# COMP 2804 Assignment 1

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### Question 1.

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### Question 2.

i) Since $k$ is equal to 1 in this case there is only one element of $L$ that can be used to label the edges of the $K_n$ graph. Therefore, every edge will have the same label. Since $n \geq 3$ there will be at least 3 cyclic edges with the same label, i.e. a boring triangle.

ii) Let $u$ be a random vertex of the $K_n$ graph.

(1) Suppose that none of the $k$ labels is used to label the edges connected to $u$ more than $(n-1)/k$ times.

There are $n-1$ edges connected to $u$ by definition of a complete graph. By (1) we have less than $k \times (n-1)/k = (n-1)$ edges connected to $u$. This is a contradiction so the amount of edges connected to $u$ with the same label must equal at least $\lceil (n-1)/k \rceil$, call this label $l$.

Consider the complete subgraph $G$ of $K_n$ which contains the vertices connected to $u$ with an edge labeled $l$.

**Case 1:**
If any edge $vw$ in the subgraph $G$ is labeled with $l$, then, since $u$ is connected to both $v$ and $w$ with an edge labeled $l$, we have found our boring triangle.

**Case 2:**
Otherwise $G$ is a complete graph with $k-1$ labels $(L \setminus l)$ and at least $\lceil (n-1)/k \rceil$ vertices.

$$\begin{aligned} Note: &\lceil (n-1)/k \rceil \geq \lceil ((3(k!)) - 1)/k \rceil = \lceil (3(k-1)!) - 1/k \rceil = 3(k-1)! \\ &\lceil (n-1)/k \rceil \geq 3(k-1)! \end{aligned}$$

Since $\lceil (n-1)/k \rceil \geq 3(k-1)!$, therefore by the induction hypothesis $G$ has a boring triangle. $G$ is a subgraph of $K_n$ so that same boring triangle is part of $K_n$.

### Question 3.

Let $k \geq 2$ and $n \geq 2$ be integers.
Let $A$ be a set of all $n$ length sequences of $a_1, a_2, \ldots, a_n$ where each element $a_i \in \{1, 2, \ldots, k\}$ and no two consecutive elements $a_i, a_{i+1}$ are the same.

In order to create a sequence in $A$ we must first choose a value for $a_i$, there are $k$ choices for this. Then for $i = 2, 3, \ldots, n$ we must choose a value for $a_i \in S \setminus a_{i-1}$, there are $k-1$ choices for this for all $n-1$ values of $i$. Therefore by the product rule

$$|A| = k \times (k-1)^{n-1}$$

### Question 4.

i) Zoltan's meal deals with distinct cheeses and beers:

Steps to generate a Zoltan's meal deal:

1. Choose dough $N_1 = 2$
2. Choose meat $N_2 = 4$
3. Choose sauce $N_3 = 5$
4. Choose any subset of 20 different toppings.
   We have seen in class the the number of subsets of a set of size $n$ is $2^n$. $N_4 = 2^{20}$
5. Choose 3 distinct cheeses from a set of 7 $N_5 = \binom{7}{3}$
6. Choose 4 distinct beers from a set of 9 $N_6 = \binom{9}{4}$

Therefore, by the product rule the number of ways to generate a Zoltan's meal deal with distinct cheeses and beers is

$$2 \times 4 \times 5 \times 2^{20} \times \binom{7}{3} \times \binom{9}{4}$$

ii) Zoltan's meal deals with not necessarily distinct cheeses and beers:

Steps to generate a Zoltan's meal deal:

1. Choose dough $N_1 = 2$
2. Choose meat $N_2 = 4$
3. Choose sauce $N_3 = 5$
4. Choose any subset of 20 different toppings.
   We have seen in class the the number of subsets of a set of size $n$ is $2^n$. $N_4 = 2^{20}$
5. For $i \in \{1, 2, \ldots, 7\}$, let $x_i$ represent the amount of cheese $i$ to put on the pizza, $x_i \geq 0$. So

   $$x_1 + x_2 + \cdots + x_7 = 3 \tag{1}$$

   Theorem 3.9.1 in the textbook states that the number of solutions to an equation of the form $x_1 + x_2 + \cdots + x_k = n$, where $x_1 \geq 0, x_2 \geq 0, \ldots, x_k \geq 0$ is

   $$\binom{n+k-1}{k-1}$$

   where $k$ is the number of terms in the equation and $n$ is the sum. So the number of solutions to (1) is

   $$\binom{3+7-1}{7-1} = \binom{9}{6}$$

   $N_5 = \binom{9}{6}$
6. For $i \in \{1, 2, \ldots, 9\}$, let $x_i$ represent the amount of beer $i$ you want with your order, $x_i \geq 0$. So

