Security Analysis of Filecoin's Expected Consensus in the Byzantine vs Honest Model

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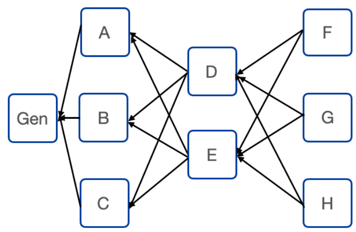
Abstract. Filecoin is the largest storage-based blockchain, both by storage capacity (>18EiB) and market capitalization. This paper provides the first formal security analysis of Filecoin's consensus protocol, Expected Consensus (EC). Specifically, we show that EC is secure against an arbitrary adversary that controls a fraction β of the total storage for $\beta m < 1 - e^{-(1-\beta)m}$, where m is a parameter that corresponds to the expected number of blocks per round, currently $m = 5$ in Filecoin. We then present an attack, the n-split attack, where an adversary splits the honest miners between multiple chains, and show that it is successful for $\beta m \geq 1 - e^{-(1-\beta)m}$, thus proving that $\beta m = 1 - e^{-(1-\beta)m}$ is the tight security threshold of EC. This corresponds roughly to an adversary with 20% of the total storage pledged to the chain. Finally, we propose two improvements to Filecoin's security that would increase this threshold. The Filecoin team is currently working on deploying one of these fixes.

1 Introduction

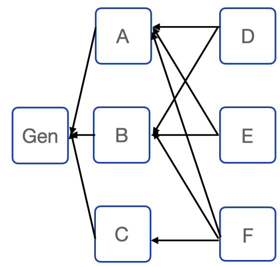
Filecoin is the largest storage-based blockchain in terms of both market cap [2] and total storage capacity (>18EiB) [4]. In Filecoin, miners called Storage Providers gain the right to participate in the consensus protocol and to create blocks by pledging storage capacity to the chain. They are in return compensated with a financial reward in the form of newly minted FIL, the cryptocurrency underlying Filecoin, whenever their blocks are included on-chain. The Filecoin consensus mechanism Storage Power Consensus (SPC) consists mainly of two components: first, a *Sybil-resistance mechanism* that keeps an accurate map of the storage pledged by each storage provider; and second, a consensus protocol that can be run by any set of weighted participants and outputs an ordered list of transactions. In this paper, we ignore the mechanisms that keep the mapping between miners and their respective storage accurate (i.e., the Sybil-resistance mechanism) and focus on the sub-protocol run by the weighted miners to produce an ordered list of transactions. This sub-protocol is called Expected Consensus (EC) and each participant is weighted according to their storage. We assume that the weighted list of miners is accurately maintained and given as an input to EC. EC is a longest-chain style protocol [1] (or, more accurately, a heaviest-chain protocol). At a high level, it operates by running a leader election at every time slot in which, on expectation, m participants may be eligible to submit a block, where m is a parameter currently equal to 5. Each participant is elected with probability proportional to their weight (i.e., in Filecoin, their fraction of total storage pledged). All valid blocks submitted in a given round form a *tipset*, which is a set of blocks sharing the same height (i.e., round number) and parent tipset. In EC, the blockchain is a chain of tipsets (i.e., a directed acyclic graph [DAG] of blocks) rather than a chain of blocks. For example, in Figure 1a blocks {A,B,C}, {D,E} and {F,G,H} each form a different tipset. Every block in a tipset adds weight to its chain of tipsets, while the fork choice rule is to choose the heaviest tipset. EC works in a very similar fashion as longest-chain protocols like Bitcoin do, but it uses tipsets instead of blocks. EC's security has, until now, only been argued informally, as with Bitcoin in its early days. Intuitively, tipsets make it harder for an adversary with less storage to form a competing chain of tipsets with more blocks than the main chain, as miners can create a number of blocks proportional to their storage. Specifically, assuming that two competing chains of tipsets are growing, with different amount of storage pledged to each, since in EC more than one block can be appended to a chain at each round, the difference between the number of blocks created on each chain will grow roughly m times faster than in the case without tipset (i.e., where each chain can grow by at most one block at each round).

In this work, we conduct a formal security analysis of EC and prove that EC is secure against any adversary that owns a fraction β of the total storage pledged to the chain for $\beta m < 1 - e^{-(1-\beta)m}$ (Section 5). To do so, we carefully extend the proof technique developed in [9] to EC: the key step is to identify the sufficient condition for a block to stay in the chain forever, regardless of the complex DAG structure in EC. Following similar literature [10, 13], we consider a rather strong adversary, which we specify in Section 2, that has "full control" over the network. We then propose an attack, the n-split attack, in Section 6 in which an adversary with power β such that $\beta m \geq 1 - e^{-(1-\beta)m}$ can break the security of EC, effectively proving that the security threshold $\beta m = 1-e^{-(1-\beta)m}$ is tight for EC. With current parameters, this means that EC is secure against an adversary that holds roughly 20% of the total storage pledged to the chain. In our attack, an elected leader, controlled by the adversary, equivocates by sending different blocks to different miners at each round with the aim to split the honest miners into different chains and thus reducing the weight of each tipsets' chain. While the honest participants are split and all mine on potentially many different forks, the adversary can construct a *private* tipsets' chain on the side, i.e., a chain that does not include any block mined by an honest miner (called for simplicity *honest block*) and that will not be broadcasted to any honest miner until the end of the attack. Finally, we propose in Section 7 two mitigations against this attack that would allow an increased security threshold of Filecoin. We have disclosed this vulnerability to the Filecoin team and they are currently implementing one of our suggested mitigations.

Related Work This work is directly inspired by the line of work formally analyzing longest-chain protocols either in the proof-of-work case [10, 14, 19] or proof-of-stake [6, 9, 13, 17]. We adapt the technique in [9] to account for tipsets



(a) Example of a tipset chain: {A,B,C} is the first tipset after Genesis, and the parent tipset of {D,E}, which is itself the parent tipset of tipset {F,G,H}.



(b) Example of two tipsets with equal weight. Tipset {D,E} and tipset {F} both have a weight of 5. (roughly, the weight is calculated as the sum of the blocks in all the tipsets, see Section 3.3 for details).

Fig. 1: Tipset structure in EC.

instead of blocks (see Lemma 1). The main difference between using a chain of tipsets and a chain of blocks is that in the tipset case, the number of blocks in the chain at each round can increase by any finite integer value and also depends on the structure of the DAG. By contrast, in the chain-of-blocks case, the number of blocks of any honest chain increases by zero or one at each round. This makes the tipset analysis more complex.

Similar attacks to the one we describe in Section 6 were proposed by Bagaria et al. [6] in the context of proof-of-stake and by Natoli and Gramoli [16] in the proof-of-work context. In both these works, the attacks are described on a DAG-based blockchain, where each new block can include any previous block as its parent. We adapt it to the tipset case, which is slightly different: in the case of tipsets, a block may have multiple parents, but only blocks that have themselves the same set of parents can be referenced by the child block. The idea behind these attacks is to have an adversary publish its blocks in a timely manner on different forks to ensure that honest miners keep on extending two or even more chains instead of one, effectively spreading their *power* (be it stake, computation or storage) on different chains.

2 Model

In this section we present our model and assumptions.

2.1 Participants

Filecoin requires participants to pledge storage capacity to the chain to be added to the list of participants. Following the work in [10,13] we consider a flat model, meaning that each participant accounts for one unit of storage. This could easily be extended to a non-flat model by considering that a participant holding x units of storage controls x "flat" participants. We consider a static model wherein the set of participants is fixed during the execution of the protocol. We assume that each participant i possesses a key pair $(\mathsf{sk}_i, \mathsf{pk}_i)$ and that every participant is aware of the other participants and their respective public keys.

We consider a static adversary that corrupts a fraction β of the participants at the beginning of the execution of the protocol. In order to defend against an adaptive adversary who can corrupt honest nodes on the fly (i.e., dynamically, at the start of each round), one can either use key evolving signature schemes [8] or checkpointing [5]. However, in order to keep the problem simple, we do not consider an adaptive adversary in this paper.

2.2 Network assumptions

We consider the lock-step synchronous network model adopted in [10, 13]. Time is divided into synchronized rounds, each indexed with an integer in $\mathbb{N}$. Following Filecoin's terminology [1], we refer to each time slot as an *epoch*. Each epoch has a fixed duration (currently 30 seconds in Filecoin). To abstract the underlying peer-to-peer gossip network in Filecoin, we simply assume that all messages sent by honest nodes are broadcast to all nodes and that all honest nodes re-broadcast any message they deliver. All network messages are delivered by the adversary, and we allow the adversary to selectively delay messages sent by honest nodes, with the following restrictions: (i) the messages sent in an epoch must be delivered by the end of the current epoch; and (ii) the adversary cannot forge or alter any message sent by an honest node. The adversary does not suffer any network delay. Note that the adversary can selectively send its message only to a subset of honest nodes. However, due to the re-broadcast mechanism, all honest nodes will receive the message by the end of the next epoch.

2.3 Randomness

Random beacon A random beacon [18] is a system that emits a random number at regular intervals. EC relies on drand [3], a decentralized random beacon, to provide miners a different random number at each epoch. This service is run by a set of 16 independent institutions that run a multi-party protocol to output, at regular intervals, a fresh random number. We assume that this random number is unbiasable (i.e., truly random) and unpredictable before the beginning of the epoch. We also assume that each miner in Filecoin has the same view of each drand output, i.e., that drand is secure. We denote drand_i the random beacon emitted by drand and used by Filecoin miners at epoch i.

