Group #5 @1pm
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**Lecture Worksheet 10**

# Task 1

- Discuss in the group your personal understanding of what independence of events should mean

  *We all thought of something along the lines of the relationship where the outcome of one event does not affect the outcome of the other event(s).*

- Give the mathematical definition of independence for events E,F of a probability space and discuss how much it captures your understanding

  *The definition of independence concerning probability of events E,F is given by:*

  $$P(E|F) = P(E) \text{ or } P(F|E) = P(E)$$

  *Our initial guess somewhat aligns with the mathematical. In the first equation, the probability $P(F)$ plays no role.*

- Consider 3 different experiments and, for each, give an example of two independent and two dependent events. First try and find such pairs of events by guessing based on your intuitive understanding and then verify your guess by using the mathematical definition

  *Experiment: Buying 1 ticket for Lottery A and 1 ticket for Lottery B and seeing if we won*

  **E: the event of winning Lottery A**
  **F: the event of winning Lottery B**

  *Assuming the lotteries aren't associated with each other, we find that the outcome of Lottery A/B does not affect the outcome of Lottery B/A so we conclude $E, F$ are independent.*

  *We can try to attach probabilities to these events. Say the winning ticket is chosen randomly from a pile of 100 tickets for*

*each lottery.*
*Then we can verify that the math tells the same story:*
*Consider* $P(E) = \frac{1}{100}$ *and* $P(F) = \frac{1}{100}$

$$P(E \cap F) = \frac{1}{10000} = P(E)P(F)$$

*Now if let's consider that Lottery A and Lottery B are grouped into 1 big jackpot with a total of 2 winning tickets and 200 total tickets.*
*Then the probabilities become*
$P(E) = \frac{2}{200}$ *and* $P(F) = \frac{2}{200}$
*However if we try to find the probability of getting both winning tickets,*

$$P(E \cap F) = \frac{2}{200}\frac{1}{199} \neq P(E)P(F)$$

*Mathematically, we find that* $E, F$ *are now dependent. This makes sense, if we draw the winning ticket for Lottery A, then the total pile has 1 less ticket and also 1 less winning ticket.*

*Experiment: Checking the weather, and drinking iced or hot coffee:*

**E: the event of raining in Tokyo**
**F: the event of getting hot coffee**

*Almost immediately we would associate* $E, F$ *as being independent. We would conclude the weather in another country would have practically no influence on what we have for lunch.*

**G: the event of raining in Irvine**
**H: the event of getting hot coffee**

*Personally, we would prefer to have hot coffee over iced on a rainy day. Thus we may associate the relationship between* $G, H$

*as dependent.*

*Still, this is just reasoning from our intuition. If we did have numerical probabilities attached to these events then it is still possible that the math depicts a dependence relation counter to our first guess.*

*Experiment: Going to a classroom building and asking students if their major is Math*

**E: the event of going to Humanities classroom**
**F: the event of people answering "yes"**

*Going into a Humanities classroom yields a more likely result that we will limit our sample to students with Humanities-related majors. Thus $E, F$ are dependent as the outcome of F depends on E.*

**G: the event of going to Anteater Learning Pavilion (ALP)**
**H: the event of people answering "yes"**

*Various classes are held at ALP, math and non-math related, so our sample does not have an obvious bias towards a certain subject. Thus, $G, H$ may be independent events.*

## Task 2

- Show that, if E,F are independent events of a probability space, so are

$$E^c, F \text{ as well as } E, F^c \text{ and } E^c, F^c$$

*Assuming E,F are independent, to prove that $E^c, F$ are independent, we want to show:*

$$P(E^c \cap F) = P(E^c)P(F)$$

*In order to do so, we make use of the identity:*

$$P(F) = P(F \cap E) + P(F \cap E^c)$$

*Starting with the RHS:*

$$\begin{aligned} P(E^c)P(F) &= (1 - P(E))P(F) \\ &= P(F) - P(F)P(E) \\ &= [P(F \cap E) + P(F \cap E^c)] - P(E \cap F) \\ &= P(F \cap E^c) \end{aligned}$$

