Sinyaller ve Sistemler

$$X_1(s) = \int_0^{\infty} x(t)e^{-st}\,dt \qquad s = \overbrace{\sigma + j\omega}^{\text{Kompleks değişken}}$$

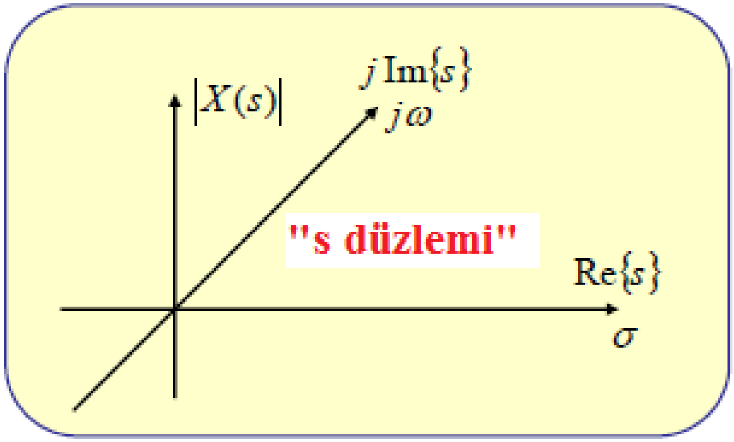
$$x(t) \neq 0,\ t < 0$$

$t = 0$ anında $x(t)$ girişi uygulanmaktadır $\Rightarrow$ $\underline{x(t) = 0 \quad t < 0}$ sistem nedensel

$$\boxed{\text{FT}:\ X(\omega) = \int_0^{\infty} x(t)e^{-j\omega t}\,dt} \qquad \boxed{\text{LT}:\ X(\sigma + j\omega) = \int_0^{\infty} x(t)e^{-(\sigma + j\omega)t}\,dt}$$

$j\omega \longrightarrow s$

$X(s)$: $\begin{cases} \text{kompleks değerli bir fonksiyon} \\ \text{kompleks değişken} \quad s = \sigma + j\omega \end{cases}$



$x(t) = e^{-bt}u(t)$ $\quad b > 0$ için $\quad$ Nedensel sinyal olsun

Fourier dönüşümü ile $\boxed{X(\omega) = \frac{1}{b + j\omega}}$

$$X(s) = \int_0^{\infty} e^{-bt} e^{-st} dt = \int_0^{\infty} e^{-(s+b)t} dt$$

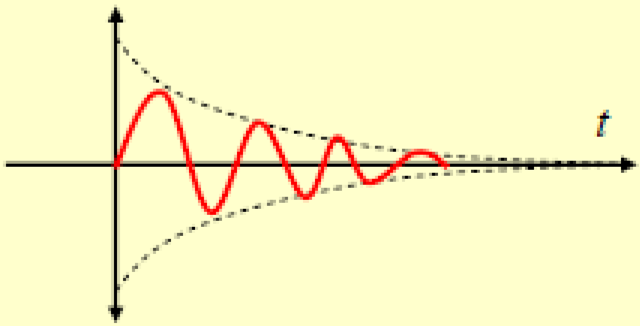
$$X(s) = \frac{-1}{s+b}\left[e^{-(s+b)t}\right]_{t=0}^{t=\infty} = \frac{-1}{s+b}\left[\underbrace{\lim_{t\to\infty} e^{-(s+b)t}} - 1\right]$$

Limit yakınsama yapmaz ise integral değeri yoktur.
İntegralin var olduğu durum incelenecek

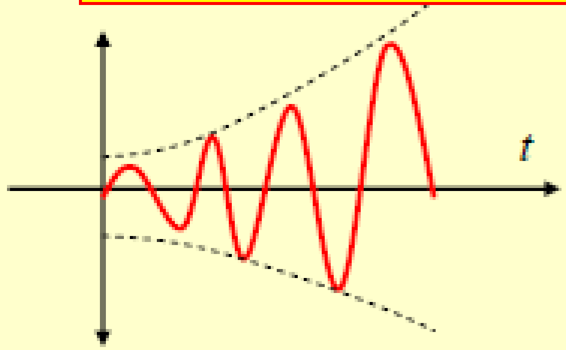
$$e^{-(s+b)t} = e^{-[(\sigma+b)+j\omega]t}$$

