Relations - II

Zero-one Matrices and Directed graphs

Consider the set $A = \{a_1, a_2 \dots, a_m\}$ and $B = \{b_1, b_2, \dots b_n\}$ of orders m and n respectively.

Then $A \times B$ consists of all ordered pairs of the form (a_i, b_j), $1 \le i \le m$, $1 \le j \le n$, which are "mn" in number.

Let R be a relation from A to B so that R is a subset of $A \times B$

Now put $m_{ij} = (a_i, b_j)$ and assign the values '1' or '0' to m_{ij} according to the following

Rule:

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

The $m \times n$ matrix formed by these m_{ij}'s is called the matrix of the relation R or the relation matrix for R, and is denoted by M_R or $M(R)$.
Since $M(R)$ contains only 0 & 1 and its elements $M(R)$ is also called the Zero-one matrix for R.

Eg Let $A = \{1, 2, 3, 4\}$ and let R be the relation on A defined by xRy if and only if "x divides y", written $x | y$.

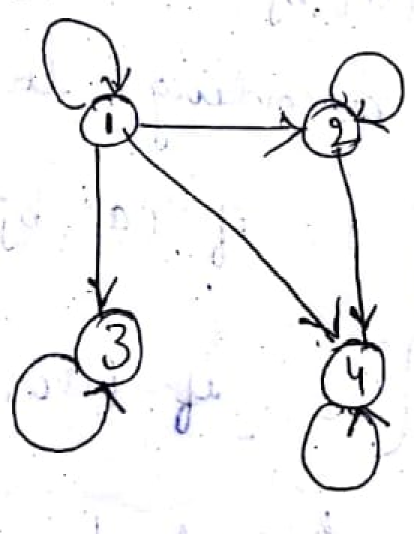
a) write down R as a set of ordered pairs

b) Draw the diagraph of R

c) Determine the indegree & outdegree of the vertices in the ~~diagram~~ diagraph.

Soln (a). $R = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (2,2)\ (2,4)\ (3,3)\ (4,4)\}$

(b) diagraph



c)

Vertex	1	2	3	4
Indegree	1	2	2	3
Out-degree	4	2	1	1

Operations on Relations

1. Union & Intersection of Relations
2. Complement of a Relation.
3. Converse of a Relation.

① $R_1 \cup R_2 \Rightarrow (a,b) \in R_1$ (OR) $(a,b) \in R_2$

$R_1 \cap R_2 \Rightarrow (a,b) \in R_1$ (AND) $(a,b) \in R_2$

② Complement of R : $\overline{R}$

$(a,b) \in \overline{R}$, if and only if $(a,b) \notin R$.

③ Converse of a Relation

Given a relation R from a set A to a set B, denoted by R^c, is defined as the converse of R, a relation from B to A with the property that $(a,b) \in R^c$ if and only if $(b,a) \in R$.

(i) If M_R is the matrix of R, then $(M_R)^T$, the transpose of M_R, is the matrix of R^c and

(ii) $(R^c)^c = R$

Eg $A = \{a,b,c\}$ $B = \{1,2,3\}$ $R = \{(a,1)(b,1)(c,2)(c,3)\}$ from A to B.
and $S = \{(a,1)(a,2)(b,1)(b,2)\}$
Determine $\overline{R}$, $\overline{S}$, $R \cup S$, $R \cap S$, R^c and S^c

Soln $A \times B = \{(a,1)(a,2)(a,3),(b,1)(b,2)(b,3)(c,1)(c,2)(c,3)\}$

$\therefore$ $\overline{R}$ = Complement of R in $A \times B = (A \times B) - R$
$= \{(a,2)(a,3)(b,2)(b,3)(c,1)\}$

$\overline{S} = (A \times B) - S = \{(a,3)(b,3),(c,1)(c,2)(c,3)\}$

$R \cup S = \{(a,1)(b,1)(a,2)(b,2)(c,2)(c,3)\}$

$R \cap S = \{(a,1)(b,1)\}$

$R^c = \{(1,a)(1,b)(2,c)(3,c)\}$ $S^c = \{(1,a)(2,a)(1,b)(2,b)\}$

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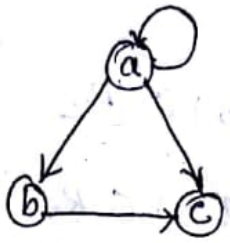
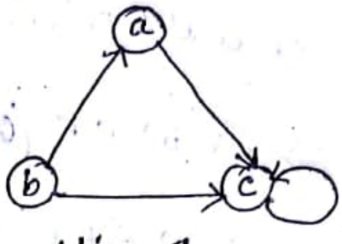
Eg 2 The diagraphs of two relations R and S on the set $A = \{a, b, c\}$ are given below. Draw the diagraph of $\overline{R}$, $R \cup S$, $R \cap S$ and R^c.



(i) : R



(ii) : S

Soln

From diagraphs

$R = \{(a,a)\ (a,b)\ (a,c)\ (b,c)\}$ & $S = \{(a,c)\ (b,a)\ (b,c)\ (c,c)\}$

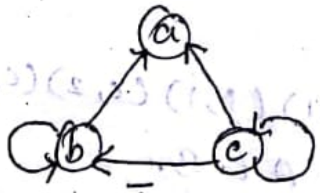
$\therefore\ \overline{R} = \{(b,a), (b,b), (c,a), (c,b)\ (c,c)\}$

$R \cup S = \{(a,a)\ (a,b)\ (a,c)\ (b,c)\ (b,a)\ (c,c)\}$

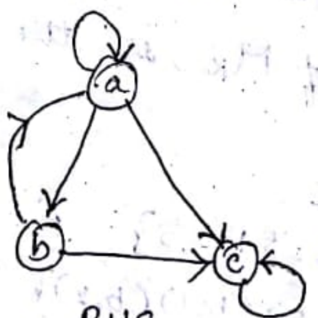
$R \cap S = \{(a,c)\ (b,c)\}$

$R^c = \{(a,a)\ (b,a)\ (c,a)\ (c,b)\}$

<u>diagraphs</u>

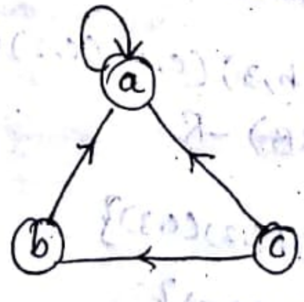


$\overline{R}$



$R \cup S$

$R \cap S$



R^C

Composition of Relations

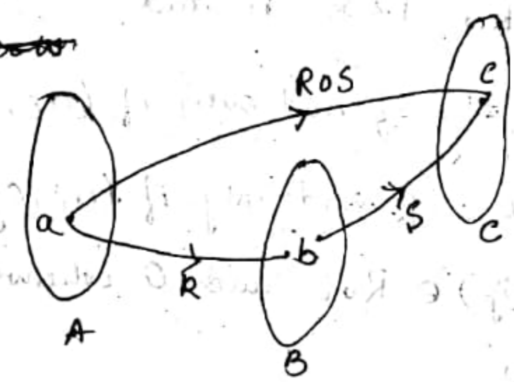
Let R, be a relation from a Set $A \to B$ and
S : relation from Set B to C.

