B.A./B.Sc. 5th Semester (General) Examination, 2023 (CBCS)

Subject : Mathematics

Course : BMG5DSE1A1

(Matrices)

Time : 3 Hours **Full Marks : 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) Find whether the vectors $(2,4,0),(0,1,0)$ and $(2,6,2)$ are linearly independent in the real vector space $\mathbb{R}^3$.

(b) Examine whether the set $W=\{(x,2y,3z) : x,y,z \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$.

(c) Find the image of the point $(2,-4)$ under the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x,y)=(x+3y,y-x), \forall (x,y) \in \mathbb{R}^2$.

(d) Under what condition (on x) the rank of the following matrix A is less than 2?

$$A=\begin{pmatrix} 1 & 2 \\ 0 & x \end{pmatrix}$$

(e) Find the eigenvalues of the matrix: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$.

(f) Examine whether the following system has solution(s) or not:

$$4x+6y=5$$
$$6x+9y=7$$

(g) When we call a matrix is diagonalizable?

(h) Find k so that the vectors $(1,-1,0)$, $(0,k,3)$ and $(0,2,3)$ are linearly dependent.

(i) Express $(5,6,9)$ as linear combination of $(-1,2,0)$, $(0,-1,1)$ and $(3,-4,2)$ in the vector space $\mathbb{R}^3$ over real field.

(j) What do you mean by rank of an $m \times n$ matrix?

(k) Find the normal form of the matrix: $A=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

(l) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x,y)=(2x-y,x+y), \forall (x,y) \in \mathbb{R}^2$. Find image of $(1,2)$ under T.

(m) Find the inverse of A if $A^2-A+I=0$.

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(n) If $A = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$, then find A^{100}.

(o) Find the matrix A, where $A^{-1} = \begin{pmatrix} 3 & -1 & 1 \\ 1 & -2 & -3 \\ 3 & -3 & 4 \end{pmatrix}$.

2. Answer *any four* questions: 5×4=20

(a) Show that $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x,y) = (x, x+y, x+2y), \forall\, (x,y) \in \mathbb{R}^2$ is a linear transformation. Find the representative matrix of T with respected to the ordered basis $\{(1,0),(0,1)\}$ of $\mathbb{R}^2$ and $\{(1,0,0),(0,1,0)(0,0,1)\}$ of $\mathbb{R}^3$.

(b) Find Kernel, image, nullity and rank of the following linear transformation.

$T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x,y) = (x,0), \forall\, (x,y) \in \mathbb{R}^2$.

(c) Determine the rank of the following matrix:

$$A = \begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$$

(d) Solve the following system of equations:

$$x + 3y - 2z = 0$$
$$2x - y + 4z = 0$$
$$x - 11y + 14z = 0$$

(e) Diagonalise the following matrix:

$$\begin{pmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{pmatrix}$$

(f) Using Elementary row operations, find the inverse of the matrix A, where

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 5 & 6 \\ 3 & 3 & 8 \end{pmatrix}$$

3. Answer *any two* questions: 10×2=20

(a) (i) Given the system of equations:

$$x + 4y + 2z = 1$$
$$2x + 7y + 5z = 2k$$
$$4x + my + 10z = 2k + 1$$

Find for what values of k and m the system will have (I) unique solution (II) no solution and (III) many solutions.

(ii) Reduce the following matrix to its normal form:

$$A = \begin{pmatrix} 2 & 2 & 2 \\ 1 & 2 & 1 \\ 3 & 4 & 3 \end{pmatrix}$$

Hence, find its rank. 5+(4+1)

(b) (i) Find a basis of the following subspace S of $\mathbb{R}^3$:

$$S = \{(x,y,z) \in \mathbb{R}^3 : 2x + y - z = 0\}.$$

Hence, find the dimension of the subspace.

(ii) Find the condition so that the planes $bx + ay - z = 0$, $cy + bz - x = 0$, $az + cx - y = 0$ intersect in a line. (4+1)+5

(c) (i) Find the eigenvalues of the 2-rowed orthogonal matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and verify that they are of unit modulus.

