NOTES ON
3D GEOMETRY FOR COMPUTER VISION AND SLAM

CONTENTS

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1.1.1 Motivation

Homogeneous coordinates (H.C.) are a way of representing points and lines. In the homogeneous space, it is easier to formalise some mathematical derivations than in the Euclidean space. Homogeneous coordinates are often used in computer vision mainly in two broad applications.

1. They allow us to *compactly represent point transforms*. For instance, they allow us to represent the affine transform (rotation and translation) as a matrix multiplication $\mathbf{y} = \mathbf{A}\mathbf{x}$ as opposed to $\mathbf{y} = \mathbf{R}\mathbf{x} + \mathbf{T}$, where $\mathbf{A}$ (affine matrix) embeds $\mathbf{R}$ (rotation) and $\mathbf{T}$ (translation). To see why this is the case, consider the rotation and translation mapping of a point $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$ given by:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}_{\mathbf{R}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}}_{\mathbf{T}}$$

   This can be written as a single matrix multiplication, which is more compact and also allows computer programs to optimise it.

   The trick is to pack $\mathbf{R}$ and $\mathbf{T}$ into one 3×4 matrix $\begin{bmatrix} \mathbf{R} \mid \mathbf{T} \end{bmatrix}$ by appending $\mathbf{T}$ to $\mathbf{R}$:

$$\begin{bmatrix} \mathbf{R} \mid \mathbf{T} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix}$$

   This is the so-called affine matrix $\mathbf{A}$. However, the multiplication of $\mathbf{A}$ with $\mathbf{x}$ is undefined as $\mathbf{x}$ is 3×1. The can augment $\mathbf{x}$ to the size 4×1 by introducing a new (redundant) coordinate $x_4 = 1$. But now the product $\begin{bmatrix} \mathbf{R} \mid \mathbf{T} \end{bmatrix}\mathbf{x}$ is of size 3×1, which certainly does not belong in the homogeneous space of the original vector of length 4. So to keep $\begin{bmatrix} \mathbf{R} \mid \mathbf{T} \end{bmatrix}\mathbf{x}$ an inner operation and to satisfy $\mathbf{x}_{hom} = \mathbf{I}_{hom}\mathbf{x}_{hom}$ we add a fourth redundant row $\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$. The final matrix

   We can convert a Euclidean point to hom. by appending a new extra coordinate with value 1.

$$\mathbf{A} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   is an affine matrix. $\begin{bmatrix} x_1 & x_2 & x_3 & 1 \end{bmatrix}$ is the H.C. form the Euclidean point $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$. Using the affine matrix, translation can also be expressed as part of a matrix product.

$$\mathbf{y} = \mathbf{A}\mathbf{x} = \begin{bmatrix} & \mathbf{R} & \mid & \mathbf{T} \\ 0 & 0 & 0 & 1 \end{bmatrix}\mathbf{x}$$

2. They can *represent points at the projective plane*, i.e. 3D points from the "real world" mapped into a 2D plane (screen). They can even represent points at infinity via finite scalars; without running into divisions by zero. Projections ubiquitously take place in camera models, the simplest one being a pinhole camera.

   Camera models, such as the pinhole map a 3D point (x, y, z) in the "real world" onto a 2D plane (camera film/screen) using linear transforms. The linear transform depends on the internal manufacturing characteristics (a.k.a. intrinsics) of the camera and the externals (extrinsics), such as its translation and rotation. For the pinhole model, the only intrinsic we consider is the focal length f[1]. It can be easily derived (see A.1) that the pinhole camera projection is described by the following transform.

$$\begin{bmatrix} x\prime \\ y\prime \end{bmatrix} = \begin{bmatrix} \frac{fx}{z} \\ \frac{fy}{z} \end{bmatrix} = \frac{1}{z} \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{1.1}$$

   H.C. allow us to express Eq. (1.1) as a linear transform, hence more concisely. Although Eq. (1.1) is simple, camera models can get complicated.

