COMP 2804 Assignment 3

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Question 1.

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Question 2.

For each different letter $\alpha \in \{A - Z\}$, let T_α represent a set containing the tiles of the letter α. For example, the set T_A is equal to the all the tiles with the letter A on them, $|T_A| = 9$. Define the set U that is a union of all the sets of letter tiles, $|U| = 98$.

Define the sample space S

$$S = \{(l_1, l_2, \dots, l_7) : \text{for each } i \in \{1, \dots, 7\}, l_i \in U \setminus \{l_1, \dots, l_{i-1}\}\}$$

$|S| = \binom{98}{7}$ and $Pr(\omega) = \frac{1}{|S|} = \frac{1}{\binom{98}{7}}$

For each event, assume that order the letters are chosen does not matter.
Event 1: $A =$ "scrabble hand contains the word OCTAGON"

$$A = \{(o_1, o_2, c, t, a, g, n) : o_1 \in T_O, o_2 \in T_O \setminus o_1, c \in T_C, t \in T_T, a \in T_A, g \in T_G, n \in T_N\}$$

By the product rule, $|A| = \binom{8}{2} \times 2 \times 6 \times 9 \times 3 \times 6$.

$$\begin{aligned} Pr(A) &= \frac{|A|}{|U|} \\ &= \frac{\binom{8}{2} \times 2 \times 6 \times 9 \times 3 \times 6}{\binom{98}{7}} \\ &\approx 3.93 \times 10^{-6} \end{aligned}$$

Therefore the probability that a hand contains the word OCTAGON is approximately equal to 3.93×10^{-6}
Event 2: $B =$ "scrabble hand contains the word DOODLES"

$$B = \{(o_1, o_2, d_1, d_2, l, e, s) : o_1 \in T_O, o_2 \in T_O \setminus o_1, d_1 \in T_D, d_2 \in T_D \setminus d_1, l \in T_L, e \in T_E, s \in T_S\}$$

By the product rule, $|B| = \binom{8}{2} \times \binom{4}{2} \times 4 \times 12 \times 4$.

$$\begin{aligned} Pr(B) &= \frac{|B|}{|U|} \\ &= \frac{\binom{8}{2} \times \binom{4}{2} \times 4 \times 12 \times 4}{\binom{98}{7}} \\ &\approx 2.33 \times 10^{-6} \end{aligned}$$

Therefore the probability that a hand contains the word DOODLES is approximately equal to 2.33×10^{-6}
Event 3: $C =$ "scrabble hand contains the word SMOKO"

$C_1 =$ "hand is SMOKOOO"
$C_2 =$ "hand is SMOKOOM"
$C_3 =$ "hand is SMOKOOS"
$C_4 =$ "hand is SMOKOOα"
$C_5 =$ "hand is SMOKOSS"
$C_6 =$ "hand is SMOKOSM"
$C_7 =$ "hand is SMOKOSα"
$C_8 =$ "hand is SMOKOMα"
$C_9 =$ "hand is SMOKO$\alpha\beta$"
(where α and β are not Ms S' or Os)
$C = C_1 \cup C_2 \cup \cdots \cup C_9$

