B.A./B.Sc. 6th Semester (Honours) Examination, 2024 (CBCS)

Subject : Mathematics

Course : BMH6CC-13

(Metric Space and Complex Analysis)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) Examine whether the set $\{(x,y) \in \mathbb{R}^2 : x^2+y^2=4,\ x>0,\ y>0\}$ is compact in $\mathbb{R}^2$ equipped with usual metric.

(b) Prove that the function $f(x) = \sin x^2, 0 \le x < \infty$ is not uniformly continuous on $[0, \infty)$.

(c) Prove that the set Q_u of all rational numbers with usual metric is not complete.

(d) Define a contraction mapping. Is $f(x) = x^2$ a contraction mapping on $(\mathbb{R}, du)$? Where *du* is the usual metric on $\mathbb{R}$.

(e) Is the set $\{1960, 1961, \dots, 2024\}$ compact in $(\mathbb{R}, du)$? Justify. (*du* is the usual metric on $\mathbb{R}$)

(f) Let $f(z) = \begin{cases} \frac{\bar{z}}{z}, & z \neq 0, z \in \mathbb{C} \\ 0, & z = 0. \end{cases}$

Prove that f is not continuous at $z = 0$. 2

(g) Find $\int_c \frac{z+2}{z}\, dz$, where c is the semi-circle $z = 2e^{i\theta}$ $(0 \le \theta \le \pi)$ in the counter clockwise direction.

(h) If f and g are analytic functions in a region D of $\mathbb{C}$ and $f'(z) = g'(z)\ \forall\, z \in D$, then show that $f - g$ is a constant function in D. 2

(i) Evaluate $\int_C \frac{z\, dz}{z^2+9}$, where C is the circle: $|z - i| = 3$.

(j) Let $\mathbb{R}$ be equipped usual metric space and $f: \mathbb{R} \to \mathbb{R}$ be a continuous function, if $f(x) = 0, \forall x \in \mathbb{Q}$, the set of rationals, prove that $f(x) = 0, \forall x \in \mathbb{R}$.

(k) Let $X = (0,1]$. Set $A_1 = \left(\frac{1}{2}, 2\right)$ and $A_n = \left(\frac{1}{n+1}, \frac{1}{n-1}\right)$, for each integer $n > 1$. Prove that $\{A_n\}_{n=1}^{\infty}$ is an open cover of X and that no finite subfamily of it covers X.

(l) Examine for convergence of the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}, z \in \mathbb{C}$.

(m) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{n\sqrt{2}+i}{1+2ni} z^n$.

(n) If $f(z)$ is differentiable in a region G and $R\,f(z)$ is a constant then prove that $f(z)$ is a constant in G.

(o) Show that $\int_C f(z)dz = 0$ where $f(z) = \frac{z^2}{z-4}$ and C is the positively oriented unit circle $|z| = 1$.

2. Answer *any four* questions: 5×4=20

(a) Prove that the space $C[a,b]$, the collection of all real valued continuous functions over the closed interval $[a,b]$ is a complete metric space with respect to the sup metric.

(b) State and prove Cantor's intersection theorem in a metric space. 1+4

(c) Prove that a function $f: (X, d) \to (Y\rho)$ is continuous if and only if $\overline{f^{-1}(B)} \subset f^{-1}(\bar{B})$ for every subset B of Y. 5

(d) Prove that the function

$$f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & \text{if } x^2 + y^2 \neq 0 \\ 0, & \text{if } x^2 + y^2 = 0 \end{cases}$$

satisfies Cauchy–Riemann equations at the origin but f is not differentiable at $z = 0$.

(e) Prove that a compact subset of a metric space is closed and bounded.

(f) State and prove Liouville's theorem. 1+4

3. Answer *any two* questions: 10×2=20

(a) (i) Let (X, d) be a compact metric space and let $f: X \to X$ be such that $d(f(x), f(y)) < d(x, y) \forall x, y \in X$ with $x \neq y$. Show that f has a fixed point. Is the fixed point unique? Justify your answer.

(ii) Let (X, d) be a complete metric space and let $Y \subset X$. Then prove that Y is complete if and only if it is closed. (3+2)+(2+3)

(b) (i) Prove that a subset $A \subset \mathbb{R}$ is connected if and only if it is an interval. (Here $\mathbb{R}$ is equipped with usual metric)

(ii) Show that continuous function on a compact set is uniformly continuous. 6+4

(c) (i) Show that the function $f(z) = \sqrt{|xy|}, z = x + iy$, is not analytic at the origin, although the Cauchy-Riemann equations are satisfied at the point.

(ii) Let f be analytic in a simply connected region D and let C be a smooth closed curve contained in D. Then prove that $\int_C f(z)dz = 0$. 4+6

(d) (i) Find the analytic function of which the real part is $e^{-x}\{(x^2 - y^2)\cos y + 2xy \sin y\}$.

(ii) State Cauchy Integral formula.

(iii) If $\sum_{n=0}^{\infty} a_n z^n$ converges to a function $\phi(z)$ for $|z| < R$, then prove that $\phi'(z) = \sum_{n=1}^{\infty} n\, a_n z^{n-1}$ for $|z| < R$. Deduce that under the hypothesis $\phi(z)$ may be differentiated any number of times for $|z| < R$. 4+2+(3+1)

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