Week 1 Homework:

P1 .1 .13

If you invest a dollar at 6% interest compounded monthly , it amounts to $(1.005)^n$ after n months. If you invest \$10 at the beginning of each month for 10 years (120 months) how much will you have at the end of 10 years?

We can see this as a geometric series where $a = 1.005$, $r = 1.005$

The formula to find the n-sum is given by $S_n = a\left(1 - r^n\right) / (1 - r)$

Here, we have $n = 120$ months.

NOTE: The key is that we invest an additional \$10 every time, so each sum is multiplied by \$10.

Also, “a” in this formula represents S_1

In[ ]:=
```
10 * (1.005 * (1 - 1.005^120) / (1 - 1.005))
```

Out[ ]= 1646.99

In[ ]:=
```
10 * Sum[(1.005)^i, {i, 0120}]
```

Out[ ]= 1646.99

We can apply geometric series to finding definite fractions for certain repeating decimals.

The formula to find the total sum of the geometric series is by taking the limit:

$\lim [S_n, n \to \infty] = a / (1 - r)$

For example, for 0.55555

we have $a = 0.5$, $r = 1/10$

In[ ]:=
```
1 / 2 / (1 - 1 / 10)
```

Out[ ]= $\frac{5}{9}$

P1 .6 .24

Use the ratio test to determine if $\sum_{n=0}^{\infty} \frac{3^{2n}}{2^{3n}}$ converges

We can apply the ratio test of doing $\text{Limit}\left[\frac{a_{n+1}}{a_n}, n \to \infty\right]$

In[ ]:=
$$\text{Limit}\left[\frac{\frac{3^{2\,(n+1)}}{2^{3\,(n+1)}}}{\frac{3^{2\,n}}{2^{3\,n}}},\ n \to \infty\right]$$

Out[ ]=
$$\frac{9}{8}$$

P1 .13 .20

Find the first 3 terms in the Maclaurin series for e^x Sin[x]. Explicitly generate the sum defined in Eq. 1.12.9 , first by writing out each term. Use /.x->0 to evaluate expressions at zero. Do it again using Sum; you will have to write an expression which evaluates to the nth term. Finally, check your result with Series.

A MacLaurin Series is a Taylor Series evaluated at $x_0 = 0$

The formula is given by $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$

First we define a variable to represent our function:

In[ ]:= func = e^x Sin[x]

Out[ ]= e^x Sin[x]

Explicitly computing the first 3 terms:

In[ ]:=
$$a0 = \frac{(D[\text{func}, \{x, 0\}]\ /.\ x \to 0)}{0\,!}\ (x - 0)^0$$

Out[ ]= 0

In[ ]:=
$$a1 = \frac{(D[\text{func}, \{x, 1\}]\ /.\ x \to 0)}{1\,!}\ (x - 0)^1$$

Out[ ]= x

In[ ]:=
$$a2 = \frac{(D[\text{func}, \{x, 2\}]\ /.\ x \to 0)}{2\,!}\ (x - 0)^2$$

Out[ ]= x^2 Log[e]

In[ ]:=
$$a3 = \frac{(D[\text{func}, \{x, 3\}]\ /.\ x \to 0)}{3\,!}\ (x - 0)^3$$

Out[ ]=
$$\frac{1}{6} x^3 \left(-1 + 3\,\text{Log}[e]^2\right)$$

Adding all the terms above, we get our MacLaurin Series:

In[ ]:=
```
MacSeries = a1 + a2 + a3
```
Out[ ]= $x + x^2 \,\mathrm{Log}[e] + \frac{1}{6} x^3 \left(-1 + 3\,\mathrm{Log}[e]^2\right)$

Now using Sum to compute the MacLaurin Series:

In[ ]:= $\texttt{MacSeriesSum = Sum}\left[\frac{\texttt{D[func, \{x, i\}] /. x} \to \texttt{0}}{\texttt{i!}} \texttt{(x - 0)}^{\texttt{i}}\texttt{, \{i, 0, 3\}}\right]$

Out[ ]= $x + x^2 \,\mathrm{Log}[e] + \frac{1}{6} x^3 \left(-1 + 3\,\mathrm{Log}[e]^2\right)$

Using Series[] to check our answer:

In[ ]:=
```
check = Series[func, {x, 0, 3}] // Normal // Simplify
```
Out[ ]= $x + x^2 \,\mathrm{Log}[e] + \frac{1}{6} x^3 \left(-1 + 3\,\mathrm{Log}[e]^2\right)$

In[ ]:=
```
MacSeries == MacSeriesSum == check
```
Out[ ]= True

P1 .13 .36

Find the first 3 non-vanishing terms of the Maclaurin series for $F[u] = \int_0^u \frac{\mathrm{Sin}[x]}{\sqrt{1-x^2}} \, dx$

This problem illustrates that you can find the series expansion for an integral, even if you cannot evaluate the integral in closed f,orm.

Defining a variable to represent our function:

In[ ]:= $\texttt{func = Integrate}\left[\frac{\texttt{Sin[x]}}{\texttt{Sqrt}\left[\texttt{1 - x}^2\right]}\texttt{, \{x, 0, u\}}\right]$

Out[ ]= $\int_0^u \frac{\mathrm{Sin}[x]}{\sqrt{1 - x^2}} \, dx$

In[ ]:=
```
nterms = 6
```
Out[ ]= 6

Note that the function is in terms of u as the integrals boundaries are from $0 \to u$
Thus we derive the function wrt u and do the power series in terms of u as well.

In[ ]:= $\texttt{Sum}\left[\frac{\texttt{D[func, \{u, n\}] /. u} \to \texttt{0}}{\texttt{n!}} \texttt{(u - 0)}^{\texttt{n}}\texttt{, \{n, 0, nterms\}}\right]$

Out[ ]= $\frac{u^2}{2} + \frac{u^4}{12} + \frac{u^6}{20}$

Using Series[] to check our answer

In[ ]:= `Series[func, {u, 0, nterms}] // Normal`

Out[ ]= $\frac{u^2}{2}+\frac{u^4}{12}+\frac{u^6}{20}$

P1 .16 .20

Find the first 3 terms of the Taylor's series of $\sqrt[3]{x}$ valid near x=8. Make sure to <u>not</u> include O[x-8] terms in your answer, i.e. put a simple expression into the input field.

In[ ]:= `func = x^(1/3)`

Out[ ]= $x^{1/3}$

In[ ]:= $\text{TaylorFunc[n\_]} := \frac{\text{D[func, \{x, n\}] /. x} \to 8}{n!}(x-8)^n$

In[ ]:= `TSeries = Sum[TaylorFunc[i], {i, 0, 3}]`

Out[ ]= $2+\frac{1}{12}(-8+x)-\frac{1}{288}(-8+x)^2+\frac{5(-8+x)^3}{20736}$

P2 .9 .20

Express $\left(\frac{\sqrt{2}}{i-1}\right)^{10}$ in x+ i y (cartesian) form by expressing the fraction in polar form and then calculating the power. Check by brute force calculation of the power

Defining a variable to represent the given fraction:

In[ ]:= $\text{ogfrac} = \frac{\text{Sqrt[2]}}{i-1}$

Out[ ]= $-\frac{1+i}{\sqrt{2}}$

Multiplying our fraction by the complex conjugate to get the denominator to have only real numbers. Note this is now in Cartesian form, where $X = \frac{-1}{\text{Sqrt[2]}}$, $y = \frac{1}{\text{Sqrt[2]}}$

In[ ]:= $\text{frac} = \text{ogfrac} * \frac{i+1}{i+1}$

Out[ ]= $-\frac{1+i}{\sqrt{2}}$

Using Abs[] and Arg[] to find the radius and angle of our fraction:

```
In[]:= r = Abs[frac]
Out[]= 1
```

```
In[]:= θ = Arg[frac]
```

$$Out[]= -\frac{3\pi}{4}$$

Now that we have the values for r and θ, we can substitute them into Euler's Formula:

```
In[]:= euler = r ⅇ^(ⅈ θ)
```

$$Out[]= e^{-\frac{3 i \pi}{4}}$$

The beautiful part of converting into this exponential form is that we can exponentiate the 10 easily.

```
In[]:= answer = euler^10
Out[]= ⅈ
```

We can check our answer just by directly computing the 10th power of the given fraction:

```
In[]:= check = (ogfrac)^10
Out[]= ⅈ
```

P2 .10 .9
Find all values of $\sqrt[5]{1}$. Do it first by expressing 1 in polar form and then computing the roots. Check your answer by solving $z^5 - 1 = 0$ and using ComplexExpand. The answer should be a list of the 5 fifth roots.

Like the previous problem, we convert the expression inside the power from Cartesian to Polar Form, then to
Exponential Form using Euler's Formula. Once it's in exponential form, we can find it's nth-root easily.

```
In[]:= cartform = x + y ⅈ /. {x → 1, y → 0}
Out[]= 1
```

Using Abs[] and Arg[] to find the radius and angle:

```
In[]:= r = Abs[cartform]
Out[]= 1
```

```
In[]:= θ = Arg[cartform]
Out[]= 0
```

We have to keep in mind that there are infinitely many solutions to obtain this angle.
Thus we define a general solution to account for any additional multiple of 2π :

In[]:=
```
θgen = 0 + 2 π n
```
Out[]= $2\,n\,\pi$

Then we plug our general θ and our r into the equation $r\,e^{i\,\theta}$, however we take the root by multiplying/dividing
the exponent by whatever root they want:

In[]:=
```
expform = r^(1/5) e^(i θgen/5)
```
Out[]= $e^{\frac{2\,i\,n\,\pi}{5}}$

In[]:=
```
ComplexExpand[1^(1/5) e^(i 2 π/5)]
```
Out[]= $-\frac{1}{4}+\frac{\sqrt{5}}{4}+i\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}$

In[]:=
```
Solve[z^5 - 1 == 0, z] // ComplexExpand
```
Out[]= $\left\{\{z\to 1\},\left\{z\to -\frac{1}{4}-\frac{\sqrt{5}}{4}-i\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}\right\},\left\{z\to -\frac{1}{4}+\frac{\sqrt{5}}{4}+i\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}\right\},\left\{z\to -\frac{1}{4}+\frac{\sqrt{5}}{4}-i\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}\right\},\left\{z\to -\frac{1}{4}-\frac{\sqrt{5}}{4}+i\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}\right\}\right\}$

Then we use Table[] to iterate through each root:

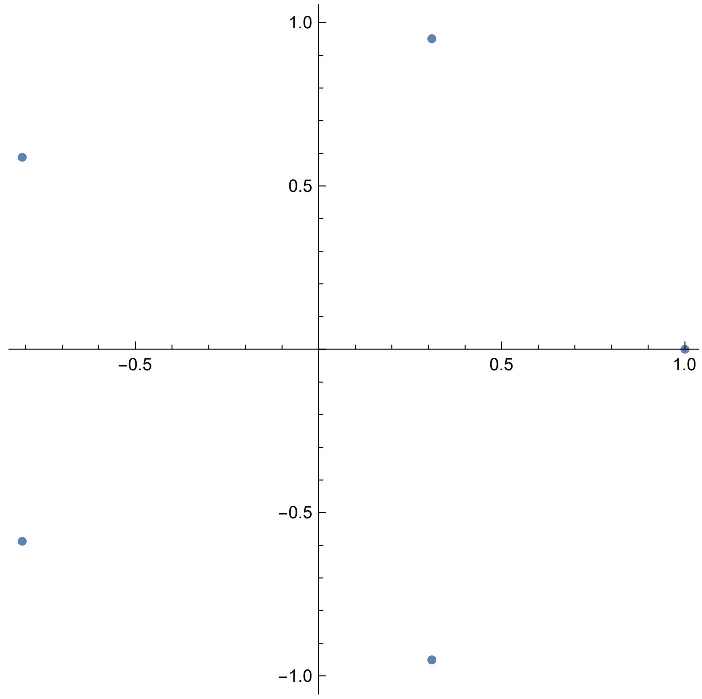
In[]:=
```
roots = Table[expform, {n, 0, 4}] // ComplexExpand
```
Out[]= $\left\{1,-\frac{1}{4}+\frac{\sqrt{5}}{4}+i\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}},-\frac{1}{4}-\frac{\sqrt{5}}{4}+i\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}},-\frac{1}{4}-\frac{\sqrt{5}}{4}-i\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}},-\frac{1}{4}+\frac{\sqrt{5}}{4}-i\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}\right\}$

Using Thread[] to put the Table[] results in the format of {Re[Z],Im[Z]} for us to plot:

In[]:=
```
points = Thread[{Re[roots], Im[roots]}]
```
Out[]= $\left\{\{1,0\},\left\{-\frac{1}{4}+\frac{\sqrt{5}}{4},\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}\right\},\left\{-\frac{1}{4}-\frac{\sqrt{5}}{4},\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}\right\},\left\{-\frac{1}{4}-\frac{\sqrt{5}}{4},-\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}\right\},\left\{-\frac{1}{4}+\frac{\sqrt{5}}{4},-\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}\right\}\right\}$

In[ ]:= `ListPlot[points, AspectRatio → 1]`

Out[ ]=



P2 .14 .4

Evaluate Log[i-1] in x+ i y (Cartesian) form , first by converting to polar form, and then by using ComplexExpand

In[ ]:= `eq = -1 + i`

Out[ ]= $-1 + i$

In[ ]:= `r = Abs[eq]`

Out[ ]= $\sqrt{2}$

In[ ]:= `θ = Arg[eq]`

Out[ ]= $\frac{3\pi}{4}$

In[ ]:= `expform = r e^(i θ)`

Out[ ]= $\sqrt{2}\, e^{\frac{3 i \pi}{4}}$

In[ ]:= `logform = Log[r] + i θ`

Out[ ]= $\frac{3 i \pi}{4} + \frac{\text{Log}[2]}{2}$

In[ ]:=
```
check = Log[ⅈ - 1] // ComplexExpand
```
Out[ ]= $\frac{3\,i\,\pi}{4}+\frac{\text{Log}[2]}{2}$

Week 2 Homework

Wk2.P1

Find all of the square roots of i. Do it by explicitly converting i into polar form (with an additional factor of $1=e^{2\pi n i}$) and then evaluating the result. Check using ComplexExpand. Your answer should be a list containing the square roots.

