Sinusoidal Signals

The sinusoidal signal is of major importance since if we know the response of a system to sinusoids of all frequencies, we have a complete description of the system. The continuous-time version of this signal is written as

$$x(t) = A\cos(\omega_0 t + \theta)$$

where A is the amplitude, ω_0 is the frequency in radians per second, and θ is the phase angle in radians. The period of the signal is $T = \frac{2\pi}{\omega_0}$. The manipulation and analysis of sinusoidal signals is simplified by dealing with their amplitude and phase angles or their phasor representation.

From Euler's identity, the complex exponential function is expressed as

$$e^{j\beta} = \cos\beta + j\sin\beta \quad \text{or}$$

$$\cos\beta = \Re e^{j\beta} \quad \text{and} \quad \sin\beta = \Im e^{j\beta}$$

Thus the sinusoidal signal $x(t)$ may be written as

$$x(t) = \Re\{Ae^{j(\omega_0 t+\theta)}\} = \Re\{Ae^{j\theta}e^{j\omega_0 t}\} = \Re\{\vec{X}e^{j\omega_0 t}\}$$

Note that $Ae^{j\theta}$ indicates the magnitude and phase angle of the sinusoidal signal $x(t)$. This complex number is called *phasor* and is shown by $\vec{X} = Ae^{j\theta}$, or $\vec{X} = A\angle\theta$. We say that we transform a sinusoidal signal from time domain into phasor domain, and we write

$$\begin{aligned}\mathcal{P}\{A\cos(\omega_0 t+\theta)\} &= Ae^{j\theta} \\ &= A\angle\theta\end{aligned}$$

We define the complex signal as

$$\tilde{x}(t) = Ae^{j(\omega_0 t+\theta)} = \vec{X}e^{j\omega_0 t}$$

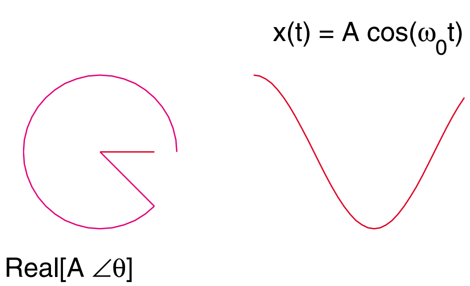
$\vec{X}$ is a rotating vector or a phasor whose real part generates the cosine signal $x(t)$. Rewriting the complex signal $\tilde{x}(t)$ and its conjugate as

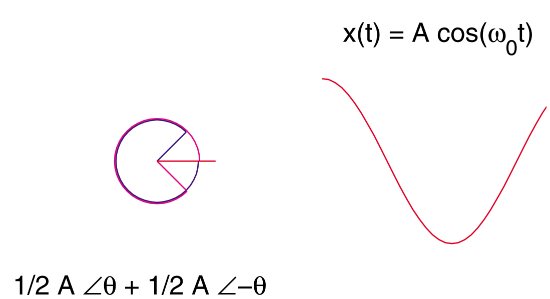
$$\begin{aligned}\tilde{x}(t) &= Ae^{j(\omega_0 t+\theta)} = A[\cos(\omega_0 t+\theta) + j\sin(\omega_0 t+\theta)] \\ \tilde{x}^*(t) &= Ae^{-j(\omega_0 t+\theta)} = A[\cos(\omega_0 t+\theta) - j\sin(\omega_0 t+\theta)]\end{aligned}$$

We conclude that the cosine signal $x(t)$ can be expressed as one-half of the sum of the complex signal $\tilde{x}(t)$ with positive frequency ω_0 and its conjugate $\tilde{x}^*(t)$, with negative frequency $-\omega_0$, i.e.,

$$x(t) = \frac{1}{2}\left[(\tilde{x}(t) + \tilde{x}^*(t)\right]$$

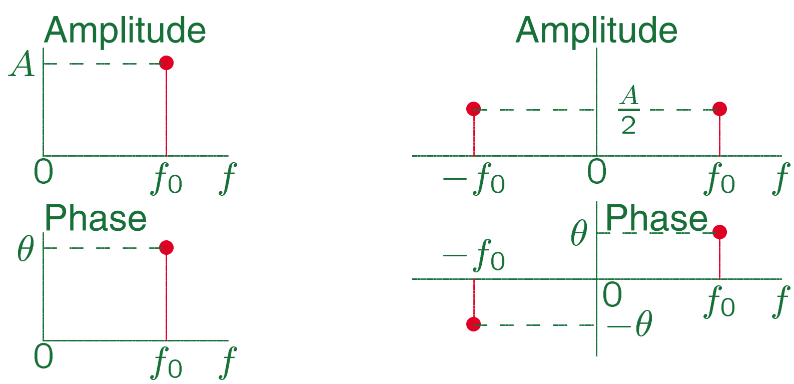
Therefore, the oppositely rotating phasors $\frac{1}{2}\vec{X} = \frac{1}{2}A\angle\theta$ and $\frac{1}{2}\vec{X}^* = \frac{1}{2}A\angle{-\theta}$ will generate the cosine signal $x(t)$. The script file **rotphasor** illustrates how the real part of a rotating phasor generates a cosine signal, and the file **rot2phasor** demonstrates how the sum of two half-amplitude oppositely rotating phasors generates the cosine signal. To see a demonstration type rotphasor, and rot2phasor at the MATLAB prompt.





