COMP 4804 Assignment 1

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Question 1.

Define random variable X ="the number of fixed points in the permutation π"

For $\forall i \in \{1, 2, ..., n\}$:

$$\text{define } X_i = \begin{cases} 1 & \text{if } \pi(i) = i \\ 0 & \text{otherwise} \end{cases}$$

$$X = \sum_{i=1}^{n} X_i \Rightarrow E(X) = E\Big(\sum_{i=1}^{n} X_i\Big) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} Pr(X_i = 1)$$

$$Pr(X_i = 1) = Pr(\text{"}i \text{ is is a fixed point"}) = \frac{(n-1)!}{n!} = \frac{1}{n}$$

$$E(X) = \sum_{i=1}^{n} Pr(X_i = 1) = \sum_{i=1}^{n} \frac{1}{n} = \frac{n}{n} = 1$$

Therefore, the expected number of fixed points in a random permutation is 1.

Question 2.

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1. Since the coin is fair, the probability of the coin landing heads is equal to the probability of it landing tails, which is $\frac{1}{2}$. That means, with equal probability, an element of N is either in X or is not in X. We do this for every element of N in order to generate X. This means, for any $X \subseteq N$, the probability of generating that particular X is equal to $\frac{1}{2^n}$. Therefore, the coin flip experiment generates any on the possible 2^n subsets of N with equal probability.

2. Define X' as a sequence $x_1, x_2, \ldots, x_n$ where for $\forall i \in \{1, \ldots, n\}$ x_i is equal to 1 if the i^{th} element in N is contained in the subset X and 0 otherwise. Define Y' in the same way for the subset Y.

   i) $Pr(X \subseteq Y)$
   Let us note that $X \subseteq Y$ if and only if Y' a 1 in every index that X' has a 1 (every element in X is also in Y). The remaining indices of Y' do not matter. Since both X' and Y' can only contain 0's and 1's, we can state $X \subseteq Y \iff \forall i \in \{1, \ldots, n\}$ $x_i \leq y_i$. Then:

$$\begin{aligned} Pr(X \subseteq Y) &= Pr\big[(x_1 \leq y_1) \cap (x_2 \leq y_2) \cap \cdots \cap (x_n \leq y_n)\big] \\ &= Pr\Big[\bigcap_{i=1}^{n} x_i \leq y_i\Big] = \prod_{i=1}^{n} Pr(x_i \leq y_i) \end{aligned}$$

$$\text{Define: } S = \{(x,y) : x, y \in \{0,1\}\}$$

$$Pr(x \le y) = \frac{|\{(0,0),(0,1),(1,1)\}|}{|\{(0,0),(0,1),(1,0),(1,1)\}|} = \frac{3}{4} = Pr(x_i \le y_i)$$

$$Pr(X \subseteq Y) = \prod_{i=1}^{n} Pr(x_i \le y_i) = \prod_{i=1}^{n} \frac{3}{4} = \left(\frac{3}{4}\right)^n$$

Therefore, $Pr(X \subseteq Y) = \left(\frac{3}{4}\right)^n$.

ii) $Pr(X \cup Y = N)$
Let us make use of X' and Y' again. We can note that $X \cup Y = N$ if and only if Y' has a 1 in every index that X' has a 0. This is slightly complicated, so let's break it down further. For $\forall i \in \{1, \ldots, n\}$ if $x_i = 0$, then $y_i > x_i$. Otherwise, if $x_i = 1$ then $y_i \le x_i$. From here we can construct the equation:

$$\begin{aligned} Pr(X \cup Y = N) &= Pr\left[\bigcap_{i=1}^{n} \Big((y_i \le x_i | x_i = 1 \cap x_i = 1) \cup (y_i > x_i | x_i = 0 \cap x_i = 0)\Big)\right] \\ &= \prod_{i=1}^{n} \Big[Pr(y_i \le x_i | x_i = 1) \cdot Pr(x_i = 1) + Pr(y_i > x_i | x_i = 0) \cdot Pr(x_i = 0)\Big] \\ &= \prod_{i=1}^{n} \left[1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}\right] = \prod_{i=1}^{n} \frac{3}{4} = \left(\frac{3}{4}\right)^n \end{aligned}$$

Therefore, $Pr(X \cup Y = N) = \left(\frac{3}{4}\right)^n$.

Question 3.

1. Define random variable $X =$" number of coin flips until you see a heads"
   Also define the random variable:

$$X_1 = \begin{cases} 1 & \text{if the first coin flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} E(X) &= \sum_{i=0}^{1} E(X | X_1 = i) \cdot Pr(X_1 = i) = \frac{1}{2} \sum_{i=0}^{1} E(X | X_1 = i) \\ &= \frac{1}{2} E(X | X_1 = 1) + \frac{1}{2} E(X | X_1 = 0) = \frac{1}{2} \cdot (1) + \frac{1}{2} \cdot (1 + E(X)) \\ &= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \cdot E(X) \\ \frac{1}{2} \cdot E(X) &= 1 \\ E(X) &= 2 \end{aligned}$$

Therefore, the expected number of coin flips before you see a heads is 2.

