

B.A./B.Sc 4th Semester (Honours) Examination, 2019 (CBCS)**Subject : Mathematics****Paper : BMH4 CC09****(Multivariate Calculus)****Time: 3 Hours****Full Marks: 60***The figures in the margin indicate full marks.**Candidates are required to give their answers in their own words
as far as practicable.*

[Notations and Symbols have their usual meaning.]

Group-A

Marks : 20

1. Answer any ten questions:

2×10=20

- (a) Let $f(x, y) = \frac{xy}{x^2+y^2}$, for $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$. Find $f_x(0, 0)$ and $f_y(0, 0)$ and examine for continuity of f at $(0, 0)$.
- (b) Prove that if a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable at a point $\xi = (x_1, x_2, \dots, x_n)$ then it is continuous there.
- (c) Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(x, y) = \sin x, \forall (x, y) \in \mathbb{R}^2$. Find $f'(1, 2)$, where $f'(1, 2)$ is the derivative of f at $(1, 2)$.
- (d) Find the directional derivative of $f(x, y, z) = xy^2 - 5xz + 7y$ at $(1, 0, -1)$ in the direction $\vec{u} = \frac{1}{\sqrt{3}}(1, 1, -1)$.
- (e) Find the equation of the tangent plane and normal line of the surface $z = x^2 + y^2$ at the point $(3, 4, 25)$.
- (f) Evaluate $\iint_R (x^2 + y^2) dx dy$, where R is the region bounded by the straight lines $x = 0, y = 1$ and the parabola $y = x^2$.
- (g) Evaluate : $\int_0^{\frac{\pi}{2}} dx \int_0^{\cos x} x^2 dy$
- (h) Evaluate : $\int_0^1 dx \int_0^y \sqrt{x^2 + y^2} dy$ by transforming to polar coordinates
- (i) Evaluate : $I = \iiint_V (xy^2 z^3)^2 dx dy dz$ over the rectangular parallelopiped V bounded by the planes $x = \pm a, y = \pm b, z = \pm c$.
- (j) Show that if $\phi(x, y, z)$ is any solution of Laplace's equation then $\vec{\nabla}\phi$ is a vector which is both solenoidal and irrotational.

- (k) If $\vec{F}(x, y, z) = xy\hat{i} - z\hat{j} + x^2\hat{k}$ and C is the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$, evaluate the line integral $\int_C \vec{F} \times d\vec{r}$.
- (l) Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$.
- (m) Evaluate : $\iint_S \vec{A} \cdot \hat{n} ds$ where $\vec{A} = y\hat{i} + 2x\hat{j} - z\hat{k}$ and S is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$.
- (n) Use Green's Theorem to find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
- (o) If the vectors \vec{A} and \vec{B} be irrotational then show that the vector $\vec{A} \times \vec{B}$ is solenoidal.

Group-B

Marks : 20

2. Answer *any four* questions from the following:

5x4=20

- (a) Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(0, 0) = 0$ and $f(x, y) = \frac{x^2y}{x^4+y^2}$ if $(x, y) \neq (0, 0)$. Show that all directional derivatives of f exist at $(0, 0)$, but f is not differentiable at $(0, 0)$.
- (b) (i) if $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$ then prove that

$$\frac{\partial x}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial x}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} + \frac{\partial x}{\partial \phi} \cdot \frac{\partial \phi}{\partial x} = 1.$$
- (ii) Let z be a differentiable function of x and y and let $x = r\cos\theta$, $y = r\sin\theta$. Prove that

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$
 3+2=5
- (c) (i) Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(x, y) = \frac{xy}{(x^2+y^2)^2}$, for $(x, y) \neq (0, 0)$
 $= 0$, for $(x, y) = (0, 0)$
Show that the repeated integrals of f over $E = \{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$ exist and are equal but $\iint_E f(x, y) dx dy$ does not exist.
- (ii) Evaluate : $\int_0^\pi \int_x^\pi \frac{\sin y}{y} dy dx.$ 3+2=5
- (d) (i) Let $u = f(x, y) = \sin^{-1} \left(\frac{x+y}{\sqrt{x+y}} \right)$, prove that $x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = -\frac{\sin u \cos 2u}{4 \cos^3 u}$.
- (ii) Evaluate : $\iint_Q [x+y] dx dy$, where $Q = [0, 2] \times [0, 2]$ and $[x]$ is the greatest integer $\leq x$. 3+2=5
- (e) Prove that $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force field. Find the scalar potential V such that $\vec{F} = \vec{\nabla}V$. Also find the work done in moving an object in this field from $(1, -2, 1)$ to $(3, 1, 4)$. 1+2+2=5

