Group #5 @1pm
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Lecture Worksheet 4

Task 1

State the Inclusion-Exclusion Principle and prove it

- *This principle is essential when it comes to counting the union of sets and finding their probability.Generally, when we want to find the union of sets, it is not as simple as adding up the individual sets together as we may be over-counting and thus obtaining the wrong probability.*
- *One way to fix this is by subtracting the over-counted elements, which are the intersections between sets. Still, we would have to be selective in subtracting otherwise we would under-count as well.*
- *The Inclusion-Exclusion Principle gives us a general idea of how to go at obtaining the union by including/excluding certain intersections.*

Consider a bunch of sets A_i where $i = 1, 2, ..., n$. Finding the union of the sets for $n = 2$ and $n = 4$ gives us a good idea of how to prove the Inclusion-Exclusion Principle.

For $n = 2$, we have

$$\begin{aligned} P(A_1 \cup A_2) = \quad & P(A_1) + P(A_2) \\ & - P(A_1 \cap A_2) \end{aligned}$$

For $n = 4$, we have

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3 \cup A_4) = \quad & + P(A_1) + P(A_2) + P(A_3) + P(A_4) \\ & - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_3 \cap A_4) - P(A_1 \cap A_3)... \\ & + P(A_1 \cap A_2 \cap A_3) + P(A_2 \cap A_3 \cap A_4) + P(A_1 \cap A_3 \cap A_4)... \\ & - P(A_1 \cap A_2 \cap A_3 \cap A_4) \end{aligned}$$

We wanted to do $n = 4$ to further demonstrate the pattern.
Given a total of n sets, we can describe the Inclusion-Exclusion Principle as:

- *First adding all the sets individually*

$$\sum_{i=0}^{n} P(A_i)$$

- *Subtracting the intersections of 2 sets at a time*

$$-\sum_{i,j}^{n} P(A_i \cap A_j) \text{ where } i \neq j$$

- *Adding the intersection of 3 sets at a time*

$$+\sum_{i,j,k}^{n} P(A_i \cap A_j \cap A_k) \text{ where } i \neq j \neq k$$

- *So on... Adding/Subtracting with increasingly more sets at a time*

$$... \pm \sum_{i_1,i_2,...,i_k}^{n} P(A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_k}) \text{ where } i_1 \neq i_2 \neq ... \neq i_k$$

- *Add/Subtract the last term being the intersection of n sets*

$$\pm P(A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_n})$$

Mathematically, the Inclusion-Exclusion Principle can be stated as:

$$\begin{aligned} P(A_1 \cup A_2 \cup ... \cup A_n) = \quad & \sum_{i=0}^{n} P(A_i) \\ & -\sum_{i,j}^{n} P(A_i \cap A_j) \\ & ... + (-1)^{k+1} \sum_{i_1,i_2,...,i_k}^{n} P(A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_k}) + ... \\ & ... + (-1)^{n+1} P(A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_n}) \end{aligned}$$

Task 2

What is the probability that a randomly picked permutation of the numbers $\{1, 2, ..., n\}$ for $n \in \mathbb{N}$ has at least 1 fixed point?

How many permutations are there in total (the sample space)?

n! total permutations

How many permutations hold a fixed $j \in \{1, 2, ..., n\}$ points (call the subset of such permutations F_j)?

Given j fixed points, there are $F_j = (n-j)!$ *permutations*

How can you describe the event F that a permutation have at least a fixed point in terms of the events F_j for $j = 1, 2, ..., n$?

$$F = F_1 \cup ... \cup F_n$$

What is the probability of F_j and that of F?

$$P(F_1 \cap ... \cap F_j) = \tfrac{(n-j)!}{n!}$$
$$P(F) = P(F_1 \cup ... \cup F_n)$$

We can now use the Inclusion-Exclusion Principle to expand the union $F_1 \cup ... \cup F_n$

$$\begin{aligned}
P(F_1 \cup ... \cup F_n) &= \sum_{k=1}^{n} (-1)^{k+1} \sum_{j_1, j_2, ..., j_k} P(F_{j_1} \cup ... \cup F_{j_k}) \\
&= \binom{n}{1}\frac{(n-1)!}{n!} - \binom{n}{2}\frac{(n-2)!}{n!} + \binom{n}{3}\frac{(n-3)!}{n!} ... \\
&= 1 - \frac{1}{2!} + \frac{1}{3!!} \mp ... + (-1)^{n+1}\frac{1}{n!} \\
&\approx 1 - \frac{1}{e} \text{ because the 2nd term onward resembles } e^{-x} \text{ where } x = 1 \\
&\approx 63\%
\end{aligned}$$