Group #5 @1pm
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Lecture Worksheet 13

Task 1

- State the definition of conditional probability of an event E given an event F

  The mathematical definition is given by:

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

  where $P(F) \neq 0$, *otherwise if P(F) were to be zero, then it'd be impossible and conditioning on an "impossible" event is absurd.*

- Explain in lay terms when conditional probability is a useful tool

  We can think of conditional probability as a mathematical tool to somewhat categorize our sample space. This helps to derive identities for the probabilities of certain events based on other events.
  Knowing that some other event(s) has occurred, then the desired probability may have been influenced so that it's easier to be computed.

- Prove that $P(\cdot|F) : 2^S \to \mathbb{R}, E \mapsto P(E|F)$ is itself a probability function

  We can first list out the appropriate axioms concerning a probability function:

  i) $0 \leq \frac{P(E \cap F)}{P(F)} \leq \frac{P(F)}{P(F)} = 1$

  ii) $P(S|F) = \frac{P(S \cap F)}{P(F)} = 1$

$$\textit{iii)}\ P(\bigcup_{j=1}^{\infty} E_j|F) = \frac{P(\bigcup_{j=1}^{\infty} E_j \cap F)}{P(F)}$$

$$= \frac{P(\bigcup_{j=1}^{\infty} (E_j \cap F))}{P(F)}$$

$$= \frac{\sum_{j=1}^{\infty} P(E_j \cap F)}{P(F)} \qquad \textit{Since } E_j \cap F \textit{ are disjoint}$$

$$= \sum_{j=1}^{\infty} \frac{P(E_j \cap F)}{P(F)} = \sum_{j=1}^{\infty} P(E_j|F)$$

Therefore, since the function obeys the 3 main probability axioms, $P(\cdot|F)$ is a probability function.

Task 2

- State Bayes' Formula for a probability Q conditioning a generic event E on the occurrence of a fixed event F.

$$Q(E) = Q(E|F) \cdot Q(F) + Q(E|F^c) \cdot Q(F^c)$$

- Now assume that, for fixed F, we define $Q(E) = P(E|F)$ for any event E. Given an event G, show that

$$Q(E|G) = P(E|G \cap F)$$

We can prove this by pushing the definition of Conditional

Probability and simplifying the result:

$$
\begin{aligned}
Q(E|G) &= \frac{Q(E\cap G)}{Q(G)} && \textit{by def. of Cond. Prob.}\\
&= \frac{P(E\cap G|F)}{P(G|F)} && \textit{by def. of } Q(E)\\
&= P(E\cap G|F)\cdot\Big(P(G|F)\Big)^{-1}\\
&= \frac{P(E\cap F\cap G)}{P(F)}\cdot\left(\frac{P(F\cap G)}{P(F)}\right)^{-1} && \textit{by def. of Cond. Prob.}\\
&= \frac{P(E\cap F\cap G)}{P(F\cap G)}\\
&= P(E|F\cap G) && \textit{by def. of Cond. Prob.}
\end{aligned}
$$

- Rewrite Bayes' Formula for the (conditional) probability $Q(\cdot) = P(\cdot|F)$ in terms of the probability P and the formula you derived

Baye's Formula from part 1, we have:

$$Q(E) = Q(E|F)\cdot Q(F) + Q(E|F^c)\cdot Q(F^c)$$

We can rewrite this equation in terms of P to obtain a general form to condition a conditional probability. However, since we conditioned on F in part (a), we get the not as useful result:

$$
\begin{aligned}
Q(E) = P(E|F) &= P(E|F\cap F)\cdot P(F|F) + P(E|F\cap F^c)\cdot P(F^c|F)\\
&= P(E|F)\cdot 1 + P(E)\cdot 0 = P(E|F)
\end{aligned}
$$

However if we change our equation from part(a) to a condition on an event G, then we get something like:

$$
\begin{aligned}
Q(E) &= Q(E|G)\cdot Q(G) + Q(E|G^c)\cdot Q(G^c)\\
&= P(E|F\cap G)\cdot P(G|F) + P(E|F^c\cap G^c)\cdot Q(G^c|F)
\end{aligned}
$$

Task 3

- There are $k+1$ coins in a box. When flipped, the i-th coin will turn up heads with probability $\frac{i}{k}$, $i = 0, 1, ..., k$. A coin is randomly selected from the box and is then repeatedly flipped. If the first n flips all yield heads, what is the conditional probability that the $(n+1)st$ flip will also be heads?

Let's define

$$E = \text{ event of n heads}$$

$$F = \text{ event of n+1 heads}$$

$$C = \text{ event of getting the i-th coin} = \frac{1}{k+1}$$

$$C_i \text{ event of getting heads with the i-th coin} = \frac{j}{k}$$

We can find the desired probability by first conditioning on all the different coins and their respective probabilities.

$$P(E) = \Big(P(C_1)\Big)^n P(C) + \Big(P(C_2)\Big)^n P(C) + ... + \Big(P(C_k)\Big)^n P(C)$$

$$= P(C)\Big[\Big(P(C_1)\Big)^n + \Big(P(C_2)\Big)^n + ... + \Big(P(C_k)\Big)^n\Big]$$

$$= P(C) \cdot \sum_{i=0}^{k} \Big(P(C_i)\Big)^n$$

$$= \frac{1}{k+1} \cdot \sum_{i=0}^{k} \left(\frac{i}{k}\right)^n$$

It follows that:

$$P(F) = P(C)\Big[\Big(P(C_1)\Big)^{n+1} + \Big(P(C_2)\Big)^{n+1} + ... + \Big(P(C_k)\Big)^{n+1}\Big]$$

$$= P(C) \cdot \sum_{i=0}^{k} \Big(P(C_i)\Big)^{n+1}$$

$$= \frac{1}{k+1} \cdot \sum_{i=0}^{k} \left(\frac{i}{k}\right)^{n+1}$$

Then the desired probability is given by:

$$P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{P(F)}{P(E)}$$

$$= \frac{\frac{1}{k+1} \cdot \sum_{i=0}^{k} \left(\frac{i}{k}\right)^{n+1}}{\frac{1}{k+1} \cdot \sum_{i=0}^{k} \left(\frac{i}{k}\right)^{n}}$$

$$= \frac{\sum_{i=0}^{k} \left(\frac{i}{k}\right)^{n+1}}{\sum_{i=0}^{k} \left(\frac{i}{k}\right)^{n}}$$