COMP 2804 Assignment 4

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Question 1.

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Question 2.

Define the sample space S,

$$S = \{(d_1, d_2, \dots, d_6) : \text{for each } i \in \{1, \dots, 6\}, d_i \in \{i, 0, 0, 0, 0, 0\}\}$$

For any die D_i, the probability that the outcome (d_i) is 0 is $\frac{5}{6}$. Additionally, the probability that d_i equals i is $\frac{1}{6}$.

The random variable $X(d_1, \dots, d_6) = d_1 + \cdots + d_6$.
Define the random variables $X_1, \dots, X_6$, where for each $i \in \{1, \dots, 6\}, X_i(d_1, \dots, d_6) = d_i$.
Therefore, $X = X_1 + \cdots + X_6$ and $\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_6)$.

For any $1 \leq i \leq 6$, the random variable X_i only has 2 possible values: 0 and i. So

$$\begin{aligned}
\mathbb{E}(X_i) &= Pr(X_i = 0) \times 0 + Pr(X_i = i) \times i \\
&= \frac{5}{6} \times 0 + \frac{1}{6} \times i \\
&= \frac{i}{6}
\end{aligned}$$

By linearity of expectation

$$\begin{aligned}
\mathbb{E}(X_1 + \cdots + X_6) &= \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_6) \\
&= \sum_{i=1}^{6} \mathbb{E}(X_i) \\
&= \sum_{i=1}^{6} \frac{i}{6} \\
&= \frac{21}{6} \\
&= 3.5
\end{aligned}$$

Therefore, the expected value of X is 3.5.

Question 3.

Define the sample space S,

$$S = \{(r, b) : r, b \in \{1, \dots, 6\}\}$$

$|S| = 6^2$, therefore, $Pr(\omega) = \frac{1}{36}$ for each $\omega \in S$.

i) Define the random variable $R =$"the result of the red die"
Define the random variable $B =$"the result of the blue die"

We have seen in class that the expected value of a standard die is 3.5, therefore the $\mathbb{E}(R) = 3.5 = \mathbb{E}(B)$.

$X = R + B$, $Y = R - B$.

$$\begin{aligned}
\mathbb{E}(X \times Y) &= \mathbb{E}((R + B) \times (R - B)) \\
&= \mathbb{E}(R^2 - B^2) \\
&= \mathbb{E}(R^2) - \mathbb{E}(B^2) \\
&= 3.5^2 - 3.5^2 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(X) \times \mathbb{E}(Y) &= \mathbb{E}(R + B) \times \mathbb{E}(R - B) \\
&= (\mathbb{E}(R) + \mathbb{E}(B)) \times (\mathbb{E}(R) - \mathbb{E}(B)) \\
&= 7 \times 0 \\
&= 0
\end{aligned}$$

Since both equations equate to zero, therefore the equations $\mathbb{E}(X \times Y) = \mathbb{E}(X) \times \mathbb{E}(Y)$.

ii) For X and Y to independent random variables, $Pr(X = x \cap Y = y) = Pr(X = x) \times Pr(Y = y)$ for every $x, y \in \mathbb{R}$.

Lets test this for $X = 1$ and $Y = 0$.

$$\begin{aligned}
Pr(X = 1 \cap Y = 0) &= Pr(\{(0, 1), (1, 0)\} \cap \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}) \\
&= Pr(\emptyset) \\
&= 0 \\
&\neq \frac{2}{36} \times \frac{6}{36} = \frac{1}{108} = Pr(X = 1) \times Pr(Y = 0)
\end{aligned}$$

Since $Pr(X = 1 \cap Y = 0) = Pr(X = 1) \times Pr(Y = 0)$, therefore X and Y are not independent random variables.

Question 4.

