Braeden Hall
101~~M~~3403

Q1.

1) Let X = "# of rolls until you see 4"

Let X_1 = the value of the first roll

$$E(X) = \sum_{i=1}^{6} E(X \mid X_1 = i) \cdot P_r(X_i = i)$$
$$= \frac{1}{6} E(X \mid X_1 = 4) + \frac{1}{6}\sum_{i=1}^{3} E(X \mid X_1 = i) + \frac{1}{6}\sum_{i=5}^{6} E(X \mid X_1 = i)$$
$$= \frac{1}{6} \cdot 1 + \frac{1}{6}(3(E(X)+1)) + \frac{1}{6}(2(E(X)+1))$$
$$= \frac{1}{6} + \frac{5E(X)}{6} + \frac{5}{6}$$
$$\frac{E(X)}{6} = 1$$
$$E(X) = 6$$ $\therefore$ expected rolls until 4 is 6.

2) Let Y = "number of time you see 4 in n rolls"

Let $Y_i = \begin{cases} 1 & \text{if roll } i \text{ is a } 4 \\ 0 & \text{otherwise} \end{cases}$ for $\forall i \in \{1, \ldots, n\}$

$$E(Y) = E\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} E(Y_i) = \sum_{i=1}^{n} P_r(Y_i = 1) = \sum_{i=1}^{n} \frac{1}{6} = \frac{n}{6}$$

$\therefore$ expected # of 4 in n rolls is $\frac{n}{6}$

$P_r(Y_i = 1) = \frac{1}{6}$

3) Let event A_i = "the i^{th} roll is 4"   Let event A = "you see no 4's in n rolls"

$P_r(A_i) = \frac{1}{6}$ $P_r(\bar{A}_i) = 1 - \frac{1}{6} = \frac{5}{6}$

Note: $A_i \cup A_j = \emptyset$ for $\forall i \neq j$

$$A = \bar{A}_1 \cup \bar{A}_2 \cup \cdots \cup \bar{A}_n$$
$$P_r(A) = P_r(\bar{A}_1) + P_r(\bar{A}_2) + \cdots + P_r(\bar{A}_n)$$
$$= n\left(\frac{5}{6}\right)$$
$$= \frac{5n}{6}$$

Q1 4) $X=$ "number of 4 you see"

$$Pr\left(X \geq \frac{n}{3}\right) \leq \frac{E(X)}{n/3} = \frac{n/6}{n/3} = \frac{\cancel{n}}{\cancel{6}_2} \cdot \frac{\cancel{3}}{\cancel{n}} = \frac{1}{2}$$

$$\leq \frac{1}{2}$$

Q2 1) $\pi(n)$ cannot be a peak because $\pi(n+1)$ is not defined. Therefore $\pi(n)$ cannot be bigger than $\pi(n+1)$.

$Pr(\text{"}\pi(n) \text{ is a peak"}) = 0$

2) Define $A=$ "$\pi(6)$ is a peak"
Define $B=$ "$\pi(6)$ is a record"
Define $C=$ "$\pi(7)$ is a record"

$$\begin{aligned} Pr(A) &= Pr(B \cap \bar{C}) \\ &= Pr(B) \cdot Pr(\bar{C}) \\ &= \frac{1}{6} \cdot \left(1 - \frac{1}{7}\right) \\ &= \frac{1}{\cancel{6}} \cdot \frac{\cancel{6}}{7} \\ &= \frac{1}{7} \end{aligned}$$

we have seen in class the probability the i^{th} index of a list is a record is equal to $\frac{1}{i}$

3) If $\pi(7)$ is a peak, then $\pi(7) > \pi(8)$. This means $\pi(8)$ cannot be a peak.