$$\begin{aligned}
C_1 &= \{(o_1, o_2, m, k, s, o_3, o_4) : o_1 \in T_O, o_2 \in T_O \setminus o_1, o_3 \in T_O \setminus \{o_1, o_2\},\\
&\quad o_4 \in T_O \setminus \{o_1, o_2, o_3\}, m \in T_M, k \in T_K, s \in T_S\}\\
C_2 &= \{(o_1, o_2, m_1, k, s, o_3, m_2) : o_1 \in T_O, o_2 \in T_O \setminus o_1, o_3 \in T_O \setminus \{o_1, o_2\},\\
&\quad , m_1 \in T_M, m_2 \in T_M \setminus m_1, k \in T_K, s \in T_S\}\\
C_3 &= \{(o_1, o_2, m, k, s_1, o_3, s_2) : o_1 \in T_O, o_2 \in T_O \setminus o_1, o_3 \in T_O \setminus \{o_1, o_2\},\\
&\quad m \in T_M, k \in T_K, s_1 \in T_S, s_2 \in T_S \setminus s_1\}\\
C_4 &= \{(o_1, o_2, m, k, s, o_3, \alpha) : o_1 \in T_O, o_2 \in T_O \setminus o_1, o_3 \in T_O \setminus \{o_1, o_2\},\\
&\quad m \in T_M, k \in T_K, s \in T_S, \alpha \in U \setminus \{T_O, T_M, T_S, T_K\}\}\\
C_5 &= \{(o_1, o_2, m, k, s_1, s_2, s_3) : o_1 \in T_O, o_2 \in T_O \setminus o_1, m \in T_M, k \in T_K,\\
&\quad s_1 \in T_S, s_2 \in T_S \setminus s_1, s_3 \in T_S \setminus \{s_1, s_2\}\}\\
C_6 &= \{(o_1, o_2, m_1, k, s_1, s_2, m_2) : o_1 \in T_O, o_2 \in T_O \setminus o_1, m_1 \in T_M,\\
&\quad m_2 \in T_M \setminus m_1, k \in T_K, s_1 \in T_S, s_2 \in T_S \setminus s_1\}\\
C_7 &= \{(o_1, o_2, m, k, s_1, s_2, \alpha) : o_1 \in T_O, o_2 \in T_O \setminus o_1, m \in T_M, k \in T_K,\\
&\quad s_1 \in T_S, s_2 \in T_S \setminus s_1, \alpha \in U \setminus \{T_O, T_M, T_S, T_K\}\}\\
C_8 &= \{(o_1, o_2, m_1, m_2, k, s, \alpha) : o_1 \in T_O, o_2 \in T_O \setminus o_1, m_1 \in T_M,\\
&\quad m_2 \in T_M \setminus m_1, k \in T_K, s_1 \in T_S, \alpha \in U \setminus \{T_O, T_M, T_S, T_K\}\}\\
C_9 &= \{(o_1, o_2, m, k, s, \alpha_1, \alpha_2) : o_1 \in T_O, o_2 \in T_O \setminus o_1, m \in T_M, k \in T_K,\\
&\quad s \in T_S, \alpha_1 \in U \setminus \{T_O, T_M, T_S, T_K\}, \alpha_2 \in U \setminus \{T_O, T_M, T_S, T_K, \alpha_1\}\}
\end{aligned}$$

By the product rule, $|C_1| = \binom{8}{4} \times 2 \times 1 \times 4$, $|C_2| = \binom{8}{3} \times \binom{2}{2} \times 1 \times 4$, $|C_3| = \binom{8}{3} \times 2 \times 1 \times \binom{4}{2}$, $|C_4| = \binom{8}{3} \times 2 \times 1 \times 4 \times (98 - 15)$, $|C_5| = \binom{8}{2} \times 2 \times 1 \times \binom{4}{3}$, $|C_6| = \binom{8}{2} \times \binom{2}{2} \times 1 \times \binom{4}{2}$, $|C_7| = \binom{8}{2} \times 2 \times 1 \times \binom{4}{2} \times (98 - 15)$, $|C_8| = \binom{8}{2} \times \binom{2}{2} \times 1 \times 4 \times (98 - 15)$, $|C_9| = \binom{8}{2} \times 2 \times 1 \times 4 \times \binom{98-15}{2}$.

$$\begin{aligned}
Pr(C) &= \frac{|C|}{|U|}\\
&= \frac{|C_1| + |C_2| + \cdots + |C_9|}{\binom{98}{7}} && \text{by the sum rule}\\
&= \frac{560 + 224 + 672 + 37184 + 224 + 168 + 27888 + 9296 + 762272}{\binom{98}{7}}\\
&= \frac{838488}{\binom{98}{7}}\\
&\approx 6.06 \times 10^{-5}
\end{aligned}$$

Therefore the probability that a hand contains the word SMOKO is approximately equal to 6.06×10^{-5}

Question 3.

