Sinyaller ve Sistemler

Fourier Dönüşümleri

Periyodik bir sinyal trigonometrik yada kompleks trigonometrik Fourier Serileri ile temsil edilebilir.

$$x(t) = \sum_{k=-N}^{N} c_k e^{jk\omega_0 t} \qquad x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(k\omega_0 t + \theta_k)$$

Periyodik olmayan sinyaller için ne söylenebilir? $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_k t}$?

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega}$$

Fourier İntegrali

"Ters Fourier Dönüşümü"

c_k ların rolünü üstlenir

$e^{jk\omega_0 t}$

Integral toplam ile yerdeğiştirmiştir frekensaların sürekliliği toplanır

$$\boxed{X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt}$$ Fourier Dönüşümü

$\underbrace{X(\omega) \text{ kompleks değerli bir fonksiyon}}$ $\omega \in (-\infty, \infty)$

$|X(\omega)|$ $\angle X(\omega)$

Fourier Serileri: Periyodik sinyaller için

Fourier Dönüşümü : Periyodik olmayan sinyaller için

	Sentez	Analiz
Fourier Serileri	$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$ Fourier Serileri	$c_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt$ Fourier Katsayıları
Fourier Dönüşümü	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$ Ters Fourier Dönüşümü	$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ Fourier Dönüşümü

FS katsayıları c_k k değeri için kompleks değerli bir fonksiyon

FT $X(\omega)$ ω değişkeninin kompleks değerli bir fonksiyon

FS: $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$

$|c_k|$ $k\omega_0$ frekansındaki sinyalin genliğini gösterir

$\angle c_k$ $k\omega_0$ frekansındaki faz farkını gösterir

FT: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$

$|X(\omega)|$ ω frekansındaki sinyalin genliğini gösterir

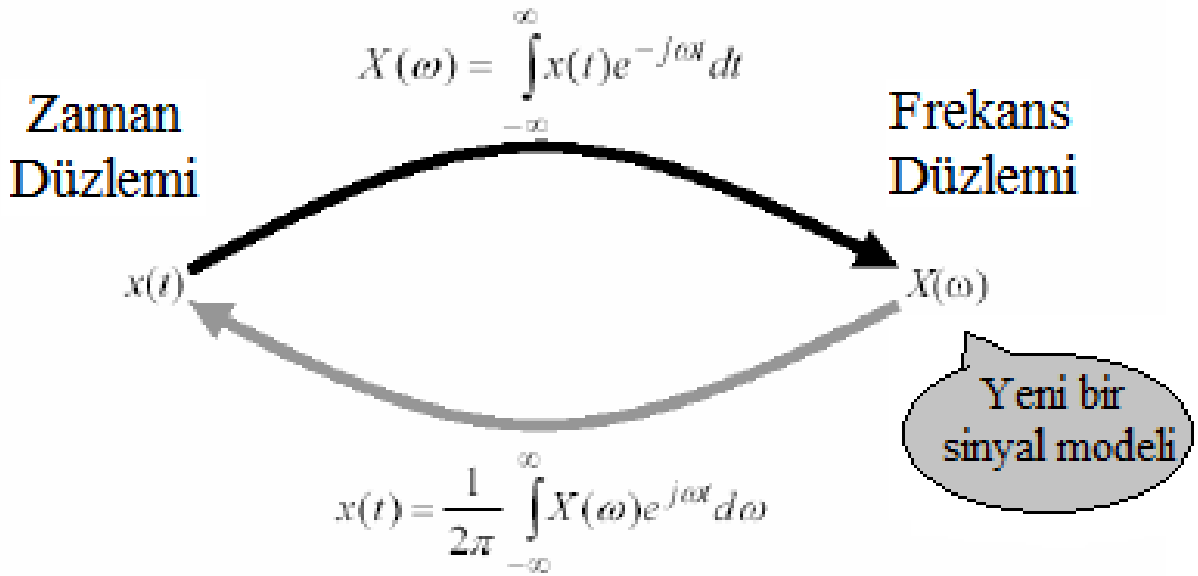
$\angle X(\omega)$ ω frekansındaki faz farkını gösterir

Fourier Dönüşüm Notasyonu

1. $x(t) \leftrightarrow X(\omega)$

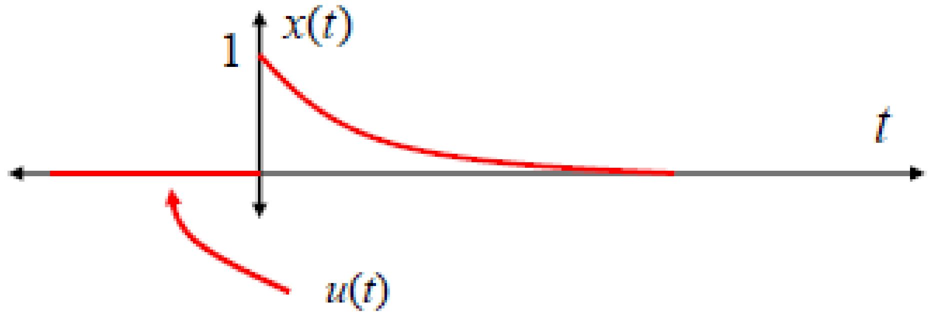
2. $X(\omega) = \mathfrak{F}\{x(t)\} \quad \Rightarrow \quad \mathfrak{F}\{\ \}$ bir operatördür $x(t) \Rightarrow X(\omega)$