Then Composition of $R \circ S$ is defined as.

If $a \in A$ and $c \in C$, then $(a,c) \in R \circ S$
if and only if there is some b in B such that
$(a,b) \in R$ and $(b,c) \in S$.

i.e $R \circ S = \{(a,c) \mid a \in A, c \in C$, and there exists
$b \in B$ with $(a,b) \in A$ and $(b,c) \in B\}$

~~Illustration~~



$R \circ S \neq S \circ R$

$R \circ R$ is a Relation on A.
$R^2 \circ R$ " " " " A
R^3
$\vdots$
$R^n = R \circ R^{n-1}$ for $n \geq 2$

Theorem 1: Let R be a relation from a set $A = \{a_1, a_2 \ldots a_m\}$ to a set $B = \{b_1, b_2 \ldots b_n\}$ and S be a relation from the set B to a set $C = \{c_1, c_2 \ldots c_p\}$. Then the matrices of R, S and $R \circ S$ satisfy the identity

$$M(R) \times M(S) = M(R \circ S)$$

Proof: Let $M(R) = r_{ij}$ $M(S) = s_{ij}$ & $M(R \circ S) = P_{ij}$

Then $r_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R \\ 0 \text{ if } (a_i, b_j) \notin R \end{cases}$ $\begin{matrix} 1 \leq i \leq m \\ 1 \leq j \leq n \end{matrix}$

$$s_{ij} = \begin{cases} 1 & \text{if } (b_i, c_j) \in S \\ 0 & \text{if } (b_i, c_j) \notin S \quad 1 \le i \le n, \; 1 \le j \le p \end{cases}$$

say p_{ij}

By the rule of matrix multiplication, we note that the $(i,j)^{th}$ element of the matrix product $M(R) \times M(S)$ is

$$q_{ij} = \sum_{k=1}^{n} r_{ik} s_{kj} \quad 1 \le i \le m \quad 1 \le j \le p$$

We note that q_{ij} is equal to 1 if $r_{ik} = 1$ and $s_{kj} = 1$ for some k, $1 \le k \le n$ and 0 otherwise.

But $r_{ik} = 1$ and $s_{kj} = 1$ if and only if $(a_i, b_k) \in R$ and $(b_k, c_j) \in S$; ie if and only if $(a_i, c_j) \in R \circ S$.

Thus $q_{ij} = 1$ if $(a_i, c_j) \in R \circ S$ and 0 otherwise.

This means $q_{ij} \equiv p_{ij}$

Thus $M(R) \times M(S) = [q_{ij}] = [p_{ij}] = M(R \circ S)$

Problem 1. Let $A = \{1, 2, 3, 4\}$ $B = \{w, x, y, z\}$ and $C = \{5, 6, 7\}$. Also, Let R_1 be a relation from A to B defined by $R_1 = \{(1, x)\ (2, x)\ (3, y), (3, z)\}$ and R_2 and R_3 be relations from B to C, defined by

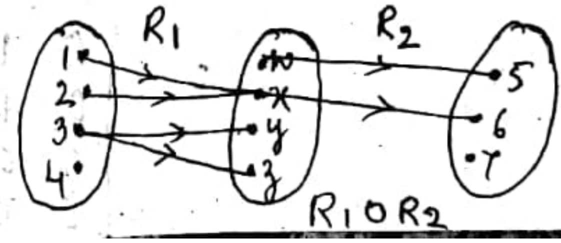
$R_2 = \{(w, 5), (x, 6)\}$ $R_3 = \{(w, 5), (w, 6)\}$.

Find $R_1 \circ R_2$ and $R_1 \circ R_3$.

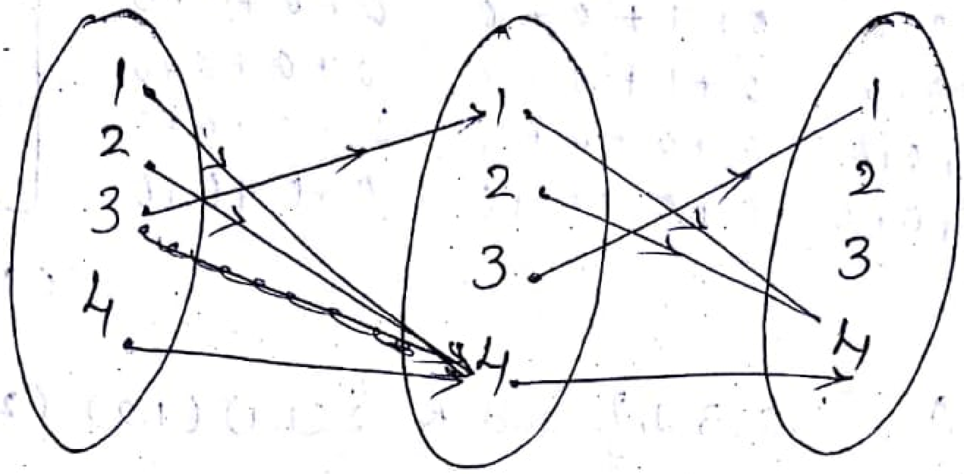
Soln: $(1, x) \in R_1$ and $(x, 6) \in R_2$; $\therefore (1, 6) \in R_1 \circ R_2$

$(2, x) \in R_1$ and $(x, 6) \in R_2$ $\therefore (2, 6) \in R_2 \circ R_1$

$\therefore R_1 \circ R_2 = \{(1,6)\ (2,6)\}$

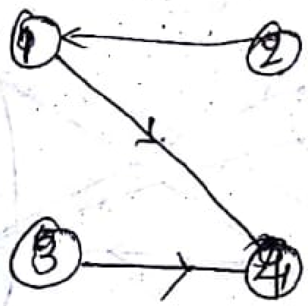


$(S \circ S) = S^2$

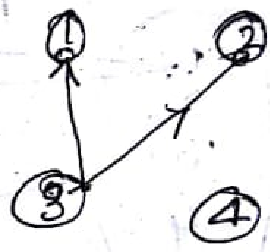


$S^2 = \{(1,4)\ (2,4)\ (3,4)\ (4,4)\}$

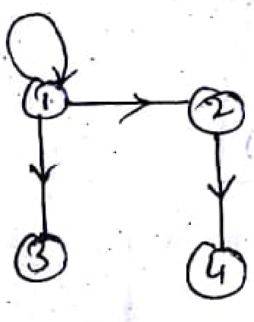
<u>$R \circ S$</u>



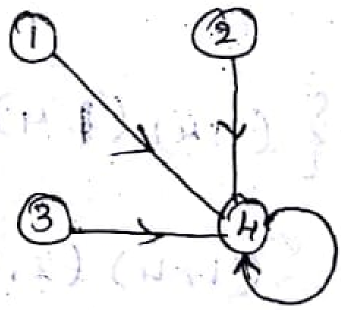
$S \circ R$



$R^2 =$



$S^2 =$



Properties of Relations

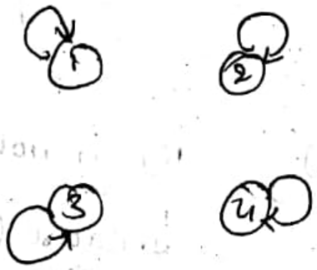
A relation R on a set A is said to be **reflexive** if $(a,b) \in R$, for all $a \in A$

It follows that R is not reflexive if there is some $a \in A$ such that $(a,a) \notin R$.

