MONTE-CARLO SIMULATION OF ISING MODEL

A PREPRINT

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ABSTRACT

Exact simulation of Ising model is intractable because of the enormous amount of computation. MC method (especially MCMC) serves as a perfect alternative here, and its result matches the theoretical analysis of Ising model.

Keywords Ising model · Monte-Carlo methods

1 Introduction

1.1 Ising model

Ising model was originally proposed in physics as a model for magnetism (Dobrow [2016]).

Consider a graph consisting of sites (vertices), in which each site v is assigned a spin of $+1$ or -1. A configuration σ is an assignment of spins to each site. That is, $\sigma_v = \pm 1$, for all v. We will assume a square $n \times n$ grid of sites, where each site is connected to four neighbors (up, down, left, and right), except at the boundary. Thus, there are n^2 sites and 2^{n^2} possible configurations. Associated with each configuration σ is its energy, defined as

$$E(\boldsymbol{\sigma}) = -\sum_{v \sim w} \sigma_v \sigma_w$$

where the sum is over all pairs of sites v and w, which are neighbors. Note that if most neighbors have similar spins, the energy is negative; if most neighbors have different spins, the energy is positive; and for a uniformly random assignment of spins, the energy is about 0.

The order parameter:

$$m = \frac{\langle \sigma \rangle}{n}$$

for this system is the average magnetization. The order parameter distinguishes the two phases realized by the systems. It is zero in the disordered state, while non-zero in the ordered, ferromagnetic, state.

1.2 Gibbs distribution

The Gibbs distribution is a probability distribution on the set of configurations, defined by

$$\pi_\sigma = \frac{e^{-\beta E(\sigma)}}{\sum_\tau e^{-\beta E(\tau)}}$$

where the sum in the denominator is over all configurations. The parameter β has a physical interpretation as the reciprocal of temperature. If $\beta = 0$ (infinite temperature), the distribution is uniform on the set of configurations. For $\beta > 0$, the Gibbs distribution puts more mass on low-energy configurations, which favors neighbors of similar spin. For $\beta < 0$, the distribution puts more mass on high-energy configurations.

2 Algorithm

Metropolis algorithm is a special type of MCMC methods. The algorithm for simulating the Ising model can be defined as follows(Shekaari and Jafari [2021]):

1. Prepare an initial configuration of N spins

2. Flip the spin of a randomly chosen lattice site.

3. Calculate the change in energy dE.

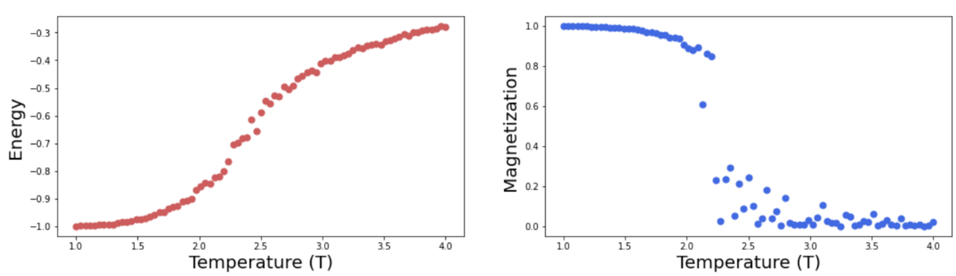
4. If $dE < 0$, accept the move. Otherwise accept the move with probability $exp^{-dE/T}$.

5. Repeat 2-4 until convergence.

The code for the simulation can be found in the attachment. Here is the result of simulation.

