Generative Models for Classification

In the case of Logistic Regression we directly model $\Pr(Y = k|X = x)$ using the Sigmoid Function (Logistic Function) its a simple conditional probability approach **Predicting Y given X**.
Considering the alternative Probability of the the the predictors X given a class Y which is the logic behind **Bayes Theorem**, When the distribution of X is normal the model turn to be very similar to Logistic Regression.

Why its needed?

- If There is a substantial separation between the two classes the **parameter estimates** for logistic regression model will be unstable
- If the predictors X are normally distributed and the sample size n is small Using **Generative Models** will be more accurate than Logistic Regression

Suppose that we want to classify an observation into on of the K classes where$K \geq 2$, means the Response Y can take K possible classes.

- Let π_k be the **Prior Probability** that the a random Observation comes from the $k\text{th}$ class $P(Y = k)$
- Let $f_k(X) = \Pr(X|Y = k)$ is the **Density Function** of X
  - $f_k(x)$ will be large if there is **high probability** of that observation being in the $k\text{th}$ class

$$\Pr(Y = k|X = x) = \frac{\pi_k f_k(x)}{\sum^K \pi_i f_i(x)}$$

- This is just the Bayes' theorem Formula
- Let $p_k(x) = \Pr(Y = k|X = x)$ and will also be called the **Posterior** probability
- So our goal is to know the probability of the Observation being in the $k\text{th}$ class *Given* the class k aka Response Y

Linear Discriminant Analysis Vs Logistic Regression

First before diving into **LDA** ,Lets establish a clear difference between What Logistic Regression Does and What **LDA Does** , cause at the end they both Classify a new observation into a class.
What Logistic Regression Does

- It tries to directly model the **posterior probability**

$$P(Y = k|X = x) = \frac{e^{\vec{X}\vec{\beta}}}{1 + e^{\vec{X}\vec{\beta}}}$$

- You just want to separate the classes,Its **Discriminative** just making distinctions
- *We don't care how the data X looks like inside each class*(We make no assumption about the distribution of X, We aim class each observation
- We also don't care about how the data was generated

What LDA Does

- Its a **generative**, it models how the data X looks within each class
- With the assumption of the data X taking a normal distribution form

$$P(X|Y = k) = \mathcal{N}\left(\mu_k, \sum\right)$$

- Then use Bayes' theorem to compute

$$P(Y = k|X = x) = \frac{P(X|Y = k)P(Y = k)}{P(X)}$$

- Then classify x by choosing the class with the highest **posterior** $P(Y = k|X = x)$
- Simply its saying *This new point looks most like the kind of data that class k generates*

Aspect	Logistic Regression	LDA
Type	Discriminative ,**Making distinctions**	Generative
Models	$P(Y\|\|X)$	$P(X\|\|Y), P(Y)$
Assumes	No assumptions	$X\|\|Y$ is Normally Distributed, All classes variances are equal
Uses	Decision boundaries	Bayes's Classifying

Linear Discriminant Analysis for $p = 1$

For this section we will assume that $p = 1$ only one predictor, The goal is :

- Obtain estimate for $f_k(x) \to \text{The update}$
- We plug it into the Bayes' theorem formula to get the estimate for $p_k(x) \to \text{ The posterior}$
- Then we **Classify** an Observation to the class with the highest $p_k(x)$ The posterior

Estimating $f_k(x)$

Also known as the The update noted $P(X|Y)$ the **easy to measure** probability used to **update** the The prior π_k

- We assume that $f_k(x)$ is **normal or Gaussian**

$$f_k(x) = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-1/2\sigma_k^2(x-\mu_k)^2}$$

- $\mu_k \to$ The mean for the $k\text{th}$ class
- $\sigma_k^2 \to$ the variance for the $k\text{th}$ class

Assuming that the variance of all classes K are equal so $\sigma_k^2 = \sigma^2$ for simplicity

$$P(Y = k|X) = p_k(x) = \frac{\pi_k \frac{1}{\sqrt{2\pi}\sigma} e^{-1/2\sigma^2(x-\mu_k)^2}}{\sum^k \pi_i \frac{1}{\sqrt{2\pi}\sigma} e^{-1/2\sigma^2(x-\mu_k)^2}}$$

- This is the **Posterior** with the assumption of all **variances are equal** and that $f_k(x)$ is **normally distributed**