2. Define random variable Y ="number of coin flips until you see a heads followed by 2 tails in a row" Also define the random variable:

$$Y_i = \begin{cases} 1 & \text{if the } i^{th} \text{ coin flip is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} E(Y) &= E(Y|Y_1 = 0) \cdot Pr(Y_1 = 0) + E(Y|Y_1 = 1) \cdot Pr(Y_1 = 1) \\ &= \frac{1}{2} \cdot \big(1 + E(Y)\big) + \frac{1}{2} \cdot E(Y|Y_1 = 1) \\ &= \frac{1}{2} + \frac{E(Y)}{2} + \frac{E(Y|Y_1 = 1)}{2} \end{aligned}$$

$$\begin{aligned} E(Y|Y_1 = 1) &= E(Y|Y_1 = 1 \cap Y_2 = 1) \cdot Pr(Y_2 = 1) + E(Y|Y_1 = 1 \cap Y_2 = 0) \cdot Pr(Y_2 = 0) \\ &= \frac{1}{2} \cdot \big(1 + E(Y|Y_1 = 1)\big) + \frac{1}{2} \cdot E(Y|Y_1 = 1 \cap Y_2 = 0) \end{aligned}$$

$$\begin{aligned} E(Y|Y_1 = 1 \cap Y_2 = 0) =& E(Y|Y_1 = 1 \cap Y_2 = 0 \cap Y_3 = 0) \cdot Pr(Y_3 = 0) + \\ & E(Y|Y_1 = 1 \cap Y_2 = 0 \cap Y_3 = 1) \cdot Pr(Y_3 = 1) \\ =& \frac{1}{2} \cdot (3) + \frac{1}{2} \cdot \big(2 + E(Y|Y_1 = 1)\big) \\ =& \frac{5}{2} + \frac{E(Y|Y_1 = 1)}{2} \end{aligned}$$

$$\begin{aligned} E(Y|Y_1 = 1) &= \frac{1}{2} \cdot \big(1 + E(Y|Y_1 = 1)\big) + \frac{1}{2} \cdot E(Y|Y_1 = 1 \cap Y_2 = 0) \\ &= \frac{1}{2} \cdot \big(1 + E(Y|Y_1 = 1)\big) + \frac{1}{2} \cdot \left(\frac{5}{2} + \frac{E(Y|Y_1 = 1)}{2}\right) \\ &= \frac{7}{4} + \frac{3 \cdot E(Y|Y_1 = 1)}{4} \\ \frac{E(Y|Y_1 = 1)}{4} &= \frac{7}{4} \Rightarrow E(Y|Y_1 = 1) = 7 \end{aligned}$$

$$\begin{aligned} E(Y) &= \frac{1}{2} + \frac{E(Y)}{2} + \frac{E(Y|Y_1 = 1)}{2} \\ &= \frac{1}{2} + \frac{E(Y)}{2} + \frac{7}{2} \\ \frac{E(Y)}{2} &= 4 \Rightarrow E(Y) = 8 \end{aligned}$$

Therefore, the expected number of coin flips until you see a heads followed by 2 tails in a row is 8.

3. Scheme: Flip the coin twice. If the result is HT then report the result as heads. If the result is TH then report the result as tails. Otherwise, result HH or TT, repeat the experiment (i.e. flip the coin another 2 times.

   This scheme will simulate an unbiased coin because $Pr(HT) = p \cdot (1-p) = (1-p) \cdot p = Pr(TH)$. In other words, the probability of the result heads is equal to that of tails. This works because we repeat the experiment if we see the same result twice (HH or TT).

   Define random variable X ="The number of tosses of the biased coin to generate 1 unbiased flip".

   Define random variable Y ="the integer representation of the binary result of the first 2 coin flips, where $H \to 1$ and $T \to 0$". I.e., if the result of the first 2 coin flips was HT then $Y = 2$ (10 in binary). Note that Y can take on values $y \in \{0, 1, 2, 3\}$.

$$
\begin{aligned}
E(X) &= \sum_{y=0}^{3} \Big(E(X|Y=y) \cdot Pr(Y=y) \Big) \\
&= \sum_{y=1}^{2} \Big(E(X|Y=y) \cdot p(1-p) \Big) + E(X|Y=3) \cdot p^2 + E(X|Y=0) \cdot (1-p)^2 \\
&= 2\Big(2 \cdot p(1-p)\Big) + \Big((2+E(X)) \cdot p^2\Big) + \Big((2+E(X)) \cdot (1-p)^2\Big) \\
E(X) &= 2 + E(X) - 2 \cdot p \cdot E(X) + 2 \cdot p^2 \cdot E(X) \\
0 &= 2\Big(1 - p \cdot E(X) + p^2 \cdot E(X)\Big) \\
-1- &= -p \cdot E(X) + p^2 \cdot E(X)\Big) \\
-1 &= -p\Big(E(X) - p \cdot E(X)\Big) \\
\frac{1}{p} &= E(X)\big(1-p\big) \\
E(X) &= \frac{1}{p(1-p)}
\end{aligned}
$$

   Therefore, this scheme simulates an unbiased flip in $\frac{1}{p(1-p)}$ expected bias flips.

