B.A./B.Sc. 3rd Semester (General) Examination, 2022 (CBCS)

Subject : Mathematics

Course : CC-1C/GE-3

(Real Analysis)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
Symbols and notation have their usual meaning.

1. Answer *any ten* questions from the following: 2×10=20

(a) State order completeness property of the set of reals $\mathbb{R}$.

(b) If $x \in \mathbb{R}$ and $x > 0$, then prove that $m - 1 \leq x < m$ for a natural number m.

(c) When is a subset A of the set of real numbers said to be an interval?

(d) Find the infimum and supremum of the set $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. 1+1

(e) Give an example of monotone increasing sequence which is not bounded above.

(f) Prove that every convergent sequence is Cauchy.

(g) Define null sequence. If $\{U_n\}$ be a null sequence, then prove that $\{|U_n|\}$ is a null sequence. 1+1

(h) Let $\sum U_n$ be a convergent series. Then prove that $\lim U_n = 0$. Does the converse of the above statement hold? Justify your answer. 1+1

(i) Let A be a countable set. Show that $A \times A$ is countable set.

(j) Examine the convergence of the series $x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots, x > 0$.

(k) Is every convergent series absolutely convergent? Justify your answer.

(l) Find the limit points of $A = [0, 1]$.

(m) Show that the series $(1 - x) + x(1 - x) + x^2(1 - x) + \cdots$ is not uniformly convergent on $[0, 1]$.

(n) Find the radius of convergence of the power series $\sum \frac{2^n x^n}{n!}$.

(o) Show that the sequence $\{x^2 e^{-nx}\}, x \geq 0$ is point-wise convergent in $[0, \infty)$.

2. Answer *any four* questions from the following: 5×4=20

(a) Prove that a necessary and sufficient condition for a sequence $\{f_n\}$ of functions defined in a set S to be uniformly convergent is that for each $\varepsilon > 0$, there corresponds m such that $\forall\, n \geq m, \forall\, p \geq 1$ and $\forall\, x \in S$, $|f_{n+p}(x) - f_n(x)| < \varepsilon$.

(b) (i) What do you mean by alternating series?

(ii) Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\log(n+1)}$ is convergent. Examine its absolute convergence.

1+(2+2)

(c) State monotone convergence theorem. Using this theorem, show that the sequence $\left\{\frac{2n-7}{3n+2}\right\}$ is convergent. 1+4

(d) Using Cauchy's criterion of convergence, examine the convergence of the sequence $\{U_n\}$, where $U_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$.

(e) (i) State Bolzano–Weierstrass theorem and hence verify it for the set $S = \left\{\frac{n-1}{n+1}; n \in \mathbb{N}\right\}$.

(ii) Show that the sequence $\left\{\frac{(-1)^n}{n}\right\}$ is a Cauchy sequence. 3+2

(f) Define rational number. If x, y are real numbers with $x < y$, then prove that there is a rational number r such that $x < r < y$. 1+4

3. Answer *any two* questions from the following: 10×2=20

(a) (i) Prove that the set of natural numbers $\mathbb{N}$ is not bounded above.

(ii) Show that a finite set has no limit point.

(iii) Let m and M be the infimum and supremum of the subset A of $\mathbb{R}$. Find infimum and supremum of $B = \{-x : x \in A\}$. 4+3+3

(b) (i) State and prove Cauchy's first theorem on limits.

(ii) State Sandwich theorem. Using this theorem, prove that $\lim_{n\to\infty}(\sqrt{n+1} - \sqrt{n}) = 0$.

5+(1+4)

(c) (i) State and prove D'Alembert's ratio test.

(ii) Test the convergence of the series $\frac{1}{2} + \frac{2}{3}x + \left(\frac{3}{4}\right)^2 x^2 + \left(\frac{4}{5}\right)^3 x^3 + \cdots, x > 0$, using Cauchy's root test. (1+4)+5

(d) (i) Prove that the series $x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$, converges if and only if $-1 \leq x \leq 1$.

(ii) Prove that the series $\sum \frac{x}{n+n^2x^2}$ is uniformly convergent for all real x. 5+5