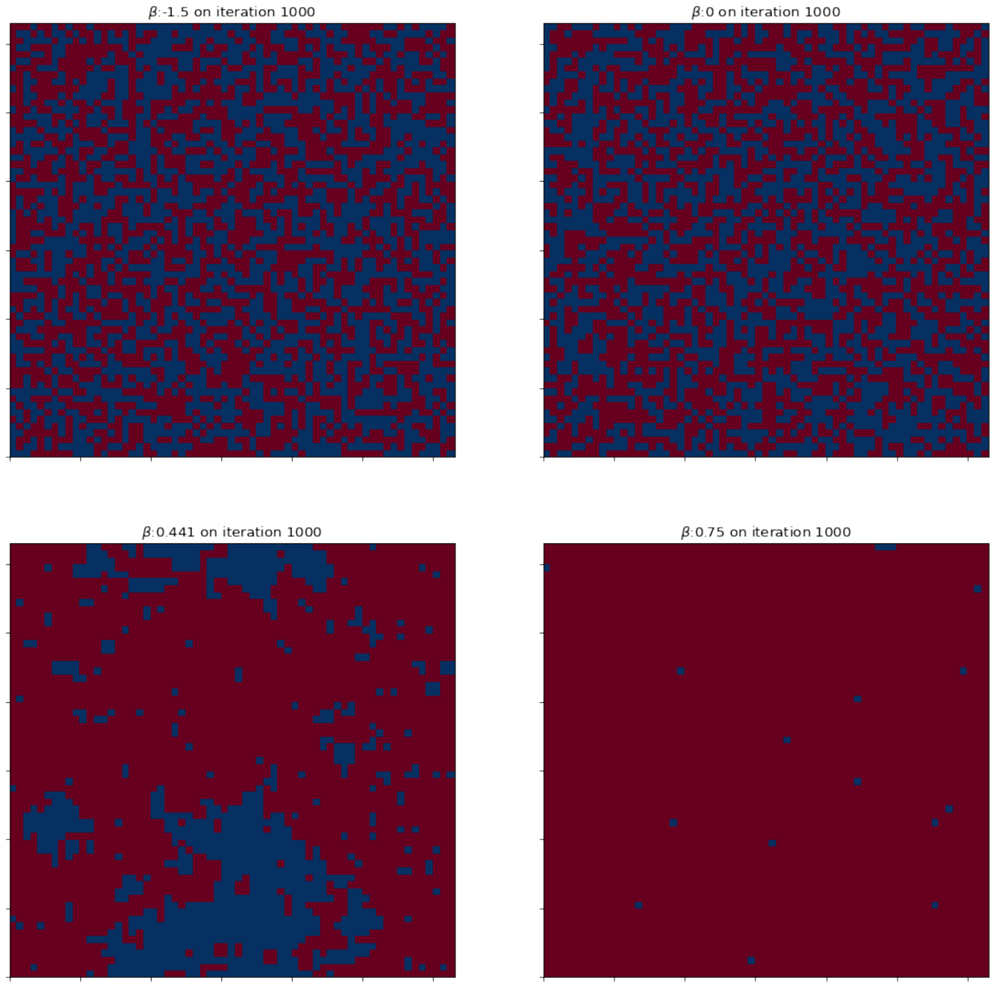
2.1 Simulation across different temperature

The following figure is generated with temperature ranging from 1 to 4, and the energy and magnetization factor is calculated as described in the last section. We can see that the energy increases monotonically against temparature, but the magnetization suddenly disappeared at around 2.2. In fact, the critical temperature is $1/\beta = 2/\ln(1+\sqrt{2}) \approx 2.269(\beta = 0.441)$ for 2D Ising model((rajeshri [2014])).

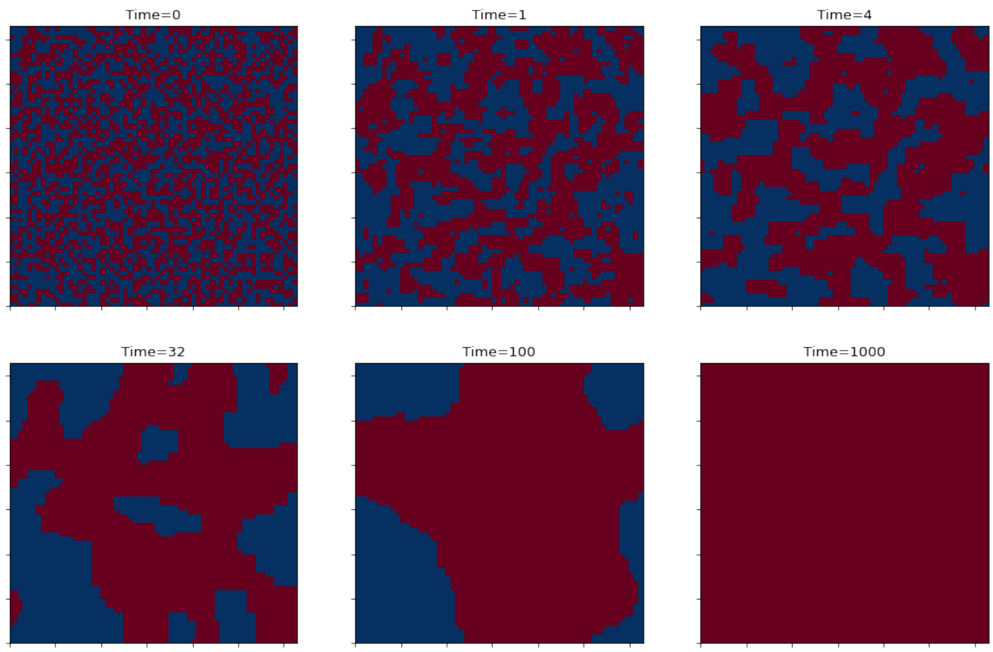


2.2 Visualization of balanced Ising model

The following figure is generated for balanced Ising model with $\beta = -1.5, 0, 0.441, 0.75$. The balanced state is defined to be the state after $1k$ iterations (which is suffice for a 10*10 grid). We can see that larger β (lower temparature) yields same spin patterns (ferromagnetic).



Another figure shows the transition (snapshots) of the Ising model with $\beta = 0.4$. We can see that eventually the grid yields the same pattern as expected.



References

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