B.A./B.Sc. 6th Semester (Honours) Examination, 2025 (CBCS)

Subject : Mathematics

Course : BMH6 CC14

(Ring Theory and Linear Algebra-II)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

1. Answer *any ten* of the following questions: 2×10=20

(a) Show that $\sqrt{-3}$ is a prime element in the integral domain $\mathbb{Z}[\sqrt{-3}]$.

(b) Show that $x^2 + x + \bar{1}$ is irreducible in $\mathbb{Z}_2$.

(c) Examine whether $Q[x]/(x^2 + x + 1)$ is a field.

(d) Define $T: \mathbb{R}^3 \to \mathbb{R}^2$ by $T(x, y, z) = (x - y, 2z)$. Find $N(T)$.

(e) Show that the characteristic polynomial of any diagonalizable linear operator splits.

(f) If V be an inner product space over F, then show that $\forall x, y \in V, \| x + y \| \leq \| x \| + \| y \|$.

(g) Let T be a linear operator on an inner product space V and suppose that $\| T(x) \| = \| x \| \; \forall x \in V$. Prove that T is one-one.

(h) Let β be a basis of a finite dimensional inner product space V, $x \in V$ and $< x, z > = 0 \; \forall \, z \in \beta$. Show that $x = 0$.

(i) If $V = \mathbb{R}^3$ and $S = \{(0, 0, 1)\}$, then find $S^{\perp}$.

(j) Let V be an inner product space over the field F and T be a normal operator on V. Then prove that T–CI is normal for every c ∈ F.

(k) Let f be a linear function on $\mathbb{R}^2$ defined by $f(1, 3) = -4$ and $f(2, 1) = 7$. Find $f(x, y)$ for all $(x, y) \in \mathbb{R}^2$.

(l) Find the dual basis of the basis $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ of $\mathbb{R}^3(\mathbb{R})$.

(m) Show that the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable over the field $\mathbb{C}$.

(n) Let R be a ring with identity. Prove that $R[x]$ is a PID if R is a field.

(o) Prove that the integral domain $(\mathbb{Z}, +, \cdot)$ is a principal ideal domain (PID).

2. Answer *any four* of the following questions: 5×4=20

(a) Show that $\mathbb{Z}[\sqrt{-5}]$ is not a unique factorisation domain (UFD).

(b) Show that in a principal ideal domain D, a non-null ideal (p) is maximal if and only if p is irreducible in D.

(c) Let $U: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ and $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be linear transformations defined by $U(f(x)) = f'(x)$ and $T(f(x)) = \int_0^x f(t)dt$. If α, β be the standard ordered bases of $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$ respectively, then find $[T]_\beta^\alpha$ and $[U]_\alpha^\beta$. Here $P_n(\mathbb{R})$ denotes the vector space of all polynomials over $\mathbb{R}$ of degree $\leq n$ together with the zero polynomial.

(d) (i) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be rotation by θ in anticlockwise direction, where $0 < \theta < \pi$. Show that T is normal.

(ii) Let V be an inner product space and T be a normal operator on V. Show that $\| T(x) \| = \| T^*(x) \| \ \forall\, x \in V$. 3+2

(e) Let V be an inner product space. Then prove that $|\langle u, v\rangle| \leq \| u \| \| v \|$ for all $u, v \in V$.

(f) Suppose $V(\mathbb{R})$ is a vector space of all polynomials over the unit interval $[0, 1]$. If $f(t), g(t) \in V$ and inner product on V is defined by $\langle f(t), g(t)\rangle = \int_0^1 f(t)g(t)dt$, then find $\langle f, g\rangle$ and $\| g \|$, if $f(t) = t^2 + t - 4$, $g(t) = t - 1$.

3. Answer *any two* of the following questions: 10x2=20

(a) (i) Prove that the quotient ring $\mathbb{Z}[i]/(3)$ is a finite field.

(ii) Let D be a Euclidean domain with a Euclidean valuation ϑ. If $a|b$ and $\vartheta(a) = \vartheta(b)$, then prove that a, b are associates in D.

(iii) Let F be a field and $f(x)$ be a polynomial in $F[x]$ of degree 2 or 3. Then prove that $f(x)$ is irreducible over F if and only if it has a zero in F. 4+3+3

(b) (i) Let V be an inner product space and T be a normal operator on V. If λ_1 and λ_2 are distinct eigenvalues of T corresponding to eigenvectors x_1 and x_2, then show that x_1 and x_2 are orthogonal.

(ii) Find the orthogonal projection of the vector $(2, 1, 3)$ on the subspace $W = \{(x, y, z): x + 3y - 2z = 0\}$ of the inner product space $\mathbb{R}^3$. 5+5

(c) (i) Show that the polynomial $x^3 + x + 1$ is irreducible over $\mathbb{Z}_5$.

(ii) Prove that in a UFD, every irreducible element is a prime. 4+6

(d) (i) Diagonalise the matrix $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$

(ii) Find the characteristic and minimal polynomial of the linear operator $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y, x - y)\ \forall (x, y) \in \mathbb{R}^2$. 5+5