(ii) Show that the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalisable. 5+5

(d) (i) Prove or disprove: Union of two subspaces of the vector space $\mathbb{R}^3$ is always a subspace of $\mathbb{R}^3$.

(ii) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by $T(x,y,z) = (3x + 2y - 4z, x)$. Find the matrix representation of T relative to the basis $\{(1,1,1),(1,1,0),(1,0,0)\}$ of $\mathbb{R}^3$ and $\{(1,0),(0,1)\}$ of $\mathbb{R}^2$. 5+5

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B.A./B.Sc. 5th Semester (General) Examination, 2023 (CBCS)

Subject : Mathematics

Course : BMG5DSE1A2

(Mechanics)

Time : 3 Hours **Full Marks : 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) State a necessary and sufficient condition for the equilibrium of a system of coplanar forces.

(b) A block of wood is lying at rest on a horizontal table. Mention the forces acting on it.

(c) Give geometrical interpretation of the moment of a force about a given point.

(d) A system of coplanar forces acting on a rigid body is such that the algebraic sum of the moments about any three non-collinear points are separately zero. Will the system of forces be in equilibrium? Justify your answer.

(e) If the tangential and the normal components of acceleration of a particle moving in a plane curve are equal, find the velocity of the particle.

(f) The displacement x of a particle, moving in one-dimension, from a fixed point at any time t is given by $x = a\cos kt + b\sin kt$. Show that the particle executes a simple harmonic motion.

(g) A particle describes a curve $s = c\tan\psi$ with uniform speed v. Find the acceleration of the particle. (s, ψ) is the intrinsic coordinates.

(h) If the radial velocity is proportional to the transverse velocity, find the path in polar coordinates.

(i) For a rectilinear motion of a particle, if an impulse I changes its velocity from u to v, then find the change in kinetic energy.

(j) Define angle of friction and cone of friction, when two bodies are in contact with each other.

(k) Find the centre of gravity of the first quadrant of the circular plate $x^2 + y^2 \le a^2$.

(l) The position of a moving point (x, y) at time t is given by $x = a\cos t$ and $y = b\sin 2t$. Find the path in Cartesian coordinates.

(m) Find the central force for the central orbit described by the relation $\frac{1}{r} = 1 + e\cos\theta$ in polar coordinates (r, θ).

(n) Find the total energy of a particle which is executing a simple harmonic motion in one-dimension with amplitude a.

(o) Write down Kepler's third law.

2. Answer *any four* questions: 5×4=20

(a) A plane system of forces is equivalent to a couple of moment M, and if the forces are turned through a right angle, about their respective points of application in the same sense, they are equivalent to a couple N. Prove that when each force is turned about its point of application through an angle α in the same sense, the system will be in equilibrium if $\tan\alpha = -\frac{M}{N}$. 5

(b) Find the position of the centre of gravity of the plane lamina in the form of the cardioid $r = a(1 + \cos\theta)$. 5

(c) A ladder whose centre of gravity divides it into two portions of length 'a' and 'b' rest with one end on a horizontal floor and the other end against a rough vertical wall. If the coefficient of friction at the floor and the wall are respectively μ and μ', show that the inclination of the ladder to the floor, when the equilibrium is limiting, is $\tan^{-1}\left(\frac{a - b\mu\mu'}{\mu(a+b)}\right)$. 5

(d) A particle is projected with a velocity $\vec{u}$ in a direction inclined at an angle α to the horizon. Find the path of the particle, if it moves under the gravity which is supposed to be constant. The resistance of air can be neglected. 5

(e) If 'a' be the amplitude and 'T' be the period of a particle executing S.H.M. in a straight line, show that the time taken by the particle to travel a distance x from the centre of force is $\frac{T}{2\pi}\sin^{-1}\left(\frac{x}{a}\right)$ and the velocity in that position is $\frac{2\pi}{T}\sqrt{a^2 - x^2}$. 3+2

(f) Three forces $\vec{P}, \vec{Q}, \vec{R}$ act along the sides of the triangle formed by the lines $x + y = 1$, $y - x = 1$, $y = 2$. Find the magnitude and the equation of the line of the action of their resultant. 1+4

3. Answer *any two* questions: 10×2=20

(a) (i) Find the centre of gravity of a semicircular plate of radius 'a' whose mass per unit area at any point varies as $\sqrt{a^2 - r^2}$, where r is the distance of the point from the centre.

