Chapter 1

Bhargav Annem

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1 Derivations

Let's derive Newton's Second Law for rotational kinematics.

$$\begin{aligned} F &= ma \\ F &= F_r\hat{r} + F_\phi\hat{\phi} \end{aligned} \tag{1}$$

First, the conversions for rectangular to angular coordinates are as follows:

$$\begin{aligned} x &= r\cos(\phi) \\ y &= r\sin(\phi) \\ \phi &= \tan(\frac{y}{x}) \\ r &= \sqrt{x^2 + y^2} \end{aligned} \tag{2}$$

Also,

$$\begin{aligned} \Delta r &= \Delta\phi\hat{\phi} \\ \Delta r &= \dot{\phi}\Delta t\hat{\phi} \\ \frac{\Delta r}{\Delta t} &= \dot{\phi}\hat{\phi} \\ \frac{d\hat{r}}{dt} &= \dot{\phi}\hat{\phi} \end{aligned} \tag{3}$$

To prove this last statement more rigorously, let's decompose $\vec{r}$ into Cartesian components.

$$\vec{r} = r\cos(\phi)\hat{x} + r\sin(\phi)\hat{y}$$

Then,

$$\frac{d\vec{r}}{dr} = \cos(\phi)\hat{x} + \sin(\phi)\hat{y}$$

since

$$\frac{d\vec{r}}{dr} = \cos(\phi)\hat{x} + \sin(\phi)\hat{y}$$

and

$$|\frac{d\vec{r}}{dr}| = \sqrt{\cos(\phi)^2 + \sin(\phi)^2} = 1$$

$$\hat{r} = \cos(\phi)\hat{x} + \sin(\phi)\hat{y}$$

Similarly, solving for $\frac{d\vec{r}}{d\phi}$ gives us the following:

$$\frac{d\vec{r}}{d\phi} = r(-\sin(\phi)\hat{x} + \cos(\phi)\hat{y})$$

and

$$|\frac{d\vec{r}}{d\phi}| = r\sqrt{(-\sin(\phi)^2) + \cos(\phi)^2} = 1$$

so

$$\hat{\phi} = -\sin(\phi)\hat{x} + \cos(\phi)\hat{y}$$

Initially, we established that

$$\vec{r} = r\hat{r}$$

Then,

$$\begin{aligned}
\vec{v} = \frac{\vec{dr}}{dt} &= \frac{d}{dt}(r\hat{r}) \\
&= r\frac{d\hat{r}}{dt} + \hat{r}\dot{r} + \dot{r}\hat{r} \\
&= r(\frac{-d\phi}{dt}\sin(\phi) + \frac{d\phi}{dt}cos(\phi)) + \dot{r}\hat{r} \\
&= r(\frac{d\phi}{dt}(-\sin(\phi) + cos(\phi))) + \dot{r}\hat{r} \\
&= r(\dot{\phi}(-\sin(\phi) + cos(\phi))) + \dot{r}\hat{r} \\
&= r\dot{\phi}\hat{\phi} + \dot{r}\hat{r}
\end{aligned} \tag{4}$$

Then,

$$\begin{aligned}
\frac{d^2\vec{r}}{dt^2} &= r(\frac{d\dot{\phi}\hat{\phi}}{dt}) + \dot{\phi}\hat{\phi}\dot{r} + \dot{r}\frac{d\hat{r}}{dt} + \hat{r}\ddot{r} \\
\frac{d^2\vec{r}}{dt^2} &= r(\ddot{\phi}\hat{\phi} + \dot{\phi}\frac{d\hat{\phi}}{dt}) + \dot{\phi}\hat{\phi}\dot{r} + \dot{r}\frac{d\hat{r}}{dt} + \hat{r}\ddot{r}
\end{aligned} \tag{5}$$

And since,

$$
\begin{aligned}
\frac{d\hat{\phi}}{dt} &= -\dot{\phi}\cos(\phi)\hat{x} - \dot{\phi}\sin(\phi)\hat{y} \\
\frac{d\hat{\phi}}{dt} &= -\dot{\phi}(\cos(\phi)\hat{x} + \sin(\phi)\hat{y}) \\
\frac{d\hat{\phi}}{dt} &= -\dot{\phi}\hat{r} \\
\frac{d\hat{r}}{dt} &= -\dot{\phi}(\sin(\phi)\hat{x} - \dot{\phi}\cos(\phi)\hat{y}) \\
\frac{d\hat{r}}{dt} &= \dot{\phi}\hat{\phi}
\end{aligned} \tag{6}
$$

