Group #5 @1pm
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Lecture Worksheet 12

Task 1

- State the definition of independence for 3 or more events of a probability space

$$P(E_1 \cap E_2 \cap E_3) = P(E_1)P(E_2)P(E_3)$$

And also all 3 events are pair-wise independent:

$$P(E_1 \cap E_2) = P(E_1)P(E_2)$$
$$P(E_2 \cap E_3) = P(E_2)P(E_3)$$
$$P(E_1 \cap E_3) = P(E_1)P(E_3)$$

- Consider two independent tosses of a fair coin and the events

$H_1 =$ "First toss yields heads"

$H_2 =$ "Second toss yields heads"

$S =$ "Both tosses yield the same outcome"

Show that H_1, H_2 are independent and that so are both H_1, S and H_2, S but that H_1, H_2, S are not.

We can enumerate the probabilities for all events H_1, H_2, S

$$P(H_1) = \frac{1}{2},\ P(H_2) = \frac{1}{2},\ P(S) = \frac{1}{2}$$

We can verify these events are pair-wise independent by using the def. of independence:

$$P(H_1 \cap H_2) = \frac{1}{4} \Leftrightarrow P(H_1)P(H_2)$$
$$P(H_1 \cap S) = \frac{1}{4} \Leftrightarrow P(H_1)P(S)$$
$$P(H_2 \cap S) = \frac{1}{4} \Leftrightarrow P(H_2)P(S)$$

Now we can try to verify if all 3 events are independent:

$$P(H_1 \cap H_2 \cap S) = \frac{1}{4}$$

$$P(H_1)P(H_2)P(S) = \frac{1}{8}$$

Since $P(H_1 \cap H_2 \cap S) \neq P(H_1)P(H_2)P(S)$, *the events* H_1, H_2, S *are dependent.*

Task 2

Urn U_1 contains 2 white and 1 black balls whereas urn U_2 contains 1 white and 5 black balls. A ball is drawn at random from urn U_1 and moved to urn U_2. A ball is then drawn from urn U_2, which happens to be white.

- What is the probability that the ball transferred was white? *Let's define:*

$$W = \text{ the event of pulling a White out of Urn 2}$$
$$T = \text{ the event of transferring a White from Urn 1 to Urn 2}$$
$$T^c = \text{ the event of transferring a Black from Urn 1 to Urn 2}$$

The desired probability can be denoted as: $P(T|W)$, but we first need to find other pieces of information.
Given these events, we can condition W on whether a black or white ball was transferred.

$$\begin{aligned} P(W) &= P(W|T)P(T) + P(W|T^c)P(T^c) \\ &= \frac{2}{7} \cdot \frac{2}{3} + \frac{1}{7} \cdot \frac{1}{3} \\ &= \frac{4}{21} + \frac{1}{21} = \frac{5}{21} \end{aligned}$$

Now, we can rewrite $P(T|W)$ using Bayes' Formula:

$$\begin{aligned} P(T|W) &= \frac{P(T \cap W)}{P(W)} \\ &= \frac{P(W|T)P(T)}{P(W)} \\ &= \frac{2/7 \cdot 2/3}{5/21} = \frac{4}{5} = 80\% \end{aligned}$$

Task 3

- A fair die is rolled twice. What is the probability that the second roll yields a higher number than the first?

  Let's define:

  $E_i =$ *the event of getting i on the first roll for $i = 1, 2, 3, 4, 5, 6$*

  $F =$ *the event of getting a higher number on the second roll*

  We condition F on all the occurrences of E_i

  Then the desired probability is given by:

$$P(F) = \sum_{i=1}^{6} P(F|E_i)P(E_i)$$

$$= \frac{1}{6}\sum_{i=1}^{6} P(F|E_i) \qquad \textit{Since } P(E_i) = \frac{1}{6} \textit{ for all } i$$

$$= \frac{1}{6}\left[\frac{5}{6} + \frac{4}{6} + \frac{3}{6} + \frac{2}{6} + \frac{1}{6} + 0\right]$$

$$\approx 42\%$$

- Suppose that an experiment yields the outcome i with probability p_i for $i = 1, 2, ..., n$. If two such experiments are run independently, what is the probability that the second outcome be larger than the first?

  We can think of this as a n-sided die with sides that give values of i with probability p_i for $i = 1, 2, ..., n$.

  The experiment would be tossing this die twice, and the desired probability would be the probability of landing on a higher number on the second toss.

  Let's define:

  $E_i =$ *the event of the outcome with value i with probability p_i*

  $F =$ *the event of the second outcome being larger than the first*

  Similarly, we condition F on all the occurrences of E_i

The desired probability is given by:

$$
\begin{aligned}
P(F) &= \sum_{i=1}^{n} P(F|E_i)P(E_i) \\
&= \sum_{i=1}^{n} P(F|E_i) \cdot p_i \\
&= \left[\frac{n-1}{n} \cdot p_1 + \frac{n-2}{n} \cdot p_2 + ... + \frac{n-(n-1)}{n} \cdot p_{n-1} + \frac{n-n}{n} \cdot p_n\right] \\
&= \left[\frac{n-1}{n} \cdot p_1 + \frac{n-2}{n} \cdot p_2 + ... + \frac{1}{n} \cdot p_{n-1} + 0 \cdot p_n\right] \\
&= \sum_{i=1}^{n} \left(\frac{n-i}{n}\right) \cdot p_i
\end{aligned}
$$