

B.A./B.Sc. 3rd Semester (General) Examination, 2023 (CBCS)

Subject : Mathematics

Course : BMG3CC-1C & Math GE-3

(Real Analysis)

Time: 3 Hours

Full Marks: 60

*The figures in the margin indicate full marks.**Candidates are required to give their answers in their own words
as far as practicable.**Notation and symbols have their usual meaning.*

1. Answer any ten questions: 2×10=20

- (a) Define limit point of a set in \mathbb{R} . Find the derived set of the set of all natural numbers \mathbb{N} .
- (b) Show that $\sqrt{2}$ is not a rational number.
- (c) Using Archimedean property, show that for any real number a , $\lim_{n \rightarrow \infty} \frac{a}{n} = 0$.
- (d) Prove that the sequence $\left\{ \frac{4n+1}{2n+3} \right\}_{n \in \mathbb{N}}$ is bounded.
- (e) Show that the series $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} + \dots$ is convergent and find its limit.
- (f) Assuming the convergence of the sequence $\{x_n\}_{n \in \mathbb{N}}$ find its limit, where $\{x_n\}_{n \in \mathbb{N}}$ is defined as $x_1 = \sqrt{6}$ and $x_{n+1} = \sqrt{6 + x_n}$, $n \geq 1$.
- (g) State Cauchy's general principle of convergence for sequence.
- (h) Give an example to show that a conditionally convergent series may not be absolutely convergent.
- (i) Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of functions defined on $[0, 1]$ as, $f_n(x) = \frac{x}{1+nx^2}$, $x \in [0, 1]$, $n \in \mathbb{N}$.
Examine whether $\{f_n\}_{n \in \mathbb{N}}$ is uniformly convergent or not.
- (j) Let $S = \left\{ \frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N} \right\}$. Show that 0 is a limit point of S .
- (k) Give an example to show that the union of arbitrary collection of closed sets in \mathbb{R} may not be a closed set.
- (l) Find the upper and lower limits of the set $\left\{ (-1)^n + \sin \frac{n\pi}{4} : n \in \mathbb{N} \right\}$.
- (m) Show that the series $\sum_{n=1}^{\infty} \frac{1}{an+b}$ ($a, b > 0$) diverges.

(n) Find $\lim_{n \rightarrow \infty} \frac{1+\sqrt{2}+\sqrt[3]{3}+\sqrt[4]{4}+\dots+\sqrt[n]{n}}{n}$.

(o) Test the convergence of the series $1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \dots$.

2. Answer *any four* questions:

5×4=20

(a) State and prove Archimedean property of \mathbb{R} .

(b) If $\lim_{n \rightarrow \infty} u_n = l$, then prove that $\lim_{n \rightarrow \infty} \frac{u_1+u_2+\dots+u_n}{n} = l$

(c) (i) Test the convergence of the series: $\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \dots$

(ii) Find the supremum and infimum of the set $A = \left\{ \frac{n+(-1)^n}{n} : n \in \mathbb{N} \right\}$.

3+2

(d) (i) Test the convergence of the series:

$$\frac{1}{1+2^{-1}} + \frac{1}{1+2^{-2}} + \frac{1}{1+2^{-3}} + \dots$$

(ii) Prove that the series $\frac{1}{1+a^2} - \frac{1}{2+a^2} + \frac{1}{3+a^2} - \dots$ is convergent.

3+2

(e) (i) Find the radius of convergence of the power series $1 + \frac{1}{2} \cdot x + \frac{1.3}{2.4} \cdot x^2 + \frac{1.3.5}{2.4.6} \cdot x^3 + \dots$

(ii) Test the convergence of the series $\sum_{n=1}^{\infty} \sin \frac{1}{n}$.

3+2

(f) Prove that a sequence of functions $\{f_n\}_{n \in \mathbb{N}}$ is uniformly convergent on $[a, b]$ to a function f if and only if $\lim_{n \rightarrow \infty} M_n = 0$, where $M_n = \sup_{x \in [a, b]} |f_n(x) - f(x)|$, $n \in \mathbb{N}$.

5

3. Answer *any two* questions:

10×2=20

(a) (i) Determine the radius of convergence of the power series

$$x + \frac{(2!)^2}{4!} x^2 + \frac{(3!)^2}{6!} x^3 + \dots + \frac{(n!)^2}{(2n)!} x^n + \dots$$

(ii) State Leibnitz Theorem of convergence for an alternating series. Use it to show that the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$ is convergent. Does the series converge uniformly? — Justify.

3+(2+3+2)

(b) (i) Prove that a convergent sequence is bounded. Does every bounded sequence converge? Support your answer.

(ii) Show that the sequence $\{x_n\}_{n \in \mathbb{N}}$, where $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n}$, $\forall n \in \mathbb{N}$ is convergent.

(4+2)+4

- (c) (i) Show that the set of all rational numbers is countable.
- (ii) Let $\sum_{n=1}^{\infty} a_n$ be a series of positive reals and let $l = \lim_{n \rightarrow \infty} \sqrt[n]{a_n}$. Then prove that the series converges if $l < 1$ and diverges if $l > 1$. 5+5

- (d) (i) Test whether the convergence is uniform or not:

$$x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \dots, x \in [0,1].$$

- (ii) Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of continuous functions converges uniformly to f on $[a, b]$.
Prove that f is continuous on $[a, b]$.

- (iii) Show that $\left\{ \frac{x^n}{1+x^n} \right\}_{n \in \mathbb{N}}$ is not uniformly convergent on $[0, 2]$. 3+5+2
