Problem

Find the volume of the solid generated by a rotation of the region enclosed by the curve $y = x^3 - x$ and the line $y = x$ about the line $y = x$ as the axis of rotation.

This problem is from AoPS forums and I had solved it in 2013 directly by the "washer method" around the line $y = x$. Although it's an easy problem, there's another, nicer, way to solve it.

The volume generated by the rotation of $y = x^3 - x$ about $y = x$ remains invariant under linear transformation of the coordinates. Specifically, we want to rotate both graphs by -45° so $y = x$ aligns with the x-x axis. Then, we can apply the "washer method"out of the box (Fig. 1).

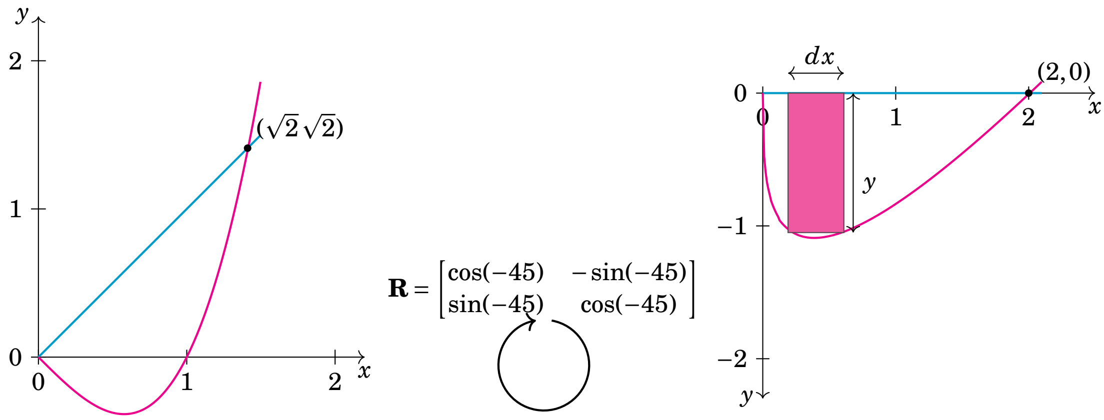

Fig. 1. The two functions before and after rotating clockwise by 45°.

Rotate the coordinates by multiplying them the 2D rotation matrix;

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{R}\begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} \cos(-45) & -\sin(-45) \\ \sin(-45) & \cos(-45) \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow$$

$$x' = \frac{1}{\sqrt{2}}(x + y) \tag{1}$$

$$y' = \frac{1}{\sqrt{2}}(-x + y) \tag{2}$$

Note that the intersection $(\sqrt{2}, \sqrt{2})$ gets transformed to $(2, 0)$.

Now we simply have to compute the volume of $y'(x')$ about the x-axis from 0 to 2. Using the washer method, it is equal to the sum of the volumes of infinitesimal cylinders with height dx' and height y' (Fig. /reffig:plotsrotation), i.e.

$$V = \int_0^2 \pi\big(y'(x')\big)^2 dx' \tag{3}$$

From Eq. (1), $dx' = dx$. From Eq. (1) and (2), we compute y' in terms of x'.

$$x' = 2^{-\frac{1}{2}}(x + y) = 2^{-\frac{1}{2}}x^3 \Rightarrow x = 2^{\frac{1}{6}}x'^{\frac{1}{3}}$$

$$\therefore y' = 2^{-\frac{1}{2}}(x^3 - 2x) = 2^{-\frac{1}{2}}\left(x'2^{\frac{1}{2}} - 2^{\frac{7}{6}}x'^{\frac{1}{3}}\right) \tag{4}$$

Substituting Eq. (4) in Eq. (3) we can calculate the volume:

$$V = \frac{\pi}{2}\int_0^2 (x'2^{\frac{1}{2}} - 2^{\frac{7}{6}}x'^{\frac{1}{3}})^2 dx' = \ldots = \frac{64\pi}{105}$$