Chapter 1

Bhargav Annem

May 2023

1 Derivations

Let's derive Newton's Second Law for rotational kinematics.

$$\begin{aligned} F &= ma \\ F &= F_r\hat{r} + F_\phi\hat{\phi} \end{aligned} \tag{1}$$

First, the conversions for rectangular to angular coordinates are as follows:

$$\begin{aligned} x &= r\cos(\phi) \\ y &= r\sin(\phi) \\ \phi &= \tan(\frac{y}{x}) \\ r &= \sqrt{x^2+y^2} \end{aligned} \tag{2}$$

Also,

$$\begin{aligned} \Delta r &= \Delta\phi\hat{\phi} \\ \Delta r &= \dot{\phi}\Delta t\hat{\phi} \\ \frac{\Delta r}{\Delta t} &= \dot{\phi}\hat{\phi} \\ \frac{d\hat{r}}{dt} &= \dot{\phi}\hat{\phi} \end{aligned} \tag{3}$$

To prove this last statement more rigorously, let's decompose $\vec{r}$ into Cartesian components.

$$\vec{r} = r\cos(\phi)\hat{x} + r\sin(\phi)\hat{y}$$

Then,

$$\frac{d\vec{r}}{dr} = \cos(\phi)\hat{x} + \sin(\phi)\hat{y}$$

since

$$\frac{d\vec{r}}{dr} = \cos(\phi)\hat{x} + \sin(\phi)\hat{y}$$

and

$$|\frac{d\vec{r}}{dr}| = \sqrt{\cos(\phi)^2 + \sin(\phi)^2} = 1$$

$$\hat{r} = \cos(\phi)\hat{x} + \sin(\phi)\hat{y}$$

Similarly, solving for $\frac{d\vec{r}}{d\phi}$ gives us the following:

$$\frac{d\vec{r}}{d\phi} = r(-\sin(\phi)\hat{x} + \cos(\phi)\hat{y})$$

and

$$|\frac{d\vec{r}}{d\phi}| = r\sqrt{(-\sin(\phi)^2) + \cos(\phi)^2} = 1$$

so

$$\hat{\phi} = -\sin(\phi)\hat{x} + \cos(\phi)\hat{y}$$

Initially, we established that

$$\vec{r} = r\hat{r}$$

Then,

$$\begin{aligned}
\vec{v} = \frac{\vec{dr}}{dt} &= \frac{d}{dt}(r\hat{r}) \\
&= r\frac{d\hat{r}}{dt} + \hat{r}\dot{r} + \dot{r}\hat{r} \\
&= r(\frac{-d\phi}{dt}\sin(\phi) + \frac{d\phi}{dt}cos(\phi)) + \dot{r}\hat{r} \\
&= r(\frac{d\phi}{dt}(-\sin(\phi) + cos(\phi))) + \dot{r}\hat{r} \\
&= r(\dot{\phi}(-\sin(\phi) + cos(\phi))) + \dot{r}\hat{r} \\
&= r\dot{\phi}\hat{\phi} + \dot{r}\hat{r}
\end{aligned} \tag{4}$$

Then,

$$\begin{aligned}
\frac{d^2\vec{r}}{dt^2} &= r(\frac{d\dot{\phi}\hat{\phi}}{dt}) + \dot{\phi}\hat{\phi}\dot{r} + \dot{r}\frac{d\hat{r}}{dt} + \hat{r}\ddot{r} \\
\frac{d^2\vec{r}}{dt^2} &= r(\ddot{\phi}\hat{\phi} + \dot{\phi}\frac{d\hat{\phi}}{dt}) + \dot{\phi}\hat{\phi}\dot{r} + \dot{r}\frac{d\hat{r}}{dt} + \hat{r}\ddot{r}
\end{aligned} \tag{5}$$

And since,

$$\begin{aligned}
\frac{d\hat{\phi}}{dt} &= -\dot{\phi}\cos(\phi)\hat{x} - \dot{\phi}\sin(\phi)\hat{y} \\
\frac{d\hat{\phi}}{dt} &= -\dot{\phi}(\cos(\phi)\hat{x} + \sin(\phi)\hat{y}) \\
\frac{d\hat{\phi}}{dt} &= -\dot{\phi}\hat{r} \\
\frac{d\hat{r}}{dt} &= -\dot{\phi}(\sin(\phi)\hat{x} - \dot{\phi}\cos(\phi)\hat{y}) \\
\frac{d\hat{r}}{dt} &= \dot{\phi}\hat{\phi}
\end{aligned} \tag{6}$$

