Logistic Regression

The question is how should we model the relationship between

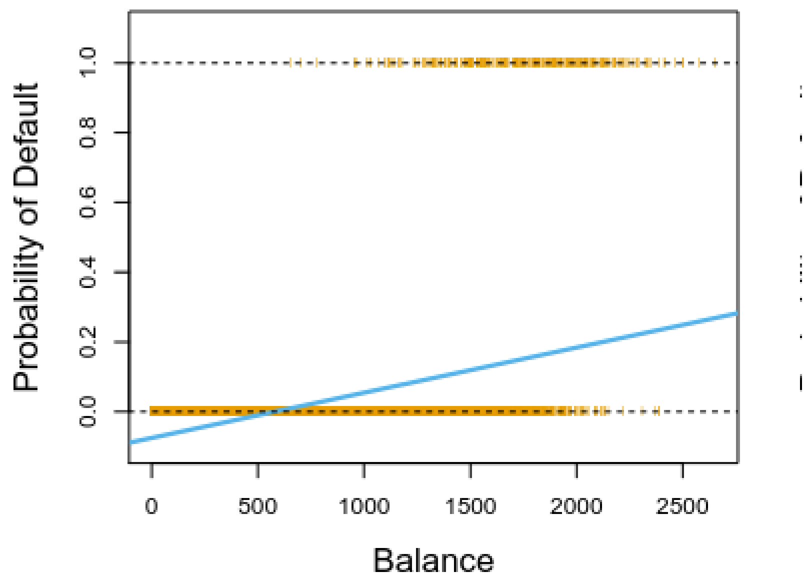
$$P(X) = \Pr(Y = 1|X)$$

- Relationship between the **Probability** of X and the **Classifying** Prediction for X
- Using 1 and 0 for the Response

Using Linear Regression model to represent these probabilities

$$p(X) = \beta_0 + \beta_1 X$$

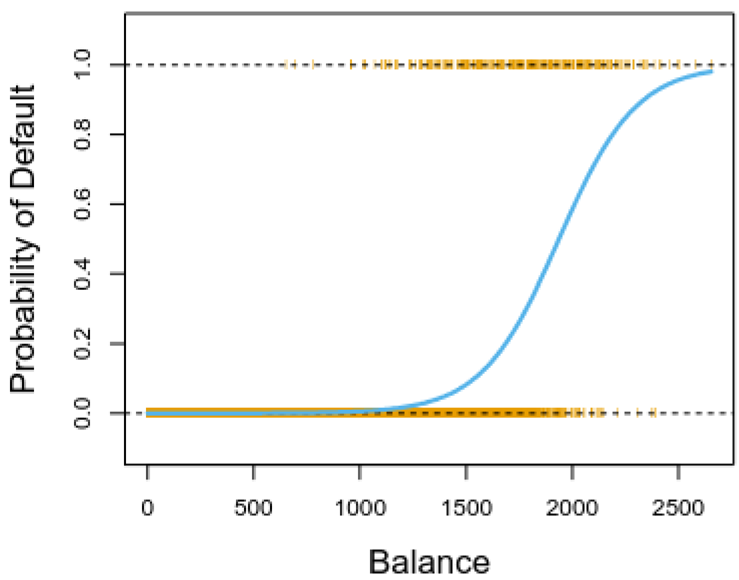
- If we fit the line to predict the **Probability**



- Notice that the `Balance` Lower than 500 our prediction for the probability is **negative**

To avoid this problem we model $p(X)$ to only fall between 1 and 0 for all values X **Logistic Function** which is a Sigmoid Function

$$p(X) = \frac{e^{\beta_0+\beta_1 X}}{1+e^{\beta_0+\beta_1 X}} = \frac{1}{e^{-(\beta_0+\beta_1 X)}+1}$$



