B.A./B.Sc. 6th Semester (Honours) Examination, 2024 (CBCS)

Subject : Mathematics

Course : BMH6 CC14

(Ring Theory and Linear Algebra-II)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) Find the zeros of the polynomial $x^3 + \bar{2}x^2 + x + \bar{1}$ in $\mathbb{Z}_5[x]$.

(b) Examine if the polynomial $2x^{10} - 25x^7 + 10x^4 + 5x^2 + 20$ is irreducible over Q.

(c) Show that $\bar{2}$ is a prime element in $\mathbb{Z}_{10}$, but not irreducible in $\mathbb{Z}_{10}$.

(d) In $\mathbb{R}^3$, if P is the subspace generated by (1, 1, 0) and (0, 1, 1), then find the orthogonal complement $P^{\perp}$ of P in $\mathbb{R}^3$.

(e) If T_1 and T_2 are two linear operators on a finite dimensional inner product space $V(F)$, then show that $(T_1T_2)^* = T_2^*T_1^*$.

(f) If α, β are two orthogonal vectors in a Euclidean space V, then show that $\| \alpha + \beta \|^2 = \| \alpha \|^2 + \| \beta \|^2$.

(g) If V is a finite dimensional vector space over a field F, then show that $\dim(V) = \dim(V^*)$, where V^* is the dual space of V.

(h) Find the characteristic polynomial of the linear operator $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$T(x, y) = (2x + 5y, x - 3y)$.

(i) If α and β be vectors in an inner product space, then show that $\| \alpha + \beta \|^2 - \| \alpha - \beta \|^2 = 4 \langle \alpha, \beta \rangle$.

(j) If $\mathbb{R}^2$ is a vector space over $\mathbb{R}$ and $B = \{(1,3), (3,1)\}$ is a basis of $\mathbb{R}^2$, find the dual basis of B.

(k) If $f \in (\mathbb{R}^2)'$ is defined by $f(x, y) = x + 2y$ and the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(x, y) = (x, 2x + 3y)$, then compute $T^t(f)$, where $(\mathbb{R}^2)'$ is the dual of $\mathbb{R}^2$ and T^t is the transpose of T.

(l) Find the basis of the annihilator W^0 of the subspace W of $\mathbb{R}^4$ spanned by $\{(1, 2, -3, 4), (0, 1, 4, -1)\}$.

(m) Is the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ diagonalizable? Justify your answer.

(n) Prove that $\mathbb{Z}[x]$ is not PID.

(o) Let R be an integral domain. Then prove that every prime element is irreducible.

2. Answer *any four* questions: 5×4=20

(a) Prove that the ring $\mathbb{Z}[i]$ of Gaussian integers is PID.

(b) Prove that in a UFD, any two non-zero elements have a gcd.

(c) Suppose V is a finite dimensional vector space and W is a subspace of V. Then prove that $\dim W + \dim W^0 = \dim V$, where W^0 is the annihilator of W.

(d) Find the minimal polynomial of the matrix $\begin{pmatrix} 2 & 2 & -5 \\ 3 & 7 & -15 \\ 1 & 2 & -4 \end{pmatrix}$.

(e) Let P_2 be the real inner product space consisting of all polynomials over $\mathbb{R}$ of degree ≤ 2 with respect to the inner product $\langle f, g\rangle = \int_0^2 f(t)g(t)dt$. Apply Gram-Schmidt process to the set of vectors $\{1, t, t^2\}$ to obtain an orthonormal basis of P_2.

(f) If $\{\beta_1, \beta_2, \ldots, \beta_r\}$ is an orthonormal set of vectors in a Euclidean space V, then prove that for any vector $\alpha \in V$, $\| \alpha \|^2 \geq C_1^2 + C_2^2 + \cdots + C_r^2$, where C_i is the scalar component of α along β_i, $i = 1, 2, 3, \cdots, r$.

3. Answer *any two* questions: 10×2=20

(a) (i) Find the gcd of 2 and $1 + i\sqrt{5}$ in the domain $\mathbb{Z}[i\sqrt{5}]$.

(ii) Let R be a commutative ring with 1. If $R[x]$ is a principal ideal domain, then show that R is a field. 5+5

(b) (i) Find the characteristic and minimal polynomial of the linear operator $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (2x - y, x + y + z, 2z)\ \forall\ (x, y, z) \in \mathbb{R}^3$.

(ii) Let V be a finite dimensional vector space over a field F and $B = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ an ordered basis of V. If $\{a_1, a_2, \ldots, a_n\}$ is an ordered set of n scalars, then show that there exists a unique functional ϕ on V such that $\phi(\alpha_i) = a_i$ for $i = 1, 2, \cdots, n$. (3+2)+5

(c) (i) If V is finite dimensional vector space, then show that there exists a canonical isomorphism of V onto V'', where V'' represents the double dual of V.

(ii) Apply Gram-Schmidt process to the set of vectors $\{(1, 0, 1), (1, 0, -1), (1, 3, 4)\}$ to obtain an orthonormal basis for $\mathbb{R}^3$ with the standard inner product. 5+5

(d) (i) Show that 3 is irreducible but not prime in $\mathbb{Z}[i\sqrt{5}]$.

(ii) Prove that $F[x]$ is an Euclidian domain for a field F. 5+5