İki durum var

if $\sigma + b > 0 \;\Rightarrow\; \sigma > -b$



if $\sigma + b < 0 \;\Rightarrow\; \sigma < -b$



$\lim_{t\to\infty} e^{-(s+b)t}$ tanımlıdır $\sigma > -b$

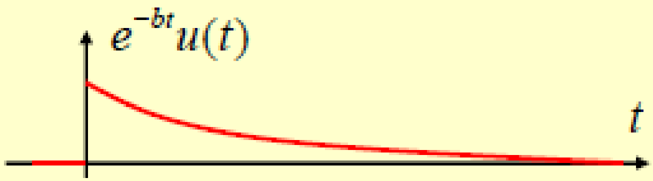
Her bir $X(s)$ için s değerleri ile tanımlı olduğu bölgeye:
<u>"Region of Convergence" (ROC)</u>

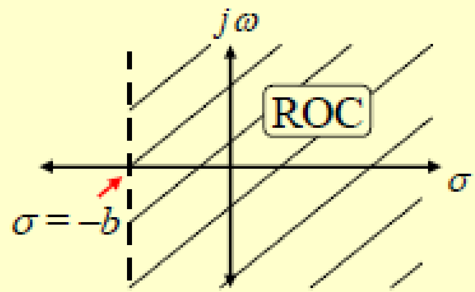
$$x(t) = e^{-bt}u(t) \qquad X(s) = \frac{1}{s+b} \quad \mathrm{Re}\{s\} > -b$$

Şart: $\mathrm{Re}\{s\} > -b$

If $b > 0$ $x(t)$ "azalandır"

$b > 0$, $-b$ negatif

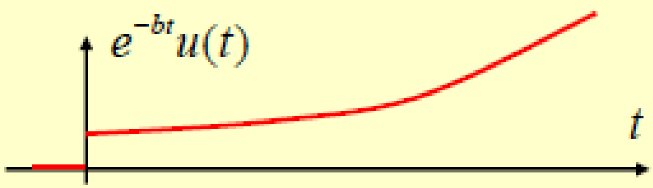


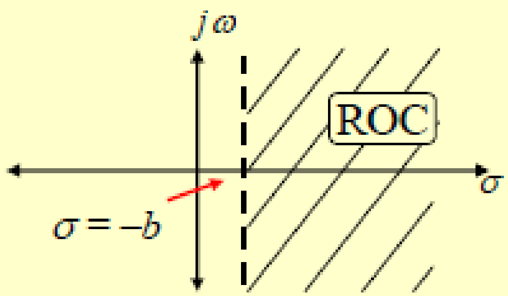


Bu durumda hem FT hem de LT hesaplanabilir

If $b < 0$ $x(t)$ "artandır"

