Assignment 1 Solutions

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1 ID

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2 Decimal Strings

A *dec-string* is a sequence of characters from the 10-character alphabet $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. For example, these are dec-strings:

```
0
36562342320
49548362729
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Q2.1: What is the number of dec-strings of length n?

Let U_n be the set of dec-strings $d_1, \ldots, d_n$ of length n. We can determine $|U_n|$ by the most straightforward application of the Product Rule: For each $i \in \{1, \ldots, n\}$ there are 10 choices for d_i, so there are 10^n dec-strings of length n.

Q2.2: What is the number of dec-strings $d_1, \ldots, d_n$ of length n such that $d_1 d_2 \neq 00$. In other words, what is the number of dec-strings of length n that don't begin with 00?

Let S_n be the set of dec-strings $d_1, \ldots, d_n$ that don't begin with $d_1 d_2 = 00$.

Any dec-string of length $n < 2$ does not begin with 00, so $|S_0| = 1$ and $|S_1| = 10$.

For $n = 2$, each of the $10^2 = 100$ dec-strings of length 2 is valid except for 00, so $|S_2| = 99$. (This is an application of the Complement Rule.)

For $n \geq 3$ we can use the Product Rule with a 2-step procedure to generate an element of S_n:

Step 1. Choose the values of $d_1 d_2$. There are 99 ways to do this.

Step 2. Choose $d_3, \ldots, d_n$, a dec-string of length $n - 2$. There are 10^{n-2} ways to do this.

Therefore $|S_n| = 99 \cdot 10^{n-2}$.

Q2.3: What is the number of dec-strings $d_1, \ldots, d_n$ of length n such that $d_1 d_2 \neq 00$ *and* $d_2 d_3 \neq 11$?

Let T_n be the set of dec-strings $d_1, \ldots, d_n$ of length n that don't have $b_1b_2 = 00$ and don't have $b_2b_3 = 11$. For $n \in \{0,1,2\}$, this problem is easy, $|T_n| = 10^n$ for $n \in \{0,1,2\}$.

For $n \geq 3$ we will use the Complement Rule. We will start with the set U_n of all dec-string of length n and remove those that start with $b_1b_2 = 00$ and remove those that have $b_2b_3 = 11$.

First, consider the case $n = 3$. Let A be the set of dec-strings $b_1b_2b_3$ of length 3 that begin with $b_1b_2 = 00$ and let B be the set of length-3 dec-strings that have $d_2d_3 = 11$. Notice that A and B are disjoint (all elements in A have $b_2 = 0$ and all elements in B have $b_2 = 1$), so by the Sum Rule:

$$|A \cup B| = |A| + |B| \ .$$

It's easy to see that $|A| = 10$ and $|B| = 10$ since each element in A has 10 options for b_3 (and $b_1b_2 = 00$) while each element in B has 10 options for b_1 (and $b_2b_3 = 11$). So

$$|A \cup B| = |A| + |B| = 10 + 10 = 20 \ .$$

Finally, let U_3 be the set of *all* dec-strings of length 3. Then

$$|T_3| = |U_3 \setminus (A \cup B)| = |U| - |A \cup B| = 10^3 - 20 \ = 980.$$

As in the previous question, for $n > 3$ we can generate any element of T_n by a two step procedure by first choosing $b_1b_2b_3$ from T_3 and then choosing $d_4, \ldots, d_n$ from U_{n-3}. Therefore, for $n \geq 3$,

$$|T_n| = |T_3| \cdot |U_{n-3}| = 980 \cdot 10^{n-3} \ .$$

Q2.4: What is the number of dec-strings $d_1, \ldots, d_n$ of length n such that $d_1d_2 \neq 00$ *and* $d_2d_3 \neq 01$?

Let S_n be the set of dec-strings $d_1, \ldots, d_n$ of length n such that $d_1d_2 \neq 00$ *and* $d_2d_3 \neq 01$? This looks a lot like the previous question, and we will tackle it the same way. The cases $n = \{0,1,2\}$ are boring.

