3 Yr. Degree/4 Yr. Honours 1st Semester Examination, 2024 (CCFUP)

Subject : Mathematics

Course : MATH1011 (MAJOR)

(Calculus, Geometry and Vector Calculus)

Time: 3 Hours **Full Marks: 60**

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notation and symbols have their usual meaning.

1. Answer *any ten* questions: 2×10=20

(a) If $y = \cos h(Sin^{-1}x)$, prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} = (n^2+1)y_n$.

(b) Evaluate $\lim\limits_{x\to 0+} \left(1+\frac{1}{x}\right)^x$ using L'Hospital's rule.

(c) If $y = \frac{1}{ax+b}$, then prove that $y_n = \frac{(-1)^n . n!}{(ax+b)^{n+1}} . a^n$.

(d) Find the perimeter of the cardioid $r = a(1 - \text{Cos }\theta)$.

(e) Find the perimeter of the circle $r = a$.

(f) Find the parametric equation of $x^2 = 4ay$.

(g) If $f''(x)$ exists in the neighbourhood of a point 'a', then prove that $\lim\limits_{h\to 0} \frac{f(a+h)+f(a-h)-2f(a)}{h^2} = f''(a)$.

(h) If the axes are rotated through an angle 45° without changing the origin, then find the form of the equation $x^2 - y^2 = a^2$ in the new coordinate system.

(i) Find the equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = 5$ at (2,0,1).

(j) Find the equation of the sphere through the four points (0,0,0), (a,0,0), (0,b,0) and (0,0,c).

(k) Find the nature of the conic $3x^2 + 2xy + 3y^2 - 16x + 20 = 0$.

(l) Determine the constant 'a' so that the vector $\bar{V} = (x+3y)i + (y-2z)j + (x+az)k$ is solenoidal.

(m) Show that a necessary and sufficient condition for vector $\hat{u}$ has a constant length is that $\hat{u}.\frac{d\hat{u}}{dt} = 0$.

(n) If $A = x^2yi - 2xzj + 2xyk$, find $\nabla \times (\nabla \times A)$.

(o) Show that $\bar{\nabla}Q$ is a vector perpendicular to the surface $Q(x,y,z) = c$.

2. Answer *any four* questions: 5×4=20

(a) Find the values of a and b such that $\lim_{x\to 0} \frac{x(1-a\cos x)+b\sin x}{x^3} = \frac{1}{3}$.

(b) (i) Find $\int_0^{\frac{\pi}{2}} \cos^n x\, dx$ by using reduction formula.

(ii) If $I_{m,n} = \int \cos^m x \ \sin nx\, dx$; m, n being positive integers, then prove that

$$I_{m,n} = -\frac{Cos^m x\ Cos\ nx}{m+n} + \frac{m}{m+n} I_{m-1,\,n-1}.$$

2+3

(c) If by a rotation of rectangular coordinate axes about the origin $(ax^2 + 2hxy + by^2)$ reduces into $(a'x'^2 + 2h'x'y' + b'y'^2)$, then prove that $a + b = a' + b'$ and $ab - h^2 = a'b' - h'^2$.

(d) Find the equation to the generating lines of the hyperboloid $yz + 2zx + 3xy + 6 = 0$ which passes through the point (–1,0,3).

(e) If $\bar{v} = \bar{\omega} \times \bar{r}$, then prove that $\bar{\omega} = \frac{1}{2}\,\text{curl}\ \bar{v}$, where $\bar{\omega}$ is a constant vector and

$$\bar{r} = xi + yj + zk$$

(f) Show that $Q(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ is a solution of $\overline{\nabla}^2 Q = 0$.

3. Answer *any two* questions: 10×2=20

(a) (i) Find the asymptotes of $x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0$.

(ii) If $I_n = \int_0^{\pi/2} x^n \sin x\, dx,\ \ n > 1$, show that $I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$. 5+5

(b) (i) Find the total length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$.

(ii) If $y = e^m \tan^{-1} x$, prove that $(1 - x^2)y_{n+2} - (2n+1)y_{n+1} - (n^2 + m^2)y_n = 0$. 5+5

(c) (i) If two conics $\frac{l_1}{r} = 1 - e_1 \text{Cos}\ \theta$ and $\frac{l_2}{r} = 1 - e_2\ \text{Cos}\ (\theta - \alpha)$ touch one another, then show that $l_1^2(1 - e_2^2) + l_2^2(1 - e_1^2) = 2l_1l_2(1 - e_1e_2 \cos\alpha)$.

(ii) Find the points on $\frac{8}{r} = 3 - \sqrt{2}\cos\theta$ whose radius vector is 4.

(iii) Find the equation of the sphere for which the circle $x^2 + y^2 + z^2 + 2x - 4y + 2z + 5 = 0$, $x - 2y + 3z + 1 = 0$ is a great circle. 4+2+4

(d) (i) Find the constants a, b, c so that $\bar{V} = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$ is irrotational.

For those values of a, b, c, show that $\bar{V}$ can be expressed as the gradient of a scalar function θ.

(ii) Prove that grad $\left(\frac{f}{g}\right) = \frac{g\ grad(f) - f\ grad(g)}{g^2}$. (3+4)+3

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