Let M be a set of n students who write the exam

i) $K =$ "Students X and Y write the exact same exam"
Define U_1 as the set of all possible exams

$$U_1 = \{(q_1, a_1, q_2, a_2, \dots, q_{17}, a_{17}) : \text{for each } i \in \{1, \dots, 17\}, q_i \text{ is a unique element of } B,$$
$$a_i \text{ is an ordered subset of } A_{q_i} \text{ containing the correct answer and 3 other unique elements}\}$$

There are $\binom{200}{17}$ ways to choose the different questions, since order matters we must choose an order for the questions. There are 17! different choices. For the answers, the number of ways to get an ordered 4 element subset of a set of size 10 is $\binom{9}{3} \times 4!$, we must do this for all 17 different questions. Therefore,

$$|U_1| = \binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17}$$

Let the sample space S_1 be the set of functions $f : M \to U_1$, $|S_1| = \left(\binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17}\right)^n$.
$n = 2$, so $|S_1| = \left(\binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17}\right)^2$

$$K = \{(x, y) : x, y \in U_1, x = y\}$$

$|K| = |U_1| = \binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17}$, Therefore,

$$\begin{aligned} Pr(K) = \sum_{\omega \in K} Pr(\omega) &= \sum_{\omega \in K} \frac{1}{\left(\binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17}\right)^2} \\ &= \frac{\binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17}}{\left(\binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17}\right)^2} \\ &= \frac{1}{\binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17}} \end{aligned}$$

ii) $L =$ "Students X and Y have exams with exactly the same questions"
Define U_2 as a set of exams with the same 17 questions

$$U_2 = \{(q_1, q_2, \dots, q_{17}) : \text{for each } i \in \{1, \dots, 17\}, q_i \in B \setminus \{q_1, \dots, q_{i-1}\}\}$$

$$|U_2| = \binom{200}{17}$$

Let the sample space S_2 be the set of functions $f : M \to U_2$, $|S_2| = \binom{200}{17}^n$
$n = 2$ so $|S_2| = \binom{200}{17}^2$

$$L = \{(x, y) : x, y \in U_2, x = y\}$$

$|L| = |U_2| = \binom{200}{17}$, Therefore,

$$\begin{aligned} Pr(L) &= \sum_{\omega \in L} Pr(\omega) = \sum_{\omega \in L} \frac{1}{\binom{200}{17}^2} \\ &= \frac{\binom{200}{17}}{\binom{200}{17}^2} \\ &= \frac{1}{\binom{200}{17}} \end{aligned}$$

iii) Let the sample space S_3 be the set of all 17 question exams, $|S_3| = \binom{200}{17}$

$J =$ "Students X and Y write exams with at least one of the same questions"
$\overline{J} =$ "Students X's exam has completely different questions than student Y's"
In other words, $\overline{J}$ is the set of exams student X can write after student Y's (e_y) questions have been chosen

$$\overline{J} = \{(e_x) : e_x \in B \setminus e_y\}$$

$|\overline{J}| = \binom{200-17}{17} = \binom{183}{17}$.
By the complement rule,

$$\begin{aligned} Pr(J) &= 1 - Pr(\overline{J}) \\ &= 1 - \frac{|\overline{J}|}{|S_3|} \\ &= 1 - \frac{\binom{183}{17}}{\binom{200}{17}} \\ &\approx 1 - (0.2066) \\ &\approx 0.793 \end{aligned}$$

iv) Let U_4 be the set of all possible exams

$$|U_4| = \binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17}$$

The sample space S_4 is the same as the sample space from part i, except $n = 500$.

$$|S_4| = \left(\binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17}\right)^{500}$$

