Group #5 @1pm
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Lecture Worksheet 21

Task 1

- A random variable X is called continuous random variable if there is a probability density function: $f_X : \mathbb{R} \mapsto [0, \infty)$, which is integrable and satisfies

$$P(X \in B) = \int_B f_X(x)dx \text{ for } B \subset \mathbb{R}$$

- The cumulative distribution function F_X of X is given by $F_X(x) = P(X \leq x)$ for $x \in \mathbb{R}$. How can this function be written in terms of the corresponding probability density function f_X?

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

- List properties of f_X and F_X. What is the relation between f_X and F_X?

Since F_X composes of a probability function, it should comply with the probability axioms such as summing up to 1 for all x:

$$F_X(\infty) = \int_{-\infty}^{\infty} f_X(t)dt = 1$$

where t is just a dummy variable for integration.

We can relate F_X and f_X by doing:

$$f_x = F'(x) = \frac{d}{dx}F$$

Task 2

Fill in the table

Discrete Random Variable	Continuous Random Variable
probability mass function p_X	probability density function f_X
cumulative mass function F_X	cumulative density function F_X
$F_X(x) = P(X \leq x) = \sum\limits_{\{i \mid x_i < x\}} p_X(x_i)$	$F_X(x) = P(X \leq x) = \int\limits_{-\infty}^{x} f(x)dx$
$E(X) = \sum\limits_i x_i p_X(x_i)$	$E(X) = \int\limits_{-\infty}^{\infty} x f_X(x)dx$
$Var(X) = E[X^2] - E[X]^2$	$Var(X) = \int\limits_{-\infty}^{\infty} (x - E[X])^2 f(x)dx$
M_X is defined by $F_X(M_X) = 1/2$	M_X is defined by $F_X(M_X) = 1/2$
$P(a \leq X \leq b) = \sum\limits_{\{i \mid a < x_i < b\}} p_X(x_i)$	$P(a \leq X \leq b) = \int\limits_a^b f_X(x)dx$

- Why are the words *mass* and *density* function for discrete and continuous random variables, respectively?

  Given a continuous variable, it is impossible to assign a discrete number to every possible value of the variable since there are infinitely many possibilities.

  Instead we can indicate the probability of the continuous random variable in units of probability per interval. Then in order to isolate the probability we would multiply the density by the interval.

  Now the reason why we call this "density" is because naturally we think of quantities that represent some unit per some other unit as a sort of density. When we refer to an object as being "dense", we typically attribute it's mass distribution to being very concentrated.

  The mass of an object takes on a discrete number, as opposed to an infinite interval of numbers. We can define individual, finite points. Therefore, we define the probability functions for discrete variables analogously to the mass of an object, and continuous functions analogous to the density of the object.

- What pattern do you observe in the table?

We see that the continuous version of the formulae essentially convert the summations from the discrete version into integrals instead. Additionally, it also translates points of interest into intervals of interest.

Task 3

- How would you define a random variable Y so that it is uniformly distributed over the interval $[a, b]$ for $-\infty < a < b < \infty$? Indicate its probability density function $f_Y : \mathbb{R} \mapsto \mathbb{R}$

  Imagine we are at In-N-Out and we just ordered some french fries. The cashier makes it known that our order will be ready within 5 minutes. Let Y represent the minutes it takes for our order to be ready. Therefore, we can describe Y as:

$$f_Y(t) = \begin{cases} 1 & 0 < t < 5 \\ 0 & \textit{otherwise} \end{cases}$$

- A stick of length 1 is split at a point X that is uniformly distributed over $(0, 1)$. Determine the expected length $L_p(X)$ of the piece that contains the point $p \in (0, 1)$.

  Firstly, let's define the length of the piece of stick that contains the point as a piece-wise function:

$$L_p(X) = \begin{cases} X & X > p \\ 1 - X & X < p \end{cases}$$

  The expected value is then given by:

$$\begin{aligned} E[L_p(X)] &= \int_0^1 L_p(X) \\ &= \int_0^p (1 - X)dX + \int_p^1 XdX \\ &= \frac{1}{2} + p(1 - p) \end{aligned}$$

NOTE: The expected value reaches it's maximum when $p = 1/2$