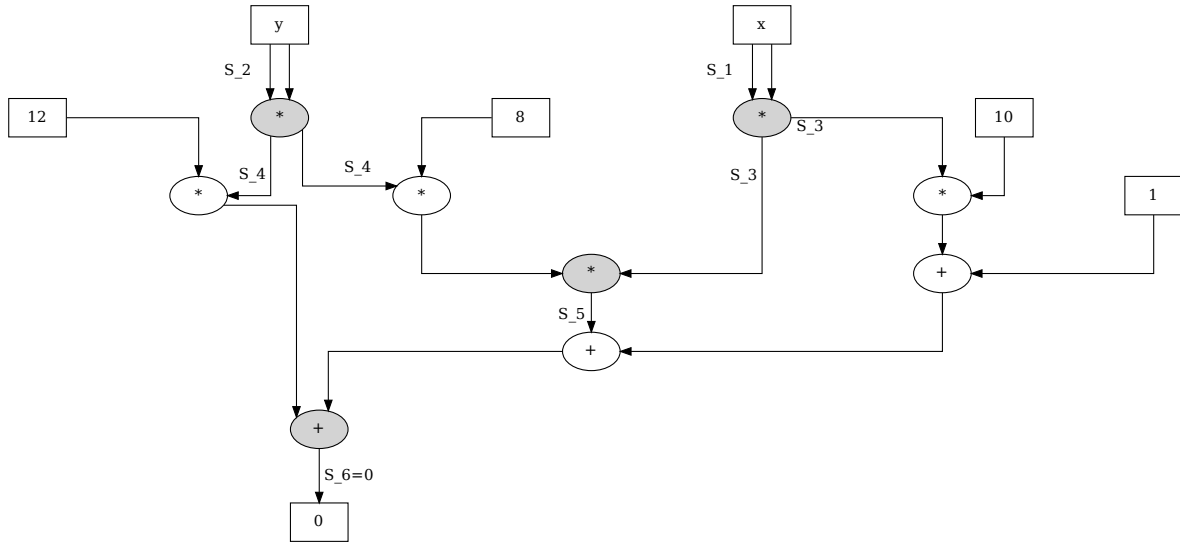
$$\begin{aligned} W_4 \cdot W_3 &= I_1 \\ W_2 \cdot W_1 &= W_4 \end{aligned}$$

This system is equivalent to the R1CS we derived in example 113. The languages $L_{3.fac_zk}$ and $L_{3.fac_circ}$ are therefore equivalent and both the circuit as well as the R1CS are just two different ways of expressing the same language.

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4710 *Example 123.* To consider a more general transformation, we consider the tiny-jubjub circuit
 4711 from example 116 again. A proper circuit is given by

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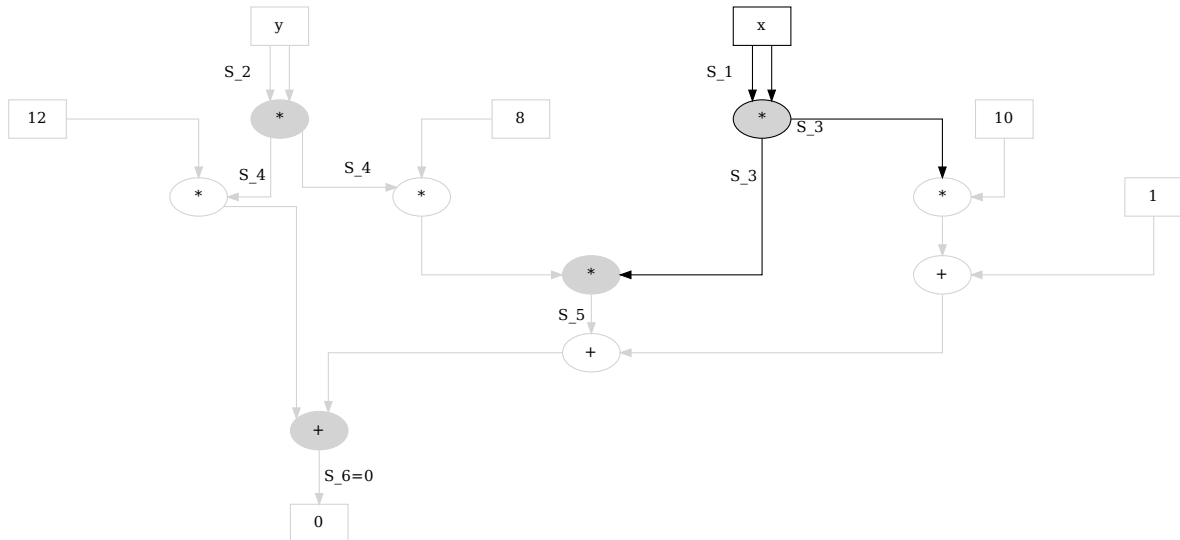
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4715 To compute the number of constraints, observe that we have 3 multiplication gates that have
 4716 labels on their outgoing edges and 1 addition gate that has a label on its outgoing edge. We
 4717 therefore have to compute 4 quadratic constraints.

4718 In order to derive the associated R1CS, we have start with an empty R1CS and then iterate
 4719 over the set $\{S_1, S_2, S_3, S_4, S_5, S_6 = 0\}$ of all edge labels, in order to generate the constraints.

4720 Considering edge label S_1 , we see that the associated edges are not outgoing edges of any
 4721 algebraic gate, and we therefore have to add no new constraint to the system. The same holds
 4722 true for edge label S_2 . Looking at edge label S_3 , we see that the associated edges are outgoing
 4723 edges of a multiplication gate and that the associated subgraph is given by:

4724



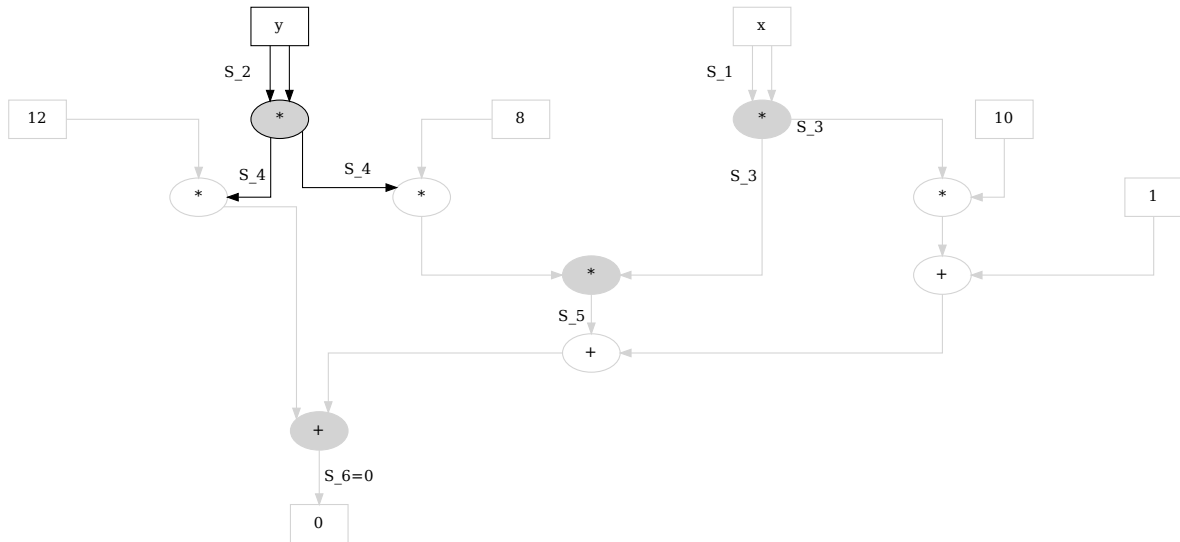
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Both the left and the right input to this multiplication gate are labeled by S_1 . We therefore have to add the following constraint to the system:

$$S_1 \cdot S_1 = S_3$$

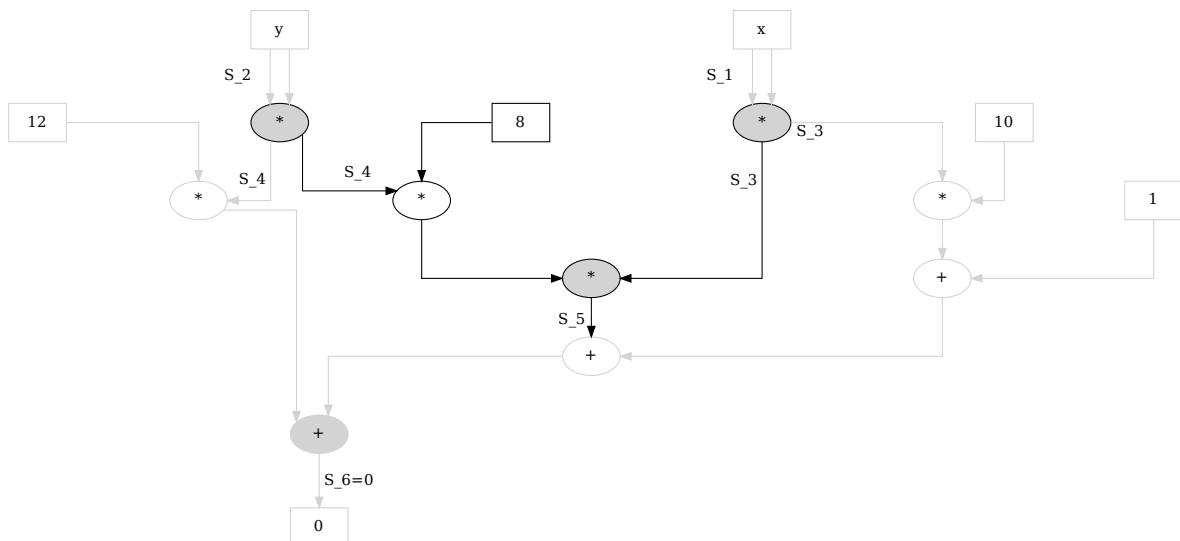
Looking at edge label S_4 , we see that the associated edges are outgoing edges of a multiplication gate and that the associated subgraph is given by:



Both the left and the right input to this multiplication gate are labeled by S_2 and we therefore have to add the following constraint to the system:

$$S_2 \cdot S_2 = S_4$$

Edge label S_5 is more interesting. To see if it implies a constraint, we have to construct the associated subgraph first, which consists of all edges and all nodes in all path starting either at a constant input or a labeled edge. We get



4737

The right input to the associated multiplication gate is given by the labeled edge S_3 . However, the left input is not a labeled edge, but has a labeled edge in one of its path. This implies that we have to add a constraint to the system. To compute the left factor of that constraint, we have to compute the output of subgraph associated to the left edge, which is $8 \cdot W_2$. This gives the constraint

$$(S_4 \cdot 8) \cdot S_3 = S_5$$

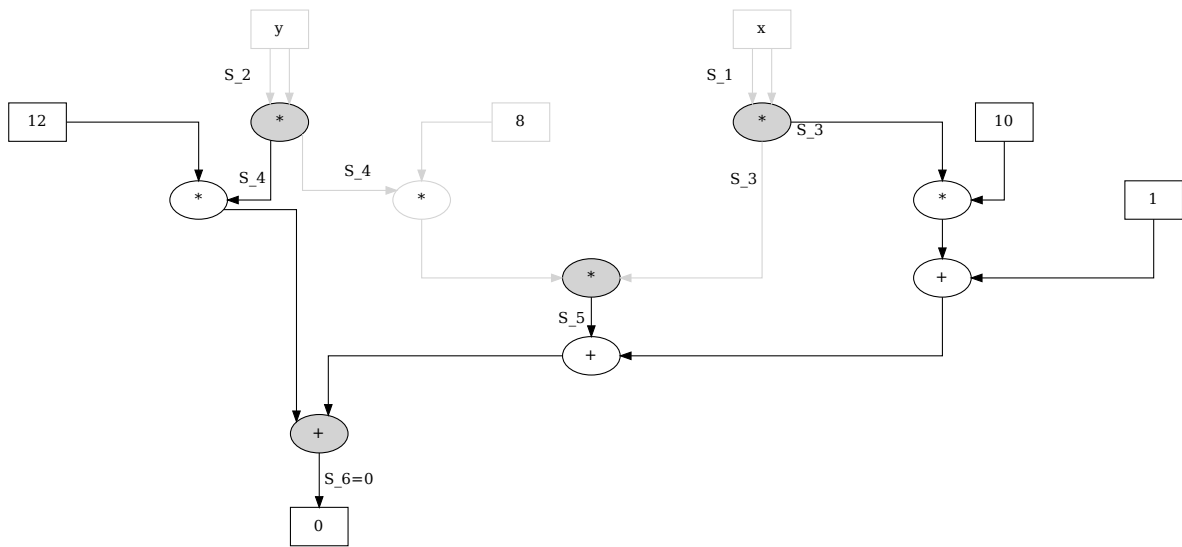
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The last edge label is the constant $S_6 = 0$. To see if it implies a constraint, we have to construct the associated subgraph, which consists of all edges and all nodes in all path starting either at a constant input or a labeled edge. We get

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Both the left and the right input are unlabeled, but have a labeled edges in their path. This implies that we have to add a constraint to the system. Since the gate is an addition gate, the right factor in the quadratic constraint is always 1 and the left factor is computed by symbolically executing all inputs to all gates in sub-circuit. We get

$$(12 \cdot S_4 + S_5 + 10 \cdot S_3 + 1) \cdot 1 = 0$$

Since there are no more labeled outgoing edges, we are done deriving the constraints. Combining all constraints together, we get the following R1CS:

$$S_1 \cdot S_1 = S_3$$

$$S_2 \cdot S_2 = S_4$$

$$(S_4 \cdot 8) \cdot S_3 = S_5$$

$$(12 \cdot S_4 + S_5 + 10 \cdot S_3 + 1) \cdot 1 = 0$$

4744

4745

4746

which is equivalent to the R1CS we derived in example 116. The languages L_{3, fac_zk} and L_{3, fac_circ} are therefore equivalent and both the circuit as well as the R1CS are just two different ways to express the same language.

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6.2.3 Quadratic Arithmetic Programs

We have introduced algebraic circuits and their associated rank-1 constraints systems as two particular models able to represent space- and time-bounded computation. Both models define formal languages, and associated membership as well as knowledge claims can be constructively proofed by executing the circuit in order to compute solutions to its associated R1CS.

One reason why those systems are useful in the context of succinct zero-knowledge proof systems is because any R1CS can be transformed into another computational model called **quadratic arithmetic programs** (QAP), which serve as the basis for some of the most efficient succinct non-interactive zero-knowledge proof generators that currently exist.

As we will see, proving statements for languages that have checking relations defined by quadratic arithmetic programs can be achieved by providing certain polynomials, and those proofs can be verified by checking a particular divisibility property.

QAP representation To understand what quadratic arithmetic programs are in detail, let \mathbb{F} be a field and R a rank-1 constraints system over \mathbb{F} such that the number of non-zero elements in \mathbb{F} is strictly larger than the number k of constraints in R . Moreover, let a_j^i, b_j^i and $c_j^i \in \mathbb{F}$ for every index $0 \leq j \leq n + m$ and $1 \leq i \leq k$, be the defining constants of the R1CS and m_1, \dots, m_k be arbitrary, invertible and distinct elements from \mathbb{F} .

Then a **quadratic arithmetic program** [QAP] of the R1CS is the following set of polynomials over \mathbb{F} :

$$QAP(R) = \left\{ T \in \mathbb{F}[x], \{A_j, B_j, C_j \in \mathbb{F}[x]\}_{h=0}^{n+m} \right\} \quad (6.12)$$

In the equation above, $T(x) := \prod_{l=1}^k (x - m_l)$ is a polynomial of degree k , called the **target polynomial** of the QAP and A_j, B_j as well as C_j are the unique degree $k - 1$ polynomials defined by the following equation:

$$A_j(m_i) = a_j^i \quad B_j(m_i) = b_j^i \quad C_j(m_i) = c_j^i \quad j = 1, \dots, n + m + 1, i = 1, \dots, k \quad (6.13)$$

Given some rank-1 constraint system, an associated quadratic arithmetic program is therefore nothing but a set of polynomials, computed from the constants in the R1CS. To see that the polynomials A_j, B_j and C_j are uniquely defined by the equations in XXX, recall that a polynomial of degree $k - 1$ is completely determined on k evaluation points and the equation 4 precisely determines those k evaluation points.

Since we only consider polynomials over fields, Lagrange's interpolation method from 3.30 in chapter ?? can be used to derive the polynomials A_j, B_j and C_j from their defining equations XXX. A practical method to compute a QAP from a given R1CS therefore consists of two steps. If the R1CS consists of k constraints, first choose k invertible and mutually different points from the underlying field. Every choice defines a different QAP for the same R1CS. Then use Lagrange's method and equation XXX to compute the polynomials A_j, B_j and C_j for every $1 \leq j \leq k$.

Example 124 (Generalized factorization SNARK). To provide a better intuition of quadratic arithmetic programs and how they are computed from their associated rank-1 constraint systems, consider the language L_{3, fac_zk} from example 108 and its associated R1CS:

$$\begin{array}{ll} W_1 \cdot W_2 = W_4 & \text{constraint 1} \\ W_4 \cdot W_3 = I_1 & \text{constraint 2} \end{array}$$

In this example we want to transform this R1CS into an associated QAP. In a first step, we have to compute the defining constants a_j^i, b_j^i and c_j^i of the R1CS. According to XXX, we have

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$$\begin{array}{cccccc} a_0^1 = 0 & a_1^1 = 0 & a_2^1 = 1 & a_3^1 = 0 & a_4^1 = 0 & a_5^1 = 0 \\ a_0^2 = 0 & a_1^2 = 0 & a_2^2 = 0 & a_3^2 = 0 & a_4^2 = 0 & a_5^2 = 1 \end{array}$$

$$\begin{array}{cccccc} b_0^1 = 0 & b_1^1 = 0 & b_2^1 = 0 & b_3^1 = 1 & b_4^1 = 0 & b_5^1 = 0 \\ b_0^2 = 0 & b_1^2 = 0 & b_2^2 = 0 & b_3^2 = 0 & b_4^2 = 1 & b_5^2 = 0 \end{array}$$

$$\begin{array}{cccccc} c_0^1 = 0 & c_1^1 = 0 & c_2^1 = 0 & c_3^1 = 0 & c_4^1 = 0 & c_5^1 = 1 \\ c_0^2 = 0 & c_1^2 = 1 & c_2^2 = 0 & c_3^2 = 0 & c_4^2 = 0 & c_5^2 = 0 \end{array}$$

Since the R1CS is defined over the field \mathbb{F}_{13} and has two constraining equations, we need to choose two arbitrary but distinct elements m_1 and m_2 from \mathbb{F}_{13} . We choose $m_1 = 5$, and $m_2 = 7$ and with this choice we get the target polynomial

$$\begin{aligned} T(x) &= (x - m_1)(x - m_2) && \# \text{ Definition of } T \\ &= (x - 5)(x - 7) && \# \text{ Insert our choice} \\ &= (x + 8)(x + 6) && \# \text{ Negatives in } \mathbb{F}_{13} \\ &= x^2 + x + 9 && \# \text{ expand} \end{aligned}$$

4781 Then we have to compute the polynomials A_j , B_j and C_j by their defining equation from the
4782 R1CS coefficients. Since the R1CS has two constraining equations, those polynomials are of
4783 degree 1 and they are defined by their evaluation at the point $m_1 = 5$ and the point $m_2 = 7$.

At point m_1 , each polynomial A_j is defined to be a_j^1 and at point m_2 , each polynomial A_j is defined to be a_j^2 . The same holds true for the polynomials B_j as well as C_j . Writing all these equations now, we get:

$$\begin{array}{l} A_0(5) = 0, \quad A_1(5) = 0, \quad A_2(5) = 1, \quad A_3(5) = 0, \quad A_4(5) = 0, \quad A_5(5) = 0 \\ A_0(7) = 0, \quad A_1(7) = 0, \quad A_2(7) = 0, \quad A_3(7) = 0, \quad A_4(7) = 0, \quad A_5(7) = 1 \end{array}$$

$$\begin{array}{l} B_0(5) = 0, \quad B_1(5) = 0, \quad B_2(5) = 0, \quad B_3(5) = 1, \quad B_4(5) = 0, \quad B_5(5) = 0 \\ B_0(7) = 0, \quad B_1(7) = 0, \quad B_2(7) = 0, \quad B_3(7) = 0, \quad B_4(7) = 1, \quad B_5(7) = 0 \end{array}$$

$$\begin{array}{l} C_0(5) = 0, \quad C_1(5) = 0, \quad C_2(5) = 0, \quad C_3(5) = 0, \quad C_4(5) = 0, \quad C_5(5) = 1 \\ C_0(7) = 0, \quad C_1(7) = 1, \quad C_2(7) = 0, \quad C_3(7) = 0, \quad C_4(7) = 0, \quad C_5(7) = 0 \end{array}$$

4784 Lagrange's interpolation implies that a polynomial of degree k , that is, that zero on $k + 1$ points
4785 has to be the zero polynomial. Since our polynomials are of degree 1 and determined on 2
4786 points, we therefore know that the only non-zero polynomials in our QAP are A_2 , A_5 , B_3 , B_4 ,
4787 C_1 and C_5 , and that we can use Lagrange's interpolation to compute them.