- (f) (i) Verify Green's theorem in the plane for $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$, where C is the boundary of the region defined by $y = \sqrt{x}$, $y = x^2$.

- (ii) If S is any closed surface enclosing a volume V and $\vec{A} = ax\hat{i} + by\hat{j} + cz\hat{k}$, prove that

$$\iint_S \vec{A} \cdot \hat{n} ds = (a + b + c)V.$$

3+2=5

Group-C

Marks : 20

3. Answer *any two* questions from the following:

10×2=20

- (a) (i) Let $S \subset \mathbb{R}^n$ be an open ball and let $f: S \rightarrow \mathbb{R}$ has partial derivatives f_{x_1}, \dots, f_{x_n} at each point of S . If f_{x_1}, \dots, f_{x_n} are continuous at a point $\xi \in S$, then prove that f is differentiable at ξ .

- (ii) If $ux = vy = wz = x^2 + y^2 + z^2$ then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(x^2+y^2+z^2)^3}{x^2y^2z^2}$, where left hand side of the above relation represents the Jacobian of the functions u, v, w w.r.t. x, y, z .

- (iii) Examine for extreme values of $f(x, y) = x^3y^3(12 - 3x - 4y)$ 5+3+2=10

- (b) (i) Find the volume of the solid in the first octant bounded by the cylinder $x = 4 - y^2$ and the planes $z = y$, $x = 0$, $z = 0$.

$$(ii) \text{ Show that } \int_0^1 dx \int_0^{1-x} e^{\frac{y}{x+y}} dy = \frac{e-1}{2}.$$

- (iii) State Schwarz's theorem and show that the function $f(x, y) = \frac{xy(x^2-y^2)}{x^2+y^2}$, for $(x, y) \neq (0, 0)$, $f(0, 0) = 0$ does not satisfy its hypothesis. 3+4+3=10

- (c) (i) Find the circulation of \vec{F} round the curve C where

$$\vec{F} = (2x - y + 4z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z^3)\hat{k} \text{ and } C \text{ is the circle}$$

$$x^2 + y^2 = 9, z = 0$$

- (ii) Verify the divergence theorem for $\vec{A} = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$ taken over the region in the first octant bounded by $y^2 + z^2 = 9$ and $x = 2$.

- (iii) Verify Stoke's theorem for $\vec{F} = xz\hat{i} - y\hat{j} + x^2y\hat{k}$ taken over the surface S of the region bounded by

$$x = 0, y = 0, z = 0, 2x + y + 2z = 8$$

which is not included in the xz -plane.

3+3+4=10

- (d) (i) Find the maximum value of $(xyz)^2$ subject to the condition $x^2 + y^2 + z^2 = 1$ using the method of Lagrange multipliers.

(ii) Evaluate : $\iiint_E \sqrt{a^2b^2c^2 - b^2c^2x^2 - c^2a^2y^2 - a^2b^2z^2} dx dy dz$ where E is the region bounded by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. 5+5=10

(iii) Use Green's Theorem to find the

- (iv) If the vectors \vec{A} and \vec{B} be irrotational then show that the vector $\vec{A} \times \vec{B}$ is irrotational. 5+5=10

Ques-3: Let $\vec{A} = (x^2 + y^2 + z^2)^{-\frac{1}{2}}(x, y, z)$ and $\vec{B} = \frac{1}{2}\nabla(x^2 + y^2 + z^2)^{-\frac{1}{2}}$ be two vector fields such that "if $\vec{A} \times \vec{B}$ is irrotational then every unit $\vec{r} \times \vec{B}$ along \vec{r} is irrotational" then prove that \vec{A} is irrotational. 10

