B.A./B.Sc. 6th Semester (Honours) Examination, 2024 (CBCS)

Subject : Mathematics

Course : BMH6DSE31

(Mathematical Modelling)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) Discuss transient states of a queuing system.

(b) Write down the logistic model for a single-species population.

(c) What do you mean by orbit of the system?

(d) What are the advantages of mathematical modelling?

(e) When a critical point is said to be asymptotically stable?

(f) What do you mean by effective arrival rate of a queueing system with finite capacity?

(g) What are the state variables for the dynamical models of ecosystem?

(h) Define growth rate of single species population model.

(i) Define stability and instability of a fixed point of the difference equation $x_{n+1} = f(x_n)$.

(j) What do you mean by "carrying capacity" of a logistic growth model?

(k) Write down two limitations of the logistic model.

(l) Write down the confined exponential growth model for a single-species population.

(m) Explain the Poisson axioms of departures of a queueing system.

(n) Write down two limitations of prey-predator model.

(o) Consider the queueing model $(M/M/1):(\infty/FCFS/\infty)$. Find the expected number of customers in the system.

2. Answer *any four* questions: 5×4=20

(a) Explain the concept of Lyapunov stability of the equilibrium state of the differential equation $\frac{dx}{dt} = f(x)$.

(b) Show that the system $\frac{dx}{dt} = -y + x\,(x^2 + y^2 - 1)$ and $\frac{dy}{dt} = x + y\,(x^2 + y^2 - 1)$ is an unstable limit cycle, where $x = r\cos\theta, y = r\sin\theta$ and $r = 1$.

(c) Discuss Gompertz population model.

(d) Discuss the Allee effect of $\frac{dP}{dt} = P[r_0 - \alpha(P-\eta)^2]$, $\left(\eta < \sqrt{\frac{r_0}{\alpha}}\right)$ where α, r_0 and η are positive constants. Can you relate $r(P)$ corresponding to this situation?

(e) Consider the set of nonlinear differential equations $\frac{dx}{dt} = x - xy$ and $\frac{dy}{dt} = -x + xy$.

(i) Show that the origin and the point (1, 1) are equilibrium points of the above system.

(ii) Show that (0, 0) is a saddle point and (1, 1) is a centre of the above system. 3+2

(f) If the arrivals are completely random, then the probability distribution of the number of arrivals in a fixed time interval follows Poisson distribution.

3. Answer *any two* questions: 10×2=20

(a) Discuss Malthus model of population growth. Solving Malthus growth equation with a given initial condition, show that a population satisfying this equation undergoes exponential growth or decay. Determine the population doubling time for Malthus model. What are the drawbacks of Malthus model? Describe a model in which these drawbacks have been overcome.

(b) One improvement in the predator-prey model is to modify the equation for the prey so that it has the form of a logistic equation of the predator. Write down the system of equations for this model. Determine the critical points of the system and discuss their nature and stability. 6+4

(c) (i) Consider the system of equations $\frac{dx}{dt} = x$ and $\frac{dy}{dt} = -x + 2y$. Find the critical point of the system. Discuss the type and stability of the critical point. Write down the general solution of the system.

(ii) For the model $\frac{dP}{dt} = r_1 P\left(1 - \frac{P}{K}\right) - EP, P(0) = K$ where r_1, E and K are constants, determine $P(t)$ explicitly. Verify from the form of the solution that $P > K\left(1 - \frac{E}{r_1}\right)$ if $E \le r_1$, then $P(t) > K\left(1 - \frac{E}{r_1}\right)$ as $t \to \infty$ whereas if $E > r_1$, then $P(t) \to 0$ exponentially as $t \to \infty$. 6+4

(d) Arrivals at a car servicing centre are considered to have Poisson distribution with an average time of 50 minutes between one arrival and the next. The length of a car servicing is assumed to be distributed exponentially with mean 30 minutes.

(i) What is the probability that a car arriving at the servicing centre will have to wait?

(ii) What is the average length of queues that form from time to time?

(iii) The car servicing company will install a second service centre when convinced that an arrival would expect to have to wait at least 3 minutes for the service. By how much time should the flow of arrivals be increased to justify a second service centre?

(iv) Find the average number of cars in the system. 2½+2½+2½+2½

B.A./B.Sc. 6th Semester (Honours) Examination, 2024 (CBCS)

Subject : Mathematics

Course : BMH6DSE32

(Industrial Mathematics)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) With a graph, explain a typical seismic image depicting various earth layers.

(b) Physically interpret Dirac delta function.

(c) Show that back projection is linear.

(d) What is Shepp-Logan filter? Explain mathematically.

(e) What is W-interpolation of a discrete function?

(f) Define full width half maximum of the function φ.

(g) Why does Radon transformation show harmonic property?

(h) Set an example of back projection.

(i) What is the typical clinical range of a CT scan between air and bone?

(j) Why optoacoustic tomography is important to study back projection?

(k) State Rayleigh-Plancherel theorem.

(l) State Nyquist's theorem.

(m) How Gauss distribution can be used to explain Radon transformation?

(n) What are the main characteristics of cylindrical samples?

(o) What are the differences between filtered and unfiltered back projections?

2. Answer *any four* questions: 5×4=20

(a) Why $l_{t,\theta} = l_{-t,\pi+\theta}$, for all t and all θ?

(b) (i) Define periodicity of a wave.

