Chapter 1

Bhargav Annem

May 2023

1 Derivations

Let's derive Newton's Second Law for rotational kinematics.

$$\begin{aligned} F &= ma \\ F &= F_r\hat{r} + F_\phi\hat{\phi} \end{aligned} \tag{1}$$

First, the conversions for rectangular to angular coordinates are as follows:

$$\begin{aligned} x &= r\cos(\phi) \\ y &= r\sin(\phi) \\ \phi &= \tan(\frac{y}{x}) \\ r &= \sqrt{x^2 + y^2} \end{aligned} \tag{2}$$

Also,

$$\begin{aligned} \Delta r &= \Delta\phi\hat{\phi} \\ \Delta r &= \dot{\phi}\Delta t\hat{\phi} \\ \frac{\Delta r}{\Delta t} &= \dot{\phi}\hat{\phi} \\ \frac{d\hat{r}}{dt} &= \dot{\phi}\hat{\phi} \end{aligned} \tag{3}$$

To prove this last statement more rigorously, let's decompose $\vec{r}$ into Cartesian components.

$$\vec{r} = r\cos(\phi)\hat{x} + r\sin(\phi)\hat{y}$$

Then,

$$\frac{d\vec{r}}{dr} = \cos(\phi)\hat{x} + \sin(\phi)\hat{y}$$

since

$$\frac{d\vec{r}}{dr} = \cos(\phi)\hat{x} + \sin(\phi)\hat{y}$$

and

$$|\frac{d\vec{r}}{dr}| = \sqrt{\cos(\phi)^2 + \sin(\phi)^2} = 1$$

$$\hat{r} = \cos(\phi)\hat{x} + \sin(\phi)\hat{y}$$

Similarly, solving for $\frac{d\vec{r}}{d\phi}$ gives us the following:

$$\frac{d\vec{r}}{d\phi} = r(-\sin(\phi)\hat{x} + \cos(\phi)\hat{y})$$

and

$$|\frac{d\vec{r}}{d\phi}| = r\sqrt{(-\sin(\phi)^2) + \cos(\phi)^2} = 1$$

so

$$\hat{\phi} = -\sin(\phi)\hat{x} + \cos(\phi)\hat{y}$$

Initially, we established that

$$\vec{r} = r\hat{r}$$

Then,

$$\begin{aligned}
\vec{v} = \frac{d\vec{r}}{dt} &= \frac{d}{dt}(r\hat{r}) \\
&= r\frac{d\hat{r}}{dt} + \hat{r}\dot{r} + \dot{r}\hat{r} \\
&= r(\frac{-d\phi}{dt}\sin(\phi) + \frac{d\phi}{dt}cos(\phi)) + \dot{r}\hat{r} \\
&= r(\frac{d\phi}{dt}(-\sin(\phi) + cos(\phi))) + \dot{r}\hat{r} \\
&= r(\dot{\phi}(-\sin(\phi) + cos(\phi))) + \dot{r}\hat{r} \\
&= r\dot{\phi}\hat{\phi} + \dot{r}\hat{r}
\end{aligned} \tag{4}$$

Then,

$$\begin{aligned}
\frac{d^2\vec{r}}{dt^2} &= r(\frac{d\dot{\phi}\hat{\phi}}{dt}) + \dot{\phi}\hat{\phi}\dot{r} + \dot{r}\frac{d\hat{r}}{dt} + \hat{r}\ddot{r} \\
\frac{d^2\vec{r}}{dt^2} &= r(\ddot{\phi}\hat{\phi} + \dot{\phi}\frac{d\hat{\phi}}{dt}) + \dot{\phi}\hat{\phi}\dot{r} + \dot{r}\frac{d\hat{r}}{dt} + \hat{r}\ddot{r}
\end{aligned} \tag{5}$$

And since,

$$\begin{aligned}
\frac{d\hat{\phi}}{dt} &= -\dot{\phi}\cos(\phi)\hat{x} - \dot{\phi}\sin(\phi)\hat{y} \\
\frac{d\hat{\phi}}{dt} &= -\dot{\phi}(\cos(\phi)\hat{x} + \sin(\phi)\hat{y}) \\
\frac{d\hat{\phi}}{dt} &= -\dot{\phi}\hat{r} \\
\frac{d\hat{r}}{dt} &= -\dot{\phi}(\sin(\phi)\hat{x} - \dot{\phi}\cos(\phi)\hat{y}) \\
\frac{d\hat{r}}{dt} &= \dot{\phi}\hat{\phi}
\end{aligned} \tag{6}$$

We end with,

$$\begin{aligned}\frac{d^2\vec{r}}{dt^2} &= r(\ddot{\phi}\hat{\phi} - \dot{\phi}\dot{\phi}\hat{r}) + \dot{\phi}\hat{\phi}\dot{r} + \dot{r}\dot{\phi}\hat{\phi} + \ddot{r}\hat{r} \\ \vec{a} = \frac{d^2\vec{r}}{dt^2} &= (\ddot{r} - r\dot{\phi}^2\hat{r})\hat{r} + (2r\dot{\phi} + \ddot{\phi})\hat{\phi}\end{aligned} \tag{7}$$

From here, we conclude with:

$$\boxed{\vec{F} = m\vec{a} = m(\ddot{r} - r\dot{\phi}^2\hat{r})\hat{r} + m(2r\dot{\phi} + \ddot{\phi})\hat{\phi}} \tag{8}$$