- Any output of the **Logistic Function** Falls between 1 or 0

$$\frac{p(X)}{1-p(X)} = \frac{\frac{e^{\beta_0+\beta_1 X}}{1+e^{\beta_0+\beta_1 X}}}{1-\frac{e^{\beta_0+\beta_1 X}}{1+e^{\beta_0+\beta_1 X}}} = \frac{e^{\beta_0+\beta_1 X}}{1+e^{\beta_0+\beta_1 X}} \times \frac{1}{\frac{e^{\beta_0+\beta_1 X}}{1+e^{\beta_0+\beta_1 X}} = \frac{1}{1+e^{\beta_0+\beta_1 X}}} = \frac{e^{\beta_0+\beta_1 X}}{1+e^{\beta_0+\beta_1 X}} \times 1 + e^{\beta_0+\beta_1 X}$$

$$\text{odds} = \frac{p(X)}{1-p(X)} = e^{\beta_0+\beta_1 X}$$

- the **quantity** $p(X)/[1-p(X)]$ is called the $odds$ can can only take values between 0 to ∞
- Values close to 0 indicates low probability
- Values close to ∞ indicate higher probability

- if $p(X) = 0.5$ the $\text{odds} = 1$ equal chance
- if $p(X) = 0.8$ the $\text{odds} = 4$ 4x success chances

Probabilities vs Odds

- The odds are the **ratio** of something happening *divided by* something not happening
- The probability is the **ratio** of something happening *divided by* to everything could happen

$$\text{Probability} = \frac{\text{Number of favorable outcomes}}{\text{Total number of possible outcomes}}$$

$$\text{Odds} = \frac{\text{Probability of even occurs}}{\text{Probability event does not occur}} = \frac{p}{1-p}$$

- In the Logistic Regression setting the **odds** are just an alternative representation for the classification problem
- **Odds** are preferred cause they allow us to transform the the odds which are a correct and alternative representations to the probabilities of each class into a **Linear Combination of Feature**

Odds in logistic regression

Taking logarithm of both sides : log odds or logit

$$\log\left(\frac{p(X)}{1-p(X)}\right) = \beta_0 + \beta_1 X$$

- the $\log$ of odds gives us a **Linear Combination of Features** which is easy to model and **interpret**

Why we use the odds?

- **Probabilities** lives in the interval $[0, 1]$
- Linear combinations like $\beta_0 + \beta_1 X$ lives on $(-\infty, +\infty)$
- The **odds** solves that by being in $(0, +\infty)$

Coefficients interpretability

In the **Linear Regression** β_1 gives the average change in Y associated with one unit increase in X
In the **Logistic Regression** β_1 does not correspond directly to the change in $p(X)$, if β_1 is positive increasing one unit in X will increase the **Probability** $p(X)$ but the increase depends on the current value of $p(X)$

$$\beta_1 > 0 \to \text{One unit increase X} \to \text{Increase } p(X)$$

$$\beta_1 < 0 \to \text{One unit increase X} \to \text{Decrease } p(X)$$

- The amount "Degree" of increase in the **Probability** $p(X)$ depends on the current value of X

Why the effect of the probability depends on current $p(X)$

- if $p(X) = 0.5$ the Sigmoid Function curve is steep, small changes in $X \to$ big changes in $p(X)$
- if $p(X) \approx 0.01 \text{ or } 0.09$ the **Sigmoid Function** curve is flat, changes in $X \to$ small changes in $p(X)$

Statistical Inference

The **Log odds** results in a linear equation which will allow us to preform the **Linear Regression** inference and tests :

- **Hypothesis Testing**
  - F-test
  - T-test
- Construct **Confidence Intervals**
- **MLE** for optimization

Estimating The Regression Coefficients

In **Linear Regression** we use the least squares to estimate the coefficients, in the logistic regression the **Maximum Likelihood**.
Intuition :

- We seek estimates for $\hat{\beta}_0, \hat{\beta}_1$ such that we maximize the probability $\hat{p}(x_i)$
- More explanation in Maximum Likelihood Estimation

$$\mathcal{L}(\beta_0, \beta_1) = \prod_{i:y_{i=1}} p(x_i) \prod (1 - p(x_{i'}))$$