[1]The projected coordinates are in distance units for simplicity, so that we can stick to one instrinsic – the focal length. If they were measured in pixels, we would include two more intrinsic parameters in the matrix.

Apart from points, H.C. also represent lines (via their coefficients). Lines appear virtually everywhere in images (e.g. edges). Homogeneous coordinates help us answer questions about the correspondences of lines between the 3D world and the image place, such as;

- Do parallel lines in the real world meet in the projection?
- Do angles between lines in the real world get preserved in the projection?
- Do straight lines stay straight?

Spoiler; the answers are yes, no, and yes (at least for the pinhole model).

Fig. 1. Two properties of the pinhole-based projection; parallel lines meet, straight lines remain straight (project to straight lines).

1.1.2 Definition

Homogeneous coordinates are useful for representing an object up to a scale. Mathematically, if $\mathbf{x}$ represents a point in the H.C. space, then $\lambda\mathbf{x}$, $\lambda \neq 0$ must represent the same point, i.e. $\mathbf{x}$ and $\lambda\mathbf{x}$ must be equivalent ($\sim$) in H.C. We denote equivalencies in H.C. with $\sim$, therefore $\mathbf{x} \sim \lambda\mathbf{x}$. A point in the H.C. space is defined only by its direction and not by its magnitude. The H.C. for 3D points are hence defined as follows.

Upright style $\mathbf{x}$ will be used for H.C. vectors and italic $\boldsymbol{x}$ for Euclidean.

Definition 1.1 (Homogeneous Coordinates of a 3D vector). *A Euclidean 3D point $\boldsymbol{x} = (x, y, z)^\top$ is represented in h o m o g e n e o u s   c o o r d i n a t e s by ANY 4-vector $(x_1, x_2, x_3, x_4)^\top$ such that:*

$$x = \frac{x_1}{x_4}, \quad y = \frac{x_2}{x_4}, \quad z = \frac{x_3}{x_4}, \quad \text{in} \quad \text{H.C.:} \quad \mathbf{x} = (x_1, x_2, x_3, x_4)^\top \tag{1.2}$$

x_4 can attain any value. For instance for $x_4 = 1$ the H.C. of (x, y, z) are $(x, y, z, 1)$. The following lemma is therefore straightforward.

Converting from E.C.to H.C. is NOT unique! Converting H.C. to E.C. is unique.

Lemma 1.1. *In H.C., $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^\top$ and $\lambda \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^\top$, $\lambda \neq 0$ represent the same point, i.e.*

$$\mathbf{x} \sim \lambda\mathbf{x}, \quad \mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^\top, \quad \lambda \neq 0 \tag{1.3}$$

For example, $(2,3,4,1) \sim (6,9,12,3) \sim (2,3,4)$, and $(4,8,12,2) \sim (2,4,6)$.

1.1.3 Converting between Euclidean and homogeneous coordinates

From Lemma 1.1, it is clear that to convert a point (x, y, z) from Euclidean to H.C. we simply add one more ordinate $w = 1$ and obtain $(x, y, z, 1)$. From now on black dot ($\bullet$) denotes H.C. and white ($\circ$) E.C.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \circ\!\!-\!\!\bullet \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix} \tag{1.4}$$

Inversely, to "dehomogenise" a point $(x_1, x_2, x_3, x_4)^\top$, $x_4 \neq 0$, we first divide its elements by x_4 such that the last element is 1. We then discard the last element to obtain $(x_1/x_4, x_2/x_4, x_3/x_4)$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \bullet\!-\!\circ \begin{bmatrix} x_1/x_4 \\ x_2/x_4 \\ x_3/x_4 \end{bmatrix}, \quad x_4 \neq 0 \tag{1.5}$$

For example, $(1,2,3,4)^\top \sim (3,6,9,12)^\top \sim (-2,-4,-6,-8) \bullet\!-\!\circ (0.25, 0.5, 0.75)$ and they all represent the 3D point Also $(0,0,2) \sim (0,0,\pi) \bullet\!-\!\circ (0,0)$.

