3 Yr. Degree/4 Yr. Honours 3rd Semester Examination, 2024 (CCFUP)

Subject : Mathematics

Course : MATH3011 (MAJOR)

(Real Analysis I)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

1. Answer *any ten* of the following questions: 2×10=20

(a) Let $S = \left\{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\right\}$. Show that 0 is a limit point of S.

(b) Prove that between any two real numbers, there exists a rational number.

(c) Prove that every convergent sequence of real numbers is bounded.

(d) If $b \in \mathbb{R}$ be such that $0 < b < 1$ then show that $\lim_{n\to\infty} nb^n = 0$.

(e) Find the upper and lower limits of the sequence $\left\{\sin\frac{n\pi}{3}\right\}$.

(f) Let S be a closed and bounded subset of $\mathbb{R}$. If each point of S is an isolated point of S, then show that S is a finite set.

(g) Give an example to show that arbitrary intersection of open sets may not be an open set in $\mathbb{R}$.

(h) Let $A \subset \mathbb{R}$. Prove that the derived set of A is a closed set.

(i) Prove that $\lim\limits_{x\to\infty} \frac{[x]}{x} = 1$, $[x]$ denotes the greatest integer not exceeding x.

(j) Prove that the function $f(x) = \sin\frac{1}{x}$, $x \in (0,1)$ is not uniformly continuous on $(0,1)$.

(k) Prove that the equation $x = \cos x$ has a root in $\left(0, \frac{\pi}{2}\right)$.

(l) Test the convergence of the series $\sum\limits_{n=2}^{\infty} \frac{1}{\log n}$.

(m) If $\sum\limits_{n=1}^{\infty} a_n$ is a convergent series, prove that $\lim\limits_{n\to\infty} a_n = 0$.

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(n) If $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers then prove that $\sum_{n=1}^{\infty} a_n^2$ is convergent.

(o) Prove that $\lim_{n\to\infty} \frac{1+2^{\frac{1}{2}}+3^{\frac{1}{3}}+\ldots+n^{\frac{1}{n}}}{n}=1$.

2. Answer *any four* of the following questions: 5×4=20

(a) Prove that every bounded sequence has a convergent subsequence. 5

(b) State the least upper bound axiom. State and prove Archimedean property of real numbers. (1+1+3)

(c) Prove that the complement of an open set in $\mathbb{R}$ is a closed set. Hence show that for any $x > 0$, the set $\{nx : n \in \mathbb{N}\}$ is a closed set. (3+2)

(d) A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $f(x+y) = f(x) + f(y)\ \forall\, x, y \in \mathbb{R}$. Prove that f is a linear function. 5

(e) Test the convergence of the series $\left(\frac{1}{2}\right)^3 + \left(\frac{1.4}{2.5}\right)^3 + \left(\frac{1.4.7}{2.5.8}\right)^3 + \cdots$ 5

(f) Let $f : [0,1] \to [0,1]$ be continuous on $[0,1]$. Then prove that $\exists\, c \in [0,1]$ such that $f(c) = c$. 5

3. Answer *any two* of the following questions: 10×2=20

(a) (i) Show that the sequence $\left\{\sqrt{7}, \sqrt{7+\sqrt{7}}, \sqrt{7+\sqrt{7+\sqrt{7}}}, \ldots\right\}$ converges to the positive root of $x^2 - x - 7 = 0$. 5

(ii) A function $f : [0,1] \to \mathbb{R}$ is defined by $f(x) = \begin{cases} x, & \text{if } x \text{ is rational in } [0,1] \\ 1-x, & \text{if } x \text{ is irrational in } [0,1] \end{cases}$.

Show that f is continuous at $\frac{1}{2}$ and discontinuous at all other points in $[0,1]$. 3

(iii) Give example with proper justification of two discontinuous functions whose product is a continuous function. 2

(b) (i) Define a Cauchy sequence in $\mathbb{R}$. Prove that a sequence $\{x_n\}$ of real numbers is convergent if and only if it is Cauchy. 1+4

(ii) Prove that every bounded infinite subset of $\mathbb{R}$ has at least one limit point. Is the condition of boundedness necessary? Justify your answer. 4+1

(c) (i) Prove that intersection of an arbitrary number of closed sets in $\mathbb{R}$ is a closed set in $\mathbb{R}$. Is arbitrary union of closed sets in $\mathbb{R}$ a closed set? Justify your answer. 2+2

(ii) Let $\{x_n\}$ be a bounded sequence. Then prove that $\underline{\lim}\, x_n \leq \overline{\lim}\, x_n$. 4

(iii) Show that the series $\sum_{n=1}^{\infty} \sin\frac{1}{n}$ is divergent. 2

(d) (i) Define absolute convergence of a series. Prove that every absolutely convergent series of real numbers is convergent. Give an example to show that the converse is not always true. 1+3+2

(ii) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and $f(a) \neq f(b)$. Then prove that f attains every value between $f(a)$ and $f(b)$. 4

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