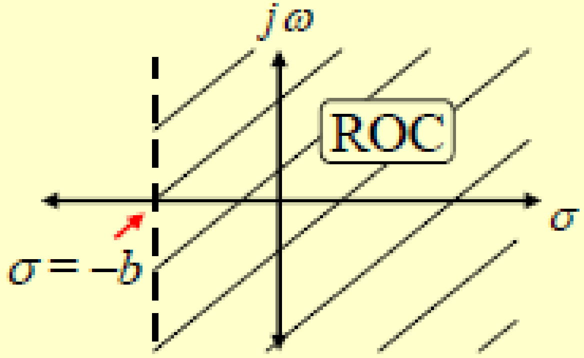
$b < 0$, $-b$ pozitif





Bu durumda FT hesaplanamaz LT hesaplanabilir

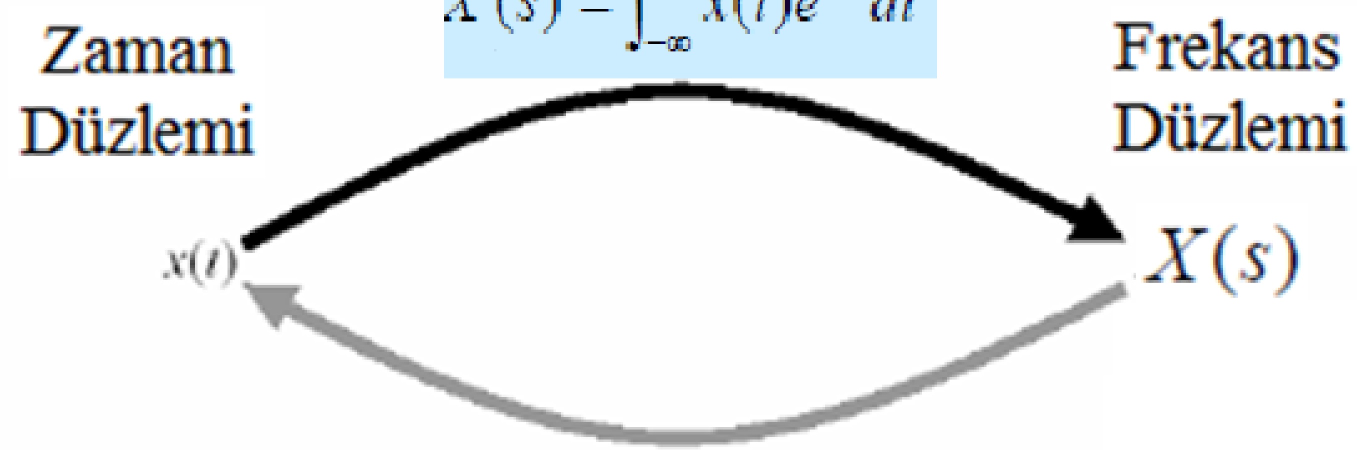
$$x(t) = e^{-bt}u(t) \quad \leftrightarrow \quad X(s) = \frac{1}{s+b} \quad \text{Re}\{s\} > -b$$



$$\Rightarrow X(s)\Big|_{s=j\omega} = \left[\frac{1}{s+b}\right]_{s=j\omega} = \underbrace{\left[\frac{1}{j\omega+b}\right]}_{\boxed{\text{FT ile aynı}}}$$

Laplace Dönüşüm Notasyonu

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$



$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s) e^{st} ds$$

$$\mathcal{L}\{x(t)\} \Rightarrow X(s)$$

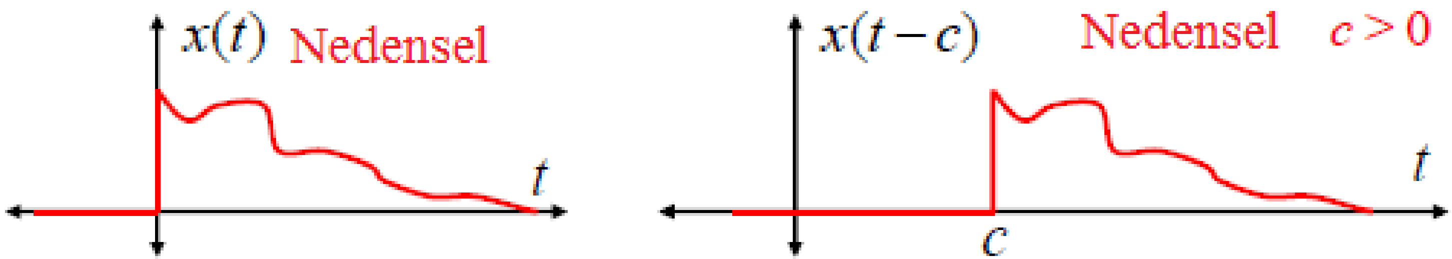
$$\mathcal{L}^{-1}\{X(s)\} \Rightarrow x(t)$$

Laplace Dönüşümünün Özellikleri

1-Doğrusallık:

$$\boxed{ax(t) + by(t) \quad \leftrightarrow \quad aX(s) + bY(s)}$$

2-Zamanda Öteleme:



$$x(t-c) \leftrightarrow e^{-cs} X(s)$$

$$(c > 0,\ x(t) \text{ causal})$$

3- Ölçekleme:

$$\boxed{x(at) \leftrightarrow \frac{1}{a} X\left(\frac{s}{a}\right)} \quad \boxed{a > 0}$$

$a < 0$ değerleri için $x(at)$ nedensel değildir

4- t^n ile çarpma:

$$\boxed{t^n x(t) \leftrightarrow (-1)^N \frac{d^N}{ds^N} X(s)}$$

5- Sinüs ile çarpma:

$$\boxed{x(t)\sin(\omega_0 t) \leftrightarrow \frac{j}{2}[X(s + j\omega_0) - X(s - j\omega_0)]}$$

$$\boxed{x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(s + j\omega_0) + X(s - j\omega_0)]}$$

6- Eksponensiyel ile çarpma:

$$\boxed{e^{at} x(t) \leftrightarrow X(s - a)}$$

s-düzleminde öteleme

7- Türev:

$$\boxed{\dot{x}(t) \leftrightarrow sX(s) - x(0)}$$

Başlangıç şartları
(IC's)

