B.A./B.Sc. 4th Semester (General) Examination, 2023 (CBCS)
Subject : Mathematics
Course : BMG4CC1D & Math GE-4
(Algebra)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
Notation and symbols have their usual meaning.

Group-A
(Marks : 20)

1. Answer *any ten* questions: 2×10=20

(a) Give an example of a commutative group of order 6.

(b) For any two elements a and b of a group $(G,\circ)$ show that $(a\circ b)^2 = a^2\circ b^2$ if $a\circ b = b\circ a\ \forall\, a,b\ \in G$.

(c) Show that the group S_3 is non-abelian.

(d) Show that the group $(G,\circ)$ when $G=\{1,\omega,\omega^2\}$, ω is a cube root of unity, is cyclic.

(e) Show that the permutation $\begin{pmatrix}1&2&3&4&5\\3&2&1&5&4\end{pmatrix}$ is even.

(f) Find all the subgroups of a group G of prime order.

(g) Show that the union of any two subgroups of a group is not necessarily a subgroup.

(h) Show that every cyclic group is abelian.

(i) Find all cyclic subgroups of the group $(\mathbb{Z}_4,+)$.

(j) Show that A_3 is a normal subgroup of S_3.

(k) Give an example of a commutative ring with unity element.

(l) Define an ideal of a ring.

(m) Prove that in an integral domain $(R,+,\cdot)$, $a^2=a \Rightarrow$ either $a=0$ or $a=e$.

(n) Find the elements in $(\mathbb{Z}_6,\ +,\ \cdot)$ which are divisor of zero.

(o) Show that $4\mathbb{Z}$ is an ideal of the ring $(2\mathbb{Z},+,\ \cdot)$.

Group-B

(Marks : 20)

2. Answer *any four* questions: 5×4=20

(a) Show that the set of all n-th roots of unity is an abelian group under usual multiplication.

(b) (i) If $a^2 = e$ for all $a \in G$, prove that G is commutative.

(ii) In a commutative group $(G,\circ)$, prove that $(a \circ b)^{-1} = a^{-1} \circ b^{-1}\ \forall\, a, b \in G$. 3+2

(c) Prove that every subgroup of a cyclic group is cyclic.

(d) Prove that the intersection of two subrings is a subring.

(e) A commutative ring R with unit element is an integral domain if and only if for every non-zero element $a \in R, a.u = a.v \Rightarrow u = v$ for all $u, v \in R$.

(f) State and prove Lagrange's theorem for finite group.

Group-C

(Marks : 20)

3. Answer *any two* questions: 10×2=20

(a) (i) State and prove a necessary and sufficient condition for a non-empty subset H of a group $(G,\circ)$ to be a subgroup of G.

(ii) Show that the centre $Z(G)$ of a group G is a normal subgroup of G. (1+4)+5

(b) (i) Define Quotient group. Prove that every quotient group of an abelian group is abelian.

(ii) Show that every proper subgroup of a group of order 6 is cyclic. (2+3)+5

(c) (i) Show that the set $\frac{G}{H}$ of all cosets of a normal subgroup H in G is a group with respect to the operation defined by $(aH)(bH) = abH, \forall a, b \in G$.

(ii) Prove that the set $M_2(3\mathbb{Z}) = \left\{\begin{pmatrix} 3a & 3b \\ 3c & 3d \end{pmatrix} : a, b, c, d \in \mathbb{Z}\right\}$ is a subring of the matrix ring $M_2(\mathbb{Z})$, where $M_2(\mathbb{Z}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}\right\}$ with respect to matrix addition and multiplication. 5+5

(d) (i) Prove that every finite integral domain is a field.

(ii) Show that the set $S = \left\{\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} : x, y \text{ are integers}\right\}$ is a ring but not a field under usual matrix addition and multiplication. 5+5

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