In[ ]:=
```
cartform = x + y ⅈ /. {x → 0, y → 1}
```
Out[ ]= i

In[ ]:=
```
r = Abs[cartform]
θ = Arg[cartform]
```
Out[ ]= 1

Out[ ]= $\frac{\pi}{2}$

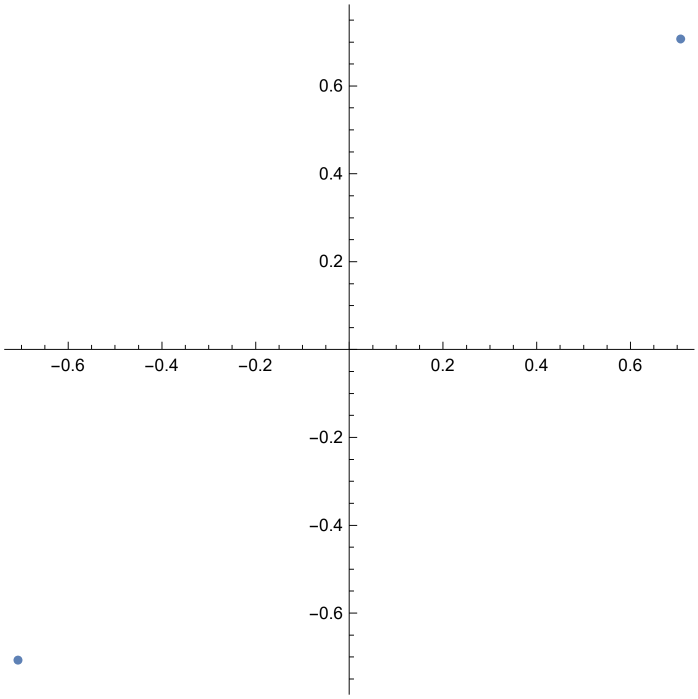
In[ ]:=
```
θgen = θ + 2 π n
```
Out[ ]= $\frac{\pi}{2}+2\,n\,\pi$

In[ ]:=
```
expform = r^(1/2) ⅇ^(θgen ⅈ /2)
```
Out[ ]= $e^{\frac{1}{2}\,i\,\left(\frac{\pi}{2}+2\,n\,\pi\right)}$

In[ ]:=
```
roots = Table[expform, {n, 0, 1}] // ComplexExpand
```
Out[ ]= $\left\{\frac{1+i}{\sqrt{2}},-\frac{1+i}{\sqrt{2}}\right\}$

In[ ]:=
```
points = Thread[{Re[roots], Im[roots]}]
```
Out[ ]= $\left\{\left\{\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right\},\left\{-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right\}\right\}$

In[ ]:=
```
ListPlot[points, AspectRatio → 1]
```

Out[ ]=


Wk2.P2

Find i^i . Do it by explicitly converting each term into polar form and then evaluating the result. Check using ComplexExpand.

In[ ]:=
```
cartform = x + y i /. {x → 0, y → 1}
```

Out[ ]= i

In[ ]:=
```
r = Abs[cartform]
θ = Arg[cartform]
```

Out[ ]= 1

Out[ ]= $\frac{\pi}{2}$

In[ ]:=
```
expform = r^i e^(i θ i)
```

Out[ ]= $e^{-\pi/2}$

In[ ]:=
```
ComplexExpand[i^i]
```

Out[ ]= $e^{-\pi/2}$

P2 .17 .26

Find $\left| \frac{2 e^{i\theta} - i}{i e^{i\theta} + 2} \right|$

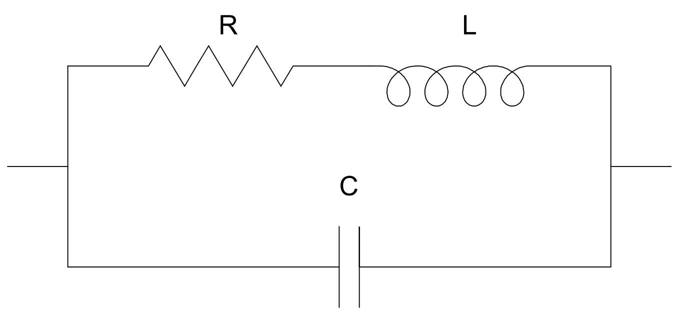
In[ ]:=
```
Abs[(2 ⅇ^(ⅈ θ) - ⅈ)/(ⅈ ⅇ^(ⅈ θ) + 2)]
```

Out[ ]= 1

P2 .16 .8

Find the impedance of the circuit in the figure below. A circuit is said to be in resonance if Z is real and ω>0; find the resonant frequency ω_{res} in terms of R, L, and C (treating these as symbols, not numbers) . You can use the fact that parallel and series combinations of complex impedances obey the same formulas as parallel and series combinations of resistors. For numerical values R=50 ohms, L=5 milli Henrys and C=0.1 micro Farad, make a plot of Abs[Z] and Arg[Z]

Out[ ]=



In[ ]:=
```
ClearAll["Global`*"]
```

First we define variables to represent the voltage going through the Resistor, Inductor, and Capacitor. Boas gives the formulas to define each component:

In[ ]:=
```
Vr = Res Cur;
Vl = ⅈ ω Ind Cur;
Vc = 1/(ⅈ ω Cpct) Cur;
```

Since these are in series, we can just add the voltages of the resistor and the inductor.

In[ ]:=
```
Vrl = Vr + Vl // Simplify
```

Out[ ]= Cur (Res + ⅈ Ind ω)

We are also given that the complex impedance is: $V = Z I$, where Z is the complex impedance, I is the current.

Our next step is to set up these equations and solve for the complex impedances of individual circuit

components.
We can use the given property of parallel and series combinations of complex impedances.

This equation represents the complex impedance for the resistor and the inductor.

```
In[ ]:= eq1 = Vrl == Zrl Cur
```

Out[ ]= $\text{Cur}\ (\text{Res} + i\ \text{Ind}\ \omega) == \text{Cur Zrl}$

Solving for Z, we get:

```
In[ ]:= ZsolRL = Solve[eq1, Zrl] // Flatten
```

Out[ ]= $\{\text{Zrl} \to \text{Res} + i\ \text{Ind}\ \omega\}$

This equation represents the complex impedance for the capacitance.

```
In[ ]:= eq2 = Vc == Zc Cur
```

Out[ ]= $-\frac{i\ \text{Cur}}{\text{Cpct}\ \omega} == \text{Cur Zc}$

Solving for Z, we get:

```
In[ ]:= ZsolC = Solve[eq2, Zc] // Flatten
```

Out[ ]= $\left\{\text{Zc} \to -\frac{i}{\text{Cpct}\ \omega}\right\}$

Now we apply the property of adding complex impedances in series, which is the inverse sum of the inverse impedances.
The total impedance is:

In[ ]:= $\text{Ztot} = \left(\frac{1}{\text{Zrl}} + \frac{1}{\text{Zc}}\right)^{-1}$ /. ZsolRL /. ZsolC // Simplify

Out[ ]= $\frac{1}{i\ \text{Cpct}\ \omega + \frac{1}{\text{Res} + i\ \text{Ind}\ \omega}}$

Using ComplexExpand[], we can force our expression into Cartesian form:

```
In[ ]:= ComplexExpand[Ztot]
```

Out[ ]= $$\frac{\text{Res}}{(\text{Res}^2 + \text{Ind}^2\ \omega^2)\left(\frac{\text{Res}^2}{(\text{Res}^2+\text{Ind}^2\ \omega^2)^2} + \left(\text{Cpct}\ \omega - \frac{\text{Ind}\ \omega}{\text{Res}^2+\text{Ind}^2\ \omega^2}\right)^2\right)} +$$
$$i\left(-\frac{\text{Cpct}\ \omega}{\frac{\text{Res}^2}{(\text{Res}^2+\text{Ind}^2\ \omega^2)^2} + \left(\text{Cpct}\ \omega - \frac{\text{Ind}\ \omega}{\text{Res}^2+\text{Ind}^2\ \omega^2}\right)^2} + \frac{\text{Ind}\ \omega}{(\text{Res}^2 + \text{Ind}^2\ \omega^2)\left(\frac{\text{Res}^2}{(\text{Res}^2+\text{Ind}^2\ \omega^2)^2} + \left(\text{Cpct}\ \omega - \frac{\text{Ind}\ \omega}{\text{Res}^2+\text{Ind}^2\ \omega^2}\right)^2\right)}\right)$$

The prompt tells us that a circuit is in resonance if Z is real and $\omega > 0$
This is equivalent to saying that the imaginary part of Z is zero.
Thus we can set up another equation that equates Im[Z] = 0, and then solve for ω:

In[ ]:= ZtotIm = ComplexExpand[Im[Ztot]]

Out[ ]= $$-\frac{\text{Cpct}\,\omega}{\frac{\text{Res}^2}{\left(\text{Res}^2+\text{Ind}^2\,\omega^2\right)^2}+\left(\text{Cpct}\,\omega-\frac{\text{Ind}\,\omega}{\text{Res}^2+\text{Ind}^2\,\omega^2}\right)^2}+\frac{\text{Ind}\,\omega}{\left(\text{Res}^2+\text{Ind}^2\,\omega^2\right)\left(\frac{\text{Res}^2}{\left(\text{Res}^2+\text{Ind}^2\,\omega^2\right)^2}+\left(\text{Cpct}\,\omega-\frac{\text{Ind}\,\omega}{\text{Res}^2+\text{Ind}^2\,\omega^2}\right)^2\right)}$$

Solve[ZtotIm == 0, ω] // Flatten

Out[ ]= $$\left\{\omega\to 0,\ \omega\to -\frac{\sqrt{\text{Ind}-\text{Cpct}\,\text{Res}^2}}{\sqrt{\text{Cpct}}\,\text{Ind}},\ \omega\to \frac{\sqrt{\text{Ind}-\text{Cpct}\,\text{Res}^2}}{\sqrt{\text{Cpct}}\,\text{Ind}}\right\}$$

P3 .5 .24

Find the parametric equation (of the form $\vec{r}=\overrightarrow{r0}+\vec{A}$ t as in Eq 3.5.8) for the line formed by the intersection of planes defined by 3x+6y-3z=10 and 5x+2y-z=12

In[ ]:= n1 = {3, 6, -3}

Out[ ]= {3, 6, -3}

In[ ]:= n2 = {5, 2, -1}

Out[ ]= {5, 2, -1}

In[ ]:= A = Cross[n1, n2]

Out[ ]= {0, -12, -24}

```
In[ ]:= eq1 = 3 x + 6 y - 3 z == 10;
eq2 = 5 x + 2 y - z == 12;
xysol = Solve[{eq1, eq2} /. z → 0, {x, y}] // Flatten
```

Out[ ]= $\left\{x\to\frac{13}{6},\ y\to\frac{7}{12}\right\}$

In[ ]:= r0 = {x, y, z} /. z → 0 /. xysol

Out[ ]= $\left\{\frac{13}{6},\ \frac{7}{12},\ 0\right\}$

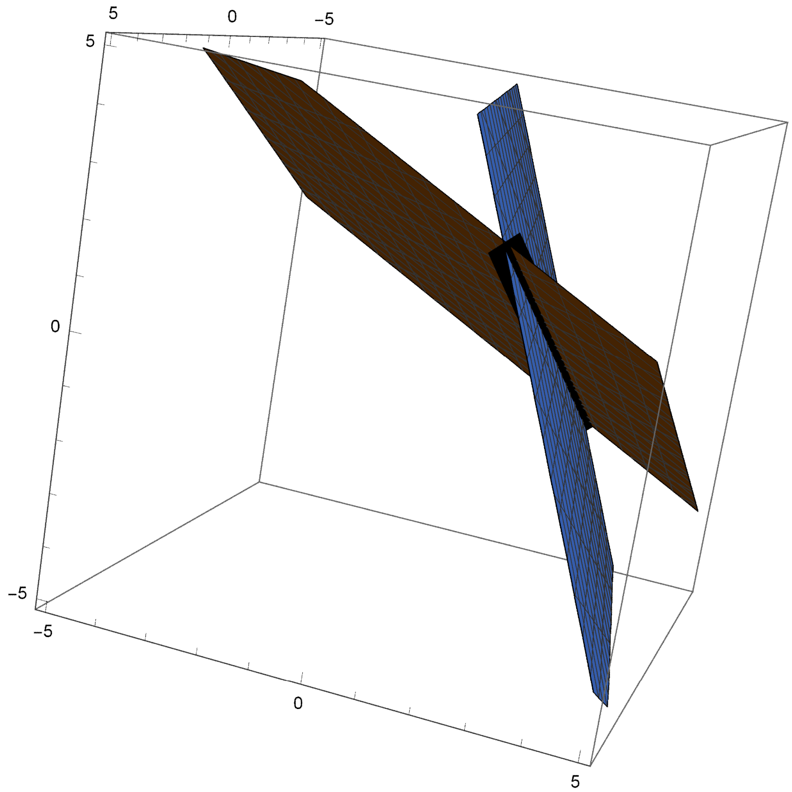
In[ ]:= r = r0 + A t

Out[ ]= $\left\{\frac{13}{6},\ \frac{7}{12}-12\,t,\ -24\,t\right\}$

```
In[ ]:= planeplot =
  ContourPlot3D[{3 x + 6 y - 3 z == 10, 5 x + 2 y - z == 12}, {x, -5, 5}, {y, -5, 5}, {z, -5, 5}];
lineplot = Graphics3D[{Thickness[0.05], Line[{r /. t → -1, r /. t → 0, r /. t → 1}]}];
```

In[ ]:= `Show[planeplot, lineplot]`

Out[ ]=



Week 3 Homework

P3 .5 .32
Find the distance from {3,-1,2} to the plane 5x-y-z=4

Defining a variable to represent the normal vector of the given plane:

In[ ]:= `n = {5, -1, -1}`

Out[ ]= {5, −1, −1}

Defining a variable to represent the given point:

In[ ]:= `point = {3, -1, 2}`

Out[ ]= {3, −1, 2}

Finding an arbitrary point on the given plane, setting x, $y \to 0$:

In[ ]:= `plane = 5 x - y - z == 4;`

In[ ]:=
```
zsol = Solve[plane /. {x → 0, y → 0}, z] // Flatten
```
Out[ ]=
```
{z → -4}
```

