DISTRIBUTIVITY OF ORDINAL ARITHMETIC

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Abstract. In this paper we will investigate left-distributivity and right-distributivity of ordinal multiplication by proving two theorems. We prove Theorem 1 using the method of transfinite induction. This theorem states that for all ordinals α, β, γ, that $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$ (left-distributivity). We prove Theorems 2 by way of counter example. This theorem states that $(\beta + \gamma) \cdot \alpha = \beta \cdot \alpha + \gamma \cdot \alpha$ does not hold for all ordinals α, β, γ (right-distributivity).

Contents

1. Lemmas

We first introduce six lemmas needed to prove Theorem 1.

Lemma 1.1. *For all ordinals α, β, δ such that $\beta \leq \delta$ we have*

$$\alpha + \beta \leq \alpha + \delta.$$

Lemma 1.2. *For all ordinals α, β, δ such that $\beta < \delta$ we have*

$$\alpha + \beta < \alpha + \delta.$$

Lemma 1.3. *For all ordinals α, β, δ such that $\beta \leq \delta$ we have*

$$\alpha \cdot \beta \leq \alpha \cdot \delta.$$

Lemma 1.4. *For all ordinals α, β, δ such that $\beta < \delta$ we have*

$$\alpha \cdot \beta < \alpha \cdot \delta.$$

Lemma 1.5. *For all ordinals α, β, we have*

$$\alpha + \bigcup_{\psi < \beta} \psi = \bigcup_{\psi < \beta} \alpha + \psi.$$

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Proof. Let α, β be ordinals. Let

$$A = \{\gamma \in ON \mid \gamma < \alpha + \bigcup_{\psi<\beta} \psi\}.$$

and

$$B = \{\gamma \in ON \mid \gamma < \bigcup_{\psi<\beta} \alpha + \psi\}.$$

We will show that $A = B$. Let $\gamma \in B$. We will show $\gamma \in A$. Since $\gamma \in B$, we have $\gamma < \bigcup_{\psi<\beta} \alpha + \psi$. Hence, $\gamma < \alpha + \hat{\psi}$ for some $\hat{\psi} \in \beta$. Let $\hat{\psi} \in \beta$ such that $\gamma < \alpha + \hat{\psi}$. By Lemma 1.1 we know that $\alpha + \hat{\psi} \leq \alpha + \bigcup_{\psi<\beta} \psi$. Hence we have $\gamma < \alpha + \hat{\psi} \leq \alpha + \bigcup_{\psi<\beta} \psi$. Hence, $\gamma \in A$ which gives us $B \subseteq A$. We will now show that $A \subseteq B$. Let $\gamma \in A$. We will show that $\gamma \in B$. Since $\gamma \in A$ we have that $\gamma < \alpha + \bigcup_{\psi<\beta} \psi$.

Successor Case: Suppose β is a successor ordinal. Let $\kappa = \bigcup_{\psi<\beta} \psi$. Since β is a successor ordinal we have κ is a successor ordinal. Further, we know $\kappa \in \beta$. Hence there exists a $\hat{\psi} \in \beta$ such that $\hat{\psi} = \kappa$. Let $\hat{\psi} \in \beta$ such that $\hat{\psi} = \kappa$. So, we have

$$\gamma < \alpha + \bigcup_{\psi<\beta} \psi = \alpha + \kappa = \alpha + \hat{\psi}.$$

So, $\gamma < \alpha + \hat{\psi} \leq \bigcup_{\psi<\beta} \alpha + \psi$. Hence, $\gamma \in B$.

Limit Case: Suppose β is a limit ordinal. Since $\kappa = \bigcup_{\psi<\beta} \psi$, we have that κ is a limit ordinal. So, by definition $\kappa = \bigcup_{\xi<\kappa} \xi$. Notice since $\gamma \in A$, we have

$$\gamma < \alpha + \bigcup_{\psi<\beta} \psi = \alpha + \kappa = \alpha + \bigcup_{\xi<\kappa} \xi.$$

Since κ is a limit ordinal, we have

$$\alpha + \bigcup_{\xi<\kappa} \xi = \bigcup_{\xi<\kappa} \alpha + \xi.$$

Thus $\gamma < \alpha + \hat{\xi}$ for some $\hat{\xi} \in \kappa$. Since $\hat{\xi} < \kappa$, we have $\hat{\xi} < \bigcup_{\psi<\beta} \psi$. Therefore $\hat{\xi} < \hat{\psi}$ for some $\hat{\psi} \in \beta$. Hence,

$$\gamma < \alpha + \hat{\xi} < \alpha + \hat{\psi}$$

for some $\hat{\psi} \in \beta$. Therefore $\gamma < \alpha + \hat{\psi} \leq \bigcup_{\psi<\beta} \alpha + \psi$. So we have $\gamma \in B$, hence $A \subseteq B$. $\square$

