

**B.A./B.Sc. 6th Semester (Honours) Examination, 2025 (CBCS)****Subject : Mathematics****Course : BMH6 CC13****(Metric Space and Complex Analysis)****Time: 3 Hours****Full Marks: 60***The figures in the margin indicate full marks.**Candidates are required to give their answers in their own words as far as practicable.**Notations and symbols have their usual meaning.***1. Answer any ten questions:****2×10=20**

- (a) Let  $(X, \rho)$  and  $(Y, \sigma)$  be two metric spaces and  $E \subset X$ . Show by an example that if  $E$  is not compact then  $f(E)$  need not to be compact where  $f: X \rightarrow Y$  is continuous.
- (b) Let  $\mathbb{N}$  be the set of all natural numbers and  $d$  be the metric defined by  

$$d(m, n) = \left| \frac{1}{m} - \frac{1}{n} \right| \quad \forall m, n \in \mathbb{N}.$$
 Prove that  $(\mathbb{N}, d)$  is an incomplete metric space.
- (c) Find out one example of an infinite subset of  $\mathbb{R}$  which is compact and connected and one example of an infinite subset of  $\mathbb{R}$  which is neither compact nor connected. **1+1**
- (d) Let  $S = \{(x, y): x^2 + y^2 = 1\} \cup \{(x, 0): 1 < x < 2\}$ . Examine whether  $S$  is connected in  $\mathbb{R}^2$  with its usual metric.
- (e) Prove that a sequence in a discrete metric space is a Cauchy sequence if and only if it is eventually constant.
- (f) Show that the space  $\mathcal{P}[a, b]$  of all polynomials defined on  $[a, b]$  with the uniform metric  $d_\infty$  is not complete where  $d_\infty(x, y) = \max_{t \in [a, b]} |x(t) - y(t)|$ ,  $x, y \in \mathcal{P}[a, b]$ .
- (g) Show that the set of all real numbers  $\mathbb{R}$  with its usual metric  $d_u$  is not compact.
- (h) Prove that the function  $\arg: \mathbb{C} \setminus \{0\} \rightarrow (-\pi, \pi]$  is not a continuous function.
- (i) Show that the function  $f(z) = \operatorname{Re} z$  is continuous everywhere on  $\mathbb{C}$  but differentiable nowhere on  $\mathbb{C}$ .
- (j) Show that an analytic function with constant modulus in a domain is constant.
- (k) Find the radius of convergence of the power series  $\sum \left(1 + \frac{1}{n}\right)^{n^2} z^n$ .
- (l) Define the complex exponential function  $e^z$  and prove that  $\frac{d}{dz}(e^z) = e^z$ . **1+1**

(m) Is it possible to construct an analytic function in some domain with real part  $x(x + 3y) + 4$ ? If not, why?

(n) Evaluate  $\int_C \bar{z} dz$  where  $C$  is the right half of the circle  $|z| = 3$  from  $z = -3i$  to  $z = 3i$ .

(o) Define an entire function. State Liouville's Theorem.

1+1

2. Answer *any four* questions:

5×4=20

(a) Prove that the space  $l_p$  ( $1 \leq p < \infty$ ) is complete with respect to a suitable metric to be defined by you.

(b) If  $f(z)$  is an analytic function of  $z$ , prove  $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4|f'(z)|^2$ .

(c) Prove that a Cauchy sequence  $\{x_n\}$  in a metric space  $(X, d)$  is convergent if and only if it has a convergent subsequence.

(d) Show that a metric space is sequentially compact if and only if it has the BWP (Bolzano-Weierstrass Property).

(e) Let  $f: G \rightarrow \mathbb{C}$  where  $f(z) = f(x + iy) = u(x, y) + iv(x, y)$  be a function of complex variable  $z$  defined on a region  $G$ , which is differentiable at  $z_0 \in G$ . Then show that the functions  $u(x, y)$  and  $v(x, y)$  are differentiable at  $(x_0, y_0)$  and satisfy Cauchy-Riemann equations at  $z_0 = x_0 + iy_0$ .

(f) State and prove the fundamental theorem of algebra.

3. Answer *any two* questions:

10×2=20

(a) (i) Let  $f: (X, d) \rightarrow (Y, \rho)$  be continuous, where  $(X, d)$  is compact. Prove that for every subset  $A$  of  $X$ ,  $f(\bar{A}) = \overline{f(A)}$ .

(ii) Let  $(X, d)$  be a metric space and  $A$  be a subset of  $X$ . Let  $f: X \rightarrow \mathbb{R}$  be given by  $f(x) = d(x, A)$ ,  $x \in X$ .

Prove that  $f$  is uniformly continuous on  $X$  and  $f(x) = 0$  if and only if  $x \in \bar{A}$ .

4+(3+3)

(b) (i) Prove that the continuous image of a connected set in a metric space is connected.

(ii) Let  $(X, d)$  be a metric space. Then prove that  $X$  is totally bounded if and only if every sequence in  $X$  contains a Cauchy subsequence.

5+5

(c) (i) Find the Laurent series expansion of  $f(z) = \frac{1}{z(z-1)(z-2)}$  in  $2 < |z| < \infty$ .

(ii) State and prove Cauchy-Hadamard Theorem for power series of complex numbers.

4+(1+5)

- (d) (i) Let  $f(z)$  be an entire function. If the real part of  $f(z)$  is bounded, then prove that  $f(z)$  is constant. (3)
- (ii) Without evaluating the integral, show that  $\left| \int_C \frac{dz}{z^2-1} \right| \leq \frac{\pi}{3}$  where,  $C$  is the arc of the circle  $|z| = 2$  from  $z = 2$  to  $z = 2i$  that lies in the first quadrant.
- (iii) Evaluate  $\int_C f(z)dz$  where,  $C$  is the positively oriented circle  $C: |z - i| = 2$  and  $f(z) = \frac{1}{(z^2+4)^2}$ .

4+3+3