Verifiable Random Function A Verifiable Random Function (VRF) [15] is a function that outputs a random number in a verifiable way, i.e., everyone can verify that the output is indeed random and was generated correctly. A VRF is composed of two polynomial-time algorithms: $\mathsf{VRF.Proof}$ and $\mathsf{VRF.Verify}$ (we omit the key generation). $\mathsf{VRF.Proof}$ takes as inputs a seed seed and a secret key sk and outputs a tuple $(y = G_{\mathsf{sk}}(\mathsf{seed}), p = \pi_{\mathsf{sk}}(\mathsf{seed}))$ where y is a random number and p is a proof that can be used to verify the correctness of y. $\mathsf{VRF.Verify}$

takes as input a tuple (seed, y, p) and uses p to verify that $y = G_{\mathsf{sk}}(\mathsf{seed})$, in which case it outputs 1; otherwise, it outputs 0. A VRF is correct if:

1. if $(y, p) = \mathsf{VRF.Proof}_{\mathsf{sk}}(\mathsf{seed})$ then $\mathsf{VRF.Verify}(\mathsf{seed}, y, p) = 1$;
2. for all $(\mathsf{sk}, \mathsf{seed})$ there is a unique y s.t. $\mathsf{VRF.Verify}(\mathsf{seed}, y, \pi_{\mathsf{sk}}(\mathsf{seed})) = 1$;
3. $G_{\mathsf{sk}}(\mathsf{seed})$ is computationally indistinguishable from a random number for any probabilistic polynomial-time adversary.

Throughout the rest of this paper, we assume the existence of a correct VRF.

3 Filecoin's Expected Consensus (EC)

Filecoin's consensus protocol, EC, consists of three main components: a leader election sub-protocol, a mining algorithm and a fork choice rule. Briefly, at the beginning of each epoch, participants will check their eligibility to produce a block by running the leader election. If they are elected, they use the fork choice rule to select a tipset and include it as their *parent* before broadcasting their block. We define the protocol more formally in this section. However, we intentionally omit some details, such as those regarding how participants must continually post proofs related to their pledged storage, as they are not relevant to our analysis. Instead, we assume that all participants continuously maintain one unit of storage. Furthermore, in practice in Filecoin [1] a participant with x unit storage that is elected twice in the same epoch will create only one block that *weighs* twice more. This is not relevant to our analysis, so we ignore it and prefer a flat model wherein a participant elected twice simply creates two blocks under two different identities. Such a model will favor an adversary as we illustrate in Appendix D and hence renders our analysis stronger.

Due to space limitations, a pseudocode representation of the algorithms described in this section can be found in Appendix A.

3.1 Leader Selection Protocol

EC's leader election is inspired by Algorand's cryptographic sortition [11]. Briefly, the leader selection relies on a Verifiable Random Function (VRF) [15] that takes as input the drand output value for that epoch. In each epoch, each participant will compute $\mathsf{VRF.Proof}_{\mathsf{sk}}(\mathsf{seed}) = (y = G_{\mathsf{sk}}(\mathsf{seed}), p = \pi_{\mathsf{sk}}(\mathsf{seed}))$ where seed is the drand value. If y is below a predefined value target that is a parameter of the protocol, then that participant is elected leader. Any other participant can then use p in order to verify that the random value y was computed correctly (i.e., $\mathsf{VRF.Verify}(\mathsf{seed}, y, p) = 1$) and that the participant is indeed an elected leader. The value of target is chosen such that on expectation m leaders are elected in each epoch. m is a parameter of the EC protocol currently set to $m = 5$.

Proving that the leader selection mechanism is secure is outside the scope of this paper, as similar results were already proven in, e.g., Algorand [11]. Instead, we assume that in each epoch, there is an independent random number of participants that are elected leaders and that the number of leaders in each epoch follows a Poisson distribution of parameter m. For a coalition that consists of a fraction α of all the participants, their number of elected leaders in an epoch follows a Poisson distribution of parameter $\alpha \times m$.

3.2 Block and Tipset Structure

A block is composed of a header and a payload. The payload includes transactions as well as other messages necessary for maintaining the set of participants up to date. We omit its content in this analysis.

When a participant is elected to create a block, they include in the *header* of the block their proof of eligibility (i.e., the VRF proof), an epoch number (the epoch at which the block was created), a proof of storage called $\mathsf{WinningPost}$ to prove that they maintain the storage they have pledged (we omit the details of such proof) and finally a pointer to a set of *parent* blocks. For a block $\mathcal{B}$, we denote $\mathcal{B}.\mathsf{parent}$ its parents set and $\mathcal{B}.\mathsf{epoch}$ its epoch number. The parents of a block must satisfy certain conditions. First, they must all be in the same epoch, and that epoch needs to be smaller than the block's epoch. Second, all parent blocks need to have the same set of parents themselves. Each set of blocks that are in the same epoch and have the same set of parents is called a *tipset* and denoted $\mathcal{T}$. Formally, a tipset $\mathcal{T}$ is a non-empty set of blocks: $\mathcal{T} = \{\mathcal{B}_1, \cdots, \mathcal{B}_r\}$, each of which belongs to the same epoch, i.e., $\mathcal{B}_1.\mathsf{epoch} = \cdots = \mathcal{B}_r.\mathsf{epoch}$ and has the same set of parents, i.e., $\forall(\mathcal{B}_i, \mathcal{B}_j) \in \mathcal{T}^2 : \ \ \mathcal{B}_i.\mathsf{parent} = \mathcal{B}_j.\mathsf{parent}$. Since all blocks in a tipset have the same parent, we abuse the notation and denote $\mathcal{T}.\mathsf{parent}$ to denote the parent of tipset $\mathcal{T}$. Similarly, $\mathcal{T}.\mathsf{epoch}$ denotes the tipset epoch. We note that $\mathcal{T}.\mathsf{parent}$ is a tipset itself.

Since each block references a set of blocks, a Directed Acyclic Graph (DAG) structure can be inferred from each block or tipset, where the blocks are the vertices and the references to parents are the edges. Similarly, the set of tipsets referencing each other as parents form a *chain*. For example, Figure 1a represents a chain of 4 tipsets (including the genesis) and a blockDAG of 9 blocks. Formally, a chain $\mathcal{C}$ is then a set of ordered tipsets $\mathcal{C} = \{\mathcal{T}_0, \mathcal{T}_1, \cdots, \mathcal{T}_l\}$ such that $\mathcal{T}_i.\mathsf{parent} = \mathcal{T}_{i-1}$ for all $i > 1$. By convention, we have $\mathcal{T}_1.\mathsf{parent} = \mathcal{T}_0 = \{\text{Genesis block}\}$. We note $\mathcal{C}[\mathcal{T}_i] = \{\mathcal{T}_0, \mathcal{T}_1, \cdots, \mathcal{T}_i\}$. Similarly, for a tipset $\mathcal{T}$, we can infer the associated chain, denoted $\mathcal{C}[\mathcal{T}]$ as follows: $\mathcal{C}[\mathcal{T}] = \{\mathcal{T}_0, \cdots, (\mathcal{T}.\mathsf{parent}).\mathsf{parent}, \mathcal{T}.\mathsf{parent}, \mathcal{T}\}$.

3.3 Fork Choice Rule and Weight Function

In order to decide which tipset to include as its parents, EC provides a *weight function* that assigns a weight to different tipsets. The fork choice rule will then consist of choosing the tipset with the heaviest weight. In practice, EC's weight function [1] is a complex function of (1) the number of blocks in the chain and (2) the total amount of storage committed to the chain. Moreover, the total amount of storage is taken far in the past to ensure that everyone agrees on it. Since in our analysis we assume a static model where the set of participants is fixed during the execution of the protocol, we only take into consideration the number of blocks in the chain. We discuss the impact of this simplification in Section 8. Formally, for a tipset $\mathcal{T}$, we denote its weight $w(\mathcal{T})$ and have:

$$w(\mathcal{T}) = \sum_{\mathcal{T}_i \in \mathcal{C}[\mathcal{T}]} |\mathcal{T}_i|$$

In the case of a tie between two chains, a deterministic tie-breaker is used. In practice, the tipset that contains the smallest VRF value is chosen. However, in our analysis we consider a powerful adversary that has the power to decide on ties. See Figure 1b for a visual representation of two tipsets with equal weight.

3.4 Mining Algorithm

We describe the mining algorithm that miners in Filecoin run continuously. At each epoch $i > 0$ each participant with key pair $(\mathsf{sk}, \mathsf{pk})$ performs the following:

1. Fetch the drand value for epoch i and verify eligibility by checking

$$G_{\mathsf{sk}}(\mathrm{drand}_i) \overset{?}{\leq} \mathsf{target},$$

   where target is chosen such that on expectation m leaders are elected (with $m = 5$ in the current implementation).
2. If elected leader, create a block as follows:
   - Choose the tipset with the highest weight (i.e., the most blocks) and reference it as the block's parent.
   - Include a proof of eligibility (i.e., the VRF value: $\mathsf{VRF.Proof}_{\mathsf{sk}}(\mathsf{drand}_i) = (y, p)$), a $\mathsf{WinningPost}$ to prove storage maintenance, as well as the payload.
3. Broadcast the block newly created.

In parallel, whenever they receive a new block in epoch i, participants verify its validity and, if it is valid, add it to their blockDAG. A block is valid if and only if:

1. The election proof (y, p) is valid i.e.,: $\mathsf{VRF.Verify}(\mathsf{drand}_i, y, p) = 1$ and $y \leq \mathsf{target}$.
2. The $\mathsf{WinningPost}$ is valid (details omitted).
3. All the transactions in the payload are valid (details omitted).
4. All its parent blocks form a valid tipset, i.e.:
   - They all belong to the same epoch.
   - They all have the same parents.
   - They are all valid blocks.