For $n = 3$, define A as before, so $|A| = 10$. Define C as the set of length-3 dec-strings $b_1b_2b_3$ with $b_2b_3 = 01$. Again $|C| = 10$. The difference now is that A and C are not disjoint: $A \cap C = \{001\}$, so $|A \cap C| = 1$. Therefore, by the Principle of Inclusion-Exclusion

$$|A \cup C| = |A| + |B| - |A \cap C| = 10 + 10 - 1 = 19 \ .$$

Therefore

$$|S_3| = |U_3 \setminus (A \cup C)| = |U_3| - |A \cup C| = 10^3 - 19 = 981 \ .$$

For $n > 3$ we apply the Product Rule with the same 2 step procedure, except now there are 981 ways to execute Step 1, so $|S_n| = 981 \cdot 10^{n-3}$.

Q2.5: What is the number of dec-strings $d_1, \ldots, d_n$ of length n such that $d_1d_2 = 00$ *or* $d_1d_2d_3 = 111$?

Let X_n be the set of dec-strings of length n that begin with 00. For $n \geq 2$, each such string consists of 00 followed by an element of U_{n-2}, so $|X_n| = |U_{n-2}| = 10^{n-2}$.

Let Y_n be the set of dec-strings of length n that begin with 111. For $n \geq 3$, each such string consists of 111 followed by an element of U_{n-3}, so $|Y_n| = |U_{n-3}| = 10^{n-3}$.

Now, the question asks about the size of $X_n \cup Y_n$ and X_n and Y_n are disjoint so, by the Sum Rule,

$$|X_n \cup Y_n| = |X_n| + |Y_n| = 10^{n-2} + 10^{n-3} \ ,$$

for any $n \geq 3$. (For values of $n < 3$, the question is vague, so I'm willing to accept any answer.)

Q2.6: What is the number of dec-strings $d_1, \ldots, d_n$ of length $n \geq 4$ such that $d_1d_2 \neq 00$ *or* $d_3d_4 \neq 11$?

Let P_n be the set of dec-strings $d_1, \ldots, d_n$ of length n such that $d_1d_2 \neq 00$. We've already studied these in Part 2, and we know that $|P_n| = |S_n| = 99 \cdot 10^{n-2}$.

Let Q_n be the set of dec-strings $d_1, \ldots, d_n$ of length $n \geq 4$ such that $d_3d_4 \neq 11$. Again, we can determine $|Q_n|$ easily using the Product Rule: There are 99 choices for d_3d_4 and 10^{n-2} choices for $d_1, d_2, d_5, d_6, \ldots, d_n$, so $|Q_n| = 99 \cdot 10^{n-2}$.

The question asks about $|P_n \cup Q_n|$. We can't use the Sum Rule because P_n and Q_n are *not* disjoint: $P_n \cap Q_n$ contains any string that has $d_1d_2 \neq 00$ and has $d_3d_4 \neq 11$. But it's easy to compute $|P_n \cap Q_n|$ using the Product Rule with the following procedure:

Step 1. Choose b_1b_2. There are 99 ways to do this (only 00 is not allowed).

Step 2. Choose b_3b_4. There are 99 ways to do this (only 11 is not allowed).

Step 3. Choose $b_5, \ldots, b_n$. There are 10^{n-4} ways to do this.

Thererefore $|P_n \cap Q_n| = 99^2 \cdot 10^{n-4}$

Therefore, by the Principle of Inclusion-Exclusion

$$
\begin{aligned}
|P_n \cup Q_n| &= |P_n| + |Q_n| + |P_n \cap Q_n| \\
&= 99 \cdot 10^{n-2} + 99 \cdot 10^{n-2} - 99^2 \cdot 10^{n-4} \\
&= 198 \cdot 10^{n-2} - 9801 \cdot 10^{n-4} \\
&= 19800 \cdot 10^{n-4} - 9801 \cdot 10^{n-4} \\
&= 9999 \cdot 10^{n-4} \quad .
\end{aligned}
$$

Q2.7: A dec-string $d_1, \ldots, d_n$ is *bad* if $d_i = d_{i+1}$ or $d_i + d_{i+1} = 9$ for at least one $i \in \{1, \ldots, n-1\}$ and it is *good* otherwise. What is the number of good dec-strings of length n?

