COMP 2804 Assignment 2

October 11, 2020

Question 1.

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Question 2.

The area of a rectangle is defined as $l \times w$ which in this case is $20 \times 30 = 600$. So the area of R is $600m^2$. We can divide R into 600 $1m \times 1m$ boxes and call the number of boxes n, so $n = 600$. The set S contains 601 points, $p = 601$, to be divided into the n boxes.

$$\left\lceil \frac{p}{n} \right\rceil = \left\lceil \frac{601}{600} \right\rceil = 2$$

Therefore, by the pigeon hole principle there is at least one box that contains at least 2 points. To achieve maximum spacing between these 2 points in the same box you would want to place them in corners diagonally across from one another. Call the distance between these 2 points d, d can be calculated using Pythagorean's theorem as follows:

$$d = \sqrt{1^2 + 1^2}$$
$$d = \sqrt{2}$$

Since $d = \sqrt{2}$m and $\sqrt{2} < 2$ these 2 points will be less than 2m apart. Therefore, there will be at least 2 covidiots in the set S.

Question 3.

i) Define the set of pigeons as $\alpha_1, \alpha_2 \ldots, \alpha_7$, so $p = 7$. The interval $[-\pi/2, \pi/2]$ has a total length of π. This interval can be divided into 6 equal disjoint parts of length $\pi/6$, this defines our number of holes, $n = 6$. The points $\alpha_1, \alpha_2 \ldots, \alpha_7$ are all contained in the interval $[-\pi/2, \pi/2]$ so each point will fall into one of the 6 sections.

$$\left\lceil \frac{p}{n} \right\rceil = \left\lceil \frac{7}{6} \right\rceil = 2$$

Therefore, by the pigeon hole principle there is at least one interval of length $\pi/6$ that contains at least 2 points, say α_i and α_j. Since α_i and α_j are contained within the same section of length $\pi/6$, the distance between them must be between 0 and $\pi/6$. Assuming, without loss of generality, that α_i is greater than α_j, then $0 \leq \alpha_i - \alpha_j \leq \pi/6$.

ii) For each $i \in \{1, 2, \ldots, 7\}$, let p_i be the point with coordinates $(1, a_i)$. Call the angle between the vector from the origin to p_i and the x-axis b_i.

$$tan(b_i) = \frac{a_i}{1} = a_i \tag{1}$$

Because the x coordinate of each point p_i is positive, we know that each angle b_i is within the interval $[-\pi/2, \pi/2]$. By part i, we know that there exists 2 indices, i, j such that $0 \leq b_i - b_j \leq \pi/6$. Now, $tan(0) = 0$

and $tan(\pi/6) = \frac{1}{\sqrt{3}}$, so $0 \leq tan(b_i - b_j) \leq \frac{1}{\sqrt{3}}$.

$$\begin{aligned} tan(b_i - b_j) &= \frac{tan(b_i) - tan(b_j)}{1 + tan(b_i)tan(b_j)} \\ &= \frac{a_i - a_j}{1 + a_i a_j} && \text{(by (1))} \end{aligned}$$

Therefore, $0 \leq \frac{a_i - a_j}{1+a_i a_j} \leq \frac{1}{\sqrt{3}}$.

Question 4.

Base case: $n = 0$
Sub $n = 0$ into the equation $f(n) = 7 \times 5^n - 3n^2$:

$$\begin{aligned} f(0) &= 7 \times 5^0 - 3(0)^2 \\ &= 7 \times 1 - 0 \\ &= 7 \end{aligned}$$

This is the defined value of $f(0)$.

Inductive hypothesis: Assume $f(k) = 7 \times 5^k - 3k^2$ for all $k \in \{0, 1, \ldots, n-1\}$.
Inductive case:
Sub inductive hypothesis into $f(n) = 5 \times f(n-1) + 12n^2 - 30n + 15$

$$\begin{aligned} f(n) &= 5 \times (7 \times 5^{n-1} - 3(n-1)^2) + 12n^2 - 30n + 15 \\ &= 7 \times 5^n - 15(n-1)^2 + 12n^2 - 30n + 15 \\ &= 7 \times 5^n - 15(n^2 - 2n + 1) + 12n^2 - 30n + 15 \\ &= 7 \times 5^n - 15n^2 \cancel{+30n} \cancel{-15} + 12n^2 \cancel{-30n} \cancel{+15} \\ &= 7 \times 5^n - 3n^2 \end{aligned}$$

Therefore, QED.

