DYNAMICAL SYSTEMS BUILT FROM CURVATURE AND TORSION OF PLANAR CURVES

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Abstract. Stereographic Projection is a smooth bijective mapping from a plane to a sphere. The mapping is a conformal projection preserving angles and circles. That is the image of a circle on the sphere is a circle in the plane and the angle between two lines on the sphere is the same as the angle between their images in the plane. A smooth curve lying on the plane can be projected to the sphere. Any smooth planar curve has zero torsion and some curvature. As a result of the preservative properties of the projection mapping we are able to make a bijective mapping of the curvature and torsion of a curve to the curvature and torsion of its projected image. Such mapping allows for a dynamical system to be built. Each iterate of the system takes in a smooth parametric planar curve and outputs a smooth parametric curve. The parametric curve created from the output of each iterate is defined by the mapping of the projected curvature and torsion of the input curve.

Summary

1. Definitions

Definition 1. *Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ such that $\alpha(t) = (x(t), y(t))$ be a planar curve on the plane $z = -1$. Let $N = (0, 0, 1)$ be the "north pole" of the unit sphere centered at the origin. For any point P along α the stereographic projection of P is defined as the unique point P' such that P' is the intersection of the line through N and P with the unit sphere. Let $\beta : \mathbb{R}^2 \to S^2 - N$ be a stereographic projection of $\alpha(t)$ onto the unit sphere. We define $\beta \cdot \alpha(t)$ as follows:*

$$\beta \cdot \alpha(t) = \beta \cdot \alpha \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{4x(t)}{x(t)^2+y(t)^2+4} \\ \frac{4y(t)}{x(t)^2+y(t)^2+4} \\ \frac{x(t)^2+y(t)^2-4}{x(t)^2+y(t)^2+4} \end{pmatrix} \tag{1}$$

Which gives us the image of the curve $\alpha(t)$ onto the unit sphere from the plane $z = -1$.

Definition 2. *Let $\beta : \mathbb{R}^2 \to S^2 - N$ be a stereographic projection. Let α be a planar curve in $\mathbb{R}^2$ lying on the plane $z = -1$. Let $\beta \cdot \alpha(t)$ be the image of α on the unit sphere. Suppose $c_\alpha(t)$ defines the osculating circle along the curve α at any point t. We will define*

$$\gamma_{c_{\alpha(t)}}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix} = \begin{pmatrix} v(x_0 + rcos(u)) \\ v(y_0 + rsin(u)) \\ 1 - 2v \end{pmatrix} \tag{2}$$

such that $0 \le v \le 1$ and $0 \le u \le 2\pi$ to be the parametric surface of a cone from $(0, 0, 1)$ to the the plane $z = -1$. The terms (x_0, y_0) correspond to the center of the osculating circle of α. The term r corresponds to the radius of the osculating circle of α at some point t.

2. Theorems

Theorem 1. *Let $\beta : \mathbb{R}^2 \to S^2 - N$ be a stereographic projection. Let α be a planar curve in $\mathbb{R}^2$ lying on the plane $z = -1$. Let $\beta \cdot \alpha(t)$ be the image of α on the unit sphere. Suppose $c_\alpha(t)$ represents the osculating circle along the curve α. Let $\gamma : \mathbb{R}^2 \to \mathbb{R}^3$ be a parametric surface of an oblique cone. The surface $\gamma \cdot c_\alpha$ will represent represent an oblique cone with base c_α, and vertex the north pole of the unit sphere. Let $\alpha(t_0)$ be some point along the curve α.*

We propose that the intersection of $\gamma \cdot c_{\alpha(t_0)}$ with the unit sphere defines the osculating circle of $\beta \cdot \alpha(t_0)$. That is, the intersection of

$\gamma \cdot c_{\alpha(t_0}$ will be the osculating circle of the image of α on the unit sphere. Hence, the Frenet apparatus T, N, B is preserved and such intersection will yield the curvature of the image of α. The torsion of the image of α can be found utilizing a differential equation which will hold true for all curves contained on the surface of a sphere.