Let G_n be the set of good dec-strings of length n. We use the Product Rule with an n-step procedure.

(a) In Step 1, we choose d_1 and there are 10 ways to do this.

(b) In Step $i+1$, for $i \in \{1, \ldots, n-1\}$, we choose d_{i+1} such that $d_{i+1} \neq d_i$ and $d_{i+1} \neq 9 - d_i$. Notice that $9 - d_i \neq d_i$ since 9 is an odd number.[1] Therefore, there are $10 - 2 = 8$ ways to perform Step $i+1$ for each $i \in \{1, \ldots, n-1\}$.

Therefore, by the Product Rule,

$$|G_n| = 10 \cdot 8^{n-1} \quad ,$$

for $n \geq 1$.

Q2.8: A dec-string $d_1, \ldots, d_n$ is *2-bad* if, $d_i = d_j$ or $d_i + d_j = 9$ for some $i < j \leq i+2$ and it is *2-good* otherwise. What is the number of 2-good dec-strings?

Let H_n be the set of 2-good dec-strings of length n. We use the Product Rule with an n-step procedure:

(a) In Step 1 we choose d_1. There are 10 ways to do this.

(b) In Step 2 we choose d_2 so that $d_2 \neq d_1$ and $d_2 \neq 9 - d_1$. As discussed above, there are 8 ways to do this.

(c) In Step j, for $j \geq 3$, we choose d_j so that $d_j \notin X_2$, where $X_2 = \{d_{j-1}, 9 - d_{j-1}, d_{j-2}, 9 - d_{j-2}\}$.

[1] If $9 - d_i = d_i$ then $9 = 2d_i$, which is impossible since 9 is odd and $2d_i$ is even.

Notice that $|X_2| = 4$. (Do you see why?) Therefore, there are $10 - 4 = 6$ ways to perform Step j for each $j \in \{3, \dots, n\}$.

Therefore, by the Product Rule,

$$|H_n| = 10 \cdot 8 \cdot 6^{n-2} \ ,$$

for $n \geq 2$.

For small values of n we have $|H_0| = 1$ and $|H_1| = 10$.

Q2.9: A dec-string $d_1, \dots, d_n$ is k-*bad* if $d_i = d_j$ or $d_i + d_j = 9$ for some $i < j \leq i + k$ and is k-*good* otherwise. What is the number of k-good dec-strings of length n?

Let K_n be the set of k-good dec-strings of length n. This question is just a generalization of the last two questions, and we attack it the same way. The only difference is that

- there are $10 - 2(i - 1)$ choices for d_i, for $i \in \{1, \dots, k\}$; and
- there are $10 - 2k$ choices for d_i when $i \in \{k + 1, \dots, n\}$.

Therefore, the number of k-good strings of length n is

$$|K_n| = \prod_{i=1}^{k}(10 - 2(i - 1)) \cdot (10 - 2k)^{n-k} = 2^k \cdot (5!/(5 - k)!) \cdot (10 - 2k)^{n-k} \ .$$

for all $n \geq k$. (Notice that this product is 0 for any $k \geq 5$.) For $n < k \leq 5$, we have $|K_0| = 1$, $|K_1| = 10$, $|K_2| = 10 \cdot 8 = 80$, $|K_3| = 10 \cdot 8 \cdot 6 = 480$, $|K_4| = 10 \cdot 8 \cdot 6 \cdot 4 = 1920$, $|K_5| = 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2 = 3840$.

If you like numerology and technology you've now learned that the horizontal resolution in the Full-HD TV standard (1920×1080) is actually $2^4 \cdot 5! \times 1080$

3 Collective Arts

Collective Arts Brewing currently makes 30 types of IPA and 6 types of Lager.

Q3.1: The manager at Mike's Place needs to choose 4 types of IPA and 4 types of Lager. How many options does the manager have?

We can solve this using the Product Rule with a 2-step procedure:

Step 1. Choose 4 IPA. There are $\binom{30}{4}$ ways to do this.