We end with,

$$
\begin{aligned}
\frac{d^2\vec{r}}{dt^2} &= r(\ddot{\phi}\hat{\phi} - \dot{\phi}\dot{\phi}\hat{r}) + \dot{\phi}\hat{\phi}\dot{r} + \dot{r}\dot{\phi}\hat{\phi} + \ddot{r}\hat{r} \\
\vec{a} = \frac{d^2\vec{r}}{dt^2} &= (\ddot{r} - r\dot{\phi}^2\hat{r})\hat{r} + (2r\dot{\phi} + \ddot{\phi})\hat{\phi}
\end{aligned} \tag{7}
$$

From here, we conclude with:

$$
\boxed{\vec{F} = m\vec{a} = m(\ddot{r} - r\dot{\phi}^2\hat{r})\hat{r} + m(2r\dot{\phi} + \ddot{\phi})\hat{\phi}} \tag{8}
$$

2 Problem Solutions

2.0.1

1.1 Given the two vectors $b = \hat{x} + \hat{y}$ and $c = \hat{x} + \hat{z}$ find $b + c$, $5b + 2c$, $b \cdot c$, and $b \times c$

Solution:

$$
\begin{aligned}
b + c &= 2\hat{x} + \hat{y} + \hat{z} \\
5b + 2c &= 7\hat{x} + 5\hat{y} + 2\hat{z} \\
b \cdot c &= 1 \\
b \times c &= \begin{cases} 1, 1, 0 \\ 1, 0, 1 \end{cases} \\
&=< 1, 1, -1 > \\
&= \hat{x} - \hat{y} - \hat{z}
\end{aligned} \tag{9}
$$

2.0.2

1.3 By applying Pythagoras's theorem (the usual two-dimensional version) twice over, prove that the length r of a three-dimensional vector $r = (x, y, z)$ statisfies $r^2 = x^2 + y^2 + z^2$

Solution:

$$\begin{aligned} h^2 &= x^2 + y^2 \\ r^2 &= h^2 + z^2 \\ r^2 &= x^2 + y^2 + z^2 \end{aligned} \tag{10}$$

2.0.3

Find the angle between a body diagonal of a cube and any of its face diagonals. [*Hint* : Choose a cube with side 1 adn with one corner at O and the opposite corner at the point (1, 1, 1). Write down the vector that represents a body diagonal and another that represents a face diagonal, and then find the angle between them as in Problem 1.4].

Solution:

$$\begin{aligned} f_{body} &=< 1, 1, 1 > \\ f_{face} &=< 1, 1, 0 > \\ f_{body} \cdot f_{face} &= 2 = |f_{body}||f_{face}| \cos(\phi) \\ \sqrt{3}\sqrt{2} \cos(\phi) &= 2 \\ \phi &= \arccos(\frac{2}{\sqrt{3}\sqrt{2}}) \end{aligned} \tag{11}$$

$$\boxed{35.26 \, \text{deg}}$$

2.0.4

Prove that the two definitions of the scalar product $r \cdot s$ as $rs \cos(\phi)$ and $\sum r_i s_i$ are equal. One way to do this is to choose your x-axis along the direction of $\boldsymbol{r}$

Solution:
If r lays along x, then $\cos(\phi) = 1$ since $\phi = 0$. Since $\sum r_i s_i = \sum r_i \cdot \sum s_i = rs$, the two statements are equivalent. The longer way is to use the law of cosines to expand $\cos(\phi)$ in terms of r and s, which will give you an expression that evaluates to a summation.

2.0.5

In elementary trigonometry, you probably learned the law of cosines for a triangle of sides a, b, c that $c^2 = a^2+b^2-2ab\cos(\phi)$ where ϕ is the angle between the sides a and b. Show that the law of cosines is an immediate consequence of the identity $(a+b)^2 = a^2+b^2+2a\cdot b$.

Solution:

$$2a\cdot b = 2ab\cos(\phi) \tag{12}$$

Since ϕ represents the angle between a and b (which is the external angle of the triangle $\pi-\phi$), then $\cos(\phi) \rightarrow -\cos(\phi)$. Let $c = a+b$, then we get $c^2 = a^2+b^2-2ab\cos(\phi)$.