To compute A_2 we note that the set S in our version of Lagrange's method is given by $S = \{(x_0, y_0), (x_1, y_1)\} = \{(5, 1), (7, 0)\}$. Using this set we get:

$$\begin{aligned} A_2(x) &= y_0 \cdot l_0 + y_1 \cdot l_1 \\ &= y_0 \cdot \left(\frac{x - x_1}{x_0 - x_1} \right) + y_1 \cdot \left(\frac{x - x_0}{x_1 - x_0} \right) = 1 \cdot \left(\frac{x - 7}{5 - 7} \right) + 0 \cdot \left(\frac{x - 5}{7 - 5} \right) \\ &= \frac{x - 7}{-2} = \frac{x - 7}{11} && \# 11^{-1} = 6 \\ &= 6(x - 7) = 6x + 10 && \# -7 = 6 \text{ and } 6 \cdot 6 = 10 \end{aligned}$$

To compute A_5 , we note that the set S in our version of Lagrange's method is given by $S = \{(x_0, y_0), (x_1, y_1)\} = \{(5, 0), (7, 1)\}$. Using this set we get:

$$\begin{aligned}
 A_5(x) &= y_0 \cdot l_0 + y_1 \cdot l_1 \\
 &= y_0 \cdot \left(\frac{x - x_1}{x_0 - x_1}\right) + y_1 \cdot \left(\frac{x - x_0}{x_1 - x_0}\right) = 0 \cdot \left(\frac{x - 7}{5 - 7}\right) + 1 \cdot \left(\frac{x - 5}{7 - 5}\right) \\
 &= \frac{x - 5}{2} \quad \# 2^{-1} = 7 \\
 &= 7(x - 5) = 7x + 4 \quad \# -5 = 8 \text{ and } 7 \cdot 8 = 4
 \end{aligned}$$

4788 Using Lagrange's interpolation, we can deduce that $A_2 = B_3 = C_5$ as well as $A_5 = B_4 = C_1$,
 4789 since they are polynomials of degree 1 that evaluate to same values on 2 points. Using this, we
 4790 get the following set of polynomials

$A_0(x) = 0$	$B_0(x) = 0$	$C_0(x) = 0$
$A_1(x) = 0$	$B_1(x) = 0$	$C_1(x) = 7x + 4$
$A_2(x) = 6x + 10$	$B_2(x) = 0$	$C_2(x) = 0$
$A_3(x) = 0$	$B_3(x) = 6x + 10$	$C_3(x) = 0$
$A_4(x) = 0$	$B_4(x) = 7x + 4$	$C_4(x) = 0$
$A_5(x) = 7x + 4$	$B_5(x) = 0$	$C_5(x) = 6x + 10$

4792 We can invoke Sage to verify our computation. In sage every polynomial ring has a function
 4793 `lagrange_polynomial` that takes the defining points as inputs and the associated Lagrange
 4794 polynomial as output.

```

4795 sage: F13 = GF(13)                                     643
4796 sage: F13t.<t> = F13[]                                   644
4797 sage: T = F13t((t-5)*(t-7))                             645
4798 sage: A2 = F13t.lagrange_polynomial([(5, 1), (7, 0)])    646
4799 sage: A5 = F13t.lagrange_polynomial([(5, 0), (7, 1)])    647
4800 sage: T == F13t(t^2 + t + 9)                             648
4801 True                                                    649
4802 sage: A2 == F13t(6*t + 10)                               650
4803 True                                                    651
4804 sage: A5 == F13t(7*t + 4)                                652
4805 True                                                    653

```

4806 Combining this computation with the target polynomial we derived earlier, a quadratic arith-
 4807 metic program associated to the rank-1 constraint system $R_{3.fac_zk}$ is given as follows:

$$\begin{aligned}
 QAP(R_{3.fac_zk}) &= \{x^2 + x + 9, \\
 \{0, 0, 6x + 10, 0, 0, 7x + 4\}, \{0, 0, 0, 6x + 10, 7x + 4, 0\}, \{0, 7x + 4, 0, 0, 0, 6x + 10\}\}
 \end{aligned} \quad (6.14)$$

4808 **QAP Satisfiability** One of the major points of quadratic arithmetic programs in proofing sys-
 4809 tems is that solutions of their associated rank-1 constraints systems are in 1:1 correspondence
 4810 with certain polynomials P such that P is divisible by the target polynomial T of the QAP if and
 4811 only if the solution is a solution. Verifying solutions to the R1CS and hence, checking proper
 4812 circuit execution is then achievable by polynomial division of P by T .

4813 To be more specific, let R be some rank-1 constraints system with associated assignment
 4814 variables $(I_1, \dots, I_n; W_1, \dots, W_m)$ and let $QAP(R)$ be a quadratic arithmetic program of R . Then

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the tuple $(I_1, \dots, I_n; W_1, \dots, W_m)$ is a solution to the R1CS if and only if the following polynomial is divisible by the target polynomial T :

$$P_{(I;W)} = (A_0 + \sum_j^n I_j \cdot A_j + \sum_j^m W_j \cdot A_{n+j}) \cdot (B_0 + \sum_j^n I_j \cdot B_j + \sum_j^m W_j \cdot B_{n+j}) - (C_0 + \sum_j^n I_j \cdot C_j + \sum_j^m W_j \cdot C_{n+j}) \quad (6.15)$$

Every tuple $(I; W)$ defines a polynomial $P_{(I;W)}$, and, since each polynomial A_j , B_j and C_j is of degree $k-1$, $P_{(I;W)}$ is of degree $(k-1) \cdot (k-1) = k^2 - 2k + 1$.

To understand how quadratic arithmetic programs define formal languages, observe that every QAP over a field \mathbb{F} defines a decision function over the alphabet $\Sigma_I \times \Sigma_W = \mathbb{F} \times \mathbb{F}$ in the following way:

$$R_{QAP} : \mathbb{F}^* \times \mathbb{F}^* \rightarrow \{true, false\} ; (I; W) \mapsto \begin{cases} true & P_{(I;W)} \text{ is divisible by } T \\ false & \text{else} \end{cases} \quad (6.16)$$

This means that every QAP defines a formal language, and, if the QAP is associated to an R1CS, it can be shown that the two languages are equivalent. A **statement** is a membership claim “There is a word $(I; W)$ in L_{QAP} ”. A proof to this claim is therefore a polynomial $P_{(I;W)}$, which is verified by dividing $P_{(I;W)}$ by T .

Note the structural similarity to the definition of an R1CS in 6.2.1.1 and the different ways of computing proofs in both systems. For circuits and their associated rank-1 constraints systems, a constructive proof consists of a valid assignment of field elements to the edges of the circuit, or the variables in the R1CS. However, in the case of QAPs, a valid proof consists of a polynomial $P_{(I;W)}$.

To compute a proof for a statement in L_{QAP} given some instance I , a proofer first needs to compute a constructive proof W , e.g. by executing the circuit. With $(I; W)$ at hand, the proofer can then compute the polynomial $P_{(I;W)}$ and publish it as proof.

Verifying a constructive proof in the case of a circuit is achieved by executing the circuit, comparing the result to the given proof, and verifying the same proof in the R1CS picture means checking if the elements of the proof satisfy all equation.

In contrast, verifying a proof in the case of a QAP is done by polynomial division of the proof P by the target polynomial T of the QAP. The proof checks out if and only if P is divisible by T .

Example 125. Consider the quadratic arithmetic program $QAP(R_{3, fac_zk})$ from example XXX, and its associated R1CS from equation 6.1.0.1. To give an intuition of how proofs in the language $L_{QAP(R_{3, fac_zk})}$ let's consider the instance $I_1 = 11$. As we know from example XXX, $(W_1, W_2, W_3, W_5) = (2, 3, 4, 6)$ is a proper witness, since $(I_1; W_1, W_2, W_3, W_5) = (11; 2, 3, 4, 6)$ is a valid circuit assignment and hence, a solution to R_{3, fac_zk} and a constructive proof for language $L_{R_{3, fac_zk}}$.

In order to transform this constructive proof into a membership proof in language $L_{QAP(R_{3, fac_zk})}$ a proofer has to use the elements of the constructive proof, to compute the polynomial $P_{(I;W)}$.

In the case of $(I_1; W_1, W_2, W_3, W_5) = (11; 2, 3, 4, 6)$, the associated proof is computed as fol-

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lows:

$$\begin{aligned}
P_{(I;W)} &= (A_0 + \sum_j^n I_j \cdot A_j + \sum_j^m W_j \cdot A_{n+j}) \cdot (B_0 + \sum_j^n I_j \cdot B_j + \sum_j^m W_j \cdot B_{n+j}) - (C_0 + \sum_j^n I_j \cdot C_j + \sum_j^m W_j \cdot C_{n+j}) \\
&= (2(6x + 10) + 6(7x + 4)) \cdot (3(6x + 10) + 4(7x + 4)) - (11(7x + 4) + 6(6x + 10)) \\
&= ((12x + 7) + (3x + 11)) \cdot ((5x + 4) + (2x + 3)) - ((12x + 5) + (10x + 8)) \\
&= (2x + 5) \cdot (7x + 7) - (9x) \\
&= (x^2 + 2 \cdot 7x + 5 \cdot 7x + 5 \cdot 7) - (9x) \\
&= (x^2 + x + 9x + 9) - (9x) \\
&= x^2 + x + 9
\end{aligned}$$

4848 Given instance $I_1 = 11$ a proofer therefore provides the polynomial $x^2 + x + 9$ as proof. To verify
4849 this proof, any verifier can then look up the target polynomial T from the QAP and divide $P_{(I;W)}$
4850 by T . In this particular example, $P_{(I;W)}$ is equal to the target polynomial T , and hence, it is
4851 divisible by T with $P/T = 1$. The verification therefore checks the proof.

```

4852 sage: F13 = GF(13) 654
4853 sage: F13t.<t> = F13[] 655
4854 sage: T = F13t(t^2 + t + 9) 656
4855 sage: P = F13t((2*(6*t+10)+6*(7*t+4))*(3*(6*t+10)+4*(7*t+4)) 657
4856             -(11*(7*t+4)+6*(6*t+10)))
4857 sage: P == T 658
4858 True 659
4859 sage: P % T # remainder 660
4860 0 661

```

To give an example of a false proof, consider the tuple $(I_1; W_1, W_2, W_3, W_4) = (11, 2, 3, 4, 8)$. Executing the circuit, we can see that this is not a valid assignment and not a solution to the R1CS, and hence, not a constructive knowledge proof in $L_{3.fac_zk}$. However, a proofer might use these values to construct a false proof $P_{(I;W)}$:

$$\begin{aligned}
P'_{(I;W)} &= (A_0 + \sum_j^n I_j \cdot A_j + \sum_j^m W_j \cdot A_{n+j}) \cdot (B_0 + \sum_j^n I_j \cdot B_j + \sum_j^m W_j \cdot B_{n+j}) - (C_0 + \sum_j^n I_j \cdot C_j + \sum_j^m W_j \cdot C_{n+j}) \\
&= (2(6x + 10) + 8(7x + 4)) \cdot (3(6x + 10) + 4(7x + 4)) - (8(6x + 10) + 11(7x + 4)) \\
&= 8x^2 + 6
\end{aligned}$$

Given instance $I_1 = 11$, a proofer therefore provides the polynomial $8x^2 + 6$ as proof. To verify this proof, any verifier can look up the target polynomial T from the QAP and divide $P_{(I;W)}$ by T . However, polynomial division has the following remainder:

$$(8x^2 + 6)/(x^2 + x + 9) = 8 + \frac{5x + 12}{x^2 + x + 9}$$

4861 This implies that $P_{(I;W)}$ is not divisible by T , and hence, the verification does not check the
4862 proof. Any verifier can therefore show that the proof is false.

```

4863 sage: F13 = GF(13) 662
4864 sage: F13t.<t> = F13[] 663
4865 sage: T = F13t(t^2 + t + 9) 664
4866 sage: P = F13t((2*(6*t+10)+8*(7*t+4))*(3*(6*t+10)+4*(7*t+4))-( 665
4867             8*(6*t+10)+11*(7*t+4)))

```

4868	sage: $P == F13t(8*t^2 + 6)$	666
4869	True	667
4870	sage: $P \% T$ # remainder	668
4871	$5*t + 12$	669

Chapter 7

Circuit Compilers

As we have seen in the previous chapter, statements can be formalized as membership or knowledge claims in formal language, and algebraic circuits as well as rank-1 constraint systems are two practically important ways to define those languages.

However, both algebraic circuits and rank-1 constraint systems are not ideal from a developers point of view, because they deviate substantially from common programming paradigms. Writing real-world applications as circuits and the associated verification in terms of rank-1 constraint systems is at least as troublesome as writing any other low-level language like assembler code. To allow for complex statement design, it is therefore necessary to have some kind of compiler framework, capable of transforming high-level languages into arithmetic circuits and associated rank-1 constraint systems.

As we have seen in chapter 6 and in 6.2.1.1., both arithmetic circuits and rank-1 constraint systems have a modularity property by which it is possible to synthesize complex circuits from simple ones. A basic approach taken by many circuit/R1CS compilers is therefore to provide a library of atomic and simple circuits and then define a way to combine those basic building blocks into arbitrary complex systems.

In this chapter, we provide an introduction to basic concepts of so-called **circuit compilers** and derive a toy language which we can “compile” in a pen-and-paper approach into algebraic circuits and their associated rank-1 constraint systems.

We start with a general introduction to our language, and then introduce atomic types like booleans and unsigned integers. Then we define the fundamental control flow primitives like the if-then-else conditional and the bounded loop. We will look at basic functionality primitives like elliptic curve cryptography. Primitives like these are often called **gadgets** in the literature.

7.1 A Pen-and-Paper Language

To explain basic concepts of circuit compilers and their associated high-level languages, we derive an informal toy language and associated “brain-compiler” which we name PAPER (**Pen-And-Paper Execution Rules**). PAPER allows programmers to define statements in Rust-like pseudo-code. The language is inspired by ZOKRATES and `circom`.

7.1.1 The Grammar

In PAPER, any statement is defined as an ordered list of functions, where any function has to be declared in the list before it is called in another function of that list. The last entry in a statement has to be a special function, called `main`. Functions take a list of typed parameters as inputs

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and compute a tuple of typed variables as output, where types are special functions that define how to transform that type into another type, ultimately transforming any type into elements of the base field where the circuit is defined over.

Any statement is parameterized over the field that the circuit will be defined on, and has additional optional parameters of unsigned type, needed to define the size of array or the counter of bounded loops. The following definition makes the grammar of a statement precise using a command line language like description:

```
statement <Name> {F:<Field> [ , <N_1: unsigned>, ... ] } {
  [fn <Name>([[pub]<Arg>:<Type>, ...]) -> (<Type>, ...)] {
    [let [pub] <Var>:<Type> ; ... ]
    [let const <Const>:<Type>=<Value> ; ... ]
    Var<==>(fn ([<Arg>|<Const>|<Var>, ...]) | (<Arg>|<Const>|<Var>)) ;
    return (<Var>, ...) ;
  } ; ...]
  fn main([[pub]<Arg>:<Type>, ...]) -> (<Type>, ...) {
    [let [pub] <Var>:<Type> ; ... ]
    [let const <Const>:<Type>=<Value> ; ... ]
    Var<==>(fn ([<Arg>|<Const>|<Var>, ...]) | (<Arg>|<Const>|<Var>)) ;
    return (<Var>, ...) ;
  } ;
}
```

Function arguments and variables are private by default, but can be declared as public by the `pub` specifier. Declaring arguments and variables as public always overwrites any previous or conflicting private declarations. Every argument, constant or variable has a type, and every type is defined as a function that transforms that type into another type:

```
type <TYPE>( t1 : <TYPE_1>) -> TYPE_2{
  let t2: TYPE_2 <== fn(TYPE_1)
  return t2
}
```

Many real-world circuit languages are based on a similar, but of course more sophisticated approach than PAPER. The purpose of PAPER is to show basic principles of circuit compilers and their associated high-level languages.

Example 126. To get a better understanding of the grammar of PAPER, the following constitutes proper high-level code that follows the grammar of the PAPER language, assuming that all types in that code have been defined elsewhere.

```
statement MOCK_CODE {F: F_43, N_1 = 1024, N_2 = 8} {
  fn foo(in_1 : F, pub in_2 : TYPE_2) -> F {
    let const c_1 : F = 0 ;
    let const c_2 : TYPE_2 = SOME_VALUE ;
    let pub out_1 : F ;
    out_1<== c_1 ;
    return out_1 ;
  } ;

  fn bar(pub in_1 : F) -> F {
    let out_1 : F ;
    out_1<==foo(in_1);
    return out_1 ;
  } ;
}
```

```

4953   } ;
4954
4955   fn main(in_1 : TYPE_1) -> (F, TYPE_2) {
4956       let const c_1 : TYPE_1 = SOME_VALUE ;
4957       let const c_2 : F = 2;
4958       let const c_3 : TYPE_2 = SOME_VALUE ;
4959       let pub out_1 : F ;
4960       let out_2 : TYPE_2 ;
4961       c_1 <== in_1 ;
4962       out_1 <== foo(c_2) ;
4963       out_2 <== TYPE_2 ;
4964       return (out_1, out_2) ;
4965   } ;
4966 }

```

7.1.2 The Execution Phases

In contrast to normal executable programs, programs for circuit compilers have two modes of execution. The first mode, usually called the **setup phase**, is executed in order to generate the circuit and its associated rank-1 constraint system, the latter of which is then usually used as input to some zero-knowledge proof system.

The second mode of execution is usually called the **prover phase**. In this phase, a prover usually computes a valid assignment to the circuit. Depending on the use case, this valid assignment is then either directly used as constructive proof for proper circuit execution or is transferred as input to the proof generation algorithm of some zero-knowledge proof system, where the full-sized, non hiding constructive proof is processed into a succinct proof with various levels of zero knowledge.

Modern circuit languages and their associated compilers abstract over those two phases and provide a unified **interphase** to the developer, who then writes a single program that can be used in both phases.

To give the reader a clear, conceptual distinction between the two phases, PAPER keeps them separated. Code can be “brain-compiled” during the **setup-phase** in a pen-and-paper approach into visual circuits. Once a circuit is derived, it can be executed in a **prover phase** to generate a valid assignment. The valid assignment is then interpreted as a constructive proof for a knowledge claim in the associated language.

The Setup Phase In PAPER, the task of the setup phase is to compile code in the PAPER language into a visual representation of an algebraic circuit. Deriving the circuit from the code in a pen-and-paper style is what we call **brain-compiling**.

Given some statement description that adheres to the correct grammar, we start circuit development with an empty circuit, compile the main function first and then inductively compile all other functions as they are called during the process.

For every function we compile, we draw a box-node for every argument, every variable and every constant of that function. If the node represents a variable, we label it with that variable’s name, and if it represents a constant, we label it with that constant’s value. We group arguments into a subgraph labeled “inputs” and return values into a subgraph labeled “outputs”. We then group everything into a subgraph and label that subgraph with the function’s name.

After this is done, we have to do a consistency and type check for every occurrence of the

assignment operator `<==`. We have to ensure that the expression on the right side of the operator is well defined and that the types of both side match.

Then we compile the right side of every occurrence of the assignment operator `<==`. If the right side is a constant or variable defined in this function, we draw a dotted line from the box-node that represents the left side of `<==` to the box node that represents the right side of the same operator. If the right side represents an argument of that function we draw a line from the box-node that represents the left side of `<==` to the box node that represents the right side of the same operator.

If the right side of the `<==` operator is a function, we look into our database, find its associated circuit and draw it. If no circuit is associated to that function yet, we repeat the compilation process for that function, drawing edges from the function's argument to its input nodes and from the functions output nodes to the nodes on the right side of `<==`.

During that process, edge labels are drawn according to the rules from 6.2.2.1. If the associated variable represents a private value, we use the *W* label to indicate a witness, and if it represents a public value, we use the *I* label to indicate an instance.

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Once this is done, we compile all occurring types in a function, by compiling the function of each type. We do this inductively until we reach the type of the base field. Circuits have no notion of types, only of field elements; hence, every type needs to be compiled to the field type in a sequence of compilation steps.

The compilation stops once we have inductively replaced all functions by their circuits. The result is a circuit that contains many unnecessary box nodes. In a final optimization step, all box nodes that are directly linked to each other are collapsed into a single node, and all box nodes that represent the same constants are collapsed into a single node.

Of course, PAPER's brain-compiler is not properly defined in any formal manner. Its purpose is to highlight important steps that real-world compilers undergo in their setup phases.

Example 127 (A trivial Circuit). To give an intuition of how to write and compile circuits in the PAPER language, consider the following statement description:

```
statement trivial_circuit {F:F_13} {
  fn main{F}(in1 : F, pub in2 : F) -> (F,F) {
    let const outc1 : F = 0 ;
    let const incl : F = 7 ;
    let out1 : F ;
    let out2 : F ;
    out1 <== incl;
    out2 <== in1;
    outc1 <== in2;
    return (out1, out2) ;
  }
}
```

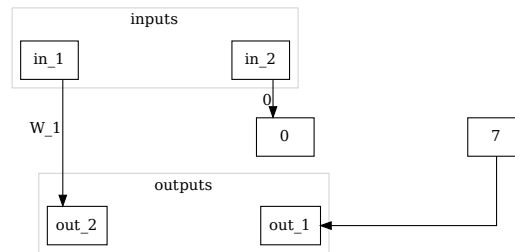
To brain-compile this statement into an algebraic circuit with PAPER, we start with an empty circuit and evaluate function `main`, which is the only function in this statement.

We draw box-nodes for every argument, every constant and every variable of the function and label them with their names or values, respectively. Then we do a consistency and type check for every `<==` operator in the function. Since the circuit only wires inputs to outputs and all elements have the same type, the check is valid.

Then we evaluate the right side of the assignment operators. Since, in our case, the right side of each operator is not a function, we draw edges from the box-nodes on the right side to the associated box node on the left side. To label those edges, we use the general rules of

algebraic circuits as defined in 6.2.2.1. According to those rules, every incoming edge of a sink node has a label and every outgoing edge of a source node has a label, if the node is labeled with a variable. Since nodes that represent constants are implicitly assumed to be private, and since the public specifier determines if an edge is labeled with W or I , we get the following circuit:

check
reference



The Prover Phase In PAPER, a so-called **prover phase** can be executed once the setup phase has generated a circuit image from its associated high-level code. This is done by executing the circuit while assigning proper values to all input nodes of the circuit. However, in contrast to most real-world compilers, PAPER does not tell the prover how to find proper input values to a given circuit. Real-world programming languages usually provide this data by computations that are done outside of the circuit.