Eg $A = \{1, 2, 3, 4\}$

$R = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (2,1)\ (2,2)\ (2,3)\ (2,4)$
$(3,1)\ (3,2)\ (3,3)\ (3,4)\ (4,1)\ (4,2)\ (4,3)\ (4,4)$

$R = \{(1,1)\ (2,2)\ (3,3)\ (4,4)\}$ – Reflexive



$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Eg R : (a is less than or equal a) aRa.

Irreflexive Relation

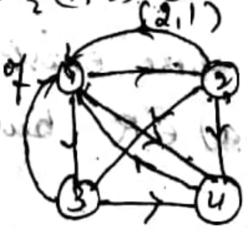
A relation on a set A is said to be irreflexive if $(a,a) \notin R$ for any $a \in A$.

ie a relation R is irreflexive if no element of A is related to itself by R.

Eg $A = \{1, 2, 3, 4\}$

$R = \{(1,2)\ (1,3)\ (1,4)\ (2,3)\ (2,4)\ (3,4)\}$
$(2,1)\ (3,1)\ (4,1)$

NO: No cycle of length 1 at any vertex



$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{vmatrix}$$

Note: have 0's on its main diagonal

Symmetric Relation

A relation R on a set is said to be symmetric if $(a,b) \in R$ i.e. $(b,a) \in R$ for all $a, b \in A$.

A Relation is not symmetric if there exist $(a,b) \in A$ such that $(a,b) \in R$ but $(b,a) \notin R$

$\therefore$ A relation which is not symmetric is called an asymmetric relation.

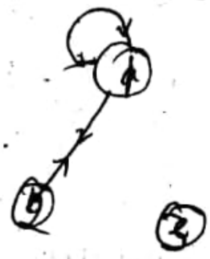
Eg $A = \{1, 2, 3\}$

$R_1 = \{(1,1)\ (1,2)\ (2,1)\}$ R_1 is symmetric.

$R_2 = \{(1,2)\ (2,1)\ (1,3)\}$ R_2 is not Symmetric because $(3,1) \notin R_2$

$$R_1 \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

Symmetrix matrix.



In diagraph the edges are always bidirectional.

Antisymmetric Relation

A relation R on a set A is said to be antisymmetric if whenever $(a,b) \in R$ and $(b,a) \in R$ then $a = b$.

→ R is not antisymmetric if there exist $a, b \in A$ such that $(a,b) \in R$ and $(b,a) \in R$ but $a \neq b$

Eg $A = \{1, 2, 3\}$

$R_1 = \{(2, 2)(3, 3)\}$ R: a is less than or equal to b.

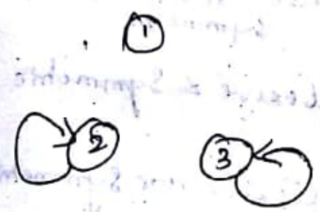
$a \le b$ $b \le a$ then $a = b$
i.e. $2 = 2$

$\therefore$ Antisymmetric, & Symmetric.

$R_2 = \{(1,3)(3,1)(1,2)\}$

neither symmetric nor antisymmetric.

Diagraph.



Note
* No bidirectional edges.

Matrix.

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

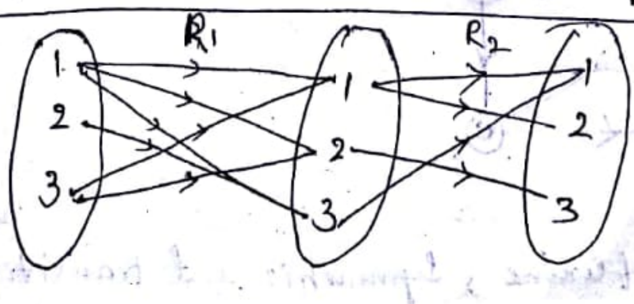
Note $i \ne j$ will have 0's.

Transitive Relation

A relation R on a set A is said to be transitive if whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$, for all $a, b, c \in A$

Eg $A = \{1, 2, 3\}$ $R_1 = \{(1,1)\ (1,2)\ (2,3)\ (1,3)\ (3,1)\ (3,2)\}$ R_1 is Transitive.

$R_2 = \{(1,2), (2,3)\ (1,3)\ (3,1)\}$ $\because$ (1,1) is not in R.



Note A relation R on a set A is transitive if and only if its matrix $M_R = [m_{ij}]$ has the following property.

$$\text{If } m_{ij} = 1 \text{ and } m_{kj} = 1, \text{ then } m_{ij} = 1$$

Eg 1 Let $A = \{1, 2, 3\}$ Determine the nature of the following relations on A.

(i) $R_1 : \{(1,2)(2,1)(1,3)(3,1)\}$ – Symmetric & irreflexive, but neither reflexive nor transitive.

(ii) $R_2 = \{(1,1), (2,2), (3,3), (2,3)\}$ – Reflexive & transitive, but not Symmetric

(iii) $R_3 = \{(1,1)(2,2)(3,3)\}$ –

(iv) $R_4 = \{(1,1)(2,2)(3,3)(2,3)(3,2)\}$

} Reflexive & Symmetric.

(v) $R_5 = \{(1,1)(2,3)(3,3)\}$ – neither Reflexive nor Symmetric

(vi) $R_6 : \{(2,3)(3,4)(2,4)\}$ – transitive & irreflexive, but not Symmetric

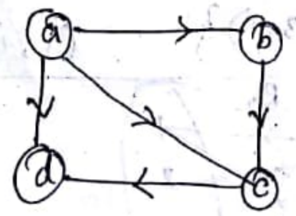
(vii) $R_7 : \{(1,3)(3,2)\}$ – irreflexive, but neither transitive nor Symmetric.

Eg 2 Let $A = \{1, 2, 3, 4\}$ Determine the nature of the following relations on A.

(1) $R_1 = \{(1,1)(1,2)(2,1)(2,2)(3,3)(3,4)(4,3)(4,4)\}$

(2) $R_2 = \{(1,2)(1,3)(3,1)(1,1)(3,3)(3,2)(1,4)(4,2)(3,4)\}$

(3) R_3 Represented by the following digraph.



Soln:

(1) R_1 is reflexive, Symmetric and transitive -

(2) R_2 is Transitive

(3) From the diagraph R_3 is both asymmetric and antisymmetric.

Eg 3 Find the nature of the relations represented by the following matrices:

(a) $\begin{vmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix}$ c $\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{vmatrix}$

soln (a) $a_{ij} = a_{ij}$ for 1, 2, 3

$\therefore$ matrix is symmetric.