8- İntegral:

$$\boxed{\int_0^t x(\lambda)d\lambda \leftrightarrow \frac{1}{s}X(s)}$$

Türev	⇨	s ile çarpım
İntegral	⇨	s ile bölme

9- Konvolüsyon: $x(t) * h(t) \quad \leftrightarrow \quad X(s)H(s)$

$h(t)$	sistemin dürtü cevabı ise
$H(\omega)$	sistemin Frekens cevabı
$H(s)$	sistemin Transfer Fonksiyonudur

Laplace Transform Table

Time Signal	Laplace Transform
$u(t)$	$1/s$
$u(t)-u(t-c), \quad c>0$	$(1-e^{-cs})/s, \quad c>0$
$t^N u(t), \quad N=1,2,3,\ldots$	$\dfrac{N!}{s^{N+1}}, \quad N=1,2,3,\ldots$
$\delta(t)$	1
$\delta(t-c), \quad c$ real	$e^{-cs}, \quad c$ real
$e^{-bt}u(t), \quad b$ real or complex	$\dfrac{1}{s+b}, \quad b$ real or complex
$t^N e^{-bt}u(t), \quad N=1,2,3,\ldots$	$\dfrac{N!}{(s+b)^{N+1}}, \quad N=1,2,3,\ldots$
$\cos(\omega_o t)u(t)$	$\dfrac{s}{s^2+\omega_o^2}$
$\sin(\omega_o t)u(t)$	$\dfrac{\omega_o}{s^2+\omega_o^2}$
$\cos^2(\omega_o t)u(t)$	$\dfrac{s^2+2\omega_o^2}{s(s^2+4\omega_o^2)}$
$\sin^2(\omega_o t)u(t)$	$\dfrac{2\omega_o^2}{s(s^2+4\omega_o^2)}$

Laplace Transform Table -2

$e^{-bt}\cos(\omega_o t)u(t)$	$\dfrac{s+b}{(s+b)^2+\omega_o^2}$
$e^{-bt}\sin(\omega_o t)u(t)$	$\dfrac{\omega_o}{(s+b)^2+\omega_o^2}$
$t\cos(\omega_o t)u(t)$	$\dfrac{s^2-\omega_o^2}{(s^2+\omega_o^2)^2}$
$t\sin(\omega_o t)u(t)$	$\dfrac{2\omega_o s}{(s^2+\omega_o^2)^2}$
$Ae^{-\zeta\omega_n t}\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t\right]u(t)$ where: $A=\dfrac{\alpha}{\omega_n\sqrt{1-\zeta^2}}$	$\dfrac{\alpha}{s^2+2\zeta\omega_n s+\omega_n^2}$
$Ae^{-\zeta\omega_n t}\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t+\phi\right]u(t)$ $A=\beta\sqrt{\dfrac{(\alpha-\zeta\omega_n)^2}{\omega_n^2(1-\zeta^2)}+1}$ $\phi=\tan^{-1}\left(\dfrac{\omega_n\sqrt{1-\zeta^2}}{\alpha-\zeta\omega_n}\right)$	$\beta\dfrac{s+\alpha}{s^2+2\zeta\omega_n s+\omega_n^2}$
$te^{-bt}\cos(\omega_o t)u(t)$	$\dfrac{(s+b)^2-\omega_o^2}{((s+b)^2+\omega_o^2)^2}$
$te^{-bt}\sin(\omega_o t)u(t)$	$\dfrac{2\omega_o(s+b)}{((s+b)^2+\omega_o^2)^2}$