2.0.6

The position of a moving particle is given as a function fo time t to be

$$r(t) = \hat{x}b\cos(\omega t) + \hat{y}c\sin(\omega t) + \hat{z}v_o t$$

where $b, c,$ and ω are constants. Describe the particle's orbit.

Solution:
The $\hat{z}v_o t$ part will cause the particle to move upwards continuously, but the two trigonometric functions of different amplitudes will create an elliptical orbit.

2.0.7

Let u be an arbitrary fixed unit vector and show that any vector b satisfies

$$b^2 = (u\cdot b)^2 + (u\times b)^2$$

Solution:

$$\begin{aligned} b^2 &= (ub\cos(\phi))^2 + (ub\sin(\phi))^2 \\ b^2 &= (ub)^2(\cos(\phi)^2 + \sin(\phi))^2 \\ b^2 &= u^2b^2 \end{aligned} \tag{13}$$

Since $|u| = 1$ because it is a unit vector, we get $b^2 = b^2$.

2.0.8

Show that the definition of the cross product is equivalent to the elementary definition that $r \times s$ is perpendicular to both r and s with magnitude $rs\sin(\phi)$ and direction given by the right-hand rule.

Solution:
Let $\vec{v} = (v_1, v_2, v_3)$ and $\vec{s} = (s_1, s_2, s_3)$, then:

$$\begin{aligned} |\vec{v_1}| &= \sqrt{v_1^2 + v_2^2 + v_3^2} \\ |\vec{s_1}| &= \sqrt{s_1^2 + s_2^2 + s_3^2} \end{aligned} \tag{14}$$

$$\begin{aligned} |(v_1, v_2, v_3) \times (s_1, s_2, s_3)| &= |(v_2s_3 - v_3s_2, v_1s_3 - v_3s_1, v_1s_2 - v_2s_1)| \\ &= \sqrt{(v_2s_3 - v_3s_2)^2 + (v_1s_3 - v_3s_1)^2 + (v_1s_2 - v_2s_1)^2} \\ &= \sqrt{\begin{array}{r} (v_2s_3)^2 + 2(v_2v_3s_2s_3) + (v_3s_2)^2 \\ +(v_1s_3)^2 + 2(v_1v_3s_1s_3) + (v_3s_1)^2 \\ +(v_1s_2)^2 + 2(v_1v_2s_1s_2) + (v_1s_2)^2 \end{array}} \\ &= \sqrt{\begin{array}{r} (v_1^2 + v_2^2 + v_3^2)(s_1^2 + s_2^2 + s_3^2) \\ -((v_1s_1)^2 + (v_2s_2)^2 + (v_3s_3)^2 + \\ 2(v_2v_3s_2s_3 + v_1v_3s_1s_3 + v_1v_2s_1s_2)) \end{array}} \end{aligned} \tag{15}$$

And since $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc)$, we get:

$$\begin{aligned} &\sqrt{(v_1^2 + v_2^2 + v_3^2)(s_1^2 + s_2^2 + s_3^2) - (v_1s_1 + v_2s_2 + v_3s_3)^2} \\ =&\sqrt{||\vec{v}||||\vec{s}|| - (\vec{v} \cdot \vec{s})^2} \end{aligned} \tag{16}$$

Since $\vec{v} \cdot \vec{s} = ||\vec{v}||||\vec{s}|| \cos(\phi)$, we get:

$$\begin{aligned} &=\sqrt{||\vec{v}||^2||\vec{s}||^2 - (\vec{v} \cdot \vec{s})^2} \\ &=\sqrt{||\vec{v}||^2||\vec{s}||^2(1 - \cos^2(\phi))} \\ &=\sqrt{||\vec{v}||^2||\vec{s}||^2(\sin^2(\phi))} \\ &=\boxed{||\vec{v}||||\vec{s}||(\sin(\phi))} \end{aligned} \tag{17}$$

2.0.9

(**a**) Defining the scalar product $\boldsymbol{r} \cdot \boldsymbol{s}$ by Equation (1.7), $\boldsymbol{r} \cdot \boldsymbol{s} = \sum r_i s_i$, show that Pythagoras's theorem implies that the magnitude of any vector $\mathbf{r}$ is $r = \sqrt{\boldsymbol{r} \cdot \boldsymbol{r}}$. (**b**) It is clear that the length of a vector does not depend on our choicee of coordinate axes. Thus the result of part (a) guarantees that the scalar product $\boldsymbol{r} \cdot \boldsymbol{r}$, as defined by (1.7), is the same for any choice of orthogonal axes. Use this to prove that $\boldsymbol{r} \cdot \boldsymbol{s}$, as defined by (1.7), is the same for any choice of orthogonal axes. [*Hint:* Consider the length of the vector $\boldsymbol{r} + \boldsymbol{s}$]

