Group #5 @1pm:
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Lecture 3 Worksheet

Task 1

1. What is the sample space of an experiment? State a short definition and give at least 3 distinct examples of experiment and their sample space?
   Experiment: Delivering babies, determining newborn's gender
   Sample Space: {Boy, Girl}
   Experiment: Taking an exam, determining letter grade
   Sample Space: {A,B,C,D,F}
   Experiment: Determining the weather
   Sample Space: {Sunny, Rainy, Cloudy, ...}

2. What is an event in the context of a sample space of an experiment? We should first define what an outcome is.

$$\omega = Outcome, \text{ where } \omega \text{ is an element}$$

   Then an event is something where an outcome occurs. In other words, it contains the outcome.

$$E = Event = \{\omega\}, \text{ where E is a subset and } \omega \in \text{E}$$

Task 2

1. Translate the following statements between Math and English. Capital letters such as E and F denote events in a sample space

English	Math
E does not occur	E^c
E or F occurs	$E \cup F$
E and F occurs	$E \cap F$
Only one of E or F occurs	$(E \setminus F) \cup (F \setminus E)$
If F occurs, then so does E	$F \in E$
No event occurs	$\emptyset$
E does not occur or F occurs	$E^c \cup F$
E and F do not occur	$E^c \cap F$

Task 3

1. Given n $\geq$ 1 and events $E_1, E_2, ..., E_n$ of a sample space, prove that:

$$(\bigcup_{j=1}^{n} E_j)^c = \bigcap_{j=1}^{n} E_j^c$$

Showing that the LHS is a subset of the RHS:
Starting with

$$x \in (\bigcup_{j=1}^{n} E_j)^c$$

$$\Rightarrow x \in \{E_j\}^c \textit{ for } j = 1, 2, ..., n$$

$$\Rightarrow x \notin \{E_j\} \textit{ for } j = 1, 2, ..., n$$

This says that x is not in the sample space of $E_1, E_2, ..., E_n$
In other words, x is in the complement of each event $E_1^c, E_2^c, ...E_n^c$

$$\Rightarrow x \in \{E_1^c \wedge E_2^c \wedge ... \wedge E_n^c\}$$

$$\Rightarrow x \in \bigcap_{j=1}^{n} E_j^c$$

We can apply this logic to all $x \in (\bigcup_{j=1}^{n} E_j)^c$,

Therefore, $(\bigcup_{j=1}^{n} E_j)^c \subset \bigcap_{j=1}^{n} E_j^c$

Now, showing that the RHS is a subset of the LHS:
Starting with

$$x \in \bigcap_{j=1}^{n} E_j^c$$

$$\Rightarrow x \in \{E_1^c \wedge E_2^c \wedge ... \wedge E_n^c\}$$

In other words, x is not in any of the events

$$\Rightarrow x \notin \{E_1, E_2, ..., E_n\}$$

$$\Rightarrow x \notin \bigcup_{j=1}^{n} E_j$$

$$\Rightarrow x \in (\bigcup_{j=1}^{n} E_j)^c$$

Therefore, since the two sets are subsets of each other

$$(\bigcup_{j=1}^{n} E_j)^c = \bigcap_{j=1}^{n} E_j^c$$

Task 4

1. A probability function $P : 2^s \to \mathbb{R}$
   (P1) For any $E \subset S,\ 0 \leq P(E) \leq 1$
   (P2) P(S) = 1
   (P3) P($\bigcup_{j=1}^{\infty} E_j) = P(E_1) + P(E_2) + ... + P(E_\infty)$, if $E_j \cap E_k = \emptyset$ for j,k $\in \mathbb{N}$

2. Show that: $P(E \cup F) + P(E \cap F) = P(E) + P(F)$

   This equation holds for general sets E,F so E and F may or may not have common elements.
   We can define

$$P(E \cup F) = \underbrace{P(E \setminus F)}_{E\ without\ F} + \underbrace{P(F \setminus E)}_{F\ without\ E} + \underbrace{P(E \cap F)}_{E\ and\ F}$$

   Here we are trying to construct the probability of E or F $P(E \cup F)$ by separating it into different parts. The first 2 terms are disjoint.
   Then we account for any similarities by adding in the probability of the intersection $P(E \cap F)$
   Also we can define and justify the following

$$\begin{aligned} P(E) &= P(E \setminus F) + P(E \cap F) \\ &= P(E \cap F^c) + P(E \cap F) \text{ by def. of E} \setminus \text{F} \\ &= P((E \cap F^c) \cup (E \cap F) \text{ summed by their union} \\ &= P(E \cup (F \cap F^c)) \text{ by set properties} \\ &= P(E \cup \emptyset) \\ &= P(E) \end{aligned}$$

   A similar argument is formed to show that $P(F) = P(F \setminus E) + P(F \cap E)$.

Plugging these into our original equation,
For the LHS:

$$\begin{aligned} P(E \cup F) + P(E \cap F) = & \\ &= [P(E \setminus F) + P(F \setminus E) + P(E \cap F)] + P(E \cap F) \\ &= P(E \setminus F) + P(F \setminus E) + 2P(E \cap F) \end{aligned}$$

Now plugging into the RHS:

$$\begin{aligned} P(E) + P(F) = & \\ &= [P(E \setminus F) + P(E \cap F)] + [P(F \setminus E) + P(E \cap F)] \\ &= P(E \setminus F) + P(F \setminus E) + 2P(E \cap F) \end{aligned}$$

Therefore, the 2 quantities are equal.

3. Show that: $P(\emptyset) = 0$
   We can prove this by rewriting the probability of the event and using set axioms.
   Let $S =$ Sample Space
   Then $P(S) = 1$ by (P2)

$$\begin{aligned} P(S) = & \\ &= P(S \cup \emptyset) \\ &= P(S) + P(\emptyset) \text{ since S and } \emptyset \text{ are disjoint} \\ &= 1 + P(\emptyset) = 1 \text{ by (P2)} \end{aligned}$$

   Therefore, $P(\emptyset) = 0$ has to hold

4. Show that: $P(E^c) = 1 - P(E)$
   In this context, E denotes some event and E^c denotes all the events other than E.
   From this, we can relate their union to the sample space S

$$E \cup E^c = S$$

   Now, since E and E^c are disjoint sets by definition of the complement, we can add their individual probabilities to get the probability of the union, without over-counting common elements.

$$P(E) + P(E^c) = P(E \cup E^c) = P(S)$$

Rearranging this equation, we get:

$$P(E^c) = P(S) - P(E)$$

By (P3): $P(S) = 1$ *the following equation holds:*

$$P(E^c) = 1 - P(E)$$