Question 5.

Mystery function:

$$\begin{aligned} f(1) &= 1 \\ f(n) &= n + (n/2) && \text{(when } n \geq 2) \end{aligned}$$

Claim: $f(n) = 2n - 1$
Base case:
Sub $n = 1$ into $f(n) = 2n - 1$

$$\begin{aligned} f(1) &= 2(1) - 1 \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

This is the defined value of $f(1)$.

Inductive hypothesis: Assume that $f(k) = 2k - 1$ for each $k \in \{1, 2, 4, \ldots, n/2\}$
Inductive case:

Sub inductive hypothesis into $f(n) = n + (n/2)$

$$\begin{aligned} f(n) &= n + \left(\cancel{2}\left(\frac{n}{\cancel{2}}\right) - 1\right) \\ &= n + n - 1 \\ &= 2n - 1 \end{aligned}$$

Therefore, QED.

Question 6.

i) B is a set of (x, y) where both x and y are 00-free bitstrings of length $n-1$. The number of 00-free bitstrings of length $n-1$ is $f_{n-1+2} = f_{n+1}$. The number of combinations of (x, y) is equal to the number of ways get x times the number of ways to get y. So by the product rule $|B| = f_{n+1} \times f_{n+1} = f_{n+1}^2$.

ii) Since x and y are both bitstrings of length $n-1$ the concatenation of them is of length $2n-2$. Each different element (x, y) of B will generate a different bitstring of length $2n-2$, so there exists a bijection between bitstrings of length $2n-2$ and elements (x, y) of B. Therefore, the number of elements (x, y) of B where the concatenation xy is 00-free is equal to the number of 00-free bitstrings of length $2n-2$ which is $f_{2n-2+2} = f_{2n}$.

iii) For the concatenation xy to not be 00-free the last bit of x must be 0 and the first bit of y must also be 0.

- For x to be a 00-free bitstring of length $n-1$ that ends with 0 the string must start with a 00-free bitstring of length $n-3$ followed by the 2 bits 10. The number of ways to generate a string x is $f_{n-3+2} = f_{n-1}$.
- For y to be a 00-free bitstring of length $n-1$ that starts with 0 the string must start with the 2 bits 01 followed by a 00-free bitstring of length $n-3$. The number of ways to generate a string y is $f_{n-3+2} = f_{n-1}$.

Therefore, by the product rule the number of non 00-free concatenations of elements (x, y) of B is
$f_{n-1} \times f_{n-1} = f_{n-1}^2$

iv) B is the set of all elements (x, y) where both x and y are 00-free bitstrings of length $n-1$.
Let S be the set of 00-free concatenations xy of elements (x, y) of B.
So $B \setminus S$ is the set non 00-free concatenations xy of elements (x, y) of B.

By part i, we know $|B| = f_{n+1}^2$. From part ii, we know that $|S| = f_{2n}$ and from part iii, we know that $|B \setminus S| = f_{n-1}^2$.
So by the complement rule

$$\begin{aligned} |S| &= |B| - |B \setminus S| \\ f_{2n} &= f_{n+1}^2 - f_{n-1}^2 \end{aligned}$$

Question 7.

i) Base case: $n = 0$
The bitstring s_0 is defined as 1 so it is 00-free.

General case: $n > 0$
In order to obtain s_n you must use s_{n-1} and replace each 1 with 10 and each 0 with 1. There is no possible sequence of bits in s_{n-1} that will generate a 00 in s_n. The sequence 11 will become 1010, the sequence 10

will become 101 and the sequence 01 will become 110. Even the sequence 00, which is impossible in s_{n-1}, would generate 11. Therefore, for every value of $n \geq 0$ s_n will be 00-free.

ii)$L_0 = 1$
$O_0 = 1$
$L_1 = 2$
$O_1 = 1$

iii) The bitstring s_n is generated using s_{n-1} and every bit of s_{n-1} generates a sequence of bits in s_n that contains exactly one 1. So the number of 1's in s_n, or O_n, is equal to the length of s_{n-1}, or L_{n-1}.

