3 Yr. Degree/4 Yr. Honours 2nd Semester Examination, 2024 (CCFUP)

Subject : Mathematics

Course : MATH2011 (MAJOR)

(Introductory Algebra and Number Theory)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

(Notion and symbols have their usual meaning.)

1. Answer *any ten* questions: 2×10=20

(a) If $2\cos\theta = x + \frac{1}{x}$ and θ is real, prove that $2\cos n\theta = x^n + \frac{1}{x^n}$, n being an integer.

(b) If α, β, γ be the roots of the equation $x^3 + qx + r = 0$, find the value of $\frac{1}{\alpha^3+r} + \frac{1}{\beta^3+r} + \frac{1}{\gamma^3+r}$.

(c) Examine if $(x^2+x+1)^4 + (x^2-x+1)^4 + x^8 + 1 = 0$ is a reciprocal equation.

(d) Give an example of a partial order relation which is not a total order relation.

(e) If an abelian group G of order 10 contains an element of order 5, then prove that G is cyclic.

(f) Find the number of generators of the cyclic group $(\mathbb{Z}_{100}, \oplus)$.

(g) Let S be an ideal of a ring R such that S contains a unit of R, show that $S = R$.

(h) If a, b, c, d are integers and m is a positive integer such that $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then show that $ac \equiv bd \pmod{m}$.

(i) Let p be a prime number such that $p|ab$. Then show that either $p|a$ or $p|b$.

(j) If a, b, c, d are positive real numbers, not all equal, prove that

$a^5 + b^5 + c^5 + d^5 > abcd\,(a+b+c+d)$.

(k) If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, find the equation whose roots are $\alpha\beta + \beta\gamma, \beta\gamma + \gamma\alpha, \gamma\alpha + \alpha\beta$.

(l) Let $(G,*)$ be a group and $a \in G$ with $0(a) = 60$. Find $0(a^{28})$.

(m) Prove that $23^8 \equiv 1 \pmod{265}$.

(n) Find $\phi(2000)$, ϕ being the Euler's phi function.

(o) By an example, show that union of two ideals in a ring may not be an ideal.

2. Answer *any four* questions: 5×4=20

(a) Find the number and position of the real roots of the equation $x^4 - 6x^3 + 10x^2 - 6x + 1 = 0$.

(b) Let $f(x) = x^4 + 6x^3 + 14x^2 + 22x + 5$. Find α, β, λ so that $f(x)$ may be expressed in the form $(x^2 + 3x + \lambda)^2 - (\alpha x + \beta)^2$. Hence solve the equation.

(c) (i) Define $\leq$ on $\mathbb{N}$ by $a \leq b$ if $a|b$. Examine whether $(\mathbb{N}, \leq)$ is a distributive lattice.

(ii) Let $S = \mathbb{N} \cup \{0\}$. Define $\leq$ on S by $a \leq b$ if $a|b$. Then find the greatest element of S if exists. 3+2

(d) Show that in S_n, the number of odd permutations is equal to the number of even permutations. Examine whether $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 1 & 3 & 7 & 5 & 6 & 9 & 8 \end{pmatrix}$ is even or odd. 3+2

(e) (i) If a is the only element of order n for some $n \in \mathbb{N}$ in a group G, show that $a \in Z(G)$.

(ii) Show that every group of order 4 is abelian. 2+3

(f) Show that $4(29)! + 5!$ Is divisible by 31.

3. Answer *any two* questions: 10×2=20

(a) (i) Let G be a group in which $(ab)^3 = a^3b^3$ and $(ab)^5 = a^5b^5 \ \forall \ a, b \in G$. Prove that G is abelian.

(ii) Prove that the order of r-cycle is r.

(iii) Let G be an infinite cyclic group generated by a. Prove that G has exactly two generators. 4+3+3

(b) (i) Solve the equation $x^4 - 4x^3 - 4x^2 - 4x - 5 = 0$, given that two roots α, β are connected by the relation $2\alpha + \beta = 3$.

(ii) Solve the equation $x^3 - 27x - 54 = 0$.

(iii) Solve the equation $x^4 + 12x - 5 = 0$ by Ferrari's method. 4+3+3

(c) (i) Show that every finite integral domain is a field.

(ii) Define $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. Examine whether $(\mathbb{Q}(\sqrt{2}), +, \cdot)$ is a field, where $\mathbb{Q}$ is the set of all rational numbers.

(iii) Find four consecutive integers divisible by 3, 4, 5, 7 respectively. 4+3+3

(d) (i) If a and b are integers, not both zero, prove that $\exists$ integers u and v such that $gcd(a, b) = au + bv$.

(ii) Show that $a^{12} - b^{12}$ is divisible by 91 if a and b are both prime to 91.

(iii) Use theory of congruences to prove that $17 | (2^{3n+1} + 3.5^{2n+1})$ for all integers $n \geq 1$. 4+3+3

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