(b) Main diagonal elements are 1 $\therefore$ it is reflexive and symmetric

(c) Not symmetric.
$\therefore$ The corresponding relation is not symmetric
Further, the presence of 1 in $(1,4)^{th}$ & $(4,1)^{th}$
position indicates it is not antisymmetric

Eg 4 Let R be a relation on a set A. Prove the following
(1) R is reflexive if and only if $\overline{R}$ is irreflexive.
(2) If R is reflexive, so is R^c
(3) If R is symmetric, so are R^c and $\overline{R}$
(4) If R is transitive, so is R^c

Soln
(1) Suppose R is reflexive. Then $(a, a) \in R$ for every $a \in A$

Properties of Relation

Relation (R)	Definition	Matrix M(R)	Diagraph (G)
Reflexive	$(a,a) \in R$ $\forall a \in R$	1 in the principal diagonal	Every Vertex has a loop.
Irreflexive	$(a,a) \notin R$ $\forall a \in R$	0 in the principal diagonal	No loop for any vertex
Symmetric	$(a,b) \in R$ & $(b,a) \in R$	Matrix is Symmetric	Two Vertices are connected by arrow in both the directio[n]
Asymmetric	$(a,b) \in R$ & $(b,a) \notin R$	Not a Symmetric matrix	No arrows in both the direction connecting two Vertices.
Anti Symmetric	$(a,b) \in R$ & $(b,a) \in R$ $\Rightarrow a = b$	Not in any specific form	only loops are formed on Vertices
Transitive	If $(a,b) \in R$ & $(b,c) \in R$ then $(a,c) \in R$	Not in any Specific form	Not in any specific form
Non transitive	If $(a,b) \in R$ & $(b,c) \in R$ then $(a,c) \notin R$	Not in any Specific form	NOT in any Specific form
Equivalence Relation	Reflexive Symmetric transitive		

Equivalence Relations

A relation R on a set A is said to be an equivalence relation on A if (i) R is reflexive (ii) R is symmetric and (iii) R is transitive on A.

Eg 1 Let $A = \{1, 2, 3, 4\}$ and

$$R = \{(1,1)(1,2)(2,1)(2,2)(3,4)(4,3)(3,3)(4,4)\}$$

be a relation on A. Verify that R is an equivalence relation.

Soln

Reflexive : $(1,1)(2,2)(3,3)(4,4) \in R$

$\therefore$ R is reflexive.

Symmetric : $(1,2)(2,1)(3,4)(4,3) \in R$

$\therefore$ R is Symmetric

Transitive : $(1,2)(2,1)(1,1) \in R$ $(2,1)(1,2)(2,2) \in R$

$(3,4)(4,3)(3,3) \in R$, $(4,3)(3,4)(4,4) \in R$.

$\therefore$ R is transitive.

$\therefore$ R is equivalence relation.

Problem 2.

Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. On this set define the relation R by $(x, y) \in R$ if and only if $x - y$ is a multiple of 5. Verify that R is an equivalence Relation.

Soln (1) For any $x \in A$,

$x - x = 0 \Rightarrow$ which is multiple of 5 [because $0 = 5 \cdot 0$]

$\therefore (x, x) \in R$ is reflexive.

(2) For any $x, y \in A$ if $(x, y) \in R$ then $x - y = 5k$ for some integer k.

lly $y - x = -5k$ so that $(y, x) \in R$

$\therefore R$ is Symmetric.

(3) For any $x, y, z \in A$ if $(x, y) \in R$ and $(y, z) \in R$

then $\left.\begin{array}{l} x - y = 5k_1 \\ y - z = 5k_2 \end{array}\right\}$ for some integer k_1 & k_2

$\therefore x - z = (x - y) + (y - z)$

$= 5k_1 + 5k_2$

$x - z = 5(k_1 + k_2) \in R$

$\therefore R$ is transitive.

$\therefore R$ is Equivalence Relation.

Eg3. For a fixed integer $n > 1$. prove that the relation "congruent modulo n" is an equivalence relation on the set of all integers, $\mathbb{Z}$.

Soln. For $a, b \in \mathbb{Z}$, we say that "a is congruent to b modulo n" $[a \equiv b \pmod{n}]$

if $a-b$ is a multiple of n, or equivalently,

$a-b = kn$ for some $k \in \mathbb{Z}$

$aRb \Rightarrow a \equiv b \pmod{n}$ hence P.T R is equivalence Relation

$a \in \mathbb{Z}$, $a-a=0$ is a multiple of n.

ie $a \equiv a \pmod{n}$

aRa

$\therefore$ R is reflexive.

Let $a, b \in \mathbb{Z}$

$$
\begin{aligned}
aRb &\Rightarrow a \equiv b \pmod{n} \\
&\Rightarrow (a-b) \text{ is a multiple of } n \\
&\Rightarrow (b-a) \text{ is a multiple of } n \\
&\Rightarrow b \equiv a \pmod{n} \\
&\Rightarrow bRa
\end{aligned}
$$

$\therefore$ R is Symmetric.

Let $a, b, c \in \mathbb{Z}$

$$
\begin{aligned}
aRb \text{ and } bRc &\Rightarrow a \equiv b \pmod{n} \text{ \& } b \equiv c \pmod{n} \\
&\Rightarrow a-b \text{ and } b-c \text{ are multiple of 'n'} \\
&\Rightarrow (a-b)+(b-c) = (a-c) \text{ is a multiple of } n \\
&\Rightarrow a \equiv c \pmod{n} \\
&\Rightarrow aRc
\end{aligned}
$$

$\therefore$ R is transitive

Hence R is equivalence Relation.

Equivalence Classes

$A = \{1, 2, 3\}$

Eg 1 $R = \{(1,1)\ (1,3)\ (2,2)\ (3,1)\ (3,3)\}$

Eg 2: $A = \{1,2,3,4,5,6\}$

$R\ \{(1,1)\ (1,3)\ (1,5)\ (3,1)\ (3,3)\ (3,5)$

$(5,1)\ (5,3)\ (5,5)\ (2,2)\ (2,6)\ (6,2)$

$(6,4)\ (4,4)$

$[a] = x$

$[1] = \{x \in A \mid xR1\}$

$(x, 1) \in R$

Second no in ordered pair.

