

B.A./ B. Sc 4th Semester (Honours) Examination, 2022 (CBCS)

Subject: Mathematics

Course: BMH4CC08

(Riemann Integration and Series of Functions)

Time: 3 Hours

Full Marks: 60

The figures in the margin indicate full marks.

Candidates are required to write their answers in their own words as far as practicable.

[Notation and Symbols have their usual meaning]

1. Answer any ten questions:

$10 \times 2 = 20$

- (a) Define $f: [0,1] \rightarrow \mathbb{Z}$ by $f(x) = [3x]$. Examine whether f is R-integrable. Evaluate $\int_0^1 f(x) dx$, if it exists. [2]
- (b) Let f be continuous on $[a,b]$ and let $\int_a^b f(x) dx = 0$. Show that the set $\{x \in [a,b]: f(x) = 0\}$ is non-empty. [2]
- (c) Let $\phi(t) = \int_0^t te^{\sin t} dt, t \in [0, \pi]$. Find $\phi''(t)$, if exists. [2]
- (d) Define $f: [0, \pi] \rightarrow \mathbb{R}$ by $f(x) = \sin x$. Find $\text{Sup } \{|f(a) - f(b)|: a, b \in [0, \pi]\}$. [2]
- (e) Find $B(m, 1)$ for $m > 0$. [2]
- (f) Test the convergence of $\int_5^\infty \frac{dx}{x^n}$. [2]
- (g) Show that $\int_3^\infty \frac{dx}{x \log x}$ diverges. [2]
- (h) Evaluate whether $\int_{\frac{\pi}{2}}^\infty \cos x dx$ is convergent. [2]
- (i) Let $\{f_n\}$ be any sequence of functions on $[a, b]$. Is $\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b \lim_{n \rightarrow \infty} f_n(x) dx$? Give reason. [2]
- (j) Examine the pointwise convergence of $\sum_{n=1}^{\infty} e^{nx} \sin nx$. [2]
- (k) Is the series $\sum_{n=1}^{\infty} \cos nx$ a Fourier series? Justify your answer. [2]
- (l) Find the sum of the series $\sum_{n=0}^{\infty} (2^n + 3^n)x^n$, with its range of validity. [2]
- (m) If $\sum_{n=0}^{\infty} a_n x^n$ converges for all real numbers x , then find $\lim_{n \rightarrow \infty} \text{Sup } |a_n|^{\frac{1}{n}}$. [2]

(n) Find the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. [2]

(o) Find the radius of convergence of $\sum_{n=0}^{\infty} 2^n x^{2n}$. [2]

2. Answer any four questions: $4 \times 5 = 20$

(a) (i) Let f be continuous on $[a, b]$ such that $f(x) \geq 0, \forall x \in [a, b]$. If $\int_a^b f(x) dx = 0$, prove that $f(x) = 0, \forall x \in [a, b]$. [3]

(ii) Define f on $[0, 1]$ by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational in } [0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

Examine whether f is R-integrable.

(b) (i) Let f and g be real valued continuous functions on $[a, b]$ such that [3]

$\int_a^b f(x) dx = \int_a^b g(x) dx$. Show that there is a $c \in [a, b]$ such that $f(c) = g(c)$.

(ii) Define $f: [0, 1] \rightarrow \mathbb{R}$ by [2]

$$f(x) = \begin{cases} \frac{1}{a^n}, & \text{if } \frac{1}{a^n} < x \leq \frac{1}{a^{n-1}}, n \in \mathbb{N}, a > 1 \\ 0, & \text{if } x = 0. \end{cases}$$

Examine whether f is R-integrable.

(c) Let $x \in (-1, 1]$. Show that $\int_0^x \frac{t^m}{t+1} dt \rightarrow 0$ as $m \rightarrow \infty$. [5]

(d) A function f is defined on $[0, 1]$ by $f(0) = 0$ and [5]

$$f(x) = (-1)^{n+1} (n+1), \text{ for } \frac{1}{n+1} < x \leq \frac{1}{n}; n = 1, 2, 3, \dots$$

Examine the convergence of $\int_0^1 f(x) dx$ and $\int_0^1 |f(x)| dx$.

(e) Show that the sequence of functions $\{f_n\}$ defined by [5]

$$f_n(x) = \frac{nx}{1+nx}, x \geq 0$$
 is not uniformly convergent on $[0, \infty)$.

Examine the uniform convergence of $\{f_n\}$ on $[a, \infty)$ if $a > 0$.

(f) If $f(x) = \{\pi - |x|\}^2$ on $[-\pi, \pi]$, prove that Fourier Series of f is given by [5]

$$\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx.$$

Hence deduce that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

3. Answer any two questions:

2×10 = 20

- (a) (i) State and prove the Second Mean Value Theorem of Integral Calculus in Bonnet's form. [6]
- (ii) Evaluate $\lim_{x \rightarrow 0} \frac{1}{x^2} \int_0^{x^2} e^{\sqrt{t+1}} dt$. [4]
- (b) (i) Show that $\int_0^\infty x^{m-1} e^{-x} dx$ is convergent if and only if $m > 0$. [5]
- (ii) Test the convergence of $\int_0^1 \frac{\sin \frac{1}{x}}{\sqrt{x}} dx$. [5]
- (c) (i) The sequence of functions $\{f_n\}$ is defined on $[0,1]$ by $f_n(x) = x^n - x^{n-1}$, $x \in [0,1]$. Examine the uniform convergence of $\{f_n\}$ on $[0,1]$. [4]
- (ii) For each n , $f_n(x) = 1 - nx$, for $0 \leq x \leq \frac{1}{n}$
 $= 0$, for $\frac{1}{n} < x \leq 1$. [6]
- Examine the pointwise convergence and the uniform convergence of $\{f_n\}$ on $[0,1]$.
- (d) (i) Show that $\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \dots = \frac{1}{1-x}$, $\forall x, -1 < x < 1$. [5]
- (ii) Expand in a series of sines and cosines of multiples in x of the function given by
 $f(x) = x - \pi$, when $-\pi < x < 0$
 $= \pi - x$, when $0 < x < \pi$.
What is the sum of the series for $x = \pm\pi$ and $x = 0$?