Laplace Transform Properties

Property Name	Property	
Linearity	$ax(t)+bv(t)$	$aX(s)+bV(s)$
Right Time Shift (Causal Signal)	$x(t-c), \quad c>0$	$e^{-cs}X(s)$
Time Scaling	$x(at), \quad a>0$	$\frac{1}{a}X(s/a), \quad a>0$
Multiply by t^n	$t^n x(t), \quad n=1,2,3,\ldots$	$(-1)^n \frac{d^n}{ds^n}X(s), \quad n=1,2,3,\ldots$
Multiply by Exponential	$e^{at}x(t), \quad a$ real or complex	$X(s-a), \quad a$ real or complex
Multiply by Sine	$\sin(\omega_o t)x(t)$	$\frac{j}{2}[X(s+j\omega_o)-X(s-j\omega_o)]$
Multiply by Cosine	$\cos(\omega_o t)x(t)$	$\frac{1}{2}[X(s+j\omega_o)+X(s-j\omega_o)]$
Time Differentiation 2nd Derivative n^{th} Derivative	$\dot{x}(t)$ $\ddot{x}(t)$ $x^{(N)}(t)$	$sX(s)-x(0)$ $s^2X(s)-sx(0)-\dot{x}(0)$ $s^N X(s)-s^{N-1}x(0)-s^{N-2}\dot{x}(0)-$ $\cdots - sx^{(N-2)}(0)-x^{(N-1)}(0)$
Time Integration	$\int_{-\infty}^{t} x(\lambda)d\lambda$	$\frac{1}{s}X(s)$
Convolution in Time	$x(t)*h(t)$	$X(s)H(s)$
Initial-Value Theorem	$x(0)=\lim_{s\to\infty}[sX(s)]$ $\dot{x}(0)=\lim_{s\to\infty}[s^2X(s)-sx(0)]$ $x^{(N)}(0)=\lim_{s\to\infty}[s^{N+1}X(s)-s^N x(0)-s^{N-1}\dot{x}(0)-\cdots-sx^{(N-1)}(0)]$	
Final-Value Theorem	If $\lim_{t\to\infty} x(t)$ exists, then $\lim_{t\to\infty} x(t)=\lim_{s\to 0} sX(s)$	

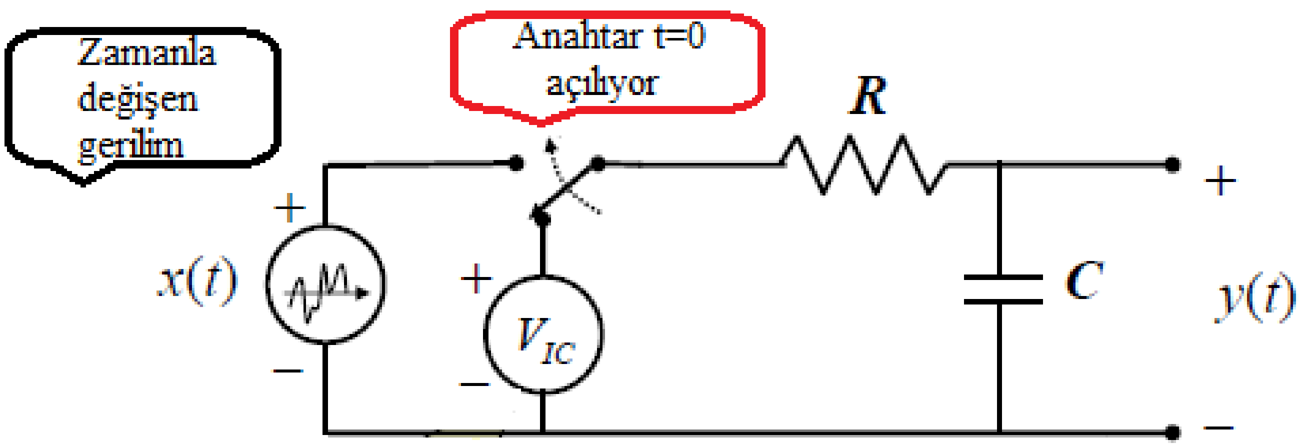
Fark Denklemlerinin LT ile Çözümü

$$y_{toplam}(t) = \underbrace{y_{zi}(t)} + \underbrace{y_{zs}(t)}$$

Karakteristik denklem ile çözüm (Homojen çözüm)
Sunu-2

Konvolüsyon / dürtü cevabı ile çözüm
Sunu-4

1.Dereceden fark denkleminin çözümü:



-1.Dereceden fark denklemine sahip basit bir RC devresi
- Nedensel bir giriş uygulansın t=0 anında
-Anahtar açılmadan önce kondansatör şarzlı olsun(sıfır olamayan başlangıç şartları)

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

IC $y(0^-) = V_{IC}$

$x(t) = 0, \ t < 0$

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

LT ile çözüm adımları:
-Denklemin her iki tarafında Laplace dönüşümü yapılır
-Uygun Laplace özellikleri kullanılır
-Y(s) için cebirsel denklemin sonucu hesaplanır
-Y(s) için ters Laplace alınarak *y(t)* hesaplanır.