In[ ]:=
```
pointonplane = {x, y, z} /. {x → 0, y → 0} /. zsol
```
Out[ ]=
```
{0, 0, -4}
```

Finding the vector connecting these two points:

In[ ]:=
```
vec = point - pointonplane
```
Out[ ]=
```
{3, -1, 6}
```

The distance is given by the scalar projection of our vector and the plane’s normal vector:

In[ ]:= $\text{distance} = \dfrac{\text{vec.n}}{\text{Sqrt[n.n]}}$

Out[ ]= $\dfrac{10}{3\sqrt{3}}$

We can check using the “distance formula” given by: $\dfrac{ax+by+cz+d}{(a^2+b^2+c^2)^{1/2}}$ where the numerator is our plane equation and
$\{x, y, z\}$ are plugged in as the given point:

In[ ]:= $\text{check} = \dfrac{(5x - y - z - 4)\ /.\ \{x \to 3,\ y \to -1,\ z \to 2\}}{\text{Sqrt[n.n]}}$

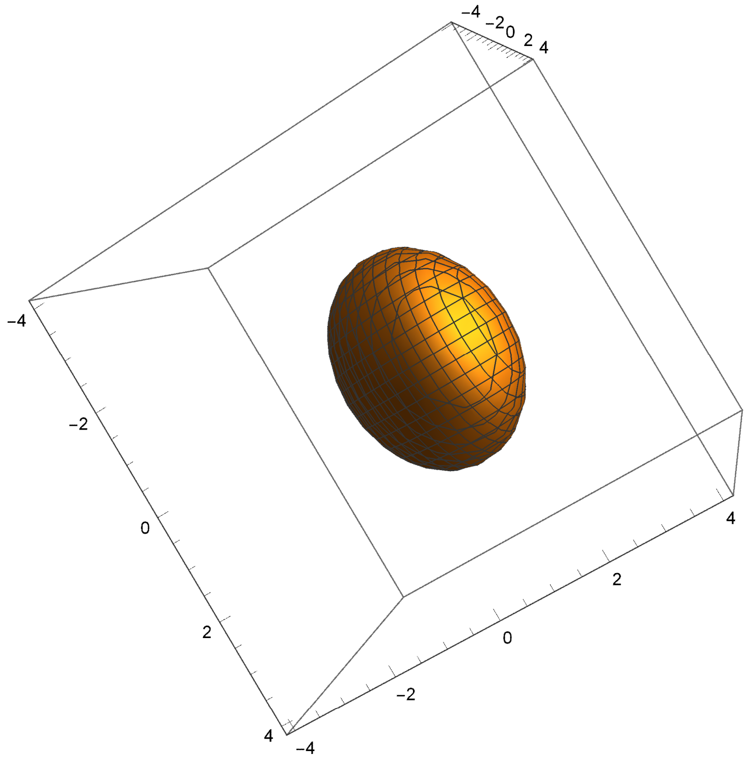
Out[ ]= $\dfrac{10}{3\sqrt{3}}$

P3 .12 .5
Find the equations of $5x^2 + 3y^2 + 2z^2 + 4xz = 14$ relative to the principal axes. (See Boas 3.12)
In the primed principal axis coordinate system, the surface has equation $a\,\text{xp}^2 + b\,\text{yp}^2 + c\,\text{zp}^2 = 14$; what are a, b, and c

In[ ]:=
```
ClearAll["Global`*"]
```

In[ ]:=
```
surface =
 ContourPlot3D[5 x^2 + 3 y^2 + 2 z^2 + 4 x z == 14, {x, -4, 4}, {y, -4, 4}, {z, -4, 4}]
```

Out[ ]=



The given quadratic equation $5x^2 + 3y^2 + 2z^2 + 4xz = 14$ can be represented in matrix form, the way to do this is

We look at the non-cross terms (x^2, y^2, z^2) and we plug their coefficients in the diagonal entries directly.

The next step is to look at the cross terms (xy, yz, xz) and we plug in their coefficients divided by 2 in the non-diagonal entries.

For example, if we were given $5x^2 + 3y^2 + 2z^2 + 3\,xy + 6\,yz + 4\,xz = 14$ instead

our matrix would look like $\begin{pmatrix} 5 & 3 & 2 \\ 3/2 & 3 & 3/2 \\ 2 & 3 & 2 \end{pmatrix}$

For the given equation, the matrix looks like:

In[ ]:=
$$\text{mat} = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

Out[ ]=
```
{{5, 0, 2}, {0, 3, 0}, {2, 0, 2}}
```

Next step is to find the eigenvectors:

NOTE: We transpose the result of Eigenvectors[] because we want the columns to be the eigenvectors, not the rows.

In[ ]:=
```
U = Eigenvectors[mat] // Transpose
```
Out[ ]=
```
{{2, 0, -1}, {0, 1, 0}, {1, 0, 2}}
```

We have to make sure they're normalized:

In[ ]:=
```
U = Map[Normalize, U];
MatrixForm[U]
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{pmatrix}$$

The reason why we want the eigenvectors is that we're trying to diagonalize the given matrix:

In[ ]:=
```
Ut = Transpose[U];
MatrixForm[Ut]
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \end{pmatrix}$$

Diagonalizing the given matrix, we get:

In[ ]:=
```
diag = Ut.mat.U // Simplify;
MatrixForm[diag]
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiplying by {xp, yp, zp} on both sides, we get the quadratic form without any cross-terms:

In[ ]:=
```
{xp, yp, zp}.diag.{xp, yp, zp} // Simplify
```
Out[ ]=
```
{xp, yp, zp}.Umat.{{5, 0, 2}, {0, 3, 0}, {2, 0, 2}}.tUmat.{xp, yp, zp}
```

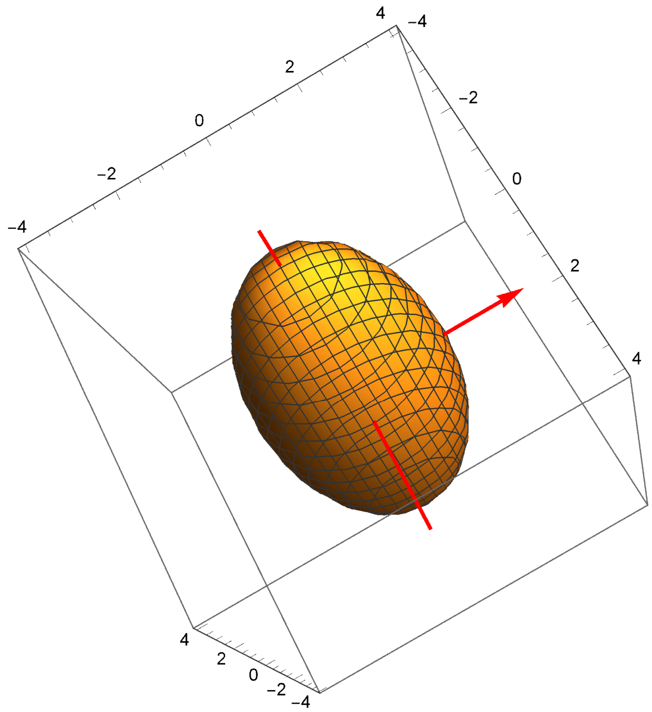
Plotting our new basis:

In[ ]:=
```
evec = Eigenvectors[mat]
```
Out[ ]=
```
{{2, 0, 1}, {0, 1, 0}, {-1, 0, 2}}
```

In[ ]:=
```
newbasis = {Graphics3D[{Thick, Red, Arrow[{{0, 0, 0}, 4 evec[[1]]}]}],
   Graphics3D[{Thick, Red, Arrow[{{0, 0, 0}, 4 evec[[2]]}]}],
   Graphics3D[{Thick, Red, Arrow[{{0, 0, 0}, 4 evec[[3]]}]}]};
```

In[ ]:= `Show[surface, newbasis]`

Out[ ]=



P3 .7 .29
Construct the matrix corresponding to a rotation of 90° about the y axis together with a reflection through the (x,z) plane. Use the standard right handed $\hat{x}, \hat{y}, \hat{z}$ basis .

The standard matrix for rotations about the y-axis by θ degrees is given by:

NOTE:

The y-value is preserved whereas the other components are multiplied by combinations of Sin[] and Cos[]

Also, the determinant is unsurprisingly 1.

In[ ]:= rotmat = $\begin{pmatrix} \text{Cos}[\theta] & 0 & \text{Sin}[\theta] \\ 0 & 1 & 0 \\ -\text{Sin}[\theta] & 0 & \text{Cos}[\theta] \end{pmatrix}$

Out[ ]= `{{Cos[θ], 0, Sin[θ]}, {0, 1, 0}, {-Sin[θ], 0, Cos[θ]}}`

In this case, we specifically want a 90 degree rotation, so we sub in $\theta \to \pi/2$:

In[ ]:=
```
rotmat90 = rotmat /. θ → π / 2;
MatrixForm[rotmat90]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

This makes somewhat good sense because our y-value is still being preserved and our x and z values are switched and negated.

Now, we also need a standard matrix that represents reflections through the (x,z) plane.
This means the x-z values are preserved whereas the y-values are reversed (negated):

In[ ]:=

$$\text{refmat} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Out[ ]= {{1, 0, 0}, {0, -1, 0}, {0, 0, 1}}

The composite matrix corresponding to several transformations is simply the product of the individual standard matrices:

In[ ]:=
```
finalmat = rotmat90.refmat;
MatrixForm[finalmat]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Wk3.P2

Consider the vector space of cubic polynomials with basis $\{1,x,x^2,x^3\}$. Let T be the operator that returns x p’[x] for any polynomial p[x] . What is the matrix representation of T in this basis?

We can see that T resembles the differentiation operator, however, it has an additional multiple of x in the end.
We can deduce the matrix representation of T by examining its effects on the basis vectors, in this case our basis is $\{1, x, x^2, x^3\}$:

For the basis of 1, we get:

```
basis1 = 1;
D[basis1, x] * x
```

Out[ ]= 0

This is what the first column of our standard matrix looks like.

For the basis of x, we get:

In[ ]:=
```
basisx = x;
D[basisx, x] * x
```

Out[ ]= x

This is what the second column of our standard matrix looks like.

For x^2, we get:

In[ ]:=
```
basisx2 = x^2;
D[basisx2, x] * x
```

Out[ ]= $2\,x^2$

This is what the third column of our standard matrix looks like.

For x^3, we get:

In[ ]:=
```
basisx3 = x^3;
D[basisx3, x] * x
```

Out[ ]= $3\,x^3$

This is what the fourth column of our standard matrix looks like.
Putting all the columns together, we get:

In[ ]:=
```
col1 = {0, 0, 0, 0};
col2 = {0, 1, 0, 0};
col3 = {0, 0, 2, 0};
col4 = {0, 0, 0, 3};
```

In[ ]:=
```
mat = {col1, col2, col3, col4};
MatrixForm[mat]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Wk3.P3

Verify that $\begin{pmatrix} 3 & 1+i & i \\ 1-i & 1 & 3 \\ -i & 3 & 1 \end{pmatrix}$ is Hermitian. Find a unitary matrix which will diagonalize it via a similarity transformation. Use N to get numerical values. Use Chop (look it up) to get rid of roundoff error.

Defining a variable to represent the given matrix:

In[ ]:=
```
Hmat = (3     1+i  i
        1-i   1    3
        -i    3    1)
```

Out[ ]=
```
{{3, 1 + i, i}, {1 - i, 1, 3}, {-i, 3, 1}}
```

We can verify that the given matrix is Hermitian by equating it to its conjugate transpose:

In[ ]:=
```
Hmatct = ConjugateTranspose[Hmat];
MatrixForm[Hmatct]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 3 & 1+i & i \\ 1-i & 1 & 3 \\ -i & 3 & 1 \end{pmatrix}$$

And we do see that this matrix is Hermitian:

In[ ]:=
```
Hmatct == Hmat
```

Out[ ]=
```
True
```

We can also verify that the matrix is Hermitian by equating its transpose and its conjugate:

In[ ]:=
```
Conjugate[Hmat] == Transpose[Hmat]
```

Out[ ]=
```
True
```

In[ ]:=
```
U = Map[Normalize, Eigenvectors[Hmat]] // Transpose // N // Chop;
MatrixForm[U]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0.230473 + 0.551826\, i & 0.148661 - 0.005275\, i & -0.478368 - 0.625625\, i \\ 0.579937 + 0.0768244\, i & -0.709578 + 0.0495538\, i & 0.355511 - 0.159456\, i \\ 0.547851 & 0.686961 & 0.477435 \end{pmatrix}$$

In[ ]:=
```
Uct = ConjugateTranspose[U] // Chop;
MatrixForm[Uct]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0.230473 - 0.551826\, i & 0.579937 - 0.0768244\, i & 0.547851 \\ 0.148661 + 0.005275\, i & -0.709578 - 0.0495538\, i & 0.686961 \\ -0.478368 + 0.625625\, i & 0.355511 + 0.159456\, i & 0.477435 \end{pmatrix}$$

In[ ]:=
```
diag = Uct.Hmat.U // Chop;
MatrixForm[diag]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 5.18296 & 0 & 0 \\ 0 & -2.10645 & 0 \\ 0 & 0 & 1.92349 \end{pmatrix}$$

Diagonalization Example:

Defining the given matrix:

In[ ]:=
```
mat = {{4, -3, -3}, {3, -2, -3}, {-1, 1, 2}};
MatrixForm[mat]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix}$$

Using Eigenvectors[] and Transpose[] to generate a matrix U whose columns are the eigenvectors of the given matrix:

In[ ]:=
```
U = Eigenvectors[mat] // Transpose;
MatrixForm[U]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -3 & 1 & 1 \\ -3 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Using Inverse[] to find the inverse of the given matrix:

In[ ]:=
```
Ut = Inverse[U];
MatrixForm[Ut]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ -3 & 4 & 3 \end{pmatrix}$$

Applying the diagonalization formula: $D = U^{-1} M U$

In[ ]:=
```
diag = Ut.mat.U;
MatrixForm[%]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The entries of our new diagonal matrix ought to match the eigenvalues of the given matrix:

In[ ]:=
```
Eigenvalues[mat]
```

Out[ ]= {2, 1, 1}

We can apply the reverse of the formula to obtain original matrix by doing $M = U D U^{-1}$

In[ ]:=
```
ogmat = U.diag.Ut;
MatrixForm[ogmat]
```

Out[ ]//MatrixForm=

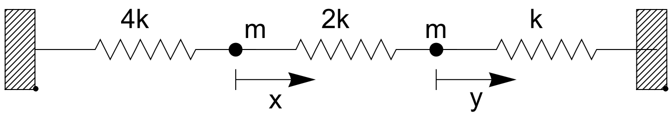
$$\begin{pmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix}$$

We see that the diagonalization formula: $D = U^{-1} M U$ works for D is a diagonal matrix and M is a “diagonalizable” matrix.

