B.A./B.Sc. 5th Semester (Honours) Examination, 2024 (CBCS)

Subject : Mathematics

Course : BMH5DSE11

(Linear Programming)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) Show that the set $X = \{(x_1, x_2): x_1^2 + x_2^2 = 16\}$ is not a convex set.

(b) Explain Maxmin and Minmax principle used in Game theory.

(c) If 4th constraint is equation and 3rd variable is unrestricted in sign of the primal L.P.P. then what will be the nature of 4th variable and 3rd constraint of the corresponding dual problem?

(d) When an L.P.P. has an unbounded solution?

(e) Is (0, 4, 0) a basic solution to the following set of equations:
$2x + y - z = 4,\ 5x + 2y + z = 8$? Justify your answer.

(f) Can you consider an assignment problem as a transportation problem? If so, state the nature of the optimal solution.

(g) Show that the following vectors are linearly dependent:
(1, –2, 3, 4), (–2, 4, –1, –3), (–1, 2, 7, 6)

(h) What are the drawbacks of Big-M method?

(i) Define simplex and give an example.

(j) Find the basic solution of the system $x_1 + 2x_3 = 1$,
$x_2 + x_3 = 4$.

(k) Define extreme point of a convex set with example.

(l) Find the range of values of p and q for which (2, 2) will be the saddle point for the following game:

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 2 & 4 & 5 \\ 10 & 7 & q \\ 4 & p & 6 \end{bmatrix} \end{array}$$

(m) 'The solution of a transportation problem is never unbounded'.—Justify.

(n) Find the optimal strategies and the value of the game whose pay-off matrix is as follows:

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

(o) Solve the L.P.P.: Minimize $Z = 3x_1 + x_2$

subject to $2x_1 + 3x_2 \leq 6,$

$x_1 + x_2 \geq 1,$

$x_1, x_2 \geq 0.$

2. Answer *any four* questions: 5×4=20

(a) Use Big-M method to solve the following L.P.P.:

Maximize $Z = 6x_1 + 4x_2$

subject to $2x_1 + 3x_2 \leq 30$

$3x_1 + 2x_2 \leq 24$

$x_1 + x_2 \geq 3$

$x_1, x_2 \geq 0$

Is the solution unique? Justify your answer. 3+1+1

(b) Write down the following transportation problem as an L.P.P.:

10	20	15	45
20	15	25	25
40	25	35	

What are the properties of a loop in a transportation problem? 3+2

(c) Use duality to solve the L.P.P.:

Minimize $Z = x_1 - x_2$

subject to $2x_1 - x_2 \geq 2$

$-x_1 - x_2 \geq 1$

$x_1, x_2 \geq 0$

(d) Two players A and B match coins in respect of Head (H) and Tail (T). If the coins match, then A wins one unit of value; if the coins do not match, then B wins one unit of value. Find the pay-off matrix for the game and determine the optimum strategies for the players and the value of the game. 1+2+2

(e) Find all basic feasible solutions of the equations:

$$x_1 + 2x_2 + 4x_3 + x_4 = 7$$
$$2x_1 - x_2 + 3x_3 - 2x_4 = 4$$

(f) Use dominance property to reduce the following game to 2×2 game and hence find the optimal strategies and the value of the game whose pay-off matrix is given below:

		Player B			
		I	II	III	IV
	I	3	2	4	0
Player A	II	2	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

3. Answer *any two* questions: 10×2=20

(a) (i) Solve the following transportation problem to maximize the profit:

		Sales Depots			
		S_1	S_2	S_3	Availability
	F_1	6	6	1	10
Factories	F_2	–2	–2	–4	150
	F_3	3	2	2	50
	F_4	8	5	3	100
Requirement		80	120	150	

(ii) Prove that if the primal problem is feasible, then it has an unbounded optimum if and only if the dual has no feasible solution and vice versa. 6+4

(b) (i) Use dual-simplex method to solve the following problem:

$$\text{Maximize } Z = -3x_1 - x_2$$
$$\text{subject to } x_1 + x_2 \geq 1$$
$$2x_1 + 3x_2 \geq 2$$
$$x_1, x_2 \geq 0$$