Laplace Dönüşümü:
Fark denklemini
Cebirsel denkleme dönüştürdü

Zordur

Kolaydır

$$\mathcal{L}\left\{\frac{dy(t)}{dt}+ay(t)\right\}=\mathcal{L}\{bx(t)\}$$

Her iki tarafın LT 'si alınır

$$\mathcal{L}\left\{\frac{dy(t)}{dt}\right\}+a\mathcal{L}\{y(t)\}=b\mathcal{L}\{x(t)\}$$

LT 'nin doğrusallık özelliği

$$\left[sY(s)-y(0^-)\right]+aY(s)=bX(s)$$

LT 'nin türev özelliği

$$Y(s)=\boxed{\frac{y(0^-)}{s+a}}+\boxed{\frac{b}{s+a}X(s)}$$

Y(s) için cebirsel denklem

Başlangıç şartlarının etkisi

"Sıfır-giriş çözümü"

Giriş etkisi

"Sıfır-durum çözümü"

Girişe birim basamak uygulansın $x(t) = u(t)$ $a = \dfrac{1}{RC}$

$$\boxed{\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)} \Rightarrow \boxed{Y(s) = \frac{y(0^-)}{s+1/RC} + \underbrace{\left[\frac{1/RC}{s+1/RC}\right]}X(s)}$$

"Transfer Fonksiyonu"
(Tüm başlangıç şartları sıfır olduğunda)

Girişin LT 'sine ihtiyaç var...

LT tablosundan: $\boxed{x(t) = u(t) \quad \leftrightarrow \quad X(s) = \frac{1}{s}}$

$$\boxed{Y(s) = \frac{y(0^-)}{s+1/RC} + \left[\frac{1/RC}{(s+1/RC)}\right]\frac{1}{s}}$$

Ters Laplace Dönüşümü uygulanırsa:

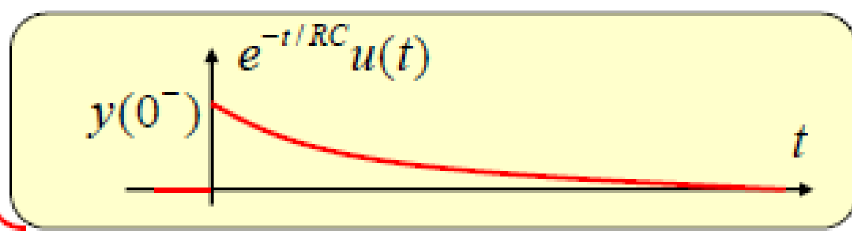
$$\mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{y(0^-)}{s+1/RC} + \left[\frac{1/RC}{(s+1/RC)s}\right]\right\}$$

Her iki tarafın ters LT 'si alınır

$$y(t) = \underbrace{\mathcal{L}^{-1}\left\{\frac{y(0^-)}{s+1/RC}\right\}}_{\text{"Tabloda var"}} + \underbrace{\mathcal{L}^{-1}\left\{\left[\frac{1/RC}{(s+1/RC)s}\right]\right\}}_{\text{"Tabloda yok"}}$$

Doğrusallık

$$\mathcal{L}^{-1}\left\{\frac{y(0^-)}{s+1/RC}\right\} = y(0^-)e^{-(t/RC)}u(t)$$



Girişin etkisini (sıfır-durum cevabını) bulalım:

$$\boxed{y(t) = \mathcal{L}^{-1}\left\{\frac{y(0^-)}{s+1/RC}\right\} + \mathcal{L}^{-1}\left\{\left[\frac{1/RC}{(s+1/RC)s}\right]\right\}}$$

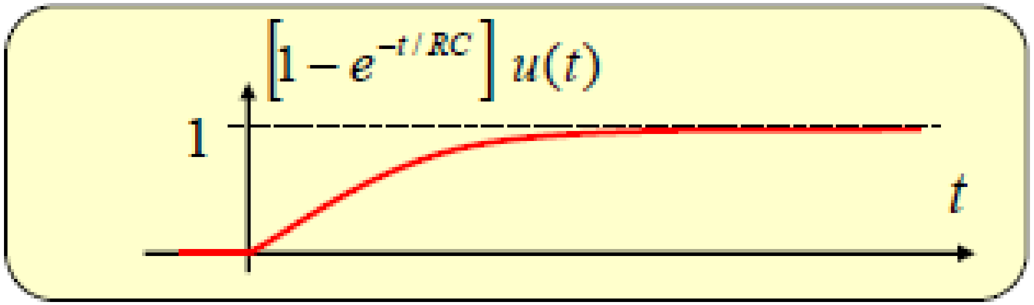
Çarpanlarına ayrılırsa;

$$\boxed{\mathcal{L}^{-1}\left\{\left[\frac{1/RC}{(s+1/RC)s}\right]\right\} = \mathcal{L}^{-1}\left\{\left[\frac{1}{s} - \frac{1}{s+1/RC}\right]\right\}}$$

$$\underbrace{\underbrace{= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}}_{=u(t)} - \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s+1/RC}\right\}}_{=e^{-(t/RC)}u(t)}}_{=\left[1-e^{-(t/RC)}\right]u(t)}$$