We end with,

$$\begin{aligned}
\frac{d^2\vec{r}}{dt^2} &= r(\ddot{\phi}\hat{\phi} - \dot{\phi}\dot{\phi}\hat{r}) + \dot{\phi}\hat{\phi}\dot{r} + \dot{r}\dot{\phi}\hat{\phi} + \ddot{r}\hat{r} \\
\vec{a} = \frac{d^2\vec{r}}{dt^2} &= (\ddot{r} - r\dot{\phi}^2\hat{r})\hat{r} + (2r\dot{\phi} + \ddot{\phi})\hat{\phi}
\end{aligned} \tag{7}$$

From here, we conclude with:

$$\boxed{\vec{F} = m\vec{a} = m(\ddot{r} - r\dot{\phi}^2\hat{r})\hat{r} + m(2r\dot{\phi} + \ddot{\phi})\hat{\phi}} \tag{8}$$

2 Problem Solutions

1. 1.1 Given the two vectors $b = \hat{x} + \hat{y}$ and $c = \hat{x} + \hat{z}$ find $b + c$, $5b + 2c$, $b \cdot c$, and $b \times c$

Solution:

$$\begin{aligned}
b + c &= 2\hat{x} + \hat{y} + \hat{z} \\
5b + 2c &= 7\hat{x} + 5\hat{y} + 2\hat{z} \\
b \cdot c &= 1 \\
b \times c &= \begin{cases} 1, 1, 0 \\ 1, 0, 1 \end{cases} \\
&=< 1, 1, -1 > \\
&= \hat{x} - \hat{y} - \hat{z}
\end{aligned} \tag{9}$$

2. 1.3 By applying Pythagoras's theorem (the usual two-dimensional version) twice over, prove that the length r of a three-dimensional vector $r = (x, y, z)$ statisfies $r^2 = x^2 + y^2 + z^2$

Solution:

$$\begin{aligned}
h^2 &= x^2 + y^2 \\
r^2 &= h^2 + z^2 \\
r^2 &= x^2 + y^2 + z^2
\end{aligned} \tag{10}$$

3. Find the angle between a body diagonal of a cube and any of its face diagonals. [*Hint* : Choose a cube with side 1 adn with one corner at O and the opposite corner at the point (1, 1, 1). Write down teh vector that represents a body diagonal and another that represents a face diagonal, and tehn find teh angle between them as in Problem 1.4].

Solution:

$$\begin{aligned}
f_{body} &=< 1, 1, 1 > \\
f_{face} &=< 1, 1, 0 > \\
f_{body} \cdot f_{face} &= 2 = |f_{body}||f_{face}| \cos(\phi) \\
\sqrt{3}\sqrt{2} \cos(\phi) &= 2 \\
\phi &= \arccos(\frac{2}{\sqrt{3}\sqrt{2}})
\end{aligned} \tag{11}$$

$$\boxed{35.26\,\text{deg}}$$

4. Prove that the two definitions of the scalar product $r \cdot s$ as $rs\cos(\phi)$ and $\sum r_i s_i$ are equal. One way to do this is to choose your x-axis along the direction of $\boldsymbol{r}$

Solution:
If r lays along x, then $\cos(\phi) = 1$ since $\phi = 0$. Since $\sum r_i s_i = \sum r_i \cdot \sum s_i = rs$, the two statements are equivalent. The longer way is to use the law of cosines to expand $\cos(\phi)$ in terms of r and s, which will give you an expression that evaluates to a summation.

5. In elementary trigonometry, you probably learned the law of cosines for a triangle of sides a, b, c that $c^2 = a^2 + b^2 - 2ab\cos(\phi)$ where ϕ is the angle between teh sides a and b. Show that the law of cosines is an immediate consequence of the identity $(a + b)^2 = a^2 + b^2 + 2a \cdot b$.