We analyze the backbone of EC in a static setting and hence omit some details of the protocol. For example, in practice, the leader election mechanism uses a *lookback parameter*, meaning that only participants who pledged their storage sufficiently in the past are eligible for block creation. Because we consider a flat and static model, these details are not relevant to our analysis.

4 Security Definitions

Security properties. We consider the standard security properties of *robust transaction ledgers* defined for blockchain systems [9,10]. We start by defining a transaction ledger and confirmed transactions in the ledger.

Definition 1 (Transaction ledger generated by a chain $\mathcal{C}$). *Given a chain $\mathcal{C}$, a transaction ledger $\mathcal{L}$ generated by C is a deterministic, totally-ordered and append-only list of transactions. In particular, if $\mathcal{C}_1$ is a prefix of $\mathcal{C}_2$, then $\mathcal{L}_1$ generated by $\mathcal{C}_1$ is a prefix of $\mathcal{L}_2$ generated by $\mathcal{C}_2$.*

For example, one way to generate a transaction ledger from a chain $\mathcal{C}$ is to order the transactions from $\mathcal{C}$ by order of chronological appearance (i.e., epoch number where they appeared in the chain) and lexicographical order. Any deterministic rule is however valid and we leave this unspecified.

Definition 2 (Confirmed transaction parameterized by $\tau \in \mathbb{R}$). *If a transaction* tx *in the ledger appears in a block which is mined in epoch $j \leq i - \tau$, then* tx *is said to be τ-*confirmed *in epoch i.*

Our goal is to generate a transaction ledger that satisfies *persistence* and *liveness* as defined in [9,10]. Together, persistence and liveness guarantee a robust transaction ledger; transactions will be adopted to the ledger and be immutable.

Definition 3 (Robust transaction ledger from [9, 10]). *A blockchain protocol Π maintains a robust transaction ledger if the generated ledger satisfies the following two properties:*

- *(Persistence) Parameterized by $\tau \in \mathbb{R}$. If a transaction* tx *becomes τ-confirmed at epoch i in the view of one honest node, then* tx *will be at least τ-confirmed in the same position in the ledger by all honest nodes for every epoch $j \geq i$.*
- *(Liveness) Parameterized by $u \in \mathbb{R}$, if a transaction* tx *is received by all honest nodes at epoch i, then after epoch $i + u$ all honest nodes will contain* tx *in the same place in the ledger forever.*

Notations. We then define random variables and stochastic processes of interest and their properties.

Let α and β be the collective fraction of storage power controlled by honest nodes and malicious nodes, respectively ($\alpha + \beta = 1$). We follow the notations of [7]. Let $H[r]$ and $Z[r]$ be the number of blocks mined by the honest nodes and by the malicious nodes in epoch r, then $H[r]$, $Z[r]$ are independent Poisson random variables with means $(1 - \beta)m$ and βm respectively [1] (the value of the target parameter is chosen to ensure this). In addition, the random variables $\{H[0], H[1], \cdots\}$ and $\{Z[0], Z[1], \cdots\}$ are independent of one another, since the value provided by drand to feed the leader election is random. We now define the auxiliary random variables $X[r]$ and $Y[r]$ as follows. If at epoch r an honest node mines at least one block (i.e., $H[r] \geq 1$), then $X[r] = 1$ and epoch r is called a *successful* epoch, otherwise $X[r] = 0$. If at epoch r honest nodes mine exactly one block (i.e., $H[r] = 1$), then $Y[r] = 1$ and epoch r is called a *uniquely successful* epoch, otherwise $Y[r] = 0$. Epoch r is called an *isolated successful* epoch if it further satisfies that there is no honest block in epoch $r-1$ (i.e., $H[r-1] = 0$ and $Y[r] = 1$). Further, $X[r_1, r_2]$ and $Y[r_1, r_2]$ are the number of successful and uniquely successful epochs, respectively, in the interval $(r_1, r_2]$, and $H[r_1, r_2]$ and $Z[r_1, r_2]$ are the number of blocks mined by honest nodes and by the adversary respectively in the interval $(r_1, r_2]$.

In EC, chains may have equal weights. For simplicity and generality, we assume tie-breaking always favors the adversary. This means that the persistence will be broken as long as there are two sufficiently long forks with equal weights.

Given a chain of tipsets $\mathcal{C}$, let $\mathcal{C}[r]$ be the chain truncated up to blocks in epoch r. Further, let $w(\mathcal{C})$ be the weight of $\mathcal{C}$. Let $W_{\max}[r]$ and $W_{\min}[r]$ be the maximum and minimum weights of chains adopted by honest nodes at the end of epoch r. Then, by our network model, we have:

$$W_{\min}[r] \leq W_{\max}[r] \leq W_{\min}[r+1]. \tag{1}$$

Even if some honest nodes' chains are "behind" in epoch r, by our re-broadcast mechanism, their view for epoch r will catch up with the rest of the honest nodes in epoch $r+1$. Furthermore, honest participants always extend the heaviest chain they are aware of, hence the inequality above.

We also have the following minimum honest chain growth property, which is essential to our proof. For $t \geq r+1$,

$$W_{\min}[t] \geq W_{\min}[r+1] + X[r+1, t] \geq W_{\max}[r] + X[r+1, t]. \tag{2}$$

This inequality again follows from the fact that honest participants always extend the heaviest chain they know of. However, it could be that different honest participants have different views and thus create blocks on different chains, hence why we consider X in the inequality above and not H.

5 Security Proof

In this section, we prove our main theorem, Theorem 1 stated below, parameterized by the security parameter κ. The proof will proceed in multiple steps. Due to space limitations, all the proofs appear in the appendix.We extend the technique of Nakamoto blocks developed in [9]. We first define the notion of Nakamoto epochs in EC and prove that the honest blocks mined in Nakamoto epochs remain in the heaviest chain forever. Then we show that Nakamoto epochs exist and appear frequently regardless of the adversarial strategy. Straightforwardly, the protocol satisfies liveness and persistence: transactions can enter the ledger frequently through the Nakamoto epochs, and once they enter, they remain at a fixed location in the ledger.

Theorem 1. *If $\beta m < 1 - e^{-(1-\beta)m}$, then EC generates a robust transaction ledger that satisfies* persistence *(parameterized by $\tau = \kappa$) and* liveness *(parameterized by $u = \kappa$) in Definition 3 with probability at least $1 - e^{-\Omega(\kappa^{1-\epsilon})}$, for any $0 < \epsilon < 1$.*

5.1 Nakamoto epochs

Let us define the events:

$$E_{rs} = \{\text{event that } Z[r-1,t] < X[r+1,t] \text{ for all } t \geq s\},$$

$$F_s = \bigcap_{0 \leq r \leq s-2} E_{rs},$$

$$\begin{aligned} U_s &= \{\text{event that epoch } s \text{ is an isolated successful epoch}\} \\ &= \{H[s-1] = 0, Y[s] = 1\}, \end{aligned}$$

and

$$G_s = F_s \cap U_s.$$

We will call epoch s a *Nakamoto epoch* if the event G_s occurs. And we have the following lemma. The proof appears in Appendix B.1.

Lemma 1. *If epoch s is a Nakamoto epoch, then the unique honest block mined in epoch s is contained in any future chain $\mathcal{C}[t]$, $t \geq s$.*

Note that Lemma 1 implies that if G_s occurs, then the entire chain leading to the unique honest block mined in epoch s from the genesis is stabilized after epoch s.

5.2 Occurrence of Nakamoto epochs

Although the existence of Nakamoto epochs ensures that the block at this epoch will be finalized, i.e., it will appear in every honest future chain, the question now remains whether Nakamoto epochs exist at all and, if so, at what frequency they appear. We start answering this question by proving in the next lemma that Nakamoto epochs have a strictly positive probability of happening, i.e., the probability of each epoch being a Nakamoto epoch is strictly positive. Due to space limitations, the proof is left to Appendix B.2.

Lemma 2. *If $\beta m < 1 - e^{-(1-\beta)m}$, then there exists $p > 0$ such that $P(G_s) \geq p$ for all s.*

5.3 Waiting time for Nakamoto epochs

We have established the fact that the event G_s has $P(G_s) \geq p > 0$ for all s. But how long do we need to wait for such an epoch to occur? We answer this question in the following lemma, wherein we provide a bound on the probability that in a interval $(j, j+k]$ of k consecutive epochs, there are no Nakamoto epochs, i.e., a bound on:

$$q(j, j+k] := P(\bigcap_{s=j+1}^{j+k} G_s^c),$$

where G_s^c is the complement of G_s. Proofs for the following two lemmas can be found in Appendix B.3 and Appendix B.4.

Lemma 3. *If $\beta m < 1 - e^{-(1-\beta)m}$, then there exist constants α, A so that for all $j, k \geq 0$,*

$$q(j, j+k] \leq A \exp(-\alpha\sqrt{k}). \tag{3}$$

We can also tighten the exponent, but at the cost of larger constants in the bound.

Lemma 4. *If $\beta m < 1 - e^{-(1-\beta)m}$, then there exist constants $\alpha_\epsilon, A_\epsilon$ so that for all $j, k \geq 0$,*

$$q(j, j+k] \leq A_\epsilon \exp(-\alpha_\epsilon k^{1-\epsilon}), \tag{4}$$

for any $0 < \epsilon < 1$.

5.4 Persistence and liveness

Equipped with all the previous lemmas, we can now prove the persistence and liveness properties of EC for $\beta m < 1 - e^{-(1-\beta)m}$.