(ii) Define Heaviside function. 2+3

(c) (i) Find full width half maximum of the function (FWMF) for a tent function.
(ii) Write the properties of a band limited function. 2+3

(d) (i) Set an example of Ram-Lak filter.
(ii) With an example, analyse irradiation frequency. 2+3

(e) Write a short note on Tikhonov regularization.

(f) Prove that for a given values of a, b and θ, the $l_{a\cos\theta+b\sin\theta,\theta}$ passes through the point (a, b).

3. Answer *any two* questions: 10×2=20

(a) (i) Let the function f be defined in the plane, let a and b be arbitrary real numbers, and let c be a positive real number. Define the function g by $g(x,y) = f(x-a, y-b)$ and the function h by $h(x,y) = f(cx, cy)$. Then for all real numbers t and θ, $Rg(t,\theta) = Rf(t - a\cos\theta - b\sin\theta, \theta)$ and $Rh(t,\theta) = \frac{1}{c} Rf(ct,\theta)$

R is Radon transformation.

(ii) Show that $\int_0^\infty e^{-\tau x} e^{-i\omega x}dx = \frac{\tau - i\omega}{\tau^2-\omega^2}$, where ω and τ are real numbers with $\tau > 0$. 5+5

(b) (i) State and prove central slice theorem.
(ii) Establish the filtered back projection formula.
(iii) Why is a Ecocardiogram report can be decoded using Radon transformation? 4+3+3

(c) (i) For suitable functions $g(t,\theta)$ and $f(x,y)$and arbitrary real numbers X and Y, show that

$$B(g*f)(X,Y) = B(g*Rf)\ (X,Y),$$

R is Radon transformation and B is back projection.

(ii) For a signal S consisting of a few cycles of a cosine wave and a filter ϕ in the form of a tent, are the crests of the original cosine wave still distinct in the convolution? 5+5

(d) (i) If the linear system $Ax = p$ has at least one solution, then prove that Keczmarz's method converges to a solution of this system. Moreover, show that, if x_0 is in the range of the transpose A, then Keczmarz's method converges to the solution of minimum norm.

(ii) Construct the Bloch equation in the context of an MRI machine. 5+5

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B.A/B.Sc. 6th Semester (Honours) Examination, 2024 (CBCS)

Subject : Mathematics

Course : BMH6DSE33

(Group Theory–II)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) Find the number of group homomorphisms from $\mathbb{Z}_5$ into $\mathbb{Z}_2 \times \mathbb{Z}_3$.

(b) Let G be a group and H be a normal subgroup of G. Show that $*: (g,h) \to ghg^{-1}$ is a left action of G on H.

(c) Show that no group of order 10 is simple.

(d) Prove that Aut $(\mathbb{Z}_2 \times \mathbb{Z}_2) \simeq S_3$.

(e) Show that A_4 is not simple.

(f) Cyclic group of order 4 cannot be expressed as an internal direct product of two subgroups of order 2. — Justify.

(g) Show that a simple group of order 63 cannot contain a subgroup of order 21.

(h) Prove that the group G of order p^n, where p is prime, has a non-trivial centre.

(i) Any characteristic subgroup of a group is normal. — Justify.

(j) Show that if a group of order 105 contains a unique Sylow 3-subgroup, then G is abelian.

(k) Let G be a finite abelian group of order n and let m be a positive integer prime to n. Show that the mapping $\sigma: x \to x^m$ is an automorphism of G.

(l) Let G be a finite abelian group. Prove that number of solutions of $x^n = e$, where $n > 0$ and n divides $|G|$, is a multiple of n.

(m) Commutator subgroup of a group is normal. — Justify.

(n) How many Sylow 7-subgroups of the simple group G of order 168 are there?

(o) Show that for any group G, Inn(G) is a normal subgroup of Aut(G).

2. Answer *any four* questions: 5×4=20

(a) Let G be a group of order p^2, where $p \geq 2$ is prime. Show that G is cyclic or isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$.

(b) Suppose H is a subgroup of a finite group G and H acts on G under $*: H \times G \to G$ such that $*(h, g) = hg$. Prove that $O(H)/O(G)$.

(c) Prove that any group of order 255 is cyclic.

(d) Let G be a group of order $2m$, where m is an odd integer. Show that G has a normal subgroup of order m.

(e) Prove that a group of order 36 is not simple.

(f) Show that any group of order 100 having a unique Sylow 2-subgroup is abelian.

3. Answer *any two* questions: 10×2=20

(a) (i) Determine up to isomorphism all groups of order 70.

(ii) Find the class equation of S_3. 7+3

(b) (i) Let G be a group of order pq, where p and q are distinct primes, $p > q$ and q does not divide $p - 1$. Prove that G is cyclic.

(ii) Let G be a group and f be an automorphism of G. Show that the set $\{a \in g : f(a) = a\}$ forms a subgroup of G. 7+3

(c) (i) Let G be a group and S be a G-set. Prove that the left action of G on S induces a homomorphism from G to $A(S)$, where $A(S)$ is the group of all permutations of S.

(ii) Show that the group G is isomorphic to a subgroup of $A(G)$. 5+5

(d) (i) Show that there does not exist any group G with $|Inn(G)| = 77$.

(ii) Let G be a cyclic group generated by $a \in G$. Show that a homomorphism $f : G \to G$ is an automorphism of G if and only if $f(a)$ is generator of G. 5+5

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