   $$x_1 + x_2 + \cdots + x_9 = 4 \tag{2}$$

   So, similarly to the last step, the number of solutions to (2) is

   $$\binom{4+9-1}{9-1} = \binom{12}{8}$$

   $N_6 = \binom{12}{8}$

Therefore, by the product rule the number of ways to generate a Zoltan's meal deal with not necessarily distinct cheeses and beers is

$$2 \times 4 \times 5 \times 2^{20} \times \binom{9}{6} \times \binom{12}{8}$$

**Question 5.**

Let $A$ be the set of students who do not where pants while watching lectures.
Let $B$ be the set of students who drink beer while watching lectures.

$|A| = 150$ and $|B| = 110$. The total number of students in the class is 230, so $|A \cup B| \leq 230$. The principle of inclusion/exclusion states that $|A \cup B| = |A| + |B| - |A \cap B|$, using algebra this can be re-arranged to $|A \cap B| = |A| + |B| - |A \cup B|$.

$$|A \cap B| \geq 150 + 110 - 230 = 30$$

Therefore, the best possible lower bound for students who drink beer while not wearing pants during lectures is 30.

**Question 6.**

i) Let $A$ be the set of all solutions to $x_1 + x_2 + x_3 = 99$, where $x_1 \geq 40, x_2 \geq 0, x_3 \geq 0$
Let $x_1 = 40 + y_1, y_1 \geq 0$

$$\begin{aligned} x_1 + x_2 + x_3 &= 99 \\ 40 + y_1 + x_2 + x_3 &= 99 \\ y_1 + x_2 + x_3 &= 59 \end{aligned}$$

Theorem 3.9.1 in the textbook states that the number of solutions to an equation of the form $x_1 + x_2 + \cdots + x_k = n$, where $x_1 \geq 0, x_2 \geq 0, \ldots, x_k \geq 0$ is

$$\binom{n+k-1}{k-1}$$

where $k$ is the number of terms in the equation and $n$ is the sum. Therefore

$$|A| = \binom{59+3-1}{3-1} = \binom{61}{2} \tag{3}$$

ii) Let $A$ be the set of all solutions to $x_1 + x_2 + x_3 = 99$, where $x_1 \geq 0, x_2 \geq 50, x_3 \geq 0$
Let $x_2 = 50 + y_2, y_2 \geq 0$

$$\begin{aligned} x_1 + x_2 + x_3 &= 99 \\ x_1 + 50 + y_2 + x_3 &= 99 \\ x_1 + y_2 + x_3 &= 49 \end{aligned}$$

By the same theorem used in the part i

$$|A| = \binom{49+3-1}{3-1} = \binom{51}{2} \tag{4}$$

iii) Let $A$ be the set of all solutions to $x_1 + x_2 + x_3 = 99$, where $x_1 \geq 0, x_2 \geq 0, x_3 \geq 55$
Let $x_3 = 55 + y_3, y_3 \geq 0$

$$\begin{aligned} x_1 + x_2 + x_3 &= 99 \\ x_1 + x_2 + 55 + y_3 &= 99 \\ x_1 + y_2 + x_3 &= 44 \end{aligned}$$

By the same theorem used in the part i

$$|A| = \binom{44+3-1}{3-1} = \binom{46}{2} \tag{5}$$

iv) Let $A$ be the set of all solutions to $x_1 + x_2 + x_3 = 99$, where $x_1 \geq 40, x_2 \geq 50, x_3 \geq 0$
Let $x_1 = 40 + y_1$ and $x_2 = 50 + y_2, y_1 \geq 0$ and $y_2 \geq 0$

$$\begin{aligned} x_1 + x_2 + x_3 &= 99 \\ 40 + y_1 + 50 + y_2 + x_3 &= 99 \\ y_1 + y_2 + x_3 &= 9 \end{aligned}$$

By the same theorem used in the part i

$$|A| = \binom{9+3-1}{3-1} = \binom{11}{2} \tag{6}$$

v) Let $U$ be the set of all solutions to $x_1 + x_2 + x_3 = 99$, where $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$
Let $S$ be the set of all solutions to $x_1 + x_2 + x_3 = 99$, where $0 \leq x_1 \leq 39, 0 \leq x_2 \leq 49, 0 \leq x_3 \leq 54$
So $U \setminus S$ is the set of all solutions to $x_1 + x_2 + x_3 = 99$, where $x_1 \geq 40, x_2 \geq 50, x_3 \geq 55$