Proof of Theorem 1. Suppose current epoch is r. Then by Lemma 4, with probability at least $1 - e^{-\Omega(\kappa^{1-\epsilon})}$, there is at least one Nakamoto epoch in the interval $(r-\kappa, r]$. Let epoch $s \in (r-\kappa, r]$ be a Nakamoto epoch. Then by Lemma 1, the chain up to epoch $s-1$ is permanent since the unique honest block in epoch s never leaves the heaviest chain. Hence EC is persistent with probability at least $1 - e^{-\Omega(\kappa^{1-\epsilon})}$. The liveness of EC is simply a consequence of the frequent occurrence of Nakamoto epochs. Particularly, for each honest transaction, either it will be included by an honest block B in a Nakamoto epoch or it has already been included by B's ancestors. □

6 n-split Attack

In order to confirm whether an adversary with power $\beta m \geq 1 - e^{-(1-\beta)m}$ can indeed break the persistence and liveness properties of the system, we consider the following n-split attack.

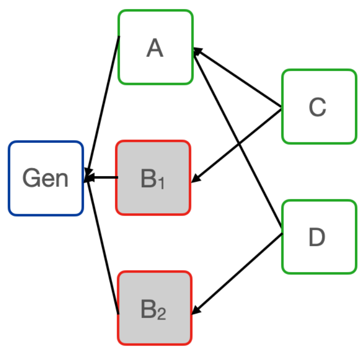
6.1 Attack description

The attacker tries to split the honest participants among n chains such that in each epoch, at most one honest block is added to each chain (i.e., no two honest players mine on the same chain). To do this, the attacker creates n copies of one of its block (each copy has the same election proof, but different payloads) and sends one different block to each of the n honest players; see illustration in Figure 2a. To maintain the split for a long period, the adversary must repeat the attack at every epoch where at least one honest block is mined. In this case, the weight of the chain of each honest player will increase by two: one honest block and one adversarial block. For example in Figure 2a, since blocks C and D are both honest (i.e., created by honest miners), by the next epoch, epoch 3, all the participants will have received them and use the deterministic tie breaker to all decide to mine on the same tipset, say {D}. Hence the adversary must create equivocating blocks in epoch 2 as well in order to ensure that in epoch 3, honest miners all choose different tipsets to append their block to. In Figure 2b, the adversary sends equivocating blocks E_1 and E_2 to prevent blocks H and F from being appended to the same chain. These figures include only two honest blocks at epoch 2 and 3 for clarity. In practice, the adversary will create as many equivocating blocks as there are honest miners to ensure that everyone sees a different block and that no two honest participants mine on the same tipset.

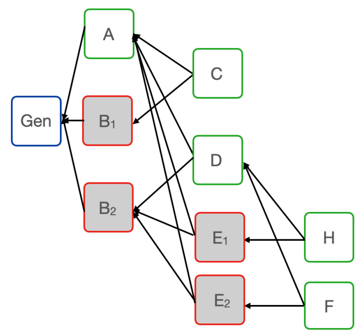
Whenever there is no honest block mined in an epoch, the attacker does nothing. In this case, the weight of the chain of each honest player will not increase. Meanwhile, the attacker also reuses all its blocks to build a private chain, i.e., a chain that it does not broadcast to other participants and that does not include any honest blocks. The expected chain growth of the adversary's private chain is βm.

The weight of the honest chain increases by two if there is at least one honest block mined in an epoch (which happens with probability $1 - e^{-(1-\beta)m}$), and 0 otherwise (which happens with probability $e^{-(1-\beta)m}$). Hence, the expected chain growth of the honest chain is $2(1 - e^{-(1-\beta)m})$. Therefore, this attack succeeds with non-negligible probability when $\beta m > 2(1 - e^{-(1-\beta)m})$, i.e., when the adversarial chain grows at a higher rate than the honest split chains. As we see this threshold is however different from the threshold we found in Theorem 1. Some numerical results: for $m = 3$, we have $\beta > 0.512$; for $m = 5$, we have $\beta > 0.382$; and $\beta > 0.284$ for $m = 7$. Approximately, $\beta \gtrsim 2/m$. Intuitively, with this attack, the threshold is inversely proportional to m, as with a bigger m, the adversary is elected leader more often and thus has more opportunities to keep the network split. If the adversary were elected leader on a less regular basis, it would be harder to keep sending equivocating blocks and thus keep the network split for longer. Note that this threshold is valid only for the attack we have just described. Different thresholds may exist for different attacks.

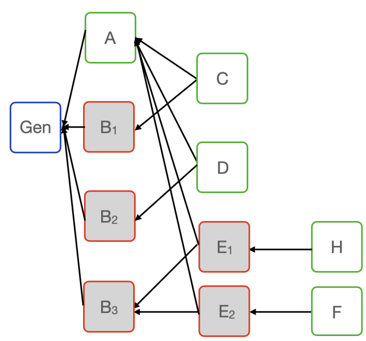
Following the standard model of longest-chain analysis [8–10], we give the power of tie-breaking to the attacker (i.e., tie-breaking always favors the attacker's block). For example in Figure 2c, if we assume that the adversarial blocks (in red) will always be favored in the case of two chains with the same number of blocks, then the adversary does not need to create a block on the same tipset as honest blocks (as in Figure 2b). The adversary will instead create another block B_3 and mine yet on another tipset than the honest participant in epoch 2. In epoch 3 the adversarial chains ending in tipsets $\{E_1\}$ and $\{E_2\}$ are preferred over $\{C\}$ or $\{D\}$, hence honest blocks H and F are mined on different forks and each fork's weight increased by only one in epoch 2, as opposed to 2 in Figure 2b. By repeating this attack at each epoch, the weight of the chain of each honest player will only increase by one when there is at least one honest block mined. Therefore, the threshold is now derived by $\beta m > 1 - e^{-(1-\beta)m}$. We notice that this matches the threshold in Theorem 1, hence proving that $\beta m = 1 - e^{-(1-\beta)m}$ is the tight threshold of the protocol in our security model



(a) The adversary sends two different blocks B_1, B_2 in epoch 1 such that in epoch 2, honest blocks C, D have different parents and hence cannot be included in a tipset. The honest power is thus split between different tipsets' chains.



(b) The adversary keeps the network split in epoch 3 by creating two equivocating blocks: E_1 and E_2 in epoch 2. In epoch 3, honest blocks H and F are mined on two different tipsets.



(c) If ties are always broken in favor of the adversary, the adversary can instead create another block B_3 in epoch 1. In epoch 2, blocks E_1 and E_2 are preferred to C and D, hence in epoch 2, all the tipsets' chains increase by only one block. The attack is repeated in the next epoch, as long as the adversary has enough blocks to create a fork as heavy as the honest chains.

Fig. 2: n-split attack. Each block filled in grey is an equivocating block, meaning they were created by the adversary using the same leader election proof in one epoch. Each green block is an honest block.

as defined in Section 2. Indeed, Theorem 1 proves that no adversary below this threshold can break the security of the EC, and the n-chain split attack just described proves that an adversary above this threshold can indeed break the persistence and liveness of the EC.

Some numerical results: for $m = 3$, we have $\beta \simeq 0.293$; for $m = 5$, we have $\beta \simeq 0.196$; and $\beta \simeq 0.143$ for $m = 7$. Approximately, $\beta \simeq 1/m$. As remarked before, the threshold is inversely proportional to the number of leaders elected as, intuitively, being elected more often gives more opportunities to an adversary.

We discuss practical aspects of this attack (e.g., its rationality) in Appendix D.

7 Mitigations

We propose two possible mitigations to the n-split attack described in Section 6 which could also help increase the security threshold of EC.

7.1 Replace EC by a Longest-chain Protocol in SPC

One solution to the n-split attack is to remove the notion of tipsets and instead change EC to a longest-chain protocol, i.e., where one block has exactly one parent. With the longest-chain setting, the effort of an adversary to split the network would have much less impact on the overall security of the protocol since each chain can increase by one block at each epoch at most anyway. Furthermore, moving to a longest-chain setting allows for inheritance of all the security properties (e.g., a security threshold of 50%) of all proof-of-stake protocols based on that setting [6, 8, 9]. Dembo et al. [9] indeed showed that in the longest-chain case, the worst attack is the private attack. Hence, the n-split attack described in Section 6, or its variant, would not be the worst attack anymore.

7.2 Consistent Broadcast

Another solution is to use a form of consistent or reliable broadcast [12]. This type of broadcast prevents an adversary from equivocating (i.e., creating two blocks with the same leader election proof but different contents). The network could, however, still be split between the nodes that accept the adversarial block vs those that do not. We conjecture that the security threshold of EC with $m = 5$ under a non-equivocating adversary is around 40%, as, roughly, in this case the adversary creates strictly less than two blocks per epoch on average and hence cannot sustain a split over many epochs. We leave a formal proof as future work.

7.3 Responsible disclosure

We contacted the Filecoin team to present the n-split attack, together with our proposed mitigations. The team decided to implement a consistent broadcast protocol for blocks propagation in order to mitigate the attack.

8 Limitations and Future Work

For practical reasons, this work made a few simplifying assumptions. We discuss them here.

Incentive consideration This work considers the classic model of honest vs malicious participants and does not address the rationality of participants. A formal study of incentive compatibility is also important for understanding the security of EC. However, we leave this for future work. Assuming a fully malicious adversary that is willing to lose money to attack the system, a scenario we consider in this paper, makes for a stronger proof than assuming a rational adversary. In Appendix D, we discussed why it is realistic to consider an irrational adversary for the n-split attack we proposed, as slashing may not always be possible if the adversary has the ability to censor transactions. It still remains to show that the honest strategy is compatible with a rational strategy even in the presence of an adversary. We leave this for future work.