$\bullet$ for H.C., $\circ$ for E.C.

EXAMPLE 1.1. *Calculate the midpoint between* $\mathbf{p}_1 = (5,7,8)$ *and* $\mathbf{p}_2 = (4,-6,1)$.

SOLUTION 1.1. *We convert to E.C. to find the midpoint.*

Distance ratios are not preserved in when converting between E.C. and H.C.

$$\mathbf{p}_1 = (5,7,8) \bullet\!-\!\circ (5/8, 7/8) := \boldsymbol{p}_1,$$

$$\mathbf{p}_2 = (4,-6,1) \bullet\!-\!\circ (4,-6) := \boldsymbol{p}_2$$

The midpoint in E.C. is $(\boldsymbol{p}_1 + \boldsymbol{p}_2)/2 = (37/16, -41/16) \bullet\!-\!\circ (37/16, -41/16, 1) \sim (37, -41, 16)$

1.1.4 Geometric interpretation of homogeneous points

If we were to convert a 2D point in E.C (x_1, y_1) to H.C., we would augment:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \circ\!-\!\bullet \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$$

Therefore in a 3D plane, the point $(x_1, x_2, 1)$ in H.C. can be geometrically determined by looking at the direction (x_1, x_2) and at height $x_3 = 1$ from the origin. Of course, because $(x_1, x_2, 1) \sim \lambda(x_1, x_2, 1)$, point $(x_1, x_2, 1)$ is also found at $(2x_1, 2x_2, 2)$, $(2.5x_1, 2.5x_2, 2.5)$, etc. but as a convention we will be taking $x_3 = 1$.

Projecting onto $x_3 = 1$ helps us also find the E.C. representation of the projected point – if $x_3 = 1$, then the E.C. are simply (x_1, x_2). Because the projected point is expressed as $(x/w, y/w, 1)$ (Fig. 2), if we plug in $x = 0$, $y = 0$, then we observe that the origin $O_2(0,0)$ in E.C. gets mapped to $O_3(0,0,1)$ in H.C.

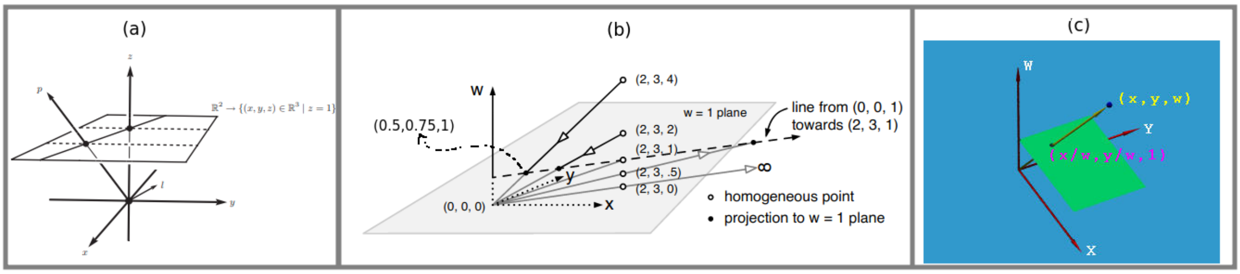

Fig. 2. (a): The set of homogeneous points (x, y, z) in the $\mathbb{R}^3$ space. (b): Numerical example of homogeneous points projected on $z = 1$. (c): To project to $z = 1$, we divide all ordinates by w.

