Group #5 @1pm
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# Lecture Worksheet 19

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## Task 1

Let $X$ be a random variable on a probability space.

- Give a precise definition of $P(X \leq x)$ for $x \in \mathbb{R}$.

  *Consider a sample space $S$,*

  $$P(X \leq x) = P(\{\omega \in S \mid X(\omega) \leq x\}) = P(X^{-1}((-\infty, x]))$$

  *We can think of $P(X \leq x)$ as the disjoint union of the probabilities of the events where the random variable outputs a value less than $x$.*

- What are the following expressions for $x_0 \in \mathbb{R}$:

  1. $\bigcap_{x \in \mathbb{R}} [X \leq x]$

     *We can rewrite $[X \leq x]$ as $[X(\omega) \leq x]$. Since $X : S \mapsto \mathbb{R}$, we are trying to find a number $X(\omega) \in \mathbb{R}$ such that $X(\omega) < x$ for all $x \in \mathbb{R}$. No such number should exist, so thus we have:*

     $$\bigcap_{x \in \mathbb{R}} [X \leq x] = \emptyset$$

  2. $\bigcup_{x \in \mathbb{R}} [X \leq x]$

     *Considering a sample space $S$, Here we are trying to find the set of all $X \in S$ such that $X \leq x$ for $x \in \mathbb{R}$. It turns out that the union is just the entire sample space itself since $x$ runs through all of $\mathbb{R}$, we can find an $x$ such that all the elements of the sample space is less than $x$.*

     $$\bigcup_{x \in \mathbb{R}} [X \leq x] = S$$

3. $\bigcap_{x>x_0} [X \leq x]$

*The index in the intersection runs through all the possible values of $x$ such that $x > x_0$. The resulting set of outcomes contains elements that are all greater than $x_0$, making $x_0$ a lower bound. Then we can just rewrite this set as the set of all outcomes that are less than $x_0$.*

$$\bigcap_{x>x_0} [X \leq x] = [X \leq x_0]$$

- What is the relation between $P(X \leq -\infty)$, $P(X \leq x_1)$, $P(X \leq x_2)$, and $P(X \leq \infty)$ when $x_1 \leq x_2 \in \mathbb{R}$? Can you identify any of these values independently of $X$?

  *We can first look at the extreme cases including $\infty$ and $-\infty$.*

  *The probability $P(X \leq -\infty)$ comes out to be just 0 because no number exists that is smaller than $-\infty$.*
  *The probability $P(X \leq \infty)$ comes out to be 1 because all finite numbers should be less than $\infty$ and we are given that $x_1, x_2 \in \mathbb{R}$.*

  *Since $x_1 \leq x_2$, we also have $P(X \leq x_1) \leq P(X \leq x_2)$. This is because $x_2$ is larger than $x_1$ so the event of $X \leq x_2$ is larger compared to $X \leq x_1$*

  *Without involving the random variable $X$, we can state the relation:*

  $$-\infty < x_1 \leq x_2 \leq \infty$$

- If a function $F_X : \mathbb{R} \mapsto \mathbb{R}$ is defined by $F_X(x) = P(X \leq x)$, what properties does it have?

  *The function $F_X$ is generally increasing as we increase $x$ in $X \leq x$, where $0 \leq F_X(x) \leq 1$. Consequently, $\lim_{x \to -\infty} F_X = 0$ and $\lim_{x \to \infty} F_X = 1$.*

## Task 2

- A discrete random variable is Poisson distributed with parameter $\lambda \in (0, \infty)$ iff

  $$X(S) = \{x_1, x_2, ..., x_n\} \text{ and } p_X(i) = P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}$$

- What is the role of the parameter $\lambda$? How do the values of $p_X$ depend on $\lambda$? How do the values of $p_X$ depend on the other variables?

  *The role of $\lambda$ is to denote the number of occurrences. It determines the distribution of the data points.*

  *Using the equation above, we can find the dependence relation of $\lambda$ and $p_X$.*

$$\begin{aligned} p_X(k+1) &= e^{-\lambda}\frac{\lambda^{k+1}}{(k+1)!} \\ &= e^{-\lambda}\frac{\lambda \cdot \lambda^k}{(k+1)\cdot k!} \\ &= \frac{\lambda}{k+1}\cdot p_X(k) \end{aligned}$$

  *From this we can see that $p_X$ is increasing for $0 \le k \le \lambda$ and decreasing for $k \ge \lambda$*

- Computed the expected value and the variance of a Poisson distributed random variable.

