Group #5 @1pm
Mimi Morales, Yiyi Zhu, Vi Pham, Brian Fong April 25^{th}, 2019

Lecture Worksheet 11

Consider the Gambler's Ruin game described in Lecture 11 where two players take turns flipping a coin for which heads comes up with probability $p \in (0,1)$. If heads comes up, player A receives a chip from player B, else she gives on to player B. Assume that there are a total of N chips in the game, m of which are owned by player A initially. Clearly, $0 \leq m \leq N$. The game is played until one player loses all her chips.

Task 1

- How do possible games look like? How can you describe them concisely? How do winning games (from the point of view of player A) look like? How about losing games?

  Possible outcomes may look like sequences of Heads and Tails.
  A winning game for Player A would have the sequence ending with tails and the number of heads being $\#H = N - m$
  In contrast, a losing game for Player A would end the sequence with Tails and the number of tails being $\#T = m$

- What are the possible lengths of a game?

  The game's duration could theoretically go on forever, but the average duration of a game should be within a finite range.

- What is the sample space of the game?

  The sample space could be the all the sequences of Heads and Tails, where Player A wins/loses.

- What is the probability of each possible game? What is the probability of winning?

  Given n trials (coin flips), then we can call E the game, or event, where there are k heads and n-k tails:

$$P(E) = p^k(1-p)^{n-k}$$

Let F be the event of Player A winning, then the probability is when there are N-m heads. However, there can be N-m heads plus an additional equal number of heads and tails to offset and elongate the game.

We can call:

$$t = Additional\ \#\ of\ Heads = Additional\ \#\ of\ Tails$$

So thus the probability of winning is proportional to:

$$P(F) \propto p^{N-m+t}(1-p)^t$$

But still, since we are working with sequences and the order may sometimes not matter, the probability may be any permutation of N-m+t heads and t tails so

$$P(F) \propto \binom{n}{N-m+t} p^{N-m+t} \binom{n}{t} (1-p)^t$$

Yet again, this still overcounts certain "invalid" permutations where we get too many consecutive heads/tails.

Task 2

- Consider the probability $P_m = P(W_m)$ of the event W_m of winning the above game when starting with m chips. What are P_0 and P_N?

  If we start with 0 chips, then we have already lost the game,

  $$P_0 = 0\%$$

  If we start with N chips, then we have already won the game,

  $$P_N = 100\%$$

- Compute the probability of W_m conditioning on the outcome of the first flip. What does that give you?

  Let $H =$ the Event of getting heads first

  By Bayes' Formula, we can rewrite $P(W_m) = P_m$ as:

  $$P_m = P(W_m|H)P(H) + P(W_m|H^c)P(H^c)$$

  Given that the $P(H) = p$, we have:

  $$P_m = p \cdot P(W_m|H) + (1-p) \cdot P(W_m|H^c)$$

- Conclude by computing P_m and compare to the computation you performed in Task 1

  The next step in computing P_m is finding $P(W_m|H)$

  After the first coin flip, Player A either has $m+1$ or $m-1$ chips. Since each coin flip is independent, we can treat this second coin flip (and the following flips) as a new probability except with a different starting amount. Then we have:

  $$P(W_m|H) = P_{m+1} \textit{ and } P(W_m|H^c) = P_{m-1}$$

  Where $m = 1, 2, ..., N-1$ Plugging this back into the equation:

  $$P_m = p \cdot P_{m+1} + (1-p) \cdot P_{m-1}$$

  Now if we let $q = 1-p$, we can simplify:

  $$P_{m+1} - P_m = \frac{q}{p}\big(P_m - P_{m-1}\big)$$

Testing cases for $m = 1, 2, ..., i$

For $m = 1$*:*

$$P_2 - P_1 = \frac{q}{p}\big(P_1 - P_0\big) = \frac{q}{p}P_1$$

For $m = 2$*:*

$$P_3 - P_2 = \frac{q}{p}\big(P_2 - P_1\big) = \Big(\frac{q}{p}\Big)^2 P_1$$

For $m = i - 1$*, where* i *is some constant:*

$$P_i - P_{i-1} = \frac{q}{p}\big(P_{i-1} - P_i\big) = \Big(\frac{q}{p}\Big)^{i-1} P_1$$

Now if we were to add up the LHS of these equations, this would give us:

$$\Big(P_m - P_{m-1}\Big) + \Big(P_{m-2} - P_{m-3}\Big) + ... + \Big(P_2 - P_1\Big)$$

$$= P_1 \cdot \Big[\Big(\frac{q}{p}\Big)^1 + \Big(\frac{q}{p}\Big)^2 + ... + \Big(\frac{q}{p}\Big)^{m-1}\Big]$$

