B.A./B.Sc. 5th Semester (Honours) Examination, 2023 (CBCS)

Subject : Mathematics

Course : BMH5CC11

(Partial Differential Equations and Applications)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols bear usual meaning.

1. Answer *any ten* questions: $2\times10=20$

(a) Find the differential equation of the set of all right circular cones whose axes coincide with z-axis.

(b) Define order and degree of a partial differential equation with example.

(c) Solve the partial differential equation $u_x^2 + u_y^2 = u$ using $u(x,y) = f(x) + g(x)$.

(d) Find the partial differential equation of the family of planes, the sum of whose x, y, z intercepts is equal to unity.

(e) Classify the following partial differential equations with proper reason whether they all linear, non-linear, semi-linear or quasi-linear: 1+1

(i) $xzp + x^2yz^2q = xy$

(ii) $xyp + x^2yq = x^2y^2z^2$

(f) Find the characteristic curve for the equation $x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x} = u$ in x-y plane.

(g) Find the general integral of $\frac{y^2z}{x}\,p + xzq = y^2$.

(h) Determine the region where the given partial differential equation $yu_{xx} + xu_{yy} = 0$ is hyperbolic in nature.

(i) Changing the independent variables by taking $u = y - x$ and $v = \frac{1}{2}(y^2 - x^2)$, find the value of $\frac{\partial^2 z}{\partial x\, \partial y}$.

(j) Find the family of surfaces orthogonal to the family of surfaces whose PDE is $(y+z)p + (z+x)q = x + y;\ p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}$.

(k) Show that $u(x,t) = \phi(x+ct) + \psi(x-ct)$ is a solution of the equation $c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, where ϕ and ψ are arbitrary functions.

(l) If $u = x\sin^{-1}\frac{y}{x} + y\tan^{-1}\frac{x}{y}$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ is $\frac{3\pi}{4}$ at (1, 1).

(m) Write down one-dimensional heat equation and indicate its nature.

(n) Solve $x^3\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} = 0$ using method of separation of variables if $u(0, y) = 10\,e^{5/y}$.

(o) Solve: $y^2(x-y)p + x^2(y-x)q = z(x^2+y^2)$

2. Answer *any four* questions: 5×4=20

(a) Find the integral surface of the differential equation $2y(z-3)p + (2x-z)q = y(2x-3)$ which passes through the circle $z = 0,\ x^2+y^2 = 2x$.

(b) Prove that the general solution of the semi linear partial differential equation $Pp + Qq = R$ is $F(u, v) = 0$ where u and v such that $u = u(x, y, z) = c_1$ and $v = v(x, y, z) = c_2$ are solution of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ $[c_1, c_2$ are constants$]$.

(c) Solve the Cauchy problem by method of characteristics $p - zq + z = 0$, for all y and $x > 0$ for the initial data curve $c: x_0 = 0, y_0 = t, z_0 = -2t, -\infty < t < \infty$.

(d) Reduce the equation $\frac{\partial^2 z}{\partial x^2} = (1+y)^2\frac{\partial^2 z}{\partial y^2}$ to canonical form.

(e) Solve the partial differential equation by the method of separation of variables:

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

(f) Find a complete integral of $x(1+y)p = y(1+x)q$.

3. Answer *any two* questions: 10×2=20

(a) (i) Consider partial differential equation of the form $ar + bs + ct + f(x, y, z, p, q) = 0$ in usual notation, where a, b, c are constants. Show how the equation can be transformed into its canonical form when $b^2 - 4ac = 0$.

(ii) Solve: $p + 3q = z + \cot(y - 3x)$ 6+4

(b) (i) Reduce the following to canonical form and hence solve:

$$x^2 r + 2xys + y^2 t = 0 \quad \left(r \equiv \frac{\partial^2 z}{\partial x^2}, s \equiv \frac{\partial^2 z}{\partial x \partial y}, t \equiv \frac{\partial^2 z}{\partial y^2}\right)$$

(ii) Find the characteristics of $\frac{\partial^2 z}{\partial x^2} + y^2\frac{\partial^2 z}{\partial y^2} = y$. (5+2)+3

(c) (i) Obtain D'Alembert's solution of following Cauchy problem of an infinite string:

$$u_{tt} - c^2 u_{xx} = 0,\ x \in \mathbb{R}, t > 0$$
$$u(x, 0) = f(x)$$
$$u_t(x, 0) = g(x)\ \forall x \in \mathbb{R}$$

(ii) Solve the following problem by method of characteristics: 5+5

$$z_x + zz_y = 1$$
$$z(0, y) = ay,\ a = \text{const.}$$

(d) (i) Use the method of separation of variable to solve the equation $\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}$, given that $v = 0$ when $t \to \infty$ as well as $v = 0$ at $x = 0$ and $x = l$.

(ii) Verify that $z = f(y + ix) + g(y - ix) - (m^2 + n^2)^{-1}$ is a solution of $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \cos mx\ .\cos ny$. 5+5

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