3. $x(t) = \mathfrak{F}^{-1}\{X(\omega)\} \quad \Rightarrow \mathfrak{F}^{-1}\{\ \}$ bir operatördür $X(\omega) \Rightarrow x(t)$



Örnek

$x(t) = e^{-bt}u(t)$ $\quad b > 0$ için $\quad X(\omega)$ 'yi hesaplayın



$\mathcal{F}\{\, x(t) = e^{-bt}u(t) \,\} = ?$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$X(\omega) = \int_{-\infty}^{\infty} \underbrace{e^{-bt}u(t)e^{-j\omega t}}_{\text{integral} = 0 \quad t<0} dt = \int_{0}^{\infty} e^{-bt} e^{-j\omega t} dt = \int_{0}^{\infty} e^{-(b+j\omega)t} dt$$

$u(t)$

Örnek

$$=\left[\frac{-1}{b+j\omega}e^{-(b+j\omega)t}\right]_{t=0}^{t=\infty}=\frac{-1}{b+j\omega}\left[e^{-(b+j\omega)\infty}-e^{-(b+j\omega)0}\right]$$

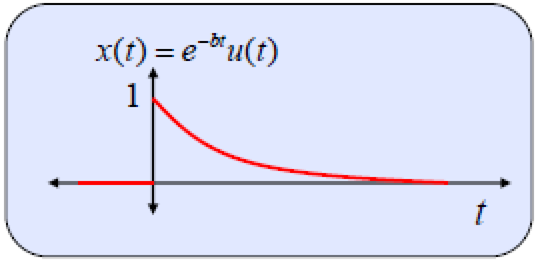
$$=\frac{-1}{b+j\omega}\left[\underbrace{e^{-b\infty}e^{-j\omega\infty}}_{=0}-\underbrace{e^{0}}_{=1}\right]=\frac{-1}{b+j\omega}[0-1]$$

$$=\frac{1}{b+j\omega}$$

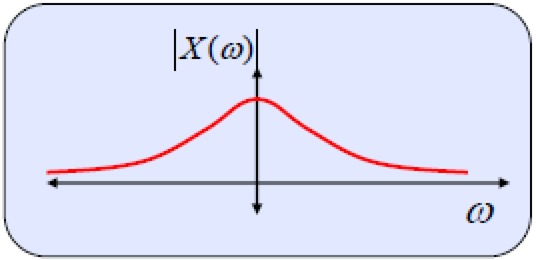
$$\boxed{x(t)=e^{-bt}u(t)} \longleftrightarrow \boxed{X(\omega)=\frac{1}{b+j\omega}}$$

$$\boxed{|X(\omega)|=\frac{1}{\sqrt{b^2+\omega^2}}}\ \text{Genlik}$$

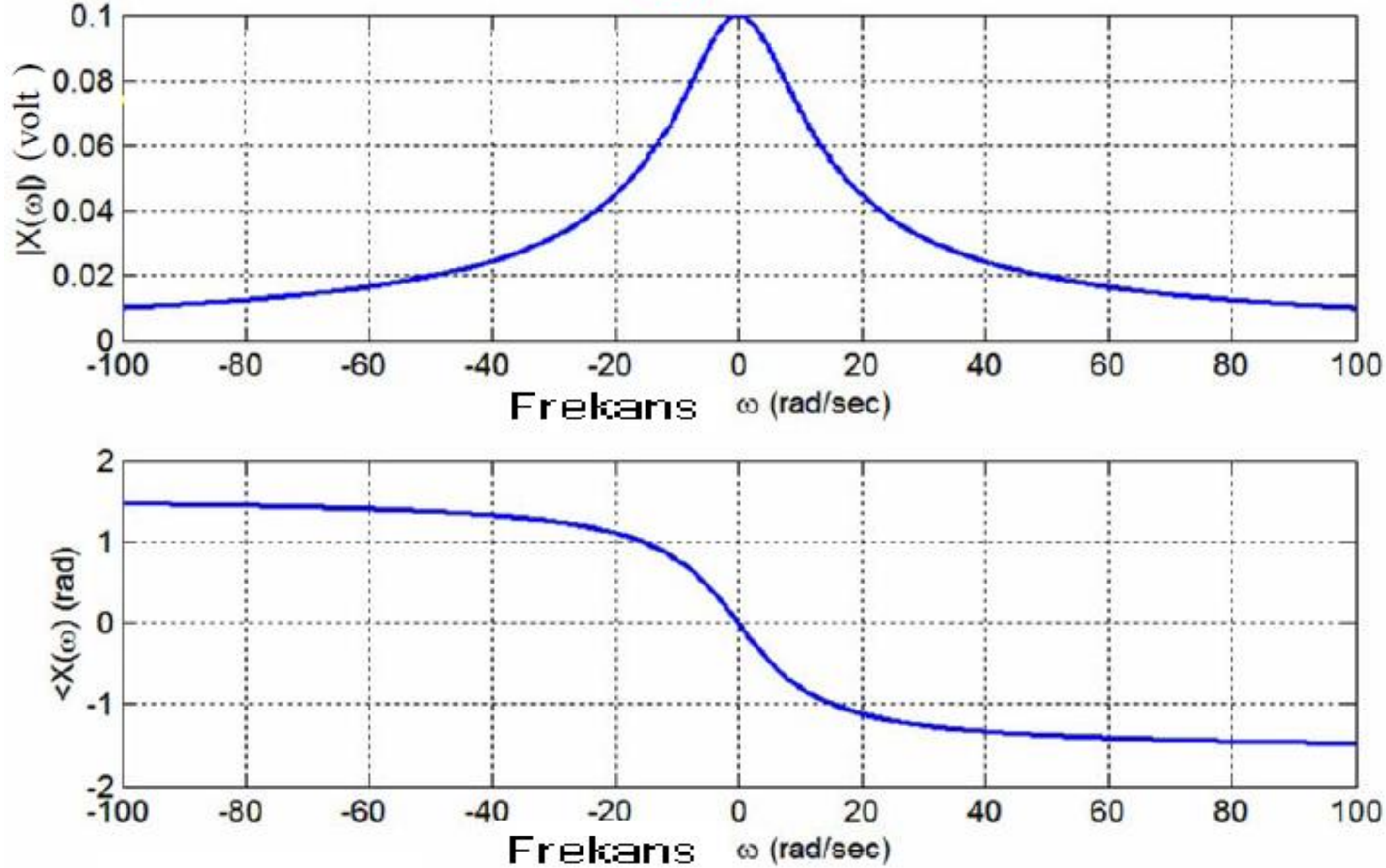
$$\boxed{\angle X(\omega)=-\tan^{-1}\left(\frac{\omega}{b}\right)}\ \text{Faz}$$



