Solutions to Review: Ch 14 & Ch 15

Note: Midterm 1 is on Tuesday (04/27/2021), covering chapters: 14, 15, 16, & 31

Objective:

- Destroy this upcoming Midterm 1. Let's gooooo!!!!

  Okay but for real tho, let's try to nail down the following:

  - ☐ Apply the **displacement function** and **Cons. of Energy** for a spring-mass
  - ☐ Finding the **resonant frequencies** (harmonics & overtones) for a guitar string.

Contextualizing the Formula Sheet (solutions) [10mins]

1) This equation represents **displacement** of the object as a function of **time**

$$x(t) = A\cos\omega t + \phi$$

2) The angular frequency ω can be expressed/rewritten in the following ways

$$\omega = \underbrace{\frac{2\pi}{T} = 2\pi f}_{\text{universal}} = \underbrace{\sqrt{\frac{k}{m}}}_{\text{spring}} = \underbrace{\sqrt{\frac{g}{\ell}}}_{\text{pendulum}} = \underbrace{\frac{\omega}{k}}_{\text{traveling wave}}$$

where k in the last equality is the **wave number**, NOT the **spring constant**.

3) The **total energy** E_{total} of an oscillating spring-mass system can be written as

$$E_{\text{total}} = U + K$$

However, the total energy can also be expressed as

$$E_{\text{total}} = U_{\text{max}} = K_{\text{max}} \qquad \text{where} \quad U_{\text{max}} = \frac{1}{2}kx^2 \quad \text{and} \quad K_{\text{max}} = \frac{1}{2}mv^2$$

This principle is called *Buy 1 Get 2 Free*

Group Activity (Leader - Student) [10mins]

Oscillating Spring-Mass

A $1.0\,\mathrm{kg}$ block oscillates on a spring with spring constant $20\,\mathrm{N/m}$. At $t = 0\,\mathrm{s}$, the block is $20\,\mathrm{cm}$ to the right of the equilibrium position and moving to the left at a speed of $100\,\mathrm{cm/s}$.

(a) Determine the period

(b) Determine the amplitude

Solution

a) We are given the mass m and the spring constant k. We can use the 2 equations for angular frequency ω and algebraically solve for the period T

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

Solving for T, we get

$$\implies T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{(1.0\,\mathrm{kg})}{(20\,\mathrm{N/m})}} = 1.40\,\mathrm{s}$$

b) Using Kinematics (displacement function)

In general form, the displacement function is given by

$$x(t) = A\cos(\omega t + \phi)$$

Taking the derivative, we also have the velocity function

$$v(t) = -A\omega\sin(\omega t + \phi)$$

We can plug in the given initial conditions:

At $t = 0$, the block is $20\,\mathrm{cm}$ to the right of the origin & traveling to the left at $100\,\mathrm{cm/s}$.

and solve for the unknown variables: amplitude A and phase-shift ϕ

$$x(t=0) = A\cos(\phi) = +20\,\mathrm{cm}$$
$$v(t=0) = -A\omega\sin(\phi) = -100\,\mathrm{cm/s}$$

Now we have 2 equations and 2 unknowns. We can solve for 1 unknown and plug it into the other.

Consider this cool algebra shortcut:

$$\frac{v(t=0)}{x(t=0)} = \frac{-A\omega sin(\phi)}{A\cos(\phi)} = -\frac{100\,\mathrm{cm/s}}{20\,\mathrm{cm}} \implies \omega\tan(\phi) = 5\,\mathrm{s}^{-1}$$

Solving for ϕ, we get

$$\phi = \tan^{-1}\left[\frac{5}{\omega}\right] = \tan^{-1}\left[5\sqrt{\frac{m}{k}}\right] = \tan^{-1}\left[5\sqrt{\frac{(1.0\,\mathrm{kg})}{(20\,\mathrm{N/m})}}\right] = 0.841\,\mathrm{rad}$$

Now that we have ϕ, we can solve for A. There's multiple ways to do this, here's one way:

$$x(t=0) = A\cos(\phi) = 20\,\mathrm{cm} \implies A = \frac{20\,\mathrm{cm}}{\cos(\phi)} = \frac{20\,\mathrm{cm}}{\cos(0.841\,\mathrm{rad})} = 30\,\mathrm{cm}$$

b) Using Conservation of Energy

There's many ways that we can express Conservation of Energy. Here's the most common way:

$$E_{\text{initial}} = E_{\text{final}}$$

However, in this case, we want to use this version of Cons. of Energy:

$$E_{\text{total}} = U + K = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

and we also have (look at Content Review):

$$E_{\text{total}} = U_{\text{max}} = \frac{1}{2}kA^2$$

Rearranging the equations and solving for the amplitude A, we get

$$\begin{aligned}
\implies U_{\text{max}} &= E_{\text{total}} \\
\frac{1}{2}kA^2 &= \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \\
\implies A &= \sqrt{x^2 + \frac{m}{k}v^2} \\
&= \sqrt{(20\,\text{cm})^2 + \frac{1.0\,\text{kg}}{20\,\text{N/m}}(100\,\text{cm/s})^2} = 30\,\text{cm}
\end{aligned}$$

Group Activity (Student - Leader) [5mins]

Guitar String

A guitar string is $90\,\mathrm{cm}$ long and has a mass of $3.16\,\mathrm{g}$. From the bridge to the support post, the length of the string is $60\,\mathrm{cm}$ and the string is under a tension of $520\,\mathrm{N}$.

- Determine the fundamental frequency and the first two overtones.

Solution

We can find the linear mass density μ by dividing the string's total mass by the string's total length.

$$\mu = \frac{m}{\ell} = \frac{0.003\,16\,\mathrm{kg}}{0.90\,\mathrm{m}} = 3.51 \times 10^{-3}\,\mathrm{kg/m^3}$$

NOTE: it would be incorrect to use the smaller length $60\,\mathrm{cm}$ since we don't know the corresponding mass.

Now that we have μ, we can find the string's velocity using

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{520\,\mathrm{N}}{3.51 \times 10^{-3}\,\mathrm{kg/m^3}}} = 385\,\mathrm{m/s}$$

It is safe to conclude that the velocity of the traveling waves on the string is CONSTANT, since v only depends on the tension T and the mass density μ, which are both constant.

The resonant frequencies for waves on a string (as well as for open tubes) are given by

$$f_n = n\,\frac{v}{2L} \quad \text{where} \quad n = 1, 2, 3, \dots$$

The fundamental frequency is the lowest resonant frequency $(n = 1)$

$$f_1 = 1 \cdot \frac{v}{2L} = \frac{385\,\mathrm{m/s}}{2 \cdot 0.60\,\mathrm{m}} = 321\,\mathrm{Hz}$$

The first two overtones correspond to $n = 2$ and $n = 3$, respectively.

There's a shortcut to finding these frequencies:

$$f_2 = 2 \cdot f_1 = 2 \cdot 321\,\mathrm{Hz} = 642\,\mathrm{Hz}$$

$$f_3 = 3 \cdot f_1 = 3 \cdot 321\,\mathrm{Hz} = 963\,\mathrm{Hz}$$

KEY: Resonant frequencies are integer multiples of the fundamental frequency.