Example 128. Consider the circuit from example 127. Valid assignments to this circuit are constructive proofs that the pair of inputs (S_1, S_2) is a point on the tiny-jubjub curve. However, the circuit does not provide a way to actually compute proper values for S_1 and S_2 . Any real-world system therefore needs an auxiliary computation that provides those values.

check
reference

7.2 Common Programing concepts

In this section, we cover concepts that appear in almost every programming language, and see how they can be implemented in circuit compilers.

7.2.1 Primitive Types

Primitive data types like booleans, (unsigned) integers, or strings are the most basic building blocks one can expect to find in every general high-level programming language. In order to write statements as computer programs that compile into circuits, it is therefore necessary to implement primitive types as constraint systems, and define their associated operations as circuits.

In this section, we look at some common ways to achieve this. After a recapitulation of the atomic type of prime field elements, we start with an implementation of the boolean type and its associated boolean algebra as circuits. After that, we define unsigned integers based on the boolean type, and leave the implementation of signed integers as an exercise to the reader.

It should be noted, however, that while primitive data types in common programming languages (like C, Go, or Rust) have a one-to-one correspondence with objects in the computer's memory, this is not the case for most languages that compile into algebraic circuits. As we will see in the following paragraphs, common primitives like booleans or unsigned integers require many constraints and memory. Primitives different from the underlying field elements can be expensive.

The base-field type

Since both algebraic circuits and their associated rank-1 constraint systems are defined over a finite field, elements from that field are the atomic informational units in those models. In this sense, field elements $x \in \mathbb{F}$ are for algebraic circuits what bits are for computers.

In PAPER, we write \mathbb{F} for this type and specify the actual field instance for every statement in curly brackets after the name of that statement. Two functions are associated to this type, which are induced by the **addition** and **multiplication** law in the field \mathbb{F} . We write

$$\text{MUL} : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F} ; (x, y) \mapsto \text{MUL}(x, y) \quad (7.1)$$

$$\text{ADD} : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F} ; (x, y) \mapsto \text{ADD}(x, y) \quad (7.2)$$

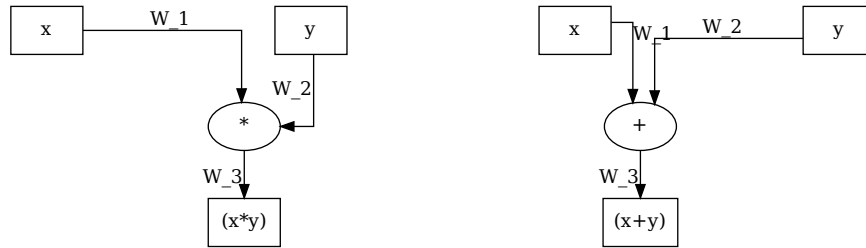
Circuit compilers have to compile these functions into algebraic gates, as explained in 6.2.2.2. Every other function has to be expressed in terms of them and proper wiring.

To represent addition and multiplication in the PAPER language, we define the following two functions:

```
fn MUL(x : F, y : F) -> (MUL(x, y) : F) {}
```

```
fn ADD(x : F, y : F) -> (ADD(x, y) : F) {}
```

The compiler then compiles every occurrence of the MUL or the ADD function into the following circuits:



Example 129 (Basic gates). To give an intuition of how a real-world compiler might transform addition and multiplication in algebraic expressions into a circuit, consider the following PAPER statement:

```
statement basic_ops {F:F_13} {
  fn main(in_1 : F, pub in_2 : F) -> (out_1:F, out_2:F) {
    out_1 <== MUL(in_1, in_2) ;
    out_2 <== ADD(in_1, in_2) ;
  }
}
```

To compile this into an algebraic circuit, we start with an empty circuit and evaluate the function `main`, which is the only function in this statement.

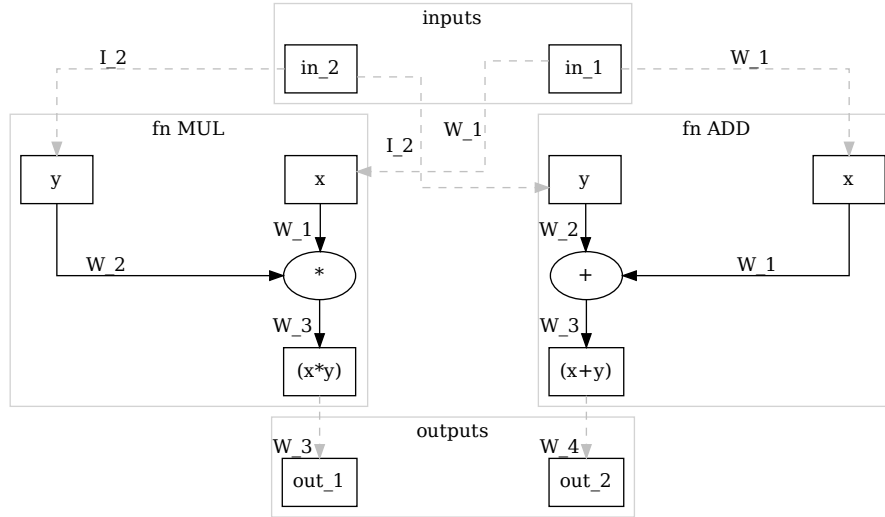
We draw an inputs subgraph containing box-nodes for every argument of the function, and an outputs subgraph containing box-nodes for every factor in the return value. Since all of these nodes represent variables of the `field` type, we don't have to add any type constraints to the circuit.

We check the validity of every expression on the right side of every `<==` operator including a type check. In our case, every variable is of the `field` type and hence the types match the types of the `MUL` as well as the `ADD` function and the type of the left sides of `<==` operators.

check
reference

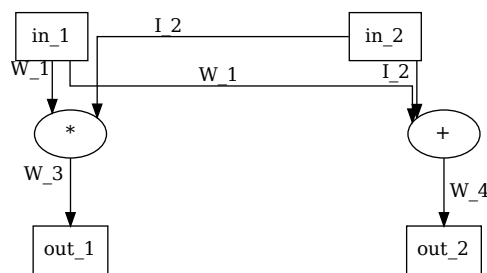
We evaluate the expressions on the right side of every `<==` operator inductively, replacing every occurrence of a function with a subgraph that represents its associated circuit.

According to PAPER, every occurrence of the `public` specifier overwrites the associate `private` default value. Using the appropriate edge labels we get:



Any real-world compiler might process its associated high-level language in a similar way, replacing functions, or gadgets by predefined associated circuits. This process is often followed by various optimization steps that try to reduce the number of constraints as much as possible.

In PAPER, we optimize this circuit by collapsing all box nodes that are directly connected to other box nodes, adhering to the rule that a variable's `public` specifier overwrites any `private` specifier. Reindexing edge labels, we get the following circuit as our pen and pencil compiler output:



Example 130 (3-factorization). Consider our 3-factorization problem from example 108 and the associated circuit $C_{3.\text{fac_zk}}(\mathbb{F}_{13})$ we provided in equation 6.8. To understand the process of replacing high-level functions by their associated circuits inductively, we want define a PAPER statement that we brain-compile into an algebraic circuit equivalent to $C_{3.\text{fac_zk}}(\mathbb{F}_{13})$:

```
statement 3_fac_zk {F:F_13} {
  fn main(x_1 : F, x_2 : F, x_3 : F) -> (pub 3_fac_zk : F) {
    f_3.fac_zk <== MUL( MUL( x_1 , x_2 ) , x_3 ) ;
  }
}
```

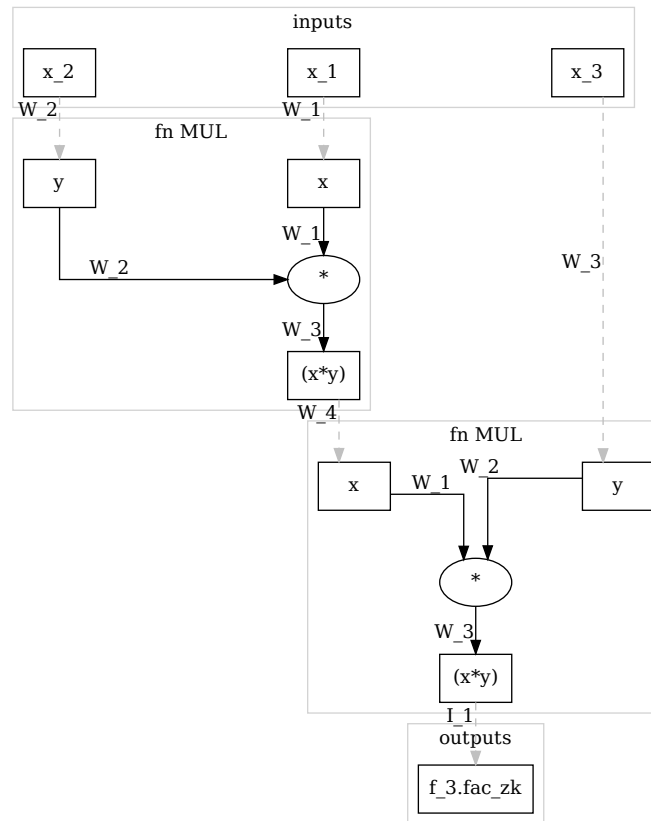
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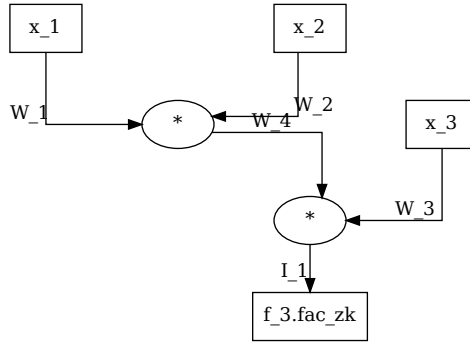
Using PAPER, we start with an empty circuit and then add 3 input nodes to the input subgraph as well as 1 output node to the output subgraph. All these nodes are decorated with the associated variable names. Since all of these nodes represent variables of the `field` type, we don't have to add any type constraints to the circuit.

We check the validity of every expression on the right side of the single `<==` operator including a type check.

We evaluate the expressions on the right side of every `<==` operator inductively. We have two nested multiplication functions and we replace them by the associated multiplication circuits, starting with the most outer function. We get:



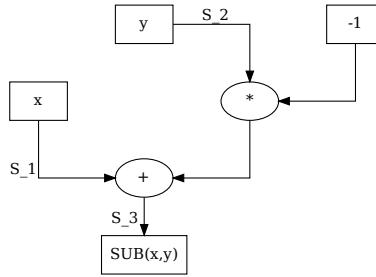
In a final optimization step, we collapse all box nodes directly connected to other box nodes, adhering to the rule that a variable's `public` specifier overwrites any `private` specifier. Reindexing edge labels we get the following circuit:



5149

5150 **The Subtraction Constraint System** By definition, algebraic circuits only contain addition
 5151 and multiplication gates, and it follows that there is no single gate for field subtraction, despite
 5152 the fact that subtraction is a native operation in every field.

5153 High-level languages and their associated circuit compilers, therefore, need another way to
 5154 deal with subtraction. To see how this can be achieved, recall that subtraction is defined by addi-
 5155 tion with the additive inverse, and that the inverse can be computed efficiently by multiplication
 5156 with -1 . A circuit for field subtraction is therefore given by



5157

5158 Using the general method from 6.2.1.1, the circuits associated rank-1 constraint system is given
 5159 by:

$$(S_1 + (-1) \cdot S_2) \cdot 1 = S_3 \quad (7.3)$$

5160 Any valid assignment $\{S_1, S_2, S_3\}$ to this circuit therefore enforces the value S_3 to be the differ-
 5161 ence $S_1 - S_2$.

5162 Real-world compilers usually provide a gadget or a function to abstract over this circuit
 5163 such that programers can use subtraction as if it were native to circuits. In PAPER, we define
 5164 the following subtraction function that compiles to the previous circuit:

```
5165 fn SUB(x : F, y : F) -> (SUB(x,y) : F) {
5166   constant c : F = -1 ;
5167   SUB <== ADD(x , MUL( y , c ) );
5168 }
```

5169 In the setup phase of a statement, we compile every occurrence of the SUB function into an
 5170 instance of its associated subtraction circuit, and edge labels are generated according to the
 5171 rules from 6.2.2.1.

check
reference

check
reference

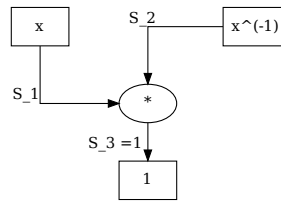
The Inversion Constraint System By definition, algebraic circuits only contain addition and multiplication gates, and it follows that there is no single gate for field inversion, despite the fact that inversion is a native operation in every field.

If the underlying field is a prime field, one approach would be to use Fermat’s little theorem 3.3 to compute the multiplicative inverse inside the circuit. To see how this works, let \mathbb{F}_p be the prime field. The multiplicative inverse x^{-1} of a field element $x \in \mathbb{F}$ with $x \neq 0$ is then given by $x^{-1} = x^{p-2}$, and computing x^{p-2} in the circuit therefore computes the multiplicative inverse.

Unfortunately, real-world primes p are large and computing x^{p-2} by repeated multiplication of x with itself is infeasible. A “double and multiply” approach (as described in XXX) is faster, as it computes the power in roughly $\log_2(p)$ steps, but still adds a lot of constraints to the circuit.

Computing inverses in the circuit makes no use of the fact that inversion is a native operation in any field. A more constraints friendly approach is therefore to compute the multiplicative inverse outside of the circuit and then only enforce correctness of the computation in the circuit.

To understand how this can be achieved, observe that a field element $y \in \mathbb{F}$ is the multiplicative inverse of a field element $x \in \mathbb{F}$ if and only if $x \cdot y = 1$ in \mathbb{F} . We can use this, and define a circuit that has two inputs, x and y , and enforces $x \cdot y = 1$. It is then guaranteed that y is the multiplicative inverse of x . The price we pay is that we can not compute y by circuit execution, but auxiliary data is needed to tell any prover which value of y is needed for a valid circuit assignment. The following circuit defines the constraint



Using the general method from 6.2.1.1, the circuit is transformed into the following rank-1 constraint system:

$$S_1 \cdot S_2 = 1 \quad (7.4)$$

Any valid assignment $\{S_1, S_2\}$ to this circuit enforces that S_2 is the multiplicative inverse of S_1 , and, since there is no field element S_2 such that $0 \cdot S_2 = 1$, it also handles the fact that the multiplicative inverse of 0 is not defined in any field.

Real-world compilers usually provide a gadget or a function to abstract over this circuit, and those functions compute the inverse x^{-1} as part of their witness generation process. Programmers then don’t have to care about providing the inverse as auxiliary data to the circuit. In PAPER, we define the following inversion function that compiles to the previous circuit:

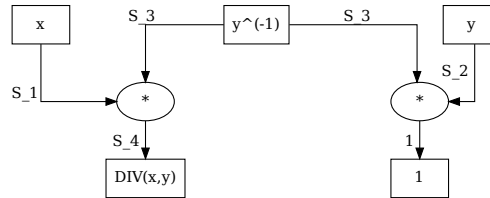
```
fn INV(x : F, y : F) -> (x_inv : F) {
  constant c : F = 1 ;
  c <== MUL( x , y ) ;
  x_inv <== y ;
}
```

As we see, this functions takes two inputs, the field value and its inverse. It therefore does not handle the computation of the inverse by itself. This is to keep PAPER as simple as possible.

In the setup phase, we compile every occurrence of the INV function into an instance of the inversion circuit XXX, and edge labels are generated according to the rules from XXX.

The Division Constraint System By definition, algebraic circuits only contain addition and multiplication gates, and it follows that there is no single gate for field division, despite the fact that division is a native operation in every field.

Implementing division as a circuit, we use the fact that division is multiplication with the multiplicative inverse. We therefore define division as a circuit using the inversion circuit and constraint system from the previous paragraph. Expensive inversion is computed outside of the circuit and then provided as circuit input. We get



Using the method from 6.2.1.1, we transform this circuit into the following rank-1 constraint system:

$$\begin{aligned} S_2 \cdot S_3 &= 1 \\ S_1 \cdot S_3 &= S_4 \end{aligned}$$

Any valid assignment $\{S_1, S_2, S_3, S_4\}$ to this circuit enforces S_4 to be the field division of S_1 by S_2 . It handles the fact that division by 0 is not defined, since there is no valid assignment in case $S_2 = 0$.

In PAPER, we define the following division function that compiles to the previous circuit:

```
fn DIV(x : F, y : F, y_inv : F) -> (DIV : F) {
  DIV <== MUL( x , INV( y, y_inv ) ) ;
}
```

In the setup phase, we compile every occurrence of the binary INV operator into an instance of the inversion circuit.

Exercise 55. Let F be the field \mathbb{F}_5 of modular 5 arithmetics from example 14. Brain-compile the following PAPER statement into an algebraic circuit:

```
statement STUPID_CIRC {F: F_5} {
  fn foo(in_1 : F, in_2 : F)->(out_1 : F, out_2 : F){
    constant c_1 : F = 3 ;
    out_1<== ADD( MUL( c_1 , in_1 ) , in_1 ) ;
    out_2<== INV( c_1 , in_2 ) ;
  } ;

  fn main(in_1 : F, in_2 : F)->(out_1 : F, out_2 : TYPE_2){
    constant (c_1,c_2) : (F,F) = (3,2) ;
    (out_1,out_2) <== foo(in_1, in_2) ;
  } ;
}
```

Exercise 56. Consider the tiny-jubjub curve from example 68 and its associated circuit 127. Write a statement in PAPER that brain-compiles the statement into a circuit equivalent to the one derived in XXX, assuming that curve points are instances and every other assignment is a witness.

5245 *Exercise 57.* Let $F = \mathbb{F}_{13}$ be the modular 13 prime field and $x \in F$ some field element. Define a
 5246 statement in PAPER such that given instance x a field element $y \in F$ is a witness for the statement
 5247 if and only if y is the square root of x .

5248 Brain-compile the statement into a circuit and derive its associated rank-1 constraint system.
 5249 Consider the instance $x = 9$ and compute a constructive proof for the statement.

5250 The boolean Type

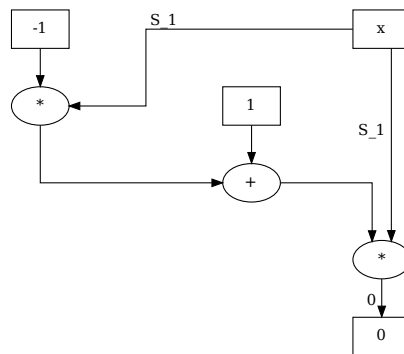
5251 Booleans are a classical primitive type, implemented by virtually every higher programing lan-
 5252 guage. It is therefore important to implement booleans in circuits. One of the most common
 5253 ways to do this is by interpreting the additive and multiplicative neutral element $\{0, 1\} \subset \mathbb{F}$ as
 5254 the two boolean values such that 0 represents *false* and 1 represents *true*. boolean operators
 5255 like *and*, *or*, or *xor* are then expressible as algebraic computations inside \mathbb{F} .

5256 Representing booleans this way is convenient, because the elements 0 and 1 are defined in
 5257 any field. The representation is therefore independent of the actual field in consideration.

5258 To fix boolean algebra notation, we write 0 to represent *false* and 1 to represent *true*, and
 5259 we write \wedge to represent the boolean AND as well as \vee to represent the boolean OR operator.
 5260 The boolean NOT operator is written as \neg .

5261 **The boolean Constraint System** To represent booleans by the additive and multiplicative
 5262 neutral elements of a field, a constraint is required to actually enforce variables of boolean type
 5263 to be either 1 or 0. In fact, many of the following circuits that represent boolean functions are
 5264 only correct under the assumption that their input variables are constrained to be either 0 or 1.
 5265 Not constraining boolean variables is a common problem in circuit design.

5266 In order to constrain an arbitrary field element $x \in \mathbb{F}$ to be 1 or 0, the key observation is
 5267 that the equation $x \cdot (1 - x) = 0$ has only two solutions 0 and 1 in any field. Implementing this
 5268 equation as a circuit therefore generates the correct constraint:



5269

Using the method from 6.2.1.1, we transform this circuit into the following rank-1 constraint system:

$$S_1 \cdot (1 - S_1) = 0$$

check
reference

5270 Any valid assignment $\{S_1\}$ to this circuit enforces S_1 to be either 0 or 1.

5271 Some real-world circuit compilers (like ZOKRATES or BELLMAN) are typed, while others
 5272 (like circom) are not. However, all of them have their way of dealing with the binary con-
 5273 straint. In PAPER, we define the following boolean type that compiles to the previous circuit:

```

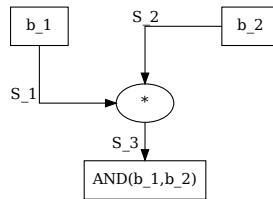
5274 type BOOL(b : BOOL) -> (x : F) {
5275     constant c1 : F = 0 ;
5276     constant c2 : F = 1 ;
5277     constant c3 : F = -1 ;
5278     c1 <== MUL( x , ADD( c2 , MUL( x , c3 ) ) ) ;
5279     x <== b ;
5280 }

```

5281 In the setup phase of a statement, we compile every occurrence of a variable of boolean type
 5282 into an instance of its associated boolean circuit.

5283 **The AND operator constraint system** Given two field elements b_1 and b_2 from \mathbb{F} that are
 5284 constrained to represent boolean variables, we want to find a circuit that computes the logical
 5285 **and** operator $AND(b_1, b_2)$ as well as its associated R1CS that enforces $b_1, b_2, AND(b_1, b_2)$ to
 5286 satisfy the constraint system if and only if $b_1 \wedge b_2 = AND(b_1, b_2)$ holds true.

