Group #5 @1pm
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Lecture Worksheet 20

Task 1

A communications channel transmits the digits 0 and 1. However due to static, the digit transmitted is incorrectly received with probability 0.2. Suppose that we want to transmit an important message consisting of one binary digit. To reduce the chance of error, we transmit 00000 instead of 0 and 11111 instead of 1.

- If the receiver of the message uses majority decoding, what is the probability that the message will be wrong when decoded?

  Based on majority decoding, if there are at least 3 out of 5 of a given signal, then that signal will be considered as the overall message. Since there are only 2 outcomes: success/failure, we can adopt a binomial random variable approach, X B(5,0.2).
  Let X denote the number of wrong messages, then the probability is given by:

$$\begin{aligned} P(3 \leq X \leq 5) &= P(X=3) + P(X=4) + P(X=5) \\ &= \binom{5}{3} 0.2^3 \cdot 0.8^2 + \binom{5}{4} 0.2^4 \cdot 0.8 + \binom{5}{5} 0.2^5 \\ &\approx 6\% \end{aligned}$$

- What independence assumptions are you making?

  Here we are assuming that each individual digit is not dependent on anything else; the probability of being incorrectly received is fixed at 0.2.

Task 2

- A random variable X is called discrete if

$$X(S) = \{x_1, x_2, ..., x_n\} \text{ where } x \in \mathbb{R} \text{ and } n \in \mathbb{N}$$

- The expected value $E[X]$ of a discrete random variable X is given by:

$$\sum_{i=1}^{n} x_i \cdot p_X(x_i)$$

- Prove that $E[X] = \sum_{s \in S} X(S)P(\{s\})$ for any discrete random variable X.

let $S_i = \{s \in S | X(s) = x_i$. This set filters out the outcomes in the sample space that correspond to a value x_i.

Starting with the basic expected value formula:

$$\begin{aligned}
E[X] &= \sum_{i=1}^{n} x_i \cdot p_X(x_i) \\
&= \sum_{i=1}^{n} x_i \cdot P(S_i) && \textit{since } p_X(x_i) = P(S_i) \\
&= \sum_{i=1}^{n} x_i \cdot \sum_{s_i \in S} p(s) && \textit{indexing for each element in } S \\
&= \sum_{i=1}^{n} \sum_{s_i \in S} X(s) \cdot P(s) && \textit{since } X(s) = x_i \\
&= \sum_{s \in S} X(s) \cdot p(s) && \textit{just re-indexing the sums} \\
&= \sum_{s \in S} X(s) \cdot P(\{s\}) && \textit{by def. } P(\{s\}) = p(s)
\end{aligned}$$

Task 3

Assume that a random variable X is given with values $r, r+1, ...$ for $r \geq 1$ and satisfying

$$P(X = n) = \binom{n-1}{r-1} p^r (1-p)^{n-r}, n \geq r$$

- What do you need to check to make sure this definition makes sense? Check that.

  Firstly, we should make sure that $n-1$ and $r-1$ are non-negative values and that $n-1$ is larger than $r-1$ in order to satisfy the "choose" function's parameter conditions.

  $$n - 1 \geq r - 1 \geq 1 - 1 = 0 \textit{ since } r \geq 1$$

  Additionally, we can also check if this probability function satisfies the probability axioms, such as summing up to 1.

- Can you identify an experiment the outcomes of which are described by this random variable? *Think of $p^r(1-p)^{n-r}$ as $p^{r-1}(1-p)^{n-1-(r-1)}p$*

  According to our text, this probability mass function is for what's called a "negative binomial" random variable, where the random variable represents the trial number of the r-th success given that the other trials are independent successes.

  I found this definition a bit confusing. My attempt at identifying an example experiment would be similar to the examples we've done where we play a game and we continue conducting the experiment until we encounter the first failure.

  For example, flipping a coin with probability p of getting heads. We stop flipping until we see tails. Then $P(X = n)$ may represent the probability that the r-th trial is a success given that we have r-1 successes and n total trials.