Group #5 @1pm
Mimi Morales, Yiyi Zhu, Vi Pham, Brian Fong May 30^{th}, 2019

Lecture Worksheet 22

Task 1

- What is the probability density function of a normal random variable X with a mean μ and variance σ^2?

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- Let $X \sim N(\mu, \sigma^2)$ and show by direct computation that $\frac{X-\mu}{\sigma} \sim N(0,1)$

  We want to show that the expected value is zero. Using properties of expected value, we have

$$\begin{aligned} E[\frac{X-\mu}{\sigma}] &= \frac{1}{\sigma}\Big[E[X] - E[\mu]\Big] \\ &= \frac{1}{\sigma}\Big[\mu - \mu\Big] \\ &= \frac{1}{\sigma} \cdot 0 = 0 \end{aligned}$$

  We want to show that the variance is equal to 1. Using properties of variance, we have

$$\begin{aligned} Var(\frac{X-\mu}{\sigma}) &= \frac{1}{\sigma^2} Var(X-\mu) \\ &= \frac{1}{\sigma^2} Var(X) \\ &= \frac{1}{\sigma^2}\sigma^2 = 1 \end{aligned}$$

  Since $\frac{X-\mu}{\sigma}$ *corresponds to an expectation of* 0 *and variance of* 1*, we have that* $\frac{X-\mu}{\sigma} \sim N(0,1)$

- State the definition of expectation and variance for a continuous random variable X and compute $E(X)$ and $Var(X)$ for $X \sim N(0,1)$. What are the expectation and variance if $X \sim N(\mu, \sigma^2)$ for general μ and σ?

For a continuous random variable $X \sim N(\mu, \sigma^2)$
The definition of the expected value is given by

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

The definition of the variance is given by

$$Var(X) = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x) dx = \int_{-\infty}^{\infty} (x - E[X])^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Now for a continuous random variable $X \sim N(0, 1)$
The expectation is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 0$$

Since the integrand consists of a product with anti-symmetry and consequently integrated from $-\infty$ to $+\infty$, we get zero.
The variance is

$$Var(X) = \int_{-\infty}^{\infty} (x - E[X])^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_{-\infty}^{\infty} (x)^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$$

Task 2

A radar unit is used to measure speeds of cars on a freeway. The speeds are normally distributed with a mean of $65mph$ and a standard deviation of $5mph$.

- What is the probability that a car picked at random is travelling at more than $75mph$?

  Let X be a continuous random variable denoting the speed of cars on a freeway. We're given that $\mu = 65mph$ and $\sigma = 5mph$.

  We can create a new random variable Z such that $Z \sim N(0, 1)$
  Let $Z = \frac{X-\mu}{\sigma}$. Now Z is a standard normal distribution

  The CDF of a standard normal distribution is given by:

$$F_Z(X) = P(Z \leq z) = \int_{-\infty}^{z} f_Z(y) dy = \int_{-\infty}^{z} \frac{y}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dy$$

The probability that a car picked at random is traveling over 75mph is given by

$$P(X > 75) = 1 - P(Z \leq z = \frac{75 - 65}{5} = 2) = 1 - F_Z(2)$$
$$= 1 - \int_{-\infty}^{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$
$$\approx 2.28\%$$

- What is the probability that a car picked at random is travelling at a speed between 60mph and 70mph?

  The probability is given by

$$P(60 < X < 70) = P(-1 < X < 1) = \int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$
$$\approx 68.3\%$$

Task 3

Admission to certain University is determined by a national test. The scores on this test are normally distributed with a mean of 500 and a standard deviation of 100. Tom wants to be admitted to this university and he knows that he must score better than at least 70% of the students who took the test. Tom takes the test and scores 585.

- What is the score that guarantees admission on average? Will he be admitted to this university?

  Let X be a continuous random variable denoting the test score of a random student.
  We are given that $\mu = 500$ and $\sigma = 100$.
  Now let Z be another random variable so that $Z = \frac{X-\mu}{\sigma}$ and $Z \sim N(0,1)$.
  If we set up the equation

$$P(X \leq x) = F_X(x) = \int_{-\infty}^{x} \frac{y}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dy$$

Then we let the LHS equal to 70%*, then we try to solve for* x *in the RHS.*

$$\begin{aligned} P(Z \le z) &= \int_{-\infty}^{z} \frac{y}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dy \\ \implies 70\% &= \int_{-\infty}^{z} \frac{y}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dy \\ \implies z &= 0.524 \qquad\qquad \textit{using Mathematica} \end{aligned}$$

Translating our value of $z = 0.542$ *into test scores, we just solve for* X *in the equation*

$$z = 0.542 = \frac{x - 500}{100}$$

$$\implies x = x_{cutoff} = 552$$

This tell us that given a limit of 70% *on test scores, we can find the cut-off score for Tom, which turns out to be* 552*. Since Tom scored* 585*, his score is greater than the cut-off and thus he will be admitted.*