5287 The key insight here is that, given three boolean constraint variables b_1, b_2 and b_3 , the
 5288 equation $b_1 \cdot b_2 = b_3$ is satisfied in \mathbb{F} if and only if the equation $b_1 \wedge b_2 = b_3$ is satisfied in
 5289 boolean algebra. The logical operator \wedge is therefore implementable in \mathbb{F} by field multiplication
 5290 of its arguments and the following circuit computes the \wedge operator in \mathbb{F} , assuming all inputs are
 5291 restricted to be 0 or 1:



5292

5293 The associated rank-1 constraint system can be deduced from the general process 6.2.1.1 and
 5294 consists of the following constraint:

$$S_1 \cdot S_2 = S_3 \quad (7.5)$$

check
reference

5295 Common circuit languages typically provide a gadget or a function to abstract over this circuit
 5296 such that programers can use the \wedge operator without caring about the associated circuit. In
 5297 PAPER, we define the following function that compiles to the \wedge -operator's circuit:

```

5298 fn AND(b_1 : BOOL, b_2 : BOOL) -> AND(b_1, b_2) : BOOL{
5299     AND(b_1, b_2) <== MUL( b_1 , b_2 ) ;
5300 }

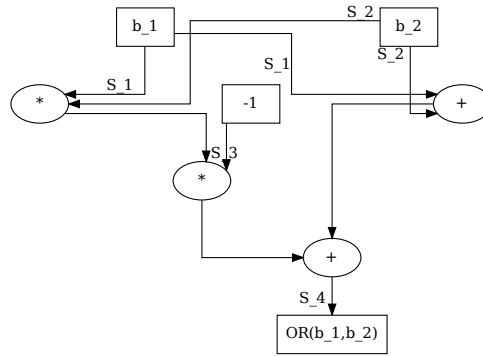
```

5301 In the setup phase of a statement, we compile every occurrence of the AND function into an
 5302 instance of its associated \wedge -operator's circuit.

5303 **The OR operator constraint system** Given two field elements b_1 and b_2 from \mathbb{F} that are
 5304 constrained to represent boolean variables, we want to find a circuit that computes the logical
 5305 **or** operator $OR(b_1, b_2)$ as well as its associated R1CS that enforces $b_1, b_2, OR(b_1, b_2)$ to satisfy
 5306 the constraint system if and only if $b_1 \vee b_2 = OR(b_1, b_2)$ holds true.

5307 Assuming that three variables b_1, b_2 and b_3 are boolean constraint, the equation $b_1 + b_2 - b_1 \cdot$
 5308 $b_2 = b_3$ is satisfied in \mathbb{F} if and only if the equation $b_1 \vee b_2 = b_3$ is satisfied in boolean algebra.
 5309 The logical operator \vee is therefore implementable in \mathbb{F} by the following circuit, assuming all
 5310 inputs are restricted to be 0 or 1:

"constraints"
or "con-
strained"?



5311

The associated rank-1 constraint system can be deduced from the general process 6.2.1.1 and consists of the following constraints:

check
reference

$$\begin{aligned} S_1 \cdot S_2 &= S_3 \\ (S_1 + S_2 - S_3) \cdot 1 &= S_4 \end{aligned}$$

5312 Common circuit languages typically provide a gadget or a function to abstract over this circuit
5313 such that programers can use the \vee operator without caring about the associated circuit. In
5314 PAPER, we define the following function that compiles to the \vee -operator's circuit:

```
5315 fn OR(b_1 : BOOL, b_2 : BOOL) -> OR(b_1,b_2) : BOOL{
5316   constant c1 : F = -1 ;
5317   OR(b_1,b_2) <== ADD(ADD(b_1,b_2),MUL(c1,MUL(b_1,b_2))) ;
5318 }
```

5319 In the setup phase of a statement, we compile every occurrence of the OR function into an
5320 instance of its associated \vee -operator's circuit.

5321 *Exercise 58.* Let \mathbb{F} be a finite field and let b_1 as well as b_2 two boolean constraint variables from
5322 \mathbb{F} . Show that the equation $OR(b_1, b_2) = 1 - (1 - b_1) \cdot (1 - b_2)$ holds true.

5323 Use this equation to derive an algebraic circuit with ingoing variables b_1 and b_2 and out-
5324 going variable $OR(b_1, b_2)$ such that b_1 and b_2 are boolean **constraint** and the circuit has a valid
5325 assignment, if and only if $OR(b_1, b_2) = b_1 \vee b_2$.

"constraints"
or "con-
strained"?

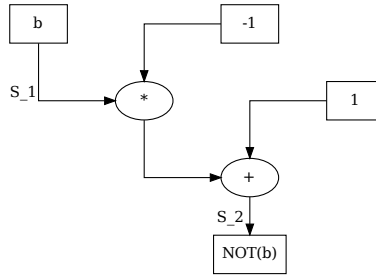
5326 Use the technique from XXX to transform this circuit into a rank-1 constraint system and
5327 find its full solution set. Define a PAPER function that brain-compiles into the circuit.

add refer-
ence

5328 **The NOT operator constraint system** Given a field element b from \mathbb{F} that is constrained to
5329 represent a boolean variable, we want to find a circuit that computes the logical **NOT** operator
5330 $NOT(b)$ as well as its associated RICS that enforces $b, NOT(b)$ to satisfy the constraint system
5331 if and only if $\neg b = NOT(b)$ holds true.

5332 Assuming that two variables b_1 and b_2 are boolean **constraint**, the equation $(1 - b_1) = b_2$ is
5333 satisfied in \mathbb{F} if and only if the equation $\neg b_1 = b_2$ is satisfied in boolean algebra. The logical
5334 operator \neg is therefore implementable in \mathbb{F} by the following circuit, assuming all inputs are
5335 restricted to be 0 or 1:

"constraints"
or "con-
strained"?



5336

The associated rank-1 constraint system can be deduced from the general process XXX and consists of the following constraints

add reference

$$(1 - S_1) \cdot 1 = S_2$$

5337 Common circuit languages typically provide a gadget or a function to abstract over this circuit
 5338 such that programers can use the \neg operator without caring about the associated circuit. In
 5339 PAPER, we define the following function that compiles to the \neg -operator's circuit:

```

5340 fn NOT(b : BOOL -> NOT(b) : BOOL{
5341   constant c1 = 1 ;
5342   constant c2 = -1 ;
5343   NOT(b_1) <== ADD( c1 , MUL( c2 , b ) ) ;
5344 }
```

5345 In the setup phase of a statement, we compile every occurrence of the NOT function into an
 5346 instance of its associated \neg -operator's circuit.

5347 *Exercise 59.* Let \mathbb{F} be a finite field. Derive the algebraic circuit and associated rank-1 constraint
 5348 system for the following operators: NOR, XOR, NAND, EQU.

5349 **Modularity** As we have seen in chapter 6,, both algebraic circuits and R1CS have a modular-
 5350 ity property, and as we have seen in this section, all basic boolean functions are expressible in
 5351 circuits. Combining those two properties, show that it is possible to express arbitrary boolean
 5352 functions as algebraic circuits.

check references

5353 This shows that the expressiveness of algebraic circuits and therefore rank-1 constraint sys-
 5354 tems is as general as the expressiveness of boolean circuits. An important implication is that
 5355 the languages $L_{R1CS-SAT}$ and $L_{Circuit-SAT}$ as defined in 4, are as general as the famous language
 5356 L_{3-SAT} , which is known to be \mathcal{NP} -complete.

check reference

5357 *Example 131.* To give an example of how a compiler might construct complex boolean expres-
 5358 sions in algebraic circuits from simple ones and how we derive their associated rank-1 constraint
 5359 systems, let's look at the following PAPER statement:

```

5360 statement BOOLEAN_STAT {F: F_p} {
5361   fn main(b_1:BOOL,b_2:BOOL,b_3:BOOL,b_4:BOOL )-> pub b_5:BOOL {
5362     b_5 <== AND( OR( b_1 , b_2 ) , AND( b_3 , NOT( b_4 ) ) ) ;
5363   } ;
5364 }
```

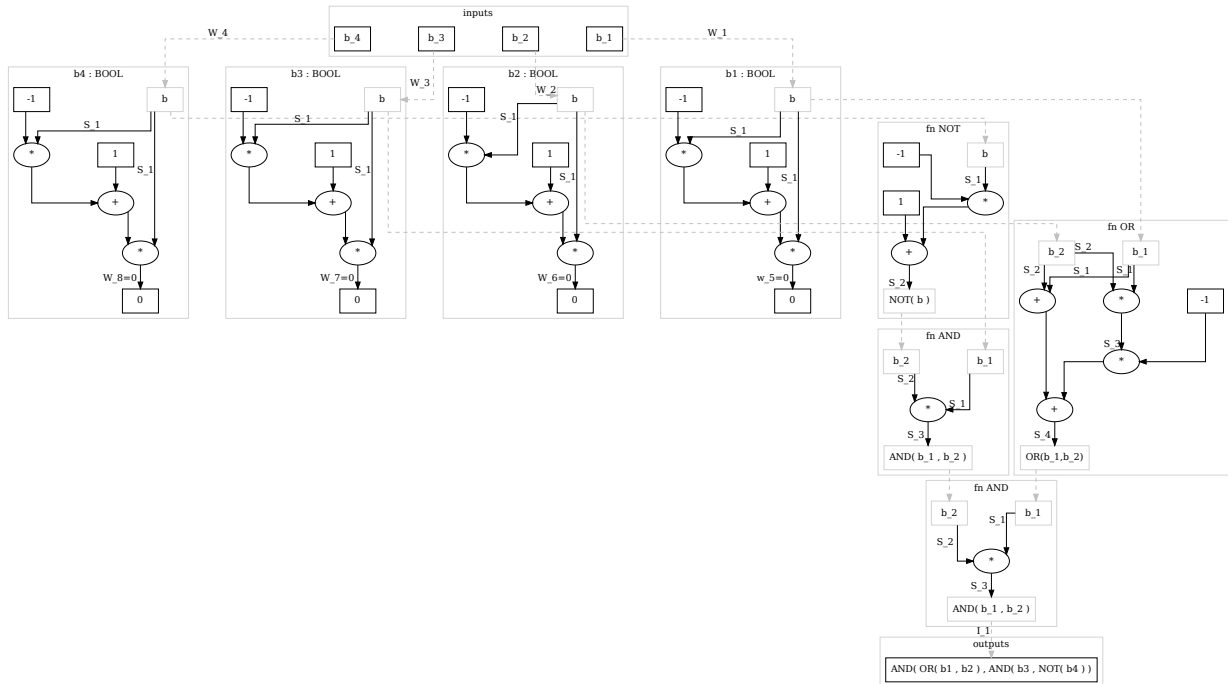
The code describes a circuit that takes four private inputs b_1, b_2, b_3 and b_4 of boolean type and computes a public output b_5 such that the following boolean expression holds true:

$$(b_1 \vee b_2) \wedge (b_3 \wedge \neg b_4) = b_5$$

During a setup-phase, a circuit compiler transforms this high-level language statement into a circuit and associated rank-1 constraint systems and hence defines a language $L_{BOOLEAN_STAT}$.

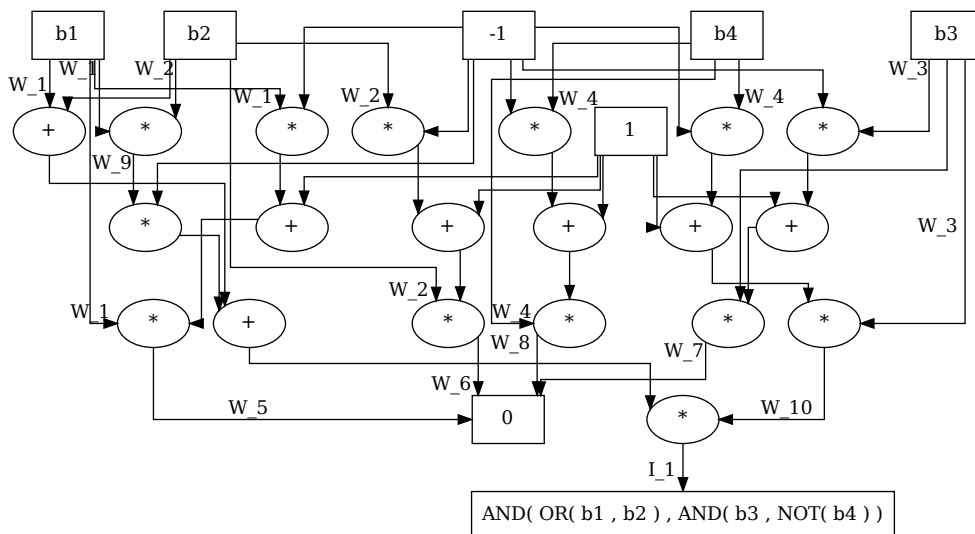
To see how this might be achieved, we use PAPER as an example to execute the setup-phase and compile `BOOLEAN_STAT` into a circuit. Taking the definition of the boolean constraint `XXX` as well as the definitions of the appropriate boolean operators into account, we get the following circuit:

add reference



Simple optimization then collapses all box-nodes that are directly linked and all box nodes that represent the same constants. After relabeling the edges, the following circuit represents the circuit associated to the `BOOLEAN_STAT` statement:

can we rotate this by 90°?



5379 Given some public input I_1 from \mathbb{F}_{13} , a valid assignment to this circuit consists of private inputs
 5380 W_1, W_2, W_3, W_4 from \mathbb{F}_{13} such that the equation $I_1 = (W_1 \vee W_2) \wedge (W_3 \wedge \neg W_4)$ holds true. In
 5381 addition, a valid assignment also has to contain private inputs W_5, W_6, W_7, W_8, W_9 and W_{10} ,
 5382 which can be derived from circuit execution. The inputs W_5, \dots, W_8 ensure that the first four
 5383 private inputs are either 0 or 1 but not any other field element, and the others enforce the boolean
 5384 operations in the expression.

To compute the associated R1CS, we can use the general method from 6.2.1.1 and look at every labeled outgoing edge not coming from a source node. We declare the edges coming from input nodes as well as the edge going to the single output node as public, and every other edge as private input. In this case we get:

check
reference

$$\begin{aligned}
 W_5 : W_1 \cdot (1 - W_1) &= 0 && \text{boolean constraints} \\
 W_6 : W_2 \cdot (1 - W_2) &= 0 \\
 W_7 : W_3 \cdot (1 - W_3) &= 0 \\
 W_8 : W_4 \cdot (1 - W_4) &= 0 \\
 W_9 : W_1 \cdot W_2 &= W_9 && \text{first OR-operator constraint} \\
 W_{10} : W_3 \cdot (1 - W_4) &= W_{10} && \text{AND(.,NOT(.))-operator constraints} \\
 I_1 : (W_1 + W_2 - W_9) \cdot W_{10} &= I_1 && \text{AND-operator constraints}
 \end{aligned}$$

5385 The reason why this R1CS only contains a single constraint for the multiplication gate in the
 5386 OR-circuit, while the general definition XXX requires two constraints, is that the second con-
 5387 straint in XXX only appears because the final addition gate is connected to an output node. In
 5388 this case, however, the final addition gate from the OR-circuit is enforced in the left factor of
 5389 the I_1 constraint. Something similar holds true for the negation circuit.

add refer-
ence

add refer-
ence

During a prover-phase, some public instance I_5 must be given. To compute a constructive proof for the statement of the associated languages with respect to instance I_5 , a prover has to find four boolean values W_1, W_2, W_3 and W_4 such that

$$(W_1 \vee W_2) \wedge (W_3 \wedge \neg W_4) = I_5$$

5390 holds true. In our case neither the circuit nor the PAPER statement specifies how to find those
 5391 values, and it is a problem that any prover has to solve outside of the circuit. This might or
 5392 might not be true for other problems, too. In any case, once the prover found those values, they
 5393 can execute the circuit to find a valid assignment.

To give a concrete example, let $I_1 = 1$ and assume $W_1 = 1, W_2 = 0, W_3 = 1$ and $W_4 = 0$. Since $(1 \vee 0) \wedge (1 \wedge \neg 0) = 1$, those values satisfy the problem and we can use them to execute the circuit. We get

$$\begin{aligned}
 W_5 &= W_1 \cdot (1 - W_1) = 0 \\
 W_6 &= W_2 \cdot (1 - W_2) = 0 \\
 W_7 &= W_3 \cdot (1 - W_3) = 0 \\
 W_8 &= W_4 \cdot (1 - W_4) = 0 \\
 W_9 &= W_1 \cdot W_2 = 0 \\
 W_{10} &= W_3 \cdot (1 - W_4) = 1 \\
 I_1 &= (W_1 + W_2 - W_9) \cdot W_{10} = 1
 \end{aligned}$$

5394 A constructive proof of knowledge of a witness, for instance, $I_1 = 1$, is therefore given by the
 5395 tuple $P = (W_5, W_6, W_7, W_8, W_9, W_{10}) = (0, 0, 0, 0, 0, 1)$.

Arrays

The `array` type represents a fixed-size collection of elements of equal type, each selectable by one or more indices that can be computed at run time during program execution.

Arrays are a classical type, implemented by many higher programming languages that compile to circuits or rank-1 constraint systems. However, most high-level circuit languages support **static** arrays, i.e., arrays whose length is known at compile time only.

The most common way to compile arrays to circuits is to transform any array of a given type τ and size N into N circuit variables of type τ . Arrays are therefore **syntactic sugar**, that is, parts of the formal language that makes the code easier for humans to read, which the compiler transforms into input nodes, much like any other variable. In PAPER, we define the following array type:

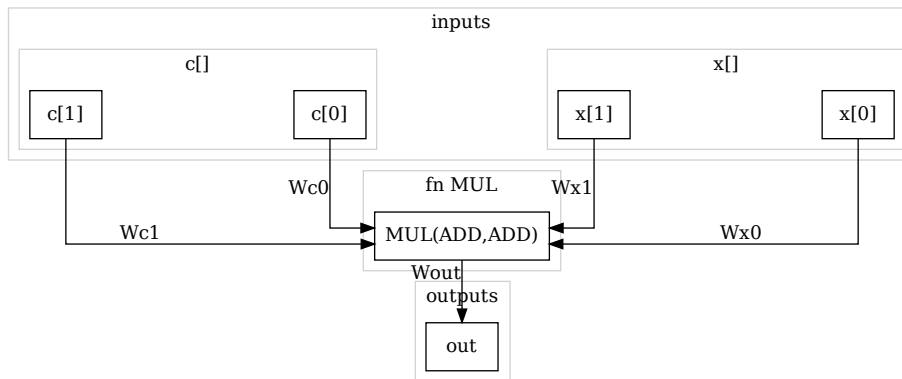
```
type <Name>: <Type>[N : unsigned] -> (Type,...) {
  return (<Name>[0],...)
}
```

In the setup phase of a statement, we compile every occurrence of an array of size N that contains elements of type `Type` into N variables of type `Type`.

Example 132. To give an intuition of how a real-world compiler might transform arrays into circuit variables, consider the following PAPER statement:

```
statement ARRAY_TYPE {F: F_5} {
  fn main(x: F[2]) -> F {
    let constant c: F[2] = [2,4] ;
    let out: F <== MUL(ADD(x[1], c[0]), ADD(x[0], c[1])) ;
    return out ;
  } ;
}
```

During a setup phase, a circuit compiler might then replace any occurrence of the array type by a tuple of variables of the underlying type, and then use those variables in the circuit synthesis process instead. To see how this can be achieved, we use PAPER as an example. Abstracting over the sub-circuit of the computation, we get the following circuit:



The Unsigned Integer Type

Unsigned integers of size N , where N is usually a power of two, represent non-negative integers in the range $0 \dots 2^N - 1$. They have a notion of addition, subtraction and multiplication, defined

5429 by modular 2^N arithmetics. If some N is given, we write uN for the associated type.

5430 **The uN Constraint System** Many high-level circuit languages define the the various uN types
 5431 as arrays of size N , where each element is of boolean type. This is similar to their representa-
 5432 tion on common computer hardware and allows for efficient and straightforward definition of
 5433 common operators, like the various **shift**, or logical operators.

shift

5434 If some unsigned integer N is known at compile time in PAPER, we define the following uN
 5435 type:

```
5436 type uN -> BOOL[N] {
5437   let base2 : BOOL[N] <== BASE_2 (uN) ;
5438   return base2 ;
5439 }
```

To enforce an N -tuple of field elements (b_0, \dots, b_{N-1}) to represent an element of type uN we therefore need N boolean constraints

$$\begin{aligned} S_0 \cdot (1 - S_0) &= 0 \\ S_1 \cdot (1 - S_1) &= 0 \\ &\dots \\ S_{N-1} \cdot (1 - S_{N-1}) &= 0 \end{aligned}$$

5440 In the setup phase of a statement, we compile every occurrence of the uN type by a size N array
 5441 of boolean type. During a=the prover phase, actual elements of the uN type are first transformed
 5442 into binary representation and then this binary representation is assigned to the boolean array
 5443 that represents the uN type.

5444 *Remark 6.* Representing the uN type as boolean arrays is conceptually clean and works over
 5445 generic base fields. However, representing unsigned integers in this way requires a lot of space
 5446 as every bit is represented as a field element and if the base field is large, those field elements
 5447 require considerable space in hardware.