iv) As s_n is made from using each bit of s_{n-1} to generate some sequence of s_n, the length of s_n must be at least the length of s_{n-1}, or L_{n-1}. Each bit equal to 1 in s_{n-1} will generate one extra bit in s_n, this will make s_n longer than s_{n-1}. To account for these extra bits in s_n we can simply add the number of 1's in s_{n-1} to its length, since each 1 generates one extra bit. The number of 1's in s_{n-1} is O_{n-1}, so $L_n = L_{n-1} + O_{n-1}$.

v) Determining L_n:

$$\begin{aligned} L_n &= L_{n-1} + O_{n-1} && \text{(by part iv)} \\ &= L_{n-1} + L_{n-2} && \text{(by part iii)} \end{aligned}$$

Define the function $f : \{0, 1, 2, \dots\} \to L_n$ as:

$$\begin{aligned} &f(0) = 1 \\ &f(1) = 2 \\ &f(n) = f(n-1) + f(n-2) && \text{(when } n \geq 2\text{)} \end{aligned}$$

We have seen in class that the Fibonacci sequence is defined $f : \{0, 1, 2, \dots\} \to \mathbb{Z}$:

$$\begin{aligned} &fib(0) = 0 \\ &fib(1) = 1 \\ &fib(n) = fib(n-1) + fib(n-2) && \text{(when } n \geq 2\text{)} \end{aligned}$$

We can see that for $n \geq 2$ $f(n)$ and $fib(n)$ have the same recursive definition. However, function f starts 2 iterations ahead of fib. So $f(n) = fib(n+2)$ for every value of n, and since every value of $f(n) = L_n$ therefore,

$$L_n = fib(n+2) \tag{2}$$

Determining O_n:

$$\begin{aligned} O_n &= L_{n-1} && \text{(by part iii)} \\ &= L_{n-2} + O_{n-2} && \text{(by part iv)} \\ &= L_{n-2} + L_{n-3} && \text{(by part iii)} \\ &= fib(n-2+2) + fib(n-3+2) && \text{(by (2))} \\ &= fib(n) + fib(n-1) \\ &= fib(n+1) \end{aligned}$$

Therefore, $O_n = fib(n+1)$.

vi) Each 1 in s_{n-1} generates exactly one 0 in s_n. So the number of 0's in s_n is equal to the number of 1's in s_{n-1}. So $Z_n = O_{n-1}$.

By part v we know that $O_n = fib(n+1)$, since $Z_n = O_{n-1}$, Z_n must be equal to $fib(n)$. Therefore, $Z_n = fib(n)$.

Question 8.

i)
$A_2 = 2$ ($\{00, 11\}$)
$A_3 = 4$ ($\{000, 011, 110, 111\}$)
$A_4 = 7$ ($\{0000, 0011, 0110, 0111, 1100, 1110, 1111\}$)
$A_5 = 12$ ($\{00000, 00011, 00110, 00111, 01100, 01110, 01111, 11000, 11011, 11100, 11110, 11111$)

ii) A string counted by A_n must either

- start with 0 followed by any string counted by A_{n-1}; OR
- start with 11 followed by some sequence of $n-2$ bits such that the resulting string is happy, this is the definition of B_n

Therefore, $A_n = A_{n-1} + B_n$.

iii) If $n \geq 4$ then $n - 1 \geq 3$. So by part ii, $A_{n-1} = A_{n-2} + B_{n-1}$.

iv) A string counted by B_n must either

- start with 110 followed by any string counted by A_{n-3}; OR
- start with 111 followed by some sequence of $n-3$ bits such that the resulting string is happy. If we only consider bits 2 though n then this is a happy bitstring of length $n-1$ that starts with 11, this is the definition of B_{n-1}.

Therefore, $B_n = A_{n-3} + B_{n-1}$.

v)

$$
\begin{aligned}
A_n &= 2A_{n-1} - A_{n-2} + A_{n-3} \\
&= 2(A_{n-2} + B_{n-1}) - A_{n-2} + A_{n-3} && \text{(by part iii)} \\
&= \cancel{2}A_{n-2} \cancel{- A_{n-2}} + 2B_{n-1} + A_{n-3} \\
&= A_{n-2} + B_{n-1} + B_n && \text{(by part iv)} \\
&= A_{n-1} + B_n && \text{(by part iii)} \\
&= A_n && \text{(by part ii)}
\end{aligned}
$$