(ii) Let us consider the following L.P.P.:

$$\text{Minimize } Z = CX$$
$$\text{subject to } AX = b,\ X \geq 0$$

Let $(X_B, 0)$ be a basic feasible solution corresponding to a basis B where $X_B = B^{-1}b$. Suppose $Z_0 = C_B X_B$ is the value of objective function such that $Z_j - C_j \leq 0$ for every column a_j in A. Show that Z_0 is the minimum value of Z of the problem and that the given basic feasible solution is optimal. 5+5

(c) (i) Solve the following game problem graphically:

		Player B		
		I	II	III
Player A	I	4	3	1
	II	0	1	2

(ii) Solve the following Travelling Salesman problem: 5+5

		TO				
		A	B	C	D	E
FROM	1	–	24	17	11	19
	2	24	–	18	16	11
	3	17	18	–	15	9
	4	11	16	15	–	21
	5	19	11	9	21	–

(d) (i) $x_1 = 1, x_2 = 2, x_3 = 1$ is a feasible solution of the following set of linearly independent equations:

$$2x_1 + 3x_2 + 5x_3 = 13$$
$$3x_1 - x_2 + 3x_3 = 4$$
$$x_1, x_2, x_3 \geq 0$$

Reduce the feasible solution to a basic feasible solution.

(ii) Find the solution of given system of equation using Simplex Method: 5+5

$$5x_1 + 2x_2 = 14$$
$$2x_1 + x_2 = 6.$$

B.A./B.Sc. 5th Semester (Honours) Examination, 2024 (CBCS)

Subject : Mathematics

Course : BMH5DSE12

(Number Theory)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) Define primitive root with an example.

(b) If $n > 2$, then prove that $\phi(n)$ is even.

(c) Find the highest power of 5 contained in 140!.

(d) Solve, if possible, $x^8 \equiv 10 \pmod{11}$.

(e) If p is a positive integer and p, $2p+1$, $4p+1$ are primes, find p.

(f) Prove that $\frac{n^7}{7} + \frac{n^3}{3} + \frac{11n}{21}$ is an integer for all $n \in \mathbb{N}$.

(g) If p is prime, prove that $2(p-3)! + 1 \equiv 0 \pmod{p}$.

(h) Show that if $(p-1)! \equiv -1 \pmod{p}$, then p must be prime.

(i) Prove that no prime factor of (n^2+1) can be of the form $(4m-1)$ where m is an integer.

(j) Show that if $F_n = 2^{2^n} + 1$, $n > 1$, n is prime, then 2 is not a primitive root of F_n.

(k) If in RSA modulus $N \equiv 15$ and encryption exponent $e = 3$, find decryption exponent d.

(l) Find the unit digit of 3^{100} by use of Fermat's theorem.

(m) Let x and y be real numbers. Prove that $[x+n] = [x] + n$ for any integer n where $[\cdot]$ denotes greatest integer function.

(n) If $\varphi(n) | (n-1)$, prove that n is a square-free integer.

(o) Show that 41 divides $2^{20} - 1$.

2. Answer *any four* questions: 5×4=20

(a) Prove that there are infinite number of primes.

(b) State and prove Fermat's Little Theorem.

(c) Prove that Möbius function $\mu(n)$ is multiplicative.

(d) Find the primitive roots of 41.

(e) Determine all solutions in the integers of the following Diophantine equation:

$$172x + 20y = 1000$$

(f) Solve the congruence $x^2 \equiv 31 \pmod{11^4}$.

3. Answer *any two* questions: 10×2=20

(a) (i) Find four consecutive integers divisible by 3, 4, 5, 7 respectively.

(ii) Let $k > 1$ and $2^k - 1$ is a prime. If $n = 2^{k-1}\left(2^k - 1\right)$ then show that n is a perfect number.

5+5

(b) (i) If p is a prime integer, then prove that:

$$(p-1)! \equiv p - 1 \pmod{(1 + 2 + 3 + \cdots + (p-1))}.$$

(ii) Prove that the quadratic congruence $x^2 + 1 \equiv 0 \pmod{p}$, where p is an odd prime, has a solution if and only if $p \equiv 1 \pmod{4}$.

5+5

(c) (i) Find the values of the Legendre symbols: $\left(\frac{180}{59}\right), \left(\frac{1236}{4567}\right)$

(ii) If $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are primes and $\alpha_i \geq 1$, for each $i = 1, 2, \cdots, k$; prove that $\sum_{d|n} \mu(d)\tau(d) = (-1)^k$.