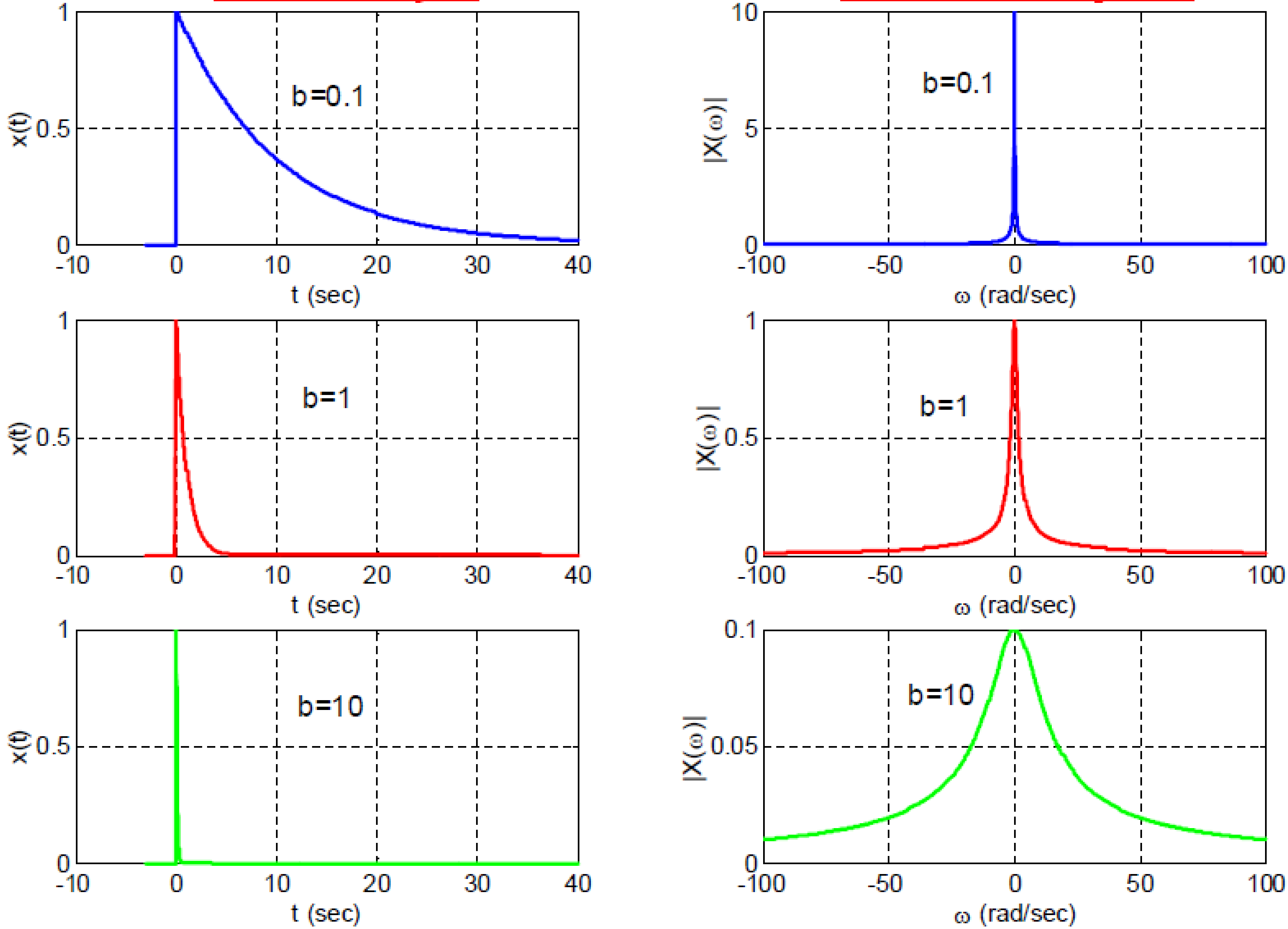
$\longleftrightarrow$



$e^{-bt}u(t) \qquad b = 10$

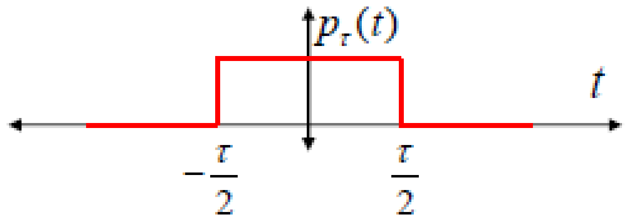


Zaman Sinyali   Fourier Dönüşümü



darbe genişliği= τ

rectangular darbe sinyali $p_\tau(t)$



$$p_\tau(t) = \begin{cases} 1, & -\frac{\tau}{2} \le t \le \frac{\tau}{2} \\ 0, & \text{aksi halde} \end{cases}$$

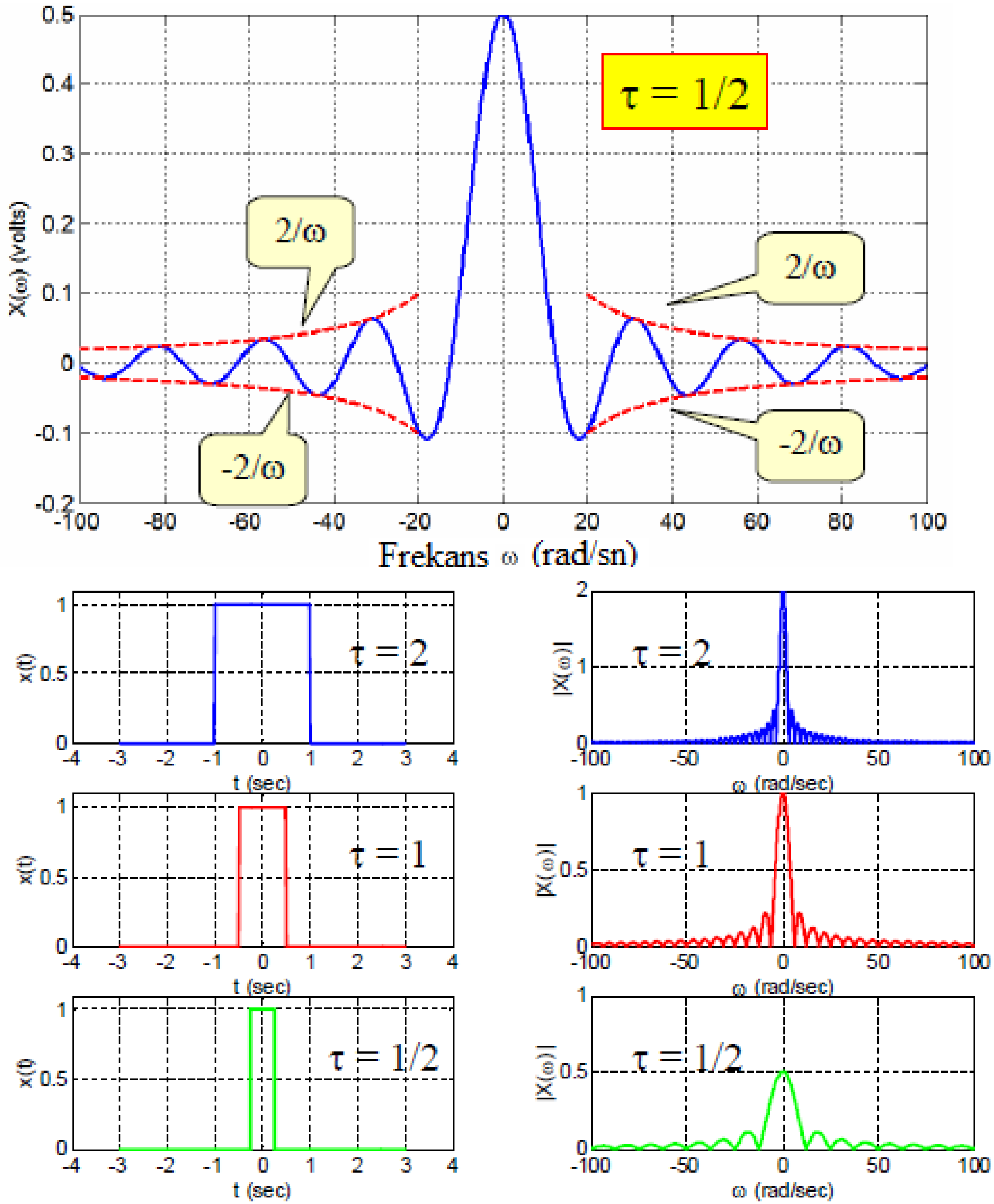
$\mathfrak{F}\{\, p_\tau(t) \,\} = ?$

$$P_\tau(\omega) = \int_{-\infty}^{\infty} p_\tau(t) e^{-j\omega t} dt = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt$$

$$= \frac{-1}{j\omega}\left[e^{-j\omega t}\right]_{\frac{-\tau}{2}}^{\frac{\tau}{2}} = \frac{2}{\omega}\underbrace{\left[\frac{e^{j\frac{\omega\tau}{2}} - e^{-j\frac{\omega\tau}{2}}}{j2}\right]}_{= \sin\left(\frac{\omega\tau}{2}\right)}$$