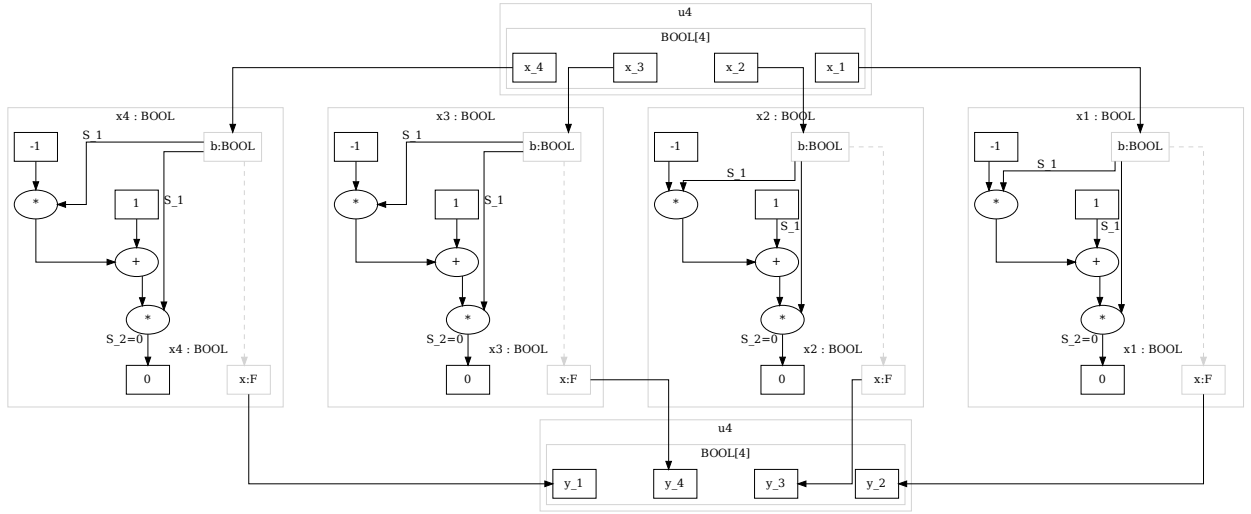
5448 It should be noted that, in some cases, there is another, more space- and constraint-efficient
 5449 approach for representing unsigned integers that can be used whenever the underlying base field
 5450 is sufficiently large. To understand this, recall that addition and multiplication in a prime field
 5451 \mathbb{F}_p is equal to addition and multiplication of integers, as long as the sum or the product does
 5452 not exceed the modulus p . It is therefore possible to represent the uN type inside the base-field
 5453 type whenever N is small enough. In this case, however, care has to be taken to never overflow
 5454 the modulus. It is also important to make sure that, in the case of subtraction, the subtrahend is
 5455 never larger than the minuend.

5456 *Example 133.* To give an intuition of how a real-world compiler might transform unsigned
 5457 integers into circuit variables, consider the following PAPER statement:

```
5458 statement RING_SHIFT{F: F_p, N=4} {
5459   fn main(x: uN) -> uN {
5460     let y:uN <== [x[1], x[2], x[3], x[0]] ;
5461     return y ;
5462   } ;
5463 }
```

5464 During the setup-phase, a circuit compiler might then replace any occurrence of the uN type
 5465 by N variables of `boolean` type. Using the definition of booleans, each of these variables is

then transformed into the `field` type and a boolean constraint system. To see how this can be achieved, we use PAPER as an example and get the following circuit:



During the prover phase, the function `main` is called with an actual input of `u4` type, say $x=14$. The high-level language then has to transform the decimal value 14 into its 4-bit binary representation $14_2 = (0, 1, 1, 1)$ outside of the circuit. Then the array of field values $x[4] = [0, 1, 1, 1]$ is used as an input to the circuit. Since all 4 field elements are either 0 or 1, the four boolean constraints are satisfiable and the output is an array of the four field elements $[1, 1, 1, 0]$, which represents the `u4` element 7.

The Unigned Integer Operators Since elements of `uN` type are represented as boolean arrays, shift operators are implemented in circuits simply by rewiring the boolean input variables to the output variables accordingly.

Logical operators, like AND, OR, or NOT are defined on the `uN` type by invoking the appropriate boolean operators bitwise to every bit in the boolean array that represents the `uN` element.

Addition and multiplication can be represented similarly to how machines represent those operations. Addition can be implemented by first defining the **full adder** circuit and then combining N of these circuits into a circuit that adds two elements from the `uN` type.

Exercise 60. Let $F = \mathbb{F}_{13}$ and $N=4$ be fixed. Define circuits and associated R1CS for the left and right **bishift** operators $x \ll 2$ as well as $x \gg 2$ that operate on the `uN` type. Execute the associated circuit for $x : u4 = 11$.

Exercise 61. Let $F = \mathbb{F}_{13}$ and $N=2$ be fixed. Define a circuit and associated R1CS for the addition operator $\text{ADD} : F \times F \rightarrow F$. Execute the associated circuit to compute $\text{ADD}(2, 7)$.

Exercise 62. Brain-compile the following PAPER code into a circuit and derive the associated R1CS.

```
statement MASK_MERGE {F:F_5, N=4} {
  fn main(pub a : uN, pub b : uN) -> F {
    let constant mask : uN = 10 ;
    let r : uN <== XOR(a, AND(XOR(a,b), mask)) ;
    return r ;
```

```
5497     }
5498 }
```

5499 Let L_{mask_merge} be the language defined by the circuit. Provide a constructive knowledge proof
5500 in L_{mask_merge} for the instance $I = (I_a, I_b) = (14, 7)$.

5501 7.2.2 Control Flow

5502 Most programming languages of the imperative or functional style have some notion of basic
5503 control structures to direct the order in which instructions are evaluated. Contemporary circuit
5504 compilers usually provide a single thread of execution and provide basic flow constructs that
5505 implement control flow in circuits.

5506 The Conditional Assignment

5507 Writing high-level code that compiles to circuits, it is often necessary to have a way for condi-
5508 tional assignment of values or computational output to variables.

5509 One way to realize this in many programming languages is in terms of the conditional
5510 ternary assignment operator $?$: that branches the control flow of a program according to some
5511 condition and then assigns the output of the computed branch to some variable.

```
5512 variable = condition ? value_if_true : value_if_false
```

5513 In this description, `condition` is a boolean expression and `value_if_true` as well as
5514 `value_if_false` are expressions that evaluate to the same type as `variable`.

5515 In programming languages like Rust, another way to write the conditional assignment oper-
5516 ator that is more familiar to many programmers is given by

```
5517 variable = if condition then {
5518     value_if_true
5519 } else {
5520     value_if_false
5521 }
```

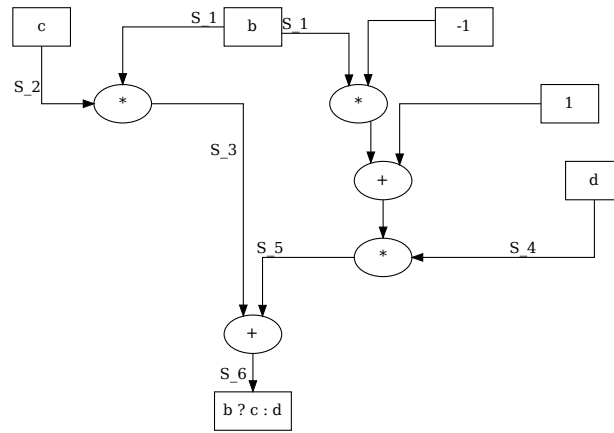
5522 In most programming languages, it is a key property of the ternary assignment operator that
5523 the expression `value_if_true` is only evaluated if `condition` evaluates to true and the
5524 expression `value_if_false` is only evaluated if `condition` evaluates to false. In fact,
5525 computer programs would turn out to be very inefficient if the ternary operator would evaluate
5526 both expressions regardless of the value of `condition`.

5527 A simple way to implement conditional assignment operator as a circuit can be achieved
5528 if the requirement that only one branch of the conditional operator is executed is dropped. To
5529 see that, let b , c and d be field elements such that b is a boolean constraint. In this case, the
5530 following equation enforces a field element x to be the result of the conditional assignment
5531 operator:

$$x = b \cdot c + (1 - b) \cdot d \quad (7.6)$$

5532 Expressing this equation in terms of the addition and multiplication operators from XXX, we
5533 can flatten it into the following algebraic circuit:

add refer-
ence



5534

5535 Note that, in order to compute a valid assignment to this circuit, both S_2 as well as S_4 are
 5536 necessary. If the inputs to the nodes c and d are circuits themselves, both circuits need valid
 5537 assignments and therefore have to be executed. As a consequence, this implementation of
 5538 the conditional assignment operator has to execute all branches of all circuits, which is very
 5539 different from the execution of common computer programs and contributes to the increased
 5540 computational effort any prover has to invest, in contrast to the execution in other programming
 5541 models.

We can use the general technique from 6.2.1.1 to derive the associated rank-1 constraint system of the conditional assignment operator. We get the following:

 check
reference

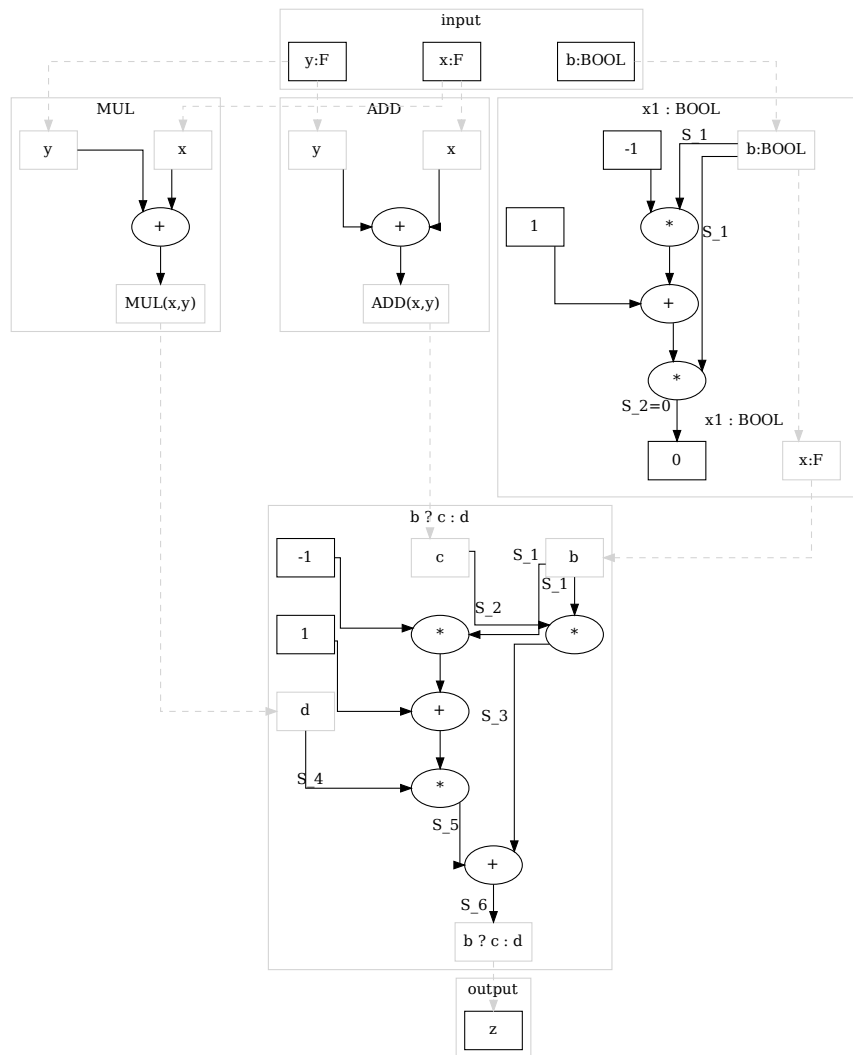
$$\begin{aligned}
 S_1 \cdot S_2 &= S_3 \\
 (1 - S_1) \cdot S_4 &= S_5 \\
 (S_3 + S_5) \cdot 1 &= S_6
 \end{aligned}$$

5542 *Example 134.* To give an intuition of how a real-world circuit compiler might transform any
 5543 high-level description of the conditional assignment operator into a circuit, consider the follow-
 5544 ing PAPER code:

```

5545 statement CONDITIONAL_OP {F:F_p} {
5546   fn main(x : F, y : F, b : BOOL) -> F {
5547     let z : F <== if b then {
5548       ADD(x,y)
5549     } else {
5550       MUL(x,y)
5551     } ;
5552     return z ;
5553   }
5554 }
```

5555 Brain-compiling this code into a circuit, we first draw box nodes for all input and output vari-
 5556 ables, and then transform the boolean type into the field type together with its associated con-
 5557 straint. Then we evaluate the assignments to the output variables. Since the conditional assign-
 5558 ment operator is the top level function, we draw its circuit and then draw the circuits for both
 5559 conditional expressions. We get the following:



5560

5561 **Loops**

5562 In many programming languages, various loop control structures are defined that allow devel-
 5563 opers to execute expressions with a specified number of repetitions or arguments. In particular,
 5564 it is often possible to implement unbounded loops like the loop structure give below, where the
 5565 number of executions depends on execution inputs and is therefore unknown at compile time:

```
5566 while true do { }
5567
```

5568 **M:** Add another example where the loop actually depends on the input.

Add ex-
ample

5569 In contrast to this, it should be noted that algebraic circuits and rank-1 constraint systems
 5570 are not general enough to express arbitrary computation, but bounded computation only. As a
 5571 consequence, it is not possible to implement unbounded loops, or loops with bounds that are
 5572 unknown at compile time in those models. This can be easily seen since circuits are acyclic
 5573 by definition, and implementing an unbounded loop as an acyclic graph requires a circuits of
 5574 unbounded size.

5575 However, circuits are general enough to express bounded loops, where the upper bound on
 5576 its execution is known at compile time. Those loop can be implemented in circuits by enrolling
 5577 the loop.

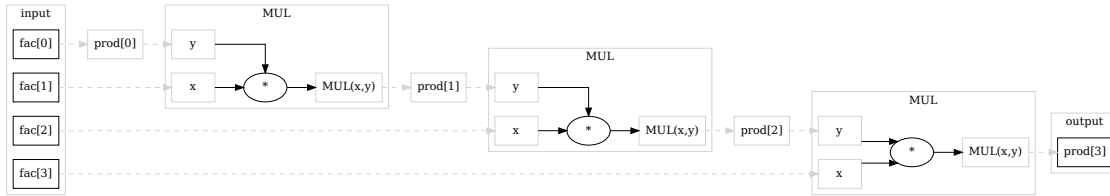
As a consequence, any programming language that compiles to algebraic circuits can only provide loop structures where the bound is a constant known at compile time. This implies that loops cannot depend on execution inputs, but on compile time parameters only.

Example 135. To give an intuition of how a real-world circuit compiler might transform any high-level description of a bounded `for` loop into a circuit, consider the following PAPER code:

```
statement FOR_LOOP {F:F_p, N: unsigned = 4} {
  fn main(fac : F[N]) -> F {
    let prod[N] : F ;
    prod[0] <== fac[0] ;
    for unsigned i in 1..N do [{
      prod[i] <== MUL(fac[i], prod[i-1]) ;
    }]
    return prod[N] ;
  }
}
```

Note that, in a program like this, the loop counter `i` has no expression in the derived circuit. It is pure syntactic sugar, telling the compiler how to unroll the loop.

Brain-compiling this code into a circuit, we first draw box nodes for all input and output variables, noting that the loop counter is not represented in the circuit. Since all variables are of `field` type, we don't have to compile any type constraints. Then we evaluate the assignments to the output variables by unrolling the loop into 3 individual assignment operators. We get:



7.2.3 Binary Field Representations

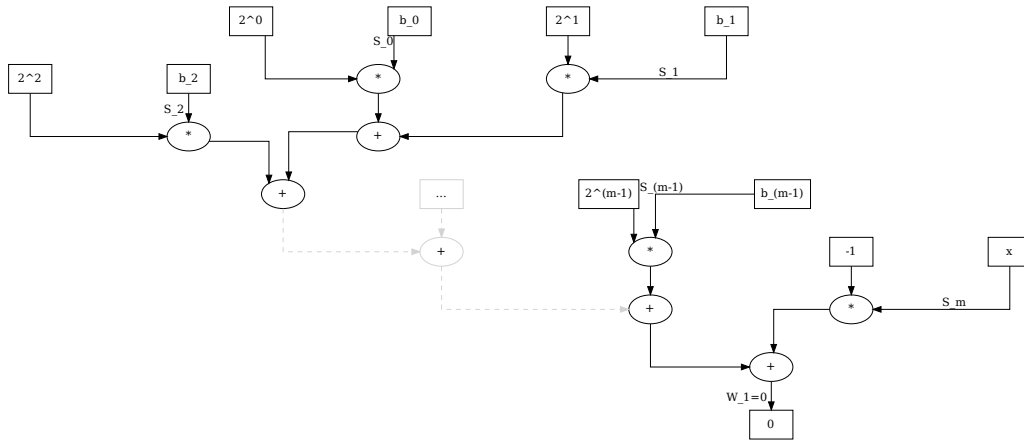
In applications, it is often necessary to enforce a binary representation of elements from the `field` type. To derive an appropriate circuit over a prime field \mathbb{F}_p , let $m = \lceil \log_2 p \rceil$ be the smallest number of bits necessary to represent the prime modulus p . Then a bitstring $(b_0, \dots, b_{m-1}) \in \{0, 1\}^m$ is a binary representation of a field element $x \in \mathbb{F}_p$, if and only if

$$x = b_0 \cdot 2^0 + b_1 \cdot 2^1 + \dots + b_{m-1} \cdot 2^{m-1}$$

In this expression, addition and exponentiation is considered to be executed in \mathbb{F}_p , which is well defined since all terms 2^j for $0 \leq j < m$ are elements of \mathbb{F}_p . Note, however, that in contrast to the binary representation of unsigned integers $n \in \mathbb{N}$, this representation is not unique in general, since the modular p equivalence class might contain more than one binary representative.

Considering that the underlying prime field is fixed and the most significant bit of the prime modulus is m , the following circuit flattens equation XXX, assuming all inputs b_1, \dots, b_m are of boolean type.

add reference



5609

Applying the general transformation rule to compute the associated rank-1 constraint systems, we see that we actually only need a single constraint to enforce some binary representation of any field element. We get

$$(S_0 \cdot 2^0 + S_1 \cdot 2^1 + \dots + S_{m-1} \cdot 2^{m-1} - S_m) \cdot 1 = 0$$

5610 Given an array `BOOL[N]` of N boolean constraint field elements and another field element x ,
 5611 the circuit enforces `BOOL[N]` to be one of the binary representations of x . If `BOOL[N]` is not
 5612 a binary representation of x , no valid assignment and hence no solution to the associated R1CS
 5613 can exist.

5614 *Example 136.* Consider the prime field \mathbb{F}_{13} . To compute binary representations of elements
 5615 from that field, we start with the binary representation of the prime modulus 13, which is $|13|_2 =$
 5616 $(1, 0, 1, 1)$ since $13 = 1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3$. So $m = 4$ and we need up to 4 bits to represent
 5617 any element $x \in \mathbb{F}_{13}$.

To see that binary representations are not unique in general, consider the element $2 \in \mathbb{F}_{13}$. It has the binary representations $|2|_2 = (0, 1, 0, 0)$ and $|2|_2 = (1, 1, 1, 1)$, since in \mathbb{F}_{13} we have

$$2 = \begin{cases} 0 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 0 \cdot 2^3 \\ 1 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3 \end{cases}$$

5618 This is because the unsigned integers 2 and 15 are both in the modular 13 remainder class of 2
 5619 and hence are both representatives of 2 in \mathbb{F}_{13} .

To see how circuit XXX works, we want to enforce the binary representation of $7 \in \mathbb{F}_{13}$. Since $m = 4$ we have to enforce a 4-bit representation for 7, which is $(1, 1, 1, 0)$, since $7 = 1 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2 + 0 \cdot 2^3$. A valid circuit assignment is therefore given by $(S_0, S_1, S_2, S_3, S_4) = (1, 1, 1, 0, 7)$ and, indeed, the assignment satisfies the required 5 constraints including the 4 boolean constraints for S_0, \dots, S_3 :

$$\begin{aligned} 1 \cdot (1 - 1) &= 0 & // \text{boolean constraints} \\ 1 \cdot (1 - 1) &= 0 \\ 1 \cdot (1 - 1) &= 0 \\ 0 \cdot (1 - 0) &= 0 \\ (1 + 2 + 4 + 0 - 7) \cdot 1 &= 0 & // \text{binary rep. constraint} \end{aligned}$$

 add refer-
ence

7.2.4 Cryptographic Primitives

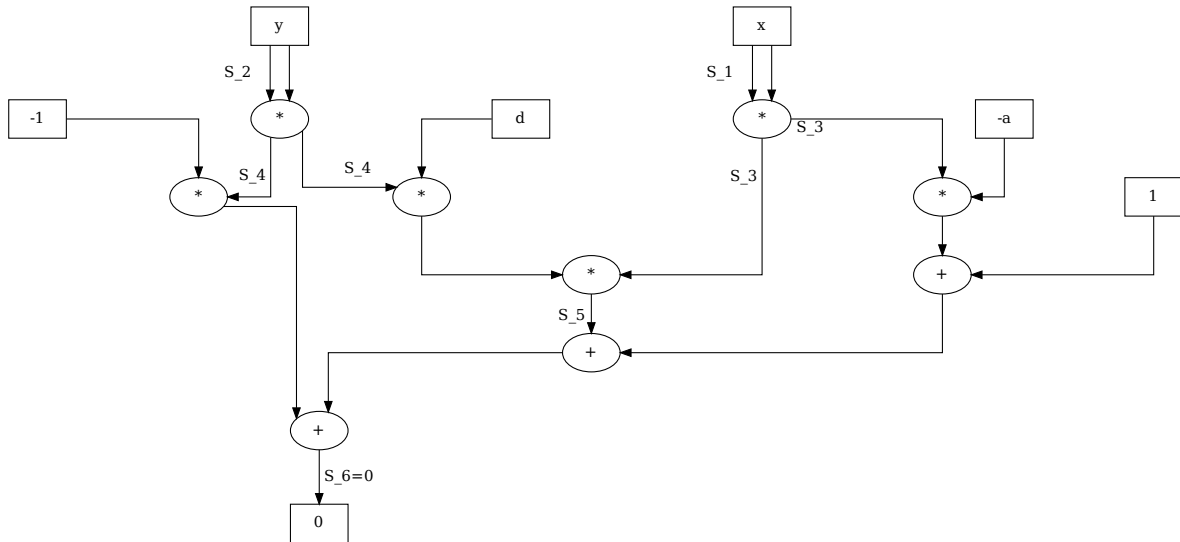
In applications, it is often required to do cryptography in a circuit. To do this, basic cryptographic primitives like hash functions or elliptic curve cryptography needs to be implemented as circuits. In this section, we give a few basic examples of how to implement such primitives.