Solution:

$$2a \cdot b = 2ab\cos(\phi) \tag{12}$$

Since ϕ represents the angle between a and b (which is the external angle of the triangle $\pi - \phi$), then $\cos(\phi) \rightarrow -\cos(\phi)$. Let $c = a + b$, then we get $c^2 = a^2 + b^2 - 2ab\cos(\phi)$.

6. The position of a moving particle is given as a function fo time t to be

$$r(t) = \hat{x} b \cos(\omega t) + \hat{y} c \sin(\omega t) + \hat{z} v_o t$$

where $b, c,$ and ω are constants. Describe the particle's orbit.

Solution:
The $\hat{z} v_o t$ part will cause the particle to move upwards continuously, but the two trigonometric functions of different amplitudes will create an elliptical orbit.

7. Let u be an arbitrary fixed unit vector and show that any vector b satisfies

$$b^2 = (u \cdot b)^2 + (u \times b)^2$$

Solution:

$$\begin{aligned} b^2 &= (ub\cos(\phi))^2 + (ub\sin(\phi))^2 \\ b^2 &= (ub)^2(\cos(\phi)^2 + \sin(\phi))^2 \\ b^2 &= u^2 b^2 \end{aligned} \tag{13}$$

Since $|u| = 1$ because it is a unit vector, we get $b^2 = b^2$.

8. Show that the definition of the cross product is equivalent to the elementary definition that $r \times s$ is perpendicular to both r and s with magnitude $rs\sin(\phi)$ and direction given by the right-hand rule.

Solution:
Let $\vec{v} = (v_1, v_2, v_3)$ and $\vec{s} = (s_1, s_2, s_3)$, then:

$$\begin{aligned} |\vec{v_1}| &= \sqrt{v_1^2 + v_2^2 + v_3^2} \\ |\vec{s_1}| &= \sqrt{s_1^2 + s_2^2 + s_3^2} \end{aligned} \tag{14}$$

$$\begin{aligned} |(v_1, v_2, v_3) \times (s_1, s_2, s_3)| &= |(v_2 s_3 - v_3 s_2, v_1 s_3 - v_3 s_1, v_1 s_2 - v_2 s_1)| \\ &= \sqrt{(v_2 s_3 - v_3 s_2)^2 + (v_1 s_3 - v_3 s_1)^2 + (v_1 s_2 - v_2 s_1)^2} \\ &= \sqrt{\begin{array}{r} (v_2 s_3)^2 + 2(v_2 v_3 s_2 s_3) + (v_3 s_2)^2 \\ +(v_1 s_3)^2 + 2(v_1 v_3 s_1 s_3) + (v_3 s_1)^2 \\ +(v_1 s_2)^2 + 2(v_1 v_2 s_1 s_2) + (v_1 s_2)^2 \end{array}} \\ &= \sqrt{\begin{array}{r} (v_1^2 + v_2^2 + v_3^2)(s_1^2 + s_2^2 + s_3^2) \\ -((v_1 s_1)^2 + (v_2 s_2)^2 + (v_3 s_3)^2 + \\ 2(v_2 v_3 s_2 s_3 + v_1 v_3 s_1 s_3 + v_1 v_2 s_1 s_2)) \end{array}} \end{aligned} \tag{15}$$

And since $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + ac + bc)$, we get:

$$\begin{aligned} &\sqrt{(v_1^2 + v_2^2 + v_3^2)(s_1^2 + s_2^2 + s_3^2) - (v_1 s_1 + v_2 s_2 + v_3 s_3)^2} \\ =&\sqrt{||\vec{v}||||\vec{s}|| - (\vec{v} \cdot \vec{s})^2} \end{aligned} \tag{16}$$

Since $\vec{v} \cdot \vec{s} = ||\vec{v}||||\vec{s}|| \cos(\phi)$, we get:

$$\begin{aligned} =&\sqrt{||\vec{v}||^2||\vec{s}||^2 - (\vec{v} \cdot \vec{s})^2} \\ =&\sqrt{||\vec{v}||^2||\vec{s}||^2(1 - \cos^2(\phi))} \\ =&\sqrt{||\vec{v}||^2||\vec{s}||^2(\sin^2(\phi))} \\ =&||\vec{v}||||\vec{s}||(\sin(\phi)) \end{aligned} \tag{17}$$