$\frac{2}{2}$ çarp

$$\boxed{P_\tau(\omega) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega}}$$



τ = 1/2
2/ω
2/ω
-2/ω
-2/ω
X(ω) (volts)
Frekans ω (rad/sn)
τ = 2
τ = 1
τ = 1/2
x(t)
t (sec)
|X(ω)|
ω (rad/sec)

Tanım: $\operatorname{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$

$$P_\tau(\omega) = \frac{2\sin\left(\dfrac{\omega\tau}{2}\right)}{\omega}$$

$$P_\tau(\omega) = \frac{2\sin\left(\dfrac{\omega\tau}{2}\right)}{\omega} = \frac{2\sin\left(\dfrac{\pi}{\pi}\dfrac{\omega\tau}{2}\right)}{\omega} = \frac{2\sin\left(\pi\dfrac{\omega\tau}{2\pi}\right)}{\omega}$$

$$= \pi\frac{\tau}{2\pi}\,\frac{2\sin\left(\pi\dfrac{\omega\tau}{2\pi}\right)}{\pi\dfrac{\tau}{2\pi}\omega} = \tau\,\frac{\sin\left(\pi\dfrac{\omega\tau}{2\pi}\right)}{\pi\dfrac{\omega\tau}{2\pi}} = \tau\operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

$\mathfrak{F}\{\,p_\tau(t)\,\} = ?$

$$P_\tau(\omega) = \tau\operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

Fourier Transform Table

Time Signal	Fourier Transform
$1, \quad -\infty < t < \infty$	$2\pi\delta(\omega)$
$-0.5 + u(t)$	$1/j\omega$
$u(t)$	$\pi\delta(\omega) + 1/j\omega$
$\delta(t)$	$1, \quad -\infty < \omega < \infty$
$\delta(t-c), \quad c$ real	$e^{-j\omega c}, \quad c$ real
$e^{-bt}u(t), \quad b>0$	$\dfrac{1}{j\omega + b}, \quad b>0$
$e^{j\omega_o t}, \quad \omega_o$ real	$2\pi\delta(\omega - \omega_o), \quad \omega_o$ real
$p_\tau(t)$	$\tau\,\mathrm{sinc}[\tau\omega/2\pi]$
$\tau\,\mathrm{sinc}[\tau t/2\pi]$	$2\pi p_\tau(\omega)$
$\left[1 - \frac{2\lvert t\rvert}{\tau}\right] p_\tau(t)$	$\frac{\tau}{2}\mathrm{sinc}^2[\tau\omega/4\pi]$
$\frac{\tau}{2}\mathrm{sinc}^2[\tau t/4\pi]$	$2\pi\left[1 - \frac{2\lvert\omega\rvert}{\tau}\right] p_\tau(\omega)$
$\cos(\omega_o t)$	$\pi[\delta(\omega + \omega_o) + \delta(\omega - \omega_o)]$
$\cos(\omega_o t + \theta)$	$\pi\left[e^{-j\theta}\delta(\omega + \omega_o) + e^{j\theta}\delta(\omega - \omega_o)\right]$
$\sin(\omega_o t)$	$j\pi[\delta(\omega + \omega_o) - \delta(\omega - \omega_o)]$
$\sin(\omega_o t + \theta)$	$j\pi\left[e^{-j\theta}\delta(\omega + \omega_o) - e^{j\theta}\delta(\omega - \omega_o)\right]$

Fourier Dönüşümünün Özellikleri

1- Doğrusallık (Linearity):

$$x(t) \leftrightarrow X(\omega) \quad \& \quad y(t) \leftrightarrow Y(\omega)$$

$$[ax(t) + by(t)] \leftrightarrow [aX(\omega) + bY(\omega)]$$

$$\mathfrak{F}\{ax(t) + by(t)\} = a\mathfrak{F}\{x(t)\} + b\mathfrak{F}\{y(t)\}$$

$$\mathfrak{F}\{ax(t) + by(t)\} = \int_{-\infty}^{\infty} [ax(t) + by(t)] e^{-j\omega t} dt$$

$$= a\underbrace{\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt}_{= X(\omega)} + b\underbrace{\int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt}_{= Y(\omega)}$$

Fourier Dönüşümünün Özellikleri

2- Zaman Ötelemesi (Time shift):

$$x(t) \leftrightarrow X(\omega) \qquad\qquad x(t-c) \leftrightarrow X(\omega)e^{-jc\omega}$$