(ii) A particle describes a path with an acceleration $\frac{\mu}{y^3}$ which is always parallel to the axis of Y and directed towards the X-axis. If the particle is projected from a point $(0, a)$ with the velocity $\frac{\sqrt{\mu}}{a}$ parallel to X-axis, show that the path described by the particle is a circle. 5+5

(b) (i) In polar coordinates (r, θ), if the path of a particle is $r = a\tan\theta$ and the acceleration is always directed towards the origin, show that the acceleration is $\frac{K^2}{r^3} = \left[3 + \frac{2a^2}{r^2}\right]$, where $K = r^2\frac{d\theta}{dt}$.

(ii) A gun of mass M fires a shell of mass m horizontally and the energy of explosion is such as would be sufficient to project the shell vertically to height h. Show that the velocity of recoil of the gun is $\left\{\frac{2m^2gh}{M(m+M)}\right\}^{1/2}$. 5+5

(c) (i) A particle is describing a circle of radius 'a' in such a way that its tangential acceleration is k times the normal acceleration, where k is a constant. If the speed of the particle at any point be u, prove that it will return to the same point after a time $\frac{a}{ku}(1 - e^{-2k\pi})$.

(ii) Prove that if the three coplanar forces acting on a rigid body be in equilibrium, they must either all three meet a point, or else all must be parallel to one another. 5+5

(d) (i) A particle is projected with velocity u at an inclination α above the horizontal in a medium whose resistance per unit mass is k times the velocity. Show that its direction will again make angle α below the horizontal after a time $\frac{1}{k}\log(1 + \frac{2Ku}{g}\sin\alpha)$.

(ii) A particle P possesses two constant velocities $\vec{u}$ and $\vec{v}$ such that $\vec{u}$ is always parallel to a fixed direction OX and $\vec{v}$ is always perpendicular to the radius vector OP. Show that the path of the particle is a conic with focus O and eccentricity ${}^{u}/_{v}$. 5+5

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B.A./B.Sc. 5th Semester (General) Examination, 2023 (CBCS)

Subject : Mathematics

Course : BMG5DSE1A3

(Linear Algebra)

Time : 3 Hours **Full Marks : 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) Use Cayley-Hamilton theorem to find A^{-1}, where $A = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$.

(b) Find all values of k such that the set $S = \{(k, 1, k), (0, k, 1), (1, 1, 1)\}$ form a basis of $\mathbb{R}^3$.

(c) If λ is an eigenvalue of an orthogonal matrix, then prove that $|\lambda| = 1$.

(d) Prove that the set $S = \{(2, 1, 1), (1, 2, 1), (1, 1, 2)\}$ is a basis of $\mathbb{R}^3$.

(e) Find dim $(W_1 \cap W_2)$, where W_1, W_2 are subspaces of $\mathbb{R}^3$, defined by $W_1 = \{(x, y, z) \in \mathbb{R}^3 : x + 2y + z = 0\}$ and $W_2 = \{(x, y, z) \in \mathbb{R}^3 : 2x + y + 3z = 0\}$.

(f) Find the eigenvalues of $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(g) Find a basis and the dimension of the subspace $W = \{(a, b, c) \in \mathbb{R}^3 : a + b + c = 0\}$ of $\mathbb{R}^3$.

(h) Check whether $W = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 5\}$ is a subspace of $\mathbb{R}^3$ or not.

(i) Prove that the mapping $T : V \to V/W$ defined by $T(\alpha) = \alpha + W;\ \alpha \in V$ is a linear transformation, where W is a subspace of V.

(j) Prove that there cannot be a one-one linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$.

(k) Determine the linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^2$ which maps the basis vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ of $\mathbb{R}^3$ to the vectors $(1, 1), (2, 3), (3, 2)$.

(l) Find the dual basis of the set $S = \{(2, 1), (3, 1)\}$ of $\mathbb{R}^2$.