1.2 Mathematics of lines and points in 3D

We have to describe the mathematics of lines in the projected space (2D) since this is what cameras use. As we will see, H.C. make it easier than Euclidean. Suppose we have a 2D point represented in E.C. as $x = (x, y)$. If this point lies on a line, then this line is defined by some a_1, a_2, a_3 such that: $a_1 x + a_2 y + a_3 = 0$, $a_1 \neq 0$ or $a_2 \neq 0$. We can rewrite this equation for in H.C. if we convert $\boldsymbol{x} = (x, y) \circ\!-\!\bullet \mathbf{x} = (x_1, x_2, x_3)$ by following Eq. (1.2):

$$x = \frac{x_1}{x_3}, \quad y = \frac{x_2}{x_3}, \quad x_3 \in \mathbb{R}*$$

Then the line is written in H.C. as:

$$a_1 x + a_2 y + a_3 = 0 \Rightarrow$$
$$a_1 \frac{x_1}{x_3} + a_2 \frac{x_2}{x_3} + a_3 = 0 \Rightarrow$$
$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0 \Rightarrow$$
$$\boldsymbol{\ell}^\top \mathbf{x} = 0, \quad \boldsymbol{\ell} = (a_1, a_2, a_3)^\top$$

$s\boldsymbol{\ell}$, $s \in \mathbb{R}^*$ and $\boldsymbol{\ell}$ encode the same line.

COROLLARY 1.1 (line equation in H.C.). *If a 2D point (x, y) in E.C. lies on a line parametrised by (a_1, a_2, a_3), then in H.C. the line equation is described by a dot product:*

$$\boldsymbol{\ell}^\top \mathbf{x} = 0, \quad \boldsymbol{\ell}^\top = (a_1, a_2, a_3)^\top, \quad a_1, a_2 \neq 0 \tag{1.6}$$

Eq. (1.6) allows us to concisely express the intersection of two lines and the line through two points.

COROLLARY 1.2 (intersection of lines in H.C.). *Consider the two parametrised lines $\boldsymbol{\ell}_1 = (a_1, a_2, a_3)^\top$, $\boldsymbol{\ell}_2 = (b_1, b_2, b_3)$ in H.C. Then they intersect at point $\mathbf{p} = \boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2$, where $\times$ denotes cross product.*

Proof. Let $\mathbf{p}$ be the point (in H.C.) where $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$ intersect. Therefore $\mathbf{p}$ satisfies the line equation Eq. (1.6) for both lines

$$\mathbf{p} \cdot \boldsymbol{\ell}_1 = 0$$
$$\mathbf{p} \cdot \boldsymbol{\ell}_2 = 0$$

, where $\mathbf{p} \cdot \boldsymbol{\ell}_1 = \mathbf{p}^\top \boldsymbol{\ell}_1$ is a dot product. We don't care about the magnitude of $\mathbf{p}$, we only need such a vector that is perpendicular to both $\boldsymbol{\ell}_1$ and $\boldsymbol{\ell}_2$. Such a vector is by its definition the cross product $\boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2$, i.e.

$$\underbrace{(\boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2)}_{\mathbf{p}} \cdot \boldsymbol{\ell}_1 = \underbrace{(\boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2)}_{\mathbf{p}} \cdot \boldsymbol{\ell}_2 = 0$$

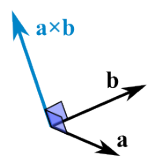

Fig. 3. The cross product $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

$\square$

COROLLARY 1.3 (line joining two points in H.C.). *If $\mathbf{p}_1, \mathbf{p}_2$ are two points in H.C., then the coefficients of the line joining them can be computed from $\boldsymbol{\ell} = \mathbf{p}_1 \times \mathbf{p}_2$.*

Proof. We want both $\mathbf{p}_1$ and $\mathbf{p}_2$ to lie on the same line, i.e. satisfy its equation:

$$\mathbf{p}_1 \cdot \boldsymbol{\ell} = 0$$
$$\mathbf{p}_2 \cdot \boldsymbol{\ell} = 0$$

As shown the previous proof, to satisfy these two conditions we can choose $\boldsymbol{\ell} = \mathbf{p}_1 \times \mathbf{p}_2$. $\square$

Note that Cor. 1.3 is *dual* to Cor. 1.2, in the sense that points and lines are swapped. This occurs because both points and lines are represented by 3D vectors. Fig. 4 visualises geometrically Cor. 1.3 and Cor. 1.2 in H.C.