Let $r(t) = \frac{1}{\kappa(t)}$ be the radius of the projected osculating circle of α at any point t. Where $\kappa(t)$ defines curvature of the image of α at some point t. Then, by a defining property of spherical curves we have that the following differential equation:

$$r(t)^2 + \frac{(r'(t))^2}{\tau(t)^2} = 1$$

will always be satisfied. Where τ is the torsion of the image of α along the parameter t. Hence, the torsion of the image of α can be calculated by simply differentiating $r(t)$ and solving the equation above for τ.

Proof. Let $\beta : \mathbb{R}^2 \to S^2 - N$ be a stereographic projection and let $\alpha(t)$ be a smooth curve in $\mathbb{R}^2$ such that $\beta \cdot \alpha(t)$ is its image on the sphere defined as

$$\beta \cdot \alpha(t) = \beta \cdot \alpha \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{4x(t)}{x(t)^2+y(t)^2+4} \\ \frac{4y(t)}{x(t)^2+y(t)^2+4} \\ \frac{x(t)^2+y(t)^2-4}{x(t)^2+y(t)^2+4} \end{pmatrix} \tag{3}$$

Let $C_{\alpha(t_0)}$ be the osculating circle of α at $t = t_0$. Let $p = (x_0, y_0)$ be the center of the osculating circle with radius $r = \frac{1}{\kappa}$ of $\alpha(t)$ at $t = t_0$. Further Let $\gamma_{C_{\alpha(t_0)}} : \mathbb{R}^2 \to S^2 - N$ be defined as follows:

$$\gamma_{C_{\alpha(t_0)}}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix} = \begin{pmatrix} v(x_0 + rcos(u)) \\ v(y_0 + rsin(u)) \\ 1 - 2v \end{pmatrix} \tag{4}$$

Where $0 \leq v \leq 1$ and $0 \leq u \leq 2\pi$. Be a surface of a cone from $(0, 0, 1)$ to the the plane $z = -1$ s.t. (x_0, y_0) is the center of the osculating circle of α and r is the radius of the osculating circle of α at $t = t_0$. Notice when $v = 1$ we have the following:

$$\gamma_{C_{\alpha(t_0)}}(u, v) = \begin{pmatrix} (x_0 + rcos(u)) \\ (y_0 + rsin(u)) \\ -1 \end{pmatrix}$$

Hence the intersection of the cone with the plane $z=-1$ gives us the osculating circle of α at $t=t_0$. Notice $\alpha'(t)=\frac{T(t)}{\|T(t)\|}=$ the unit tangent vector of α at $t=t_0$. Further $N(t)=\frac{T'(t)}{\|T'\|}$. So, we have $\kappa(t)=\| T'(t) \|$ = Curvature of $\alpha(t)$. Since $1/\kappa(t)=r$ of $C_\alpha(t)$ and $N(t)=\frac{T'(t)}{\|T'\|}$. We have $\frac{T'(t)_x}{\|T'\|}=x(t)+\frac{T'(t)_x}{\kappa(t)}$ and $y_0=y(t)+\frac{T'(t)_y}{\|T'\|}=y(t)+\frac{T'(t)_y}{\kappa(t)}$.

Substituting gives us

$$\gamma_{C_{\alpha(t_0)}}(u,v)=\begin{pmatrix} v(x(t)+\frac{T'(t_0)_x}{\kappa(t_0)}+rcos(u) \\ v(y(t)+\frac{T'(t_0)_y}{\kappa(t_0)}+rsin(u) \\ 1-2v \end{pmatrix}.$$

The intersection of the cone defined by γ with the unit sphere will be the points that satisfy

$$(v(x(t)+\frac{T'(t_0)_x}{\kappa(t_0)}+rcos(u))^2+(v(y(t)+\frac{T'(t_0)_y}{\kappa(t_0)}+rsin(u))^2+(1-2v)^2=1$$

$\Rightarrow$

$$v^2(x_0+rcos(u))^2+v^2(y_0+rsin(u))^2+1-4v+4v^2=1.$$

So, $v^2((x_0+rcos(u))^2+(y_0+rsin(u))^2+4)=4v$. Solving for v we get

$$v^*=v=\frac{4}{(x_0+rcos(u))^2+(y_0+rsin(u))^2+4} \tag{5}$$

.