$N =$ "Student X can be uniquely identified"

$$N = \{(e_1, \ldots, e_x, \ldots, e_{500}) : e_x \in U_4, e_1, \ldots, e_{x-1}, e_{x+1}, \ldots, e_{500} \in U_4 \setminus e_x\}$$

$|N| = \left(\binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17}\right) \times \left(\binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17} - 1\right)^{499}$

$$
\begin{aligned}
Pr(N) &= \frac{\left(\binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17}\right) \times \left(\binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17} - 1\right)^{499}}{\left(\binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17}\right)^{500}} \\
&= \frac{\left(\binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17}\right) \times \left(\binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17}\right)^{499} - \left(2\left(\binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17}\right)^{498} - \cdots + 1\right)}{\left(\binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17}\right)^{500}} \\
&= \frac{\left(\binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17}\right)^{500} - \left(2\left(\binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17}\right)^{498} - \cdots + 1\right)}{\left(\binom{200}{17} \times 17! \times \left(\binom{9}{3} \times 4!\right)^{17}\right)^{500}}
\end{aligned}
$$

Therefore, the probability that no other student received an exam the same as student X is close to 1.

Question 4.

Let U_B be the set of black bathing suits.
Let U_R be the set of red bathing suits.
$U = U_B \cup U_R$.
i)

$$S = \{(c_1, c_2) : c_1 \in U, c_2 \in U \setminus c_1\}$$

ii) $|S| = 100 \times 99 = 9900$, $Pr(\omega) = \frac{1}{|S|} = \frac{1}{9900}$

$$A = \{(b, c_2) : b \in U_B, c_2 \in U \setminus b\}$$
$$B = \{(c_1, r) : (c_1 \in U_B, r \in U_R) \text{ or } (c_1 \in U_R, r \in U_R \setminus c_1)$$

$|A| = 90 \times 99 = 8910$, $|B| = (90 \times 10) + (10 \times 9) = 990$
iii)

$$A \cap B = \{(b, r) : b \in U_B, r \in U_R\}$$

$|A \cap B| = 90 \times 10$

$$
\begin{aligned}
Pr(A \cap B) &= \frac{|A \cap B|}{|S|} \\
&= \frac{900}{9900} \\
&= \frac{1}{11}
\end{aligned}
$$

iv) A and B are not disjoint sets, so by the principle of inclusion exclusion,

$$\begin{aligned} Pr(A \cup B) &= Pr(A) + Pr(B) - Pr(A \cap B) \\ &= \frac{|A| + |B| - |A \cap B|}{|S|} \\ &= \frac{8910 + 990 - 900}{9900} \\ &= \frac{9000}{9900} \\ &= \frac{10}{11} \end{aligned}$$

v)

$$\begin{aligned} Pr(A) \times Pr(B) &= \frac{8910}{9900} \times \frac{990}{9900} \\ &= \frac{9}{10} \\ &\neq \frac{10}{11} = Pr(A \cap B) \end{aligned}$$

Therefore, events A and B are not independent.

Question 5.

Game A: Define sample space S_A

$$S_A = \{(r_1, r_2, \ldots, r_8) : r_1, r_2, \ldots, r_8 \in \{1, 2, \ldots, 8\}\}$$

$|S_A| = 8^8$, $Pr(\omega) = \frac{1}{8^8}$

A = "You roll at least one 8"
$\overline{A}$ = "You roll no 8s"

$$\overline{A} = \{(r_1, r_2, \ldots, r_8) : r_1, r_2, \ldots, r_8 \in \{1, 2, \ldots, 7\}\}$$

$|\overline{A}| = 7^8$

$$\begin{aligned} Pr(\overline{A}) &= \frac{|\overline{A}|}{8^8} \\ &= \frac{7^8}{8^8} \end{aligned}$$

By the complement rule,

$$\begin{aligned} Pr(A) &= 1 - Pr(\overline{A}) \\ &= 1 - \frac{7^8}{8^8} \\ &\approx 1 - 0.3436 \\ &\approx 0.656 \end{aligned}$$

Therefore, the probability you roll at least one 8 is 0.656.

Game B: Define sample space S_B

$$S_B = \{(r_1, r_2, \ldots, r_{24}) : r_1, r_2, \ldots, r_{24} \in \{1, 2, \ldots, 8\}\}$$

$|S_B| = 8^{24}$, $Pr(\omega) = \frac{1}{8^{24}}$
B = "You roll at least 3 8s"
$\overline{B}$ = "You roll less than 3 8s"
By the sum rule,

$$\begin{aligned} Pr(\overline{B}) &= \frac{|\overline{B}|}{8^{24}} \\ &= \frac{\text{"You roll exactly 0 8s"} + \text{"You roll exactly 1 8"} + \text{"You roll exactly 2 8s"}}{8^{24}} \\ &= \frac{7^{24} + (24 \times 7^{23}) + \left(\binom{24}{2} \times 7^{22}\right)}{8^{24}} \end{aligned}$$

And by the complement rule,

$$\begin{aligned} Pr(B) &= 1 - Pr(\overline{B}) \\ &= 1 - \left(\frac{7^{24} + (24 \times 7^{23}) + (\binom{24}{2} \times 7^{22})}{8^{24}}\right) \\ &\approx 1 - 0.4081 \\ &\approx 0.592 \end{aligned}$$

Therefore, the probability you roll at least three 8s is 0.592.

Question 6.

Define the sample space S,

$$S = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \{1, \ldots, n\}\}$$

$|S| = n^n$, so $Pr(\omega) = \frac{1}{n^n}$
i) A = "$x_1 = n$"

$$A = \{(x_1, x_2, \ldots, x_n) : x_1 = n, x_2, \ldots, x_n \in \{1, \ldots, n\}\}$$

$|A| = n^{n-1}$

$$\begin{aligned} Pr(A) &= \frac{n^{n-1}}{n^n} \\ &= \frac{1}{n} \end{aligned}$$

ii)B = "$max\{x_1, \ldots, x_n\} = n$"
$\overline{B}$ = "n is not in $\{x_1, \ldots, x_n\}$"

$$\overline{B} = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in \{1, \ldots, n-1\}\}$$

$|\overline{B}| = (n-1)^n$

$$\begin{aligned} Pr(B) &= 1 - Pr(\overline{B}) \\ &= 1 - \frac{|\overline{B}|}{n^n} \\ &= 1 - \frac{(n-1)^n}{n^n} \\ &= \frac{n^n - (n-1)^n}{n^n} \end{aligned}$$

iii) $C =$ "$x_1 = max\{x_1, \ldots, x_n\}$"

$$C = \{(x_1, \ldots, x_n) : \text{for each } i \in \{1, \ldots, n\}, x_1 = i, x_2, \ldots, x_n \in \{1, \ldots, i\}\}$$

You have no option for x_1 and you have i options for all $x_2, \ldots, x_n$, so for each i the number of elements in C is i^{n-1}. Therefore, $|C| = \sum_{i=1}^{n} i^{n-1}$

$$Pr(C) = \frac{\sum_{i=1}^{n} i^{n-1}}{n^n}$$

iv) $D =$ "for each $i \in \{1, \ldots, n\}, max\{x_1, \ldots, x_n\} = i$"
In other words, D is the set of all sequences $x_1, \ldots, x_n \in \{1, \ldots, i\} \backslash$ sequences $x_1, \ldots, x_n \in \{1, \ldots, i-1\}$.
$|D| = i^n - (i-1)^n$

$$Pr(D) = \frac{i^n - (i-1)^n}{n^n}$$

Question 7.