Let $A_1$ be the set of all solutions to $x_1 + x_2 + x_3 = 99$, where $x_1 \geq 40, x_2 \geq 0, x_3 \geq 0$
Let $A_2$ be the set of all solutions to $x_1 + x_2 + x_3 = 99$, where $x_1 \geq 0, x_2 \geq 50, x_3 \geq 0$
Let $A_3$ be the set of all solutions to $x_1 + x_2 + x_3 = 99$, where $x_1 \geq 0, x_2 \geq 0, x_3 \geq 55$

$|U \setminus S| = |A_1 \cup A_2 \cup A_3|$

By (3) $|A_1| = \binom{61}{2}$, by (4) $|A_2| = \binom{51}{2}$, by (5) $|A_3| = \binom{46}{2}$. By the theorem used in part i

$$|U| = \binom{99+3-1}{3-1} = \binom{101}{2}$$

By (6) $|A_1 \cap A_2| = \binom{11}{2}$. By the theorem used in part i

$$|A_1 \cap A_3| = \binom{4+3-1}{3-1} = \binom{6}{2}$$

For $A_2 \cap A_3$

$$\begin{aligned} x_1 + 50 + y_2 + 55 + y_3 &= 99 \\ x_1 + y_2 + y_3 &= -6 \end{aligned}$$

Since $n$ is negative in the equation there are no solutions, so $|A_2 \cap A_3| = 0$. $|A_1 \cap A_2 \cap A_3| = 0$ as well by the same logic.

By the principle of inclusion/exclusion

$$\begin{aligned} |A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_2| + |A_1 \cap A_2 \cap A_3| \\ |A_1 \cup A_2 \cup A_3| &= \binom{61}{2} + \binom{51}{2} + \binom{46}{2} - \binom{11}{2} - \binom{6}{2} - 0 + 0 \end{aligned}$$

By the complement rule

$$\begin{aligned} |S| &= |U| - |U \setminus S| \\ |S| &= \binom{101}{2} - \left(\binom{61}{2} + \binom{51}{2} + \binom{46}{2} - \binom{11}{2} - \binom{6}{2}\right) \\ |S| &= 980 \end{aligned}$$

Therefore, the number pf solutions to $x_1 + x_2 + x_3 = 99$, where $0 \leq x_1 \leq 39, 0 \leq x_2 \leq 49, 0 \leq x_3 \leq 54$ is 980.

## Question 7.

i) The function $f$ is defined as a bijection $f : S \to S$. So for each element $y \in S$ there is exactly one element $x \in S$ such that $f(x) = y$. So if the function $f$ is continuously applied to the value received the previous time it was applied, say $m - 1$ times, you will eventually arrive at the one value of $S$ that the function will map to $x$. Therefore, the $m^{th}$ time you apply the function, or $f^m(x)$, it will produce the value $x$.

ii) $S$ is a set with $n$ elements and every cycle of some element $x \in S$ contains some number of elements of $S$. Because the function used to generate cycles is a bijection (and therefore, one-to-one), each element of $S$ can only appear in one cycle in a cycle decomposition. That means that if there is a cycle of $k$ elements where $k > n/2$, then there will be less than $n/2$ elements remaining to make up the remaining cycles. Therefore, it is impossible to have more than one cycle of length $k$ in the same cycle decomposition.

iii) Let $k$ be an integer with $k > n/2$ and let $A$ be the set of permutations of $S$ whose cycle decomposition contains a cycle of length $k$.

Steps to generate an element of $A$:

1. Choose the $k$ elements of $S$ that will be in the $k$ length cycle $\quad N_1 = \binom{n}{k}$
2. Write down the smallest of the $k$ elements as the first number in the cycle $\quad N_2 = 1$
3. Choose an order for the remaining $k - 1$ elements and write them down in the cycle $\quad N_3 = (k - 1)!$
4. Permute the remaining $n - k$ elements in some way and write them down in the cycle decomposition. $N_4 = (n - k)!$

Therefore, by the product rule

$$|A| = \binom{n}{k} \times (k - 1)! \times (n - k)!$$

iv) Let $A$ be the same set described in the previous part.

$$|A| = \binom{n}{k} \times (k - 1)! \times (n - k)!$$

$$|A| = \frac{n!}{\cancel{(n-k)!}k!} \times (k - 1)! \times \cancel{(n - k)!}$$

$$|A| = \frac{n!(k - 1)!}{k!}$$

$$|A| = \frac{n!\cancel{(k - 1)!}}{k\cancel{(k - 1)!}}$$

$$|A| = \frac{n!}{k}$$

## Question 8.

i) Let $A$ be the set of permutations of $S$ that have at least one cool index.
Let $U$ be the set of all permutations of $S$.
So $U \setminus A$ is the set of all permutations of $S$ that have 0 cool indices.