By substituting $v*$ into (4) we get

$$\gamma_{C_{\alpha(t_0)}}(u,v^*)=\begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix}=\begin{pmatrix} \frac{4(x_0+rcos(u))}{(x_0+rcos(u))^2+(y_0+rsin(u))^2+4} \\ \frac{4(y_0+rsin(u))}{(x_0+rcos(u))^2+(y_0+rsin(u))^2+4} \\ 1-\frac{8}{(x_0+rcos(u))^2+(y_0+rsin(u))^2+4} \end{pmatrix} \tag{6}$$

Therefore,

$$\gamma_{C_{\alpha(t_0)}}(u,v*)=\begin{pmatrix} \frac{4(x(t)+\frac{T'(t_0)_x}{\kappa(t_0)}+rcos(u))}{(x(t)+\frac{T'(t_0)_x}{\kappa(t_0)}+rcos(u))^2+(y(t)+\frac{T'(t_0)_y}{\kappa(t_0)}+rsin(u))^2+4} \\ \frac{4(y(t)+\frac{T'(t_0)_y}{\kappa(t_0)}+rsin(u))}{(x(t)+\frac{T'(t_0)_x}{\kappa(t_0)}+rcos(u))^2+(y(t)+\frac{T'(t_0)_y}{\kappa(t_0)}+rsin(u))^2+4} \\ 1-\frac{8}{(x(t)+\frac{T'(t_0)_x}{\kappa(t_0)}+rcos(u))^2+(y(t)+\frac{T'(t_0)_y}{\kappa(t_0)}+rsin(u))^2+4} \end{pmatrix} \tag{7}$$

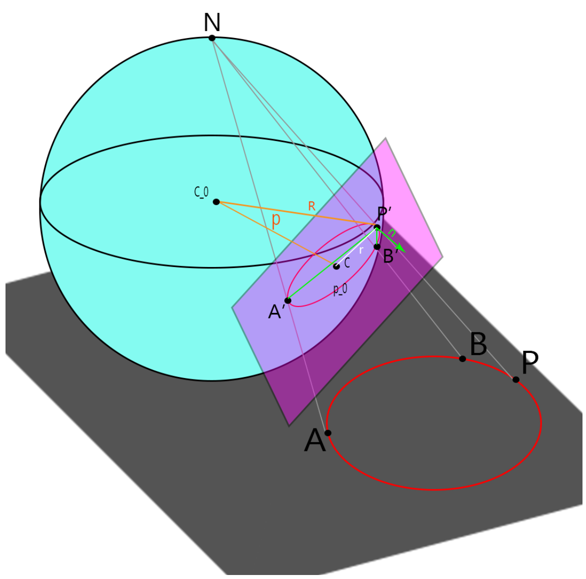
Substituting $r = \frac{1}{\kappa(t_0)}$ we get