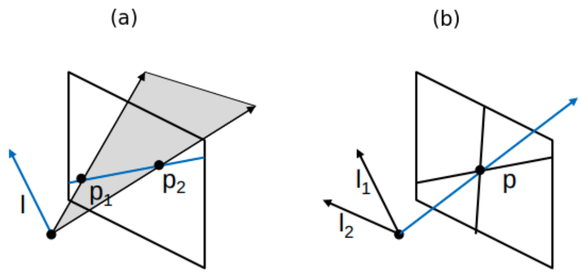

Fig. 4. (a): The vector $\mathbf{l}$ with the coefficients of the line joining $\mathbf{p}_1, \mathbf{p}_2$ is perpendicular to both $\mathbf{p}_1$ and $\mathbf{p}_2$ vectors, therefore perpendicular to the plane they span. (b): The same principle holds for a point $\mathbf{p}$ which at the intersection of lines $\mathbf{l}_1, \mathbf{l}_2$.

Points at infinity $(\pm\infty, \pm\infty)$ can be represented in H.C. by finite quantities. To verify that, consider all points starting from $\boldsymbol{r}_0 = (x_0, y_0)$ with direction $\mathbf{v} = (a, b)$. Then, points given by $\boldsymbol{r} = \boldsymbol{r}_0 + t\mathbf{v}$, $t \in \mathbb{R}$ lie along the same straight line. Each scalar component of $\boldsymbol{r}$ is given by:

$$x = x_0 + ta, \quad y = y_0 + tb \tag{1}$$

Eq. (1) is the parametric line equation of the 2D line where points $\boldsymbol{r} = (x, y, z) = (x_0 + ta, y_0 + tb, z_0 + tc)$ lie. In H.C., points of $\boldsymbol{r}$ are given by $\mathbf{r} = (x_0 + ta, y_0 + tb, 1)$. Because $\mathbf{r} = \mathbf{r}/t$ in H.C., they are also given by:

$$\mathbf{r} = (\frac{x_0}{t} + a, \frac{y_0}{t} + b, \frac{1}{t}) \tag{2}$$

The E.C. points $(x_0 + ta, y_0 + tb, z_0 + tc)$ approach infinity as $t \to \infty$. If we plug in $t \to \infty$ in Eq. (2), it turns out the points in infinity in H.C. are given by:

$$\mathbf{r} = (a, b, 0)$$

Moving in the directions of (a, b) and $(-a, -b)$ will end up at the same infinity point represented by $(a, b, 0)$ as well as $(-a, -b, 0)$. For instance, the points at infinity in the direction vector $\mathbf{v} = (-3, 2)$ are given by $(-3, 2, 0)$ in H.C. and all lie on the line $y = -2/3x$.

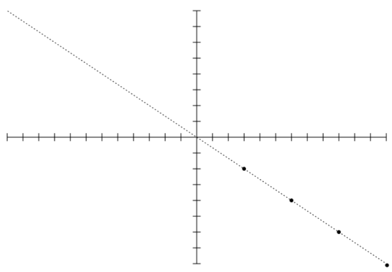

Fig. 5. Points at infinity along the direction $(-3, 2)$

COROLLARY 1.4 (point at infinity in H.C.). *In H.C., points of the form $(a, b, 0)$ represent a point that tends to infinity in E.C. in the direction of $\boldsymbol{v} = (a, b)$. They're also known as* ideal points.

