Market impact with multi-timescale liquidity

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Introduction to LLOB model

Introduction to LLOB model

The latent volume densities of limit orders in the order book $\phi_{\mathrm{b}}(x,t)$ (bid side) and $\phi_{\mathrm{a}}(x,t)$ (ask side) follows

$$\begin{aligned}\partial_t \phi_{\mathrm{b}} &= D\partial_{xx}\phi_{\mathrm{b}} - \nu\phi_{\mathrm{b}} + \lambda\Theta(x_t - x) - R_{\mathrm{ab}}(x) \\ \partial_t \phi_{\mathrm{a}} &= D\partial_{xx}\phi_{\mathrm{a}} - \nu\phi_{\mathrm{a}} + \lambda\Theta(x - x_t) - R_{\mathrm{ab}}(x)\end{aligned} \tag{1}$$

The price x_t is defined as the solution of

$$\begin{aligned}\phi(x_t,t) &= \phi_{\mathrm{b}}(x,t) - \phi_{\mathrm{a}}(x,t) = 0 \\ \partial_t \phi &= D\partial_{xx}\phi - \nu\phi + \lambda \mathrm{sign}(x_t - x)\end{aligned} \tag{2}$$

The stationary order book is given by

$$\phi^{\text{st}}(\xi) = -\frac{\lambda}{\nu}\text{sign}(\xi)\left[1 - \exp\left(-\frac{|\xi|}{\xi_c}\right)\right] \tag{3}$$

where $\xi_c = \sqrt{D\nu^{-1}}$ and $\xi = x - x_t$.

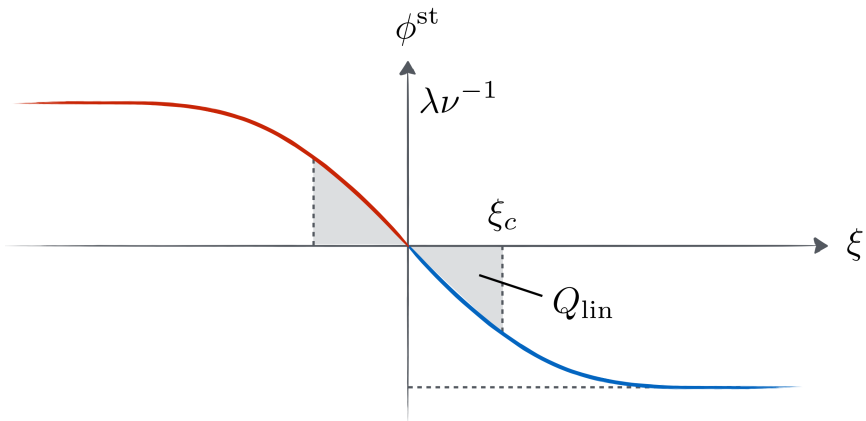

Figure 1: Stationary order book under LLOB model.

Deficiencies of LLOB model

- The original LLOB model focused on the infinite memory limit, namely $\nu, \lambda \to 0$ while keeping $\mathcal{L} \sim \lambda\nu^{-1/2}$ constant, such that the latent order book becomes exactly linear since in that limit $\xi_c \to \infty$; while in reality we are facing finite memory.
- A strict square-root law is only recovered in the limit where the execution rate m_0 of the meta-order is larger than the normal execution rate J of the market itself; whereas most meta-order impact data is in the opposite limit $m_0 \lesssim 0.1J$.
- The theoretical inverse square-root impact decay is too fast and leads to significant short time mean-reversion effects, not observed in real prices.

Price and impact profile under finite memory

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Price and impact profile under finite memory

We introduce a buy meta-order as the source of market impact with intensity rate m_t, then equation (2) becomes

$$\partial_t \phi = D\partial_{xx}\phi - \nu\phi + \lambda \mathrm{sign}(x_t - x) + m_t \delta(x - x_t)\mathbb{1}_{[0,T]} \tag{4}$$

Focusing on constant participation rates $m_t = m_0$, we consider

- Small participation rate $m_0 \ll J$ *v.s.* large participation rate $m_0 \gg J$.
- Fast execution $\nu T \ll 1$ *v.s.* slow execution $\nu T \gg 1$.
- Small meta-order volumes $Q = m_0 T \ll Q_{\text{lin}}$ *v.s.* large meta-order volumes $Q = m_0 T \gg Q_{\text{lin}}$.

