3 Yr. Degree/4 Yr. Honours 3rd Semester Examination, 2024 (CCFUP)

Subject : Mathematics

Course : MATH3012 (MAJOR)

(Linear Algebra)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations and symbols have their usual meaning.

1. Answer *any ten* questions from the following: 2×10=20

(i) Examine whether the following set of vectors is linearly independent in $\mathbb{R}^3$:
$\{(2,-3,1),(3,-1,5),(1,-4,3)\}$

(ii) Is the map $T(x,y)=(x,y+1), \forall\, x,y\in\mathbb{R}$, a linear transformation from the real vector space $\mathbb{R}^2$ into itself? Justify your answer.

(iii) Find the largest eigenvalue of the matrix $\begin{pmatrix} 2 & -1 & 3 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{pmatrix}$.

(iv) For what real values of k, does the set $S=\{(k,1,k),(0,k,1),(1,1,1)\}$ form a basis of $\mathbb{R}^3$?

(v) Show that real quadratic form $Q(x,y)=x^2-xy+y^2$ on $\mathbb{R}^2$ is positive definite.

(vi) Let β be a basis of a finite dimensional inner product space such that $\langle x,z\rangle=0\ \forall\, z\in\beta$. Then show that $x=\theta$.

(vii) Let $T:\mathbb{R}^3\to\mathbb{R}^3$ be a linear mapping on the vector space $\mathbb{R}^3$ over the field $\mathbb{R}$, which is defined as follows:
$T(x,y,z)=(x+z,\ x+y+2z,\ 2x+y+3z)\ \forall (x,y,z)\in\mathbb{R}^3$. Find the kernel of T.

(viii) Find algebraic and geometric multiplicity of each eigenvalue of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$.

(ix) Use Cayley-Hamilton theorem to find A^{50}, where $A=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(x) Find the eigenvalues of $A^2+5A+2I$, where $A=\begin{bmatrix} -2 & 6 & 6 \\ 0 & 3 & -5 \\ 0 & -3 & 1 \end{bmatrix}$.

(xi) Show that the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

(xii) Consider the system

$$2x + ky = 2 - k$$
$$kx + 2y = k$$
$$ky + kz = k - 1$$

in three unknowns and one real parameter k. Determine the values of k for which the system of linear equations is consistent.

(xiii) State the parallelogram law of an inner product space. Give an example of an inner product space where this law does not hold.

(xiv) Show that $\langle \binom{x_1}{x_2}, \binom{y_1}{y_2} \rangle = x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2$ is an inner product on $\mathbb{R}^2$.

(xv) State Sylvester's law of inertia.

2. Answer *any four* questions from the following: 5×4=20

(a) If a vector space is generated by a finite set, then prove that it has a basis.

(b) Prove that a vector space is infinite dimensional if and only if it contains an infinite linearly independent subset.

(c) Determine the conditions for which the system of linear equations

$$x + y + z = 1,$$
$$x + 2y - z = b$$

and $5x + 7y + az = b^2$

admits (i) only one solution,

(ii) no solution,

(iii) many solutions.

(d) Define direct sum of two subspaces of a vector space. Let V be the vector space of all functions from $\mathbb{R} \to \mathbb{R}$. Define $V_e = \{f \in V : f \text{ is even}\}$ and $V_O = \{f \in V : f \text{ is odd}\}$. Then show that V_e and V_O are subspaces of V and also show that $V = V_e \oplus V_O$. 1+2+2

(e) The matrix of a linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the ordered basis $\{(0,1,1),(1,0,1),(1,1,0)\}$ of $\mathbb{R}^3$ is given by $\begin{bmatrix} 0 & 3 & 0 \\ 2 & 3 & -2 \\ 2 & -1 & 2 \end{bmatrix}$.

Find T. Find also the matrix of T relative to the ordered basis $\{(2,1,1),(1,2,1),(1,1,2)\}$. 3+2

(f) Apply the Gram-Schmidt orthogonalization process to the given subset S of the inner product space V to obtain an orthogonal basis for span S: Given $V = \text{Span}(S)$ with inner product $\langle f, g \rangle = \int_0^\pi f(t) \cdot g(t)dt$ and $S = \{\sin t, \cos t, 1, t\}$.

10×2=20

3. Answer *any two* questions from the following:

(a) (i) State and prove Schwarz's inequality.

(ii) Using Gram-Schmidt technique, find an orthonormal basis for the subspace spanned by the vectors (1, – 1, 1, – 1), (5, 1, 1, 1) and (– 3, – 3, 1, – 3).

(iii) Show that an inner product on a vector space is a continuous mapping. (1+4)+3+2

(b) (i) Let T be a linear operator on $\mathbb{R}^3$ defined by $T(x\ y\ z) = (3x,\ x - y,\ 2x + y + z)$, show that T is invertible and also find T^{-1} if defined.

(ii) Find a Jordan canonical form of the matrix $\begin{bmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$. (3+3)+4

(c) (i) Show that any inner product space has an orthonormal basis.

(ii) Let V be a vector space over a field F and W be a subspace of V. Then show that $\dim \left({}^{V}/_{W}\right) = \dim V - \dim W$. 5+5

(d) (i) State and prove the extension theorem for a basis of a vector space. How does the extension theorem facilitate the construction of a basis for a vector space?

(ii) Let $U = \{(x, y, z) : x + y + z = 0\}$ and $W = \{(x, y, z) : x + 2y - z = 0\}$ be two subspaces of $\mathbb{R}^3$. Find $\dim U, \dim V, \dim (U \cap W)$ and $\dim\ (U + W)$. (1+4)+2+3