Twisted Edwards curves

Implementing elliptic curve cryptography in circuits means to implement the field equation as well as the algebraic operations of an elliptic curve as circuits. To do this efficiently, the curve must be defined over the same base field as the field that is used in the circuit.

For efficiency reasons, it is advantageous to choose an elliptic curve such that that all required constraints and operations can be implement with as few gates as possible. Twisted Edwards curves are particularly useful for that matter, since their addition law is particularly simple and the same equation can be used for all curve points including the point at infinity. This simplifies the circuit a lot.

Twisted Edwards curve constraints As we have seen in section 5.1.3, a twisted Edwards curve over a finite field F is defined as the set of all pairs of points $(x, y) \in \mathbb{F} \times \mathbb{F}$ such that x and y satisfy the equation $a \cdot x^2 + y^2 = 1 + d \cdot x^2 y^2$. As we have seen in example XXX, we can transform this equation into the following circuit:



The circuit enforces the two inputs of `field` type to satisfy the twisted Edwards curve equation and, as we know from example XXX, the associated rank-1 constraint system is given by:

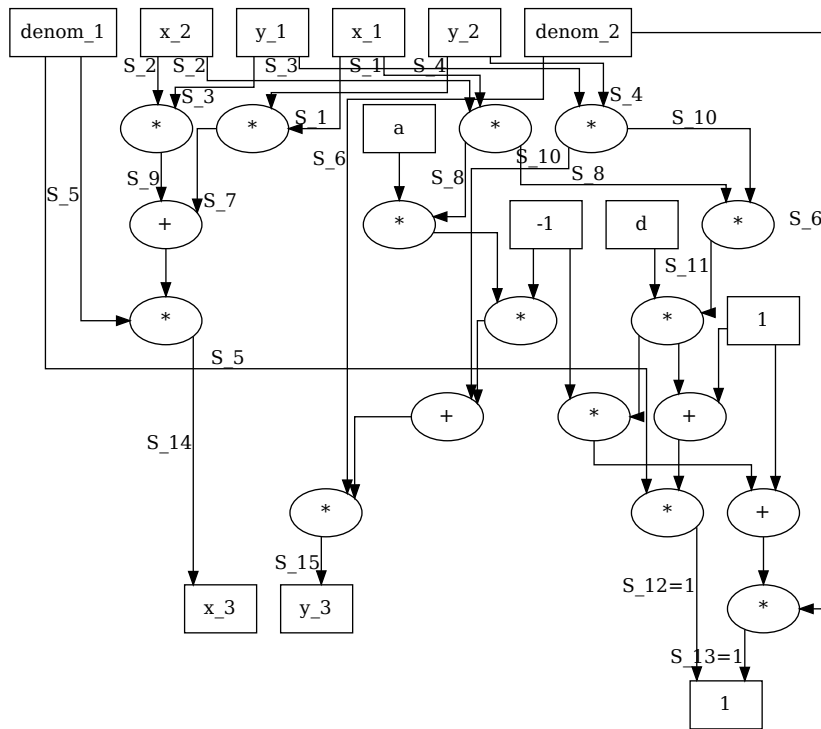
$$\begin{aligned}
 S_1 \cdot S_1 &= S_3 \\
 S_2 \cdot S_2 &= S_4 \\
 (S_4 \cdot 8) \cdot S_3 &= S_5 \\
 (12 \cdot S_4 + S_5 + 10 \cdot S_3 + 1) \cdot 1 &= 0
 \end{aligned}$$

Exercise 63. Write the circuit and associated rank-1 constraint system for a Weierstraß curve of a given field \mathbb{F} .

Twisted Edwards curve addition As we have seen in 5.1.3, a major advantage of twisted Edwards curves is the existence of an addition law that contains no branching and is valid for all curve points. Moreover, the neutral element is not “at infinity” but the actual curve point $(0, 1)$. In fact, given two points (x_1, y_1) and (x_2, y_2) on a twisted Edwards curve, their sum is defined as

$$(x_3, y_3) = \left(\frac{x_1 y_2 + y_1 x_2}{1 + d \cdot x_1 x_2 y_1 y_2}, \frac{y_1 y_2 - a \cdot x_1 x_2}{1 - d \cdot x_1 x_2 y_1 y_2} \right)$$

We can use the division circuit from XXX to flatten this equation into an algebraic circuit. Inputs to the circuit are then not only the two curve points (x_1, y_1) and (x_2, y_2) , but also the two denominators $denum_1 = 1 + d \cdot x_1 x_2 y_1 y_2$ as well as $denum_2 = 1 - d \cdot x_1 x_2 y_1 y_2$, which any prover needs to compute outside of the circuit. We get



Using the general technique from XXX to derive the associated rank-1 constraint system, we get the following result:

$$\begin{aligned} S_1 \cdot S_4 &= S_7 \\ S_1 \cdot S_2 &= S_8 \\ S_2 \cdot S_3 &= S_9 \\ S_3 \cdot S_4 &= S_{10} \\ S_8 \cdot S_{10} &= S_{11} \\ S_5 \cdot (1 + d \cdot S_{11}) &= 1 \\ S_6 \cdot (1 - d \cdot S_{11}) &= 1 \\ S_5 \cdot (S_9 + S_7) &= S_{14} \\ S_6 \cdot (S_{10} - a \cdot S_8) &= S_{15} \end{aligned}$$

5647 *Exercise 64.* Let \mathbb{F} be a field. Define a circuit that enforces field inversion for a point of a
5648 twisted Edwards curve over \mathbb{F} .

Chapter 8

Zero Knowledge Protocols

A so-called **zero-knowledge protocol** is a set of mathematical rules by which one party (usually called **the prover**) can convince another party (usually called **the verifier**) that a given statement is true, while not revealing any additional information apart from the truth of the statement.

As we have seen in chapter 6, given some language L and instance I , the knowledge claim “there is a witness W such that $(I; W)$ is a word in L ” is constructively provable by providing W to the verifier. However, the challenge for a zero-knowledge protocol is to prove knowledge of a witness without revealing any information beyond its bare existence.

In this chapter, we look at various systems that exist to solve this task. We start with an introduction to the basic concepts and terminology in zero-knowledge proving systems and then introduce the so-called Groth_16 protocol as one of the most efficient systems. We plan to update this chapter with new inventions in future versions of this book.

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8.1 Proof Systems

From an abstract point of view, a proof system is a set of rules which models the generation and exchange of messages between two parties: a prover and a verifier. Its task is to ascertain whether a given string belongs to a formal language or not.

Proof systems are often classified by certain trust assumptions and the computational capabilities of both parties. In its most general form, the prover usually possesses unlimited computational resources but cannot be trusted, while the verifier has bounded computation power but is assumed to be honest.

Proving the membership statement for some string is executed by the generation of certain messages that are sent between prover and verifier, until the verifier is convinced that the string is an element of the language in consideration.

To be more specific, let Σ be an alphabet, and let L be a formal language defined over Σ . Then a **proof system** for language L is a pair of probabilistic interactive algorithms (P, V) , where P is called the **prover** and V is called the **verifier**.

Both algorithms are able to send messages to one another, and each algorithm has its own state, some shared initial state and access to the messages. The verifier is bounded to a number of steps which is polynomial in the size of the shared initial state, after which it stops and outputs either `accept` or `reject` indicating that it accepts or rejects a given string to be in L . In contrast, there are bounds on the computational power of the prover.

When the execution of the verifier algorithm stops the following conditions are required to hold:

- (Completeness) If the tuple $x \in \Sigma^*$ is a word in language L and both prover and verifier follow the protocol, the verifier outputs `accept`.

- (Soundness) If the tuple $x \in \Sigma^*$ is not a word in language L and the verifier follows the protocol, the verifier outputs `reject`, except with some small probability.

In addition, a proof system is called **zero-knowledge** if the verifier learns nothing about x other than $x \in L$.

The previous definition of proof systems is very general, and many subclasses of proving systems are known in the field. The type of languages that a proof system can support crucially depends on the abilities of the verifier (for example, whether it can make random choices) or on the nature and number of the messages that can be exchanged. If the system only requires to send a single message from the prover to the verifier, the proof system is called **non-interactive**, because no interaction other than sending the actual proof is required. In contrast, any other proof system is called **interactive**.

A proof system is usually called **succinct** if the size of the proof is shorter than the witness necessary to generate the proof. Moreover, a proof system is called **computationally sound** if soundness only holds under the assumption that the computational capabilities of the prover are polynomially bound. To distinguish general proofs from computationally sound proofs, the latter are often called **arguments**. Zero-knowledge, succinct, non-interactive arguments of knowledge claims are often abbreviated **zk-SNARKs**.

Example 137 (Constructive Proofs for Algebraic Circuits). To formalize our previous notion of constructive proofs for algebraic circuits, let \mathbb{F} be a finite field, and let $C(\mathbb{F})$ be an algebraic circuit over \mathbb{F} with associated language $L_{C(\mathbb{F})}$. A non-interactive proof system for $L_{C(\mathbb{F})}$ is given by the following two algorithms:

Given some instance I , the prover algorithm P uses its unlimited computational power to compute a witness W such that the pair $(I; W)$ is a valid assignment to $C(\mathbb{F})$ whenever the circuit is satisfiable for I . The prover then sends the constructive proof $(I; W)$ to the verifier.

On receiving a message $(I; W)$, the verifier algorithm V assigns the constructive proof $(I; W)$ to circuit $C(\mathbb{F})$, and decides whether the assignment is valid by executing all gates in the circuit. The runtime is polynomial in the number of gates. If the assignment is valid, the verifier returns `accepts`, if not, it returns `reject`.

To see that this proof system has the completeness and soundness properties, let $C(\mathbb{F})$ be a circuit of the field \mathbb{F} , and let I be an instance. The circuit may or may not have a witness W such that $(I; W)$ is a valid assignment to $C(\mathbb{F})$.

If no W exists, I is not part of any word in $L_{C(\mathbb{F})}$, and there is no way for P to generate a valid assignment. It follows that the verifier will not accept any claimed proof sent by P , which implies that the system has **soundness**.

If, on the other hand, W exists and P is honest, P can use its unlimited computational power to compute W and send $(I; W)$ to V , which V will accept in polynomial time. This implies that the system has **completeness**.

The system is non-interactive because the prover only sends a single message to the verifier, which contains the proof itself. Because in this simple system the witness itself is the proof, the proof system is **not** succinct.

8.2 The “Groth16” Protocol

In chapter 6, we have introduced algebraic circuits, their associated rank-1 constraint systems and their induced quadratic arithmetic programs. These models define formal languages, and

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associated memberships and knowledge claims can be constructively proofed by executing the circuit to compute a solution to its associated R1CS. The solution can then be transformed into a polynomial such that the polynomial is divisible by another polynomial if and only if the solution is correct.

In [\[1\]](#), Jens Groth provides a method that can transform those proofs into zero-knowledge succinct non-interactive arguments of knowledge. Assuming that the pairing groups $(\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T, b)$ are given, the arguments are of constant size and consist of 2 elements from \mathbb{G}_1 and a single element from \mathbb{G}_2 , regardless of the size of the witness. They are zero-knowledge in the sense that the verifier learns nothing about the witness beside the fact that the instance-witness pair is a proper word in the language of the problem.

Verification is non-interactive and needs to compute a number of exponentiations proportional to the size of the instance, together with 3 group pairings in order to check a single equation.

The generated argument is perfectly zero-knowledge has perfect completeness and soundness in the generic bilinear group model, assuming the existence of a trusted third party that executes a preprocessing phase to generate a **common reference string** and a **simulation trapdoor**. This party must be trusted to delete the simulation trapdoor, since everyone in possession of it can simulate proofs.

To be more precise, let R be a rank-1 constraint system defined over some finite field \mathbb{F}_r . Then **Groth_16 parameters** for R are given by the following set:

$$\text{Groth_16} - \text{Param}(R) = (r, \mathbb{G}_1, \mathbb{G}_2, e(\cdot, \cdot), g_1, g_2) \quad (8.1)$$

In the equation above, \mathbb{G}_1 and \mathbb{G}_2 are finite cyclic groups of order r , g_1 is a generator of \mathbb{G}_1 , g_2 is a generator of \mathbb{G}_2 and $e : \mathbb{G}_1 \times \mathbb{G}_2 \rightarrow \mathbb{G}_T$ is a non-degenerate, bilinear pairing for some target group \mathbb{G}_T . In real-world applications, the parameter set is usually agreed on in advance.

Given some Groth_16 parameters, a **Groth_16 protocol** is then a quadruple of probabilistic polynomial algorithms (SETUP, PROVE, VFY, SIM) such that the following conditions hold:

- (Setup-Phase): $(CRS, \tau) \leftarrow \text{Setup}(R)$: Algorithm Setup takes the R1CS R as input and computes a common reference string CRS and a simulation trapdoor τ .
- (Prover-Phase): $\pi \leftarrow \text{Prove}(R, CRS, I, W)$: Given a constructive proof (I, W) for R , algorithm Prove takes the R1CS R , the common reference string CRS and the constructive proof (I, W) as input and computes an zk-SNARK π .
- Verify: $\{\text{accept}, \text{reject}\} \leftarrow \text{Vfy}(R, CRS, I, \pi)$: Algorithm Vfy takes the R1CS R , the common reference string CRS , the instance I and the zk-SNARK π as input and returns `reject` or `accept`.
- $\pi \leftarrow \text{Sim}(R, \tau, CRS, I)$: Algorithm Sim takes the R1CS R , the common reference string CRS , the simulation trapdoor τ and the instance I as input and returns a zk-SNARK π .

We will explain these algorithms together with detailed examples in the remainder of this section.

Assuming a trusted third party for the setup, the protocol is able to compute a zk-SNARK from a constructive proof for R , provided that r is sufficiently large, and, in particular, larger than the number of constraints in the associated R1CS.

Example 138 (The 3-Factorization Problem). Consider the 3-factorization problem from 108 and its associated algebraic circuit and rank-1 constraint system from 6.8. In this example,

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we want to agree on a parameter set $(R, r, \mathbb{G}_1, \mathbb{G}_2, e(\cdot, \cdot), g_1, g_2)$ in order to use the Groth_16 protocol for our 3-factorization problem.

To find proper parameters, first observe that the circuit XXX, as well as its associated R1CS $R_{3, \text{fac_zk}}$ ex:3-factorization-r1cs and the derived QAP 6.14, are defined over the field \mathbb{F}_{13} . We therefore have $r = 13$ and need pairing groups \mathbb{G}_1 and \mathbb{G}_2 of order 13.

We know from 5.4 that the moon-math curve BLS6_6 has two subgroups $\mathbb{G}_1[13]$ and $\mathbb{G}_2[13]$, which are both of order 13. The associated Weil pairing b (5.45) is a proper bilinear map. We therefore choose those groups and the Weil pairing together with the generators $g_1 = (13, 15)$ and $g_2 = (7v^2, 16v^3)$ of $\mathbb{G}_1[13]$ and $\mathbb{G}_2[13]$, as a parameter:

$$\text{Groth_16} - \text{Param}(R_{3, \text{fac_zk}}) = (r, \mathbb{G}_1[13], \mathbb{G}_2[13], e(\cdot, \cdot), (13, 15), (7v^2, 16v^3))$$

It should be noted that our choice is not unique. Every pair of finite cyclic groups of order 13 that has a proper bilinear pairing qualifies as a Groth_16 parameter set. The situation is similar to real-world applications, where SNARKs with equivalent behavior are defined over different curves, used in different applications.

The Setup Phase To generate zk-SNARKs from constructive knowledge proofs in the Groth16 protocol, a preprocessing phase is required. This has to be executed a single time for every rank-1 constraint system and any associated quadratic arithmetic program. The outcome of this phase is a common reference string that prover and verifier need in order to generate and verify the zk-SNARK. In addition, a simulation trapdoor is produced that can be used to simulate proofs.

To be more precise, let L be a language defined by some rank-1 constraint system R such that a constructive proof of knowledge for an instance (I_1, \dots, I_n) in L consists of a witness (W_1, \dots, W_m) . Let $QAP(R) = \{T \in \mathbb{F}[x], \{A_j, B_j, C_j \in \mathbb{F}[x]\}_{j=0}^{n+m}\}$ be a quadratic arithmetic program associated to R , and let $\{\mathbb{G}_1, \mathbb{G}_2, e(\cdot, \cdot), g_1, g_2, \mathbb{F}_r\}$ be the set of Groth_16 parameters.

The setup phase then samples 5 random, **inverible** elements $\alpha, \beta, \gamma, \delta$ and s from the scalar field \mathbb{F}_r of the protocol and outputs the **simulation trapdoor** τ :

$$\tau = (\alpha, \beta, \gamma, \delta, s) \quad (8.2)$$

In addition, the setup phase uses those 5 random elements together with the two generators g_1 and g_2 and the quadratic arithmetic program to generate a **common reference string** $CRS_{QAP} = (CRS_{\mathbb{G}_1}, CRS_{\mathbb{G}_2})$ of language L :

$$CRS_{\mathbb{G}_1} = \left\{ \begin{array}{l} g_1^\alpha, g_1^\beta, g_1^\delta, \left(g_1^{s^j}, \dots \right)_{j=0}^{deg(T)-1}, \left(g_1^{\frac{\beta \cdot A_j(s) + \alpha \cdot B_j(s) + C_j(s)}{\gamma}}, \dots \right)_{j=0}^n \\ \left(g_1^{\frac{\beta \cdot A_{j+n}(s) + \alpha \cdot B_{j+n}(s) + C_{j+n}(s)}{\delta}}, \dots \right)_{j=1}^m, \left(g_1^{\frac{s^j \cdot T(s)}{\delta}}, \dots \right)_{j=0}^{deg(T)-2} \end{array} \right\}$$

$$CRS_{\mathbb{G}_2} = \left\{ g_2^\beta, g_2^\gamma, g_2^\delta, \left(g_2^{s^j}, \dots \right)_{j=0}^{deg(T)-1} \right\}$$

Common reference strings depend on the simulation trapdoor, and are therefore not unique to the problem. Any language can have more than one common reference string. The size of a common reference string is linear in the size of the instance and the size of the witness.

If a simulation trapdoor $\tau = (\alpha, \beta, \gamma, \delta, s)$ is given, we call the element s a **secret evaluation point** of the protocol, because if \mathbb{F}_r is the scalar field of the finite cyclic groups \mathbb{G}_1 and \mathbb{G}_2 , then

5795 a key feature of any common reference string is that it provides data to compute the evaluation
 5796 of any polynomial $P \in \mathbb{F}_r[x]$ of degree $\deg(P) < \deg(T)$ at the point s in the exponent of the
 5797 generator g_1 or g_2 , without knowing s .

To be more precise, let s be the secret evaluation point and let $P(x) = a_0 \cdot x^0 + a_1 \cdot x^1 + \dots a_k \cdot x^k$ be a polynomial of degree $k < \deg(T)$ with coefficients in \mathbb{F}_r . Then we can compute $g_1^{P(s)}$ without knowing what the actual value of s is:

$$\begin{aligned} g_1^{P(s)} &= g_1^{a_0 \cdot s^0 + a_1 \cdot s^1 + \dots a_k \cdot s^k} \\ &= g_1^{a_0 \cdot s^0} \cdot g_1^{a_1 \cdot s^1} \cdot \dots \cdot g_1^{a_k \cdot s^k} \\ &= \left(g_1^{s^0}\right)^{a_0} \cdot \left(g_1^{s^1}\right)^{a_1} \cdot \dots \cdot \left(g_1^{s^k}\right)^{a_k} \end{aligned}$$

5798 In this expression, all group points $g_1^{s^j}$ are part of the common reference string, hence, they can
 5799 be used to compute the result. The same holds true for the evaluation of $g_2^{P(s)}$, since the \mathbb{G}_2 part
 5800 of the common reference string contains the points $g_2^{s^j}$.

5801 In real-world applications, the simulation trapdoor is often called the **toxic waste** of the
 5802 setup-phase, while a common reference string is also-called the pair of **prover and verifier**
 5803 **key**.

explain why

5804 In order to make the protocol secure, the setup needs to be executed in a way that guarantees
 5805 that the simulation trapdoor is deleted. Anyone in possession of it can generate arguments
 5806 without knowledge of a constructive proof. The most simple approach to achieving deletion of
 5807 the toxic waste is by a so-called **trusted third party**, where the trust assumption is that that
 5808 the party generates the common reference string precisely as defined and deletes the simulation
 5809 backdoor afterwards.

5810 However, as trusted third parties are not easy to find, more sophisticated protocols exist
 5811 in real-world applications. They execute the setup phase as a multi party computation, where
 5812 the proper execution can be publicly verified and the simulation trapdoor is deleted if at least
 5813 one participant deletes their individual contribution to the randomness. Each participant only
 5814 possesses a fraction of the simulation trapdoor, so it can only be recovered if all participants
 5815 collude and share their fraction.

