Group #5 @1pm
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Lecture Worksheet 22

Task 1

- A random variable X is called exponentially distributed with parameter $\lambda \in (0, \infty)$ in short $X \sim Exp(\lambda)$, iff

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

- Verify that f_X is indeed a probability density function

  According to one of the probability axioms, if we integrate throughout the probability space, the integral should come out to be 1.

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_0^{\infty} f_X(x)dx = \int_0^{\infty} \lambda e^{-\lambda x} dx = 1$$

- Compute expectation and variance of $E \sim Exp(\lambda)$. What interpretation can one give of the parameter λ?

  The expected value is given by

$$E[X] = \int_0^{\infty} x \cdot f_X(x)dx = \int_0^{\infty} x \cdot \lambda e^{-\lambda x} dx$$

  We can compute the integral by doing integration by parts where we set
  $u = -\lambda x \qquad du = -\lambda dx$
  $dv = e^{-\lambda x} dx \qquad v = \frac{-1}{\lambda} e^{-\lambda x}$

$$\implies \lambda \Bigg(\Big[-\lambda x \cdot \frac{-1}{\lambda} e^{-\lambda x} \Big]_0^{\infty} - \int_0^{\infty} \frac{-1}{\lambda} e^{-\lambda x} dx \Bigg)$$

$$= \lambda \Bigg(\Big[-\lambda x \cdot \frac{-1}{\lambda} e^{-\lambda x} \Big]_0^{\infty} - \int_0^{\infty} \frac{-1}{\lambda} e^{-\lambda x} dx \Bigg)$$

$$= \Big[0 - 0 \Big] + \int_0^{\infty} e^{-\lambda x} dx = \Big[\frac{-1}{\lambda} e^{-\lambda x} \Big]_0^{\infty}$$

$$= \frac{1}{\lambda}$$

We can calculate the variance by taking a shortcut and finding that the expected value with moment n can be rewritten as

$$E[X^n] = \int_0^\infty x^n \cdot \lambda e^{-\lambda x} dx = \frac{n}{\lambda} \int_0^\infty x^{n-1} \lambda e^{-\lambda x} dx = \frac{n}{\lambda} E[X^{n-1}]$$

To find the variance we need to find EX^2, which is given by

$$E[X^2] = \frac{2}{\lambda} E[X] = \frac{2}{\lambda^2}$$

The variance is given by

$$Var(X) = E[X^2] - E[X]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

We can interpret λ as a the reciprocal of the mean as well as the variation being the mean squared.

- What does an exponential random variable typically model?

  The time it takes for some event to occur is typically modeled by an exponential distribution. For example, the time it takes for the next earthquake to occur.

- State and prove the memory-less property of an exponential random variable X

$$P(X > t + s | X > t) = P(X > s) \text{ for } t, s \geq 0$$

We can rewrite the LHS of the above equation using the def. of Conditional Probability

$$\frac{P(X > t + s \text{ and } X > t)}{P(X > t)} = P(X > s)$$

We can multiply both sides by $P(X > t)$ and the numerator reduces to just $P(X > t + s)$ since $X > s + t$ implies that $X > t$

$$P(X > t + s) = P(X > t)P(X > s)$$

To show that this equation holds, we can plug in the parameters into the cumulative distribution function:

$$P(X > a) = \int_a^\infty \lambda e^{-\lambda x} dx = \Big[- e^{-\lambda x} \Big]_a^\infty = e^{-\lambda a}$$

It follows that

$$
\begin{aligned}
\implies P(X > t + s) &= e^{-\lambda(t+s)} \\
&= e^{-\lambda t} \cdot e^{-\lambda s} \\
&= P(X > t) \cdot P(X > s)
\end{aligned}
$$

Task 2

- What is probability mass function p_X of a discrete random variable X?

$$p_X(x) = P(X = x_i), \text{ where } i = 1, 2, 3, ...$$

- A discrete random variable N is Poisson distributed with parameter $\lambda \in (0, \infty)$ if

$$p_N(k) = e^{-\lambda} \frac{\lambda^i}{i!} \text{ for } i = 0, 1, 2, ...$$

- What does a Poisson random variable N typically model?

  The number of occurrences of an event within a specific time interval is typically modeled by a Poisson distribution.

- If $N(t) \sim P(\lambda t)$ (ie $N(t)$ is Poisson distributed with parameter λt) models the number of occurrences in the time interval of size $t > 0$, what event captures the fact that there is no occurrence in that time interval?

  The event where $N(t) = 0$ indicates that there is no occurrence from 0 to time t. This happens when say the random variable X denoting the time it takes is larger than λ so that $X > \lambda$.

- If X is time to the next occurrence, how can you relate $P(X \leq t)$ to $N(t)$?

  The notation $P(X \leq t)$ denotes the probability that the time to the next occurrence is under some fixed time t.

  We can relate this to $N(t)$ by the following

Since $N(t)$ is Poisson distributed with parameter λt, we have

$$P(N(t) = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, k = 0, 1, ...$$

We see that $P(X > t) = P\{N(t) = 0\} = e^{-\lambda t}$

It follows that

$$\begin{aligned} P(X > t) &= 1 - P(X \leq t) \\ \implies P(X \leq t) &= 1 - P(X > t) \\ &= 1 - P\{N(t) = 0\} = 1 - e^{-\lambda t} \end{aligned}$$

- What is the cumulative distribution function of an exponentially distributed random variable X with parameter λ?

  The cumulative distribution function is given by

$$F_X(x) = P(X \leq x) = \int_0^x \lambda e^{-\lambda x} dx$$