Dropping the denominator in the **Posterior** formula we get the Discriminant Functions which is the log of the **normalized posterior**, The denominator is constant across all classes k

$$\log(p_k(x)) = \log(\pi_k) - \frac{1}{2\sigma^2}(x - \mu_k)^2$$

- Dropping the term $\frac{1}{\sqrt{2\pi}\sigma}$ for being a constant

$$\log(p_k(x)) = \log(\pi_k) - \frac{1}{2\sigma^2}x^2 - \frac{1}{2\sigma^2}\mu_k^2 + \frac{x\mu_k}{\sigma^2}$$

$$\log(p_k(x)) = \log(\pi_k) - \frac{\mu_k}{2\sigma^2} + x\frac{\mu_k}{\sigma^2} = \delta_k$$

- Dropping x^2 will get δ which is the a **Linear Discriminant Function** of x
- δ_k is the score for class k
- We aim to assign $X = x$ to the class that gives the largest δ_k

If $K = 2$ and $\pi_1 = \pi_2$ then the Bayes decision boundary is the point where $\delta_1 = \delta_2$

$$\delta_1 = \delta_2$$

$$x\frac{\mu_1}{\sigma^2} - \frac{\mu_1^2}{2\sigma^2} + \log(\pi_1) = x\frac{\mu_2}{\sigma^2} - \frac{\mu_2^2}{2\sigma^2} + \log(\pi_2)$$

$$\log(\pi_1) = \log(\pi_2)$$

$$\begin{aligned} x\frac{\mu_1}{\sigma^2} - \frac{\mu_1^2}{2\sigma^2} &= x\frac{\mu_2}{\sigma^2} - \frac{\mu_2^2}{2\sigma^2} \\ x\frac{\mu_1 - \mu_2}{\sigma^2} &= \frac{\mu_1^2 - \mu_2^2}{2\sigma^2} \\ x = \frac{\mu_1^2 - \mu_2^2}{2(\mu_1 - \mu_2)} &= \frac{\mu_1 - \mu_2}{2} \end{aligned}$$

- If $x > \frac{\mu_1 - \mu_2}{2}$ we assign it to class 1 otherwise class 2

In practice we are not quite certain of our assumption that X is **normally distributed** and we will still need to estimate : LDA Mean And Variance Estimates

