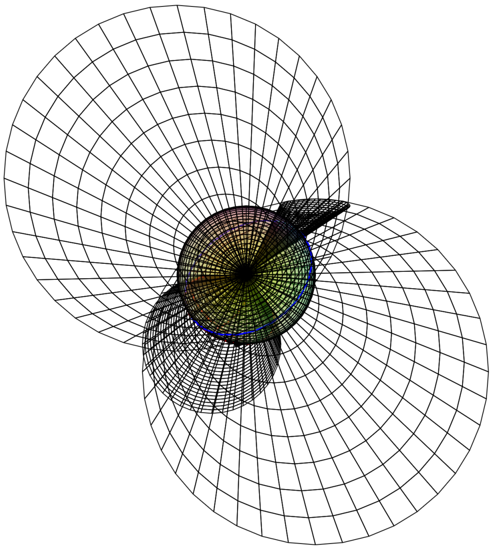

Dynamical Systems built from curvature and torsion of planar curves

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1 Abstract

Stereographic Projection is a smooth bijective mapping from a plane to a sphere. The mapping is a conformal projection preserving angles and circles. That is the image of a circle on the sphere is a circle in the plane and the angle between two lines on the sphere is the same as the angle between their images in the plane. A smooth curve lying on the plane can be projected to the sphere. Any smooth planar curve has zero torsion and some curvature. As a result of the preservative properties of the projection mapping we are able to make a bijective mapping of the curvature and torsion of a curve to the curvature and torsion of its projected image. Such mapping allows for a dynamical system to be built. Each iterate of the system takes in a smooth parametric planar curve and outputs a smooth parametric curve. The parametric curve created from the output of each iterate is defined by the mapping of the projected curvature and torsion of the input curve parametrized by the parameter of the input of the iterate.

2 Introduction

Let $\alpha : R^2 \to R^2$ s.t. $\alpha(t) = (x(t), y(t))$ be a planar curve on the plane $z = -1$.
Let $N = (0, 0, 1)$ be the "north pole" of the unit sphere centered at the origin.
For any point P along α the stereographic projection of P is defined as the unique point P' such that P' is the intersection of the line through N and P with the unit sphere.
Let $\beta : R^2 \to S^2 - N$ be a stereographic projection of $\alpha(t)$ onto the unit sphere
We define $\beta \cdot \alpha(t)$ as follows:

$$\beta \cdot \alpha(t) = \beta \cdot \alpha \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{4x(t)}{x(t)^2+y(t)^2+4} \\ \frac{4y(t)}{x(t)^2+y(t)^2+4} \\ \frac{x(t)^2+y(t)^2-4}{x(t)^2+y(t)^2+4} \end{pmatrix}$$

(1)

Which gives us the image of some curve $\alpha(t)$ onto the sphere.

3 Theorem and Proof

Theorem 1. Let $\beta : R^2 \to S^2 - N$ be a stereographic projection.

Let $\alpha(t)$ be a planar curve in R^2 on the plane $z = -1$ and $\beta \cdot \alpha(t)$ the image of $\alpha(t)$ on the unit sphere.

Let $C_{\alpha(t_0)}$ be the osculating circle of α at $t = t_0$

Let $\gamma_{C_{\alpha(t)}} : R^2 \to R^3$ be a cone with vertex the "north pole" of the sphere to the plane $z = -1$ with base $C_{\alpha(t)}$.

Let $\alpha(t_0)$ be some point on α

The intersection of $\gamma_{C_{\alpha(t_0)}}$ defines the osculating circle of $\beta \cdot \alpha(t_0)$. Therefore the Frenet apparatus T, N, B is preserved and such intersection yields the curvature of the image of $\alpha(t)$ at $t = t_0$

The torsion of the image of $\alpha(t)$ at $t = t_0$ can be found utilizing a differential equation always satisfied by a spherical curve.

Proof. Let $\beta : R^2 \to S^2 - N$ be a stereographic projection and let $\alpha(t)$ be a smooth curve in R^2 such that $\beta \cdot \alpha(t)$ is its image on the sphere defined as:

$$\beta \cdot \alpha(t) = \beta \cdot \alpha \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{4x(t)}{x(t)^2+y(t)^2+4} \\ \frac{4y(t)}{x(t)^2+y(t)^2+4} \\ \frac{x(t)^2+y(t)^2-4}{x(t)^2+y(t)^2+4} \end{pmatrix} \tag{2}$$

Let $C_{\alpha(t_0)}$ be the osculating circle of α at $t = t_0$
Let $p = (x_0, y_0)$ be the center of the osculating circle with radius $r = \frac{1}{\kappa}$ of $\alpha(t)$ at $t = t_0$.