Ideal points all lie along the same line $\boldsymbol{\ell}$. If $\boldsymbol{\ell} = (l_1, l_2, l_3)$ in H.C., then all ideal points $(a, b, 0)$ must satisfy

$$\begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^\top \begin{bmatrix} a & b & 0 \end{bmatrix} = 0 \Rightarrow a = 0, \quad b = 0, \quad c \in \mathbb{R}$$

Therefore all ideal points lie along the line with coefficients $(0, 0, c) \sim (0, 0, 1)$ in H.C. To summarise:

COROLLARY 1.5 (line at infinity). *All ideal points lie along the line $\boldsymbol{\ell}_\infty = (0, 0, 1)$ in H.C. This is known as* ideal line.

Ideal points are important because that's where lines that are parallel in E.C. meet. To verify that, consider the equations of two parallel lines:

$$\begin{cases} a_1 x + b_1 y + c_1 = 0 \\ a_1 x + b_1 y + c_2 = 0 \end{cases}, \quad c_1 \neq c_2$$

Therefore in H.C., they can be described by $\boldsymbol{\ell}_1 = (a_1, b_1, c_1)$, $\boldsymbol{\ell}_2 = (a_1, b_1, c_2)$. From Cor. 1.2, we know that they intersect at:

$$\begin{aligned}
\mathbf{p} &= \boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2 \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_1 & b_1 & c_2 \end{vmatrix} = \mathbf{i}(b_1c_2 - b_1c_1) + \mathbf{j}(a_1c_2 - a_1c1) + \mathbf{k}\cdot 0 \\
&= (b_1c_2 - b_1c_1, a_1c_2 - a_1c1, 0)
\end{aligned}$$

$(b_1c_2 - b_1c_1, a_1c_2 - a_1c1, 0)$ is an ideal point. To summarise:

Corollary 1.6 (intersection of parallel lines in H.C.). *Two parallel lines in H.C., $\boldsymbol{\ell}_1 = (a, b, c)$ and $\boldsymbol{\ell}_2 = (a, b, d)$, $\quad c \neq d$ intersect at an ideal point $\mathbf{p}$ (point at infinity) given by:*

$$\mathbf{p} = (b(d-c), a(d-c), 0) \tag{1.7}$$

In conclusion, very concept in projective geometry has its dual. If we start with a certain statement, e.g. "The line joining points $\mathbf{p}_1$, $\mathbf{p}_2$ is given by $\boldsymbol{\ell} = \mathbf{p}_1 \times \mathbf{p}_2$" and take the dual of each concept, we end up with "The point of intersection of lines $\mathbf{l}_1$, $\mathbf{l}_2$ is given by $\mathbf{p} = \boldsymbol{\ell}_1 \times \boldsymbol{\ell}_2$". For completeness, the duality table is given below.

Original	Dual
point	line
join	intersect
$\mathbf{p}$	$\boldsymbol{\ell}$

1.2.1 Geometric interpretation of conversions between E.C. and H.C.

So far 2D E.C. points were described as infinite rays in the 3D Euclidean space. In this section H.C. points are described in an alternative geometrical fashion – as straight 2D line segments. The most important thing to remember to interpret H.C. coordinates this way is that an infinite line through the origin is normal to a (3D) plane through the origin. In this section, the following are described in Euclidean space:

1. H.C. points,
2. a line intersection in H.C.,
3. and the points of the line joining two H.C. points

Below each case is derived geometrically.

1. **(H.C. point in E.C. space)** We already know that a H.C. point (ℓ_1, ℓ_2, ℓ_3) lies anywhere along the straight line from the origin to $\lambda(\ell_1, \ell_2, \ell_3)$, $\lambda \in \mathbb{R}$. The line defined by the direction of (ℓ_1, ℓ_2, ℓ_3) is normal to a plane through the origin that is normal to it, i.e. to plane $\ell_1x_1 + \ell_2x_3 + \ell_3x_3 = 0$. Also, to find the Euclidean equivalent of a H.C. point, we set $x_3 = 1$. Ultimately, the Euclidean equivalent of the H.C. point (ℓ_1, ℓ_2, ℓ_3) is determined by the intersection of the following two planes:

$$\ell_1x_1 + \ell_2x_2 + \ell_3x_3 = 0 \quad \cap \quad x_3 - 1 = 0$$

Therefore the 2D line equation that describes point (ℓ_1, ℓ_2, ℓ_3) is:

$$\ell_1x_1 + \ell_2x_2 + \ell_3 = 0$$

All this is visualised in Fig. 6 and Fig. 7.