Doğrusallık özelliği

Sistemin sıfır-durum cevabı: $\left[1-e^{-(t/RC)}\right]u(t)$

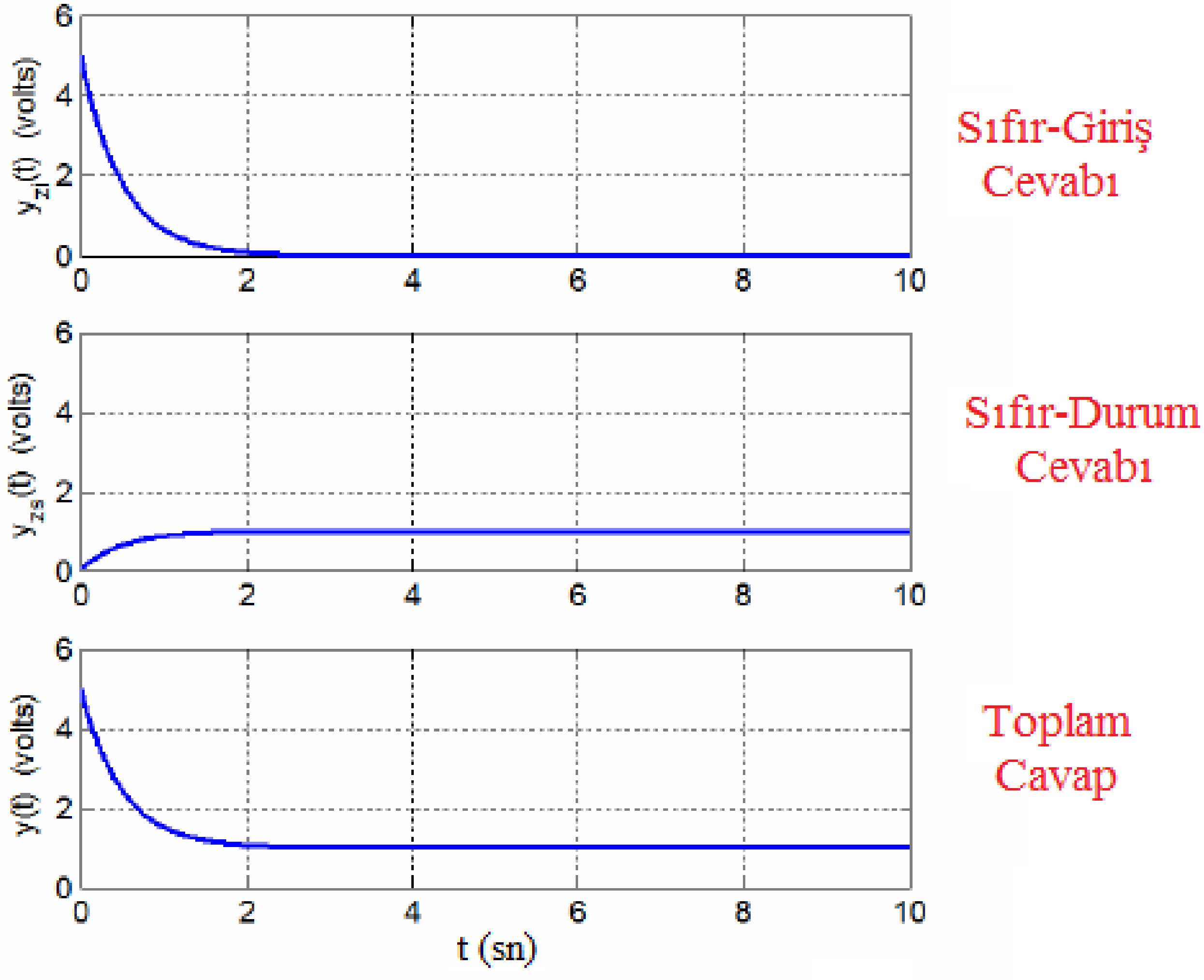


Toplam çözüm:

$$\boxed{y(t) = \underbrace{y(0^-)e^{-(t/RC)}u(t)}_{\text{IC Kısmı}} + \underbrace{\left[1-e^{-(t/RC)}\right]u(t)}_{\text{Giriş Kısmı}}}$$