Lemma 1.6. *For any ordinals α, β, γ, we have*

$$\alpha \cdot \bigcup_{\psi<\gamma} (\beta + \psi) = \bigcup_{\psi<\gamma} \alpha \cdot (\beta + \psi).$$

Proof. Let α, β, γ be ordinals. Let

$$A = \{\delta \in ON \mid \delta < \alpha \cdot \bigcup_{\psi<\gamma} (\beta + \psi)\}$$

and

$$B = \{\delta \in ON \mid \delta < \bigcup_{\psi<\gamma} \alpha \cdot (\beta + \psi)\}.$$

We will show $A = B$. Let $\delta \in B$. We will show $\delta \in A$. Since $\delta \in B$, we have $\delta < \bigcup_{\psi<\gamma} \alpha \cdot (\beta + \psi)$. Hence $\delta < \alpha \cdot (\beta + \hat{\psi})$ for some $\hat{\psi} \in \gamma$. Let $\hat{\psi} \in \gamma$ such that $\delta < \alpha \cdot (\beta + \hat{\psi})$. By Lemma 1.3 we have $\alpha \cdot (\beta + \hat{\psi}) \leq \alpha \cdot \bigcup_{\psi<\gamma} (\beta + \psi)$. Therefore

$$\delta < \alpha \cdot (\beta + \hat{\psi}) \leq \alpha \cdot \bigcup_{\psi<\gamma} (\beta + \psi).$$

Hence $\delta \in A$. It follows that $B \subseteq A$. We will now show that $A \subseteq B$. Let $\delta \in A$. We will show that $\delta \in B$. Since $\delta \in A$, we have $\delta < \alpha \cdot \bigcup_{\psi<\gamma} (\beta + \psi)$. By Lemma 1.5 we have

$$\delta < \alpha \cdot \bigcup_{\psi<\gamma} (\beta + \psi) = \alpha \cdot (\beta + \bigcup_{\psi<\gamma} \psi).$$

Let $\kappa = \bigcup_{\psi<\gamma} \psi$. So, we have

$$\delta < \alpha \cdot \bigcup_{\psi<\gamma} (\beta + \psi) = \alpha \cdot (\beta + \bigcup_{\psi<\gamma} \psi) = \alpha \cdot (\beta + \kappa).$$

We will consider the two cases when $\kappa \in \gamma$ and $\kappa \notin \gamma$. If $\kappa \in \gamma$, then we know there exists some $\hat{\psi} \in \gamma$ such that $\hat{\psi} = \kappa$. Let $\hat{\psi} \in \gamma$ such that $\hat{\psi} = \kappa$. So, we have $\delta < \alpha \cdot (\beta + \kappa) = \alpha \cdot (\beta + \hat{\psi})$. By Lemma 1.3 we have

$$\delta < \alpha \cdot (\beta + \kappa) = \alpha \cdot (\beta + \hat{\psi}) \leq \bigcup_{\psi<\gamma} \alpha \cdot (\beta + \psi).$$

So, we have $\delta < \bigcup_{\psi<\gamma}(\beta+\psi)$. Therefore $\delta \in B$. If $\kappa \notin \gamma$, then κ is a limit ordinal. Recall from above that since $\delta \in A$, we have

$$\delta < \alpha \cdot \bigcup_{\psi<\gamma}(\beta+\psi) = \alpha \cdot (\beta + \bigcup_{\psi<\gamma}\psi) = \alpha \cdot (\beta+\kappa).$$

Since κ is a limit ordinal, we have $\kappa = \bigcup_{\xi<\kappa}\xi$. So,

$$\alpha \cdot (\beta+\kappa) = \alpha \cdot (\beta + \bigcup_{\xi<\kappa}\xi).$$

By Lemma 1.5, we have

$$\alpha \cdot (\beta + \bigcup_{\xi<\kappa}\xi) = \alpha \cdot \bigcup_{\xi<\kappa}(\beta+\xi).$$

Since $\xi < \kappa$, by Lemma 1.2 we have $\beta+\xi < \beta+\kappa$. So,

$$\alpha \cdot \bigcup_{\xi<\kappa}(\beta+\xi) = \alpha \cdot \bigcup_{\beta+\xi<\beta+\kappa}(\beta+\xi).$$

Let $\sigma = \beta+\xi$, then we have

$$\alpha \cdot \bigcup_{\beta+\xi<\beta+\kappa}(\beta+\xi) = \alpha \cdot \bigcup_{\beta\leq\sigma<\beta+\kappa}(\sigma).$$

Since κ is a limit ordinal, we have

$$\alpha \cdot \bigcup_{\beta\leq\sigma<\beta+\kappa}(\sigma) = \bigcup_{\beta\leq\sigma<\beta+\kappa}(\alpha\cdot\sigma).$$

