B.A./B.Sc. 5th Semester (Honours) Examination, 2024 (CBCS)

Subject : Mathematics

Course : BMH5CC11

(Partial Differential Equations and Applications)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) Find a partial differential equation by eliminating a, b, c from $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

(b) Write down the Lagrange's linear PDE. What is the nature of this PDE?

(c) Show that the surfaces represented by $Pp+Qq=R$ are orthogonal to the surface represented by $Pdx+Qdy+Rdz=0$.

(d) Examine whether the region of the PDE $u_{xx}-\sqrt{y}\,u_{xy}+xu_{yy}=\cos(x^2-2y), y\geq 0$ is hyperbolic, parabolic or elliptic.

(e) Find the characteristic curves of the PDE $\frac{\partial^2 z}{\partial x^2}+y^2\frac{\partial^2 z}{\partial y^2}=y,\ y>0$.

(f) Explain the concepts of Cauchy problem for second order partial differential equation.

(g) When is a first order partial differential equation said to be non-linear? Give example.

(h) Find the general solution of the partial differential equation $\frac{\partial^2 z}{\partial x \partial y}=x+y$.

(i) Find the order and nature of the PDE $\frac{\partial^2 z}{\partial x^2}=x^2\frac{\partial^2 z}{\partial y^2}$.

(j) Form the PDE by eliminating the arbitrary function from $f(x+y+z, x^2+y^2+z^2)=0$.

(k) State Cauchy–Kowalewskaya theorem.

(l) Prove that the equation $\left(x^2D^2+2xy\,DD'+y^2D'^2\right)z=x^3y^2$ is reduced to linear PDE by the substitutions $x=e^u, y=e^v$; where $D\equiv\frac{\partial}{\partial x}, D'\equiv\frac{\partial}{\partial y}$.

(m) Determine whether the given PDE is linear, semi-linear, quasi-linear or non-linear.

(i) $p^2+q^2=x^2$

(ii) $xp+yq=z$

(n) Write down the Laplace equation in plane polar co-ordinates.

(o) Distinguish between Boundary value problem and Initial value problem with examples.

2. Answer *any four* questions: 5×4=20

(a) Find the equation of all surfaces whose tangent planes cut off an intercept of constant length from z-axis.

(b) Find the complete solution of $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

by the method of separation of variables.

(c) Reduce $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$ to canonical form and hence solve it. 3+2

(d) Solve : $yz\frac{\partial z}{\partial x} + xz\frac{\partial z}{\partial y} = xy$

Also find the integral surface passing through $z^2 - y^2 = 1,\ x^2 - y^2 = 4$.

(e) Solve the following problem by method of characteristics $z_x + zz_y = 1$, $z(0, y) = ay,\ a =$ constant.

(f) Solve : $\cos(x + y)\, p + \sin(x + y)\, q = z$.

3. Answer *any two* questions: 10×2=20

(a) (i) Find the solution of $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$

subject to $u(x, 0) = 6e^{-3x}$.

(ii) Prove that the general solution of the semi-linear PDE $Pp + Qq = R$ is $F(U, V) = 0$, where U and V are such that $U(x, y, z) = C_1, V(x, y, z) = C_2$ are the solutions of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, $[C_1, C_2$ are constants$]$. 5+5

(b) (i) Find the solution of the one-dimensional diffusion equation:

$$K\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

satisfying the following boundary conditions:

(I) u is bounded as $t \to \infty$

(II) $u_x(0, t) = 0$ and $u_x(a, t) = 0$ for all t

(III) $u(x, 0) = x(a - x), 0 < x < a$

(ii) Find the characteristic strips of the equation

$$xp + yq - pq = 0.$$

6+4

(c) (i) A string is stretched between two fixed points at a distance l apart. Motion is started by displacing the string in the form $y = y_0 \sin\left(\frac{\pi x}{l}\right)$ from which it is released at time $t = 0$. Find the displacement at any point at a distance x from one end at time t.

(ii) Solve by the method of separation of variables: 5+5

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$

(d) (i) Reduce the equation $r - 2s + t + p - q = 0$ to canonical form and hence solve it.

(ii) Find the surface which intersects the surface of the system $z(x + y) = K(3z + 1)$ orthogonally and which passes through the circle $x^2 + y^2 = 1; z = 1$. (4+2)+4