Further Let $\gamma_{C_{\alpha(t_0)}} : R^2 \to S^2 - N$ be defined:

$$\gamma_{C_{\alpha(t_0)}}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix} = \begin{pmatrix} v(x_0 + rcos(u)) \\ v(y_0 + rsin(u)) \\ 1 - 2v \end{pmatrix} \tag{3}$$

Where $0 \leq v \leq 1$ and $0 \leq u \leq 2\pi$

Be a surface of a cone from $(0,0,1)$ to the the plane $z = -1$ s.t. (x_0, y_0) is the center of the osculating circle of α and r is the radius of the osculating circle of α at $t = t_0$

Notice when $v = 1$ we have:

$$\gamma_{C_{\alpha(t_0)}}(u,v) = \begin{pmatrix} (x_0 + rcos(u)) \\ (y_0 + rsin(u)) \\ -1 \end{pmatrix}$$

Hence the intersection of the cone with the plane $z = -1$ gives us the osculating circle of α at $t = t_0$

Notice $\alpha'(t) = \frac{T(t}{\|T(t)\|}$ = the unit tangent vector of α at $t = t_0$

Further $N(t) = \frac{T'(t)}{\|T'\|}$

So, we have $\kappa(t) =\| T'(t) \|$ = Curvature of $\alpha(t)$

Since $1/\kappa(t) = r$ of $C_\alpha(t)$ and $N(t) = \frac{T'(t)}{\|T'\|}$

$\Rightarrow x_0 = x(t) + \frac{T'(t)_x}{\|T'\|} = x(t) + \frac{T'(t)_x}{\kappa(t)}$ and

$y_0 = y(t) + \frac{T'(t)_y}{\|T'\|} = y(t) + \frac{T'(t)_y}{\kappa(t)}$

By substituting we get:

$$\gamma_{C_{\alpha(t_0)}}(u,v) = \begin{pmatrix} v(x(t) + \frac{T'(t_0)_x}{\kappa(t_0)} + rcos(u) \\ v(y(t) + \frac{T'(t_0)_y}{\kappa(t_0)} + rsin(u) \\ 1 - 2v \end{pmatrix}$$

The intersection of the cone defined by with the unit sphere will be the points that satisfy:

$$(v(x(t) + \frac{T'(t_0)_x}{\kappa(t_0)} + rcos(u))^2 + (v(y(t) + \frac{T'(t_0)_y}{\kappa(t_0)} + rsin(u))^2 + (1 - 2v)^2 = 1$$

$$\Rightarrow$$

$$v^2(x_0 + rcos(u))^2 + v^2(y_0 + rsin(u))^2 + 1 - 4v + 4v^2 = 1$$

So,

$$v^2((x_0+rcos(u))^2+(y_0+rsin(u))^2+4)=4v$$

Solving for v we get:

$$v^* = \frac{4}{(x_0+rcos(u))^2+(y_0+rsin(u))^2+4} \tag{4}$$

By substituting $v*$ into (3) we get:

$$\gamma_{C_{\alpha(t_0)}}(u,v^*) = \begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix} = \begin{pmatrix} \frac{4(x_0+rcos(u))}{(x_0+rcos(u))^2+(y_0+rsin(u))^2+4} \\ \frac{4(y_0+rsin(u))}{(x_0+rcos(u))^2+(y_0+rsin(u))^2+4} \\ 1-\frac{8}{(x_0+rcos(u))^2+(y_0+rsin(u))^2+4} \end{pmatrix} \tag{5}$$

Therefore

$$\gamma_{C_{\alpha(t_0)}}(u,v*) = \begin{pmatrix} \frac{4(x(t)+\frac{T'(t_0)x}{\kappa(t_0)}+rcos(u))}{(x(t)+\frac{T'(t_0)x}{\kappa(t_0)}+rcos(u))^2+(y(t)+\frac{T'(t_0)y}{\kappa(t_0)}+rsin(u))^2+4} \\ \frac{4(y(t)+\frac{T'(t_0)y}{\kappa(t_0)}+rsin(u))}{(x(t)+\frac{T'(t_0)x}{\kappa(t_0)}+rcos(u))^2+(y(t)+\frac{T'(t_0)y}{\kappa(t_0)}+rsin(u))^2+4} \\ 1-\frac{8}{(x(t)+\frac{T'(t_0)x}{\kappa(t_0)}+rcos(u))^2+(y(t)+\frac{T'(t_0)y}{\kappa(t_0)}+rsin(u))^2+4} \end{pmatrix}$$

(6)

Substituting $r=\frac{1}{\kappa(t_0)}$ we get:

$$\gamma_{C_{\alpha(t_0)}}(u,v^*) = \begin{pmatrix} \frac{4(x(t)+\frac{T'(t_0)x}{\kappa(t_0)}+\frac{1}{\kappa(t_0)}cos(u))}{(x(t)+\frac{T'(t_0)x}{\kappa(t_0)}+\frac{1}{\kappa(t_0)}cos(u))^2+(y(t)+\frac{T'(t_0)y}{\kappa(t_0)}+\frac{1}{\kappa(t_0)}sin(u))^2+4} \\ \frac{4(y(t)+\frac{T'(t_0)y}{\kappa(t_0)}+\frac{1}{\kappa(t_0)}sin(u))}{(x(t)+\frac{T'(t_0)x}{\kappa(t_0)}+\frac{1}{\kappa(t_0)}cos(u))^2+(y(t)+\frac{T'(t_0)y}{\kappa(t_0)}+\frac{1}{\kappa(t_0)}sin(u))^2+4} \\ 1-\frac{8}{(x(t)+\frac{T'(t_0)x}{\kappa(t_0)}+\frac{1}{\kappa(t_0)}cos(u))^2+(y(t)+\frac{T'(t_0)y}{\kappa(t_0)}+\frac{1}{\kappa(t_0)}sin(u))^2+4} \end{pmatrix} \tag{7}$$