  *Using the general formula of expected value:*

$$E[X] = \sum i \, p_X(i)$$

  *For a Poisson distributed random variable X, the expected value is given by:*

$$\begin{aligned} E[X] &= \sum_{i=0}^{\infty} i \cdot e^{-\lambda}\frac{\lambda^i}{i!} \\ &= \lambda \cdot e^{-\lambda}\sum_{i=1}^{\infty}\frac{\lambda^{i-1}}{(i-1)!} \\ &= \lambda \cdot e^{-\lambda}\sum_{j=0}^{\infty}\frac{\lambda^j}{j!} && \textit{using } j = i-1 \\ &= \lambda \cdot e^{-\lambda}\cdot e^{\lambda} && \textit{since } \sum_{i=0}^{\infty}\frac{\lambda^i}{i!} = e^{\lambda} \\ &= \lambda \end{aligned}$$

*For the variance, we first try to compute $E[X^2]$:*

$$\begin{aligned}
E[X^2] &= \sum_{i=0}^{\infty} i^2 \cdot e^{-\lambda} \frac{\lambda^i}{i!} \\
&= \lambda \cdot e^{-\lambda} \sum_{i=1}^{\infty} i \cdot \frac{\lambda^{i-1}}{(i-1)!} \\
&= \lambda \cdot e^{-\lambda} \sum_{j=0}^{\infty} (j+1) \frac{\lambda^j}{j!} && \textit{using } j = i-1 \\
&= \lambda \cdot e^{-\lambda} \sum_{j=0}^{\infty} \left[ j \cdot \frac{\lambda^j}{j!} + \frac{\lambda^j}{j!} \right] \\
&= \lambda \cdot e^{-\lambda} \left[ \lambda \sum_{j=1}^{\infty} \frac{\lambda^{j-1}}{(j-1)!} + \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \right] \\
&= \lambda \cdot e^{-\lambda} \left[ \lambda \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} + e^{\lambda} \right] && \textit{using } k = j-1 \\
&= \lambda \cdot e^{-\lambda} \left[ \lambda \cdot e^{\lambda} + e^{\lambda} \right] \\
&= \lambda(\lambda + 1)
\end{aligned}$$

*Now the variance is given by:*

$$\begin{aligned}
Var(X) &= E[X^2] - E[X]^2 \\
&= \lambda(\lambda + 1) - \lambda^2 \\
&= \lambda^2 + \lambda - \lambda^2 \\
&= \lambda
\end{aligned}$$

*It turns out that the expected value and the variance of a Poisson distributed random variable is represented both by the parameter $\lambda$.*

## Task 3

The number of people entering a gambling casino is Poisson distributed with a rate of 1 person every 2 minutes.

- What is the probability that no one enters between 12:00 and 12:05?

  *Let random variable $X$ denote the number of people entering the casino.*
  *If we take 1 person per 2 minutes, then we can reduce this ratio to half a person per minute. The reason why we would do this is so that we can multiply by 5 to account for the given 5-minute window. Then our Poisson parameter becomes $\lambda = 5 \cdot 1/2 = 5/2$*
  *The probability is given by:*

$$P(X = 0) = e^{(-5/2)} \cdot \frac{\lambda^0}{0!} = e^{-5/2} \approx 8.21\%$$

- What is the probability that at least 4 people enter the casino during that time?

  *It is easier to compute the complement.*
  *The probability is given by:*

$$\begin{aligned} P(X \geq 4) &= 1 - P(X < 4) \\ &= 1 - \Big[P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)\Big] \\ &= 1 - \Big[e^{-5/2} + e^{-5/2}\frac{(5/2)^1}{1!} + e^{-5/2}\frac{(5/2)^2}{2!} + e^{-5/2}\frac{(5/2)^3}{3!}\Big] \\ &\approx 24.2\% \end{aligned}$$