Step 2. Choose 4 Lager. There are $\binom{6}{4}$ ways to do this.

Therefore, by the Product Rule, the Mike's Place manager has

$$\binom{30}{4} \cdot \binom{6}{4} = 411075$$

options.

In case you're interested in knowing how I evaluate expressions like $\binom{30}{4} \cdot \binom{6}{4}$, I use Python:

```
from math import factorial
binom = lambda n,k: factorial(n)//((factorial(n-k)*factorial(k)))
print(binom(30,4)*binom(6,4))
```

Q3.2: The 8 beers (4 IPA and 4 Lager) selected in the previous question must be placed in a line on a display shelf so that no two IPA are adjacent and no two Lager are adjacent. How many ways are there to do this?

We solve this using the Product Rule to produce the sequence $b_1, \dots, b_8$ of beers. We use the following procedure:

Step 1. Decide whether the first (leftmost) beer, b_1, will be an IPA or a Lager. There are 2 ways to do this.

Step 2. Choose b_1 (an IPA or a Lager). There are 4 ways to do this.

Step 3. Choose b_2 (a Lager or an IPA). There are 4 ways to do this.

Step 4. Choose b_3 (an IPA or a Lager). Since b_1 is no longer available, there are 3 ways to do this.

Step 5. Choose b_4 (a Lager or an IPA). Since b_2 is no longer available, there are 3 ways to do this.

and so on for Steps 6–9. Therefore, there are

$$2 \cdot 4 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 2 \cdot 4! \cdot 4! = 1152$$

ways to place this beers so that no two IPAs are adjacent and no two Lagers are adjacent.

Q3.3: Continuing from the previous question, suppose that two of the beers selected were All Together Now (an IPA) and Hot Pink (a Lager). Since both cans are pink, the manager doesn't want to place them adjacent to each other. How many ways are there to do this (while still alternating between IPA and Lager)?

For this we can use the Complement Rule. Let U denote the set of 1152 arrangements counted in Part 3.2. Let $B \subseteq U$ denote those arrangements in which Hot Pink is adjacent to All Together Now. By the Complement Rule, the answer to our question is

$$|U \setminus B| = |U| - |B| = 1152 - |B|$$

so we only need to figure out $|B|$.

To determine $|B|$, we use the Product Rule with the following procedure.

Step 1. Decide whether b_1 will be an IPA or a Lager. There are 2 ways to do this.

Step 2. Pick the index $i \in \{1, \dots, 7\}$ such that $\{b_i, b_{i+1}\}$ will be Hot Pink and All Together Now (which of these two comes first is determined by the choice made in Step 1.) There are 7 ways to do this.

Step 3. Place the remaining 6 beers in the remaining 6 unused locations. There are $3! \cdot 3!$ ways to do this.

Therefore, by the Product Rule,

$$|A| = 2 \cdot 7 \cdot 3! \cdot 3! = 504$$

so

$$|U \setminus B| = 1152 - 504 = 648\ .$$

Q3.4: How many of the arrangements from the previous question have the All Together Now among the 4 leftmost bottles and the bottle of Hot Pink among the 4 rightmost bottles?

For this question it's probably easiest to directly count the arrangements we're interested in. First we count the set A of arrangements $b_1, \dots, b_8$ satisfying all the requirements and where, additionally, b_1 is an IPA. We can determine $|A|$ using the Product Rule:

Step 1. Choose the location of All Together Now. It must be one of $b_1, \dots, b_4$, and All Together Now is an IPA, so it must be one of b_1 or b_3. Therefore, there are 2 ways to execute this step.

Step 2. Choose the location of Hot Pink. It must be one of $b_5, \dots, b_8$ and it is a Lager, so it must be b_6 or b_8. Therefore, there are 2 ways to execute this step.