$[1] = \{1, 3, 5\}$

$xR1$

$[2] = \{2, 6\}$

$xR2 \in R$

$[3] = \{xR3\}\quad \{3, 1, 5\} = [1]$

$[4] = \{4\}\quad xR4$

Defn :-

Let R be an equivalence relation on a set A and $a \in A$. Then the set of all those elements x of A which is related to a by R is called the equivalence class of a with respect to R.

ie $R(a)$ or $[a]$ or $\bar{a}$

$[a] = R(a) = x \in A \mid (x, a) \in R$

xRa

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Fundamental properties of equivalence classes

Theorem 1 :- Let R be an equivalence relation on a set A and $a \in A$. Then $a \in [a]$

Proof :- Since R is reflexive, we have aRa

$\therefore a \in [a]$

Theorem 2 : Let R be an equivalence relation on a set A, and let $a, b \in A$, Then aRb if and only if $[a] = [b]$

Proof :- Suppose aRb,

Let $x \in [a] \Rightarrow xRa$

Thus, we have xRa ie aRb

Since R is transitive

$\therefore xRb$

$\therefore x \in [b]$

thus $[a] \subseteq [b]$

|||ly $[b] \subseteq [a]$

Conversly, Suppose $[a] = [b]$

Since $a \in [a]$ it follows that $a \in [b]$

Thus aRb

Theorem 3 : Let R be an equivalence relation on a set A and let $a, b \in A$. Then the following result is true.

if $[a] \cap [b] \neq \phi$, then $[a] = [b]$

Proof : Suppose $[a] \cap [b] \neq \phi$. Then $x \in A$

ie $x \in [a]$

ie $x \in [b]$

So that xRa & xRb

$\Rightarrow$ aRx & xRb so that aRb [$\because$ R is Symmetric & Transitive]

$\therefore$ $[a] = [b]$

Contrapositive   if $[a] \neq [b]$ then $[a] \cap [b] = \phi$

Eg : Refer Eg1.

For the equivalence relation
$R = \{(1,1)\ (1,2)\ (2,1)\ (2,2)\ (3,4)\ (4,3)\ (3,3)\ (4,4)\}$
defined on the set $A = \{1,2,3,4\}$ determine the partition induced.

Def: Partition of a Set

Let A be a nonempty set. Suppose there exist non-empty subsets $A_1, A_2, A_3 \cdots A_K$ of A such that the following two conditions hold.

(1) A is union of $A_1, A_2, A_3 \cdots A_K$ ie, $A = A_1 \cup A_2 \cup A_3$

(2) Any two of the subsets $A_1, A_2 \cdots A_K$ are disjoint; that is $A_i \cap A_j = \phi$ for $i \neq j$.

then the set $P = \{A_1, A_2, A_3 \cdots A_K\}$ is called a partition of A. Also $A_1, A_2, A_3 \cdots A_K$ are called the blocks or cells of the partition.

Soln $A\ \{1,2,3,4\}$
Equivalence Relation w.r.t R

$[1] = \{1,2\}$

$[2] = \{1,2\}$

$[3] = \{3,4\}$

$[4] = \{3,4\}$

From the above
$[1]$ & $[3]$ are distinct

These two distinct equivalence classes constitute the partition.

$P = \{[1],[3]\}$

$= \{\{1,2\},\{3,4\}\}$

We observe that
$A = [1] \cup [3] = \{1,2\} \cup \{3,4\}$

For the set A and the relation R on A considered find the partition of A induced by R.

$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ & R is defined by $(x, y) \in R$ if & only if $x - y$ is a multiple of 5.

$(1,6)$ $(2,7)$ $(3,8)$ $(4,9)$ $(5,10)$, $(6,11)$ $(7,12)$

$(1,1)$ $(2,2)$ $(3,3)$ $(4,4)$

$[1] = [6] = [11]$

$[2] = [7] = [12]$

$[3] = [8]$,

$[4] = [9]$

$[5] = [10]$

All these classes are distinct

$\therefore$ The partition of A induced by R is

$A = \{1, 6, 11\} \cup \{2, 7, 12\} \cup \{3, 8\} \cup \{4, 9\} \cup \{5, 10\}$

Partial Orders / Partially ordered set or poset

A relation R on a set A is said to be a partial ordering relation or a partial order on A if (i) R is reflexive

(ii) R is antisymmetric

& (iii) R is transitive on A.

It is denoted by the pair (A, R).

Eg. of partial order on the set $\mathbb{Z}$ of all integers is: (i) R: "less than or equal to" $(\mathbb{Z}, \leq)$

(ii) R: "is greater than or equal to", $(\mathbb{Z}, \geq)$

(iii) R: $a|b$ on the set $\mathbb{Z}^+$ $(\mathbb{Z}^+, a|b)$

(iv) The subset relation $\subseteq$ on the power set S is a partial order $(P(S), \subseteq)$

Note: → The Relation "is less than" & "is greater than" are not partial orders on $\mathbb{Z}$. because these are not reflexive.

→ The Relation "congruent modulo n" defined on the set of all integers $\mathbb{Z}$ is also not a partial order. because this relation is not antisymmetric.

$\mathbb{Z} = \{0, 1, 2 \ldots\}$

$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.

Total order :- Let R be a partial order on a set A. Then R is called a total order on A if for all $x, y \in A$ either xRy or yRx. In this case, the poset (A, R) is called a totally ordered set.

Eg: The partial order relation " less than or equal to" is a total order on the set R. Because, for any $x, y \in R$, we have $x \leq y$ or $y \leq x$ thus $(R, \leq)$ is a totally ordered set.

Eg: $A = \{1, 2, 4, 8\}$ is a total order on A.

Hasse Diagrams :

Partial order is a relation with reflexive, antisymmetric & transitive on a set A.

* Since Partial order is reflexive, at every vertex in the diagraph, of there would be a cycle of length 1.
* Note :- while drawing the digraph, we need not exhibit such cycles explicitly; they will be automatically understood by convention.
* Since partial order is transitive, there is an edge from a vertex 'a' to 'b' and there is an edge from 'b' to 'c' ~~vertex~~, then there is an edge from a to c.

  Note :- we need not exhibit an edge from a to c explicitly;
* For simplicity, we represent the ~~edges~~ vertice by dots and draw the diagraph in such a way that all edges point upward. With this convention we need not put arrows in the edges

The digraph drawn by adopting the above conversions is called poset diagram or Hasse diagram.

Application :- use of dependency analysis in compiler design

ordering & scheduling of st of collection

Data Control.

Dependency graphs can be constructed by drawing edges connect dependent operations → these arcs impose partial ordering among operations.

Eg Let $A = \{1, 2, 3, 4\}$

$R = \{(1,1)\ (1,2)\ (2,2)\ (2,4)\ (1,3)\ (3,3)\ (3,4)$
$(1,4)\ (4,4)\}$

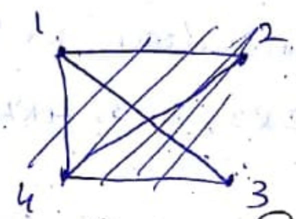
Verify that R is a partial order on A. Also, write down the Hasse diagram for R.

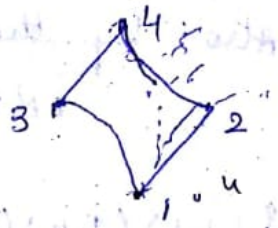
Soln Reflexive $= \{(1,1)\ (2,2)\ (3,3)\ (4,4)\}$

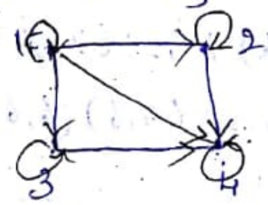
Transitive $= \{(1,2) \cdot (2,2)\ (1,2)$

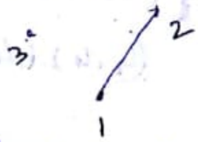
$(1,3)\ (3,3)\ (1,3)$

$(3,3)\ (3,4)\ ;\ 34$ Transitive.