5+5

(d) (i) Prove that if $p > 3$ is an odd prime, then $\left({-3}/{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{6} \\ -1 & \text{if } p \equiv 5 \pmod{6} \end{cases}$.

(ii) Using the above result show that there are infinitely many primes of the form $(6K + 1)$.

5+5

B.A./B.Sc. 5th Semester (Honours) Examination, 2024 (CBCS)

Subject : Mathematics

Course : BMH5DSE13

(Point Set Topology)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) Prove that in a topological space, Int $(A \cap B) =$ Int $A \cap$ Int B.

(b) If $f:(X,\tau) \to (Y,\tau')$ is continuous, prove that $f(\overline{A}) \subset \overline{f(A)}$, where (X,τ) and (Y,τ') are two topological spaces and $A \subset X$.

(c) Define locally compact space. Is locally compactness a hereditary property? Justify.

(d) Is the set $[0,4) \cup [4,10]$ closed in the real number space with usual topology? Justify your answer.

(e) Let X be an infinite set and $\tau = \{A \subset X : X - A \text{ is finite}\} \cup \{\phi\}$. Examine if τ is a topology on X.

(f) State Zorn's lemma.

(g) Define cardinal number of a set.

(h) Show that Int (Int A) = Int A, for any subset A in a topological space (X,τ).

(i) Define ordinal number.

(j) Give an example to show that the continuous image of a locally connected space need not be locally connected.

(k) Prove that the space $C[0,1]$ is not compact with respect to the sup metric on $C[0,1]$.

(l) Let (X,τ) be a topological space and $A \subset X$. Show that Int $(X - A) = X - \overline{A}$.

(m) Is the set $\left\{\pm 1, \pm 2, \pm\frac{1}{2}, \pm 3, \pm\frac{1}{3}, \dots, \pm 2024, \pm\frac{1}{2024}\right\}$ compact? Is it connected? 1+1

(n) If u, v and w are cardinal numbers then prove that $u(v+w) = u\,v + uw$.

(o) Find a base for the metric topology.

2. Answer *any four* questions: 5×4=20

(a) Let A be a subset of a topological space (X, τ) and $x_0 \in X$. Let $\{x_n\}$ be a sequence in A and $x_n \to x_0$ as $n \to \infty$. Prove that $x_0 \in \overline{A}$. Show by an example that the converse is not true. 2+3

(b) Prove that $f: (X, \tau) \to (Y, \tau')$ is continuous if and only if $f^{-1}(F)$ is closed in X whenever F is closed in Y, where (X, τ) and (Y, τ') are two topological spaces.

(c) If u is an infinite cardinal number, prove that $u\, u = u$.

(d) Prove that a real valued continuous function defined on a compact space is bounded and attains its bounds.

(e) Prove that exactly one of $\alpha < \beta, \alpha = \beta, \beta < \alpha$ holds for any two ordinal numbers α, β.

(f) Prove that a nonempty subset in the space of reals $\mathbb{R}$ is connected if and only if it is an interval.

3. Answer *any two* questions: 10×2=20

(a) (i) Define a Lebesgue number. Prove that in a sequentially compact metric space every open cover has a Lebesgue number.

(ii) State and prove Heine-Borel theorem.

(1+4)+(1+4)

(b) (i) Let u be the cardinal number of the set U. Prove that the cardinal number of the power set $P(U)$ of U is 2^u.

(ii) Prove that every non-degenerate interval which is either open or closed or semi open or semi closed has the same cardinality as that of $\mathbb{R}$.

(iii) Let $(A, \leq)$ be a totally ordered set. Define an initial segment Ax of A. If $x \leq y$ in A, prove that $Ax \subset Ay$.

4+3+3

(c) (i) Prove that (X, τ) is compact if and only if every family of closed sets with finite intersection property has nonempty intersection.

(ii) Let (A, τ_A) be a subspace of a topological space (X, τ). Then show that a subset F of X is closed if and only if $F = A \cap K$, where K being a closed set in (X, τ). 5+5

(d) (i) Prove that an open subset of a locally connected space is locally connected.

(ii) Give an example of a connected space which is not locally connected.

(iii) Prove that a topological space is locally connected if and only if each component of an open set is open.

3+3+4

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