# Eigenvectors & Eigenvalues

Before talking about Eigenvectors, let's establish that any matrix can be interpreted as a **Linear Transformation**, for example :

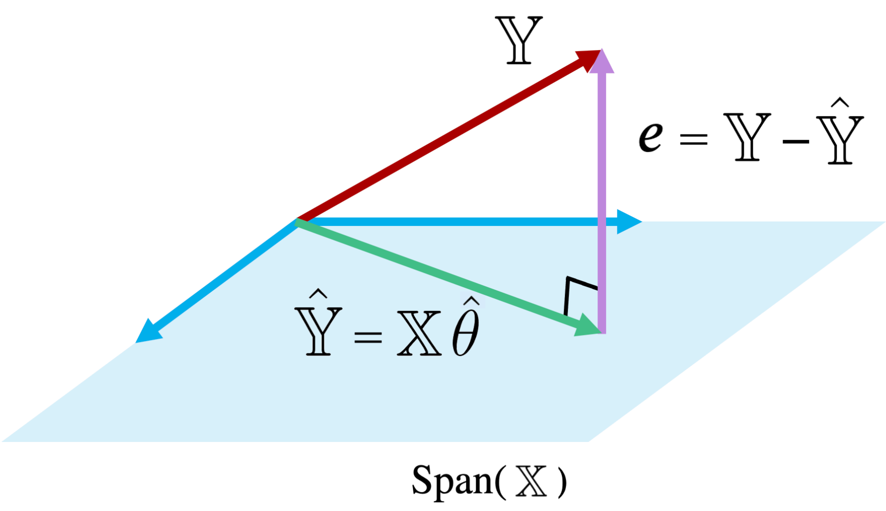
$$y = Ax$$

- The **matrix** $A$ is the Transformation matrix for the **vector** $x$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

- This matrix scales the $x$ axis by $1$ and the $y$ axis by $2$
- The columns of the matrix $A$ tells you what happen to the standard basis

The Geometry of Linear Transformations makes easier to understand as scaling , rotating ,projection :




- Same as in the Hat Matrix which Transform $Y$ into $\hat{Y}$

$$\hat{Y} = HY$$

Formally any matrix defines a **Linear Transformation**

$$T_A(x) = Ax$$

- Linearity check : $T_A(\alpha x + \beta = \alpha T_A(x) + \beta T_A(y)$

## Eigen vectors & values

Let $A$ be a square matrix $\in \mathbb{R}^{n \times n}$ and

- $\lambda$ a scalar
- $x$ a non-zero column vector $\in \mathbb{R}^n\{0\}$

$$Ax = \lambda x$$
$$Ax - \lambda x = 0$$
$$Ax - \lambda I x = 0$$
$$(A - \lambda I)x = 0$$

- $x$ is called the **Eigenvector** and can not be the zero vector
- $\lambda$ is the **Eigenvalue**
- The Transformation matrix $A$ can have multiple **Eigenvectors** (no more than $n$)
- $(A - \lambda I)$ can't be invertible cause :
  if $(A - \lambda I)$ is invertible

$$\begin{aligned} &(A - \lambda I)^{-1}(A - \lambda I)x = (A - \lambda I)^{-1}\vec{0} \\ &x = (A - \lambda I)^{-1}\vec{0} \\ &x = \vec{0} \end{aligned}$$

- Which break the rule of $x$ not being the zero vector

Since $(A - \lambda I)$ isn't invertible :

- $\det(A - \lambda I) = 0$
- $\text{rk}(A - \lambda I) < n$

Solving $|A - \lambda I| = 0$ will results in **Eigenvalue** $\lambda$

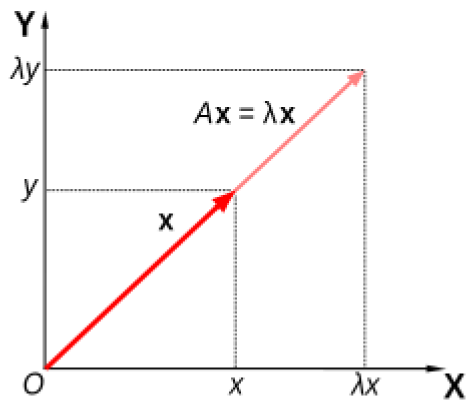
A vectorized version is written :

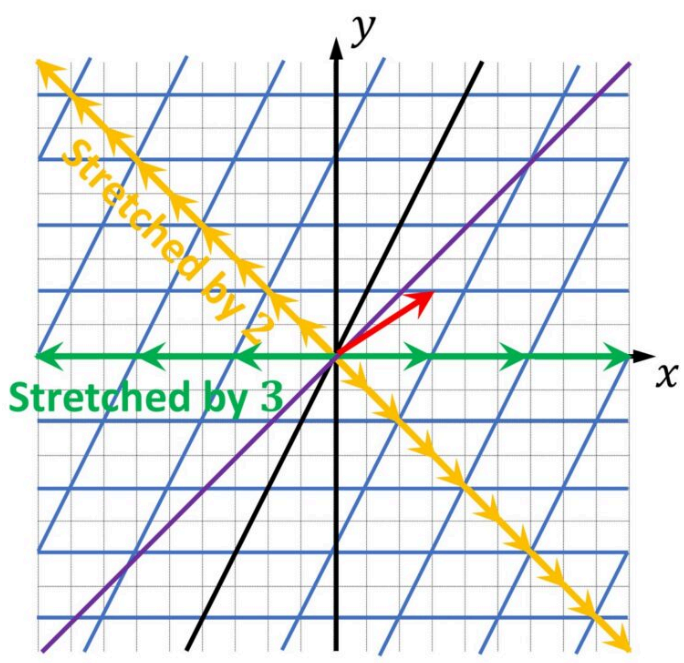
$$AT = TD$$

- $T$ the columns of this matrix are **Eigenvectors**
- $D$ is a diagonal matrix with each entry as a **Eigenvalue**

Geometrically **Eigenvectors** remain on their own span when transformed by the matrix $A$ , that's what makes them special since most of the vectors will get knocked their own span.

- The **Linear Transformation** $A$ applied to the **Eigenvector** $x$ will only scale it by an **Eigenvalue** $\lambda$




- The green and yellow vectors stays on their spans(origin) after the **Transformation** these are the **Eigenvectors**
- As for the purple vector gets knocked off it's span shown in the red vector