Örnek RC devresi:

$RC = 0.5$ sn Başlangıç şartı: $V_{IC} = 5$ volt



N. Dereceden sistem:

$$\boxed{\frac{d^N y(t)}{dt^N} + a_{N-1}\frac{d^{N-1}y(t)}{dt^{N-1}} + \ldots + a_1\frac{dy(t)}{dt} + a_0 y(t) = b_M \frac{dx^M(t)}{dt^M} + b_1\frac{dx(t)}{dt} + b_0 x(t)}$$

$$M \le N \qquad \left.\frac{d^i x(t)}{dt^i}\right|_{t=0^-} = 0 \quad i = 0,1,2,\ldots,M-1$$

Genelleştirilirse:

$$\boxed{Y(s) = \frac{IC(s)}{A(s)} + \frac{B(s)}{A(s)}X(s)}$$

$$\begin{cases} A(s) = s^N + a_{N-1}s^{N-1} + \ldots + a_1 s + a_0 & \text{" çıkış polinomu"} \\ B(s) = b_M s^M + \ldots + b_1 s + b_0 & \text{"giriş polinomu"} \\ IC(s) = \text{Başlangıç şartlarına bağlı polinom} & \end{cases}$$

$$X(s) = \frac{N_X(s)}{D_X(s)}$$

$$Y(s) = \frac{IC(s)}{A(s)} + \frac{B(s)}{A(s)} X(s) = \frac{IC(s)}{A(s)} + \frac{B(s)}{A(s)} \frac{N_X(s)}{D_X(s)}$$

$$Y(s) = \frac{IC(s)}{A(s)} + \frac{E(s)}{A(s)} + \frac{F(s)}{D_X(s)}$$

$$H(s) = \frac{B(s)}{A(s)} \qquad X(s) = \frac{N_X(s)}{D_X(s)}$$

$$Y(s) = \overbrace{\boxed{\frac{IC(s)}{A(s)}}}^{\text{Sıfır-Giriş Cevabı}} + \overbrace{\boxed{\underbrace{\frac{E(s)}{A(s)}}_{\text{Geçici Cevap}} + \underbrace{\frac{F(s)}{D_X(s)}}_{\text{Kararlı-Hal Cevabı}}}}^{\text{Sıfır-Durum Cevabı}}$$

Eğer $\underline{IC(s) = 0}$

$$\boxed{Y(s) = \underbrace{\left[\frac{B(s)}{A(s)}\right]}_{\equiv H(s)} X(s)}$$

"Transfer Fonksiyonu"

$$Y(s) = \overbrace{\underbrace{\frac{E(s)}{A(s)}}_{\text{Geçici Cevap}} + \underbrace{\frac{F(s)}{D_X(s)}}_{\text{Karalı-Hal Cevabı}}}^{\text{Sıfır-Durum Cevabı}}$$

Örnek: $Y(s) = \dfrac{3s-1}{s^2+3s+2}$ $y(t) = ?$

$$Y(s) = \frac{3s-1}{(s+2)(s+1)}$$

$$\frac{A}{(s+2)} + \frac{B}{(s+1)}$$

$$Y(s) = \frac{3s-1}{s^2+3s+2} = \frac{7}{s+2} + \frac{-4}{s+1} \qquad \mathcal{L}^{-1}\left\{ Y(s) \right\}$$

$$y(t) = 7e^{-2t}u(t) - 4e^{-t}u(t)$$

Örnek-2:

$$Y(s)=\frac{2s^2+3s-1}{s^2+3s+2} \qquad y(t)=?$$

$$Y(s)=\frac{2s^2+3s-1}{s^2+3s+2}=\frac{\mathrm{A}}{s+2}+\frac{\mathrm{B}}{s+1}+\mathrm{C}$$

$$Y(s)=\frac{-1}{s+2}+\frac{-2}{s+1}+2 \qquad \mathcal{L}^{-1}\left\{Y(s)\right\}$$

$$y(t)=-e^{-2t}u(t)-2e^{-t}u(t)+2\delta(t)$$