Weight function In our analysis, we only took into consideration the number of blocks in the chain for the weight function. We leave as future work an analysis that also considers the total storage, as specified in Expected Consensus [1]. Specifically we believe that a complex weight function allows for more vectors of attack and that an adversary could use this to try to blow the weight of its own chain. For example, the adversary could remove its storage from the main chain and thus decrease the weight of the main chain, while privately creating an alternative chain that would be heavier because it has more storage pledged. The mechanisms for maintaining and removing storage are, however, complex and ignored in this work. For simplicity, we thus consider the weight of a tipset to simply be equal to the number of blocks referenced in its blockDAG.

9 Conclusion

In this paper we presented a formal analysis of Expected Consensus, a subprotocol of Filecoin's Storage Power Consensus, and we proposed two concrete ways to improve SPC's security. One of our mitigations, using consistent broadcast, is currently being deployed by the Filecoin team. It remains an open problem to quantify the new security threshold of EC with this fix, although our proofs remain valid in this case, hence the security threshold is at least such that $\beta m < 1 - e^{-(1-\beta)m}$ as proved in Section 5. Futhermore, we made many simplifying assumptions in this work. It would be interesting to relax these in a future work; e.g., by extending this proof to the dynamic and asynchronous case, considering the more complex variant of the weight function or incorporating incentives.

References

1. Filecoin Spec. `https://spec.filecoin.io/`.

2. Coin Market Cap. `https://coinmarketcap.com/`.
3. Drand. `http://https://drand.love/`. Accessed: 2022-08-30.
4. Filfox - Filecoin Explorer. `https://filfox.info/en`.
5. Sarah Azouvi and Marko Vukolić. Pikachu: Securing PoS blockchains from long-range attacks by checkpointing into Bitcoin PoW using Taproot. *arXiv preprint arXiv:2208.05408*, 2022.
6. Vivek Bagaria, Amir Dembo, Sreeram Kannan, Sewoong Oh, David Tse, Pramod Viswanath, Xuechao Wang, and Ofer Zeitouni. Proof-of-stake longest chain protocols: Security vs predictability. *arXiv preprint arXiv:1910.02218*, 2019.
7. Vivek Bagaria, Sreeram Kannan, David Tse, Giulia Fanti, and Pramod Viswanath. Prism: Deconstructing the blockchain to approach physical limits. In *Proceedings of the 2019 ACM SIGSAC Conference on Computer and Communications Security*, pages 585–602. ACM, 2019.
8. Bernardo David, Peter Gaži, Aggelos Kiayias, and Alexander Russell. Ouroboros Praos: An adaptively-secure, semi-synchronous proof-of-stake blockchain. In *Annual International Conference on the Theory and Applications of Cryptographic Techniques*, pages 66–98. Springer, 2018.
9. Amir Dembo, Sreeram Kannan, Ertem Nusret Tas, David Tse, Pramod Viswanath, Xuechao Wang, and Ofer Zeitouni. Everything is a race and Nakamoto always wins. In *Proceedings of the 2020 ACM SIGSAC Conference on Computer and Communications Security*, pages 859–878, 2020.
10. Juan Garay, Aggelos Kiayias, and Nikos Leonardos. The bitcoin backbone protocol: Analysis and applications. In *Annual International Conference on the Theory and Applications of Cryptographic Techniques*, pages 281–310. Springer, 2015.
11. Yossi Gilad, Rotem Hemo, Silvio Micali, Georgios Vlachos, and Nickolai Zeldovich. Algorand: Scaling Byzantine agreements for cryptocurrencies. In *Proceedings of the 26th Symposium on Operating Systems Principles*, pages 51–68, 2017.
12. Rachid Guerraoui, Petr Kuznetsov, Matteo Monti, Matej Pavlovic, Dragos-Adrian Seredinschi, and Yann Vonlanthen. Scalable Byzantine reliable broadcast (extended version). *arXiv preprint arXiv:1908.01738*, 2019.
13. Aggelos Kiayias, Alexander Russell, Bernardo David, and Roman Oliynykov. Ouroboros: A provably secure proof-of-stake blockchain protocol. In *Annual International Cryptology Conference*, pages 357–388. Springer, 2017.
14. Lucianna Kiffer, Rajmohan Rajaraman, and Abhi Shelat. A better method to analyze blockchain consistency. In *Proceedings of the 2018 ACM SIGSAC Conference on Computer and Communications Security*, pages 729–744, 2018.
15. Silvio Micali, Michael Rabin, and Salil Vadhan. Verifiable random functions. In *40th Annual Symposium on Foundations of Computer Science (Cat. No. 99CB37039)*, pages 120–130. IEEE, 1999.
16. Christopher Natoli and Vincent Gramoli. The balance attack against proof-of-work blockchains: The r3 testbed as an example. *arXiv preprint arXiv:1612.09426*, 2016.
17. Rafael Pass and Elaine Shi. The sleepy model of consensus. In *International Conference on the Theory and Application of Cryptology and Information Security*, pages 380–409. Springer, 2017.
18. Michael O Rabin. Transaction protection by beacons. *Journal of Computer and System Sciences*, 27(2):256–267, 1983.
19. Ling Ren. Analysis of Nakamoto consensus. *Cryptology ePrint Archive*, 2019.

Appendix

A Pseudocode for EC

The main algorithm is presented in Algorithm 1 and the algorithm for block validation is presented in Algorithm 2.

Algorithm 1 Main algorithm

```
import
    drand
    ForkChoiceRule
    Broadcast
    VRF
    isValid (Algorithm 2)
Parameters:
    epochLength
    m
    target                ▷ Chosen such that m leaders are elected on expectation
Init:
    epochNumber ← 0
    blockDAG←{Genesis Block}
upon event time.Now() % epochLength == 0 do        ▷ Beginning of the epoch
    epochNumber ← epochNumber +1
    seed ← drand(epochNumber)
    (y, p) ← VRF.Proof_sk(seed)
    if y ≤ target then
        𝒯 ←ForkChoiceRule(blockDAG)  ▷ Choose the DAG with the most blocks
        ℬ ← CreateBlock(𝒯, (y, p), epochNumber,WinningPost payload)
        Broadcast(ℬ)
upon event Receiving block ℬ do
    if isValid(ℬ) == 1 then
        blockDAG.append(ℬ)
```

B Proofs

B.1 Proof of Lemma 1

Proof. Let b_s be the unique honest block mined in epoch s. We will argue by contradiction. Suppose G_s occurs and let $t \geq s$ be the smallest t such that b_s is not contained in $\mathcal{C}[t]$, an honest chain adopted by some honest node at the end of epoch t. Let b_r, mined in epoch r, be the last honest block on $\mathcal{C}[t]$ (which must exist, because the genesis block is by definition honest). If $r > s$, then $\mathcal{C}[r-1]$ is the prefix of $\mathcal{C}[t]$ before block b_r, and does not contain b_s (because $\mathcal{C}[r-1]$ is a

Algorithm 2 isValid($\mathcal{B}$)

Input: block $\mathcal{B}$
import
    drand
    isPayloadValid
    isStorageValid
Parse $(\mathcal{T}, (y, p)$, epochNumber,WinningPost payload$) \leftarrow \mathcal{B}$
seed $\leftarrow$ drand(epochNumber)
if VRF.Verify(seed, y, p) $== 0$ or $y >$ target **then** ▷ Check the election proof
    return 0
if isPayloadValid(payload)$==$0 **then** ▷ Check the payload
    return 0
if isStorageValid(WinningPost) $== 0$ **then** ▷ Check the storage proof
    return 0
for $\mathcal{B}_i \in \mathcal{T}$ **do** ▷ Check validity of parent blocks
    if isValid($\mathcal{B}_i$) $== 0$ **then**
        return 0
return 1

prefix of $\mathcal{C}[t]$) contradicting the minimality of t. So b_r must be mined before or in epoch s. Since epoch s is an isolated successful epoch, we further know that $r \leq s-2$. The part of $\mathcal{C}[t]$ after block b_r must consist of all malicious blocks by the definition of b_r. Note that this may also include malicious blocks in epoch r (i.e., *headstart* of the adversary). Hence, we have an upper bound for the weight of $\mathcal{C}[t]$.

$$w(\mathcal{C}[t]) \leq W_{\max}[r] + Z[r-1, t] < W_{\max}[r] + X[r+1, t], \tag{5}$$

where the first inequality is illustrated in Figure 3, and the second inequality follows from the fact that event F_s occurs. We also have a trivial lower bound: $w(\mathcal{C}[t]) \geq W_{\min}[t]$. Therefore, we have

$$W_{\min}[t] < W_{\max}[r] + X[r+1, t], \tag{6}$$

which contradicts the minimum honest chain growth property (Eqn. 2). □

B.2 Proof of Lemma 2

Proof. Let $1 - e^{-(1-\beta)m} = (1+\varepsilon)\beta m$ for some $\varepsilon > 0$. The random processes of interest start from epoch 0. To look at the system in stationarity, let us extend them to $-\infty < r < \infty$. This extension allows us to extend the definition of E_{rs} to all r, s, $-\infty < r < s < \infty$, and define $\hat{F}_s$ and $\hat{G}_s$ to be:

$$\hat{F}_s = \bigcap_{r \leq s-2} E_{rs}$$

and

$$\hat{G}_s = \hat{F}_s \cap U_s.$$

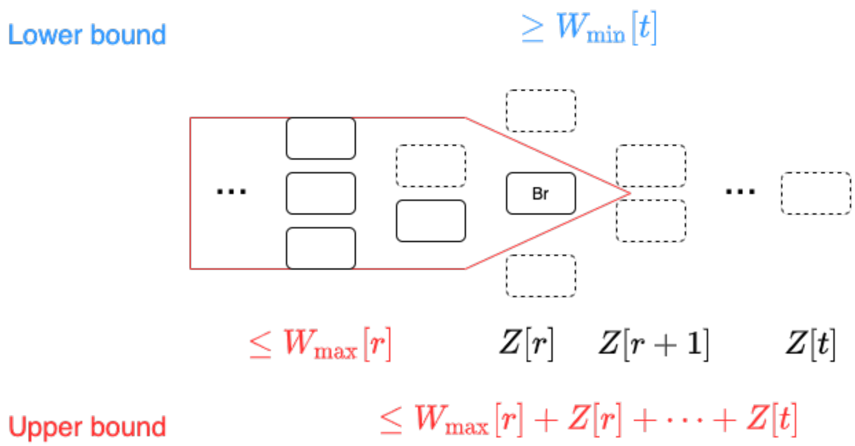


Fig. 3: Upper bound and lower bound of the weight of $\mathcal{C}[t]$ in the proof of Lemma 1. Blocks with dotted lines are adversarial blocks. The parent links are omitted for readability; each block has all blocks from the previous epoch as parents.

