Group #5 @1pm
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Lecture Worksheet 17

Task 1

Consider the experiment consisting of flipping a coin, which has probability $p \in [0, 1]$ of yielding heads, a total of n times, ($n \in \mathbb{N}$). Denote by X the number of heads observed.

- What are the possible values of X?

  Looking at the extreme cases, we may see either all haeds or all tails, and of course any number in between.

$$X = 0, 1, ..., n$$

- What are the probabilities that X assume any of its values?

  Since there are two outcomes for each experiment, we can invoke a Binomial approach towards deriving a probability for each possible value of X.

$$P(X = j) = \binom{n}{j} p^j \, (1 - p)^{n-j}$$

- What is the average number of heads you expect to see and why?

  We expect to see heads about p% of the time, so in total there should be $n \cdot p$ heads.

- Compute the expectation of X. How much deviation from this number do you expect on average (if repeating the same experiment many times)?

  The expected value for X is given by:

$$\begin{aligned} E[X] &= \sum_{j=0}^{n} x_j \cdot \binom{n}{j} p^j \, (1 - p)^{n-j} \\ &= n \cdot p \end{aligned}$$

  This expected value actually matches the mean value. Therefore, we do not anticipate the expected value to deviate too *far from the mean since they are essentially the same. The deviation can be given by:*

$$E[X-\mu] = E[X] - \mu = n \cdot p - n \cdot p = 0$$

- How can you graphically render the informational content of the random variable X in a concise way?

  We can try plotting the various values of X for several experiments on a histogram.
  This gives us an idea of how X is distributed.

Task 2

Let $X\ B(n,p)$ for $p \in (0,1)$ and $n \in \mathbb{N}$

- What is the value of p which maximizes $P(X=k)$?

  Given that $P(X=k) = \binom{n}{k}\ p^k\ (1-p)^{n-k}$

  Like any function, we can treat the Probability Function for X as a function of p. Then we can take the derivative, set it equal to zero, and solve for p as a maximum point. This gives us:

$$\frac{d}{dp}P(X=k) = 0$$

$$\Rightarrow \binom{n}{k}\left[k\ p^{k-1}\ (1-p)^{n-k} - (n-k)\ p^k\ (1-p)^{n-k-1}\right] = 0$$

$$\Rightarrow k\ p^{-1} - (n-k)\ (1-p)^{n-k-1} = 0$$

$$\Rightarrow k\ (1-p) - (n-k)\ p = 0$$

$$\Rightarrow k - kp - np + kp = 0$$

$$\Rightarrow k - np = 0$$

$$\Rightarrow p = \frac{k}{n}$$

  This value for p would maximize the probability function $P(X=k)$

Task 3

Let $X\ B(n,p)$ for $p \in (0,1)$ and $n \in \mathbb{N}$

- What is the value of $\sum_{k=0}^{n} \binom{n}{k}\ p^k\ (1-p)^{n-k}$ and why?

  Plugging this summation into Mathematica, the result turns out to be 1 or 100%.

  The reasoning behind this outcome may be that the probability mass function for x is represented by the argument inside the summation. If we sum for all possible values of k in the probability function, then they all ought to add up to 1, for k has to assume one of its possible values.

- Use your knowledge about a binomial random variable (previous question) to show the validity of

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}\ x^k\ y^{n-k} \text{ for } x, y \geq 0$$

  Since we see that the probability function ought to sum up to be 1, we find that the LHS of the above equation ought to equal 1 somehow. This makes sense because the variables "x" and "y" are analogous to "p" and "1-p", respectively.

  The RHS looks like:

$$\sum_{k=0}^{n} \binom{n}{k}\ x^k\ y^{n-k} \Longleftrightarrow \sum_{k=0}^{n} \binom{n}{k}\ p^k\ (1-p)^{n-k}$$

  And the LHS looks like:

$$(x+y)^n = ((p)+(1-p))^n = (1)^n = 1 = \sum_{k=0}^{n} \binom{n}{k}\ p^k\ (1-p)^{n-k}$$

  Thus, taking the sum of x and y resembles the sum of p and 1-p, which is just 1, and 1 raised to the n-th power is just 1, which matches our value for the summation of the probability function.