Frequency Domain Spectra

An alternative way to visualize the sinusoidal signal $x(t)$ in the frequency domain is in the form of two plots. One the amplitude A as the function of frequency f, and the other its phase angle θ as a function of f. These plots are referred to as *single-sided spectrum*. If the amplitude and phase angle plots are made for the oppositely rotating phasors we obtain the so called double-sided spectra as shown.



(a) Single-sided spectrum (b) Double-sided spectra

Note that if a signal is represented as a sine function, before finding the signal spectra it must be expressed in terms of a cosine function, $\sin(\omega_0 t + \theta) = \cos(\omega_0 t + \theta - \frac{\pi}{2})$.

Example 1.12

For the signal

$$x(t) = 6\cos(10\pi t + \frac{\pi}{6}) + 4\sin(18\pi t + \frac{\pi}{6})$$

(a) Find the period and the fundamental frequency of the signal.

(b) Write the signal as the real part of the sum of the rotating phasors.

(c) Write $x(t)$ as the sum of counter rotating phasors.

(d) plot the single-sided amplitude and phase spectra.

(e) plot the 2-sided amplitude and phase spectra.

(a) Converting the second term to a cosine function, we have

$$x(t) = 6\cos(10\pi t + \frac{\pi}{6}) + 4\cos(18\pi t - \frac{\pi}{3})$$

We have $10\pi = 2\pi m f_0$ and $18\pi = 2\pi n f_0$, where, m, and n are integers and f_0 is the largest constant that satisfies these equations. Therefore,

$$f_0 = \frac{10\pi}{2\pi m} = \frac{18\pi}{2\pi n} \qquad \Rightarrow \quad \frac{5}{m} = \frac{9}{n}$$

or

$$n = 1.8m$$

for $m = 5 \quad n = 9$, therefore

$$f_0 = \frac{10}{(2)(5)} = 1, \quad \text{and} \quad T_0 = 1.0\text{s}$$

(b) $x(t)$ in terms of the real part of the rotating phasors is

$$x(t) = \Re\left[6e^{j(10\pi t + \frac{\pi}{6})}\right] + \Re\left[4e^{j(18\pi t - \frac{\pi}{3})}\right]$$

(c) $x(t)$ in terms of the of counter rotating phasors is

$$\begin{aligned} x(t) &= \left[3e^{j(10\pi t + \frac{\pi}{6})}\right] + \left[3e^{-j(10\pi t + \frac{\pi}{6})}\right] \\ &+ \left[2e^{j(18\pi t - \frac{\pi}{3})}\right] + \left[2e^{-j(18\pi t - \frac{\pi}{3})}\right] \end{aligned}$$

(d)-(e) The single-sided amplitude spectra are 6 at $f_0 = 5$ Hz, and 4 at $f_0 = 9$ and the corresponding phase angles are $\frac{\pi}{6}$, and $-\frac{\pi}{3}$. The double-sided amplitude spectra is given by one-half the above amplitudes their mirror image. The phase spectra is obtained by taking antisymmetric mirror image.

The following MATLAB commands are used to plot the single-sided and double-sided spectra.

```
%chs1_ex12.m
f = [5, 9];                          % Fundamental frequencies
A =[6, 4];                          % Corresponding amplitudes
theta = [pi/6, -pi/3];                  % corresponding angles
subplot(2,2,1),stem(f, A, 'r')  % Single-sided amplitude
subplot(2,2,2),stem(f,theta,'m')      % and angle spectra
f = [-f f];                           %-ve and +ve frequencies
AD = [A/2 A/2];            % 1/2 A mirror image, and 1/2 A
thetaD = [-theta theta]; %antisymmetric angle, and angle
subplot(2,2,3), stem(f, AD,'r') % double-sided amplitude
subplot(2,2, 4), stem(f, thetaD, 'm')% and angle spectra
```

The result is