Define the sample space S,

$$S = \{(g), (b, g), (b, b, g), (b, b, b, g), (b, b, b, b)\}$$

Since the probability of having a girl or a boy is $\frac{1}{2}$ each time the probabilities of getting each element of the sample space are as follows:
$Pr(g) = \frac{1}{2}$, $Pr(b, g) = \frac{1}{4}$, $Pr(b, b, g) = \frac{1}{8}$, $Pr(b, b, b, g) = \frac{1}{16}$, $Pr(b, b, b, b) = \frac{1}{16}$

FX and XF will have at least 1 child and will never have more than 4 children so the possible values for C are integers in the range {1,. . . ,4}.

$$\begin{aligned}
\mathbb{E}(C) &= \sum_{x=1}^{4} x \times Pr(C = x) \\
&= \left(1 \times \frac{1}{2}\right) + \left(2 \times \frac{1}{4}\right) + \left(3 \times \frac{1}{8}\right) + \left(4 \times 2 \times \frac{1}{16}\right) \\
&= \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{8}{16} \\
&= \frac{15}{8}
\end{aligned}$$

If FX and XF's first child is a girl then they will stop having kids and therefore have no boys. Conversely, if they have no girls, after 4 boys they will stop having kids. So the possible values for B are integers in the range $\{0,\ldots,4\}$.

$$\begin{aligned}
\mathbb{E}(B) &= \sum_{x=0}^{4} x \times Pr(B = x) \\
&= \left(0 \times \frac{1}{2}\right) + \left(1 \times \frac{1}{4}\right) + \left(2 \times \frac{1}{8}\right) + \left(3 \times \frac{1}{16}\right) + \left(4 \times \frac{1}{16}\right) \\
&= \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{16} \\
&= \frac{15}{16}
\end{aligned}$$

Question 5.

For $i \in \{1, \ldots, n\}$, define the random variable X_i,

$$X_i = \begin{cases} 1 & \text{if } a^2 + b^2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$X = X_1 + X_2 + \cdots + X_n$.

If we think of a and b as coordinates of a point on a 2 dimensional plane, then, by Pythagorean's theorem, the distance from the origin to the point (a, b) is equal to $a^2 + b^2$. We would like to know the probability that $a^2 + b^2$ is less than or equal to 1, in other words, does the point (a, b) fall within the circle that starts at the origin and has a radius equal to 1. Since the maximum/minimum values that a and b can take on are 1 and -1 respectively, the point will always fall within the square with side length 2 and its midpoint at the origin. Therefore,

$$\begin{aligned}
\mathbb{E}(X_i) &= Pr(a^2 + b^2 \leq 1) \\
&= \frac{\pi \times 1^2}{2^2} \\
&= \frac{\pi}{4}
\end{aligned}$$

By linearity of expectation,

$$\begin{aligned}\mathbb{E}(X) &= \sum_{i=1}^{n} \mathbb{E}(X_i) \\ &= \sum_{i=1}^{n} \frac{\pi}{4} \\ &= \frac{n \times \pi}{4}\end{aligned}$$

Finally,

$$\begin{aligned}\mathbb{E}(Y) &= \mathbb{E}\Big(\frac{4}{n} \times X\Big) \\ &= \mathbb{E}\Big(\frac{4}{n}\Big) \times \mathbb{E}(X) \\ &= \frac{4}{n} \times \frac{n \times \pi}{4} \\ &= \pi\end{aligned}$$

Therefore, the expected value of Y is π.

Question 6.

i) Define the random variable $Z = |X \cap Y|$.
Define the sample space $S =$"Number of ways to choose two 6 element subsets of a set of size 49". $|S| = \binom{49}{6}^2$.

The maximum value of Z is 6 which happens if X and Y are the exact same subset of N. The minimum value is 0, when X and Y consist of completely different numbers. So, the value of Z is an integer in the range [0,6].

For each $i \in \{0, \dots, 6\}$, define the event $A_i = Z = i$. Assuming the that X is determined before Y, for any $0 \leq i \leq 6$, $|A_i|$ can be determined using the product rule as follows:

1. Determine the subset X $\quad D_1 = \binom{49}{6}$
2. Choose the i elements of X that will be in Y $\quad D_2 = \binom{6}{i}$
3. Choose the remaining $6 - i$ elements of Y from $N \setminus X$ $\quad D_3 = \binom{43}{6-i}$

Therefore, $|A_i| = \binom{49}{6} \times \binom{6}{i} \times \binom{43}{6-i}$.

$$\begin{aligned}\mathbb{E}(Z) &= \sum_{i=0}^{6} i \times Pr(Z = i) \\ &= \sum_{i=0}^{6} i \times \frac{\binom{49}{6} \times \binom{6}{i} \times \binom{43}{6-i}}{\binom{49}{6}^2} \\ &= \sum_{i=0}^{6} i \times \frac{\binom{6}{i} \times \binom{43}{6-i}}{\binom{49}{6}} \\ &= \frac{36}{49}\end{aligned}$$

ii) Define the random variable $P =$"the payout of $Z = i$".