Define the event $A =$ "GetBiasedBit returns 1"
This event happens if and only if $p_k = 1$.
The value of k depends on the number of coin flips required to get the first heads, so if we define the sample space S:

$$S = \{(T^{i-1}H) : i \geq 1\}$$

we can calculate that for each $\omega \in S$, $Pr(\omega) = \frac{1}{2^i}$, as seen in class. This gives us the probability that k takes on each value i, $Pr(k = i) = \frac{1}{2^i}$.
Since we assume that we know the value the bit p_k for each $k \geq 1$ and we have the value i from the number of coin flips, we just need calculate the probability $Pr(p_k = 1 | k = i)$. Given that p_k only ever takes on two values, 0 or 1, this probability is equal to p_k.
To conclude, $Pr(A) = Pr(p_k = 1 | k = i) \times Pr(k = i)$ for each value $i \geq 1$. Represented mathematically this

is:

$$\begin{aligned}
Pr(A) &= \sum_{i=1}^{\infty} Pr(p_k = 1|k = i) \times Pr(k = i) \\
&= \sum_{i=1}^{\infty} p_i \times \frac{1}{2^i} \\
&= \sum_{i=1}^{\infty} \frac{p_i}{2^i} \\
&= p
\end{aligned}$$

As noted in the question, this sum is equal to p. Therefore, the function GetBiasedBit returns 1 with probability p.

Question 8.

i) Let $P_k(x) = (1 + x + x^2 + x^3 \dots)$ for each $1 \leq k \leq n+1$

$$G_n(x) = P_1(x) \times P_2(x) \times P_3(x) \times \cdots \times P_{n+1}(x)$$

When expanding the equation $G_n(x)$, any term x^m in the final equation results from multiplying some x term in every $P_k(x)$ together.
Let q_k be the degree of the x term you take from $P_k(x)$.
Since we are looking for the coefficient of the m^{th} term in the expansion of $G_k(x)$,

$$q_1 + q_2 + q_3 + \cdots + q_{n+1} = m \tag{1}$$

$$q_k \geq 0$$

Each sum of q_1 to q_{n+1} that is equal to m adds one to the coefficient of x^m. So, counting the number of solutions to (1) is equivalent to the coefficient x^m. By Theorem 3.9.1 in the textbook the number of solutions to (1) is

$$\begin{aligned}
&\binom{(m + (n+1) - 1)}{(n+1) - 1} \\
=&\binom{m+n}{n}
\end{aligned}$$

Therefore, the coefficient of x^m is $\binom{m+n}{n}$.

ii) By part i, we know that the coefficient of any term x^m in the expansion of $G_n(x)$ is $\binom{m+n}{n}$, so it follows that

$$G_n(x) = \sum_{m=0}^{\infty} \binom{m+n}{n} x^m$$

$G_n(x)$ is also infinite geometric sum, and we have seen in class that they can be represented as the following,

$$G_n(x) = \left(\lim_{N \to \infty} \frac{1 - x^{N+1}}{1 - x} \right)^{n+1}$$

Since $0 \leq x \leq 1$, this simplifies to $\frac{1}{(1-x)^{n+1}}$ as $N \to \infty$.

As we have seen that $G_n(x)$ can be represented in these 2 forms, it is suffice to say that

$$\frac{1}{(1-x)^{n+1}} = \sum_{m=0}^{\infty} \binom{m+n}{n} x^m$$

iii) If we multiply the left side of the the equation in part ii, we get the right side of the equation in part iii,

$$\begin{aligned}
\frac{x^n}{(1-x)^{n+1}} &= x^n \times \frac{1}{(1-x)^{n+1}} \\
&= x^n \times \sum_{m=0}^{\infty} \binom{m+n}{n} x^m \\
&= \sum_{m=0}^{\infty} x^n \times \binom{m+n}{n} x^m \\
&= \sum_{m=0}^{\infty} \binom{m+n}{n} x^{m+n}
\end{aligned}$$

This final summation can be changed by substituting the variable $k = m + n$ in all places where m appears. So, $\sum_{k=n}^{\infty} \binom{k}{n} x^k$ Therefore, it has been shown that

$$\sum_{k=n}^{\infty} \binom{k}{n} x^k = \frac{x^n}{(1-x)^{n+1}}$$