Step 3. Place the remaining 6 bottles in the 6 available locations. As before, there are $3! \cdot 3!$ ways to execute this step. Therefore, by the Product Rule

$$|A| = 2 \cdot 2 \cdot 3! \cdot 3! = 144 \ .$$

Next, we count the set B of arrangements $b_1, \dots, b_8$ satisfying all the requirements and where, additionally, b_1 is a Lager. This sounds deceptively similar to the definition of A, but it's not. Now there is a danger than All Together Now and Hot Pink are adjacent, which happens when All Together Now is b_4 and Hot Pink is b_5. We can determine $|B|$ using the Product Rule:

Step 1. Choose the locations of All Together Now and Hot Pink. The allowable options are $(b_2, b_5), (b_2, b_7), (b_4, b_7)$. So there are 3 ways to do this step.

Step 2. Place the remaining 6 bottles in the 6 available locations. As before, there are $3! \cdot 3!$ ways to execute this step.

Therefore, by the Product Rule

$$|B| = 3 \cdot 3! \cdot 3! = 108 \ .$$

Now, the answer to the question is $|A \cup B|$ and A and B are disjoint (all the arrangements in A start with an IPA and all those in B start with a Lager) so we can use the Sum Rule:

$$|A \cup B| = |A| + |B| = 144 + 108 = 252 \ .$$

4 Restricted Permutations

Consider all permutations of the integers $1, \dots, 1000$.

Q4.1 In how many of these permutations do $1, 2, 3, 4$ appear consecutively and in this order?

We can construct any such a permutation $\pi_1, \dots, \pi_{1000}$ using the following procedure:

Step 1. Choose an index $i \in \{1, \dots, 997\}$ and set $\pi_i, \pi_{i+1}, \pi_{i+2}, \pi_{i+3} = 1, 2, 3, 4$. There are 997 ways to do this.

Step 2. Make $\pi_1, \dots, \pi_{i-1}, \pi_{i+4}, \dots, \pi_{1000}$ be any permutation of $5, \dots, 1000$. There are $996!$ ways to do this.

Therefore, by the Product Rule, the number of permutations of $1, \dots, 1000$ in which $1, 2, 3, 4$ appear consecutively and in this order is $997 \cdot 996!$.

Q4.2 In how many of these permutations do $1, 2, 3, 4$ appear consecutively, but not necessarily in order? (For example, they may appear as $1, 2, 3, 4$, or $4, 2, 3, 1$, or $3, 1, 2, 4$, or so on.)

We can construct any such a permutation $\pi_1, \dots, \pi_{1000}$ using the following procedure:

Step 0. Choose a permutation $x_1, \dots, x_4$ of $1, \dots, 4$. There are $4!$ ways to do this.

Step 1. Choose an index $i \in \{1, \dots, 997\}$ and set $\pi_i, \pi_{i+1}, \pi_{i+2}, \pi_{i+3} = x_1, x_2, x_3, x_4$. There are 997 ways to do this.

Step 2. Make $\pi_1, \ldots, \pi_{i-1}, \pi_{i+4}, \ldots, \pi_{1000}$ be any permutation of $5, \ldots, 1000$. There are 996! ways to do this.

Therefore, by the Product Rule, the number of permutations of $1, \ldots, 1000$ in which $1, 2, 3, 4$ appear consecutively but not necessarily in order is $4! \cdot 997 \cdot 996!$.

Q4.3 In how many of these permutations does 1 appear before 2, 2 appear before 3, and 3 appear before 4? (In other words, $1, 2, 3, 4$ appear in order, but not necessarily consecutively.)

We can construct any such a permutation $\pi_1, \ldots, \pi_{1000}$ using the following procedure:

Step 1. Choose locations for $1, 2, 3, 4$ as follows: Choose a subset $\{i_1, i_2, i_3, i_4\} \subset \{1, \ldots, 1000\}$ of size 4 and name its elements so that $i_1 < i_2 < i_3 < i_4$. Set $\pi_{i_1} = 1$, $\pi_{i_2} = 2$, $\pi_{i_3} = 3$, and $\pi_{i_4} = 4$. There are $\binom{1000}{4}$ choices for $\{i_1, \ldots, i_4\}$ so there are $\binom{1000}{4}$ ways to execute this step.