Note that $\hat{G}_s \subset G_s$, so to prove that $P(G_s) \geq p$ for all s, it suffices to prove that $P(\hat{G}_s) \geq p$ for all s. This is proved next.

Let $\mathcal{W}_r = (H[r], Z[r])$. Let's note that the variables $\{\mathcal{W}_r\}$'s are independent from each other, hence $\hat{F}_s \cap U_s = \bigcap_{r \leq s-2}(E_{rs} \cap U_s)$ has a time-invariant dependence on $\{\mathcal{W}_r\}$, which means that $P(\hat{G}_r)$ does not depend on r. Then it suffices to show $P(\hat{G}_0) > 0$.

$$\begin{aligned}
P(\hat{G}_0) &= P(\hat{F}_0|U_0)P(U_0) \\
&= P(\hat{F}_0|U_0)P(H[-1]=0)P(H[0]=1) \\
&= (1-\beta)me^{-2(1-\beta)m}P(\hat{F}_0|U_0).
\end{aligned}$$

Then, it remains to show that $P(\hat{F}_0|U_0) > 0$.

Recall that

$$\hat{F}_0 = \text{event that } Z[r-1,t] < X[r+1,t] \text{ for all } t \geq 0 \text{ and } r \leq -2.$$

Let

$$\hat{B}_{rt} = \text{event that } Z[r-1,t] \geq X[r+1,t],$$

then

$$\hat{F}_0^c = \bigcup_{t \geq 0, r \leq -2} \hat{B}_{rt}.$$

Let us fix a particular integer $n > 2$, and define:

$$M_n = \text{event that } Z[-n-1,n] = 0.$$

Then

$$
\begin{aligned}
P(\hat{F}_0|U_0) &\geq P(\hat{F}_0|U_0, M_n)P(M_n|U_0) \\
&= \left(1 - P(\cup_{t\geq 0, r\leq -2}\hat{B}_{rt}|U_0, M_n)\right) P(M_n|U_0) \\
&\geq \left(1 - \sum_{t\geq 0, r\leq -2} P(\hat{B}_{rt}|U_0, M_n)\right) P(M_n|U_0) \\
&\geq (1 - a_n - b_n - c_n)P(M_n|U_0), \qquad (7)
\end{aligned}
$$

where

$$
\begin{aligned}
a_n &:= \sum_{(r,t):-n\leq r\leq -2 \text{ and } 0\leq t\leq n} P(\hat{B}_{rt}|U_0, M_n), \\
b_n &:= \sum_{(r,t):r\leq -2 \text{ and } t>n} P(\hat{B}_{rt}|U_0, M_n), \\
c_n &:= \sum_{(r,t):r<-n \text{ and } t\geq 0} P(\hat{B}_{rt}|U_0, M_n).
\end{aligned}
$$

Note that the cases where $t > n$ and $r < -n$ are counted twice in b_n and c_n, but this is fine because we only need a lower bound. Next we will bound $P(\hat{B}_{rt}|U_0, M_n)$. Consider three cases:

Case 1: $-n \leq r \leq -2$ and $0 \leq t \leq n$:

$$
\begin{aligned}
&P(\hat{B}_{rt}|U_0, M_n) \\
&= P(Z[r-1,t] \geq X[r+1,t]|H[0] = 1, H[-1] = 0, Z[-n-1,n] = 0) \\
&= P(X[r+1,t] \leq 0|H[0] = 1, H[-1] = 0, Z[-n-1,n] = 0) \\
&\leq P(X[0] \leq 0|H[0] = 1, H[-1] = 0, Z[-n-1,n] = 0) \\
&= 0.
\end{aligned}
$$

Summing these terms, we have $a_n = 0$.

Case 2: $r \leq -2$ and $t > n$:

$$
\begin{aligned}
&P(\hat{B}_{rt}|U_0, M_n) \\
&= P(Z[r-1,t] \geq X[r+1,t]|H[0] = 1, H[-1] = 0, M_n) \\
&\leq P((Z[r-1,-2] + Z[n,t]) \geq (X[r+1,-2] + X[0,t] + 1)) \\
&\leq P((Z[r-1,-2] + Z[n,t]) \geq (X[r+1,-2] + X[0,t])); \qquad //(*) \\
&\leq P(Z[r+3,t] \geq X[r+3,t]) \qquad //n > 1 \text{ and } Z\text{'s and } X\text{'s are i.i.d r.v.s.} \\
&\leq P(Z[r+3,t] \geq X[r+3,t], X[r+3,t] \geq (1-\varepsilon/4)(1-e^{-(1-\beta)m})(t-r-3)) \\
&\quad + P(Z[r+3,t] \geq X[r+3,t], X[r+3,t] < (1-\varepsilon/4)(1-e^{-(1-\beta)m})(t-r-3)) \\
&\leq P(Z[r+3,t] \geq (1+\varepsilon/4)\beta m(t-r-3)) \\
&\quad + P(X[r+3,t] < (1-\varepsilon/4)(1-e^{-(1-\beta)m})(t-r-3)) \\
&\leq A_0 e^{-\alpha_0\varepsilon^2(t-r-3)}
\end{aligned}
$$

for some positive constants A_0, α_0 independent of n, r, t. Inequality (*) follows from counting $t-r-3$ epochs on the right hand side of the equation and $\leq t-r-3$ epochs on the left hand side. The last two inequalities follow from Lemma 6, Lemma 5 and the facts that $1 - e^{-(1-\beta)m} = (1+\varepsilon)\beta m$ and $\frac{1+\varepsilon/4}{1-\varepsilon/4} < 1+\varepsilon$. Summing these terms, we have:

$$
\begin{aligned}
b_n &= \sum_{(r,t):r\leq -2 \text{ and } t>n} P(\hat{B}_{rt}|U_0, M_n) \\
&\leq \sum_{(r,t):r\leq -2 \text{ and } t>n} \left[A_0 e^{-\alpha_0 \varepsilon^2 (t-r-3)}\right] := \bar{b}_n,
\end{aligned}
$$

which is bounded and moreover $\bar{b}_n \to 0$ as $n \to \infty$.

Case 3: $r < -n$ and $t \geq 0$:

$$
\begin{aligned}
& P(\hat{B}_{rt}|U_0, M_n) \\
&= P(Z[r-1,t] \geq X[r+1,t] | H[0]=1, H[-1]=0, M_n) \\
&\leq P((Z[r-1,-n-1] + Z[2,t]) \geq (X[r+1,-2] + X[0,t] + 1)) \\
&\leq P((Z[r-1,-n-1] + Z[2,t]) \geq (X[r+1,-2] + X[0,t])) \\
&\leq P(Z[r+3,t] \geq X[r+3,t]) \qquad //n > 2 \text{ and } Z\text{'s and } X\text{'s are i.i.d r.v.s.} \\
&\leq P(Z[r+3,t] \geq X[r+3,t], X[r+3,t] \geq (1-\varepsilon/4)(1-e^{-(1-\beta)m})(t-r-3)) \\
&\quad + P(Z[r+3,t] \geq X[r+3,t], X[r+3,t] < (1-\varepsilon/4)(1-e^{-(1-\beta)m})(t-r-3)) \\
&\leq P(Z[r+3,t] \geq (1+\varepsilon/4)\beta m(t-r-3)) \\
&\quad + P(X[r+3,t] < (1-\varepsilon/4)(1-e^{-(1-\beta)m})(t-r-3)) \\
&\leq A_0 e^{-\alpha_0 \varepsilon^2 (t-r-3)}
\end{aligned}
$$

for some positive constants A_0, α_0 independent of n, r, t. The last two inequalities follow from Lemma 6, Lemma 5 and the facts that $1 - e^{-(1-\beta)m} = (1+\varepsilon)\beta m$ and $\frac{1+\varepsilon/4}{1-\varepsilon/4} < 1+\varepsilon$. Summing these terms, we have:

$$
\begin{aligned}
c_n &= \sum_{(r,t):r< -n \text{ and } t\geq 0} P(\hat{B}_{rt}|U_0, M_n) \\
&\leq \sum_{(r,t):r< -n \text{ and } t\geq 0} \left[A_0 e^{-\alpha_0 \varepsilon^2 (t-r-3)}\right] := \bar{c}_n,
\end{aligned}
$$

which is bounded and moreover $\bar{c}_n \to 0$ as $n \to \infty$.

Substituting these bounds in (7) we finally get:

$$
P(\hat{F}_0|U_0) > (1 - \bar{b}_n - \bar{c}_n) P(M_n|U_0).
$$

By setting n sufficiently large such that $\bar{b}_n$ and $\bar{c}_n$ are sufficiently small, we conclude that $P(\hat{G}_0) > 0$.