Question 4.

1. Define the random variable X ="the degree of the root in the final tree"
   For $\forall i \in \{1, 2, \ldots, n-1\}$ define the random variables:

$$
X_i = \begin{cases} 1 & \text{if the root is selected as the second element in step } i \\ 0 & \text{otherwise} \end{cases}
$$

$$
X = \sum_{i=1}^{n-1} X_i \Rightarrow E(X) = \sum_{i=1}^{n-1} E(X_i) = \sum_{i=1}^{n-1} Pr(X_i = 1)
$$

   Define A_i ="the root is selected as the second element in step i"
   Define B_i ="the root is not selected as the first element in step i"

$$
\begin{aligned}
Pr(X_i = 1) &= Pr(A_i|B_i) \cdot Pr(B_i) = \frac{Pr(A_i \cap B_i)}{Pr(B_i)} \cdot Pr(B_i) \\
&= Pr(A_i \cap B_i) = Pr(B_i|A_i) \cdot Pr(A_i) = 1 \cdot Pr(A_i) \\
&= \frac{1}{n-i}
\end{aligned}
$$

$$
\begin{aligned}
X &= \sum_{i=1}^{n-1} Pr(X_i = 1) = \sum_{i=1}^{n-1} \frac{1}{n-i} \\
&= \frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{1}{2} + \frac{1}{1} \\
&= \sum_{i=1}^{n-1} \frac{1}{i} = H_{n-1}
\end{aligned}
$$

Therefore, the expected degree of the root is H_{n-1} or approximately $\ln(n-1)$.

2. Define the random variable $Y =$"the distance from the node with label 1 to the root"
For $\forall i \in \{1, 2, \ldots, n-1\}$ define the random variables:

$$
Y_i = \begin{cases} 1 & \text{if the node with label 1 is selected as the first element in step } i \\ 0 & \text{otherwise} \end{cases}
$$

$$
\begin{aligned}
Y &= \sum_{i=1}^{n-1} Y_i \Rightarrow E(Y) = \sum_{i=1}^{n-1} E(Y_i) = \sum_{i=1}^{n-1} Pr(Y_i = 1) = \sum_{i=1}^{n-1} \frac{1}{n-i+1} \\
&= \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{3} + \frac{1}{2} \\
&= \sum_{i=2}^{n} \frac{1}{i} = H_n - 1
\end{aligned}
$$

Therefore, the expected distance from the node with label 1 to the root is $H_n - 1$ or approximately $\ln(n) - 1$.

Question 5.

1. If a bit gets flipped an even number of times (0 being even) then it arrives correctly. So, if the bit arrives correctly then it must have traveled through an even number of faulty edges. That means for $k \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$ $2k$ of the edges must have been faulty, and the remaining $n - 2k$ were not. Since which of the n edges were faulty, we must account for all possible variations. We can do this by multiplying by $\binom{n}{2k}$. If we combine all of this we get that the probability of a bit correctly arriving at its destination is:

$$
= \binom{n}{2k} \cdot p^{2k}(1-p)^{n-2k}
$$

When we do this for all possible values of k we get the equation given in the question:

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \cdot p^{2k}(1-p)^{n-2k}$$

2. The probability of a bit being flipped when travelling through any edge is p. If an edge has bias q, then $p = \frac{(1-q)}{2}$. If a bit travels through 2 such edges, then the probability that it is flipped is $2\big(p(1-p)\big)$; i.e. 2 times the chance that 1 is faulty and the other is not.

$$\begin{aligned} 2\big(p(1-p)\big) &= 2\left(\frac{1-q}{2} \cdot \left(1 - \frac{1-q}{2}\right)\right) \\ &= 2\left(\frac{1-q}{2} - \frac{(1-q)^2}{4}\right) = 2\left(\frac{2(1-q)-(1-q)^2}{4}\right) \\ &= \frac{2-2q-(1-2q+q^2)}{2} = \frac{1-q^2}{2} \end{aligned}$$

Therefore, the probability of a bit flipping after travelling through 2 edges with bias q is $\frac{1-q^2}{2}$.

3. Using the result of part 2, let us assume that the probability a bit flips after travelling through n edges with bias q is equal to $\frac{1-q^n}{2}$ (I should probably prove this, but I do not know how. Induction?).

Define the event A ="Bit arrives correctly after travelling through n edges with bias q".

$$Pr(A) = 1 - \frac{1-q^n}{2} = \frac{2-(1-q^n)}{2} = \frac{1+q^n}{2} \tag{1}$$

Let us calculate the bias q of an edge that flips bits with probability p:

$$\begin{aligned} p &= \frac{1-q}{2} \\ 2p &= 1-q \\ q &= 1-2p \end{aligned}$$

Thus, the edges in the network defined in the question have bias $1-2p$. If we sub this value into the equation 1 we get that the probability a bit arrives correctly after travelling through n edges is equal to $\frac{1+(1-2p)^n}{2}$. This is what we wanted to show.