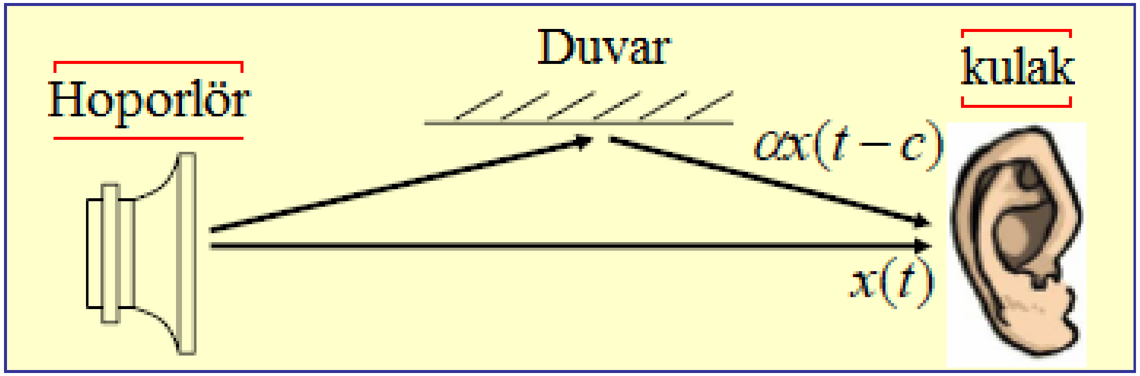
Sinyalin gecikme hali

Genlik aynıdır FT: $\left|X(\omega)e^{-j\omega c}\right| = \left|X(\omega)\right|$

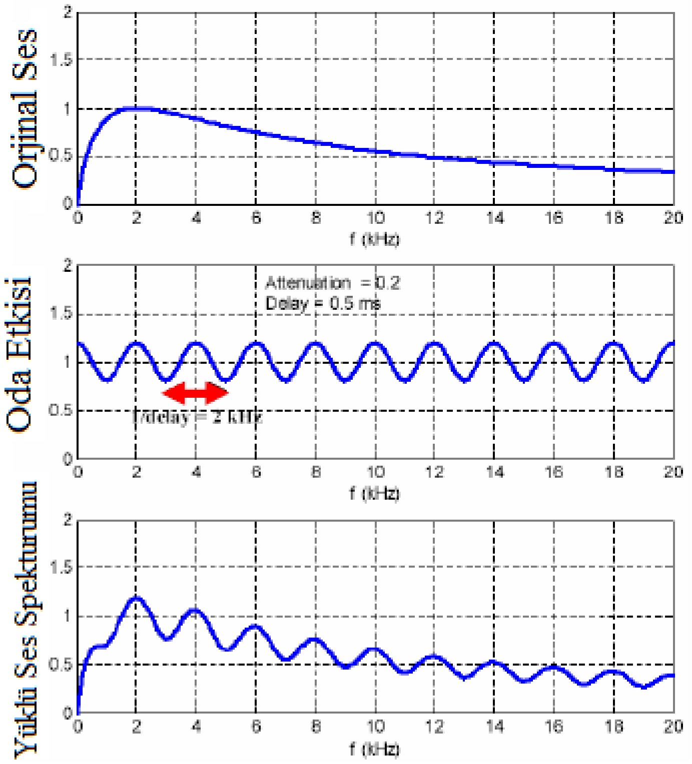
Faz değişir FT: $\angle\{X(\omega)e^{-jc\omega}\} = \angle X(\omega) + \angle e^{-jc\omega}$

$$= \angle X(\omega) + \underbrace{c\omega}_{\text{orjinal faza eklenir}}$$

Oda Akustiği



$x(t)$ yerine $\quad y(t) = x(t) + \alpha x(t-c) \quad$ duyuyoruz



$$Y(\omega) = \mathfrak{F}\{x(t) + \alpha x(t-c)\} = \mathfrak{F}\{x(t)\} + \alpha \mathfrak{F}\{x(t-c)\}$$

$$= X(\omega) + \alpha X(\omega) e^{-j\omega c}$$

$$\boxed{Y(\omega) = X(\omega)\left[1 + \alpha e^{-j\omega c}\right]}$$

$$\left|1 + \alpha e^{-jc\omega}\right| = \left|1 + \alpha\cos(c\omega) - j\alpha\sin(c\omega)\right|$$

Oda etkisi

Fourier Dönüşümünün Özellikleri

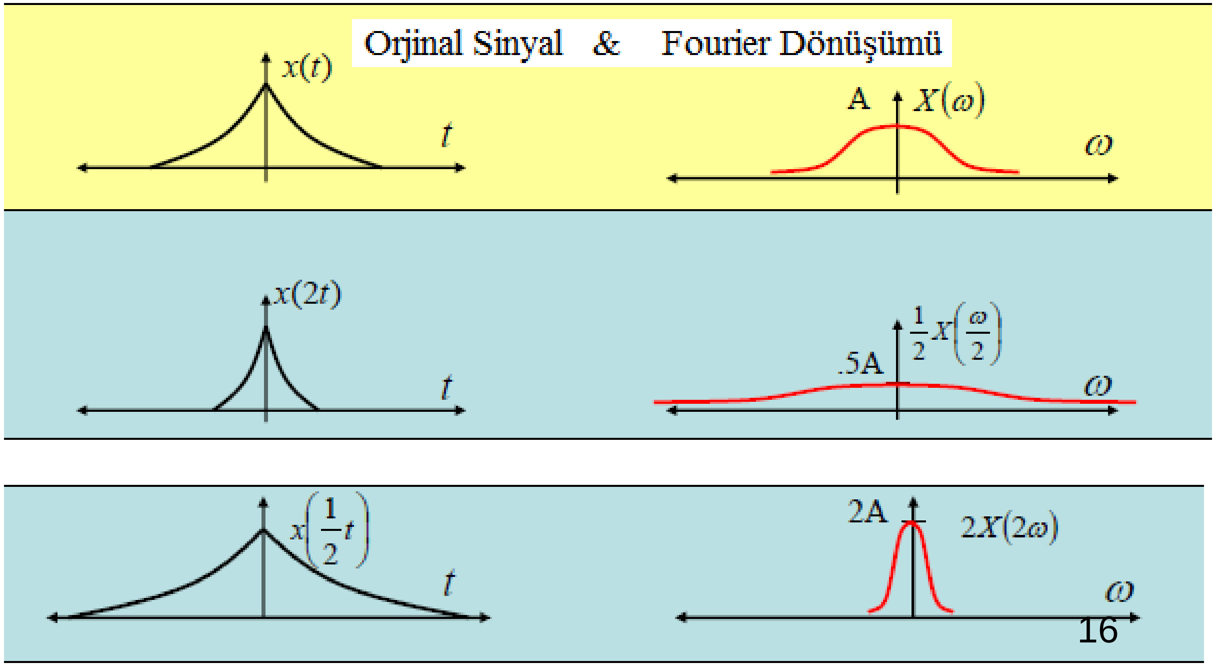
3- Zaman Ölçeklendirmesi (Time scaling):