Eg2 $R = \{(x,y) \mid x, y \in A$ and x divides $y\}$

$A = \{1, 2, 3, 4\}$

$R = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (2,2)\ (2,4)\ (3,3)\ (4,4)\}$

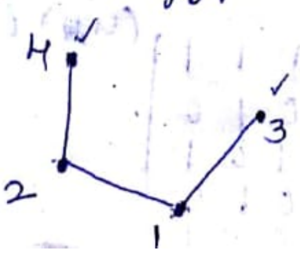
P.T (A, R) is a poset. Draw its Hasse diagram

$(a,a) \in R$ ∴ R is Reflexive

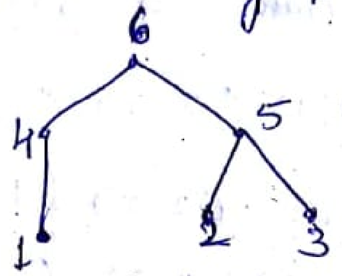
$(a,b) \in R$, $(b,c) \in R$ So $(a,c) \in R$ ∴ R is transitive.

$(a,b) \in R$; So $(b,a) \in R$ with $b = a$ ∴ antisymmetric

Hence Hasse digraph



Eg3. Hasse diagram of a partial order R on the set $A = \{1, 2, 3, 4, 5, 6\}$ is as given below. Write down R as a subset of $A \times A$. Construct its digraph



(1,4)

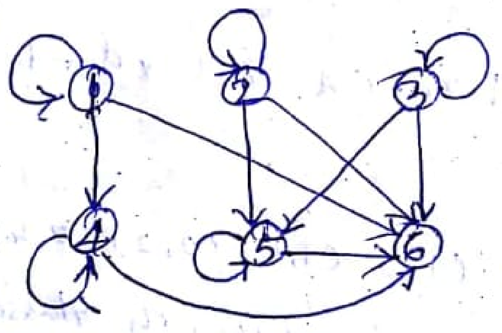
1R4, 1R6, 2R5, 2R6, 3R5, 3R6, 4R6, 5R6

Also by the convention we have

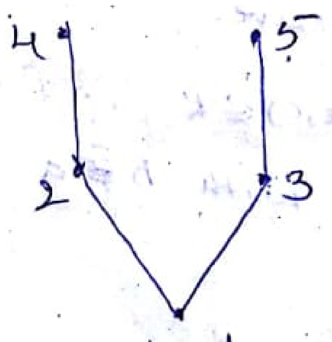
1R1, 2R2, 3R3, 4R4, 5R5 & 6R6.

$\therefore$ $R = \{(1,1)(1,4)(1,6)(2,2)(2,5)(2,6)(3,3)(3,5)(3,6)$
$(4,4)(4,6)(5,5)(5,6)(6,6)\}$

Eg



Eg4



1R1 1R2 1R4 2R2 2R4
3R3 1R3 1R5 3R5

$R = \{(1,1)(1,2)(1,4)(1,3)(1,5)$
$(2,2)(2,4)(3,3)(3,5)$
$(4,4)(5,5)\}$

$$M_R = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

Extremal elements in Posets

The Extremal elements are :

(i) Maximal element : An element $a \in A$ is called a maximal element of A if there exists no element $x \neq a$ in A such that aRx.

OR

$a \in A$ is a maximal element of A if whenever there is $x \in A$ such that aRx then $x = a$.

means : that a is a maximal element of A if and only if in the Hasse diagram of R no edge starts at a*.

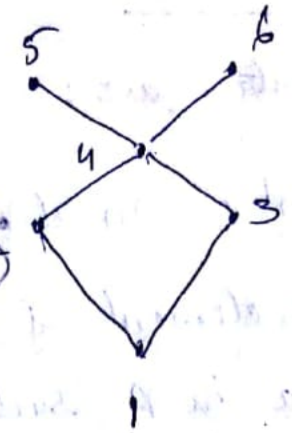
(ii) Minimal element : An element $a \in A$ is called a minimal element of A if there exists no element $x \neq a$ in A such that xRa. In other words, a is a minimal element of A if whenever there is $x \in A$ Such that xRa, then $x = a$.

ie a is a minimal element of A iff in the Hasse diagram of R no edge terminates at a.

(iii) An element $a \in A$ is called a greatest element of A if xRa for all $x \in A$.

(iv) An element $a \in A$ is called a least element of A if aRx for all $x \in A$.

Eg



5, 6 maximal elements

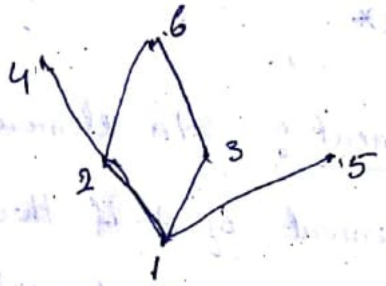
1 is a minimal element.

1 : is the least element

$a \rightarrow aRx$

Eg $\{1, 2, 3, 4, 5, 6\}$ R = {(1,2) (1,3) (1,4) (1,1) (1,5) (1,6) (2,4) (2,6) (3,6)}

R : a divides b



1|2 1|3 1|5 1|6 |4

2|6 2|6

1 least element

no greatest element

4, 6, 5 are maximal

for 2 & 3

1 is the lower bound

6 is the upper bound

Lattice: Let $[P, \leq]$ be any poset, and let $a, b \in P$ be given. Then an element $d \in P$ is called the greatest lower bound, or meet, of a and b (in symbols, $d = a \wedge b$) when

$$d \leq a, \quad d \leq b.$$

and $x \leq a$ and $x \leq b$ imply $x \leq d$.

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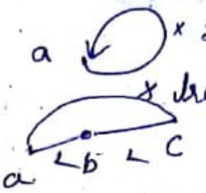
Hasse diagram

1. Create a vertex for element of your domain
2. If aRb then draw an edge from a to b. | – up, – down
3. Remove self loops & transitive edges.

Since R is reflexive a

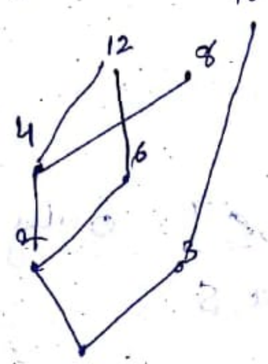


aRb & bRc

It is understood that there exist loop & transitive edge.