Solution:
(**a**)

$$r = \sqrt{\sum r_i^2} \rightarrow r^2 = \sum r_i^2$$

(**b**) The length of $\vec{r} + \vec{s}$ is $\sqrt{\sum (r_i + s_i)^2}$

$$\begin{aligned} |\vec{r} + \vec{s}|^2 &= (r_1 + s_1)^2 + (r_2 + s_2)^2 + (r_3 + s_3)^2 \\ &= ||\vec{r}||^2 ||\vec{s}||^2 + 2(r_1 s_1 + r_2 s_2 + r_3 s_3) \\ &= ||\vec{r}||^2 ||\vec{s}||^2 + 2(\vec{r} \cdot \vec{s}) \end{aligned} \tag{18}$$

$$\boxed{\vec{r} \cdot \vec{s} = \frac{1}{2}(|\vec{r} + \vec{s}|^2 - ||\vec{r}||^2 ||\vec{s}||^2)}$$

Which depends only on the lengths of $\vec{r}$ and $\vec{s}$

2.0.10

(a) Prove that the vector product $r \times s$ as defined by (1.9) is distributive; that is, that $r \times (u + v) = (r \times u) + (r \times v)$. (b) Prove the product rule:

$$\frac{d}{dt}(r \times s) = r \times \frac{ds}{dt} + \frac{dr}{dt} \times s$$

Solution:
(a)

$$\begin{aligned} r \times (u + v) &= (r_1, r_2, r_3 > \times < u_1 + v_1, u_2 + v_2, u_3 + v_3) \\ &= (r_2(u_3 + v_3) - r_3(u_2 + v_2), r_1(u_3 + v_3) - r_3(u_1 + v_1), r_2(u_3 + v_3) - r_3(u_2 + v_2)) \\ &= (r_2 u_3 + r_2 v_3 - r_3 u_2 - r_3 v_2, r_1 u_3 + r_1 v_3 - r_3 u_1 - r_3 v_1, r_2 u_3 + r_2 v_3 - r_3 u_2 - r_3 v_2) \\ &= ((r_2 u_3 - r_3 u_2) + (r_2 v_3 - r_3 v_2), (r_1 u_3 - r_3 u_1) + (r_1 v_3 - r_3 v_1), (r_2 u_3 - r_3 u_2) + (r_3 u_2 - r_3 v_2)) \\ &\boxed{= r \times u + r \times v} \end{aligned} \tag{19}$$

(b)

$$\begin{aligned}
\frac{d}{dt}(r \times s) &= \frac{d}{dt}(r_2 s_3 - r_3 s_2, r_1 s_3 - r_3 s_1, r_1 s_2 - r_2 s_1) \\
&= \begin{pmatrix} (r_2 \frac{ds_3}{dt} + \frac{dr_2}{dt} s_3) - (r_3 \frac{ds_2}{dt} + \frac{dr_3}{dt} s_2) \\ (r_1 \frac{ds_3}{dt} + \frac{dr_1}{dt} s_3) - (r_3 \frac{ds_1}{dt} + \frac{dr_3}{dt} s_1) \\ (r_1 \frac{ds_2}{dt} + \frac{dr_1}{dt} s_2) - (r_2 \frac{ds_1}{dt} + \frac{dr_2}{dt} s_1) \end{pmatrix} \\
&= \begin{pmatrix} (r_2 \frac{ds_3}{dt} - \frac{ds_2}{dt} r_3) + (s_3 \frac{dr_2}{dt} - \frac{dr_3}{dt} s_2) \\ (r_1 \frac{ds_3}{dt} - \frac{ds_1}{dt} r_3) + (s_3 \frac{dr_1}{dt} - \frac{dr_3}{dt} s_1) \\ (r_1 \frac{ds_2}{dt} - \frac{ds_1}{dt} r_2) + (s_2 \frac{dr_1}{dt} - \frac{dr_2}{dt} s_1) \end{pmatrix} \\
&\boxed{= r \times \frac{ds}{dt} + s \times \frac{dr}{dt}}
\end{aligned} \tag{20}$$