Step 2. Fill in the remaining 996 locations of the permutation as follows: Choose a permutation $x_1, \ldots, x_{996}$ of $5, \ldots, 1000$. Then set

$$\pi_1, \ldots, \pi_{i_1-1}, \pi_{i_1+1}, \ldots, \pi_{i_2-1}, \pi_{i_2+1}, \ldots \pi_{i_3-1}, \pi_{i_3+1}, \ldots, \pi_{i_4-1}, \pi_{i_4+1}, \ldots, \pi_{1000} = x_1, \ldots, x_{996} \ .$$

There are 996! choices for the permutation $x_1, \ldots, x_{996}$ so there are 996! ways to execute this step.

Therefore, the number of permutations of $1, \ldots, 1000$ in which $1, 2, 3, 4$ appear consecutively (but not necessarily in order) is

$$\binom{1000}{4} \cdot 996! = \left(\frac{1000!}{4! \cdot 996!}\right) \cdot 996! = \frac{1000!}{4!}$$

Q4.4 In how many of these permutations do $1, 2, 3, 4$ appear in order but no two are adjacent?

This is similar to the previous question except that we are restricted in our choices of $i_1 < i_2 < i_3 < i_4$. In particular, we need $i_{j+1} \geq i_j + 2$ for each $j \in \{1, 2, 3\}$. There are lots of hard ways to answer this question, but I already hinted that we can reduce it to counting the number of integer solutions to some linear equation, so lets see how.

x_1: Let $x_1 = i_1 - 1$ be the number of elements before 1.

x_2: Let $x_2 = i_2 - i_1 - 2$ be one less than the number of elements between 1 and 2.

x_3: Let $x_3 = i_3 - i_2 - 2$ be one less than the number of elements between 2 and 3.

x_4: Let $x_4 = i_4 - i_3 - 2$ be one less than the number of elements between 3 and 4.

x_5: Let $X_5 = 1000 - i_4$ be the number of elements after 4.

x_1	i_1		x_2	i_2		x_3	i_3		x_4	i_4	x_5

Now, notice that if we choose $1 \leq i_1 < i_2 < i_3 < i_4 \leq 1000$ that satisfy the conditions of the question, then each $x_1, \ldots, x_5 \geq 0$. Furthermore,

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= (i_1 - 1) + (i_2 - i_1 - 2) + (i_3 - i_2 - 2) + (i_4 - i_3 - 2) + (1000 - i_4) \\ &= 1000 - 7 = 993 \ . \end{aligned}$$

Therefore, every choice of $1 \leq i_1 < i_2 < i_3 < i_4 \leq 1000$ that satisify the conditions of the question gives a solution to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 993$$

where $x_1, \ldots, x_5 \geq 0$ are integers.

On the other hand, if we take $x_1, \ldots, x_5 \geq 0$ be any non-negative integers satisfying $x_1 + \cdots + x_5 = 993$, then we can take $i_1 = x_1 + 1$, $i_2 = x_1 + x_2 + 3$, $i_3 = x_1 + x_2 + x_3 + 5$, and $i_4 = x_1 + x_2 + x_3 + x_4 + 7$ to get locations for $1, 2, 3, 4$, respectively where no two are adjacent.

We've just established (using the Bijection Rule) that the number of non-negative integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 993$ is equal to the number of valid choices of $i_1, \ldots, i_4$.

We've seen in class (Theorem 3.9.1 in the textbook) that the number of non-negative integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 993$ is $\binom{997}{4}$.

Now we can proceed exactly as in the previous question with a 2-step procedure that first picks locations $i_1, \ldots, i_4$ for $1, \ldots, 4$ and then picks a permutation of $5, \ldots, 1000$. By applying the Product Rule we see that the number of permutations of $1, \ldots, 1000$ in which $1, 2, 3, 4$ appear in order, but no two are adjacent is

$$\binom{997}{4} \cdot 996! \ .$$

5 Drug Trials

A certain friend of mine has spent the better part of a lifetime testing recreational drugs. After thorough testing, this friend has identified 20 recreational drugs $D_1, \ldots, D_{20}$ and determined (experimentally) that any 3 of these drugs can be taken simultaneously with no adverse effects.