- $\mu_1 \ldots \mu_K$
- $\pi_1 \ldots \pi_k$
- σ^2

$$\hat{\mu_k} = \frac{1}{n_k} \sum_{i:y_i=k} x_i$$

$$\hat{\sigma}^2 = \frac{1}{n-K} \sum^K \sum_{i:y_i=k} (x_i - \hat{\mu_k})^2$$

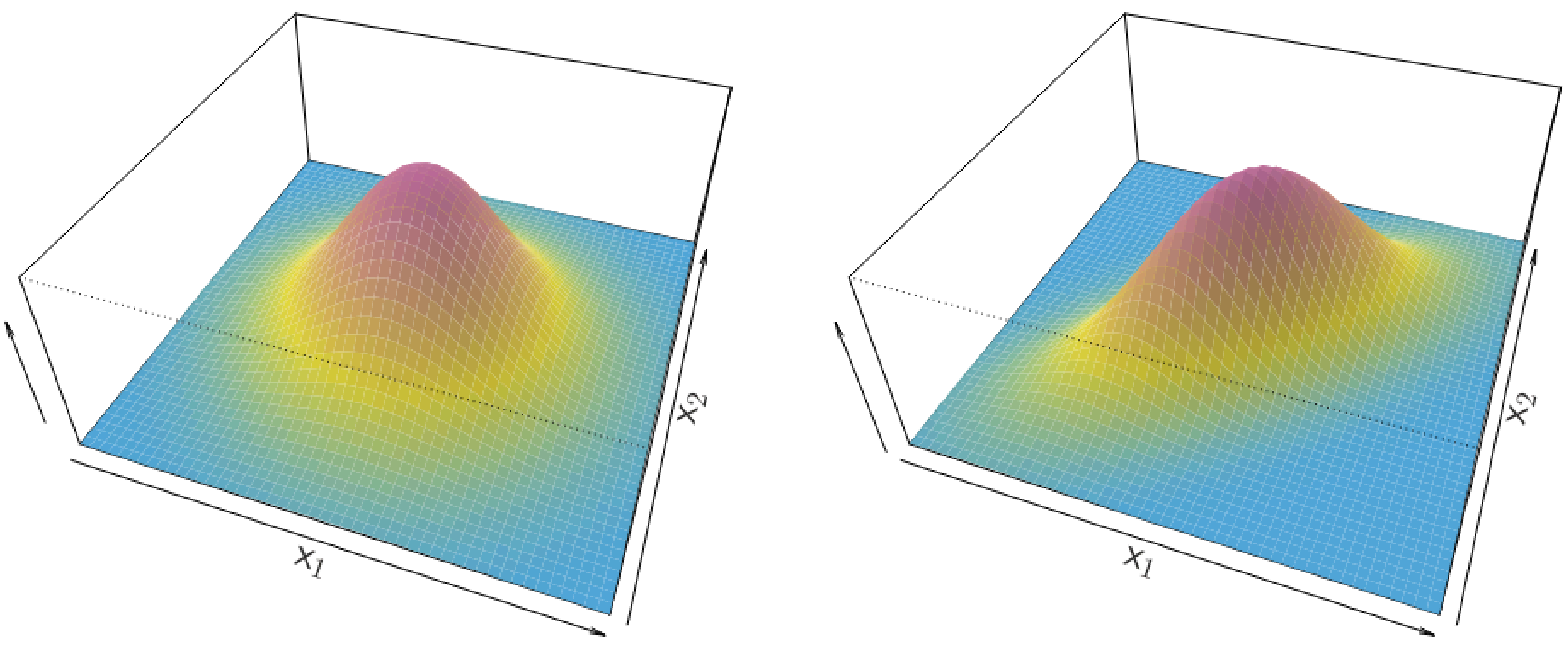
$$\hat{\pi_k} = \frac{n_k}{n}$$

The estimated discriminant function for class k is given then :

$$\hat{\delta}_k(x) = x\frac{\hat{\mu_k}}{\hat{\sigma^2}} - \frac{\hat{\mu_k}^2}{2\hat{\sigma}^2} + \log(\hat{\pi_k})$$

Linear Discriminant Analysis for $p > 1$

Here we assume that the X predictors are drawn from a Multivariate Normal Distribution with a mean for each class k μ_k and a **covariance matrix**



- The **Left Figure** shows a multivariate normal distribution of two predictors X with no **correlation**, each of the predictors is a one dimensional normal distribution
- The **Right Figure** shows the same as the left one but with a correlation value of 0.7
  The formula for Multivariate Normal Distribution is given by:

$$f_k(x) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} e^{\frac{-1}{2}(X-\mu)^T\Sigma^{-1}(X-\mu)}$$

- μ_k is a mean vector special for each k class
- Σ is a covariance matrix that is common among all classes k

Plugging it into the Bayes classifier format:

$$P(Y=k|X=x) = p_k(x) = \frac{\pi_k \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} e^{\frac{-1}{2}(X-\mu_k)^T\Sigma^{-1}(X-\mu_k)}}{\sum_i^K \pi_i \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} e^{\frac{-1}{2}(X-\mu_i)^T\Sigma^{-1}(X-\mu_i)}}$$

- Σ is the covariance matrix common with all K classes $p \times p$
- X observation vector $p \times 1$
- μ_k Mean vector for class k with $p \times 1$

From that expression we can derive and get this Discriminant Functions :

$$\log(p_k(x) = \log(\frac{\pi_k \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} e^{\frac{-1}{2}(X-\mu_k)^T\Sigma^{-1}(X-\mu_k)}}{\sum_i^K \pi_i \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} e^{\frac{-1}{2}(X-\mu_i)^T\Sigma^{-1}(X-\mu_i)}}))$$

$$\log(p_k(x)) = \log(\pi_k) - \frac{1}{2}(X-\mu_k)^T\Sigma^{-1}(X-\mu_k)$$

- The **denominator** is shared across all classes k so its can be dropped
- $\frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}}$ is a **constant** term so we can drop it

$$\log(p_k(x)) = \log(\pi_k) - \frac{1}{2}(X^T\Sigma^{-1} - \mu_k^T\Sigma^{-1})(X-\mu_k)$$

$$\log(p_k(x)) = \log(\pi_k) - \frac{1}{2}(X^T\Sigma^{-1}X - X^T\Sigma^{-1}\mu_k - \mu_k^T\Sigma^{-1}X + \mu_k^T\Sigma^{-1}\mu_k)$$