By the definition of a permutation $|U| = n!$. The only permutation of $S$ that has 0 cool indices is the permutation where $a_1 = 1, a_2 = 2, \ldots, a_n = n$. So $|U \setminus A| = 1$. Therefore, by the complement rule

$$|A| = n! - 1$$

ii) Assume index 1 is cool, so $a_1 \neq 1$ in the permutation. Since every element of $S$ must appear exactly once in the permutation there must be some index $i$, where $1 < i \leq n$, for which $a_i = 1$. This will make index $i$ a cool index. Index $i$ will always come after index 1 because, by the problem definition, index 1 is the first index. Therefore, index 1 can never be super cool.

iii) Let $B_k$ be the set of permutations of $S$ where index $k+1$ is super cool, where $1 \leq k \leq n-1$.

Steps to generate an element of $B_k$:

1. Choose a value for $a_{k+1}$ from the set $\{1, 2, \ldots, k\}$ $\quad N_1 = k$
2. Write down a permutation of the elements of the set $\{1, 2, \ldots, k+1\} \setminus a_{k+1}$ $\quad N_2 = k!$
3. Write down the value you chose for $a_{k+1}$ $\quad N_3 = 1$
4. For $i$ in range $(k+2, k+3, \ldots, n)$, write down $a_i = i$ $\quad N_4 = 1$

Therefore, by the product rule

$$|B_k| = k \times k!$$

iv) Let $C$ be the set of all permutations of $S$ that contain at least 1 cool index.
As shown in part i, $|C| = n! - 1$.

Another way to count the size of $C$ is to sum up the sizes all the subsets of $C$, where each subset contains all the permutations of $S$ where index $k+1$ is super cool. And to do this for every valid value of $k$ i.e. each $k \in \{1, 2, \ldots, n-1\}$, call this subset $C_k$. We have seen in part ii that $|C_k| = k \times k!$. $C_1, C_2 \ldots, C_{n-1}$ are all pairwise disjoint sets, so, by the sum rule

$$|C| = |C_1| + |C_2| + \cdots + |C_{n-1}|$$

$$|C| = \sum_{k=1}^{n-1} k \times k!$$

Since $|C|$ is shown to be equal to $n! - 1$ and $\sum_{k=1}^{n-1} k \times k!$, therefore

$$\sum_{k=1}^{n-1} k \times k! = n! - 1$$

**Question 9.**

i) Since the minimum value in $X$ must be equal to $k$ that means $k$ must be an element of $S$, so $0 \leq k \leq 3n$. In order for $|X| = 2n + 1$ there must be at least $2n$ elements in $S$ that are greater than $k$, i.e. $k \leq n$. Therefore, by these 2 conditions, $0 \leq k \leq n$.

ii) Let $N_k$ be a set as defined in the question where $k$ is an integer such that $0 \leq k \leq n$.
Steps to generate an element of $N_k$:

1. Add the value of $k$ to the set $X$ $\quad N_1 = 1$
2. Choose $2n$ values from the set $\{k+1, k+2, \ldots, 3n\}$ and add them to the set $X$ $\quad N_2 = \binom{3n-k}{2n}$

Therefore, by the product rule

$$|N_k| = \binom{3n-k}{2n}$$

iii) Let $N$ be the set of all subsets of $S$ that contain $2n+1$ elements.

One way to count the size of $N$ is to use the sum rule to sum up the sizes of each $N_k$. By doing this we get

$$|N| = \sum_{k=0}^{n} \binom{3n-k}{2n}$$

Values of $k > n$ can be excluded because as explained in part i: $N_k = 0$ when $k > n$.

Another way to count the size of $N$ is to use a binomial coefficient. We want to choose $2n+1$ elements from a set of size $3n+1$, the number of ways to do this is $\binom{3n+1}{2n+1}$. This can be simplified using theorem 3.7.1 from the textbook, which states that

$$\binom{n}{k} = \binom{n}{n-k}$$

for any integers n and k with $0 \leq k \leq n$. So

$$|N| = \binom{3n+1}{2n+1} = \binom{3n+1}{(3n+1)-(2n+1)} = \binom{3n+1}{n}$$
$$|N| = \binom{3n+1}{n}$$

Since $|N|$ is shown to be equal to $\sum_{k=0}^{n} \binom{3n-k}{2n}$ and $\binom{3n+1}{n}$, therefore

$$\sum_{k=0}^{n} \binom{3n-k}{2n} = \binom{3n+1}{n}$$