□

B.3 Proof of Lemma 3

Proof. Following the definition in Lemma 2, let

$$\hat{B}_{rt} = \text{event that } Z[r-1,t] \geq X[r+1,t].$$

Similar to the calculation in Lemma 2, we have

$$P(\hat{B}_{rt}) \leq A_1 e^{-\alpha_1 \varepsilon^2 (t-r)} \tag{8}$$

for some positive constants A_1, α_1 independent of r, t.

Also we have

$$G_s^c = F_s^c \cup U_s^c = \bigcup_{(r,t):r \leq s-2, t \geq s} \hat{B}_{rt} \cup U_s^c. \tag{9}$$

Divide $(j, j+k]$ into $\sqrt{k}$ sub-intervals of length $\sqrt{k}$ (assuming $\sqrt{k}$ is a integer), so that the i-th sub-interval is:

$$\mathcal{J}_i := [j+1+(i-1)\sqrt{k}, j+i\sqrt{k}].$$

Now look at the first, fourth, seventh, etc sub-intervals, i.e. all the $i = 1 \mod 3$ sub-intervals. Introduce the event that in the ℓ-th ($1 \mod 3$) sub-interval ($\mathcal{J}_{3\ell+1}$), a pure adversarial chain that is rooted at a honest block (or more accurately a tipset including at least one honest block) mined in that sub-interval ($\mathcal{J}_{3\ell+1}$) or in the previous ($0 \mod 3$) sub-interval ($\mathcal{J}_{3\ell}$) catches up with a honest block in that sub-interval ($\mathcal{J}_{3\ell+1}$) or in the next ($2 \mod 3$) sub-interval ($\mathcal{J}_{3\ell+2}$).

Formally,

$$C_\ell = \bigcap_{s \in \mathcal{J}_{3\ell+1}} \bigcup_{(r,t):r \in \mathcal{J}_{3\ell} \cup \mathcal{J}_{3\ell+1}, r \leq s-2, t \geq s, t \in \mathcal{J}_{3\ell+1} \cup \mathcal{J}_{3\ell+2}} \hat{B}_{rt} \cup U_s^c.$$

Note that for distinct ℓ, the events C_ℓ's are independent since $\hat{B}_{rt}$'s in different C_ℓ's do not have overlap (the $\mathcal{J}$ intervals were cut specifically for this purpose). Also, we have

$$P(C_\ell) \leq P(\text{no Nakamoto epoch in } \mathcal{J}_{3\ell+1}) = 1 - p < 1 \tag{10}$$

by Lemma 2.

Introduce the atypical events:

$$B = \bigcup_{(r,t):r \in (j,j+k] \text{ or } t \in (j,j+k], r<t, t-r \geq \sqrt{k}} \hat{B}_{rt} \text{ , and}$$

$$\tilde{B} = \bigcup_{(r,t):r \leq j, j+k<t} \hat{B}_{rt} \text{ .}$$

The events B and $\tilde{B}$ are the events that an adversarial chain catches up with an honest block far ahead (more than $\sqrt{k}$ epochs).

By (8) and an union bound we have that

$$
\begin{aligned}
P(B) &\le \sum_{(r,t): r\in[j+1,j+k] \text{ or } t\in[j+1,j+k], r<t, t-r\ge\sqrt{k}} A_1 e^{-\alpha_1\varepsilon^2(t-r)} \\
&\le \sum_{r=j+1}^{j+k}\Big(\sum_{t=r+\sqrt{k}}^{\infty} A_1 e^{-\alpha_1\varepsilon^2(t-r)}\Big) + \sum_{t=j+1}^{j+k}\Big(\sum_{r=0}^{t-\sqrt{k}} A_1 e^{-\alpha_1\varepsilon^2(t-r)}\Big) \\
&\le 2k\frac{A_1 e^{-\alpha_1\varepsilon^2\sqrt{k}}}{1-e^{-\alpha_1\varepsilon^2}},
\end{aligned}
$$

and

$$
\begin{aligned}
P(\tilde{B}) &\le \sum_{(r,t): r\le j, t>j+k} A_1 e^{-\alpha_1\varepsilon^2(t-r)} \\
&\le \sum_{r=0}^{j}\Big(\sum_{t=j+k+1}^{\infty} A_1 e^{-\alpha_1\varepsilon^2(t-r)}\Big) \\
&= \sum_{r=0}^{j}\frac{A_1 e^{-\alpha_1\varepsilon^2(j+k+1-r)}}{1-e^{-\alpha_1\varepsilon^2}} \\
&\le \frac{A_1 e^{-\alpha_1\varepsilon^2(k+1)}}{(1-e^{-\alpha_1\varepsilon^2})^2}.
\end{aligned}
$$

Now, we have:

$$
\begin{aligned}
& q(j, j+k] \\
& \le P(\text{no Nakamoto epoch in } \textstyle\bigcup_{\ell=0}^{\sqrt{k}/3} \mathcal{J}_{3\ell+1}) \\
& \le P(\text{no isolated successful epoch in } \textstyle\bigcup_{\ell=0}^{\sqrt{k}/3} \mathcal{J}_{3\ell+1}) \\
& + P(B) + P(\tilde{B}) + P\Big(\bigcap_{\ell=0}^{\sqrt{k}/3} C_\ell\Big) \\
& = e^{-\Omega(k)} + P(B) + P(\tilde{B}) + (P(C_\ell))^{\sqrt{k}/3} \qquad (11) \\
& \le e^{-\Omega(k)} + 2k\frac{A_1 e^{-\alpha_1\varepsilon^2\sqrt{k}}}{1-e^{-\alpha_1\varepsilon^2}} + \frac{A_1 e^{-\alpha_1\varepsilon^2(k+1)}}{(1-e^{-\alpha_1\varepsilon^2})^2} + (P(C_\ell))^{\sqrt{k}/3} \\
& \le A\exp(-\alpha\sqrt{k}) \qquad (12)
\end{aligned}
$$

for some positive constants A and α. The equality (11) is due to the independence of C_ℓ's and the inequality (12) is due to (10). Hence the lemma follows. □

B.4 Proof of Lemma 4

Proof. By Lemma 3, we know that the statement is true for $\epsilon = 1/2$ (and hence for $\epsilon > 1/2$). Now we prove for any $\epsilon > 0$ by recursively applying the bootstrapping procedure in Lemma 3. We use induction to show that the statement is true for all $\epsilon \geq m_n$, where $m_n = \frac{1}{n+1}, n \geq 1$. As a base step, this is true for $n = 1$. Now we assume it holds for $n \geq 1$ and prove for the step $n+1$.

Divide $(j, j+k]$ into $k^{\frac{1}{n+2}}$ sub-intervals of length $k^{\frac{n+1}{n+2}}$ (assuming $k^{\frac{1}{n+2}}$ is a integer), so that the i-th sub-interval is:

$$\mathcal{J}_i := [j + 1 + (i-1)k^{\frac{n+1}{n+2}}, j + ik^{\frac{n+1}{n+2}}].$$

Now look at the first, fourth, seventh, etc sub-intervals, i.e. all the $i = 1 \mod 3$ sub-intervals. Introduce the event that in the ℓ-th $(1 \mod 3)$ sub-interval $(\mathcal{J}_{3\ell+1})$, an pure adversarial chain that is rooted at a honest block (or more accurately a tipset including at least one honest block) mined in that sub-interval $(\mathcal{J}_{3\ell+1})$ or in the previous $(0 \mod 3)$ sub-interval $(\mathcal{J}_{3\ell})$ catches up with a honest block in that sub-interval $(\mathcal{J}_{3\ell+1})$ or in the next $(2 \mod 3)$ sub-interval $(\mathcal{J}_{3\ell+2})$. Formally,

$$C_\ell = \bigcap_{s \in \mathcal{J}_{3\ell+1}} \bigcup_{(r,t): r \in \mathcal{J}_{3\ell} \cup \mathcal{J}_{3\ell+1}, r \leq s-2, t \geq s, t \in \mathcal{J}_{3\ell+1} \cup \mathcal{J}_{3\ell+2}} \hat{B}_{rt} \cup U_s^c.$$

Note that for distinct ℓ, the events C_ℓ's are independent since $\hat{B}_{rt}$'s in different C_ℓ's do not have overlap. Also by the previous induction step, we have

$$\begin{aligned} P(C_\ell) &\leq P(\text{no Nakamoto epoch in } \mathcal{J}_{3\ell+1}) \\ &\leq A_{m_n} \exp(-\alpha_{m_n} (k^{\frac{n+1}{n+2}})^{1-m_n}) \\ &= A_{m_n} \exp(-\alpha_{m_n} k^{\frac{n}{n+2}}). \end{aligned} \tag{13}$$

Introduce the atypical events:

$$\begin{aligned} B &= \bigcup_{(r,t): r \in (j, j+k] \text{ or } t \in (j, j+k], r < t, t - r \geq k^{\frac{n+1}{n+2}}} \hat{B}_{rt}\ , \text{ and} \\ \tilde{B} &= \bigcup_{(r,t): r \leq j, j+k < t} \hat{B}_{rt}\ . \end{aligned}$$

