Group #5 @1pm
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Lecture Worksheet 9

Task 1

In a certain population, the event of a newborn reaching the age of n years is denoted by A_n. Assume that $P(A_{50}) = 0.913$, that $P(A_{55}) = 0.881$ and that $P(A_{65}) = 0.746$?

- What is the probability that a 50 year old person reaches the age of 65?

  This probability can be denoted as $P(A_{65}|A_{50})$*.*

  Using a useful identity that relates two events:

  $$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

  We have that:

  $$P(A_{65}|A_{50}) = \frac{P(A_{50}|A_{65})P(A_{65})}{P(A_{50})}$$

  We can think of the probability $P(A_{50}|A_{65})$ *as the probability of reaching 50 years old, given having already reached 65 years old. This amounts to* 100% *since the person MUST have reached 50 in order to have reached 65.*

  Given that, we have:

  $$P(A_{65}|A_{50}) = \frac{P(A_{50}|A_{65})P(A_{65})}{P(A_{50})} = \frac{P(A_{65})}{P(A_{50})} \approx 82\%$$

- What is the probability of a 50 year old person to die before reaching age 55?

  This probability can be denoted as $P(A_{55}^c|A_{50})$*.*

We can rewrite the complement of the Conditional Probability as:

$$\begin{aligned} P(A_{55}^c|A_{50}) &= 1 - P(A_{55}|A_{50}) \\ &= 1 - \frac{P(A_{50}|A_{55})P(A_{55})}{P(A_{50})} \\ &= 1 - \frac{P(A_{55})}{P(A_{50})} \\ &\approx 3.5\% \end{aligned}$$

Claim: $P(A_{55}^c|A_{50}) = 1 - P(A_{55}|A_{50})$
Proof: Assuming a sample space S

$$P(A_{55}^c|A_{50}) = 1 - P(A_{55}|A_{50})$$

$$\Rightarrow \frac{P(A_{55}^c \cap A_{50})}{P(A_{50})} = 1 - \frac{P(A_{55} \cap A_{50})}{P(A_{50})}$$

$$\Rightarrow P(A_{55}^c \cap A_{50}) = P(A_{50}) - P(A_{55} \cap A_{50})$$

$$\Rightarrow P(A_{55} \cap A_{50}) + P(A_{55}^c \cap A_{50}) = P(A_{50})$$

We can rewrite the LHS as a union of disjoint sets

$$\Rightarrow P((A_{55} \cap A_{50}) \cup (A_{55}^c \cap A_{50})) = P(A_{50})$$

Using set properties, we have:

$$\Rightarrow P((A_{55} \cup A_{55}^c) \cap A_{50}) = P(A_{50})$$

$$\Rightarrow P(S \cap A_{50}) = P(A_{50})$$

$$\Rightarrow P(A_{50}) = P(A_{50})$$

- Assume that a 65 year old dies within 5 years with probability 0.16. What is $P(A_{70})$?

  This probability can be denoted as $P(A_{70}^c|A_{65}) = 0.16$

We can rewrite the probability as:

$$P(A_{70}|A_{65}) = 1 - P(A_{70}^c|A_{60})$$
$$= 84\%$$

Using the definition of Conditional Probability:

$$P(A_{70}|A_{65}) = \frac{P(A_{65}|A_{70})P(A_{70})}{P(A_{65})}$$
$$= \frac{P(A_{70})}{P(A_{65})}$$

Solving for $P(A_{70}) = P(A_{70}|A_{65}) \cdot P(A_{65}) \approx 63\%$

Task 2

- For events, E,F of a probability space, find a relation between $P(E|F)$ and $P(F|E)$. Assume that $P(E) \neq P(F) \neq 0$.

  We use the definition of Conditional Probability to find:

  $$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{P(F|E)P(E)}{P(F)}$$

  Simplifying:

  $$P(E|F)P(F) = P(F|E)P(E)$$

Suppose a voter poll is taken in 3 states. In State A, 50% of voters support the liberal candidate. In State B, 60% of voters support the liberal candidate. In State C, 35% of voters support the liberal candidate. Of the total population of the 3 states, 40% live in State A, 25% live in State B, and 35% live in State C.

- Given that a voter supports the liberal candidate, what is the probability that she lives in State B?

  We can denote this probability as $P(B|L)$

  Using Bayes' Formula and Conditional Probabilities, we have:

  $$P(B|L) = \frac{P(B \cap L)}{P(L)}$$
  $$= \frac{P(L|B)P(B)}{\sum_{j=1}^{3} P(L|S_j)P(S_j)}$$

  where $\{S_1, S_2, S_3\} = \{\textit{State A, State B, State C}\}$

Plugging in the numbers, we get:

$$P(B|L) = \frac{0.25 \cdot 0.6}{0.4 \cdot 0.5 + 0.25 \cdot 0.6 + 0.35 \cdot 0.35} \approx 32\%$$

- What is the sample space for this problem? Can you translate it into an urn and colored balls problem? How many urns? How many colors?

  In a general viewpoint, the sample space can be seen as {Liberal, Non-Liberal}

  Yes, in this problem, we can view each state as its own urn and the balls they contain are represented by the people. The colors are the people's party favor, either {Liberal, Non-Liberal}.

Task 3

Suppose that we have 3 cards that are identical in form, except that both sides of the first card are colored red, both sides of the second card are colored black, and one side of the third card is colored red and the other black. The 3 cards are mixed in a hat, and 1 card is randomly selected and placed on the table.

- If the upper side of the chosen card is colored red, what is the probability that the other side is colored black?

  Let's call Card A the card with both red (R), Card B with both Black (B), and Card C with R and B.
  We can denote the desired probability as $P(C|R)$.

  Using Bayes' Formula and the definitions we've learned, we have:

$$
\begin{aligned}
P(C|R) &= \frac{P(C \cap R)}{P(R)} \\
&= \frac{P(C)P(R|C)}{\sum_{j=1}^{3} P(R|C_1)} \\
&= \frac{0.33 \cdot 0.5}{0.33 \cdot 0 + 0.33 \cdot 1.0 + 0.33 \cdot 0.5} \\
&= \approx 33\%
\end{aligned}
$$