$$\gamma_{C_{\alpha(t_0)}}(u, v^*) = \begin{pmatrix} \frac{4(x(t)+\frac{T'(t_0)x}{\kappa(t_0)}+\frac{1}{\kappa(t_0)}cos(u))}{(x(t)+\frac{T'(t_0)x}{\kappa(t_0)}+\frac{1}{\kappa(t_0)}cos(u))^2+(y(t)+\frac{T'(t_0)y}{\kappa(t_0)}+\frac{1}{\kappa(t_0)}sin(u))^2+4} \\ \frac{4(y(t)+\frac{T'(t_0)y}{\kappa(t_0)}+\frac{1}{\kappa(t_0)}sin(u))}{(x(t)+\frac{T'(t_0)x}{\kappa(t_0)}+\frac{1}{\kappa(t_0)}cos(u))^2+(y(t)+\frac{T'(t_0)y}{\kappa(t_0)}+\frac{1}{\kappa(t_0)}sin(u))^2+4} \\ 1 - \frac{8}{(x(t)+\frac{T'(t_0)x}{\kappa(t_0)}+\frac{1}{\kappa(t_0)}cos(u))^2+(y(t)+\frac{T'(t_0)y}{\kappa(t_0)}+\frac{1}{\kappa(t_0)}sin(u))^2+4} \end{pmatrix} \tag{8}$$

which yields the equation of the osculating circle of $\beta \cdot \alpha(t)$ at $t = t_0$ Let $P = \alpha(t_0)$. Let B, A be 2 arbitrary points on $C_{\alpha(t_0)}$. Further let A', B', P' be the image of A, B, P on the unit sphere. Because A', B', P' all lie on $\gamma_{C_{\alpha(t_0)}}(u, v^*)$. We have

$$\vec{P'A'} \times \vec{P'B'} = \vec{n}$$

the normal vector to the plane to which $\gamma_{C_{\alpha(t_0)}}(u, v^*)$ lies on. Let K be such plane defined by $Ex + Fy + Gz = D$, such that $P' = (x, y, z)$ and $\vec{n} = (E, F, G)$



Let c be the center of the circle $\gamma_{C_{\alpha(t_0)}}(u, v^*)$. Let p be the signed distance from the center of the sphere to c. Let R be the radius of the sphere

Notice that

$$p = \frac{(c_0 - P') \cdot \vec{n}}{\parallel \vec{n} \parallel} = \frac{-D}{\sqrt{E^2 + F^2 + G^2}} \tag{9}$$

and

$$r = \frac{1}{\kappa(\beta(\alpha(t_0))} = \sqrt{R^2 - p^2} \tag{10}$$

So, $\gamma_{C_{\alpha(t_0)}}(u, v^*)$ gives us a mapping of the curvature $\kappa(t)$ of $\alpha(t)$ to the curvature of its image $\beta \cdot \alpha(t)$ at $t = t_0$.Let $\tau(t_0)$ be the torsion of the image $\beta(\alpha(t))$ on the unit sphere at $t = t_0$. Because $\beta(\alpha(t))$ lies on the unit sphere we know the differential equation

$$r(t) + \frac{(r'(t))^2}{\tau(t)^2} = 1 \tag{11}$$

Will always be satisfied. Recall $r(t) = \frac{1}{\kappa(\beta(\alpha(t))} = \sqrt{R^2 - p^2} = \sqrt{1 - p^2}$ and $p = \frac{(c_0 - P')\cdot \vec{n}}{\|\vec{n}\|} = \frac{-D}{\sqrt{E^2+F^2+G^2}}$. Differentiating we get $r'(t) = \frac{(.5)}{\sqrt{(1-p^2)}} \cdot (-2p \cdot \frac{\|\vec{n(t)}\| \cdot (-P' \cdot \frac{\partial \vec{n(t)}}{\partial t} + \vec{n(t)} \cdot \frac{\partial - P'}{\partial t})}{(\|\vec{n(t)}\|)^2})$.

So, we have

$$\frac{1}{\tau(t)^2} = \frac{-(\frac{(c_0 - P')\cdot \vec{n}}{\|\vec{n}\|})^2}{(\frac{(.5)}{\sqrt{(1-p^2)}} \cdot (-2p \cdot \frac{\|\vec{n(t)}\| \cdot (-P' \cdot \frac{\partial \vec{n(t)}}{\partial t} + \vec{n(t)} \cdot \frac{\partial - P'}{\partial t})}{(\|\vec{n(t)}\|)^2}))^2} \tag{12}$$

$\square$