- This called the **Likelihood function**
- Our estimates $\hat{\beta}_0, \hat{\beta}_1$ are chosen to **maximize** this likelihood function and that **best to separates classes based on labels**
- **The least squares** is a special case of $\text{maximum Likelihood}$ Residual Sum of Squares
- Unlike Linear Regression the **Likelihood function** in logistic regression is nonlinear in the parameter β so there is no closed form solution for β Maximum Likelihood Estimator Derivation Logistic Regression

Once the **Coefficients** are estimated (Using iterative numerical optimization methods),we can calculate the predicted probability for the observation X_i

Multiple Logistic Regression

Generalizing the **Simple Logistic Regression** equation of the **log-odds**

$$\log\left(\frac{p(x)}{1 - p(x)} = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p\right)$$

While $p(X)$ can be written as :

$$p(X) = \frac{e^{\beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p}}{1 + e^{\beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p}}$$

$$P(X) = \frac{e^{X\beta}}{1 + e^{X\beta}}$$

- The **Maximization** of the coefficients is done via **MLE** Maximum Likelihood Estimator Derivation Logistic Regression
- The Response is till binary 1 or 0
- With multiple predictors **Variables** to make the prediction
- it uses **Linear decision boundary** in the log-odds space

Same case as in multiple linear regression it give different results than the simple linear regression for the same predictor cause often the predictors will be **correlated**

Multinomial Logistic Regression

Also know as **softmax regression**

It's an extension to the **Binary Logistic Regression** where the response Y has more than two classes $K > 2$, the **softmax regression** outputs a vector which we can interpret each elements of that vector as a the probability of the input of a class k

$$f(x; B) = \begin{pmatrix} P(y = 1 | X = x) \\ P(y = 2 | X = x) \\ \vdots \\ P(y = K | X = x) \end{pmatrix}$$

- The elements of this vectors are the probabilities of the data x being in that class

1. For input x , the score for class k is given by $x\beta_k$
2. Taking the exponential $e^{x\beta_k}$, so its always positive
3. Normalize it , $P(y = k | X = x) = \frac{e^{x\beta_k}}{\sum_{j=1} e^{x\beta_j}}$

$$f(x; \beta) = \frac{1}{\sum_{j=1}^{K} e^{Bj^T x}} \begin{bmatrix} e^{\beta_1^T x} \\ e^{\beta_2^T x} \\ \vdots \\ e^{\beta_k^T x} \end{bmatrix} = \text{softmax}(x\beta)$$

Note :

$$B_k = \begin{bmatrix} \dots \beta_1^T \dots \\ \dots \beta_2^T \dots \\ \vdots \\ \dots \beta_k^T \dots \end{bmatrix}$$

- B_k is a parameter matrix
- This is the **softmax regression** which does not require a **Baseline**
Example
- Studying the **CRP** and it effect on infection types **(No infection ,Viral infection, Bacterial infection)**
Here we have 3 **logistic regressions** : with $CRP = 25$
- $0 =$ (Viral ,Bacterial infections) and $1 =$ No infection, resulted in 0.009
- $0 =$ (No,Bacterial infections) and $1 =$ Viral infection, resulted in 0.360
- $0 =$(No,Viral infection) and $1 =$ Bacterial infection, resulted 0.001

These results don't sum up to one which means they we cannot be represented as **probabilities** to each class, Here using **softmax regression** is the option to go

Baseline

- To set a model as a baseline we subtract the **coefficients** of the **baseline** model from the coefficients of all model so its will be a reference for all models
- The decision of choosing the **baseline** will only effect the coefficients estimates but the predictions remains the same Baseline Models Difference
- The goal of a baseline is to get result that sum up to 1 and be interpreted as probability, so we make all the classes have a relative difference to the **baseline**