Example 139 (The 3-factorization Problem). To see how the setup phase of a Groth_16 zk-SNARK can be computed, consider the 3-factorization problem from 108 and the parameters from page 193. As we have seen in 6.14, an associated quadratic arithmetic program is given as follows:

$$\begin{aligned} QAP(R_{3, fac_zk}) &= \{x^2 + x + 9, \\ &\quad \{0, 0, 6x + 10, 0, 0, 7x + 4\}, \{0, 0, 0, 6x + 10, 7x + 4, 0\}, \{0, 7x + 4, 0, 0, 0, 6x + 10\}\} \end{aligned}$$

To transform this QAP into a common reference string, we choose the field elements $\alpha = 6$, $\beta = 5$, $\gamma = 4$, $\delta = 3$, $s = 2$ from \mathbb{F}_{13} . In real-world applications, it is important to sample those values randomly from the scalar field, but in our approach, we choose those non-random values to make them more memorizable, which helps in pen-and-paper computations. Our simulation trapdoor is then given as follows:

$$\tau = (6, 5, 4, 3, 2)$$

5816 We keep this secret in order to simulate proofs later on, but we are careful though to hide τ
 5817 from anyone who hasn't read this book. Then we instantiate the common reference string XXX

4 examples have the same title. Change it to be distinct

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5818 from those values. Since our groups are subgroups of the BLS_{6_6} elliptic curve, we use scalar
 5819 product notation instead of exponentiation.

To compute the \mathbb{G}_1 part of the common reference string, we use the logarithmic order of the group \mathbb{G}_1 XXX, the generator $g_1 = (13, 15)$, as well as the values from the simulation backdoor. Since $\deg(T) = 2$, we get the following:

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$$\begin{aligned} [\alpha]_{g_1} &= [6](13, 15) = (27, 34) \\ [\beta]_{g_1} &= [5](13, 15) = (26, 34) \\ [\delta]_{g_1} &= [3](13, 15) = (38, 15) \end{aligned}$$

To compute the rest of the \mathbb{G}_1 part of the common reference string, we expand the indexed tuples and insert the secret random elements from the simulation backdoor. We get the following:

$$\begin{aligned} \left([s^j]_{g_1}, \dots \right)_{j=0}^1 &= \left([2^0](13, 15), [2^1](13, 15) \right) \\ &= \left((13, 15), (33, 34) \right) \\ \left(\left[\frac{\beta A_j(s) + \alpha B_j(s) + C_j(s)}{\gamma} \right]_{g_1}, \dots \right)_{j=0}^1 &= \left(\left[\frac{5A_0(2) + 6B_0(2) + C_0(2)}{4} \right](13, 15), \right. \\ &\quad \left. \left[\frac{5A_1(2) + 6B_1(2) + C_1(2)}{4} \right](13, 15) \right) \\ \left(\left[\frac{\beta A_{j+n}(s) + \alpha B_{j+n}(s) + C_{j+n}(s)}{\delta} \right]_{g_1}, \dots \right)_{j=1}^4 &= \left(\left[\frac{5A_2(2) + 6B_2(2) + C_2(2)}{3} \right](13, 15), \right. \\ &\quad \left[\frac{5A_3(2) + 6B_3(2) + C_3(2)}{3} \right](13, 15), \\ &\quad \left[\frac{5A_4(2) + 6B_4(2) + C_4(2)}{3} \right](13, 15), \\ &\quad \left. \left[\frac{5A_5(2) + 6B_5(2) + C_6(2)}{3} \right](13, 15) \right) \\ \left(\left[\frac{s^j \cdot T(s)}{\delta} \right]_{g_1} \right)_{j=0}^0 &= \left(\left[\frac{2^0 \cdot T(2)}{3} \right](13, 15) \right) \end{aligned}$$

To compute the curve points on the right side of these expressions, we need the polynomials from the associated quadratic arithmetic program and evaluate them on the secret point $s = 2$.

Since $4^{-1} = 10$ and $3^{-1} = 9$ in \mathbb{F}_{13} , we get the following:

$$\begin{aligned}
 \left[\frac{5A_0(2) + 6B_0(2) + C_0(2)}{4} \right](13, 15) &= [(5 \cdot 0 + 6 \cdot 0 + 0) \cdot 10](13, 15) = [0](13, 15) = \\
 &\quad \mathcal{O} \\
 \left[\frac{5A_1(2) + 6B_1(2) + C_1(2)}{4} \right](13, 15) &= [(5 \cdot 0 + 6 \cdot 0 + (7 \cdot 2 + 4)) \cdot 10](13, 15) = [11](13, 15) = \\
 &\quad (33, 9) \\
 \left[\frac{5A_2(2) + 6B_2(2) + C_2(2)}{3} \right](13, 15) &= [(5 \cdot (6 \cdot 2 + 10) + 6 \cdot 0 + 0) \cdot 9](13, 15) = [2](13, 15) = \\
 &\quad (33, 34) \\
 \left[\frac{5A_3(2) + 6B_3(2) + C_3(2)}{3} \right](13, 15) &= [(5 \cdot 0 + 6 \cdot (6 \cdot 2 + 10) + 0) \cdot 9](13, 15) = [5](13, 15) = \\
 &\quad (26, 34) \\
 \left[\frac{5A_4(2) + 6B_4(2) + C_4(2)}{3} \right](13, 15) &= [(5 \cdot 0 + 6 \cdot (7 \cdot 2 + 4) + 0) \cdot 9](13, 15) = [10](13, 15) = \\
 &\quad (38, 28) \\
 \left[\frac{5A_5(2) + 6B_5(2) + C_5(2)}{3} \right](13, 15) &= [(5 \cdot (7 \cdot 2 + 4) + 6 \cdot 0 + 0) \cdot 9](13, 15) = [4](13, 15) = \\
 &\quad (35, 28) \\
 \left[\frac{2^0 \cdot T(2)}{3} \right](13, 15) &= [1 \cdot (2^2 + 2 + 9) \cdot 9](13, 15) = [5](13, 15) = \\
 &\quad (26, 34)
 \end{aligned}$$

Putting all those values together, we see that the \mathbb{G}_1 part of the common reference string is given by the following set of 12 points from the BLS6_6 13-torsion group \mathbb{G}_1 :

$$\text{CRS}_{\mathbb{G}_1} = \left\{ \begin{array}{l} (27, 34), (26, 34), (38, 15), \left((13, 15), (33, 34) \right), \left(\mathcal{O}, (33, 9) \right) \\ \left((33, 34), (26, 34), (38, 28), (35, 28) \right), \left((26, 34) \right) \end{array} \right\}$$

To compute the \mathbb{G}_2 part of the common reference string, we use the logarithmic order of the group \mathbb{G}_2 XXX, the generator $g_2 = (7v^2, 16v^3)$, as well as the values from the simulation backdoor. Since $\deg(T) = 2$, we get the following:

$$\begin{aligned}
 [\beta]g_2 &= [5](7v^2, 16v^3) = (16v^2, 28v^3) \\
 [\gamma]g_2 &= [4](7v^2, 16v^3) = (37v^2, 27v^3) \\
 [\delta]g_2 &= [3](7v^2, 16v^3) = (42v^2, 16v^3)
 \end{aligned}$$

To compute the rest of the \mathbb{G}_2 part of the common reference string, we expand the indexed tuple and insert the secret random elements from the simulation backdoor. We get the following:

$$\begin{aligned}
 \left([s^j]g_2, \dots \right)_{j=0}^1 &= ([2^0](7v^2, 16v^3), [2^1](7v^2, 16v^3)) \\
 &= ((7v^2, 16v^3), (10v^2, 28v^3))
 \end{aligned}$$

Putting all these values together, we see that the \mathbb{G}_2 part of the common reference string is given by the following set of 5 points from the BLS6_6 13-torsion group \mathbb{G}_2 :

$$\text{CRS}_{\mathbb{G}_2} = \left\{ (16v^2, 28v^3), (37v^2, 27v^3), (42v^2, 16v^3), (7v^2, 16v^3), (10v^2, 28v^3) \right\}$$

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Given the simulation trapdoor τ and the quadratic arithmetic program 6.14, the associated common reference string of the 3-factorization problem is as follows:

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$$\begin{aligned} CRS_{\mathbb{G}_1} &= \left\{ (27, 34), (26, 34), (38, 15), \left((13, 15), (33, 34) \right), \left(\mathcal{O}, (33, 9) \right) \right\} \\ &\quad \left\{ \left((33, 34), (26, 34), (38, 28), (35, 28) \right), \left((26, 34) \right) \right\} \\ CRS_{\mathbb{G}_2} &= \left\{ (16v^2, 28v^3), (37v^2, 27v^3), (42v^2, 16v^3), \left(7v^2, 16v^3 \right), (10v^2, 28v^3) \right\} \end{aligned}$$

5820 We then publish this data to everyone who wants to participate in the generation of a zk-SNARK
5821 or its verification in the 3-factorization problem.

5822 To understand how this common reference string can be used to evaluate polynomials at
5823 the secret evaluation point in the exponent of a generator, let’s assume that we have deleted the
5824 simulation trapdoor. In that case, we have no way to know the secret evaluation point anymore,
5825 hence, we cannot evaluate polynomials at that point. However, we can evaluate polynomials of
5826 smaller degree than the degree of the target polynomial in the exponent of both generators at
5827 that point.

To see that, consider e.g. the polynomials $A_2(x) = 6x + 10$ and $A_5(x) = 7x + 4$ from the QAP of this problem. To evaluate these polynomials in the exponent of g_1 and g_2 at the secret point s without knowing the value of s (which is 2), we can use the common reference string and equation XXX. Using the scalar product notation instead of exponentiation, we get the following:

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ence

$$\begin{aligned} [A_2(s)]g_1 &= [6 \cdot s^1 + 10 \cdot s^0]g_1 \\ &= [6](33, 34) + [10](13, 15) & \# [s^0]g_1 = (13, 15), [s^1]g_1 = (33, 34) \\ &= [6 \cdot 2](13, 15) + [10](13, 15) = [9](13, 15) & \# \text{logarithmic order on } \mathbb{G}_1 \\ &= (35, 15) \\ [A_5(s)]g_1 &= [7 \cdot s^1 + 4 \cdot s^0]g_1 \\ &= [7](33, 34) + [4](13, 15) \\ &= [7 \cdot 2](13, 15) + [4](13, 15) = [5](13, 15) \\ &= (26, 34) \end{aligned}$$

Indeed, we are able to evaluate the polynomials in the exponent at a secret evaluation point, because that point is encrypted in the curve point $(33, 34)$ and its secrecy is protected by the discrete logarithm assumption. Of course, in our computation, we recovered the secret point $s = 2$, but that was only possible because we have a group of logarithmic order in order to simplify our pen-and-paper computations. Such an order is infeasible for computing in cryptographically secure curves. We can do the same computation on \mathbb{G}_2 and get the following:

$$\begin{aligned} [A_2(s)]g_2 &= [6 \cdot s^1 + 10 \cdot s^0]g_2 \\ &= [6](10v^2, 28v^3) + [10](7v^2, 16v^3) \\ &= [6 \cdot 2](7v^2, 16v^3) + [10](7v^2, 16v^3) = [9](7v^2, 16v^3) \\ &= (37v^2, 16v^3) \\ [A_5(s)]g_2 &= [7 \cdot s^1 + 4 \cdot s^0]g_2 \\ &= [7](10v^2, 28v^3) + [4](7v^2, 16v^3) \\ &= [7 \cdot 2](7v^2, 16v^3) + [4](7v^2, 16v^3) = [5](7v^2, 16v^3) \\ &= (16v^2, 28v^3) \end{aligned}$$

5828 Apart from the target polynomial T , all other polynomials of the quadratic arithmetic pro-
 5829 gram can be evaluated in the exponent this way.

5830 **The Prover Phase** Given some rank-1 constraint system R and instance $I = (I_1, \dots, I_n)$, the
 5831 task of the prover phase is to convince any verifier that a prover knows a witness W to instance
 5832 I such that $(I; W)$ is a word in the language L_R of the system, without revealing anything about
 5833 W .

5834 To achieve this in the Groth_16 protocol, we assume that any prover has access to the rank-
 5835 1 constraint system of the problem, in addition to some algorithm that tells the prover how
 5836 to compute constructive proofs for the R1CS. In addition, the prover has access to a common
 5837 reference string and its associated quadratic arithmetic program.

5838 In order to generate a zk-SNARK for this instance, the prover first computes a valid con-
 5839 structive proof as explained in XXX, that is, the prover generates a proper witness $W = (W_1, \dots, W_m)$
 5840 such that $(I_1, \dots, I_n; W_1, \dots, W_m)$ is a solution to the rank-1 constraint system R . add refer-
ence

5841 The prover then uses the quadratic arithmetic program and computes the polynomial $P_{(I;W)}$,
 5842 as explained in 6.15. They then divide $P_{(I;W)}$ by the target polynomial T of the quadratic arith- check
reference
 5843 metic program. Since $P_{(I;W)}$ is constructed from a valid solution to the R1CS, we know from
 5844 6.15 that it is divisible by T . This implies that polynomial division of P by T generates another check
reference
 5845 polynomial $H := P/T$, with $\deg(H) < \deg(T)$.

5846 The prover then evaluates the polynomial $(H \cdot T)\delta^{-1}$ in the exponent of the generator g_1
 5847 at the secret point s , as explained in XXX. To see how this can be achieved, let $H(x)$ be the add refer-
ence
 5848 quotient polynomial P/T :

$$H(x) = H_0 \cdot x^0 + H_1 \cdot x^1 + \dots + H_k \cdot x^k \quad (8.3)$$

To evaluate $H \cdot T$ at s in the exponent of g_1 , the prover uses the common reference string and computes as follows:

$$g_1^{\frac{H(s) \cdot T(s)}{\delta}} = \left(g_1^{\frac{s^0 \cdot T(s)}{\delta}}\right)^{H_0} \cdot \left(g_1^{\frac{s^1 \cdot T(s)}{\delta}}\right)^{H_1} \dots \left(g_1^{\frac{s^k \cdot T(s)}{\delta}}\right)^{H_k}$$

After this has been done, the prover samples two random field elements $r, t \in \mathbb{F}_r$, and uses the common reference string, the instance variables I_1, \dots, I_n and the witness variables W_1, \dots, W_m to compute the following curve points:

$$\begin{aligned} g_1^W &= \left(g_1^{\frac{\beta \cdot A_{1+n}(s) + \alpha \cdot B_{1+n}(s) + C_{1+n}(s)}{\delta}}\right)^{W_1} \dots \left(g_1^{\frac{\beta \cdot A_{m+n}(s) + \alpha \cdot B_{m+n}(s) + C_{m+n}(s)}{\delta}}\right)^{W_m} \\ g_1^A &= g_1^\alpha \cdot g_1^{A_0(s)} \cdot \left(g_1^{A_1(s)}\right)^{I_1} \dots \left(g_1^{A_n(s)}\right)^{I_n} \cdot \left(g_1^{A_{n+1}(s)}\right)^{W_1} \dots \left(g_1^{A_{n+m}(s)}\right)^{W_m} \cdot \left(g_1^\delta\right)^r \\ g_1^B &= g_1^\beta \cdot g_1^{B_0(s)} \cdot \left(g_1^{B_1(s)}\right)^{I_1} \dots \left(g_1^{B_n(s)}\right)^{I_n} \cdot \left(g_1^{B_{n+1}(s)}\right)^{W_1} \dots \left(g_1^{B_{n+m}(s)}\right)^{W_m} \cdot \left(g_1^\delta\right)^t \\ g_2^B &= g_2^\beta \cdot g_2^{B_0(s)} \cdot \left(g_2^{B_1(s)}\right)^{I_1} \dots \left(g_2^{B_n(s)}\right)^{I_n} \cdot \left(g_2^{B_{n+1}(s)}\right)^{W_1} \dots \left(g_2^{B_{n+m}(s)}\right)^{W_m} \cdot \left(g_2^\delta\right)^t \\ g_1^C &= g_1^W \cdot g_1^{\frac{H(s) \cdot T(s)}{\delta}} \cdot \left(g_1^A\right)^r \cdot \left(g_1^B\right)^t \cdot \left(g_1^\delta\right)^{-r \cdot t} \end{aligned}$$

5849 In this computation, the group elements $g_1^{A_j(s)}$, $g_1^{B_j(s)}$ and $g_2^{B_j(s)}$ can be derived from the
 5850 common reference string and the quadratic arithmetic program of the problem, as we have seen
 5851 in XXX. In fact, those points only have to be computed once, and can be published and reused add refer-
ence

for multiple proof generations because they are the same for all instances and witnesses. All other group elements are part of the common reference string.

After all these computations have been done, a valid zero-knowledge succinct non-interactive argument of knowledge π in the Groth_16 protocol is given by the following three curve points:

$$\pi = (g_1^A, g_1^C, g_2^B) \quad (8.4)$$

As we can see, a Groth_16 zk-SNARK consists of 3 curve points, two points from \mathbb{G}_1 and 1 point from \mathbb{G}_2 . The argument is specifically designed this way because, in typical applications, \mathbb{G}_1 is a torsion group of an elliptic curve over some prime field, while \mathbb{G}_2 is a subgroup of a torsion group over an extension field. Elements from \mathbb{G}_1 therefore need less space to be stored, and computations in \mathbb{G}_1 are typically faster than in \mathbb{G}_2 .

Since the witness is encoded in the exponent of a generator of a cryptographically secure elliptic curve, it is hidden from anyone but the prover. Moreover, since any proof is randomized by the occurrence of the random field elements r and t , proofs are not unique to any given witness. This is an important feature because, if all proofs for the same witness would be the same, knowledge of a witness would destroy the zero-knowledge property of those proofs.

Example 140 (The 3-factorization Problem). To see how a prover might compute a zk-SNARK, consider the 3-factorization problem from 108, our protocol parameters from XXX as well as the common reference string from XXX.

Our task is to compute a zk-SNARK for the instance $I_1 = 11$ and its constructive proof $(W_1, W_2, W_3, W_4) = (2, 3, 4, 6)$ as computed in XXX. As we know from 6.15, the associated polynomial $P_{(I;W)}$ of the quadratic arithmetic program from XXX is given by the following equation:

$$P_{(I;W)} = x^2 + x + 9$$

Since $P_{(I;W)}$ is identical to the target polynomial $T(x) = x^2 + x + 9$ in this example, we know from XXX that the quotient polynomial $H = P/T$ is the constant degree 0 polynomial:

$$H(x) = H_0 \cdot x^0 = 1 \cdot x^0$$

We therefore use $[\frac{s^0 \cdot T(s)}{\delta}]_{g_1} = (26, 34)$ from our common reference string XXX of the 3-factorization problem and compute as follows:

$$\begin{aligned} [\frac{H(s) \cdot T(s)}{\delta}]_{g_1} &= [H_0](26, 34) = [1](26, 34) \\ &= (26, 34) \end{aligned}$$

In the next step, we have to compute all group elements required for a proper Groth16 zk-SNARK. We start with g_1^W . Using scalar products instead of the exponential notation, and \oplus for the group law on the BLS6_6 curve, we have to compute the point $[W]g_1$:

$$\begin{aligned} [W]g_1 &= [W_1]g_1 \frac{\beta \cdot A_2(s) + \alpha \cdot B_2(s) + C_2(s)}{\delta} \oplus [W_2]g_1 \frac{\beta \cdot A_3(s) + \alpha \cdot B_3(s) + C_3(s)}{\delta} \oplus [W_3]g_1 \frac{\beta \cdot A_4(s) + \alpha \cdot B_4(s) + C_4(s)}{\delta} \\ &\quad \oplus [W_4]g_1 \frac{\beta \cdot A_5(s) + \alpha \cdot B_5(s) + C_5(s)}{\delta} \end{aligned}$$

To compute this point, we have to remember that a prover should not be in possession of the simulation trapdoor, hence, they do not know what α , β , δ and s are. In order to compute this group element, the prover therefore needs the common reference string. Using the logarithmic order from XXX and the witness, we get the following:

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$$\begin{aligned}
[W]g_1 &= [2](33, 34) \oplus [3](26, 34) \oplus [4](38, 28) \oplus [6](35, 28) \\
&= [2 \cdot 2](13, 15) \oplus [3 \cdot 5](13, 15) \oplus [4 \cdot 10](13, 15) \oplus [6 \cdot 4](13, 15) \\
&= [2 \cdot 2 + 3 \cdot 5 + 4 \cdot 10 + 6 \cdot 4](13, 15) = [5](13, 15) \\
&= (26, 34)
\end{aligned}$$

5873 In a next step, we compute g_1^A . We sample the random point $r = 11$ from \mathbb{F}_{13} , using scalar
5874 products instead of the exponential notation, and \oplus for the group law on the BLS6_6 curve.
5875 We then have to compute the following expression:

$$\begin{aligned}
[A]g_1 &= [\alpha]g_1 \oplus [A_0(s)]g_1 \oplus [I_1][A_1(s)]g_1 \oplus [W_1][A_2(s)]g_1 \oplus [W_2][A_3(s)]g_1 \\
&\quad \oplus [W_3][A_4(s)]g_1 \oplus [W_4][A_5(s)]g_1 \oplus [r][\delta]g_1
\end{aligned}$$

Since we don't know what α , δ and s are, we look up $[\alpha]g_1$ and $[\delta]g_1$ from the common reference string. Recall from XXX that we can evaluate $[A_j(s)]g_1$ without knowing the secret evaluation point s . According to XXX, we have $[A_2(s)]g_1 = (35, 15)$, $[A_5(s)]g_1 = (26, 34)$ and $[A_j(s)]g_1 = \mathcal{O}$ for all other indices $0 \leq j \leq 5$. Since \mathcal{O} is the neutral element on \mathbb{G}_1 , we get the following:

$$\begin{aligned}
[A]g_1 &= (27, 34) \oplus \mathcal{O} \oplus [11]\mathcal{O} \oplus [2](35, 15) \oplus [3]\mathcal{O} \oplus [4]\mathcal{O} \oplus [6](26, 34) \oplus [11](38, 15) \\
&= (27, 34) \oplus [2](35, 15) \oplus [6](26, 34) \oplus [11](38, 15) \\
&= [6](13, 15) \oplus [2 \cdot 9](13, 15) \oplus [6 \cdot 5](13, 15) \oplus [11 \cdot 3](13, 15) \\
&= [6 + 2 \cdot 9 + 6 \cdot 5 + 11 \cdot 3](13, 15) = [9](13, 15) \\
&= (35, 15)
\end{aligned}$$