Week 4 Homework:

P3 .12 .16

Find the characteristic frequencies and characteristic modes of vibration for the system of masses and springs as shown in the figure.

Out[ ]=


This is our expression of the potential energy with time dependence on x and y:

In[ ]:= $\textbf{pot} = \left(\frac{1}{2} k_1 x[t]^2 + \frac{1}{2} k_2 (-x[t] + y[t])^2 + \frac{1}{2} k_3 y[t]^2\right) /. \{k_1 \to 4k, k_2 \to 2k, k_3 \to 1k\}$

Out[ ]= $2 k x[t]^2 + \frac{1}{2} k y[t]^2 + k (-x[t] + y[t])^2$

To write the equations of motion, we take the negative derivative for the potential energy with respect to distance.
The motion in x is:

```
In[ ]:= xmot = m x''[t] == Simplify[-D[pot, x[t]]]
```

Out[ ]= $m x''[t] == 2 k (-3 x[t] + y[t])$

The motion in y is:

```
In[ ]:= ymot = m y''[t] == Simplify[-D[pot, y[t]]]
```

Out[ ]= $m y''[t] == k (2 x[t] - 3 y[t])$

We can put these equations of motion in a list:

```
In[ ]:= eqmot = {xmot, ymot} // Expand;
TableForm[eqmot]
```

Out[ ]//TableForm=

$m x''[t] == -6 k x[t] + 2 k y[t]$
$m y''[t] == 2 k x[t] - 3 k y[t]$

We can confirm if these are the right equations by intuitively creating the force equations ourselves:

In[ ]:= $\textbf{xmotguess} = m x''[t] == (-k_1 x[t] - k_2 x[t] + k_2 y[t]) /. \{k_1 \to 4k, k_2 \to 2k, k_3 \to 1k\}$

Out[ ]= $m x''[t] == -6 k x[t] + 2 k y[t]$

In[ ]:= $\textbf{ymotguess} = m y''[t] == (k_2 x[t] - k_2 y[t] - k_3 y[t]) /. \{k_1 \to 4k, k_2 \to 2k, k_3 \to 1k\}$

Out[ ]= $m y''[t] == 2 k x[t] - 3 k y[t]$

Above, we do see that our guess matches the derivative of the potential energy. We generated these guesses

by focusing on just one mass, then seeing the direction and magnitude of each spring force ($F = \pm k\, x$) where
x is just the displacement of either mass.

Plugging in periodic solutions for $x[t]$ and $y[t]$ with the same time dependence, but different amplitudes. In this case, we are asked for our trial solutions to assume the form of $e^{i\,\omega\, t}$:

In[ ]:=
```
eqmot2 = eqmot /. {x → Function[t, Ax ⅇ^(ⅈ ω t)], y → Function[t, Ay ⅇ^(ⅈ ω t)]};
TableForm[eqmot2]
```

Out[ ]//TableForm=

$$-\text{Ax}\, e^{i\,t\,\omega}\, m\, \omega^2 == -6\, \text{Ax}\, e^{i\,t\,\omega}\, k + 2\, \text{Ay}\, e^{i\,t\,\omega}\, k$$
$$-\text{Ay}\, e^{i\,t\,\omega}\, m\, \omega^2 == 2\, \text{Ax}\, e^{i\,t\,\omega}\, k - 3\, \text{Ay}\, e^{i\,t\,\omega}\, k$$

We can clean up these equations a little bit by doing the substitution $t \to 0$:

In[ ]:=
```
eqmot3 = eqmot2 /. t → 0;
TableForm[eqmot3]
```

Out[ ]//TableForm=

$$-\text{Ax}\, m\, \omega^2 == -6\, \text{Ax}\, k + 2\, \text{Ay}\, k$$
$$-\text{Ay}\, m\, \omega^2 == 2\, \text{Ax}\, k - 3\, \text{Ay}\, k$$

In order to solve for ω, we can generate a coefficient matrix for Ax and Ay.
We can rewrite the above two equations in matrix form:

$$m\,\omega^2 \begin{pmatrix} \text{Ax} \\ \text{Ay} \end{pmatrix} = \begin{pmatrix} 6\,k & -2\,k \\ -2\,k & 3\,k \end{pmatrix} \begin{pmatrix} \text{Ax} \\ \text{Ay} \end{pmatrix}$$

Moving the left side to the right side:

$$\begin{pmatrix} 6\,k & -2\,k \\ -2\,k & 3\,k \end{pmatrix} \begin{pmatrix} \text{Ax} \\ \text{Ay} \end{pmatrix} - m\,\omega^2 \begin{pmatrix} \text{Ax} \\ \text{Ay} \end{pmatrix} = 0$$

Simplifying:

$$\begin{pmatrix} 6\,k - m\,\omega^2 & -2\,k \\ -2\,k & 3\,k - m\,\omega^2 \end{pmatrix} \begin{pmatrix} \text{Ax} \\ \text{Ay} \end{pmatrix} = 0$$

Defining a variable to represent our coefficient matrix:

In[ ]:=
```
mat = {{6 k - m ω^2, -2 k}, {-2 k, 3 k - m ω^2}};
MatrixForm[mat]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 6\,k - m\,\omega^2 & -2\,k \\ -2\,k & 3\,k - m\,\omega^2 \end{pmatrix}$$

Solving for the eigenvalues, we get:

In[ ]:=
```
efreq = Solve[Det[mat] == 0, ω]
```

Out[ ]=

$$\left\{\left\{\omega \to -\frac{\sqrt{2}\,\sqrt{k}}{\sqrt{m}}\right\}, \left\{\omega \to \frac{\sqrt{2}\,\sqrt{k}}{\sqrt{m}}\right\}, \left\{\omega \to -\frac{\sqrt{7}\,\sqrt{k}}{\sqrt{m}}\right\}, \left\{\omega \to \frac{\sqrt{7}\,\sqrt{k}}{\sqrt{m}}\right\}\right\}$$

The positive values are the characteristic frequency of vibration. Each frequency is associated with a mode of vibration, which can be found by its null space or eigenvector:

In[ ]:= NullSpace$\left[\text{mat /. } \omega \text{ -> } \frac{\sqrt{2}\sqrt{k}}{\sqrt{m}}\right]$

Out[ ]= $\left\{\left\{\frac{1}{2}, 1\right\}\right\}$

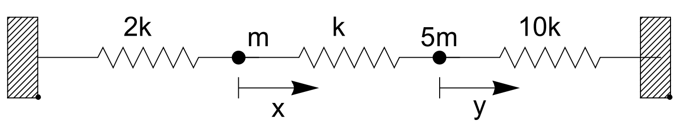
In[ ]:= NullSpace$\left[\text{mat /. } \omega \text{ -> } \frac{\sqrt{7}\sqrt{k}}{\sqrt{m}}\right]$

Out[ ]= $\{\{-2, 1\}\}$

We can see that for the first mode of vibration, they go in the same direction ie both positive. However, for the
second mode of vibration, they both go in opposite direction ie opposite signs

---

P3 .12 .18
Find the characteristic frequencies and characteristic modes of vibration for the system of masses and springs as shown in the figure.

Out[ ]= 

Writing our potential energy equation:

In[ ]:= pot = $\left(\frac{1}{2} k_1 x[t]^2 + \frac{1}{2} k_2 (-x[t] + y[t])^2 + \frac{1}{2} k_3 y[t]^2\right)$ /. $\{k_1 \to 2k, k_2 \to 1k, k_3 \to 10k\}$

Out[ ]= $k\,x[t]^2 + 5\,k\,y[t]^2 + \frac{1}{2} k\,(-x[t] + y[t])^2$

Generating equations of motion for x and y by taking the negative derivative of the potential energy with respect to distance:

```
In[ ]:= xmot = m x''[t] == -D[pot, x[t]] // Expand
```

Out[ ]= $m\,x''[t] == -3\,k\,x[t] + k\,y[t]$

```
In[ ]:= ymot = 5 m y''[t] == -D[pot, y[t]] // Expand
```

Out[ ]= $5\,m\,y''[t] == k\,x[t] - 11\,k\,y[t]$

```
In[ ]:= eqmot = {xmot, ymot};
TableForm[eqmot]
```

Out[ ]//TableForm=

$m\,x''[t] == -3\,k\,x[t] + k\,y[t]$

$5\,m\,y''[t] == k\,x[t] - 11\,k\,y[t]$

Plugging in our trial solution for x[t] and y[t] in the form of $A\,e^{i\,\omega\,t}$:

In[ ]:=
```
eqmot2 = eqmot /. {x → Function[t, Ax e^(i ω t)], y → Function[t, Ay e^(i ω t)]};
TableForm[eqmot2]
```

Out[ ]//TableForm=

$$-Ax\, e^{i t \omega}\, m\, \omega^2 == -3\, Ax\, e^{i t \omega}\, k + Ay\, e^{i t \omega}\, k$$
$$-5\, Ay\, e^{i t \omega}\, m\, \omega^2 == Ax\, e^{i t \omega}\, k - 11\, Ay\, e^{i t \omega}\, k$$

We clean up our equation by subbing in $t \to 0$

In[ ]:=
```
eqmot3 = eqmot2 /. t → 0;
TableForm[eqmot3]
```

Out[ ]//TableForm=

$$-Ax\, m\, \omega^2 == -3\, Ax\, k + Ay\, k$$
$$-5\, Ay\, m\, \omega^2 == Ax\, k - 11\, Ay\, k$$

Rewriting the above equations in matrix form:

$$\begin{pmatrix} m\,\omega^2 & 0 \\ 0 & 5\, m\, \omega^2 \end{pmatrix} \begin{pmatrix} Ax \\ Ay \end{pmatrix} = \begin{pmatrix} 3\, k & -k \\ -k & 11\, k \end{pmatrix} \begin{pmatrix} Ax \\ Ay \end{pmatrix}$$

Moving everything onto one side:

$$\begin{pmatrix} 3\, k & -k \\ -k & 11 \end{pmatrix} \begin{pmatrix} Ax \\ Ay \end{pmatrix} - \begin{pmatrix} m\,\omega^2 & 0 \\ 0 & 5\, m\, \omega^2 \end{pmatrix} \begin{pmatrix} Ax \\ Ay \end{pmatrix} = 0$$

Simplifying:

$$\begin{pmatrix} 3\, k - m\,\omega^2 & -k \\ -k & 11\, k - 5\, m\, \omega^2 \end{pmatrix} \begin{pmatrix} Ax \\ Ay \end{pmatrix} = 0$$

Our coefficient matrix now looks like:

In[ ]:=
```
mat = {{3 k - m ω^2, -k}, {-k, 11 k - 5 m ω^2}};
MatrixForm[mat]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 3\, k - m\,\omega^2 & -k \\ -k & 11\, k - 5\, m\, \omega^2 \end{pmatrix}$$

It's characteristic frequency is given by its eigenvalues:

In[ ]:=
```
Solve[Det[mat] == 0, ω] // Simplify
```

Out[ ]=
$$\left\{\left\{\omega \to -\frac{\sqrt{2}\,\sqrt{k}}{\sqrt{m}}\right\}, \left\{\omega \to \frac{\sqrt{2}\,\sqrt{k}}{\sqrt{m}}\right\}, \left\{\omega \to -\frac{4\,\sqrt{k}}{\sqrt{5}\,\sqrt{m}}\right\}, \left\{\omega \to \frac{4\,\sqrt{k}}{\sqrt{5}\,\sqrt{m}}\right\}\right\}$$

The characteristic modes are given by its eigenvectors:

In[ ]:=
```
NullSpace[mat /. ω → Sqrt[2] Sqrt[k] / Sqrt[m]]
```

Out[ ]= {{1, 1}}

In[ ]:=
```
NullSpace[mat /. ω → 4 √k / (√5 √m)]
```
$$\text{NullSpace}\left[\text{mat} \;/.\; \omega \to \frac{4\sqrt{k}}{\sqrt{5}\sqrt{m}}\right]$$

Out[ ]= {{-5, 1}}

P3 .15 .18

Find the matrix C that diagonalizes M= $\begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$. Observe that M is not symmetric and C is not orthogonal. However C does have an inverse C^{-1} ; find C^{-1} and show C^{-1}M C=D

Defining a variable to represent the given matrix:

In[ ]:=
```
mat = {{1, 0}, {3, -2}};
MatrixForm[mat]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix}$$

C is the standard matrix used to diagonalize the given matrix M.
We can construct C by having its columns be the eigenvectors of M:

In[ ]:=
```
Cmat = Eigenvectors[mat] // Transpose;
MatrixForm[Cmat]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Using Inverse[] to find its inverse:

In[ ]:=
```
invCmat = Inverse[Cmat];
MatrixForm[invCmat]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

Applying the diagonalization formula:

In[ ]:=
```
diag = invCmat.mat.Cmat;
MatrixForm[diag]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$$

We can see from earlier that because M is not symmetric, it’s matrix C is not orthogonal. If C were orthogonal then it’s inverse
would be the transpose. Furthermore, the formula to diagonalize becomes: $D = C^{-1} M C = C^T M C$

PWk4 .1

Define an inner product <f,g>= $\int_0^\infty e^{-t} f[t] \times g[t] \, dt$. Consider the functions $\{1,t,t^2,t^3,t^4\}$. Apply the Gram-Schmidt process to construct an orthonormal set of functions. Verify that they are orthonormal by constructing a table of all of the inner products . Do not use Orthogonalize (you can use it to check your answer) , but work out conditions for each e[i] .

```
In[ ]:= ClearAll["Global`*"]
```

We can start by defining our given basis:

```
In[ ]:= Thread[{old[1], old[2], old[3], old[4], old[5]} = {1, t, t^2, t^3, t^4}]
Out[ ]= {1, t, t^2, t^3, t^4}
```

Symbolically, our new orthonormal basis vectors will look like:

```
In[ ]:= Table[e[i], {i, 1, 5}]
Out[ ]= {e[1], e[2], e[3], e[4], e[5]}
```

Defining our inner product function:

```
In[ ]:= inner[f_, g_] := Integrate[e^-t f * g, {t, 0, ∞}]
```

The first orthonormal vector of our new basis can just assume the first vector of our old basis:

```
In[ ]:= e[1] = old[1] / Sqrt[inner[old[1], old[1]]]
Out[ ]= 1
```

We can define the second new basis vector as some linear combination of the first 2 old basis vectors:

```
In[ ]:= e[2] = a old[1] + b old[2]
Out[ ]= a + b t
```

where a and b are just constants.