Which yields the equation of the osculating circle of $\beta\cdot\alpha(t)$ at $t=t_0$

Let $P = \alpha(t_0)$

Let B, A be 2 arbitrary points on $C_{\alpha(t_0)}$

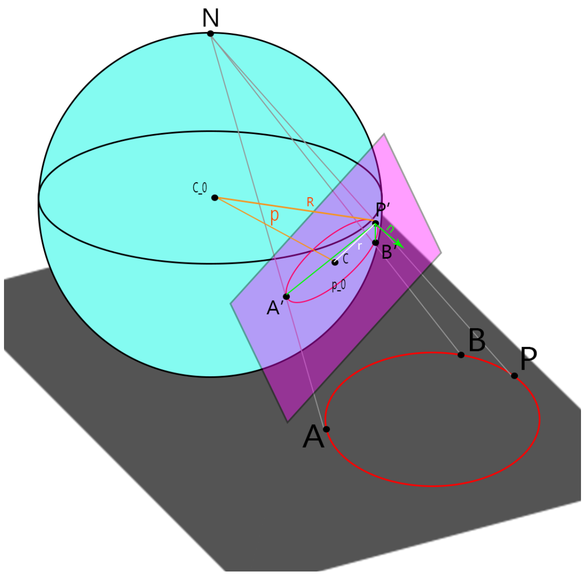
Further let A', B', P' be the image of A, B, P on the unit sphere.

Because A', B', P' all lie on $\gamma_{C_{\alpha(t_0)}}(u, v^*)$
$\Rightarrow$
$\vec{P'A'} \times \vec{P'B'} = \vec{n}$
the normal vector to the plane to which$\gamma_{C_{\alpha(t_0)}}(u, v^*)$ lies on.

Let K be such plane defined by $Ex + Fy + Gz = D$

Such that

$P' = (x, y, z)$ and $\vec{n} = (E, F, G)$



Let c be the center of the circle $\gamma_{C_{\alpha(t_0)}}(u, v^*)$
Let p be the signed distance from the center of the sphere to c
Let R be the radius of the sphere

Notice

$$p = \frac{(c_0 - P') \cdot \vec{n}}{\| \vec{n} \|} = \frac{-D}{\sqrt{E^2 + F^2 + G^2}} \tag{8}$$

and

$$r = \frac{1}{\kappa(\beta(\alpha(t_0))} = \sqrt{R^2 - p^2} \tag{9}$$

So, $\gamma_{C_{\alpha(t_0)}}(u, v^*)$ gives us a mapping of the curvature $\kappa(t)$ of $\alpha(t)$ to the curvature of its image $\beta \cdot \alpha(t)$ at $t = t_0$

Let $\tau(t_0)$ be the torsion of the image $\beta(\alpha(t))$ on the unit sphere at $t = t_0$.

Because $\beta(\alpha(t))$ lies on the unit sphere we know the differential equation

$$r(t) + \frac{(r'(t))^2}{\tau(t)^2} = 1 \tag{10}$$

Will always be satisfied.

Recall $r(t) = \frac{1}{\kappa(\beta(\alpha(t))} = \sqrt{R^2 - p^2} = \sqrt{1 - p^2}$

and $p = \frac{(c_0 - P') \cdot \vec{n}}{\|\vec{n}\|} = \frac{-D}{\sqrt{E^2 + F^2 + G^2}}$

Differentiating we get:
$r'(t) = \frac{(.5)}{\sqrt{(1-p^2)}} \cdot (-2p \cdot \frac{\|n(\vec{t})\| \cdot (-P' \cdot \frac{\partial n(\vec{t})}{\partial t} + n(\vec{t}) \cdot \frac{\partial - P'}{\partial t})}{(\|n(\vec{t})\|)^2})$

So, we have

$$\frac{1}{\tau(t)^2} = \frac{-(\frac{(c_0-P')\cdot\vec{n}}{\|\vec{n}\|})^2}{(\frac{(.5)}{\sqrt{(1-p^2)}} \cdot (-2p \cdot \frac{\|n\vec{(t)}\|\cdot(-P'\cdot\frac{\partial n\vec{(t)}}{\partial t}+n\vec{(t)}\cdot\frac{\partial -P'}{\partial t})}{(\|n\vec{(t)}\|)^2}))^2} \tag{11}$$

$\square$