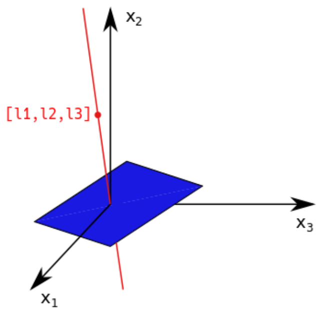

Fig. 6. The plane $\ell_1 x_1 + \ell_2 x_2 + \ell_3 x_3 = 0$

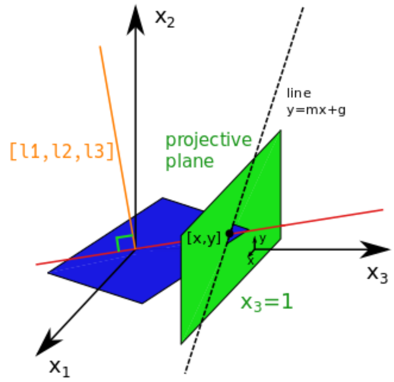

Fig. 7. The intersection of plane$\ell_1 x_1 + \ell_2 x_2 + \ell_3 x_3 = 0$ with $x_3 = 1$ describes H.C. point (ℓ_1, ℓ_2, ℓ_3) as a 2D straight line.

2. **(line intersection in H.C.)** We now know that a line along point $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell_3)$ in H.C. can be represented as the intersection of planes $\boldsymbol{\ell}^\top \mathbf{x} = 0$, $x_3 = 1$ in the 3D space. Also the intersection of two E.C. points $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell_3)$, $\mathbf{m} = (m_1, m_2, m_3)$ can be described as an infinite ray spanned by the cross product $\boldsymbol{\ell} \times \mathbf{m}$ (Cor. 1.3). Finally, to convert ray $\boldsymbol{\ell} \times \mathbf{m}$ to its E.C. equivalent, we project it onto plane $x_3 = 1$

   This is visualised in the figure below, where the red and blue planes are given, the orange ray describes their cross product, and the intersection of the orange ray with plane $x_3 = 1$ (which is a straight line) describes the intersection of the two lines defined by $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell_3)$ and $\mathbf{m} = (m_1, m_2, m_3)$ respectively. Note that the orange ray is NOT necessarily along the axis x_3, although the figure is misleading.

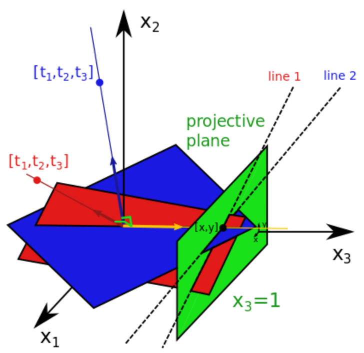

Fig. 8. Intersection of H.C. lines $\boldsymbol{\ell} = (\ell_1, \ell_2, \ell_3)$ and $\mathbf{m} = (m_1, m_2, m_3)$ described as a 2D line.

3. **(Line joining two H.C. points)** It is known from Cor. 1.3 that if two H.C. points $\mathbf{p}_1$, $\mathbf{p}_2$ are given , then the coefficients of the line through them in H.C. are given by the cross product $\boldsymbol{\ell} = \mathbf{p}_1 \times \mathbf{p}_2$. This cross product is once again a vector normal to the plane $\boldsymbol{\ell}_1 x_1 + \boldsymbol{\ell}_2 x_2 + \boldsymbol{\ell}_3 x_3 = 0$. To unproject the plane $\boldsymbol{\ell}^\top \mathbf{x} = 0$ in 2D, we project it onto $x_3 = 1$. The line projected onto $x_3 = 1$ describes the line through $\mathbf{p}_1$ and $\mathbf{p}_2$ in 2D.