Now we have a simple, symbolic expression for our new basis. We can generate conditions based on orthonormal properties that we know ought to be true, turn these conditions into equation form, and use Solve[] to find solutions to the constants (a,b):

```
In[ ]:= cond1 = inner[e[2], e[2]] == 1
Out[ ]= a^2 + 2 a b + 2 b^2 == 1
```

```
In[ ]:= cond2 = inner[e[2], e[1]] == 0
Out[ ]= a + b == 0
```

Cond1 is the normalized condition and Cond2 is the orthogonal condition.
Nonetheless, 2 equations and 2 unknowns, we can solve this bitch:

In[ ]:=
```
sol1 = Solve[{cond1, cond2}, {a, b}] // First
```
Out[ ]= $\{a \to -1, b \to 1\}$

We can post-wrap //First[] on the solution just to arbitrarily select only one of the given solutions.
Subbing in these solutions, our second new basis vector is:

In[ ]:=
```
e[2] = e[2] /. sol1
```
Out[ ]= $-1 + t$

Moving onto the next new basis vector, we can write it as a linear combination of the first 3 old basis vectors:

In[ ]:=
```
e[3] = a old[1] + b old[2] + c old[3]
```
Out[ ]= $a + b\,t + c\,t^2$

Notice that we will always have the 1 condition concerning normalization, and $n - 1$ conditions concerning orthogonality.
For instance, for the third new basis vector, it ought obey the normalization property and it ought to be orthogonal
to the first and second new basis vectors:

In[ ]:=
```
cond1 = inner[e[3], e[3]] == 1
```
Out[ ]= $a^2 + 2\,a\,b + 2\,b^2 + 4\,(a + 3\,b)\,c + 24\,c^2 == 1$

In[ ]:=
```
cond2 = inner[e[3], e[2]] == 0
```
Out[ ]= $b + 4\,c == 0$

In[ ]:=
```
cond3 = inner[e[3], e[1]] == 0
```
Out[ ]= $a + b + 2\,c == 0$

In[ ]:=
```
sol2 = Solve[{cond1, cond2, cond3}, {a, b, c}] // First
```
Out[ ]= $\left\{a \to 1, b \to -2, c \to \frac{1}{2}\right\}$

In[ ]:=
```
e[3] = e[3] /. sol2
```
Out[ ]= $1 - 2\,t + \frac{t^2}{2}$

Rinse & repeat.
We define our fourth new basis vector:

In[ ]:=
```
e[4] = a old[1] + b old[2] + c old[3] + d old[4]
```
Out[ ]= $a + b\,t + c\,t^2 + d\,t^3$

In[ ]:=
```
cond1 = inner[e[4], e[4]] == 1
```
Out[ ]= $a^2 + 2\,a\,(b + 2\,c + 6\,d) + 2\left(b^2 + 6\,b\,(c + 4\,d) + 12\left(c^2 + 10\,c\,d + 30\,d^2\right)\right) == 1$

In[ ]:=
```
cond2 = inner[e[4], e[3]] == 0
```
Out[ ]= $2\,(c + 9\,d) == 0$

In[ ]:=
```
cond3 = inner[e[4], e[2]] == 0
```
Out[ ]= $b + 4\,c + 18\,d == 0$

In[ ]:=
```
cond4 = inner[e[4], e[1]] == 0
```
Out[ ]= $a + b + 2\,c + 6\,d == 0$

In[ ]:=
```
sol3 = Solve[{cond1, cond2, cond3, cond4}, {a, b, c, d}] // First
```
Out[ ]= $\left\{a \to -1,\ b \to 3,\ c \to -\frac{3}{2},\ d \to \frac{1}{6}\right\}$

In[ ]:=
```
e[4] = e[4] /. sol3
```
Out[ ]= $-1 + 3\,t - \frac{3\,t^2}{2} + \frac{t^3}{6}$

Rinse & repeat

We define our fifth new basis vector:

In[ ]:=
```
e[5] = a old[1] + b old[2] + c old[3] + d old[4] + e old[5]
```
Out[ ]= $a + b\,t + c\,t^2 + d\,t^3 + e\,t^4$

In[ ]:=
```
cond1 = inner[e[5], e[5]] == 1
```
Out[ ]= $a^2 + 2\,a\,(b + 2\,c + 6\,d + 24\,e) + 2\,\left(b^2 + 6\,b\,(c + 4\,(d + 5\,e)) + 12\,\left(c^2 + 10\,c\,(d + 6\,e) + 30\,\left(d^2 + 14\,d\,e + 56\,e^2\right)\right)\right) == 1$

In[ ]:=
```
cond2 = inner[e[5], e[4]] == 0
```
Out[ ]= $6\,(d + 16\,e) == 0$

In[ ]:=
```
cond3 = inner[e[5], e[3]] == 0
```
Out[ ]= $2\,(c + 9\,(d + 8\,e)) == 0$

In[ ]:=
```
cond4 = inner[e[5], e[2]] == 0
```
Out[ ]= $b + 4\,c + 18\,d + 96\,e == 0$

In[ ]:=
```
cond5 = inner[e[5], e[1]] == 0
```
Out[ ]= $a + b + 2\,c + 6\,d + 24\,e == 0$

In[ ]:=
```
sol4 = Solve[{cond1, cond2, cond3, cond4, cond5}, {a, b, c, d, e}] // First
```
Out[ ]= $\left\{a \to 1,\ b \to -4,\ c \to 3,\ d \to -\frac{2}{3},\ e \to \frac{1}{24}\right\}$

In[ ]:=
```
e[5] = e[5] /. sol4
```
Out[ ]= $1 - 4\,t + 3\,t^2 - \frac{2\,t^3}{3} + \frac{t^4}{24}$

This is quite tiresome. Now we've found all the coefficients to numerically define our new basis. Our new basis looks like:

In[]:=
```
Thread[{new[1], new[2], new[3], new[4], new[5]} = {e[1], e[2], e[3], e[4], e[5]}] //
 Simplify
```

Out[]= $\left\{1,\ -1+t,\ \frac{1}{2}\left(2-4t+t^2\right),\ \frac{1}{6}\left(-6+18t-9t^2+t^3\right),\ 1-4t+3t^2-\frac{2t^3}{3}+\frac{t^4}{24}\right\}$

We can check our answer using Orthogonalize[]:

In[]:=
```
check = Orthogonalize[{old[1], old[2], old[3], old[4], old[5]}, inner] // Simplify
```

Out[]= $\left\{1,\ -1+t,\ \frac{1}{2}\left(2-4t+t^2\right),\ \frac{1}{6}\left(-6+18t-9t^2+t^3\right),\ 1-4t+3t^2-\frac{2t^3}{3}+\frac{t^4}{24}\right\}$

In[]:=
```
Table[inner[new[i], new[j]], {i, 1, 5}, {j, 1, 5}] // MatrixForm
```

Out[]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

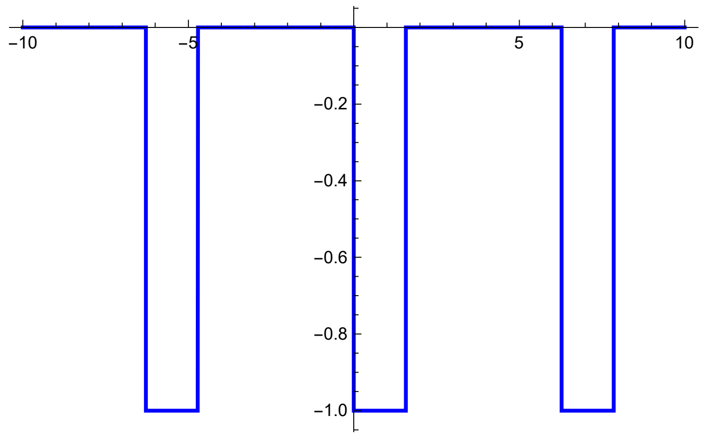
P7 .5 .5

$$f(x) = \begin{cases} 0 & , -\pi < x < 0 \\ -1 & , 0 < x < \pi/2 \\ 0 & \pi/2 < x < \pi \end{cases}$$

is periodic with period 2π. Construct the Fourier sine-cosine series (first 5 or so terms for each function) and plot f(x) and the Fourier series approximation . I used Mod[x,2π,-π] (look it up) to make a function periodic over the interval -π,π.

First, plotting our function using Which[]:

In[ ]:=
```
func = Which[-π < Mod[x, 2 π, -π] < 0, 0,
   0 < Mod[x, 2 π, -π] < π / 2, -1, π / 2 < Mod[x, 2 π, -π] < π, 0];
funcplot = Plot[func, {x, -10, 10}, Exclusions → None, PlotStyle → {Blue, Thick}]
```

Out[ ]=


We can symbolically define our Fourier Series. We will adopt the linear combinations of
A[*n*] Sin[*n x*] and *B*[*n*] Cos[*n x*]
where *A*[*n*], *B*[*n*] are the coefficients.

In[ ]:=
```
nterms = 10
```

Out[ ]= 10

In[ ]:=
```
fourser = Sum[A[i] Sin[i x] + B[i] Cos[i x], {i, 0, nterms}]
```

Out[ ]=
```
B[0] + B[1] Cos[x] + B[2] Cos[2 x] + B[3] Cos[3 x] + B[4] Cos[4 x] + B[5] Cos[5 x] +
 B[6] Cos[6 x] + B[7] Cos[7 x] + B[8] Cos[8 x] + B[9] Cos[9 x] + B[10] Cos[10 x] +
 A[1] Sin[x] + A[2] Sin[2 x] + A[3] Sin[3 x] + A[4] Sin[4 x] + A[5] Sin[5 x] +
 A[6] Sin[6 x] + A[7] Sin[7 x] + A[8] Sin[8 x] + A[9] Sin[9 x] + A[10] Sin[10 x]
```

We can now define 4 functions to represent the left-hand side and right-hand side of the equation after we take the inner
product of both sides and then integrate.
NOTE: This is unlike how we usually define just the 2 functions because we are doing a basis of both Sin[] and Cos[] so we need
2 functions for each basis.
Also keep in mind of orthogonality, we must confirm that whatever's inside the Sin[] and Cos[] ought to match our
symbolic Fourier Series

In[ ]:=
```
sinlhs[n_] := Integrate[func Sin[n x], {x, -π, π}]
sinrhs[n_] := Integrate[fourser Sin[n x], {x, -π, π}]
```

Now for the Cos[] basis:

In[ ]:=
```
coslhs[n_] := Integrate[func Cos[n x], {x, -π, π}]
cosrhs[n_] := Integrate[fourser Cos[n x], {x, -π, π}]
```

Now we use Table[] to create a list of a bunch of equations that we can use to solve for the coefficients.

We can do them individually, Sin[] and Cos[], then use Join[] to combine the 2 lists of equations.

In[ ]:=
```
sineqs = Table[sinlhs[i] == sinrhs[i], {i, 0, nterms}];
TableForm[sineqs]
```

Out[ ]//TableForm=

True

$-1 == \pi A[1]$

$-1 == \pi A[2]$

$-\frac{1}{3} == \pi A[3]$

$0 == \pi A[4]$

$-\frac{1}{5} == \pi A[5]$

$-\frac{1}{3} == \pi A[6]$

$-\frac{1}{7} == \pi A[7]$

$0 == \pi A[8]$

$-\frac{1}{9} == \pi A[9]$

$-\frac{1}{5} == \pi A[10]$

In[ ]:=
```
coseqs = Table[coslhs[i] == cosrhs[i], {i, 0, nterms}];
TableForm[coseqs]
```

Out[ ]//TableForm=

$-\frac{\pi}{2} == 2\pi B[0]$

$-1 == \pi B[1]$

$0 == \pi B[2]$

$\frac{1}{3} == \pi B[3]$

$0 == \pi B[4]$

$-\frac{1}{5} == \pi B[5]$

$0 == \pi B[6]$

$\frac{1}{7} == \pi B[7]$

$0 == \pi B[8]$

$-\frac{1}{9} == \pi B[9]$

$0 == \pi B[10]$

Solving for both coefficients A[n] and B[n]:

In[ ]:=
```
sol = Solve[Join[sineqs, coseqs],
   Join[Table[A[i], {i, 1, nterms}], Table[B[i], {i, 0, nterms}]]] // Flatten
```

Out[ ]=

$$\left\{A[1] \to -\frac{1}{\pi}, A[2] \to -\frac{1}{\pi}, A[3] \to -\frac{1}{3\pi}, A[4] \to 0, A[5] \to -\frac{1}{5\pi}, A[6] \to -\frac{1}{3\pi}, A[7] \to -\frac{1}{7\pi}, A[8] \to 0, A[9] \to -\frac{1}{9\pi}, A[10] \to -\frac{1}{5\pi}, B[0] \to -\frac{1}{4}, B[1] \to -\frac{1}{\pi}, B[2] \to 0, B[3] \to \frac{1}{3\pi}, B[4] \to 0, B[5] \to -\frac{1}{5\pi}, B[6] \to 0, B[7] \to \frac{1}{7\pi}, B[8] \to 0, B[9] \to -\frac{1}{9\pi}, B[10] \to 0\right\}$$

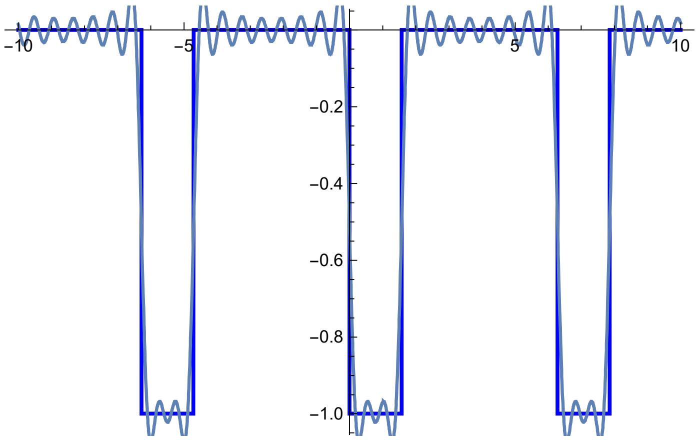
Plugging in our solution:

In[ ]:=
```
fourser2 = fourser /. sol
```

Out[ ]=
$$-\frac{1}{4}-\frac{\mathrm{Cos}[x]}{\pi}+\frac{\mathrm{Cos}[3x]}{3\pi}-\frac{\mathrm{Cos}[5x]}{5\pi}+\frac{\mathrm{Cos}[7x]}{7\pi}-\frac{\mathrm{Cos}[9x]}{9\pi}-\frac{\mathrm{Sin}[x]}{\pi}-\frac{\mathrm{Sin}[2x]}{\pi}-\frac{\mathrm{Sin}[3x]}{3\pi}-\frac{\mathrm{Sin}[5x]}{5\pi}-\frac{\mathrm{Sin}[6x]}{3\pi}-\frac{\mathrm{Sin}[7x]}{7\pi}-\frac{\mathrm{Sin}[9x]}{9\pi}-\frac{\mathrm{Sin}[10x]}{5\pi}$$

Plotting our given function and our Fourier Series:

In[ ]:=
```
fourplot = Plot[fourser2, {x, -10, 10}];
Show[funcplot, fourplot]
```

Out[ ]=


Week 5 Homework:

P7 .5 .11

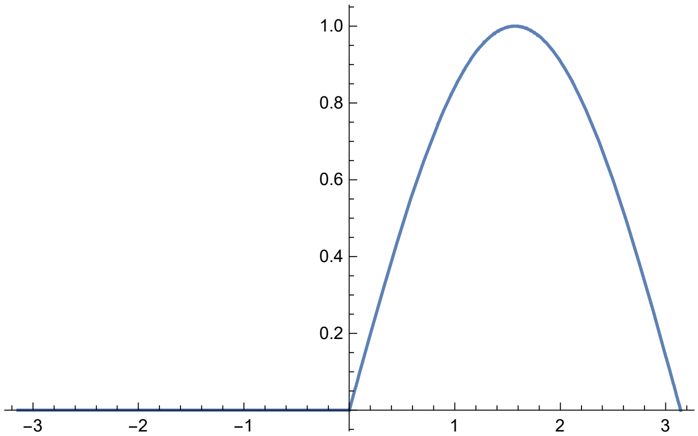
Consider the periodic function f[x] = $\begin{cases} 0 & -\pi < x < 0 \\ \mathrm{Sin}[x] & 0 < x < \pi \end{cases}$

Make a plot of the function. Find the Fourier series for the function using any convenient tools, and make a plot that shows the comparison.

We can start off by defining the given function:

In[ ]:=
```
func = Which[-π < x < 0, 0, 0 < x < π, Sin[x]];
Plot[func, {x, -π, π}]
```

Out[ ]=



In[ ]:=
```
fourser = FourierSeries[func, x, 5]
```

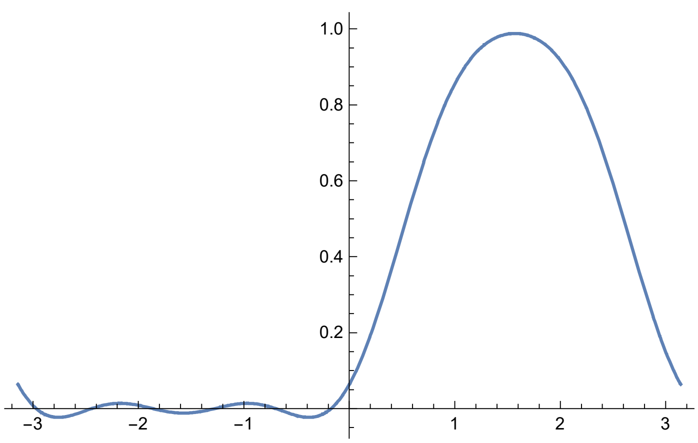
Out[ ]= $\frac{1}{4} i e^{-i x} - \frac{1}{4} i e^{i x} + \frac{1}{\pi} - \frac{e^{-2 i x}}{3 \pi} - \frac{e^{2 i x}}{3 \pi} - \frac{e^{-4 i x}}{15 \pi} - \frac{e^{4 i x}}{15 \pi}$

In[ ]:=
```
fourtrig = ExpToTrig[fourser]
```

Out[ ]= $\frac{1}{\pi} - \frac{2 \,\mathrm{Cos}[2 x]}{3 \pi} - \frac{2 \,\mathrm{Cos}[4 x]}{15 \pi} + \frac{\mathrm{Sin}[x]}{2}$

In[ ]:=
```
Plot[fourtrig, {x, -π, π}]
```

Out[ ]=



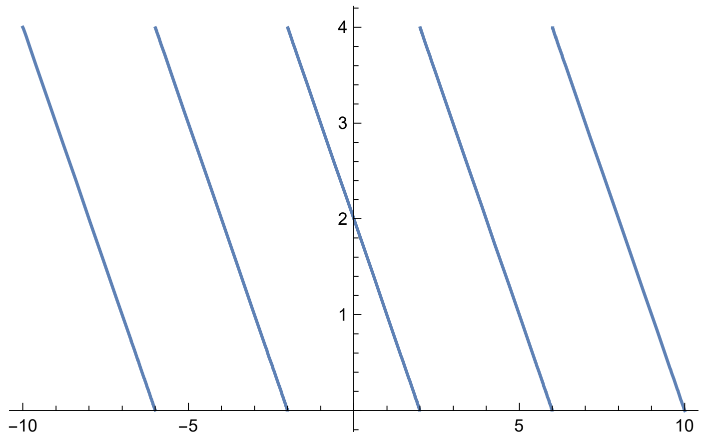
P7 .8 .13

Find the complex exponential series for f[x] =2-x -2<x<2 , periodically repeated. Explicitly calculate the integrals that define the coefficients for every exponential term. You can check your answer with the FourierSeries function. Consult the Handbook Fourier series notebook for guidance on how to do this. Use ExpToTrig to show that the complex exponential series is a conventional

trigonometric series in disguise. Make a graph of several periods of f[x] and compare your complex exponential series to it.
Recall that Mod[x,per,off] is a convenient way to generate periodic behavior.

Defining a variable to represent our function.
We use Mod[] to graph several periods.

In[ ]:=
```
func = 2 - x;
Plot[2 - Mod[x, 4, -2], {x, -10, 10}]
```

Out[ ]=


A orthonormal complex exponential basis is given by:

In[ ]:=
```
cexp2L[n_] := ⅇ^(ⅈ π n x / L) / Sqrt[2 L]
```

We also define a Hermitian inner product to be:

In[ ]:=
```
hip2L[f_, g_] := Integrate[(f /. Complex[x_, y_] → Complex[x, -y]) * g, {x, -L, L}]
```

Now that we have our basis and its proper inner product, we can construct our Fourier Series, however unlike how we usually define the symbolic Fourier Series first and then solve for the coefficients A[n].

In this case, the prompt tells us to adopt complex exponentials as a basis. We can generate the basis functions using
cexp2L[] and also we have to take the Hermitian inner product because complex inner products obey different axioms.

In[ ]:=
```
nterms = 5
```

Out[ ]= 5

In[ ]:=
```
fourser = Sum[cexp2L[i] * hip2L[cexp2L[i], func], {i, -nterms, nterms}] /. L → 2
```

Out[ ]=
$$2-\frac{2\,i\,e^{-\frac{1}{2}i\pi x}}{\pi}+\frac{2\,i\,e^{\frac{i\pi x}{2}}}{\pi}+\frac{i\,e^{-i\pi x}}{\pi}-\frac{i\,e^{i\pi x}}{\pi}-\frac{2\,i\,e^{-\frac{3}{2}i\pi x}}{3\pi}+\frac{2\,i\,e^{\frac{3i\pi x}{2}}}{3\pi}+\frac{i\,e^{-2i\pi x}}{2\pi}-\frac{i\,e^{2i\pi x}}{2\pi}-\frac{2\,i\,e^{-\frac{5}{2}i\pi x}}{5\pi}+\frac{2\,i\,e^{\frac{5i\pi x}{2}}}{5\pi}$$

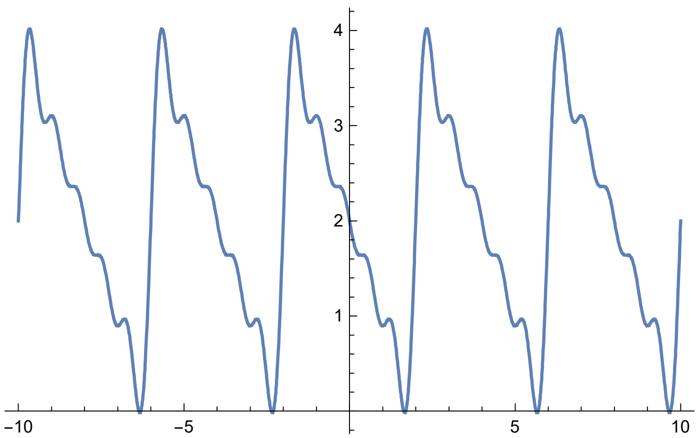
In Trig. form our series looks like:

In[ ]:=
```
ExpToTrig[fourser]
```

Out[ ]=
$$2-\frac{4\,\mathrm{Sin}\left[\frac{\pi x}{2}\right]}{\pi}+\frac{2\,\mathrm{Sin}[\pi x]}{\pi}-\frac{4\,\mathrm{Sin}\left[\frac{3\pi x}{2}\right]}{3\pi}+\frac{\mathrm{Sin}[2\pi x]}{\pi}-\frac{4\,\mathrm{Sin}\left[\frac{5\pi x}{2}\right]}{5\pi}$$

Plotting our series:

In[ ]:=
```
Plot[fourser, {x, -10, 10}]
```

Out[ ]=


Checking our answer with FourierSeries[]:

In[ ]:=
```
check = FourierSeries[func, x, nterms, FourierParameters → {1, π / 2}]
```

Out[ ]=
$$2-\frac{2\,i\,e^{-\frac{1}{2}i\pi x}}{\pi}+\frac{2\,i\,e^{\frac{i\pi x}{2}}}{\pi}+\frac{i\,e^{-i\pi x}}{\pi}-\frac{i\,e^{i\pi x}}{\pi}-\frac{2\,i\,e^{-\frac{3}{2}i\pi x}}{3\pi}+\frac{2\,i\,e^{\frac{3i\pi x}{2}}}{3\pi}+\frac{i\,e^{-2i\pi x}}{2\pi}-\frac{i\,e^{2i\pi x}}{2\pi}-\frac{2\,i\,e^{-\frac{5}{2}i\pi x}}{5\pi}+\frac{2\,i\,e^{\frac{5i\pi x}{2}}}{5\pi}$$

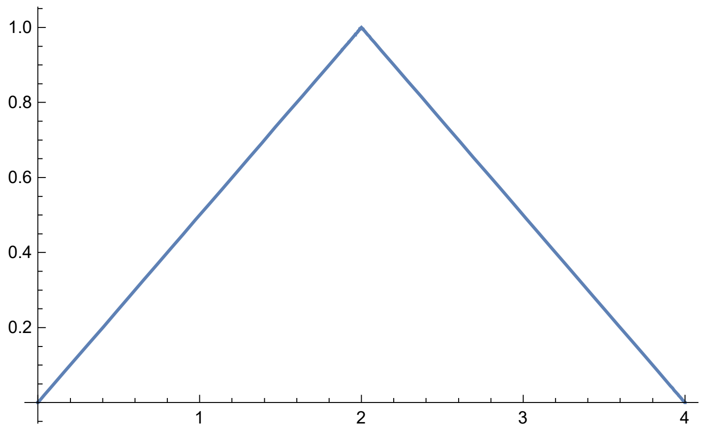
P7 .9 .23

A violin string is plucked at time t=0 with a deviation from the straight line f(x,0) as shown in the figure. To calculate the subsequent motion f[x,t], the displacement at time t at point x, we will see later that it is first necessary to expand f[x,0] in a Fourier sine series. Find the series if a string of length L is plucked at the center by a height h. Make sure it works by making a plot of the

deflection for L->4,h->1.