*Now we want to show that $E, F^c$ are independent:*

$$P(E \cap F^c) = P(E^c)P(F)$$

*Starting with the RHS:*

$$\begin{aligned} P(E)P(F^c) &= P(E)(1 - P(F)) \\ &= P(E) - P(E)P(F) \\ &= [P(E \cap F) + P(E \cap F^c)] - P(E \cap F) \\ &= P(E \cap F^c) \end{aligned}$$

*Now to show that $E^c, F^c$ are independent:*

$$P(E^c \cap F^c) = P(E^c)P(F^c)$$

*Starting with the RHS:*

$$\begin{aligned}
P(E^c)P(F^c) &= (1 - P(E))(1 - P(F)) \\
&= 1 - P(F) - P(E) + P(E)P(F) \\
&= 1 - [P(F \cap E) + P(F \cap E^c)] - [P(E \cap F) \\
&+ P(E \cap F^c)] + P(E \cap F) \\
&= 1 - P(F \cap E^c) - P(E \cap F^c) - P(E \cap F) \\
&= 1 - [P(F \cap E^c) + P(E \cap F^c)] - P(E \cap F) \\
&= 1 - [P(E \cup F)] \\
&= (P(E \cup F))^c = P(E^c \cap F^c)
\end{aligned}$$

*Above, we use the identity:*

$$P(E \cup F) = P(E \cap F^c) + P(F \cap E^c) + P(E \cap F)$$

*And for the last line, we apply the definition of the complement and DeMorgan's Law.*

*Therefore, given that $E, F$ are independent, we have that $E, F^c$ and $E^c, F$, and $E^c, F^c$ are all independent pairs as well.*

## Task 3

Consider a biased coin for which heads comes up with probability $p \neq 0.5$

- Try to find a way to use the coin in a way as to simulate the outcome of a fair coin.

*We may try to equalize the probabilities of winning/losing the coin toss by weighing the results differently depending on heads or tails.*

*Let* $P(Heads) = p$ *and* $P(Tails) = 1 - p$

*Assuming that* $p > 0.5$ *(we can account for tails by switching the inequality).*

*To create a fair game, we can wager that for every result of heads, we have an additional probability* $q$ *of accepting the result of heads (ending the bet) or denying the result and continue flipping. If the coin lands on tails, then we accept the result and the bet ends.*

*The question now is how to tailor* $q$ *so that given* $p$*, we can have the* $P(Heads) = P(Tails)$

$$E = \text{the event of getting heads (and keeping) heads}$$

$E_n$ = *the event of getting (and keeping) heads on the n-th trial*

*The probability is given by:* $P(E_n) = \left[p \cdot (1-q)\right]^{n-1} \cdot pq$

*The framework of our bet is that we will continue flipping the coin until we accept the result of heads (depending on* $q$*) or until the coin lands on tails.*

*We can express the result of keeping heads as the union of all possible* $n$

$$P(E) = P(\bigcup_{n}^{\infty} E_n)$$

*Since these events are all disjoint, we can rewrite the probability of the union as the sum of all the individual probabilities.*

$$P(E) = \sum_{n}^{\infty} E_n = \sum_{n}^{\infty} \left[p \cdot (1-q)\right]^{n-1} \cdot pq$$

*Here this is a geometric series so the limit as $n \to \infty$ is given by:*

$$P(E) = \frac{pq}{1 - \big[p \cdot (1-q)\big]} = \frac{1}{2}$$

*We set this equation equal to $\frac{1}{2}$ for a fair game.*

*Now we can solve for $q$ in terms of $p$*

$$q = \frac{1-p}{p} \Leftrightarrow \frac{P(Heads)^c}{P(Heads)}$$

*This result resembles the mathematical definition for "the odds" of an event, where we take the ratio of the probability of the event and its complement.*