So, we have

$$\delta < \bigcup_{\beta\leq\sigma<\beta+\kappa}(\alpha\cdot\sigma).$$

Recall that $\sigma = \beta+\xi$ such that $\xi < \kappa$. Hence, $\delta < \alpha \cdot (\beta+\hat{\xi})$ for some $\hat{\xi} \in \kappa$. Since $\kappa = \bigcup_{\psi<\gamma}\psi$ and $\hat{\xi} < \kappa$. Then there exists a $\hat{\psi} \in \gamma$ such that $\hat{\psi} > \hat{\xi}$. Let $\hat{\psi}$ and $\hat{\xi}$ be such ordinals. Then we have

$$\delta < \alpha \cdot (\beta+\hat{\xi}) < \alpha \cdot (\beta+\hat{\psi}) \leq \bigcup_{\psi<\gamma}\alpha\cdot(\beta+\psi).$$

Hence $\delta \in B$. Therefore $A \subseteq B$. Since $A \subseteq B$ and $B \subseteq A$, we have $A = B$. $\square$

2. Theorems

Theorem 1. *For all ordinals α, β, γ, we have*

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma.$$

That is, ordinal multiplication is left-distributive.

Proof. Let α, β, γ be ordinals. We will show by way of transfinite induction on γ that

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma.$$

Let $S = \{\gamma \in ON \mid \ \forall \alpha, \beta \in ON(\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma)\}$.

Successor Case

We begin with the case in which γ is a successor ordinal. Suppose $\gamma = 0$, then we have

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot (\beta + 0) = \alpha \cdot \beta = \alpha \cdot \beta + \alpha \cdot 0 = \alpha \cdot \beta + \alpha \cdot \gamma.$$

Hence $\gamma = 0$ belongs to our set S. Suppose $\gamma \in S$. We will show $\gamma + 1 \in S$. Let γ be an ordinal such that $\gamma \in S$. Then, by definition we have $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$. Notice

$$\alpha \cdot (\beta + (\gamma + 1)) = \alpha \cdot ((\beta + \gamma) + 1) = \alpha \cdot (\beta + \gamma) + \alpha.$$

Since $\gamma \in S$, we have

$$\alpha \cdot (\beta + \gamma) + \alpha = \alpha \cdot \beta + \alpha \cdot \gamma + \alpha = \alpha \cdot \beta + \alpha \cdot (\gamma + 1).$$

Therefore if $\gamma \in S$ then $\gamma + 1 \in S$.

Limit Case

We will now show that Theorem 1 holds when γ is a limit ordinal. Suppose γ is a limit ordinal such that for all $\psi < \gamma$ we have $\psi \in S$. Notice that since γ is a limit ordinal, we have that $\beta + \gamma$ is a limit ordinal. By definition we have $\beta + \gamma = \bigcup_{\psi < \gamma} \beta + \psi$. So,

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \bigcup_{\psi < \gamma} \beta + \psi.$$

By Lemma 1.6, we have

$$\alpha \cdot \bigcup_{\psi < \gamma} \beta + \psi = \bigcup_{\psi < \gamma} \alpha \cdot (\beta + \psi).$$

Since for all $\psi < \gamma$ we have $\psi \in S$. We know

$$\bigcup_{\psi < \gamma} \alpha \cdot (\beta + \psi) = \bigcup_{\psi < \gamma} \alpha \cdot \beta + \alpha \cdot \psi.$$

By Lemma 1.5, we have

$$\bigcup_{\psi<\gamma} \alpha \cdot \beta + \alpha \cdot \psi = \alpha \cdot \beta + \bigcup_{\psi<\gamma} \alpha \cdot \psi.$$

By definition $\bigcup\limits_{\psi<\gamma} \alpha \cdot \psi = \alpha \cdot \gamma$. Therefore

$$\alpha \cdot \beta + \bigcup_{\psi<\gamma} \alpha \cdot \psi = \alpha \cdot \beta + \alpha \cdot \gamma.$$

Hence, we can conclude that

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$$

for all ordinals α, β, γ. $\square$

Theorem 2. *For all ordinals α, β, γ, we have*

$$(\beta + \gamma) \cdot \alpha = \beta \cdot \alpha + \gamma \cdot \alpha$$

does not hold. That is, ordinal multiplication is not right-distributive.

Proof. Let α, β, γ be ordinals such that $\alpha = \omega, \beta = 1, \gamma = 1$. We will show that $(\beta + \gamma) \cdot \alpha \neq \beta \cdot \alpha + \gamma \cdot \alpha$. Notice

$$(\beta + \gamma) \cdot \alpha = (1 + 1) \cdot \omega = 2 \cdot \omega = \omega$$

and

$$\beta \cdot \alpha + \gamma \cdot \alpha = 1 \cdot \omega + 1 \cdot \omega = \omega + \omega.$$

Hence, by way of counter example we conclude that

$$(\beta + \gamma) \cdot \alpha = \beta \cdot \alpha + \gamma \cdot \alpha$$

does not hold for all ordinals α, β, γ. $\square$