The events B and $\tilde{B}$ are the events that an adversarial chain catches up with an honest block far ahead (more than $k^{\frac{n+1}{n+2}}$ epochs). Following the calculations in Lemma 3, we have

$$\begin{aligned} P(B) &\leq \sum_{(r,t): r \in [j+1, j+k] \text{ or } t \in [j+1, j+k], r < t, t - r \geq k^{\frac{n+1}{n+2}}} A_1 e^{-\alpha_1 \varepsilon^2 (t-r)} \\ &\leq \sum_{r=j+1}^{j+k} \Big(\sum_{t = r + k^{\frac{n+1}{n+2}}}^{\infty} A_1 e^{-\alpha_1 \varepsilon^2 (t-r)} \Big) + \sum_{t=j+1}^{j+k} \Big(\sum_{r=0}^{t - k^{\frac{n+1}{n+2}}} A_1 e^{-\alpha_1 \varepsilon^2 (t-r)} \Big) \\ &\leq 2k \frac{A_1 e^{-\alpha_1 \varepsilon^2 k^{\frac{n+1}{n+2}}}}{1 - e^{-\alpha_1 \varepsilon^2}}, \end{aligned}$$

and

$$
\begin{aligned}
P(\tilde{B}) &\leq \sum_{(r,t):r\leq j, t>j+k} A_1 e^{-\alpha_1 \varepsilon^2 (t-r)} \\
&\leq \sum_{r=0}^{j} \Big(\sum_{t=j+k+1}^{\infty} A_1 e^{-\alpha_1 \varepsilon^2 (t-r)} \Big) \\
&= \sum_{r=0}^{j} \frac{A_1 e^{-\alpha_1 \varepsilon^2 (j+k+1-r)}}{1-e^{-\alpha_1 \varepsilon^2}} \\
&\leq \frac{A_1 e^{-\alpha_1 \varepsilon^2 (k+1)}}{(1-e^{-\alpha_1 \varepsilon^2})^2}.
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
&q(j, j+k] \\
&\leq P(\text{no Nakamoto epoch in } \textstyle\bigcup_{\ell=0}^{k^{\frac{1}{n+2}}/3} \mathcal{J}_{3\ell+1}) \\
&\leq P(\text{no isolated successful epoch in } \textstyle\bigcup_{\ell=0}^{k^{\frac{1}{n+2}}/3} \mathcal{J}_{3\ell+1}) \\
&+ P(B) + P(\tilde{B}) + P\Big(\bigcap_{\ell=0}^{k^{\frac{1}{n+2}}/3} C_\ell\Big) \\
&= e^{-\Omega(k)} + P(B) + P(\tilde{B}) + (P(C_\ell))^{k^{\frac{1}{n+2}}/3} \qquad (14) \\
&\leq e^{-\Omega(k)} + 2k \frac{A_1 e^{-\alpha_1 \varepsilon^2 k^{\frac{n+1}{n+2}}}}{1-e^{-\alpha_1 \varepsilon^2}} + \frac{A_1 e^{-\alpha_1 \varepsilon^2 (k+1)}}{(1-e^{-\alpha_1 \varepsilon^2})^2} + (P(C_\ell))^{k^{\frac{1}{n+2}}/3} \\
&\leq A_{m_{n+1}} \exp(-\alpha_{m_{n+1}} k^{\frac{n+1}{n+2}}) \qquad (15) \\
&= A_{m_{n+1}} \exp(-\alpha_{m_{n+1}} k^{1-m_{n+1}})
\end{aligned}
$$

for some positive constants $A_{m_{n+1}}$ and $\alpha_{m_{n+1}}$. Equality (14) is due to the independence of C_ℓ's and inequality (15) is due to (13).

So we know that the statement holds for all $\epsilon \geq m_{n+1} = \frac{1}{n+2}$. And since $\lim_{n\to\infty} m_n = 0$, this concludes the lemma.

□

C Concentration Inequalities

Lemma 5 (Chernoff). *Let $X = \sum_{i=1}^{n} X_i$, where $X_i = 1$ with probability p_i and $X_i = 0$ with probability $1 - p_i$, and all X_i's are independent. Let $\mu = \mathbb{E}[X] = \sum_{i=1}^{n} p_i$. Then for $0 < \delta < 1$, $\mathbb{P}(X > (1+\delta)\mu) < e^{-\Omega(\delta^2 \mu)}$ and $\mathbb{P}(X < (1-\delta)\mu) < e^{-\Omega(\delta^2 \mu)}$.*

Lemma 6 (Poisson). *Let X be a Poisson random variable with rate μ. Then for $0 < \delta < 1$, $\mathbb{P}(X > (1+\delta)\mu) < e^{-\Omega(\delta^2 \mu)}$ and $\mathbb{P}(X < (1-\delta)\mu) < e^{-\Omega(\delta^2 \mu)}$.*

D Attack discussion

We discuss some practical aspects of the n-split attack described in Section 6.

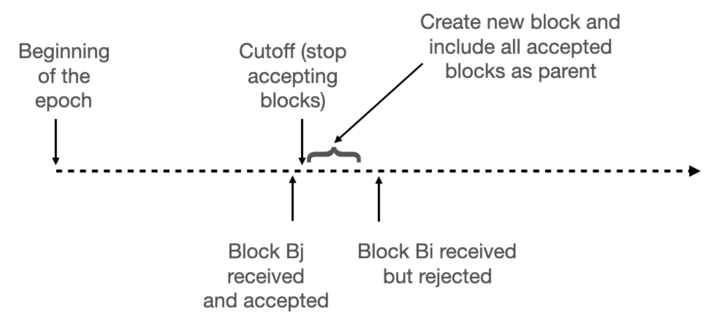
Rationality of the attack We note that this attack is detectable as everyone can see that blocks with the same proof of eligibility but different payload were created. In practice, this behaviour is slashable in Filecoin [1]. However, in Filecoin an adversary has the ability to spread its storage over multiple identities, i.e., create multiple identities that each possesses one unit of storage. For example in Filecoin the minimum unit of storage that can be pledged to the chain is 32 GiB. As of October 2022 the total storage pledged to the chain is around 18 EiB [4], hence an adversary that possesses 20% of the total power, i.e., 3.6 EiB could potentially "spread" its storage over $\frac{3.6\times10^{18}}{32\times10^{9}} \simeq 10^8$ different identities, that each possesses 32 GiB of storage. At each epoch, except with extremely small probability, the adversary will have a new "identity" elected to create a block (it is very unlikely for a miner with 32GiB of storage out of 18 EiB to be elected twice in a row). Assuming that each identity gets slashed and removed from the list of participants after equivocation, after performing the attack over 1000 epochs, the adversary will still have $9.9999\cdot10^7$ identities left out of 10^8 and will only be slashed $\frac{10^3}{10^8} = 10^{-5}$ of its total collateral. Note that it is not possible for any honest miner to know which identities belong to the adversary before the equivocation. Hence excluding equivocating participants from the protocol is not sufficient to prevent the attack.

Furthermore, we note that for the adversary to be slashed, a special transaction, *a fraud proof* transaction must be submitted on-chain by any participant. An adversary that is able to continually exclude honest blocks as is the case with the n-split attack may thus in practice never be slashed as no honest participant will get the opportunity to include the slashing transaction on-chain. This is why even when considering incentives and the slashing mechanism in place, this attack is still rational.

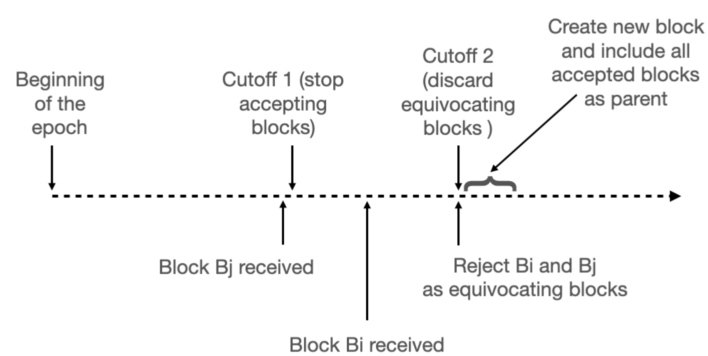
Network control In this attack, we assumed a powerful adversary that not only has the power to break ties in its favor, but has also full control of the network, as specified in Section 2.2. However we remark that for this attack to work, an adversary only needs limited power over the network. Specifically, the adversary needs to be directly connected to every participants but does not need to control the propagation time between honest participants, as we illustrate now.

In Filecoin honest miners will stop accepting blocks for an epoch after a cutoff time. For the split to happen the adversary could send each block $B_1, \ldots, B_n$ to each different participants $1, \ldots, n$ right before the cutoff, i.e., participant i receives block B_i from the adversary just before the cutoff time, ensuring block B_i is accepted by participant i. The adversary would need to know the propagation time between itself and each participant to do so, however this is easy to estimate. Since participant i, receives B_i just before the cutoff time, whatever the propagation delay between i and another honest participant j is, the adversary is guaranteed that j will received B_i from i *after* the cutoff time and hence that any honest miner $j \neq i$ will not accept B_i.

Furthermore we note that when participant j receives block B_i, after the cutoff, j will detect the equivocation as j already received block B_j from the adversary. However at that point, j has already created its block that includes B_j as a parent, hence it is too late for j to discard B_j due to equivocation. The mitigation that we propose in Section 7.2 changes this. A consistent broadcast protocol will ensure that after j received B_j from the adversary before the cutoff, j will wait for a "second" cutoff before forming its block and including B_j as a parent. When j receives B_i, j will detect the equivocation and decide not to include B_j (neither B_i), hence the attack is mitigated. This is illustrated in Figure 4.



(a) Current case.



(b) Case with a (simplified) consistent broadcast.

Fig. 4: The dashed arrow represents the arrow of time. The different cutoff and arrival time are marked with vertical arrows. The adversary ensures that j receives B_j right before the cutoff so j accepts B_j. In the first case, by the time j sees an equivocation, it is too late as B_j was already included as a parent. In the second case, assuming j receives B_i before the second cutoff, then j will discard B_j and not include it as a parent.