3 Yr. Degree/4 Yr. Honours 2nd Semester Examination, 2025 (CCFUP)

Subject : Mathematics

Course : MATH2011 (MAJOR)

(Introductory Algebra and Number Theory)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) If $\cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$, then prove that

$$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma).$$

(b) If $\alpha,\ \beta,\ \gamma,\ \delta$ are the roots of $x^4 - x^3 + 2x^2 + x + 1 = 0$, then find the value of $(\alpha + 1)(\beta + 1)(\gamma + 1)(\delta + 1)$.

(c) Apply Descartes' rule of signs to find the nature of the roots of the equation $7x^7 + 5x^5 - 3x^3 = 0$.

(d) If $\alpha,\ \beta,\ \gamma$ are the roots of $x^3 + qx + r = 0$, then find the value of $\frac{1}{\alpha^2+q} + \frac{1}{\beta^2+q} + \frac{1}{\gamma^2+q}$.

(e) Let $S = \{x \in \mathbb{N} : 1 \le x \le 10\}$. Define '$\le$' on S by $x \le y$ if x is a divisor of y. Find the maximal elements in the partially ordered set $(S, \le)$.

(f) If $a,\ b,\ c$ are all positive real numbers and $abc = k^3$, then prove that $(1 + a)(1 + b)(1 + c) \ge (1 + k)^3$.

(g) Prove that $S = \{(a, 3b) : a, b \in \mathbb{Z}\}$ is a subring of the ring $\mathbb{Z} \times \mathbb{Z}$.

(h) Prove that $\phi(n^2) = n\phi(n)\ \forall\, n \in \mathbb{N}$.

(i) Let $(G, \circ)$ be a group and $x, y \in G$ be such that $O(x) = 2 = O(y)$. If x and y commute, prove that $O(x \circ y) = 2$.

(j) Find the order of the permutation $(1\ 2\ 3)(4\ 5\ 7\ 8)$.

(k) Give an example of a non-commutative group in which every subgroup is normal.

(l) Prove that in a field F, $(-a)^{-1} = -a^{-1}\ \forall\, a \ne 0$ in F.

(m) If $d = gcd(a, b)$, show that $gcd(a^2, b^2) = d^2$.

(n) If p and $p^2 + 8$ are both primes, prove that $p = 3$.

(o) Find the least positive residue in $2^{41} (\text{mod } 23)$.

2. Answer *any four* questions: $5 \times 4 = 20$

(a) If α, β, γ are the roots of the equation $x^3 + qx + r = 0$, find the equation whose roots are $(\alpha - \beta)(\alpha - \gamma)$, $(\beta - \alpha)(\beta - \gamma)$, $(\gamma - \alpha)(\gamma - \beta)$.

(b) (i) Let P be the set of all real valued functions defined on the closed interval $\left[0, \frac{\pi}{4}\right]$. Define $\leq$ on P by $f \leq g$ if $f(x) \leq g(x)\ \forall\, x \in \left[0, \frac{\pi}{4}\right]$. Then find $u \vee v$, where $u(x) = \sin x$ and $v(x) = \cos x$.

(ii) Let M and N be normal subgroups of a group G such that $M \cap N = \{e\}$. Prove that $mn = nm\ \forall\, m \in M$ and $\forall\, n \in N$. 2+3

(c) State and prove Lagrange's theorem on groups.

(d) Solve the equation $2x^4 - 5x^3 - 15x^2 + 10x + 8 = 0$, the roots being in the geometric progression.

(e) Prove that the total number of generators of a finite cyclic group of order n is $\phi(n)$, ϕ being the Euler's phi function.

(f) State and prove Wilson's theorem. 1+4

3. Answer *any two* questions: $10 \times 2 = 20$

(a) (i) If a is an element of a group G such that $O(a) = n$, prove that $O(a^m) = \frac{n}{gcd(m,n)}, \forall m \in \mathbb{N}$.

(ii) Let G be a group in which $(ab)^3 = a^3b^3\ \forall\, a, b \in G$. Show that $H = \{x^2 : x \in G\}$ is a subgroup of G.

(iii) Define $*$ on $\mathbb{Z} \times \mathbb{Z}$ by $(a, b) * (c, d) = (ac, bd)$. Examine whether $(\mathbb{Z} \times \mathbb{Z}, *)$ is a monoid. Find the invertible elements in $(\mathbb{Z} \times \mathbb{Z}, *)$ if exists. 4+3+(2+1)

(b) (i) Solve the equation $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$ by Ferrari's method.

(ii) Find the number of real roots of the equation $x^4 + 4x^3 - x^2 - 2x - 5 = 0$.

(iii) Find the equation whose roots are the roots of the equation $x^3 + 3x^2 - 8x + 1 = 0$ each diminished by 1. 4+4+2

(c) (i) Define an integral domain. Prove that the characteristic of an integral domain is either zero or a prime number.

(ii) Let $(\mathbb{Z}_n, +, \cdot)$ be the ring of integers modulo n. Prove that $(\mathbb{Z}_n, +, \cdot)$ is a field if and only if n is a prime.

(iii) Prove that the subfield Q of rational numbers is the smallest subfield of the field of real numbers. (1+3)+4+2

(d) (i) If a and b are two integers different from zero, prove that $[a,b]\,(a,b) = |ab|$, where $[a,b]$ = the lcm of a and b and (a,b) = the gcd of a and b.

(ii) If p be a prime and K be a positive integer, prove that $\phi(p^K) = p^K\left(1-\frac{1}{p}\right)$.

(iii) Find the unit digit in 7^{99}. 4+3+3