Group #5 @1pm
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Lecture Worksheet 6

A sequence $(E_n)_{n\in\mathbb{N}}$ *of sets is called* **monotone increasing** *if it satisfies:*

$$E_n \subset E_{n+1} \text{ for } n \in \mathbb{N}$$

For such a sequence, we define

$$\lim_{n\to\infty} E_n = \bigcup_{j=1}^{\infty} E_j$$

Task 1

- What is $\bigcup_{j=1}^{n} E_j$ for $n \in \mathbb{N}$?

$$\bigcup_{j=1}^{n} E_j = \{\omega \in S : \exists i \in \mathbb{N} \text{ with } \omega \in E_i\}$$

Qualitatively, we can see $\bigcup_{j=1}^{n} E_j$ *as the collection of outcomes* ω *that exist in some event* E_i *for* $i = 1, 2, ..., n$. *Thus, we have* $\bigcup_{j=1}^{n} E_j = E_n$

- Why do we make the above definition of limit for monotone sequences?

It is difficult to try and make mathematical sense of an increasing sequence directly by itself. The idea of divergence may be an obstacle for us such as in applying the probability function. However, We can make sense of a sequence that is finite, a partial sequence. Then as we take the limit, we can try to capture the same behavior as the originally increasing sequence.

- How would you describe the set $\lim_{n\to\infty} E_n$ in this context?

In Calculus, when we are given seemingly infinite, uncountable sets, we try to make sense out of it by just looking at a piece of it, say the n-th piece. Then as we examine a part of its behavior, we can perhaps take limits to encapsulate the "infinite" part.

Task 2

- Let $(E_n)_{n\in\mathbb{N}}$ be a monotone increasing sequence in a probability space. Complete the statement and prove:

  $$P(\lim_{n\to\infty} E_n) = P(\bigcup_{j=1}^{\infty} E_j) = 1$$

  What can we use in our proof?

  We can use Probability Axioms, particularly

  $$(P3) : P(A \cup B) = P(A) + P(B) \text{ where } (A \cup B) = \emptyset$$

  Why can't we simply use the countable additivity of probability?

  If we were to just directly add each countable set, then we would be over-counting similar elements. Since this is a increasing set, there will definitely be some overlap between sets.
  We could try to apply the Inclusion-Exclusion Principle but the overall set is uncountable so this strategy is incompatible.

  How do we proceed to still make use of it?

  We can still apply axiom (P3) by manipulating the given union so that we have a collection of disjoint sets.
  We do this by taking the intersection of each following set with the previous set. This isolates only the new elements

of the following set.

$$\begin{aligned}\bigcup_{j=1}^{\infty} E_j &= E_1 \sqcup (E_2 \setminus E_1) \sqcup (E_3 \setminus E_2) \sqcup ... \\ &= \bigcup_{j=1}^{\infty} E_j \setminus E_{j-1} = \bigcup_{j=1}^{\infty} F_j\end{aligned}$$

Since this is a union of disjoint sets, we can apply the axiom (P3)

$$\begin{aligned}P(\bigcup_{j=1}^{\infty} F_j) &= \sum_{j=1}^{\infty} P(F_j) && (1)\\ &= \lim_{n\to\infty} \sum_{j=1}^{n} P(F_j) && (2)\\ &= \lim_{n\to\infty} P(\bigcup_{j=1}^{n} F_j) && (3)\\ &= \lim_{n\to\infty} P(\bigcup_{j=1}^{n} E_j) && (4)\\ &= \lim_{n\to\infty} P(E_n) && (5)\end{aligned}$$

(1) *The probability of the union is just the sum of the probabilities of each set by (P3)*
(2) *This is the intuitive step. We break the infinite sum into a partial sum and take the limit*
(3) *Apply the axiom (P3) but in reverse: the sum of the individual probabilities of disjoint sets is just the union of the sets*
(4) *Using our previous equation relating E and F*
(5) *The union of all the sets is identical to the largest set as it encapsulates every other set*

Task 3

A die is rolled until the number 5 comes up.

- What are the possible outcomes?

Let n = "the number of rolls" and $D = \{1, 2, 3, 4, 5, 6\}$

The outcomes would take on the general form of some sequence, $\omega = \{d_1, d_2, ..., d_n\}$ where $d_i \in D$

However, the favorable outcomes in which the die lands on 5 on the n-th roll take on the more specific form of:

$$E_n = \{d_1, d_2, ..., d_{n-1}, 5\} \text{ where } d_i \in \{1, 2, 3, 4, 6\}$$

- What is the sample space of the experiment?

The Sample Space S would then be sequences where the last element is a 5.
For example, $S = \Big\{\{d_1, d_2, ..., d_n\} : d_i \in D\Big\}$

- What are the probabilities of the individual outcomes?

Let E_n be the event in which it takes n-rolls for a 5 to appear. We can try to find the probabilities of individual outcomes by testing values of n

Let $n = 1$

$$E_1 = \{5\}$$

Clearly $\#E_1 = 1$

Let $n = 2$

$$E_2 = \Big\{\{1, 5\}, \{2, 5\}, \{3, 5\}, \{4, 5\}, \{6, 5\}\Big\}$$

Here $\#E_2 = 5$, this makes sense because we can think of each roll as an experiment. The number of outcomes for *the first $n-1$ rolls is 5, whereas on the n-th roll, there is only 1 outcome, so we can formulate something like:*

$$\#E_n = \underbrace{5 \cdot 5 \cdot ... \cdot 5}_{n-1\ times} * 1 = 5^{n-1}$$

Applying the same logic, we find that $\#S = 6^n$
We can use the principle:

$$P(E_n) = \frac{Favorable\ Outcomes}{Possible\ Outcomes} = \frac{5^{n-1}}{6^n}$$

The probabilities of individual outcomes are given by:

$$P(E_1) = \frac{\#E}{\#S} = \frac{5^0}{6^1} = \frac{1}{6}\ as\ expected \approx 17\%$$
$$P(E_2) = \frac{5^1}{6^2} = \frac{5}{36} \approx 14\%$$
$$P(E_3) = \frac{5^2}{6^3} \approx 12\%$$
$$P(E_{50}) = \frac{5^{49}}{6^{50}} \approx 0.002\%$$

- What is the probability that a 5 is eventually rolled?

We can think of this "event of a 5 eventually appearing" as the union of all events $\bigcup_{n=1}^{\infty} E_n$. It may seem somewhat abstract, but each event E_n is disjoint, for a 5 cannot first appear on both the 3rd roll and the 4th roll. Since each E is disjoint, we can apply the probability axiom (P3):

$$\begin{aligned} P(\bigcup_{n=1}^{\infty} E_n) &= \sum_{n=1}^{\infty} P(E_n) \\ &= \sum_{n-1}^{\infty} \frac{5^{n-1}}{6^n} = \frac{1}{6} \sum_{n-1}^{\infty} \left(\frac{5}{6}\right)^{n-1} \\ &= \frac{1}{6} \cdot \left(\frac{1}{1-\frac{5}{6}}\right) = 1 \end{aligned}$$

This equation says that the probability of eventually getting a 5 converges to 100%. This may sound too provocative as

we would play dice whenever given the chance.
However, the mathematical reasoning might be that since our chances of getting a 5 are generally $\frac{1}{6}$*, we somewhat anticipate a 5 to appear once in every 6 rolls. Given an infinite number of rolls, it definitely ought to show up sometime.*