5876 In order to compute the two curve points $[B]g_1$ and $[B]g_2$, we sample another random element
5877 $t = 4$ from \mathbb{F}_{13} . Using the scalar product instead of the exponential notation, and \oplus for the group
5878 law on the BLS6_6 curve, we have to compute the following expressions:

$$\begin{aligned}
[B]g_1 &= [\beta]g_1 \oplus [B_0(s)]g_1 \oplus [I_1][B_1(s)]g_1 \oplus [W_1][B_2(s)]g_1 \oplus [W_2][B_3(s)]g_1 \\
&\quad \oplus [W_3][B_4(s)]g_1 \oplus [W_4][B_5(s)]g_1 \oplus [t][\delta]g_1 \\
[B]g_2 &= [\beta]g_2 \oplus [B_0(s)]g_2 \oplus [I_1][B_1(s)]g_2 \oplus [W_1][B_2(s)]g_2 \oplus [W_2][B_3(s)]g_2 \\
&\quad \oplus [W_3][B_4(s)]g_2 \oplus [W_4][B_5(s)]g_2 \oplus [t][\delta]g_2
\end{aligned}$$

Since we don't know what β , δ and s are, we look up the associated group elements from the common reference string. Recall from XXX that we can evaluate $[B_j(s)]g_1$ without knowing the secret evaluation point s . Since $B_3 = A_2$ and $B_4 = A_5$, we have $[B_3(s)]g_1 = (35, 15)$, $[B_4(s)]g_1 = (26, 34)$ according to XXX, and $[B_j(s)]g_1 = \mathcal{O}$ for all other indices $0 < j < 5$. Since \mathcal{O} is the neutral element on \mathbb{G}_1 , we get the following:

$$\begin{aligned}
[B]g_1 &= (26, 34) \oplus \mathcal{O} \oplus [11]\mathcal{O} \oplus [2]\mathcal{O} \oplus [3](35, 15) \oplus [4](26, 34) \oplus [6]\mathcal{O} \oplus [4](38, 15) \\
&= (26, 34) \oplus [3](35, 15) \oplus [4](26, 34) \oplus [4](38, 15) \\
&= [5](13, 15) \oplus [3 \cdot 9](13, 15) \oplus [4 \cdot 5](13, 15) \oplus [4 \cdot 3](13, 15) \\
&= [5 + 3 \cdot 9 + 4 \cdot 5 + 4 \cdot 3](13, 15) = [12](13, 15) \\
&= (13, 28)
\end{aligned}$$

$$\begin{aligned}
 [B]g_2 &= (16v^2, 28v^3) \oplus \mathcal{O} \oplus [11]\mathcal{O} \oplus [2]\mathcal{O} \oplus [3](37v^2, 16v^3) \oplus [4](16v^2, 28v^3) \oplus [6]\mathcal{O} \oplus [4](42v^2, 16v^3) \\
 &= (16v^2, 28v^3) \oplus [3](37v^2, 16v^3) \oplus [4](16v^2, 28v^3) \oplus [4](42v^2, 16v^3) \\
 &= [5](7v^2, 16v^3) \oplus [3 \cdot 9](7v^2, 16v^3) \oplus [4 \cdot 5](7v^2, 16v^3) \oplus [4 \cdot 3](7v^2, 16v^3) \\
 &= [5 + 3 \cdot 9 + 4 \cdot 5 + 4 \cdot 3](7v^2, 16v^3) = [12](7v^2 + 16v^3) \\
 &= (7v^2, 27v^3)
 \end{aligned}$$

In a last step, we combine the previous computations to compute the point $[C]g_1$ in the group \mathbb{G}_1 as follows:

$$\begin{aligned}
 [C]g_1 &= [W]g_1 \oplus \left[\frac{H(s) \cdot T(s)}{\delta} \right]g_1 \oplus [t][A]g_1 \oplus [r][B]g_1 \oplus [-r \cdot t][\delta]g_1 \\
 &= (26, 34) \oplus (26, 34) \oplus [4](35, 15) \oplus [11](13, 28) \oplus [-11 \cdot 4](38, 15) \\
 &= [5](13, 15) \oplus [5](13, 15) \oplus [4 \cdot 9](13, 15) \oplus [11 \cdot 12](13, 15) \oplus [-11 \cdot 4 \cdot 3](13, 15) \\
 &= [5 + 5 + 4 \cdot 9 + 11 \cdot 12 - 11 \cdot 4 \cdot 3](13, 15) = [7](13, 15) \\
 &= (27, 9)
 \end{aligned}$$

Given the instance $I_1 = 11$, we can now combine these computations and see that the following 3 curve points are a zk-SNARK for the witness $(W_1, W_2, W_3, W_4) = (2, 3, 4, 6)$:

$$\pi = ((35, 15), (27, 9), (7v^2, 27v^3))$$

5879 We can now publish this zk-SNARK, or send it to a designated verifier. Note that, if we had
 5880 sampled different values for r and t , we would have computed a different SNARK for the same
 5881 witness. The SNARK, therefore, hides the witness perfectly, which means that it is impossible
 5882 to reconstruct the witness from the SNARK.

5883 **The Verification Phase** Given some rank-1 constraint system R , instance $I = (I_1, \dots, I_n)$ and
 5884 zk-SNARK π , the task of the verification phase is to check that π is indeed an argument for
 5885 a constructive proof. Assuming that the simulation trapdoor does not exist anymore and the
 5886 verification checks the proof, the verifier can be convinced that someone knows a witness $W =$
 5887 (W_1, \dots, W_m) such that $(I; W)$ is a word in the language of R .

To achieve this in the Groth16 protocol, we assume that any verifier is able to compute the pairing map $e(\cdot, \cdot)$ efficiently, and has access to the common reference string used to produce the SNARK π . In order to verify the SNARK with respect to the instance (I_1, \dots, I_n) , the verifier computes the following curve point:

$$g_1^I = \left(g_1^{\frac{\beta \cdot A_0(s) + \alpha \cdot B_0(s) + C_0(s)}{\gamma}} \right) \cdot \left(g_1^{\frac{\beta \cdot A_1(s) + \alpha \cdot B_1(s) + C_1(s)}{\gamma}} \right)^{I_1} \cdots \left(g_1^{\frac{\beta \cdot A_n(s) + \alpha \cdot B_n(s) + C_n(s)}{\gamma}} \right)^{I_n}$$

5888 With this group element, the verifier is able to verify the SNARK $\pi = (g_1^A, g_1^C, g_2^B)$ by checking
 5889 the following equation using the pairing map:

$$e(g_1^A, g_2^B) = e(g_1^\alpha, g_2^\beta) \cdot e(g_1^I, g_2^\gamma) \cdot e(g_1^C, g_2^\delta) \quad (8.5)$$

5890 If the equation holds true, the SNARK is accepted. If the equation does not hold, the
 5891 SNARK is rejected.

Remark 7. We know from chapter 5 that computing pairings in cryptographically secure pairing groups is computationally expensive. As we can see, in the Groth16 protocol, 3 pairings are required to verify the SNARK, because the pairing $e(g_1^\alpha, g_2^\beta)$ is independent of the proof, meaning that can be computed once and then stored as an amendment to the verifier key.

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In [?](#), the author showed that 2 is the minimal amount of pairings that any protocol with similar properties has to use. This protocol is therefore close to the theoretical minimum. In the same paper, the author outlined an adaptation that only uses 2 pairings. However, that reduction comes with the price of much more overhead computation. Having 3 pairings is therefore a compromise that gives the overall best performance. To date, the Groth16 protocol is the most efficient in its class.

Example 141 (The 3-factorization Problem). To see how a verifier might check a zk-SNARK for some given instance I , consider the 3-factorization problem from 108, our protocol parameters from XXX, the common reference string from XXX as well as the zk-SNARK $\pi = ((35, 15), (27, 9), (7v^2, 27v^3))$, which claims to be an argument of knowledge for a witness for the instance $I_1 = 11$.

check
reference

In order to verify the zk-SNARK for that instance, we first compute the curve point g_1^I . Using scalar products instead of the exponential notation, and \oplus for the group law on the BLS6_6 curve, we have to compute the point $[I]g_1$ as follows:

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$$[I]g_1 = \left[\frac{\beta \cdot A_0(s) + \alpha \cdot B_0(s) + C_0(s)}{\gamma} \right]_{g_1} \oplus [I_1] \left[\frac{\beta \cdot A_1(s) + \alpha \cdot B_1(s) + C_1(s)}{\gamma} \right]_{g_1}$$

To compute this point, we have to remember that a verifier should not be in possession of the simulation trapdoor, which means that they do not know what α , β , γ and s are. In order to compute this group element, the verifier therefore needs the common reference string. Using the logarithmic order from XXX and instance I_1 , we get the following:

add refer-
ence

$$\begin{aligned} [I]g_1 &= \left[\frac{\beta \cdot A_0(s) + \alpha \cdot B_0(s) + C_0(s)}{\gamma} \right]_{g_1} \oplus [I_1] \left[\frac{\beta \cdot A_1(s) + \alpha \cdot B_1(s) + C_1(s)}{\gamma} \right]_{g_1} \\ &= \mathcal{O} \oplus [11](33, 9) \\ &= [11 \cdot 11](13, 15) = [4](13, 15) \\ &= (35, 28) \end{aligned}$$

In the next step, we have to compute all the pairings involved in equation XXX. Using the logarithmic order on \mathbb{G}_1 and \mathbb{G}_2 as well as the bilinearity property of the pairing map we get the following:

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ence

$$\begin{aligned}
e([A]_{g_1}, [B]_{g_2}) &= e((35, 15), (7v^2, 27v^3)) = e([9](13, 15), [12](7v^2, 16v^3)) \\
&= e((13, 15), (7v^2, 16v^3))^{9 \cdot 12} \\
&= e((13, 15), (7v^2, 16v^3))^{108} \\
e([\alpha]_{g_1}, [\beta]_{g_2}) &= e((27, 34), (16v^2, 28v^3)) = e([6](13, 15), [5](7v^2, 16v^3)) \\
&= e((13, 15), (7v^2, 16v^3))^{6 \cdot 5} \\
&= e((13, 15), (7v^2, 16v^3))^{30} \\
e([I]_{g_1}, [\gamma]_{g_2}) &= e((35, 28), (37v^2, 27v^3)) = e([4](13, 15), [4](7v^2, 16v^3)) \\
&= e((13, 15), (7v^2, 16v^3))^{4 \cdot 4} \\
&= e((13, 15), (7v^2, 16v^3))^{16} \\
e([C]_{g_1}, [\delta]_{g_2}) &= e((27, 9), (42v^2, 16v^3)) = e([7](13, 15), [3](7v^2, 16v^3)) \\
&= e((13, 15), (7v^2, 16v^3))^{7 \cdot 3} \\
&= e((13, 15), (7v^2, 16v^3))^{21}
\end{aligned}$$

5917 In order to check equation XXX, observe that the target group \mathbb{G}_T of the Weil pairing is
 5918 a finite cyclic group of order 13. Exponentiation is therefore done in modular 13 arithmetics.
 5919 Accordingly, since $108 \bmod 13 = 4$, we evaluate the left side of equation XXX as follows:

$$e([A]_{g_1}, [B]_{g_2}) = e((13, 15), (7v^2, 16v^3))^{108} = e((13, 15), (7v^2, 16v^3))^4$$

5920 Similarly, we evaluate the right side of equation XXX using modular 13 arithmetics and the
 5921 exponential law $a^x \cdot a^y = a^{x+y}$:

$$\begin{aligned}
&e([\alpha]_{g_1}, [\beta]_{g_2}) \cdot e([I]_{g_1}, [\gamma]_{g_2}) \cdot e([C]_{g_1}, [\delta]_{g_2}) = \\
&e((13, 15), (7v^2, 16v^3))^{30} \cdot e((13, 15), (7v^2, 16v^3))^{16} \cdot e((13, 15), (7v^2, 16v^3))^{21} = \\
&e((13, 15), (7v^2, 16v^3))^4 \cdot e((13, 15), (7v^2, 16v^3))^3 \cdot e((13, 15), (7v^2, 16v^3))^8 = \\
&e((13, 15), (7v^2, 16v^3))^{4+3+8} = \\
&e((13, 15), (7v^2, 16v^3))^2
\end{aligned}$$

5922 As we can see, both the left and the right side of equation XXX are identical, which implies
 5923 that the verification process accepts the simulated proof.

5924 **NOTE: UNFORTUNATELY NOT! :-((HENCE THERE IS AN ERROR SOMEWHERE ...**
 5925 **NEED TO FIX IT AFTER VACATION**

5926 **Proof Simulation** During the execution of a setup phase, a common reference string is gen-
 5927 erated, along with a simulation trapdoor, the latter of which must be deleted at the end of the
 5928 setup-phase. As an alternative, a more complicated multi-party protocol like [XXX] can be
 5929 used to split the knowledge of the simulation trapdoor among many different parties.

5930 In this paragraph, we will show why knowledge of the simulation trapdoor is problematic,
 5931 and how it can be used to generate zk-SNARKs for a given instance without any knowledge or
 5932 the existence of associated witness.

5933 To be more precise, let I be an instance for some R1CS language L_R . We call a zk-SNARK
 5934 for L_R **forged** or **simulated** if it passes a verification but its generation does not require the
 5935 existence of a witness W such that $(I; W)$ is a word in L_R .

To see how simulated zk-SNARKs can be computed, assume that a forger has knowledge of proper Groth_16 parameters, a quadratic arithmetic program of the problem, a common reference string and its associated simulation trapdoor τ :

$$\tau = (\alpha, \beta, \gamma, \delta, s) \quad (8.6)$$

Given some instance I , the forger’s task is to generate a zk-SNARK for this instance that passes the verification process, without having access to any other zk-SNARKs for this instance and without knowledge of a valid witness W .

To achieve this in the Groth_16 protocol, the forger can use the simulation trapdoor in combination with the QAP and two arbitrary field elements A and B from the scalar field \mathbb{F}_r of the pairing groups to g_1^C compute for the instance (I_1, \dots, I_n) as follows:

$$g_1^C = g_1^{\frac{A \cdot B}{\delta}} \cdot g_1^{-\frac{\alpha \cdot \beta}{\delta}} \cdot g_1^{-\frac{\beta A_0(s) + \alpha B_0(s) + C_0(s)}{\delta}} \cdot \left(g_1^{-\frac{\beta A_1(s) + \alpha B_1(s) + C_1(s)}{\delta}} \right)^{I_1} \cdots \left(g_1^{-\frac{\beta A_n(s) + \alpha B_n(s) + C_n(s)}{\delta}} \right)^{I_n}$$

The forger then publishes the zk-SNARK $\pi_{\text{forged}} = (g_1^A, g_1^C, g_2^B)$, which will pass the verification process and is computable without the existence of a witness (W_1, \dots, W_m) .

To see that the simulation trapdoor is necessary and sufficient to compute the simulated proof π_{forged} , first observe that both generators g_1 and g_2 are known to the forger, as they are part of the common reference string, encoded as $g_1^{s^0}$ and $g_2^{s^0}$. The forger is therefore able to compute $g_1^{A \cdot B}$. Moreover, since the forger knows α, β, δ and s from the trapdoor, they are able to compute all factors in the computation of g_1^C .

If, on the other hand, the simulation trapdoor is unknown, it is not possible to compute g_1^C , since, for example, the computational Diffie-Hellman assumption makes the derivation of $g_1^{\alpha \cdot \beta}$ from g_1^α and g_1^β infeasible.

Example 142 (The 3-factorization Problem). To see how a forger might simulate a zk-SNARK for some given instance I , consider the 3-factorization problem from 108, our protocol parameters from XXX, the common reference string from XXX and the simulation trapdoor $\tau = (6, 5, 4, 3, 2)$ of that CRS.

In order to forge a zk-SNARK for instance $I_1 = 11$, we don’t need a constructive proof for the associated rank-1 constraint system, which implies that we don’t have to execute the circuit $C_{3.\text{fac}}(\mathbb{F}_{13})$. Instead, we have to choose 2 arbitrary elements A and B from \mathbb{F}_{13} , and compute g_1^A, g_2^B and g_1^C as defined in XXX. We choose $A = 9$ and $B = 3$, and, since $\delta^{-1} = 3$, we compute as follows:

$$\begin{aligned} [A]g_1 &= [9](13, 15) = (35, 15) \\ [B]g_2 &= [3](7v^2, 16v^3) = (42v^2, 16v^3) \\ [C]g_1 &= \left[\frac{A \cdot B}{\delta} \right]g_1 \oplus \left[-\frac{\alpha \cdot \beta}{\delta} \right]g_1 \oplus \left[-\frac{\beta A_0(s) + \alpha B_0(s) + C_0(s)}{\delta} \right]g_1 \oplus \\ &\quad [I_1] \left[-\frac{\beta A_1(s) + \alpha B_1(s) + C_1(s)}{\delta} \right]g_1 \\ &= [(9 \cdot 3) \cdot 9](13, 15) \oplus [-(6 \cdot 5) \cdot 9](13, 15) \oplus [0](13, 15) \oplus [11][-(7 \cdot 2 + 4) \cdot 9](13, 15) \\ &= [9](13, 15) \oplus [3](13, 15) \oplus [12](13, 15) = [11](13, 15) \\ &= (33, 9) \end{aligned}$$

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This is all we need to generate our forged proof for the 3-factorization problem. We publish the simulated zk-SNARK:

$$\pi_{fake} = ((35, 15), (33, 9), (42v^2, 16v^3))$$

Despite the fact that this zk-SNARK was generated without knowledge of a proper witness, it is indistinguishable from a zk-SNARK that proves knowledge of a proper witness.

To see that, we show that our forged SNARK passes the verification process. In order to verify π_{fake} , we proceed as in XXX, and compute the curve point g_1^I for the instance $I_1 = 11$. Since the instance is the same as in example XXX, we can parallel the computation from XXX:

$$\begin{aligned} [I]g_1 &= \left[\frac{\beta \cdot A_0(s) + \alpha \cdot B_0(s) + C_0(s)}{\gamma} \right]g_1 \oplus [I_1] \left[\frac{\beta \cdot A_1(s) + \alpha \cdot B_1(s) + C_1(s)}{\gamma} \right]g_1 \\ &= (35, 28) \end{aligned}$$

In a next step we have to compute all the pairings involved in equation XXX. Using the logarithmic order on \mathbb{G}_1 and \mathbb{G}_2 as well as the bilinearity property of the pairing map we get

$$\begin{aligned} e([A]g_1, [B]g_2) &= e((35, 15), (42v^2, 16v^3)) = e([9](13, 15), [3](7v^2, 16v^3)) \\ &= e((13, 15), (7v^2, 16v^3))^{9 \cdot 3} \\ &= e((13, 15), (7v^2, 16v^3))^{27} \\ e([\alpha]g_1, [\beta]g_2) &= e((27, 34), (16v^2, 28v^3)) = e([6](13, 15), [5](7v^2, 16v^3)) \\ &= e((13, 15), (7v^2, 16v^3))^{6 \cdot 5} \\ &= e((13, 15), (7v^2, 16v^3))^{30} \\ e([I]g_1, [\gamma]g_2) &= e((35, 28), (37v^2, 27v^3)) = e([4](13, 15), [4](7v^2, 16v^3)) \\ &= e((13, 15), (7v^2, 16v^3))^{4 \cdot 4} \\ &= e((13, 15), (7v^2, 16v^3))^{16} \\ e([C]g_1, [\delta]g_2) &= e((33, 9), (42v^2, 16v^3)) = e([11](13, 15), [3](7v^2, 16v^3)) \\ &= e((13, 15), (7v^2, 16v^3))^{11 \cdot 3} \\ &= e((13, 15), (7v^2, 16v^3))^{33} \end{aligned}$$

In order to check equation XXX, observe that the target group \mathbb{G}_T of the Weil pairing is a finite cyclic group of order 13. Exponentiation is therefore done in modular 13 arithmetics. Using this, we evaluate the left side of equation XXX as follows:

$$e([A]g_1, [B]g_2) = e((13, 15), (7v^2, 16v^3))^{27} = e((13, 15), (7v^2, 16v^3))^1$$

since $27 \bmod 13 = 1$. Similarly, we evaluate the right side of equation XXX using modular 13 arithmetics and the exponential law $a^x \cdot a^y = a^{x+y}$. We get

$$\begin{aligned} e([\alpha]g_1, [\beta]g_2) \cdot e([I]g_1, [\gamma]g_2) \cdot e([C]g_1, [\delta]g_2) &= \\ e((13, 15), (7v^2, 16v^3))^{30} \cdot e((13, 15), (7v^2, 16v^3))^{16} \cdot e((13, 15), (7v^2, 16v^3))^{33} &= \\ e((13, 15), (7v^2, 16v^3))^4 \cdot e((13, 15), (7v^2, 16v^3))^3 \cdot e((13, 15), (7v^2, 16v^3))^7 &= \\ e((13, 15), (7v^2, 16v^3))^{4+3+7} &= \\ e((13, 15), (7v^2, 16v^3))^1 & \end{aligned}$$

5969 As we can see, both the left and the right side of equation XXX are identical, which implies
5970 that the verification process accepts the simulated proof. π_{fake} therefore convinces the veri-
5971 fier that a witness to the 3-factorization problem exists. However, no such witness was really
5972 necessary to generate the proof.

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5973 **Chapter 9**

5974 **Exercises and Solutions**

5975 TODO: All exercises we provided should have a solution, which we give here in all detail.