3 Yr. Degree/4 Yr. Honours 2nd Semester Examination, 2024 (CCFUP)

Subject : Mathematics

Course : MATH2021 (MINOR)

(Introductory Algebra and Number Theory)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) If x, y, z are positive real numbers and $x + y + z = 1$, then prove that
$(1 - x)(1 - y)(1 - z) \geq 8xyz$.

(b) Prove that $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$ if n is an integer and θ is real.

(c) Apply Descarte's rule of signs to examine the nature of the roots of the equation
$x^4 + 3x^2 + 4x - 2 = 0$.

(d) Can '0' be a root of a reciprocal equation? Give reason.

(e) Let $(s, \leq)$ be a partially ordered set. Prove that if $\{a, b\} \subset S$ has a least upper bound, then it is unique.

(f) Show that $Z(G)$ is a subgroup of G, where $Z(G)$ is the centre of a group G.

(g) Show that intersection of two ideals of a ring R is also an ideal of R.

(h) If $a \equiv b \pmod{m}$, then show that $a^n \equiv b^n \pmod{m}$ $\forall n \in \mathbb{N}$.

(i) If α, β, γ be the roots of the cubic equation $x^3 + px^2 + qx + r = 0$, find the value of $\sum \alpha^2\beta$.

(j) Find the order of the permutation (1 2 4) (3 5 7 8).

(k) Justify whether the ring $\mathbb{Z}_6$ of all integers modulo 6 is an integral domain.

(l) Prove that $(Q, +)$ is not cyclic. Hence deduce that $(\mathbb{R}, +)$ is also not cyclic.

(m) Prove that $17^{20} \equiv 1 \pmod{145}$.

(n) Find ϕ (700), ϕ being the Euler's phi function.

(o) Show by an example that union of two subrings of a ring may not be a subring.

2. Answer *any four* questions: 5×4=20

(a) Define $\leq$ on $\mathbb{N}$ by $a \leq b$ if a is a divisor of b. Show that $(\mathbb{N}, \leq)$ is a lattice. Find $6 \vee 4$ and $9 \wedge 6$. Find the least element of $(\mathbb{N}, \leq)$ if exists. 2+2+1

(b) (i) Let a be an element of a group G, such that $0\ (a) = n$. Then show that $0\ (a^p) = n$, if and only if $\gcd(p, n) = 1$.

(ii) Let H be a subgroup of a group G such that $[G : H] = 2$. Then show that H is a normal subgroup of G. 3+2

(c) Solve the system of linear congruences: $x \equiv 1 (\text{mod } 3),\ x \equiv 2\ (\text{mod } 5),\ x \equiv 3\ (\text{mod } 7)$.

(d) Solve by Cardan's method: $x^3 - 27x - 54 = 0$

(e) If p be a prime, prove that $(p-1)! + 1 \equiv 0\ (\text{mod } p)$.

(f) Prove that $\left(\frac{1 + sin\theta + icos\theta}{1 + sin\theta - icos\theta}\right)^n = \cos n\left(\frac{\pi}{2} - \theta\right) + isin\, n\left(\frac{\pi}{2} - \theta\right) = (\sin\theta + icos\theta\,)^n$.

3. Answer *any two* questions: 10×2=20

(a) (i) If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4 + 3x^2 - 8 = 0$, form an equation whose roots are $2\alpha, 2\beta, 2\gamma, 2\delta$.

(ii) Prove that $f(x) = 0$ be a reciprocal equation of degree n and of the first type if and only if $f(x) = x^n f\left(\frac{1}{x}\right)$.

(iii) Solve $x^4 + px^2 + qx + r = 0$ by Ferrari's method. 2+4+4

(b) (i) Solve the equation $16x^3 - 44x^2 + 36x - 9 = 0$, given that the roots are in harmonic progression.

(ii) Show that every group of prime order is cyclic. Is the converse true? Give reason.

(iii) Give an example of a non-cyclic group of order 4 with reason. 4+(3+1)+2

(c) (i) Give an example of an integral domain which is not a field.

(ii) Show that the characteristics of an integral domain is either zero or a prime number.

(iii) Examine whether the ring $\left\{\begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a, b \in Q\right\}$ is an integral domain. 2+4+4

(d) (i) Give an example of a lattice which is not complete.

(ii) If p be a prime number and p is not a division of a, then show that $a^{p-1} \equiv 1 (\text{mod } p)$.

(iii) If n be an even integer, then show that $\phi(2n) = 2\phi(n)$.

(iv) Find $\tau(300)$ and $\sigma\ (300)$. 2+4+2+2