If Z is less than 2 then the payout is 0, and the highest payout is achieved when $Z = 6$. So P need only be defined for integers in the range $[2, 6]$.

$$\begin{aligned}
\mathbb{E}(P) &= \sum_{i=2}^{6} P(i) \times Pr(Z = i) \\
&= \sum_{i=2}^{6} P(i) \times \frac{\binom{6}{i} \times \binom{43}{6-i}}{\binom{49}{6}} \\
&= x \times \sum_{i=4}^{6} \frac{P(i)}{x} \times \frac{\binom{6}{i} \times \binom{43}{6-i}}{\binom{49}{6}} + \sum_{i=2}^{3} P(i) \times \frac{\binom{6}{i} \times \binom{43}{6-i}}{\binom{49}{6}} \\
&= \frac{x \times 683}{1400661570} + \frac{572975}{998844}
\end{aligned}$$

iii) We want to find a jackpot value x such that $\mathbb{E}(P) \geq 3$.

$$\begin{aligned}
3 &\leq \frac{x \times 683}{1400661570} + \frac{572975}{998844} \\
\frac{2423557}{998844} &\leq \frac{x \times 683}{1400661570} \\
x &\geq \frac{156331544285}{31418} \approx 4975859
\end{aligned}$$

Therefore, in order for the expected payout to be at least \$3 the jackpot must be at least about \$4,975,859.

Question 7.

i) The random variable X is equal to the value of i and the end of the loop. If x_1 gets set to n and then x_2 can take on any value but the loop will terminate regardless, therefore the loop will only iterate twice and X will be 2. Conversely, if x_i is set to 1 each time through the loop then the loop will iterate $n + 1$ times, so X will be $n + 1$. Therefore, the values X can take are integers in the range $[2, n + 1]$.

ii) Define the sample space S,

$$S = \{(x_1, \ldots, x_k) : x_1, \ldots, x_k \in \{1, \ldots, n\}\}$$

$|S| = n^k$.

Define the event $A =$"$X \geq k + 1$".

If $X \geq k + 1$ then the sum of x_1 to x_k must be less than or equal to n. Since $x_i \geq 1$ for each $i \in \{1, \ldots, k\}$, define $y_i = x_i - 1$ so that $y_i \geq 0$. So

$$\begin{aligned}
y_1 + 1 + y_2 + 1 + \cdots + y_k + 1 &\leq n \\
y_1 + y_2 + \cdots + y_k + k &\leq n \\
y_1 + y_2 + \cdots + y_k &\leq n - k
\end{aligned}$$

By Theorem 3.9.2 in the textbook, the number of solutions to this equation is

$$\binom{(n-k)+k}{k} = \binom{n}{k}$$

Therefore, $|A| = \binom{n}{k}$, and

$$\begin{aligned} Pr(X \geq k+1) &= \frac{\binom{n}{k}}{n^k} \\ &= \binom{n}{k} \times \left(\frac{1}{n}\right)^k \end{aligned}$$

iii)

$$\begin{aligned} \mathbb{E}(X) &= \sum_{k=0}^{n} Pr(X \geq k+1) \\ &= \sum_{k=0}^{n} \binom{n}{k} \times \left(\frac{1}{n}\right)^k \times (1)^{n-k} \end{aligned}$$

By Newton's Binomial Theorem the equation is equal to $\left(\frac{1}{n}+1\right)^n$. Therefore,

$$\mathbb{E}(X) = \left(\frac{1}{n}+1\right)^n$$

iv)

$$\lim_{n\to\infty} \mathbb{E}(X) = \lim_{n\to\infty} \left(\frac{1}{n}+1\right)^n$$

This value is equal to Euler's number e by definition. So $\lim_{n\to\infty} \mathbb{E}(X) = e$.

Question 8.

i)

$$\begin{aligned} \frac{a}{b} + \frac{b}{a} &\geq 2 \\ \frac{a^2+b^2}{ab} &\geq 2 \\ a^2+b^2 &\geq 2ab \\ a^2-2ab+b^2 &\geq 0 \\ (a-b)^2 &\geq 0 \end{aligned}$$

ii) The range of the random variable X is all the degrees of each different vertex u in G, i.e. Range of $X = \{deg(u) : u \text{ is a vertex of } G\}$.