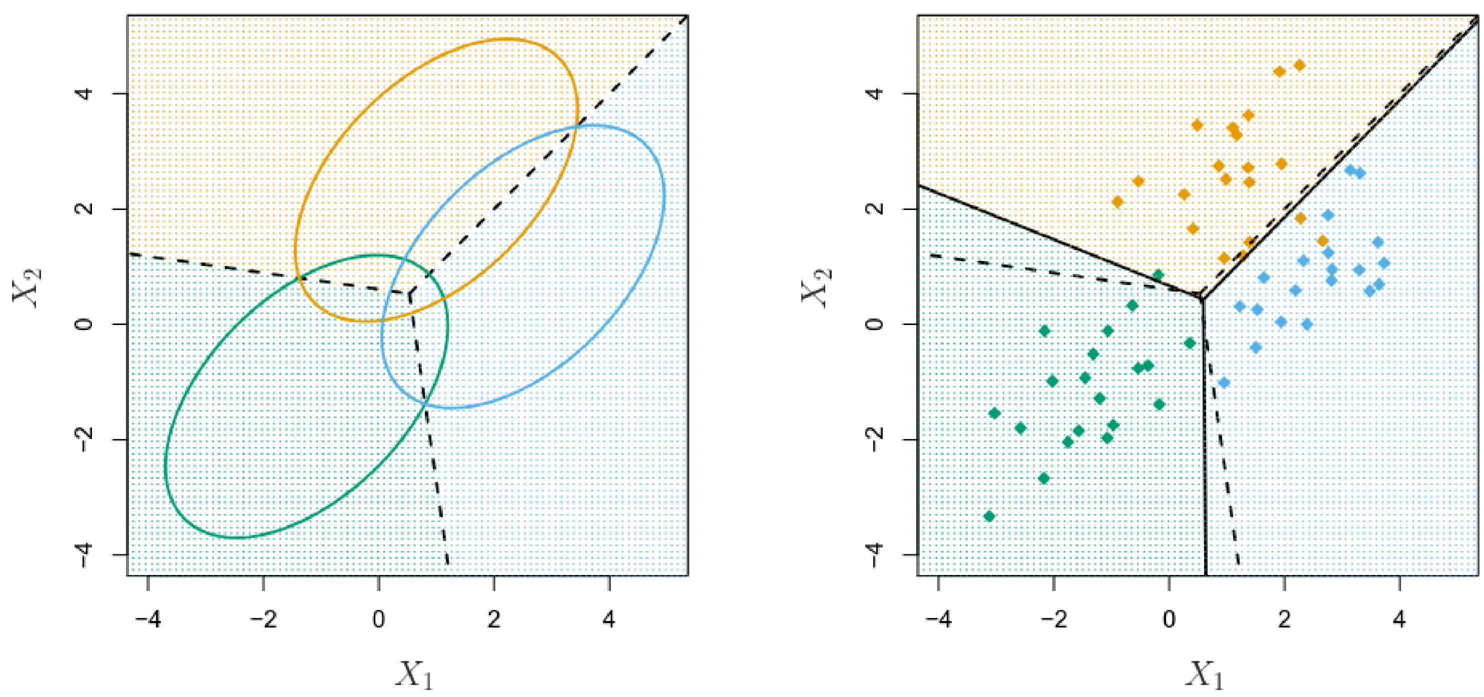
- $X^T\Sigma^{-1}X$ is a scalar and it doesn't depend on k
- $X^T\Sigma^{-1}\mu_k$ and $\mu_k^T\Sigma^{-1}X$ are both **scalars** and the same but **transposed** of each other

Note : We drop every term that doesn't depend on k

$$\log(p_k(x)) = \log(\pi_k) + X^T\Sigma^{-1}\mu_k - \frac{1}{2}\mu_k^T\Sigma^{-1}\mu_k$$

$$\delta_k(x) = X^T\Sigma^{-1}\mu_k - \frac{1}{2}\mu_k^T\Sigma^{-1}\mu_k + \log(\pi_k)$$

- $\delta_k(x)$ is the **Discriminant Function**
- We assign an observation X to the class k with the highest $\delta_k(x)$



- Dashed Lines represent where $\delta_k(x) = \delta_j(x)$, Which is the Bayes decision boundary
- The ellipses represent regions with 95% of the **probability** for each class k

- π_k is equal across all classes k
  Note : $\pi_k = P(Y = k)$ its the **Prior Probability**, The number of Observation does effect π_k, Since we estimate $\hat{\pi}_k = \frac{n_k}{n}$