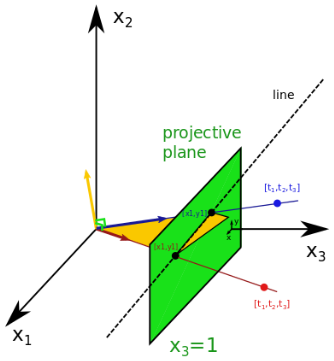

Fig. 9. The intersection of the plane $(\mathbf{p}_1\mathbf{p}_2)^\top\mathbf{x} = 0$ with $x_3 = 1$ describes the line through H.C. points $\mathbf{p}_1$ and $\mathbf{p}_2$.

1.3 The cross ratio

A Appendices

A.1 Pinhole camera model derivation

Consider the case of a *pinhole camera*, which consists of a box which a tiny aperture on the front plane and a film which captures the energy (colour) of the light on the real plane (Fig. 10). We assume that all light rays entering the camera meet at the aperture (centre of projection). The horizontal distance between the aperture and the film is called focal length f.

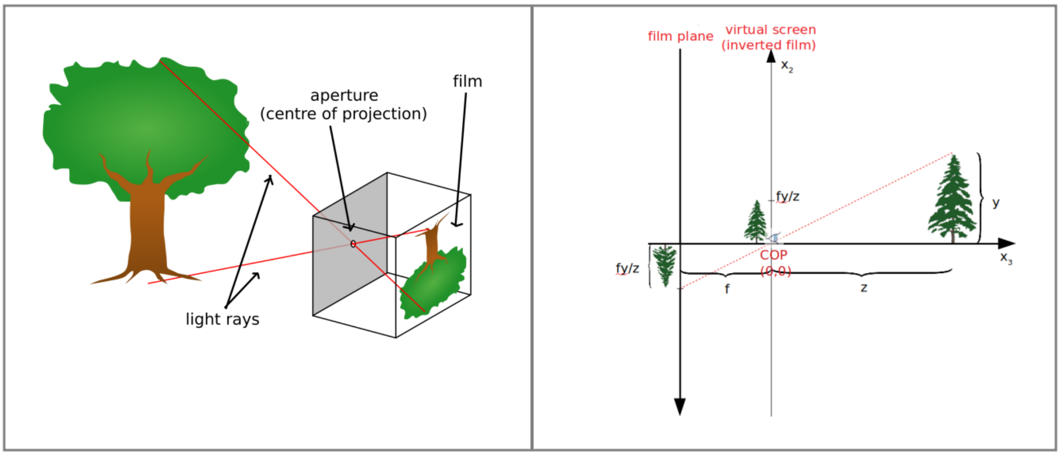

Fig. 10. Left: Pinhole camera schematic in 3D. Right: Pinhole camera model of a tree in the y dimension.

From the similar triangles in front and behind the screen in Fig. 10, we obtain for the project $y\prime$ of the object:

$$\frac{y}{z} = \frac{y\prime}{f} \Rightarrow y\prime = f\frac{y}{z}$$

Similarly for the projected $x\prime$ (not shown in Fig. 10) we obtain:

$$x\prime = f\frac{x}{z}$$

The position (x', y') on the screen is measured in distance units – not pixels.

To reiterate, pinhole camera performs the following linear mapping in Caertesian coordinates:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} \frac{fx}{z} \\ \frac{fy}{z} \end{bmatrix}, \qquad \begin{bmatrix} \frac{fx}{z} \\ \frac{fy}{z} \end{bmatrix} = \frac{1}{z}\begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$