So $Pr(\text{"}\pi(8) \text{ is a peak"} \mid \text{"}\pi(7) \text{ is a peak"}) = 0$

4) Define r.v. $X=$ "# of peaks in random perm"

Define r.v. $X_i = \begin{cases} 1 & \text{if } \pi(i) \text{ is a peak} \\ 0 & \text{otherwise} \end{cases}$

$Pr(X_i = 1) = \frac{1}{i+1}$ (from pt 2)

$$\begin{aligned} E(X) &= \sum_{i=2}^{n-1} E(X_i) = \sum_{i=2}^{n-1} \frac{1}{i+1} = \sum_{i=1}^{n-1} \frac{1}{i+1} - \frac{1}{2} \\ &= \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} \\ &= H_{n-1} - 1 - \frac{1}{2} \\ &= H_{n-1} - \frac{3}{2} \end{aligned}$$

Q3

1) Define r.v. X = "# of element in box n"

Define $X_i = \begin{cases} 1 & \text{if element } i \text{ is placed in box } n \\ 0 & \text{otherwise} \end{cases}$

$Pr(X_i = 1) = \frac{1}{n}$

$E(X) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \frac{1}{n} = \frac{n}{n}$

$= 1$

2) Use X_i from pt 1

Skipped

3) Define r.v. Y = "# of empty boxes in A"

Define $Y_i = \begin{cases} 1 & \text{if box } i \text{ is empty} \\ 0 & \text{otherwise} \end{cases}$

$Pr(Y_i = 1) = 1/e$ from pt 2

$E(Y) = \sum_{i=1}^{n} E(Y_i) = \sum_{i=1}^{n} \frac{1}{e} = \frac{n}{e}$

$= \frac{n}{e}$

Q4

1) $p = Pr(\text{"v is black"} \cap \text{"v's ccw neighbour is white"})$
$= \frac{1}{2} \cdot \frac{1}{2}$
$= \frac{1}{4}$

2) Define r.v. X = "# of good points"
Define $X_i = \begin{cases} 1 & \text{if point } i \text{ is good} \\ 0 & \text{otherwise} \end{cases}$ for $\forall i \in \{1, \ldots, n\}$

$Pr(X_i = 1) = \frac{1}{4}$ from pt 1

$E(X) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \frac{1}{4} = \frac{n}{4}$
$= \frac{n}{4}$

3) Define event A_i = "v is unmarked in iteration i"

$Pr(A_i) = 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$

Define event A = "v is unmarked after k iterations"

$Pr(A) = Pr(A_1 \cap A_2 \cap \ldots \cap A_k)$
$= Pr(A_1) \cdot Pr(A_2) \cdot \ldots \cdot Pr(A_k)$
$= \left(\frac{3}{4}\right)^k$

4) Define r.v. Y = "# of unmarked points after k iterations"
Define $Y_i = \begin{cases} 1 & \text{if point } i \text{ is unmarked after k iterations} \\ 0 & \text{otherwise} \end{cases}$

$E(Y) = \sum_{i=1}^{n} E(Y_i) = \sum_{i=1}^{n} \left(\frac{3}{4}\right)^k$
$= n\left(\frac{3}{4}\right)^k$

Q5

1) $Pr(\text{"correct bit in } N_1\text{"}) = \frac{1+(1-2(3/100))^3}{2}$ eq'n from A1 Q5.3

$\doteq 0.9152$

$Pr(\text{"correct bit } N_2) = \frac{1+(1-2(99/100))^3}{2}$

$\doteq 0.0294$

$\therefore N_1$ is the more reliable network, when considering 3 edges.

2) $Pr(\text{"correct bit in } N_1\text{"}) = \frac{1+(1-2(3/100))^2}{2}$

$= 0.9418$

$Pr(\text{"correct bit in } N_2\text{"}) = \frac{1+(1-2(99/100))^2}{2}$

$= 0.9802$

$\therefore N_2$ is the more reliable network, when considering 2 edges

3) N_2 is more reliable network for 2 edges because there is a very high probability the bit will flip twice, and if the bit flips twice then the correct bit will arrive at the destination. However, when considering 3 edges, the bit could (with high likelihood) flip the third time. This will result in the incorrect bit arriving. When considering network 1, the bit flips with low probability each edge. This means it is unlikely the bit will flip at all. However, the probability of a bit not flipping in N_2 is lower that the bit flipping in N_1. That is why the probability of a correct bit arriving when considering paths with even #'d edges is higher for N_2, but with odd #'d edges it is higher for N_1.