Group #5 @1pm:
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Lecture 2 Worksheet

Task 1

1. Derive the **Binomial Formula**:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \text{ where n} \geq 0$$

We can start by expanding

$$\underbrace{(x+y)(x+y)\cdots(x+y)}_{n\text{-}times}$$

We expect the result to be a sum whose terms consist of some product of $x's$ *and* $y's$*, where* c_i *is some constant*

$$c_1x^n + c_2x^{n-1}y + ... + c_{n-1}xy^{n-1} + c_ny^n$$

Initially, the whole quantity was raised to the nth-power. Thus each term, as we expand, should have n-factors, some of x and some of y.
If we assign k to be the number of x-factors, then y must have n-k factors, since the total number of x,y factors should add up to n.
Using summation notation:

$$\sum_{k=0}^{n} c_k x^k y^{n-k}$$

Here we can see that c_k *represents the number of terms with specifically* k_i *factors of* x *and* $n - k_i$ *factors of* y

In the process of expanding, we somehow actually sweep through all the possible combinations of factors of x,y, such that their factors add up to n. Since, we "add like-terms", the ordering of the factors of x,y are irrelevant. So, in combining like-terms, we are essentially choosing how to organize k factors of x, and correspondingly n-k factors of y, from a group of n factors. Thus c_k *is just the number of ways to choose a group of k factors of x from a total of n factors. However, we also have to choose n-k factors of y, except not from a total of n factors but n-k. There is only 1 way of choosing* *a group of n-k factors from a total of n-k factors. It may be redundant but still visually understanding.*

$$c_k = \binom{n}{k}\binom{n-k}{n-k} = \binom{n}{k}$$

Plugging this into our summation:

$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

2. How can you interpret the special case when $x = y = 1$
   When $x = y = 1$, the powers of x and y are essentially irrelevant since all integer powers of x,y will just result in 1 anyways. The summation is reduced to something like

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k}$$

   Now this equation resembles the computation in which we are trying to add up all the ways of forming groups of k from a set of n elements, where $k = 1, 2, ..., n$. In other words, trying to find the total number of subsets within a set of n elements

Task 2

1. Based on what you learned above, derive a formula for

$$(x_1 + x_2 + ... + x_m)^n \text{ where n} \geq 0 \text{ and m} \geq 3$$

Considering the case of $m = 3$,

$$(x_1 + x_2 + x_3)^n = \;\; ???$$

In expanding this expression, we expect each term to be generally of the form

$$c_i x_1^i x_2^j x_3^k \textit{ where } i + j + k = n$$

Likewise here, we can find the coefficient c_i by finding the number of ways in which we can choose a group of i factors of x_1, j factors of x_2, and k factors of x_3 from a total of n factors We multiply because the number of possible ways applies for each outcome.

$$c_i = \binom{n}{i}\binom{n-i}{j}\binom{n-i-j}{k} = \binom{n}{i}\binom{n-i}{j}$$

Since $i + j + k = 1$, the third binomial factor evaluates to just 1 since $n - i - j = k$

Plugging this into our summation:

$$(x_1 + x_2 + x_3)^n = \sum_{i+j+k=n}^{n} \binom{n}{i}\binom{n-i}{j} x_1^i x_2^j x_3^k$$

Now to generalize this, as we increase m, we would need m number of indexing variables to represent powers of each x_i.

Let's define the powers of x_i to be $n_1, n_2, ..., n_m$. The process is similar, we are still finding the number of ways to group up n_1 factors of x_1, and n_2 factors of x_2, and so forth, all from a group of n total factors. However, we still need to satisfy the condition $n_1 + n_2 + ... + n_m = n$. We multiply all these numbers to get the total number of ways to group up cetain factors of x_i

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n_m}{n_m} = \binom{n}{n_1, n_2, ..., n_m}$$

Plugging this into our summation:

$$(x_1 + x_2 + ... + x_m)^n = \sum_{n_1+n_2+...+n_m=n}^{n} \binom{n}{n_1, n_2, ..., n_m} x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$$

Task 3

1. In a game show, the winner is asked to choose one of three doors in order to determine her final prize.
   Behind two of the doors there is a goat, while behind the remaining one there is a car and a cash prize. The winner chooses one the doors and the host opens what she knows to be a door concealing a goat from the other two doors.
   She then asks the participant whether she would like to
   - stick to the initial choice or
   - switch to the other closed door.
   What would you tell the participant to do and why?

   We would advise the participant to switch to the other closed door because we have more space for error if we were to choose the wrong door initially, which is 2/3 since there are 2 wrong doors and 1 right door.
   To clarify, when the hosts reveals the other door with the goat behind it, we know that our initial choice may be right or wrong. The odds become somewhat 50/50. But we can find an objective advantage by looking at all the cases.
   If we choose the wrong door initially and we switch, then we can keep the odds of 50/50.
   But if we open the right door initially and we switch, then we have literally removed all possibility of winning.
   Since there are 2 wrong doors, our initial choice has a 2/3 probability of leading to a promising outcome.
   Alternatively, if we were to keep our initial choice in all cases, then we have only a 1/3 probability of leading to a promising outcome.