(m) Determine $ker(T)$, where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(x, y) = (\cos x, \sin y), \forall (x, y) \in \mathbb{R}^2$.

(n) If the linear operator T on a vector space V such that $T^2 - T + I = 0$, then show that T is invertible.

(o) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation defined by $T(x, y) = (x + 3y, 0, 2x - 4y)$. Find the matrix representation of T relative to the standard basis $\alpha = \{e_1, e_2\}$ of $\mathbb{R}^2$ and $\gamma = \{e_3, e_2, e_1\}$ of $\mathbb{R}^3$.

2. Answer *any four* questions: 5×4=20

(a) Find the eigenvalues and the corresponding eigenvectors of $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{pmatrix}$.

(b) State and prove Cayley-Hamilton theorem.

(c) A linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x, y, z) = (x + 2y + 3z\ ,\ 2x + 3y + z\ ,\ 3x + y + 2z)$, $(x, y, z) \in \mathbb{R}^3$. Find the matrix of T relative to the ordered basis $\{(-1, 1, 1), (1, -1, 1), (1, 1, -1)\}$ of $\mathbb{R}^3$.

(d) Let V and W be finite dimensional vector spaces over a field F and $\psi : V \to W$ be an isomorphism. Then show that S is linearly independent in V if and only if $\psi(S)$ is linearly independent in W, where S is a set of vectors in V.

(e) Let V be a vector space. Then prove that each element of $\alpha \in V$ determines a specific element of $\hat{\alpha} \in V^{**}$, where V^{**} is the double dual of V.

(f) Determine the linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^2$ which maps the basis vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ of $\mathbb{R}^3$ to the vectors $(1, 1)$, $(2, 4)$, $(3, 4)$ respectively. Find $\ker T$ and $\operatorname{Im} T$.

3. Answer *any two* questions: 10×2=20

(a) (i) Consider the subspace $W_1 = \{(x, 0, z) : x, z \in \mathbb{R}\}$ and $W_2 = \{(0, y, z) : y, z \in \mathbb{R}\}$ of the vector space $\mathbb{R}^3$. Find $W_1 + W_2$. Is $W_1 + W_2$ a direct sum? Support your answer.

(ii) Find a basis of the vector space $S = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : a + b = 0\right\}$.

(iii) Find the transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ for which the matrix of transformation T is $A = \begin{bmatrix} 1 & 2 & 5 & 1 \\ 1 & 3 & 4 & -2 \\ 3 & 8 & 11 & -3 \end{bmatrix}$ relative to the ordered basis $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)$ $(0, 0, 0, 1)\}$ of $\mathbb{R}^4$ and $\{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$. Find the dimension of $ker(T)$ and find a basis of $ker(T)$. (2+1)+2+(3+1+1)

(b) (i) Find dim $(S \cap T)$, where S and T are subspaces of the vector space $\mathbb{R}^4$ given by $S = \{(x, y, z, w) \in \mathbb{R}^4 : 2x + y + 3z + w = 0\}$, $T = \{(x, y, z, w) : x + 2y + z + 3w = 0\}$

(ii) Find the matrix of T relative to the order basis $\{(1,1,1), (1,1,0), (1,0,0)\}$ of $\mathbb{R}^3$, where $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x_1, x_2, x_3) = (x_1 + 2x_2 - x_3,\ x_1 + 4x_3,\ x_1 - 2x_2 + 3x_3)$.

(iii) In a real vector space $\mathbb{R}^3$, prove that every plane through the origin is a subspace of $\mathbb{R}^3$. 4+3+3

(c) (i) Prove that composition of two linear transformations (from $V \to V$) is another linear transformation, where V is a vector space over a field F.

(ii) Prove that for every finite dimensional vector space V over a field F has a basis. 3+7

(d) (i) The matrix of a linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^2$ relative to the ordered bases $\{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ of $\mathbb{R}^3$ and $\{(1,0), (1, 1)\}$ of $\mathbb{R}^2$ is $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$. Find T.

(ii) If T and Q be both $n \times n$ matrices and Q be non-singular, then show that T and $Q^{-1}TQ$ have the same eigenvalues. 5+5

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