$\{1, 2, 3, 4, 6, 8, 12, 15\}$

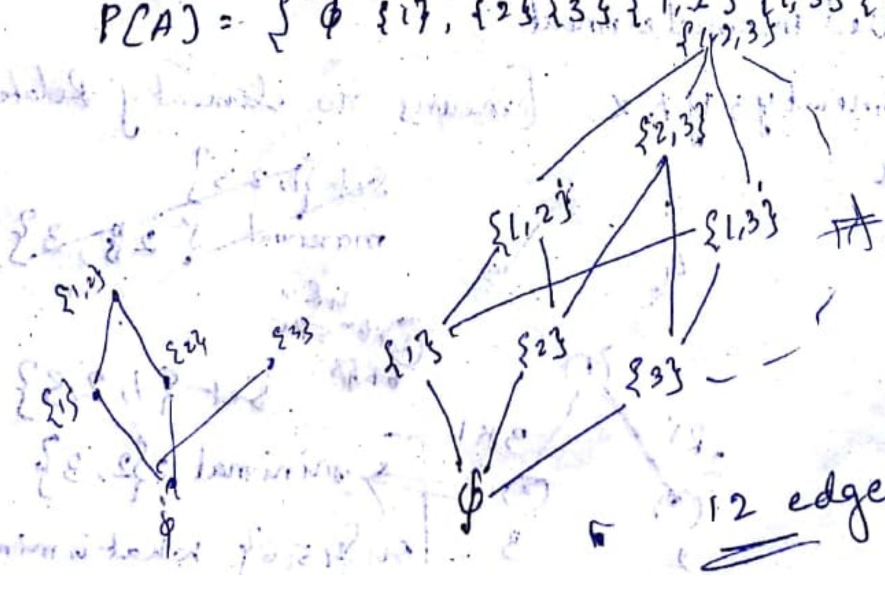
$1|2$, $1|3$, $1|4$, $1|6$, 15



Eg $A = \{1, 2, 3\}$

$[P(A), \subseteq]$

$P(A) = \{\phi\ \{1\}, \{2\} \{3\} \{1,2\} \{1,3\} \{2,3\} \{1,2,3\}\}$

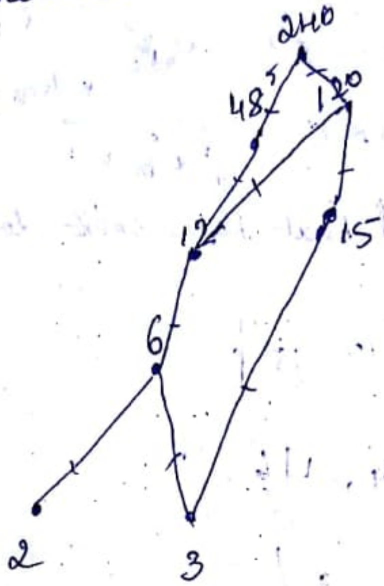


12 edges

Eg numerical example divides Relation

{[2, 3, 6, 12, 15, 48, 120, 240] ; divides}

2, 3 are unrelated



9 edges

Lattice $S \subseteq P$ $[P, \leq]$ only for Poset

→ minimal element of set S

→ maximal element of set S

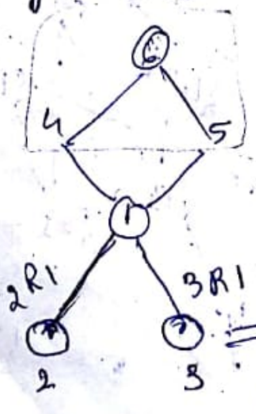
→ minimum element of set S

→ maximum element of set S

minimal element

$x \in S$ is called minimal element

There is no element y; yRx [means no element y Related to x] ie $y \neq x$



~~Set {1, 2, 3}~~

~~maximal {2, 3}~~

4R4

(4)R6 5R5

6R4

2R2

1R2

1R3

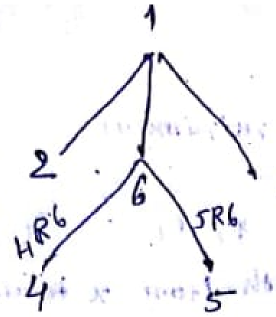
3R1

Set {1, 2, 3}

⟹ minimal = {2, 3}

on {4 5 6} what is minimal

minimal {4, 5}



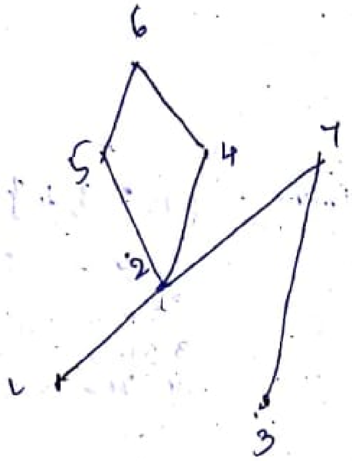
Set = {1, 2, 3}

min = {2, 3}

Set {4, 5, 6}

minimal = {4, 5}

~~Set {2, 4, 5, 6}~~

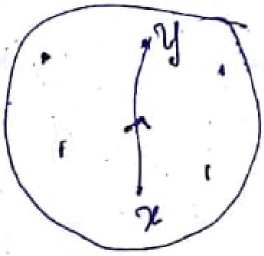


= {2, 4, 5, 6}

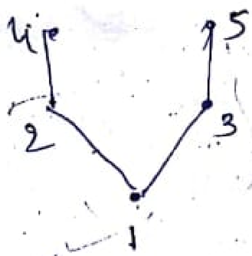
minimal = {2}

maximal element

$x \in S$ is called maximal element of S
if there is no $y \in S$, $x R y$ $(x \neq y)$



if there is a path from x to y
then x is not maximal.
$y \in S$

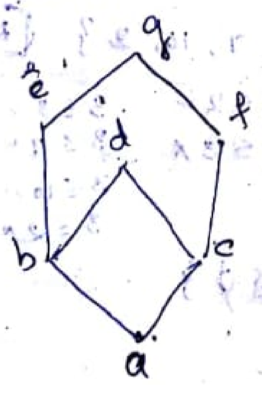


{1, 2, 3}

maximal = {2, 3}

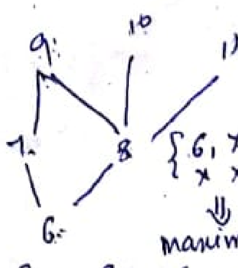
1 R 2
1 R 3

{1, 2}
1 R 2
2 R 4 but 4 is not in set
∴ 2 is maximal.