The probability that X takes the value of $deg(u)$ is equal to the chance that that specific vertex u is chosen. Therefore, $Pr(X = deg(u)) = \frac{1}{n}$, since the graph G has n vertices.

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{u\text{: vertex of } G} deg(u) \times Pr(X = deg(u)) \\
&= \sum_{u\text{: vertex of } G} deg(u) \times \frac{1}{n} \\
&= \frac{1}{n} \times \sum_{u\text{: vertex of } G} deg(u) \\
&= \frac{1}{n} \times 2m \qquad\qquad \text{by the handshake theorem} \\
&= \frac{2m}{n}
\end{aligned}$$

iii) The range of the random variable Y is the degrees of each vertex v neighbor of each vertex u in G, i.e. Range of $Y = \{deg(v) :$ for all v in G neighbor of u, for all u in $G\}$.

The probability that Y takes on the value $deg(v)$ is equal to the chance the the vertex u is chosen and v is chosen as the neighbor of u. Therefore,

$$\begin{aligned}
Pr(Y = deg(v)) &= Pr(u \cap v) \\
&= Pr(u) \times Pr(v|u) \\
&= \frac{1}{n} \times \frac{1}{deg(u)}
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(Y) &= \sum_{u\text{: vertex of } G} \left(\sum_{v\text{: neighbor of } u} deg(v) \times Pr(Y = deg(v)) \right) \\
&= \sum_{u\text{: vertex of } G} \left(\sum_{v\text{: neighbor of } u} deg(v) \times \frac{1}{n} \times \frac{1}{deg(u)} \right) \\
&= \frac{1}{n} \times \sum_{u\text{: vertex of } G} \left(\sum_{v\text{: neighbor of } u} deg(v) \times \frac{1}{deg(u)} \right) \\
&= \frac{1}{n} \times \sum_{u\text{: vertex of } G} \left(\sum_{v\text{: neighbor of } u} \frac{deg(v)}{deg(u)} \right)
\end{aligned}$$

iv) On the right hand side (RHS) of the equation, each edge contributes $\frac{deg(v)}{deg(u)} + \frac{deg(u)}{deg(v)}$ to the total sum.

On the LHS consider each vertex u and v separately. If v is a neighbor of u then u is also a neighbor of v. u and v are both vertices of G, Assume, without loss of generality, that u appears before v in the iteration of G done by the first sum. So first when the iteration is at the vertex u it will contribute $\frac{deg(v}{deg(u)}$ to the total sum, then when the iteration gets to v it will contribute $\frac{deg(u)}{deg(v)}$ to the total sum. Therefore, the edge u,v will contribute $\frac{deg(v)}{deg(u)} + \frac{deg(u)}{deg(v)}$ to the total sum. This is the same amount as the RHS does.

Since both sides of the equation contribute the same amount to the total sum for each edge and each will iterate over each edge exactly once, it is safe to say that the two sides are equal.

v)

$$\begin{aligned}
\mathbb{E}(X) &\leq \mathbb{E}(Y) \\
\frac{2m}{n} &\leq \frac{1}{n} \times \sum_{u\text{: vertex of } G} \left(\sum_{v\text{: neighbor of } u} \frac{deg(v)}{deg(u)} \right) \\
\frac{2m}{n} &\leq \frac{1}{n} \times \sum_{\{u,v\}\text{: edge in } G} \left(\frac{deg(v)}{deg(u)} + \frac{deg(u)}{deg(v)} \right) && \text{by part iv} \\
\frac{2m}{n} &\leq \frac{1}{n} \times \sum_{\{u,v\}\text{: edge in } G} 2 && \text{by part i} \\
\frac{2m}{n} &\leq \frac{1}{n} \times 2m \\
\frac{2m}{n} &\leq \frac{2m}{n}
\end{aligned}$$

Therefore, if each connected component is regular ($deg(u) = deg(v)$ for each edge $\{u, v\}$), then using the same logic we did in part i, the statement $\mathbb{E}(Y) \geq \mathbb{E}(X)$ is true.