$x(t) \leftrightarrow X(\omega)$, $x(at) \leftrightarrow ???$ $a \neq 0$

$$\boxed{x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)}$$

a kadar zaman ötelemesi olursa

Frekans Skalasında $\frac{1}{a}$ değişim olur



Fourier Dönüşümünün Özellikleri

4- Zaman Terslemesi (Time reversal):

$$\boxed{x(-t) \leftrightarrow X(-\omega)}$$

$$X(-\omega) = \int_{-\infty}^{\infty} x(t)e^{-j(-\omega)t}dt \quad = \int_{-\infty}^{\infty} x(t)e^{+j\omega t}dt$$

$$\boxed{x(-t) \leftrightarrow \overline{X(\omega)}}$$

Genlik aynıdır

$$\left|\overline{X(\omega)}\right| = \left|X(\omega)\right|$$

Fazın işareti değişir

$$\angle\overline{X(\omega)} = -\angle X(\omega)$$

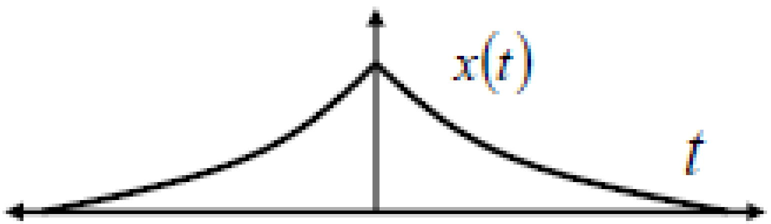
Fourier Dönüşümünün Özellikleri

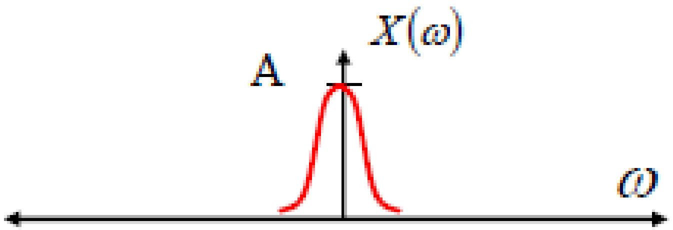
5- Modülasyon Özelliği (Modulation):

$$\boxed{x(t)e^{j\omega_0 t} \leftrightarrow X(\omega-\omega_0)}$$

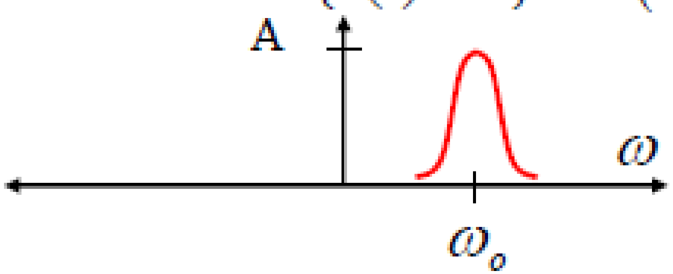
Sinyali kompleks bir sinüsoidal ile çarpmak

FT frekansında ötelenmeye neden olur





$$\mathfrak{F}\{x(t)e^{j\omega_o t}\} = X(\omega-\omega_o)$$

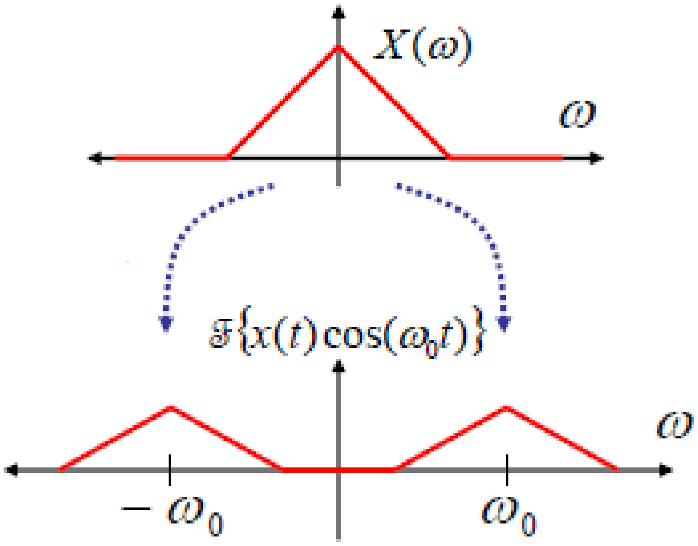


$$\mathfrak{F}\{x(t)\cos(\omega_0 t)\} = \mathfrak{F}\left\{\frac{1}{2}\left[x(t)e^{j\omega_0 t} + x(t)e^{-j\omega_0 t}\right]\right\}$$