Ques-4: Answer any four questions from the following

Ques-4: If $\vec{F}(x, y, z) = \frac{(x^2 + y^2 + z^2)^{\frac{1}{2}}}{x^2 + y^2} \hat{i} + \frac{(x^2 + y^2 + z^2)^{\frac{1}{2}}}{y^2 + z^2} \hat{j} + \frac{(x^2 + y^2 + z^2)^{\frac{1}{2}}}{x^2 + z^2} \hat{k}$ then show that \vec{F} is irrotational. 10

Ques-5: Let $f(x, y, z)$ be a function defined in \mathbb{R}^3 such that all partial derivatives of f exist at $(0, 0, 0)$ and $\lim_{(x,y,z) \rightarrow (0,0,0)} f(x, y, z) = 0$. Show that the first order partial derivatives of f exist at $(0, 0, 0)$ and $\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{\partial f}{\partial x}(x, y, z) = 0$, $\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{\partial f}{\partial y}(x, y, z) = 0$, $\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{\partial f}{\partial z}(x, y, z) = 0$. 10

Ques-6: (i) If $r = (\sqrt{x^2 + y^2 + z^2})^{\frac{1}{2}}$ then find $\frac{\partial r}{\partial x}$ and $\frac{\partial r}{\partial y}$. 5

Ques-6: (ii) Let $\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$ be a vector defined in \mathbb{R}^3 then find $\vec{r} \cdot \vec{r}$ and $\vec{r} \times \vec{r}$. 5

$$\left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 + \left(\frac{\partial r}{\partial z} \right)^2 = \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}} \right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2 + z^2}} \right)^2 + \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)^2 = 1 \quad \text{and we get (ii)}$$

Ques-7: (i) Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $f(x, y) = \sin^{-1}(\frac{y-x}{x+y})$ for all $(x, y) \in (0, 0)$

$$\text{and } \frac{\partial f}{\partial x} = \frac{y-x}{(x+y)^2} \hat{i} + \frac{y-x}{(x+y)^2} \hat{j} \text{ and show that } f \text{ is continuous at } (0, 0).$$

Ques-7: (ii) Show that the repeated integrals of $f(x, y) = \frac{1}{x^2 + y^2}$ over the region $R = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq x\}$ and are equal but $\int_R f(x, y) dxdy$ does not exist in the sense of Riemann. 10

$$\text{Solved and find that } \int_R f(x, y) dxdy = \int_0^1 \int_0^x \frac{1}{x^2 + y^2} dy dx + \int_0^1 \int_x^1 \frac{1}{x^2 + y^2} dy dx = \frac{1}{2}$$

Ques-8: (i) Let $u = f(x, y) = \sin^{-1}\left(\frac{y-x}{x+y}\right)$ prove that $\frac{\partial u}{\partial x} = \frac{y-x}{(x+y)^2} \hat{i} + \frac{y-x}{(x+y)^2} \hat{j}$ and $\frac{\partial u}{\partial y} = \frac{y-x}{(x+y)^2} \hat{i} + \frac{y-x}{(x+y)^2} \hat{j}$ and hence $\vec{F} = \vec{u}$ is irrotational. 10

Ques-8: (ii) Evaluate $\iint_D (x+1) \sinh(x^2 + y^2) dy dx$ where D is the region bounded above and below by $y = x$ and $y = -x$ and left and right by $x = 2$ and $x = -2$. 10

Ques-8: (iii) Find the volume of the solid bounded above and below by $y = x^2 + 1$ and $y = 4$ and left and right by $x = 2$ and $x = -2$. 10

Ques-9: (i) Prove that $\vec{F} = (2xy + x^2y^2 + x^2) \hat{i} + 3x^2y \hat{j}$ is a conservative field. Find the scalar potential V such that $\vec{F} = \nabla V$ and $\int_S \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$ where C_1 is the line $x = 0$ from $(0, 0)$ to $(0, 1)$ and C_2 is the line $y = 0$ from $(0, 0)$ to $(1, 0)$. 10