Price trajectories with finite cancel and deposit rates

For fast execution and small meta-order volumes, we have the price trajectory as

$$x_t = \alpha \left[z_t^0 + \sqrt{\nu} z_t^1 + \mathcal{O}(\nu) \right] \tag{5}$$

While for slow execution and/or large meta-order volumes,

$$x_t = \frac{m_0 \nu}{\lambda} t \tag{6}$$

Price trajectories with finite cancel and deposit rates

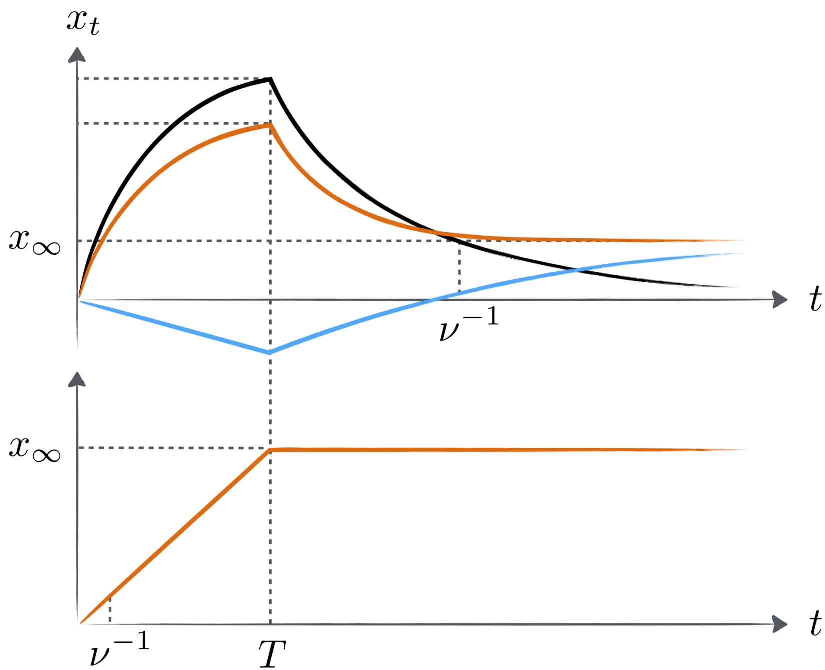

Figure 2: Top graph: Price trajectory during and after a buy meta-order execution for $\nu T \ll 1$. Bottom graph: Price trajectory for $\nu T \gg 1$.

Linear permanent impact with finite memory

We find that the permanent impact I_∞ follows

$$I_\infty = \frac{1}{2}\xi_c \frac{Q}{Q_{\text{lin}}} \tag{7}$$

which is linear in execution volume Q in both small and large participation regime. (see to x_∞ in Figure 2)

The result is dictated by non-arbitrage arguments and compatible with the classical Kyle model.

The double-frequency framework

The double-frequency framework

Consider there are two sorts of agents co-exists in the market:

- Slow agents with vanishing cancellation and deposition rates: $\nu_{\mathrm{s}} T \to 0$, while keeping the corresponding liquidity $\mathcal{L}_{\mathrm{s}} := \lambda_{\mathrm{s}} / \sqrt{\nu_{\mathrm{s}} D}$ finite.
- Fast agents with large cancellation and deposition rates: $\nu_{\mathrm{f}} T \gg 1$, such that $\mathcal{L}_{\mathrm{f}} := \lambda_{\mathrm{f}} / \sqrt{\nu_{\mathrm{f}} D} \gg \mathcal{L}_{\mathrm{s}}$.