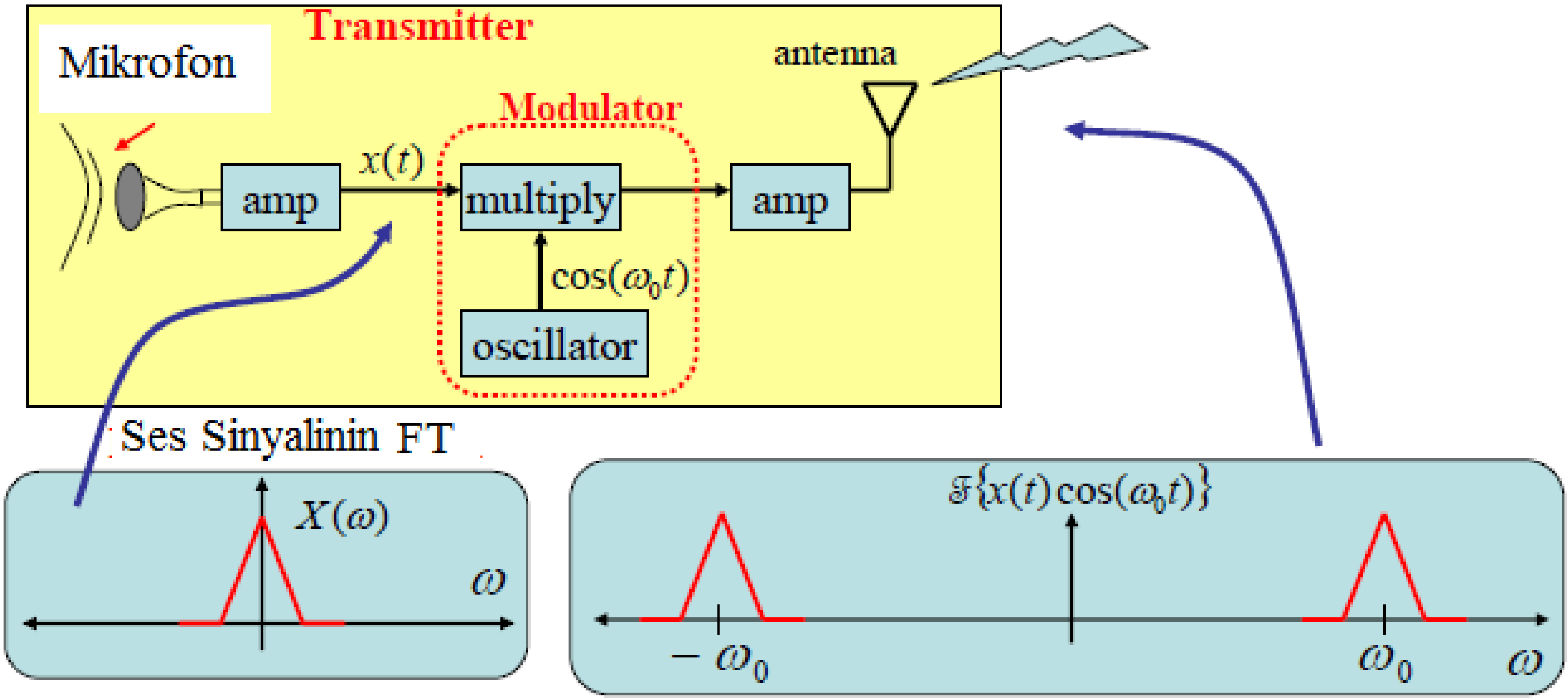
$$= \frac{1}{2}\left[\mathfrak{F}\left\{\left[x(t)e^{j\omega_0 t}\right]\right\} + \mathfrak{F}\left\{\left[x(t)e^{-j\omega_0 t}\right]\right\}\right]$$

$$= \frac{1}{2}\left[X(\omega - \omega_o) + X(\omega + \omega_o)\right]$$

$$\boxed{x(t)\cos(\omega_0 t) \quad \leftrightarrow \quad \frac{1}{2}\left[X(\omega + \omega_0) + X(\omega - \omega_0)\right]}$$



Basit Bir Radyo Haberleşmesi



FM radyo sinyalleri: ortalama 100Mhz
GSM: 900Mhz şimdi 1800Mhz

Not:10 Khz den büyük sinyallerde radyasyon etkisi başlar

Fourier Dönüşümünün Özellikleri

5- Konvolüsyon Özelliği:

$$x(t) * h(t) \quad \leftrightarrow \quad X(\omega)H(\omega)$$

$$\mathfrak{F}\{x(t) * h(t)\} = X(\omega)H(\omega)$$

LTI sistemde: