Group #5 @1pm
Mimi Morales, Yiyi Zhu, Vi Pham, Brian Fong May 12^{th}, 2019

Lecture Worksheet 15

Task 1

- When is a random variable X called **discrete**?

  X can be discrete when $X(S) = \{X(\omega) \mid \omega \in S\}$ is countable.
  In other words, X is discrete when there are at most a countable number of possible values for X.

- What is the **probability mass function** p_x of a discrete random variable X?

$$p_X(x) = P(X = x), \text{ for } x \in X(S)$$

  The probability mass function gives the probability that a discrete random variable X is equal to some value.

- What is the **expected value** of a discrete random variable?

$$E(X) = \sum_{x \in X(S)} x\, p_X(x) = \sum_{k \in \mathbb{N}} x_k\, p_X(x_k) = \sum_{k \in \mathbb{N}} x_k\, x p_x(X = x_k)$$

- What is the meaning of expected value?
  Think in terms of frequency of occurrence

  We can think of the expected value as the weighted average of all the outcomes.

Task 2

An integer N is to be selected at random from $\{1, 2, ..., 10^3\}$ (each integer has the same probability of being selected).

- What is the probability that N will be divisible by 3? by 5? by 7? by 15? by 105?

  We can find the desired probability by applying the basic principle of favorable to total outcomes.
  The favorable outcomes are $N \in \{1, 2, ..., 10^3\}$ such that N is divisible by c. We can conveniently find this number by simply dividing the max number (10^3) by c and then using the floor function to get the nearest, least integer.

$$P(c|N) = \frac{\textit{Favorable Outcomes}}{\textit{Total Outcomes}} = \frac{\lfloor \frac{10^3}{c} \rfloor}{10^3}$$

$$P(3|N) = \frac{333}{1000}$$

$$P(5|N) = \frac{200}{1000}$$

$$P(7|N) = \frac{142}{1000}$$

$$P(15|N) = \frac{66}{1000}$$

$$P(105|N) = \frac{9}{1000}$$

- How would your answer change if 10^3 is replaced by 10^k for larger and larger k?

  We can adjust our general formula by replacing the 3 in 10^3 by k.

$$P[c, k] = \frac{\textit{Favorable Outcomes}}{\textit{Total Outcomes}} = \frac{\lfloor \frac{10^c}{k} \rfloor}{10^c}$$

Task 3

- State the definition of (cumulative) **distributive function** F_x of a random variable X

$$F_X(x) = P\{X \leq x\}, \text{ for } -\infty < x < \infty$$

The distribution function gives us the probability that the random variable is less than or equal to x.

- If X has distribution function F_X, what is the distribution function of the random variable $\alpha X + \beta$, where α and β are constants and $\alpha \neq 0$?

We can approach this problem by replacing the X in our probability function with $\alpha X + \beta$

$$P(X \leq x) \to P(\alpha X + \beta \leq x)$$

Re-arranging the arguments into standard form so that X is on LHS and x is on RHS

$$\implies P(X \leq \frac{x-\beta}{\alpha}) = F_X(\frac{x-\beta}{\alpha})$$

However this holds for when $\alpha > 0$, to account for $\alpha < 0$ we would switch the inequality, giving us:

$$\implies P(X \geq \frac{x-\beta}{\alpha}) = 1 - P(X < \frac{x-\beta}{\alpha})$$

NOTE: $P(X < \frac{x-\beta}{\alpha}) \neq F_X(\frac{x-\beta}{\alpha})$ in the case of equality
We can try to apply limit properties to approximate the RHS so that the inequality becomes strictly less than.

$$X < \frac{x-\beta}{\alpha} \Rightarrow X \leq \lim_{n\to\infty} \frac{x-\beta}{\alpha} - \frac{1}{n}$$

Now we can have:

$$\begin{aligned} P(X \geq \frac{x-\beta}{\alpha}) &= 1 - P(X < \frac{x-\beta}{\alpha}) \\ &= 1 - \lim_{n\to\infty} P(X \leq \frac{x-\beta}{\alpha} - \frac{1}{n}) \\ &= 1 - \lim_{n\to\infty} F_X(\frac{x-\beta}{\alpha} - \frac{1}{n}) \end{aligned}$$

So in the end, we have a piece-wise function relating the probability and the distribution function:

$$P(\alpha X + \beta \leq x) = \begin{cases} F_X(\frac{x-\beta}{\alpha}) & \alpha > 0 \\ 1 - \lim\limits_{n \to \infty} F_X(\frac{x-\beta}{\alpha} - \frac{1}{n}) & \alpha < 0 \end{cases}$$