LLOB model with fast and slow agents

ϕ_s and ϕ_f follows equation (4) with their own coefficients ν, λ, m_t respectively. With the conditions below,

$$\begin{aligned} m_{\mathrm{s}t} + m_{\mathrm{f}t} &= m_0 \\ x_{\mathrm{s}t} = x_{\mathrm{f}t} &= x_t \end{aligned} \tag{8}$$

Then the total order book volume is given by

$$\phi^{\mathrm{st}}(x) = \phi_\mathrm{s}^{\mathrm{st}}(x) + \phi_\mathrm{f}^{\mathrm{st}}(x) \tag{9}$$

where

$$\begin{aligned} \phi_\mathrm{s}^{\mathrm{st}} &\approx -\mathcal{L}_\mathrm{s} x \\ \phi_\mathrm{f}^{\mathrm{st}} &\approx -\frac{\lambda_\mathrm{f}}{\nu_\mathrm{f}} \mathrm{sign}(x) \end{aligned} \tag{10}$$

LLOB model with fast and slow agents

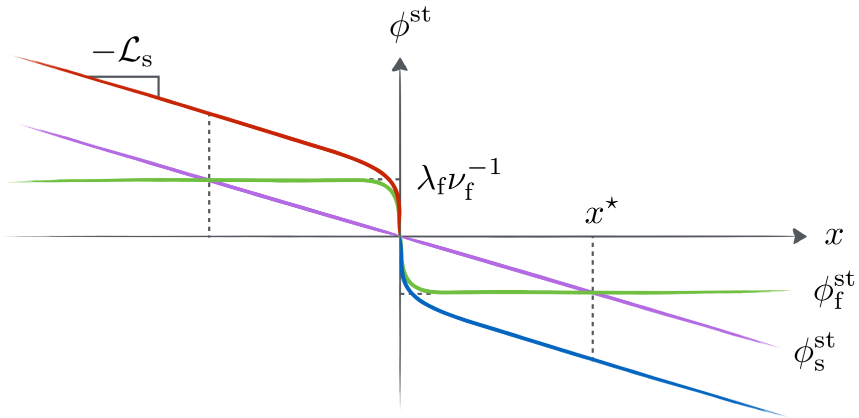

Figure 3: Stationary double-frequency order book.

From linear to square-root impact

Consider the meta-order intensity is large compared to the the average transaction rate of slow traders while small compared to the total transaction rate of the market. That is $J_\mathrm{s} \ll m_0 \ll J_\mathrm{f}$.

Equation (4) and equation (8) yield that

$$m_{\mathrm{f}t} = \frac{m_0}{\sqrt{1+\frac{t}{t^\star}}}, \quad t^\star := \frac{1}{2\nu_\mathrm{f}} \frac{J_\mathrm{f}^2}{J_\mathrm{s} m_0} \\ m_{\mathrm{s}t} = m_0 - m_{\mathrm{f}t} \tag{11}$$

From linear to square-root impact

The resulting price trajectory reads

$$x_t = \frac{\lambda_\mathrm{f}}{\mathcal{L}_\mathrm{s}\nu_\mathrm{f}}\left(\sqrt{1+\frac{t}{t^\star}}-1\right) \tag{12}$$

The most of the incoming meta-order is executed against the rapid agents for $t < t^\star$ but the slow agents then take over for $t > t^\star$.

The result leads to a market impart that crosses over from a linear regime when $t \ll t^\star$ to a square root regime for $t \gg t^\star$.

The impact decay behaves asymptotically ($t \gg T$) to zero as $x_t \sim t^{-1/2}$.