{a, b, c, d}

maximal {d}



{6, 7, 8, 9}

⇓

maximal {9}

{7, 8, 9, 10} ⇒ {9, 10}

minimum

An element $x \in S$ is called minimum

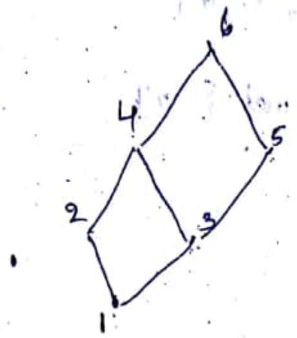
if $\forall y \in S$ xRy

Path from x to all other elements

An element $x \in S$ is called maximum

if $\forall y \in S$ yRx

Eg



maximum : $\{1, 2, 3, 4\}$

$1 \le 4$ ✓
$2 \le 4$ ✓
$3 \le 4$ ✓
$4 \le 4$ ✓

$\{4\}$ = maxi

$\{3, 4, 5\}$

$3R4$ ✓
$4R4$ ✓
$5R4$ ✗

$\{\phi\}$

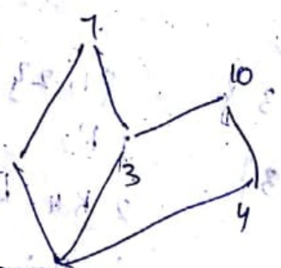
$\{1, 2, 3, 4\}$ $(1R1, 1R2, 1R3, 1R4)$, $(2R1$ ✗ $2R2$ ✓ $2R3$ $2R4$ ✓$)$.

minimum $= \{1\}$

① $e_1 R e_2$
② $e_2 R e_1$

$e_1 = e_2$

Eg



$1R1$
$1R2$
$1R3$
$1R4$

Set $\{1, 2, 3, 10\}$

minimum $= \{1\}$

Set $\{3, 7, 10, 2\}$ $[R, \le]$

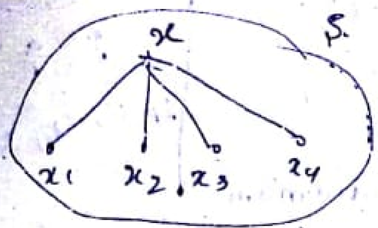
$2 \le 2$ ✓
$2 \le 3$ ✗

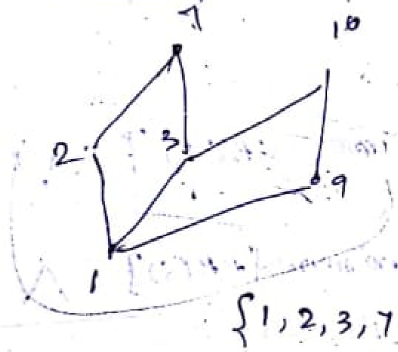
$3 \le 3$ ✓
$3 \le 7$ ✓
$3 \le 10$ ✓
$3 \le 2$ ✗

minimum $= \{\phi\}$

maximum

An element $x \in S$ is called maximum if $\forall y \in S$, yRx





Set $= \{2, 3, 9, 10\}$ maximum $= \{\phi\}$

$2R7$ ✓
$3R7$ ✓
$7R7$ ✓
$10R7$ ✗

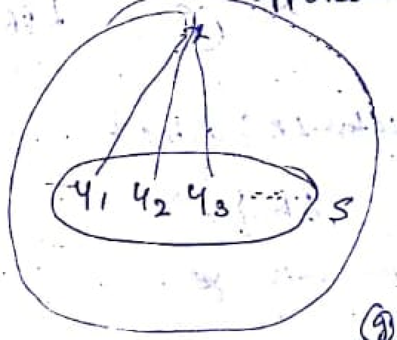
$\{1, 2, 3, 7\}$

$1R7$ ✓
$2R7$ ✓ . $\{7\}$ maximum
$3R7$ ✓
$7R7$ ✓

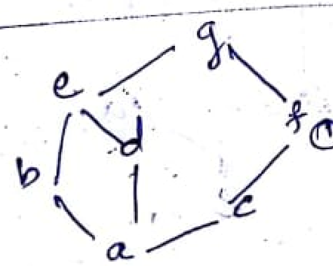
Upperbound & lower bound.

upperbound.

$x \in P$ is called $UB(S)$
if $\forall y \in S$ $y \leq x$



x many be any where in poset



(1) $UB = \{a, b, c\}$
$\{g\}$ is upperbound.

(2) $\{b, d\}$ $b \leq e \leq g$ $\therefore g$ is upperbound.
$d \leq e \leq g$

Lower bound

$x \in P$ is called $LB(S)$.
if $\forall y \in S$ $x \leq y$

(1) $\{a, b\}$ find LB

aRa ✓ $a \leq a$ ✓ | $b \leq a$ ✗
aRb ✓ $a \leq b$ ✓ | $b \leq b$ ✓

Lower Bound $\{a\}$ | $\{b\}$ is not Lower Bound

$S = \{e, g, f\}$

Single element
$S = \{g\}$

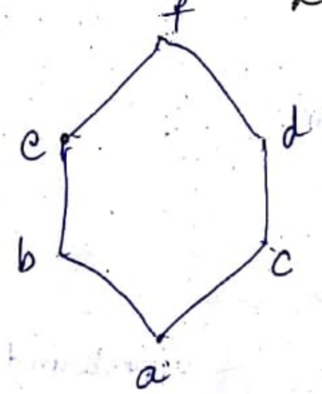
$d \le e$ ✓
$d \le g$ ✓
$d \le f$ ✗
∴ d is not minimum

$a \le e$ ✓
$a \le g$ ✓
$a \le f$ ✓
∴ a is minimum

$g \le g$ ✓
$LB = \{e, d, f, b, c$

Least upper Bound (S) = minimum {UB(S)} $\vee$ operators.

Greatest lower Bound (S) = maximum {LB(S)} $\wedge$



Least upper Bound

$S = \{b, c\}$

$UB = \{b, c\} = \underline{f}$ $LUB = f$

$LUB = \{a, b\}$

$ub = \{e, f\}$

$LUB = \{e\}$.

~~Least~~ Greatest LB {a, b c}

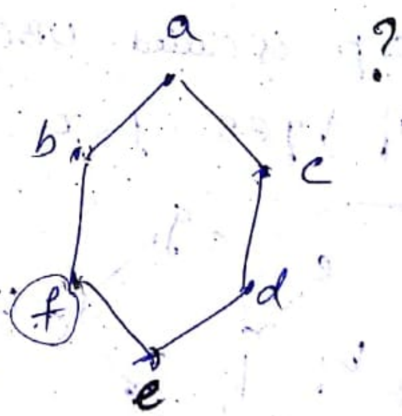
LB {e,}

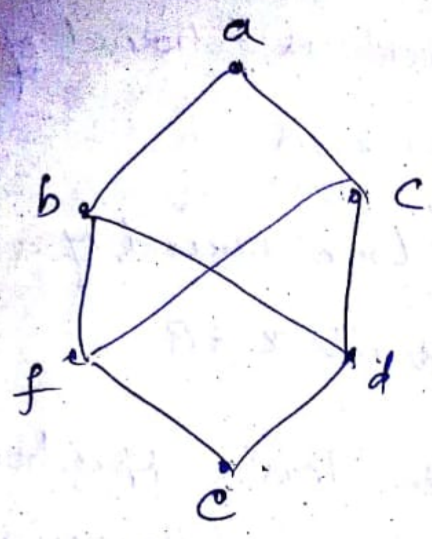
maximum LB = {e}

GLB = {b, f}

{f, e}

maximum of {f, e} = $\underline{f}$





GLB {b c}

LB·{b,c} = {.

maximum of {d, f, e}

= ϕ set

LUB {f, d}

UB {b, c, a}

Least of {b, c, a}

= ϕ.

LUB {e, f}

UB = {b, c, a, f}

Least of {b, c, a, f}

Least = minimum = {f}

GLB {a, b}

LB {f, d, e, b}

Great of {f, d, e, b}

Greatest = maximum = {b}