Plotting our function:

In[ ]:=
```
func = Which[0 ≤ x < L / 2, h/(L/2) x, x == L / 2, h, L / 2 < x ≤ L, -h/(L/2) x + 2 h];
funcplot = Plot[func /. {L → 4, h → 1}, {x, 0, 4}]
```

Out[ ]=


Clearly a Sin[] basis would work out perfect in this case, thus we being grinding out our Fourier Series routine:

In[ ]:=
```
nterms = 10
```

Out[ ]= 10

In[ ]:=
```
fourser = Sum[A[n] Sin[n π x/L], {n, 1, nterms}]
```

Out[ ]= $A[1]\,\mathrm{Sin}\left[\frac{\pi x}{L}\right] + A[2]\,\mathrm{Sin}\left[\frac{2\pi x}{L}\right] + A[3]\,\mathrm{Sin}\left[\frac{3\pi x}{L}\right] + A[4]\,\mathrm{Sin}\left[\frac{4\pi x}{L}\right] + A[5]\,\mathrm{Sin}\left[\frac{5\pi x}{L}\right] + A[6]\,\mathrm{Sin}\left[\frac{6\pi x}{L}\right] + A[7]\,\mathrm{Sin}\left[\frac{7\pi x}{L}\right] + A[8]\,\mathrm{Sin}\left[\frac{8\pi x}{L}\right] + A[9]\,\mathrm{Sin}\left[\frac{9\pi x}{L}\right] + A[10]\,\mathrm{Sin}\left[\frac{10\pi x}{L}\right]$

In[ ]:=
```
lhs[n_] := Integrate[func * Sin[n π x/L], {x, 0, L}, Assumptions → L > 0]
rhs[n_] := Integrate[fourser * Sin[n π x/L], {x, 0, L}, Assumptions → L > 0]
```

In[ ]:=
```
equations = Table[lhs[i] == rhs[i], {i, 1, nterms}]
```

Out[ ]= $\left\{\frac{4hL}{\pi^2} == \frac{1}{2} L\,A[1],\ 0 == \frac{1}{2} L\,A[2],\ -\frac{4hL}{9\pi^2} == \frac{1}{2} L\,A[3],\ 0 == \frac{1}{2} L\,A[4],\ \frac{4hL}{25\pi^2} == \frac{1}{2} L\,A[5],\ 0 == \frac{1}{2} L\,A[6],\ -\frac{4hL}{49\pi^2} == \frac{1}{2} L\,A[7],\ 0 == \frac{1}{2} L\,A[8],\ \frac{4hL}{81\pi^2} == \frac{1}{2} L\,A[9],\ 0 == \frac{1}{2} L\,A[10]\right\}$

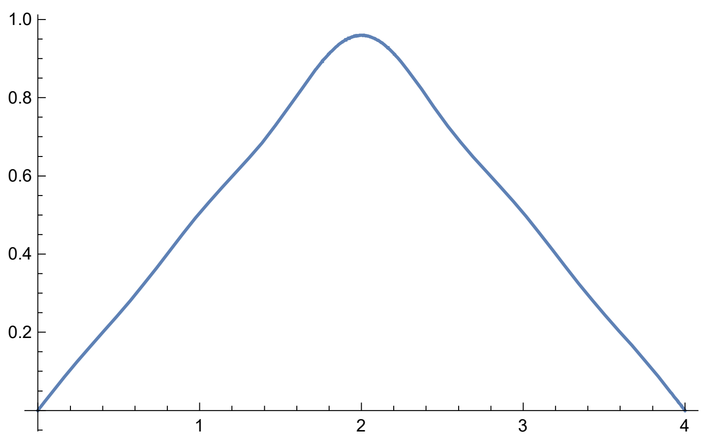
In[ ]:=
```
Asol = Solve[equations, Table[A[i], {i, 1, nterms}]] // Flatten
```

Out[ ]=
$$\left\{A[1] \to \frac{8\,h}{\pi^2},\ A[2] \to 0,\ A[3] \to -\frac{8\,h}{9\,\pi^2},\ A[4] \to 0,\ A[5] \to \frac{8\,h}{25\,\pi^2},\ A[6] \to 0,\ A[7] \to -\frac{8\,h}{49\,\pi^2},\ A[8] \to 0,\ A[9] \to \frac{8\,h}{81\,\pi^2},\ A[10] \to 0\right\}$$

In[ ]:=
```
fourser2 = fourser /. Asol
```

Out[ ]=
$$\frac{8\,h\,\mathrm{Sin}\left[\frac{\pi x}{L}\right]}{\pi^2} - \frac{8\,h\,\mathrm{Sin}\left[\frac{3\pi x}{L}\right]}{9\,\pi^2} + \frac{8\,h\,\mathrm{Sin}\left[\frac{5\pi x}{L}\right]}{25\,\pi^2} - \frac{8\,h\,\mathrm{Sin}\left[\frac{7\pi x}{L}\right]}{49\,\pi^2} + \frac{8\,h\,\mathrm{Sin}\left[\frac{9\pi x}{L}\right]}{81\,\pi^2}$$

In[ ]:=
```
Plot[fourser2 /. {L → 4, h → 1}, {x, 0, 4}]
```

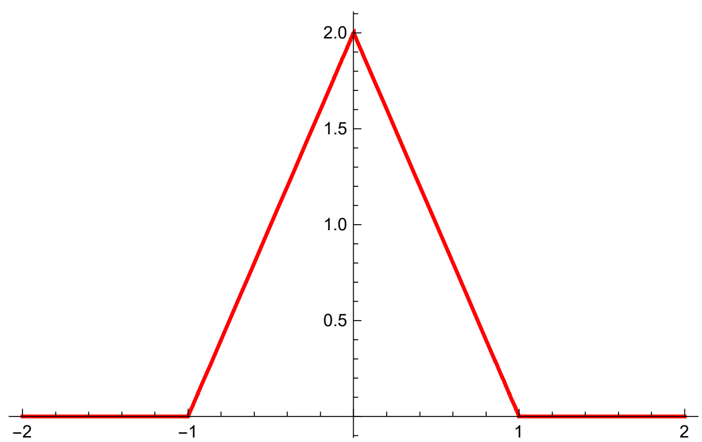
Out[ ]=


P7 .12 .9

a) Find the Fourier transform f(k) for the function in the sketch. Make a plot of the Fourier transform as a function of k.

Defining and plotting our function:

In[ ]:=
```
func = Which[x < -a, 0, -a < x < 0, 2 a + 2 x, 0 < x < a, 2 a - 2 x, x > a, 0];
Plot[func /. a → 1, {x, -2, 2}, PlotStyle → {Red, Thick}]
```

Out[ ]=


From the Mathematica Handbook, we can define a function to conveniently compute a Fourier Series transformation:

In[ ]:=
```
fourtran = Integrate[e^(i k x) * func, {x, -∞, ∞}, Assumptions → a > 0] / Sqrt[2 π] // Simplify
```

Out[ ]=
$$-\frac{\sqrt{2}\, e^{-i a k} \left(-1 + e^{i a k}\right)^2}{k^2 \sqrt{\pi}}$$

In Trig. form our series looks like:

In[ ]:=
```
ExpToTrig[fourtran] // Simplify
```

Out[ ]=
$$\frac{4 \sqrt{\frac{2}{\pi}}\, \mathrm{Sin}\left[\frac{a k}{2}\right]^2}{k^2}$$

We can also compute the Fourier Series “by hand”, we would have to break up the region of integration into 2 pieces:

In[ ]:=
```
four1 = Integrate[e^(i k x) * (2 a + 2 x), {x, -a, 0}, Assumptions → a > 0] / Sqrt[2 π]
```

Out[ ]=
$$\frac{2 - 2 e^{-i a k} - 2 i a k}{k^2 \sqrt{2 \pi}}$$

In[ ]:=
```
four2 = Integrate[e^(i k x) * (2 a - 2 x), {x, 0, a}, Assumptions → a > 0] / Sqrt[2 π]
```

Out[ ]=
$$\frac{2 - 2 e^{i a k} + 2 i a k}{k^2 \sqrt{2 \pi}}$$

In[ ]:=
```
fourtran2 = four1 + four2 // Simplify
```

Out[ ]=
$$-\frac{\sqrt{2}\, e^{-i a k} \left(-1 + e^{i a k}\right)^2}{k^2 \sqrt{\pi}}$$

In Trig form:

In[ ]:=
```
ExpToTrig[fourtran2]
```

Out[ ]=
$$-\frac{\sqrt{\frac{2}{\pi}}\, (\mathrm{Cos}[a k] - i\, \mathrm{Sin}[a k])\, (-1 + \mathrm{Cos}[a k] + i\, \mathrm{Sin}[a k])^2}{k^2}$$

We can check using FourierTransform[]:

In[ ]:=
```
FourierTransform[func, x, k, Assumptions → a > 0]
```

Out[ ]=
$$-\frac{\sqrt{2}\, e^{-i a k} \left(-1 + e^{i a k}\right)^2}{k^2 \sqrt{\pi}}$$

NOTE: a is our given parameter whereas k is the Fourier Transform pair variable:

P8 .11 .15 a & d

Compute a) $\int_0^{\pi}\text{Sin}[x]\,\delta\left(x-\frac{\pi}{2}\right)dx$ and d) $\int_0^{\pi}\text{Cosh}[x]\,\delta''(x-1)\,dx$.

Use basic properties of δ(x) to do the problem “by hand” , and then verify with DiracDelta

```
In[]:= nterms = 3
Out[]= 3
```

The mechanics of how DiracDelta functions work is that they output 1 when $x = x_0$ where x_0 is the point of interest, and they output
0 for every other x.
Because of this behavior, we can just take the Taylor Series approximation around $x_0 = \pi/2$ for the function Sin[x] particularly.

```
In[]:= approx1 = Series[Sin[x], {x, π/2, nterms}] // Normal // Simplify
```

Out[]= $1-\frac{1}{8}(\pi-2x)^2$

With the point of interesting being $x_0 = \pi/2$, we should replace $x \to x_0$:

```
In[]:= approx1 /. x → π / 2
Out[]= 1
```

We can check our answer by integrating directly:

```
In[]:= Integrate[Sin[x] DiracDelta[x - π / 2], {x, 0, π}]
Out[]= 1
```

We do a similar approach to finding part (d), however this time since we are given the 2nd derivative of the DiracDelta function
we take the 2nd derivative of our Taylor Series approximation and then sub $x \to x_0 = 1$:

```
In[]:= approx2 = Series[Cosh[x], {x, 1, nterms}] // Normal // Simplify
```

Out[]= $\text{Cosh}[1]+\frac{1}{2}(-1+x)^2\,\text{Cosh}[1]+(-1+x)\,\text{Sinh}[1]+\frac{1}{6}(-1+x)^3\,\text{Sinh}[1]$

```
In[]:= approx2 = ((-1)^n D[approx2, {x, n}]) /. n → 2
Out[]= Cosh[1] + (-1 + x) Sinh[1]
```

```
In[]:= approx2 /. x → 1
Out[]= Cosh[1]
```

Checking our answer:

```
In[]:= Integrate[Cosh[x] D[DiracDelta[x - 1], {x, 2}], {x, 0, π}]
Out[]= Cosh[1]
```

P8 .11 .21 a & b

Compute a) $\int_0^3 (5x-2)\,\delta(2-x)\,dx$ and b) $\int_0^\infty \phi(x)\,\delta(x^2-a^2)\,dx$

Use basic properties of δ(x) and change of variables to do the problem “by hand” , and then verify with DiracDelta

A standard step to doing the problem “by hand” is to take the Taylor Series approximation of our given function
around x_0 and then subbing in the point of interest for $x \to x_0$

```
In[]:= nterms = 3
Out[]= 3

In[]:= approx1 = Series[5 x - 2, {x, 2, nterms}] // Normal // Simplify
Out[]= -2 + 5 x

In[]:= approx1 /. x → 2
Out[]= 8
```

For part (b), we can do a change of variables for the function inside the DiracDelta function:

$\int_0^\infty \phi(x)\,\delta\left(x^2-a^2\right)dx$, we let $g(x) = x^2 - a^2$, $dg = 2\,x\,dx$, $x = \text{Sqrt}[g + a^2]$

Subbing in our variables:

$$\int_{-a^2}^\infty \phi(x)\,\delta(g)\,\frac{dg}{2\,x} = \int_{-a^2}^\infty \phi\,(\text{Sqrt}[g + a^2])\,\delta(g)\,\frac{dg}{2\,\text{Sqrt}[g + a^2]}$$

Since the δ function picks out the value of the integrand at the value of the integration variable which makes
the argument inside the δ function zero ($g = 0$). In this case, it is just $x = 0$, so it’s as if we just plug in zero for g:

$$\int_{-a^2}^\infty \phi\,(\text{Sqrt}[g + a^2])\,\delta(g)\,\frac{dg}{2\,\text{Sqrt}[g + a^2]} = \frac{\phi(a)}{2\,a}$$

P8 .3.11

Solve y’ +y Cos[x] = Sin[2 x]

Use the standard formula for the first order linear equation

$$y[x] = h[x] + h[x] \times \int_0^x h[-w] \times f[w]\,dw \quad \text{with } h[x] = C_1\, e^{-\int_0^x p[z]\,dz} \quad \text{and}$$

check your result with DSolve.

Defining a variable to represent the given ODE:

```
In[•]:= eq = y'[x] + y[x] Cos[x] == Sin[2 x]
Out[•]= Cos[x] y[x] + y′[x] == Sin[2 x]
```

Before we attempted to define a function to "conveniently" compute *h*[*x*] however things got too complicated.
Instead we can straight up plug everything in:

```
In[•]:= h[x] = ⅇ^(-Integrate[Cos[z], {z, 0, x}])
```

Out[•]= $e^{-\mathrm{Sin}[x]}$

```
In[•]:= sol = C[1] × h[x] + h[x] * Integrate[ⅇ^(Sin[w]) * Sin[2 w], {w, 0, x}] // Expand
```

Out[•]= $-2 + 2\, e^{-\mathrm{Sin}[x]} + e^{-\mathrm{Sin}[x]}\, C[1] + 2\, \mathrm{Sin}[x]$

Checking with DSolve[]:

```
In[•]:= DSolve[eq, y[x], x]
```

Out[•]= $\{\{y[x] \to -2 + e^{-\mathrm{Sin}[x]}\, C[1] + 2\, \mathrm{Sin}[x]\}\}$

P8 .5 .1

Solve y''+y'-2y=0 . Do this by explicitly substituting a exponential trial solution using Function to convert the DE to an algebraic equation . Solve the algebraic equation and use the results to construct the general solution of the DE.Use Function to plug in your general solution into the DE to make sure that it works. Check with DSolve.

```
In[•]:= ClearAll["Global`*"]
```

Defining a variable to represent the given ODE:

```
In[•]:= eq = y''[x] + y'[x] - 2 y[x] == 0
Out[•]= -2 y[x] + y′[x] + y″[x] == 0
```

We employ an exponential trial solution to convert the given ODE into an algebraic equation by subbing in $y \to e^\wedge(r\, x)$:

```
In[•]:= eq2 = eq /. y → Function[x, ⅇ^(r x)]
```

Out[•]= $-2\, e^{r x} + e^{r x}\, r + e^{r x}\, r^2 == 0$

In[•]:= $\mathrm{eq3} = \dfrac{\mathrm{eq2}}{e^\wedge(r\, x)}$ // Simplify

Out[•]= $e^{-r x}\left(e^{r x}\left(-2 + r + r^2\right) == 0\right)$

Simplifying the equation, we can see that the solutions are the roots to the polynomial: $r^2 + r - 2 = 0$:

In[ ]:=
```
roots = Solve[-2 + r + r^2 == 0]
```
Out[ ]=
```
{{r → -2}, {r → 1}}
```

Creating a general solution:

In[ ]:=
```
trialsol = e^(r x)
```
Out[ ]= $e^{r x}$

In[ ]:=
```
homosol = trialsol /. roots
```
Out[ ]= $\{e^{-2 x}, e^{x}\}$

In[ ]:=
```
gensol = Thread[{C[1], C[2]} * homosol] // Total
```
Out[ ]= $e^{-2 x} C[1] + e^{x} C[2]$

Testing our trial solutions:

In[ ]:=
```
eq /. y → Function[x, e^(-2 x)]
```
Out[ ]=
```
True
```

In[ ]:=
```
eq /. y → Function[x, e^(x)]
```
Out[ ]=
```
True
```

In[ ]:=
```
eq /. y → Function[x, e^(-2 x) C[1] + e^x C[2]] // Simplify
```
Out[ ]=
```
True
```

Using DSolve[] to check our answer:

In[ ]:=
```
DSolve[eq, y[x], x]
```
Out[ ]= $\{\{y[x] \to e^{-2 x} C[1] + e^{x} C[2]\}\}$

In[ ]:=
```
Integrate[e^-x, {x, 0, 1}]
```
Out[ ]= $\frac{-1 + e}{e}$

In[ ]:=
```
Integrate[e^x, {x, 0, 1}]
```
Out[ ]= $-1 + e$

In[ ]:= $\frac{-1 + e}{e} == -1 + e$

Out[ ]=
```
False
```

In[ ]:=
```
Series[Sin[x], {x, 0, 3}]
```
Out[ ]= $x - \frac{x^3}{6} + O[x]^4$

In[∘]:= `Series[1/Sqrt[1 + x], {x, 0, 3}]`

Out[∘]= $1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + O[x]^4$

P8 .5 .3

Solve y''+2y'+2y=0 . Do this by explicitly substituting an exponential trial solution using Function to convert the DE to an algebraic equation . Solve the algebraic equation and construct a general two parameter family of solutions of complex exponentials. Find an equivalent two parameter family that is manifestly real (see Eq. 8.5.16) ; use Function to plug it into the DE to verify. Check with DSolve.

