Group #5 @1pm
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Lecture Worksheet 22

Task 1

Let X be uniformly distributed on the interval $[a, b]$ for $-\infty < a < b < +\infty$

- What is the probability density function $f_X : \mathbb{R} \mapsto \mathbb{R}$ of X?

  The probability density function is given by:

$$f_X(x) = \frac{1}{b-a}$$

- What is $E(X)$? After carrying out the computation, give an argument for the answer which does not require any calculation.

  The expected value is given by:

$$\begin{aligned} E(X) &= \int_a^b x f_X(x) dx \\ &= \int_a^b x \frac{1}{b-a} dx \\ &= \frac{1}{b-a} \left[\frac{1}{2} x^2 \right]_a^b \\ &= \frac{b+a}{2} \end{aligned}$$

  This result makes sense intuitively, it essentially represents the midpoint between a and b. We would expect the random values of uniformly distributed random points in an interval to lie around the middle of the interval.

- What are $E(X^2)$ and $Var(X)$?

  $E(X^2)$ is given by:

$$\begin{aligned} E(X^2) &= \int_a^b x^2 f_X(x)dx \\ &= \frac{1}{b-a}\int_a^b x^2 dx \\ &= \frac{1}{b-a}\left[\frac{1}{3}x^3\right]_a^b \\ &= \frac{1}{3}(b^2+ab+a^2) \end{aligned}$$

  $Var(X)$ is given by:

$$\begin{aligned} Var(X) &= E(X^2) - E(X)^2 \\ &= \frac{1}{3}(b^2+ab+a^2) - \left(\frac{b+a}{2}\right)^2 \\ &= \frac{(b-a)^2}{12} \end{aligned}$$

Task 2

- What do you expect the following limit to amount to (assuming that it exists) for a continuous random variable X

$$\lim_{h\downarrow 0}\frac{1}{2h}P([x-h\leq X\leq x+h])$$

Since $h\to 0$, we expect the interval of interest to begin to resemble a point Then the probability for a point considering a continuous random variable is always zero.

$$\lim_{h\downarrow 0}\frac{1}{2h}P([x_0-h\leq X\leq x_0+h])=0$$

- Try to justify your answer rigorously. Compute using the cumulative distribution function F_X of X.

We can begin our analysis with the given set:

$$[x-h\leq X\leq x+h]$$

It follows that

$$[x-h\leq X\leq x+h]=[X\leq x+h]\setminus[X\leq x-h]$$

Since $[X\leq x+h]$ and $[X\leq x-h]$ are disjoint, we can apply the probability axiom where we can directly add disjoint sets

$$P([x-h\leq X\leq x+h])=P([X\leq x+h])-P([X\leq x-h])$$

By the definition of the cumulative distribution function F_X

$$P([x-h\leq X\leq x+h])=F_X(x+h)-F_X(x-h)$$

Now plugging this into the original equation

$$\lim_{h\downarrow 0}\frac{P([x-h\leq X\leq x+h])}{2h}=\lim_{h\downarrow 0}\frac{F_X(x+h)-F_X(x-h)}{2h}$$

The RHS resembles the limit definition of the derivative

$$\Rightarrow \frac{1}{2}\lim_{h\downarrow 0}\frac{F_X(x+h)-F_X(x-h)}{h}$$

$$\Rightarrow \frac{1}{2}\Big(\lim_{h\downarrow 0}\frac{F_X(x+h)-F_X(x)}{h}+\frac{F_X(x)-F_X(x-h)}{h}\Big)$$

$$=\frac{1}{2}\Big(F_X'(x)+F_X'(x)\Big)=F_X'(x)$$

$$=f_X(x)=0$$

Task 3

Let a continuous random variable X on a probability space S be given such that $X(S) = [0, 1]$

- Find a discrete random variable Y that approximates the continuous random variable X

  We can come up with a discrete random variable Y so that Y takes on values like

  $$Y = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\}$$
  $$\implies Y(i) = \frac{i}{n} \qquad \text{where } i = 0, , 1, ..., n$$

  Where the probability is given by

  $$P\Big(Y = \frac{k}{n}\Big) = F_X\Big(\frac{k}{n}\Big) - F_X\Big(\frac{k-1}{n}\Big), \text{ where } k = 1, 2, ..., n$$

- How close are the two random variables X and Y?

  The expected value for Y is given by:

  $$E[Y] = \sum_{k=1}^{n} k \cdot (F_X\Big(\frac{k}{n}\Big) - F_X\Big(\frac{k-1}{n}\Big))$$

  Assuming that our discrete random variable Y is modeled such that $\frac{E(X)}{E(Y)} \approx 1$ then the values of X and Y ought to be pretty close to each other.

- Can the values of Y be chosen in such a way that $E(X) = E(Y)$?

  Yes, we can approach this task with a similar method like when we try to produce a fair game of coin flips involving an unfair coin. We would "tweak" certain Y values in such a way that lower than expected values have a higher probability of being 1, so that the overall $E[Y]$ is higher and thus closer to $E[X]$. Likewise for Y values that are higher, we would decrease the probability of being 0 so that the resulting value is closer to $E[X]$.