Group #5 @1pm
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Lecture Worksheet 7

Task 1

Consider two urns containing 5 red and 10 blue balls, and 6 red and 4 blue balls, respectively. Flip a coin and draw a ball from the first urn, if heads turns up; draw it from the second, if tails turns up.

- What is the probability that the ball drawn is blue?

We can think of flipping a coin and picking out of the urn as 2 different experiments. Then we can apply the general principle of favorable over total outcomes to each experiment and multiply accordingly. Since there are 2 ways the experiments can go, we add the two different paths.

$$P(Blue) = \underbrace{\frac{1}{2}}_{\text{P(Heads)}} \cdot \overbrace{\frac{10}{15}}^{\text{P(Blue in Urn 1)}} + \underbrace{\frac{1}{2}}_{\text{P(Tails)}} \cdot \overbrace{\frac{4}{10}}^{\text{P(Blue in Urn 2)}}$$

$$= \frac{16}{30} \approx 53\%$$

Task 2

Consider now a probability space S and two events $E \subset F$

- Find an interpretation (in terms of frequency of occurrence) of the ratio $\frac{P(E)}{P(F)}$

E occurs $P(E)$ of the time and F occurs $P(F)$ of the time, then E occurs $\frac{P(E)}{P(F)}$ of the time that F occurs.

Assuming a countable sample space S, where $\#S = N$, we can take a quantitative approach.

$$Number\ of\ occurrences\ of\ E = P(E) * N$$

$$Number\ of\ occurrences\ of\ F = P(F) * N$$

Now, we can try to piece together these quantities to form

$\frac{P(E)}{P(F)}$, which happens to just be the number of occurrences of E divided by that of F.

$$\frac{\text{Number of occurrences of } E}{\text{Number of occurrences of } F} = \frac{P(E)}{P(F)}$$

Intuitively, we can interpret this fraction as the ratio of E outcomes to F outcomes. Given that F occurs, there is some probability that E may also occur, this is just given by said ratio.
Remark: since it is the ratio, the sample size N cancels out. Interestingly, the occurrence of E given F is independent on the overall sample size, almost as if $\#F$ replaces $\#S$.

Task 3

Consider now F_1, F_2 with $F_1 \cap F_2 = \emptyset$ and $F_1 \cup F_2 = S$

- Show that, for any $E \subset S$, one has that

$$P(E) = P(E \cap F_1) + P(E \cap F_2) = \frac{P(E \cap F_1)}{P(F_1)} P(F_1) + \frac{P(E \cap F)}{P(F_2)} P(F_2)$$

Typically, we have always approached finding $P(E)$ in terms of frequency of occurrence. However, it is not always feasible to compute $\frac{\#E}{\#S}$ as we are not always able to directly get $\#E$ or $\#S$. Instead we may be given bits of information with which we can tie together to obtain the equivalent.

Since F_1 and F_2 make up the entire sample space, we may find $P(E)$ by categorizing its elements into those in common with F_1 and F_2 and computing their respective probabilities.

$$P(E) = P(E \cap F_1) + P(E \cap F_2)$$

Now when it comes to finding the probability of the intersection of 2 events, we can apply the general principle of favorable outcomes to possible outcomes.

$$P(E \cap F_1) = \frac{\#(E \cap F_1)}{\#S}$$

It follows that:

$$P(E \cap F_1) + P(E \cap F_2) = \frac{\#(E \cap F_1)}{\#S} + \frac{\#(E \cap F_2)}{\#S}$$

The next step seems confusing as it seems redundant to multiply and divide by $P(F)$. However, there is a mathematical advantage to rewriting the expression.

$$\frac{P(E \cap F_1)}{P(F_1)} P(F_1) + \frac{P(E \cap F)}{P(F_2)} P(F_2)$$

$$\Rightarrow \frac{\frac{\#(E \cap F_1)}{\#S}}{\frac{\#F_1}{\#S}} \frac{\#F_1}{\#S} + \frac{\frac{\#(E \cap F_2)}{\#S}}{\frac{\#F_2}{\#S}} \frac{\#F_2}{\#S} \tag{1}$$

$$\Rightarrow \frac{\#(E \cap F_1)}{\#F_1} \frac{\#F_1}{\#S} + \frac{\#(E \cap F_2)}{\#F_2} \frac{\#F_2}{\#S} \tag{2}$$

$$\Rightarrow P(E|F_1)P(F_1) + P(E|F_2)P(F_2) \tag{3}$$

(1): We rewrote the expression in terms of frequency of occurrences.
(2): Simplifying
(3): We find that the quantity of $\frac{P(E \cap F)}{P(F)}$ is actually known as the **Conditional Probability**. *In this case, it denotes the probability of E occurring given F has occurred.*

- Discuss the relationship of the above formula to the problem of Task 1 and its solution. In particular, what are E, F_1, and F_2 in that problem?

  In the context of Task 1,

  $$E = \textit{Event of Drawing a Blue}$$
  $$F_1 = \textit{Event of Getting Heads}$$
  $$F_2 = \textit{Event of Getting Tails}$$

Using these variables to compute $P(Blue)$, we get

$$
\begin{aligned}
P(Blue) = P(E) &= \frac{P(E \cap F_1)}{P(F_1)} P(F_1) + \frac{P(E \cap F)}{P(F_2)} P(F_2) \\
&= \frac{1/2 * 10/15}{1/2} \cdot \frac{1}{2} + \frac{1/2 * 4/10}{1/2} \cdot \frac{1}{2} \\
&= \frac{1}{2} \cdot \frac{10}{15} + \frac{1}{2} \cdot \frac{4}{10} = \frac{16}{30} \approx 53\%
\end{aligned}
$$

Although it looks slightly different this matches our answer for Task 1.

- Use Task 2 to interpret the ratios in the above formula

Similarly to Task 2, the formula above can be reduced into terms of frequency of occurrence, where $\#S = N$

$$
\begin{aligned}
&\frac{P(E \cap F_1)}{P(F_1)} P(F_1) + \frac{P(E \cap F_2)}{P(F_2)} P(F_2) \\
&= \frac{\text{\# of occurrences of E and } F_1}{\text{\# of occurrences of } F_1} P(F_1) + \frac{\text{\# of occurrences of E and } F_2}{\text{\# of occurrences of } F_2} P(F_2)
\end{aligned}
$$

In order to obtain all of E, we have to put together its pieces. In this case, we only have to worry about 2 parts, one of which is a subset of F_1 and the other a subset of F_2.

We divide by the respective number of occurrences. This is because we are trying to find the probability of getting blue as if we've conducted the experiment many, many times.

We divide given that F_1 or F_2 occurs and then we multiply the sum by its probability weights ($P(F_1)$ and $P(F_2)$) to obtain the relative frequencies.