In[∘]:= `eq = y''[x] + 2 y'[x] + 2 y[x] == 0`

Out[∘]= $2\,y[x] + 2\,y'[x] + y''[x] == 0$

In[∘]:= `eq2 = eq /. y → Function[x, e^(r x)]`

Out[∘]= $2\,e^{r\,x} + 2\,e^{r\,x}\,r + e^{r\,x}\,r^2 == 0$

In[∘]:= `roots = Solve[eq2, r]`

Out[∘]= $\{\{r \to -1 - i\}, \{r \to -1 + i\}\}$

In[∘]:= `trialsol = e^(r x) /. roots`

Out[∘]= $\{e^{(-1-i)\,x}, e^{(-1+i)\,x}\}$

In[∘]:= `gensolimag = Thread[{C[1], C[2]} * trialsol] // Total`

Out[∘]= $e^{(-1-i)\,x}\,C[1] + e^{(-1+i)\,x}\,C[2]$

In[∘]:= `famsol = Function[x, e^((-1-i) x) C[1] + e^((-1+i) x) C[2]]`

Out[∘]= $\text{Function}[x, e^{(-1-i)\,x}\,C[1] + e^{(-1+i)\,x}\,C[2]]$

In[∘]:= `eq /. y → famsol // Simplify`

Out[∘]= True

We can use the equation e^(α x) (C[1] Sin[β x] + C[2] Cos[β x]) to generate a real solution:

In[∘]:= `gensolreal = e^(α x) * (C[1] Cos[β x] + C[2] Sin[β x]) /. {α → -1, β → 1}`

Out[∘]= $e^{-x}\,(C[1]\,\text{Cos}[x] + C[2]\,\text{Sin}[x])$

In[∘]:= `eq /. y → Function[x, e^-x (C[1] Cos[x] + C[2] Sin[x])] // Simplify`

Out[∘]= True

In-Class Activity Week 6:

In[ ]:= `Series[1/(1 - x), {x, 0, 1}] // Normal`

$$\text{Series}\left[\frac{1}{1-x}, \{x, 0, 1\}\right] \text{ // Normal}$$

Out[ ]= $1 + x$

In[ ]:=
$$\text{Series}\left[\frac{1}{1+x^2}, \{x, 0, 2\}\right] \text{ // Normal}$$

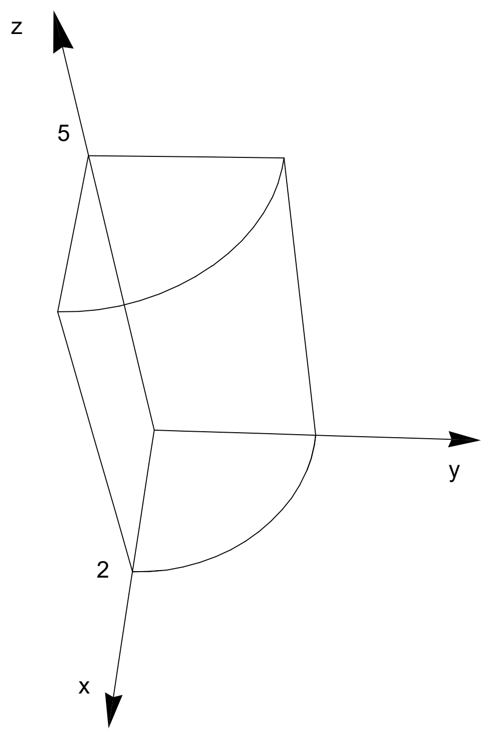
Out[ ]= $1 - x^2$

In[ ]:=
$$\text{Series}\left[\frac{1}{1+e^{-x}}, \{x, 0, 1\}\right]$$

Out[ ]=
$$\frac{1}{2} + \frac{x}{4} + O[x]^2$$

Week 9 Homework:

Out[ ]=



In[ ]:= $\text{func} = \{r\,(2 + \text{Sin}[\phi]^2),\ r\,\text{Sin}[\phi]\,\text{Cos}[\phi],\ 3\,z\}$

Out[ ]= $\{r\,(2 + \text{Sin}[\phi]^2),\ r\,\text{Cos}[\phi]\,\text{Sin}[\phi],\ 3\,z\}$

For top surface:

```
In[ ]:= Integrate[func.{0, 0, 1} * r /. z → 5, {r, 0, 2}, {ϕ, 0, π / 2}] // Simplify
Out[ ]= 15 π
```

For bottom surface:

```
In[ ]:= Integrate[func.{0, 0, -1} * r /. z → 0, {r, 0, 2}, {ϕ, 0, π / 2}]
Out[ ]= 0
```

For curved side:

```
In[ ]:= Integrate[func.{1, 0, 0} * r /. r → 2, {ϕ, 0, π / 2}, {z, 0, 5}]
Out[ ]= 25 π
```

For side along x-axis:

```
In[ ]:= Integrate[func.{0, -1, 0} * r /. ϕ → 0, {r, 0, 2}, {z, 0, 5}]
Out[ ]= 0
```

For side along y-axis:

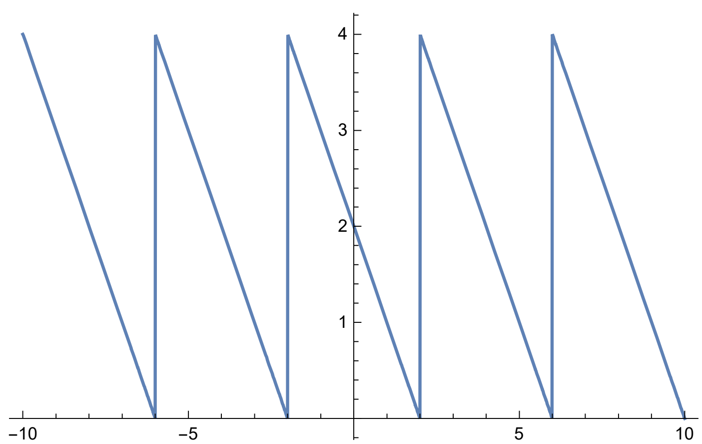
```
In[ ]:= Integrate[func.{0, 1, 0} * r /. ϕ → π / 2, {r, 0, 2}, {z, 0, 5}]
Out[ ]= 0
```

```
In[ ]:= Integrate[Div[func, {r, ϕ, z}, "Cylindrical"], {r, 0, 2}, {ϕ, 0, π / 2}, {z, 0, 5}]
Out[ ]= 40 π
```

```
In[ ]:= nterms = 5
Out[ ]= 5
```

```
In[ ]:= func = 2 - x;
funcplot = Plot[2 - Mod[x, 4, -2], {x, -10, 10}, Exclusions → None]
```

Out[ ]=



In[ ]:=
```
SymbolicFourierSeries = Sum[A[n] e^(-π i n x/L), {n, -nterms, nterms}]
```

Out[ ]=
$$e^{\frac{5 i \pi x}{L}} A[-5] + e^{\frac{4 i \pi x}{L}} A[-4] + e^{\frac{3 i \pi x}{L}} A[-3] + e^{\frac{2 i \pi x}{L}} A[-2] + e^{\frac{i \pi x}{L}} A[-1] + A[0] + e^{-\frac{i \pi x}{L}} A[1] + e^{-\frac{2 i \pi x}{L}} A[2] + e^{-\frac{3 i \pi x}{L}} A[3] + e^{-\frac{4 i \pi x}{L}} A[4] + e^{-\frac{5 i \pi x}{L}} A[5]$$

In[ ]:=
```
lhs[n_] := Integrate[SymbolicFourierSeries * e^(-π i n x/L), {x, -L, L}]
rhs[n_] := Integrate[func * e^(-π i n x/L), {x, -L, L}]
```

In[ ]:=
```
equations = Table[lhs[i] == rhs[i], {i, -nterms, nterms}];
TableForm[equations]
```

Out[ ]//TableForm=
$$2 L A[5] == -\frac{2 i L^2}{5 \pi}$$
$$2 L A[4] == \frac{i L^2}{2 \pi}$$
$$2 L A[3] == -\frac{2 i L^2}{3 \pi}$$
$$2 L A[2] == \frac{i L^2}{\pi}$$
$$2 L A[1] == -\frac{2 i L^2}{\pi}$$
$$2 L A[0] == 4 L$$
$$2 L A[-1] == \frac{2 i L^2}{\pi}$$
$$2 L A[-2] == -\frac{i L^2}{\pi}$$
$$2 L A[-3] == \frac{2 i L^2}{3 \pi}$$
$$2 L A[-4] == -\frac{i L^2}{2 \pi}$$
$$2 L A[-5] == \frac{2 i L^2}{5 \pi}$$

In[ ]:=
```
Solve[equations, Table[A[n], {n, -nterms, nterms}]] // Flatten
```

Out[ ]=
$$\left\{A[-5] \to \frac{i L}{5 \pi}, A[-4] \to -\frac{i L}{4 \pi}, A[-3] \to \frac{i L}{3 \pi}, A[-2] \to -\frac{i L}{2 \pi}, A[-1] \to \frac{i L}{\pi}, A[0] \to 2, A[1] \to -\frac{i L}{\pi}, A[2] \to \frac{i L}{2 \pi}, A[3] \to -\frac{i L}{3 \pi}, A[4] \to \frac{i L}{4 \pi}, A[5] \to -\frac{i L}{5 \pi}\right\}$$

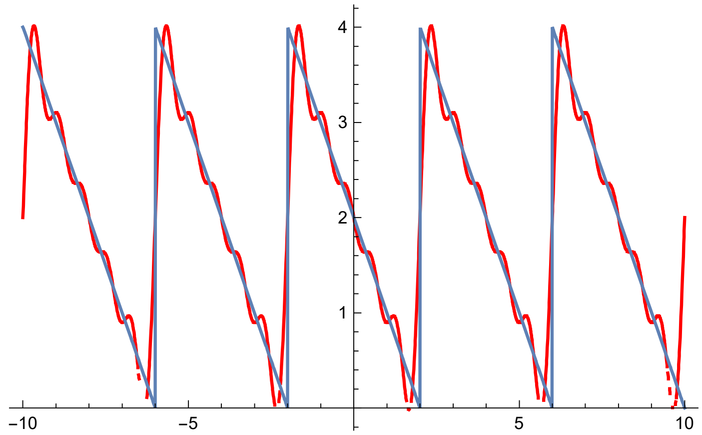
In[ ]:=
```
NumericalFourierSeries = SymbolicFourierSeries /. % /. L → 2 // Simplify
```

Out[ ]=
$$\frac{1}{30 \pi} i e^{-\frac{5}{2} i \pi x} \left(-12 + 15 e^{\frac{i \pi x}{2}} - 20 e^{i \pi x} + 30 e^{\frac{3 i \pi x}{2}} - 60 e^{2 i \pi x} + 60 e^{3 i \pi x} - 30 e^{\frac{7 i \pi x}{2}} + 20 e^{4 i \pi x} - 15 e^{\frac{9 i \pi x}{2}} + 12 e^{5 i \pi x} - 60 i e^{\frac{5 i \pi x}{2}} \pi\right)$$

In[ ]:=
```
FourierPlot = Plot[NumericalFourierSeries /. L → 2, {x, -10, 10}, PlotStyle → {Red}];
```

In[ ]:=
```
Show[FourierPlot, funcplot]
```

Out[ ]=



In[ ]:=
```
Cosh[0] // N
```

Out[ ]= 1.

In[ ]:=
```
Integrate[Cosh[x] D[DiracDelta[x], x], {x, 0, 2 π}] // N
```

Out[ ]= 0.

In[ ]:=
```
Cosh[0.5] // N
```

Out[ ]= 1.12763

In[ ]:=
```
Integrate[Cosh[x] D[DiracDelta[x - 0.5], x], {x, 0, 2 π}] // N
```

Out[ ]= -0.521095

In[ ]:=
```
Cosh[1] // N
```

Out[ ]= 1.54308

In[ ]:=
```
Integrate[Cosh[x] D[DiracDelta[x - 1], x], {x, 0, 2 π}] // N
```

Out[ ]= -1.1752

In[ ]:=
```
Cosh[π] // N
```

Out[ ]= 11.592

In[ ]:=
```
Integrate[Cosh[x] D[DiracDelta[x - π], x], {x, 0, 2 π}] // N
```

Out[ ]= -11.5487

In[ ]:=
```
Cosh[3 π / 2] // N
```

Out[ ]= 55.6634

In[ ]:=
```
Integrate[Cosh[x] D[DiracDelta[x - 3 π / 2], x], {x, 0, 2 π}] // N
```

Out[ ]= -55.6544