From linear to square-root impact

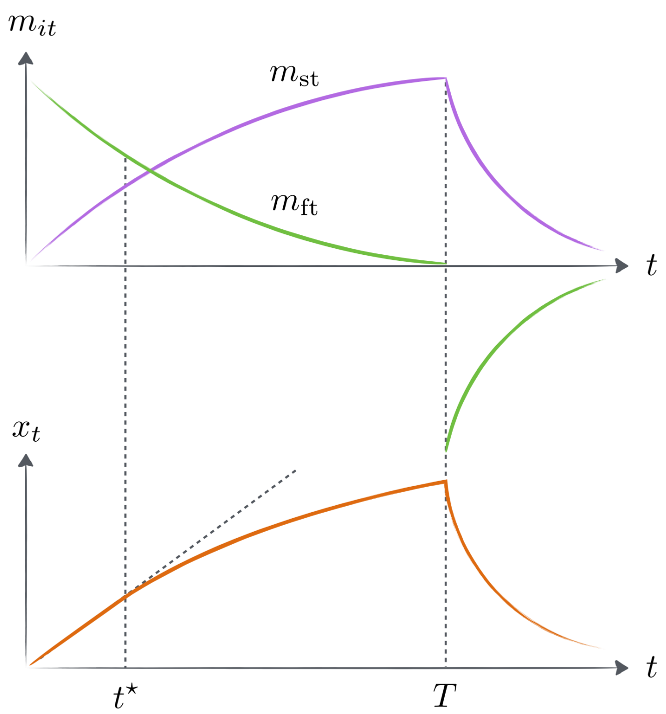

Figure 4: Execution rates m_{it} (top) and price trajectory (bottom) within the double-frequency order book model.

The multi-frequency framework

The multi-frequency framework

The double-frequency framework can be extended to the more realistic case of a continuous range of cancellation and deposition rates.

$$\partial_t \phi_\nu = D\partial_{xx}\phi_\nu - \nu\phi_\nu + \lambda_\nu \text{sign}(x_{\nu t} - x) + m_{\nu t}\delta(x - x_{\nu t}) \tag{13}$$

where $\phi_\nu(x,t)$ denotes the contribution of agents with typical frequency ν to the latent order book and $\lambda_\nu = \mathcal{L}_\nu\sqrt{\nu D}$.

Equation (13) must then be completed with:

$$\begin{aligned} \int_0^\infty \rho(\nu) m_{\nu t} \mathrm{d}\nu &= m_t \\ x_{\nu t} = x_t, &\quad \forall \nu \end{aligned} \tag{14}$$

where $\rho(\nu)$ denotes the distribution of cancellation rates ν, and where we have allowed for an arbitrary order flow m_t.

Resolution of the "diffusivity puzzle"

With $\langle x_t^2 \rangle \propto t^{1-\gamma}$, the latent liquidity in the LLOB case is too persistent and prevents the price from diffusing.

With the power-law distribution $\rho(\nu)$ as

$$\rho(\nu) = Z\nu^{\alpha-1}e^{-\nu t_c} \tag{15}$$

The price diffusion under multi-frequency framework is given by

$$\langle x_t^2 \rangle \propto t^{1+2\alpha-\gamma} \tag{16}$$

When the liquidity memory times are themselves fat-tailed, mean-reversion effects induced by a persistent order book can exactly offset trending effects induced by a persistent order flow.

Meta-order impact

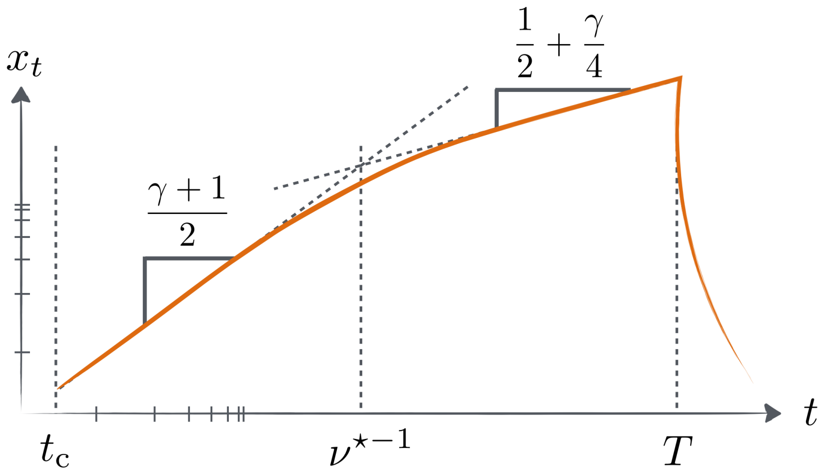

Figure 5: Price trajectory during a constant rate meta-order execution within the multi-frequency order book model. For $\gamma = 1/2$, the impact crosses over from a $t^{3/4}$ to a $t^{5/8}$ regime.

References

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Market impact with multi-timescale liquidity

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