Now we can represent the RHS as a partial geometric series:

$$P_1 \cdot \Big[\Big(\frac{q}{p}\Big)^1 + \Big(\frac{q}{p}\Big)^2 + ... + \Big(\frac{q}{p}\Big)^{m-1}\Big] = \sum_{k=1}^{m-1} P_1\Big(\frac{q}{p}\Big)^k$$

Now if we isolate P_m *we have:*

$$P_m = P_1 + \sum_{k=1}^{m-1} P_1\Big(\frac{q}{p}\Big)^k = \sum_{k=0}^{m-1} P_1\Big(\frac{q}{p}\Big)^k = P_1 \cdot \frac{1 - (q/p)^m}{1 - (q/p)}$$

NOTE: This equation only holds if $p \neq q$*, so we would have to make a piece-wise equation.*

$$P_m = \begin{cases} P_1 \cdot \frac{1-(q/p)^m}{1-(q/p)} & p \neq q \\ m * P_1 & p = q \end{cases}$$

Task 3

Consider the set $N = \{1, 2, ..., n\}$ for $n \in \mathbb{N}$ and let A, B be two randomly chosen sets (any subset has equal likelihood of being chosen, including the empty set.

- What is the probability that $A \subseteq B$?

We can condition on the number of elements of the set B, allowing us to rewrite the probability of $A \subseteq B$ as:

$$P(A \subseteq B) = \sum_{i=0}^{n} P(A \subseteq B | \#B = i) \cdot P(\#B = i)$$

We can interpret this probability as the probability that A is a subset of B given that B has i number of elements.

Here we use the equation which gives us the possible number of subsets of a set containing n elements:

$$\# \textit{ of possible subsets} = 2^n$$

The probability that B has i number of elements is given by:

$$P(\#B = i) = \frac{\binom{n}{i}}{2^n}$$

By definition, A being a subset of B means that A contains 0,1,...,i elements. The probability $P(A \subseteq B | \#B = i)$ is given by:

$$P(A \subseteq B | \#B = i) = \sum_{k=0}^{i} \frac{\binom{i}{k}}{2^n}$$

We can rewrite the equation as:

$$\Rightarrow P(A \subseteq B | \#B = i) = \frac{1}{2^n}\left[\binom{i}{0} + \binom{i}{1} + ... + \binom{i}{i}\right]$$

Using the Binomial Theorem:

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + ... + \binom{n}{n-1} x^1 y^n - 1 + \binom{n}{n} x^0 y^n$$

Now if we substitute $x = y = 1$, then we have something that resembles the RHS of the probability equation. Actually substituting into the Binomial Theorem, we have

$$(x+y)^n = (1+1)^n = \binom{n}{0} + \binom{n}{1} + ... + \binom{n}{n} = 2^n$$

Then we have:

$$P(A \subseteq B|\#B = i) = \frac{1}{2^n}\Big[\binom{i}{0} + \binom{i}{1} + ... + \binom{i}{i}\Big] = \frac{1}{2^n} \cdot 2^i$$

Putting this all back into the original equation:

$$\begin{aligned}
P(A \subseteq B) &= \sum_{i=0}^{n} P(A \subseteq B|\#B = i) \cdot P(\#B = i) \\
&= \sum_{i=0}^{n} \left[\frac{1}{2^n}2^i\right] \cdot \left[\frac{\binom{n}{i}}{2^n}\right] \\
&= \frac{1}{4^n}\sum_{i=0}^{n}\binom{n}{i}2^i \\
&= \frac{1}{4^n}\Big[\binom{n}{0}2^0 + \binom{n}{1}2^1 + ... + \binom{n}{i}2^i\Big]
\end{aligned}$$

Here we can see that the RHS resembles the Binomial Theorem where we substitute $x = 1$ *and* $y = 2$.
Actually substituting into the Binomial Theorem gives us:

$$(x + y)^n = (1 + 2)^n = \binom{n}{0}2^0 + \binom{n}{1}2^1 + ... + \binom{n}{n}2^n = 3^n$$

Then we have:

$$\Rightarrow \frac{1}{4^n} \cdot 3^n$$

Finally, the probability that A *is a subset of* B *is given by:*

$$P(A \subseteq B) = \left(\frac{3}{4}\right)^n$$