Q5.1 Assuming my friend determined this entirely by testing, how many experiments did my friend have to perform?

My friend did $\binom{20}{3}$ trials, where each trial consists of taking 3 different drugs.

Q5.2 A new designer drug called D_{21} has just hit the streets and my friend wants to know if D_{21} can be added to their list. That is, can any triple of $D_1, \ldots, D_{21}$ be safely taken together? How many *additional* experiments does my friend need to determine this?

There are two possible answers:

S1: There are $\binom{21}{3}$ triples of 21 drugs and my friend has already tested $\binom{20}{3}$ of them, so they still have

$$\binom{21}{3} - \binom{20}{3}$$

triples left to test.

S2: My friend has already tested all triples involving combinations of $D_1, \ldots, D_{20}$, so they only have to perform an additional

$$\binom{20}{2}$$

test of the form $\{D_{21}, a, b\}$ where $\{a, b\}$ is a 2-element subset of $\{D_1, \ldots, D_{20}\}$.

Both these answers are correct.

Q5.3 Suppose my friend survives the experience and D_{21} makes it onto the list. My friend takes scrupulous notes about all experiments and notices something peculiar about the answers to the preceding two questions. What combinatorial identity did my friend just discover?

Since both answers to the previous question are correct,

$$\binom{21}{3} - \binom{20}{3} = \binom{20}{2} \ .$$

We can rewrite this as $\binom{21}{3} = \binom{20}{3} + \binom{20}{2}$. This is Pascal's Identity: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Q5.4 Suppose that an impatient novice wants to repeat my friend's discovery using fewer experiments. This novice is willing to take 5 drugs at a time. What is the fewest number of experiments this novice can perform so that for any triple $D_i, D_j, D_k \in \{D_1, \dots, D_{21}\}$, at least one of this novice's experiments includes D_i, D_j, and D_k?

This is a challenging question that asks: What is the smallest value ℓ such that there exists a set of 5-element subsets $X_1, \dots, X_\ell$ of $\{D_1, \dots, D_{21}\}$ such that, for each 3-element subset $T \subseteq \{D_1, \dots, D_{21}\}$, there exists some $p \in \{1, \dots, \ell\}$ such that $T \subset X_p$? This object is called a $(21, 5, 3)$-*covering design*. The value ℓ mentioned above is denoted as $C(21, 5, 3)$.

There is an obvious (to me) lower bound on $C(21, 5, 3)$:

$$\ell = C(21, 5, 3) \geq \binom{21}{3} \Big/ \binom{5}{3} = 1330/10 = 133$$

since each of $X_1, \dots, X_\ell$ contains only $\binom{5}{3}$ 3-element subsets.

It wasn't clear to me whether or not this lower bound is achievable. If it were achievable, the resulting set of 133 5-element subsets of $D_1, \dots, D_{21}$ would be called a $(3, 5, 21)$-*Steiner System*, named after the mathematician Jakob Steiner (1976–1863).

Steiner systems have been studied for hundreds of years and covering designs have been studied intensely since at least the mid-1990s. It is known that $C(21, 5, 3) \geq 147$ (because of a more general lower bound by Schonheim) and, in 1996, Floyd L. Oats gave a list of 151 5-element subsets of $\{1, \dots, 21\}$ that contains every three element subset of $\{1, \dots, 21\}$ thereby showing that $C(21, 5, 3) \leq 151$.

To summarize: The number of experiments the newcomer must perform is some number $C(21, 5, 3) \in \{147, 148, 149, 150, 151\}$. We don't know the exact value yet. (Though given the increase in computer performance since Oats' 1996 construction, we could probably figure out the exact value today with a bit of work and AWS time.)

References

- Wikipedia article on Steiner systems: `https://en.wikipedia.org/wiki/Steiner_system`
- Page on Covering Designs: `https://www.dmgordon.org/cover/`
- Floyd L. Oats' construction: `https://ljcr.dmgordon.org/show_cover.php?v=21&k=5&t=3`