Stirling Numbers of the Second Kind

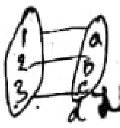
Let A and B be finite Sets with $|A| = m$ and $|B| = n$, where $m > n$, then the number of onto functions from A to B is given by the formula.

$$p(m,n) = \sum_{k=0}^{n} (-1)^k \left({}^nC_{n-k}\right)(n-k)^m \quad \text{— onto}$$

with $p(m,n)$ given by the above formula, the number $\{p(m,n)/n!\}$ is called the Stirling number of the second kind and is denoted by $S(m,n)$.

one to one fun

$$\text{ie } S(m,n) = \frac{p(m,n)}{n!} = \frac{1}{n!}\sum_{k=0}^{n} (-1)^k \left({}^nC_{n-k}\right)(n-k)^m \text{ for } m \geq n$$



This no represents the number of ways in which it is possible to assign 'm' distinct objects into n identical places with no place left empty.

$\Rightarrow$ The number of possible ways to assign m distinct objects to n identical places with empty places allowed is given by

$$P(m) = \sum_{i=1}^{n} S(m,i) \qquad \text{for } m \geq n.$$

Note The number of possible functions from A to B

$|A| = m$ & $|B| = n$

$= n^m$.

Problem

Let $A = \{1,2,3,4,5,6,7\}$ and $B = \{w,x,y,z\}$

Find the number of onto functions from A to B.

Soln Here $m = |A| = 7$

$n = |B| = 4$

$\therefore$ the no of onto functions from A to B is

$$P(7,4) = \sum_{k=0}^{4} (-1)^k \left({}^4C_{4-k}\right)(4-k)^7$$

$$= {}^4C_4 \times 4^7 + (-1)^1 \left({}^4C_{4-1}\right)(4-1)^7$$

$$+ (-1)^2\, {}^4C_{4-2}\,(4-2)^7 + (-1)^3\, {}^4C_{4-3}\,(4-3)^7$$

$$+ (-1)^4\, {}^4C_{4-4}\,(4-4)^7$$

$$= {}^4C_4 \times 4^7 - {}^4C_3\, 3^7 + {}^4C_2\, 2^7 - {}^4C_1\, 1^7 - 0$$

$$= 1 \times 4^7 - 4 \cdot 3^7 + \frac{4!}{2! \times 2!}\, 2^7 - 4 \cdot \frac{4!}{(4-1)!\,1!}$$

$$\frac{4!}{(4-3)!\,3!} \qquad 6 \cdot 2$$

$$= 1 \times 4^7 - 4 \cdot 3^7 + 6 \cdot 2^7 - 4$$

$$= \underline{\underline{8400}}$$

Problem

Let $A = \{1,2,3,4\}$ and $B = \{1,2,3,4,5,6\}$

(a) Find how many functions are there from A to B. How many of these are one-to-one? How many are onto?

(b) Find how many functions are there from B to A. How many of these are one-to-one? How many are onto?

Soln: $|A| = m = 4 \quad |B| = n = 6$

(a) The number of functions possible from A to B is

$n^m = 6^4 = 1296$

The number of one-to-one functions possible from A to B is $\frac{n!}{(n-m)!} = \frac{6!}{2!} = 360$.

There is no onto function from A to B.

[because $m < n$]

(b) The number of functions possible from B to A

is $m^n = 4^6 = 4096$

A^B

No one-to-one function from B to A.

$\frac{n!}{(n-m)!} = \frac{4!}{(4-6)!} \cdot (-2)!$

$\therefore$ n - m is negative

i.e. $4-6 = -$ve no

The number of onto functions from B to A is

$$P(6,4) = \sum_{K=0}^{4} (-1)^K \left({}^4C_{4-K}\right)(4-K)^6$$

$$= 4^6 - 4\times 3^6 + 6\times 2^6 - 4 = 1560$$

Problem.

Given that $P(6,4) = 1560$ and $P(7,4) = 8400$, evaluate $S(6,4)$ and $S(7,4)$

Soln

$$S(m,n) = \frac{P(m,n)}{n!} \quad \text{for } m \geq n.$$

$$\therefore \quad S(6,4) = \frac{P(6,4)}{4!} = \frac{1560}{24} = 65$$

$$S(7,4) = \frac{P(7,4)}{4!} = \frac{8400}{24} = 350$$

Problem - Evaluate $S(5,4)$ and $S(8,6)$

Soln $S(m,n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \left({}^nC_{n-k}\right)(n-k)^m$

$$\therefore \quad S(5,4) = \frac{1}{4!} \left\{ (-1)^0\, {}^4C_4 (4-0)^5 + (-1)^1\, {}^4C_3\, 3^5 \right.$$

$$\left. + (-1)^2\, {}^4C_2\, 2^5 + (-1)^3\, 4_1 \times 1^5 \right\}$$

$$= \frac{1}{4!} \left\{ 1 \cdot 4^5 - {}^4C_3\, 3^5 + {}^4C_2\, 2^5 - {}^4C_1\, 1^5 \right\}$$

$$= \frac{1}{4!} \left\{ 4^5 - 4 \times 3^5 + 6 \times 2^5 - 4 \right\}$$

$$= \frac{240}{4!} = 10$$

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and $S(8,6) = \frac{1}{6!}\sum_{K=0}^{6}(-1)^K\left({}^6C_{6-K}\right)(6-K)^8$

$$= \frac{1}{6!}\Big\{(-1)^0\ {}^6C_6\,(6-0)^8 + (-1)^1\ {}^6C_5\ 5^8 +$$

$$(-1)^2\ {}^6C_4 \times 4^8 + (-1)^3\ {}^6C_3\ 3^8 + (-1)^4\ {}^6C_2 \times 2^8$$

$$+ (-1)^5\ {}^6C_1 \times 1^8\Big\}$$

$$= \frac{1}{6!}\Big\{6^8 - (6 \times 5^8) + (15 \times 4^8) - (20 \times 3^8)$$

$$+ (15 \times 2^8) - 6$$

$$= \underline{\underline{266}}$$

Problem. Find the number of ways of distributing four distinct objects among three identical containers with some containers possibly empty.

Soln objects $m = 4$

Containers $n = 3$

$$P(m) = \sum_{i=1}^{n} S(m,i)$$

where $S(m,i) = \frac{1}{i!}\sum_{K=0}^{i}(-1)^K\left({}^iC_{i-K}\right)(i-K)^m \quad m \geq i$

$$P(4) = \sum_{i=1}^{n} S(4,i)$$

$$= S(4,1) + S(4,2) + S(4,3)$$

$$\therefore S(4,1) = \frac{1}{1!}\sum_{k=0}^{1}(-1)^k\left({}^1C_{1-k}\right)(1-k)^4$$

$$= \frac{1}{1!}\left\{(-1)^0\ {}^1C_1 . 1^4 + 1^1 . {}^1C_0\ 0^4\right\}$$

$$= 1 + 0 = \underline{\underline{1}}.$$

$$S(4,2) = \frac{1}{2!}\sum_{k=0}^{2}(-1)^k\left({}^2C_{2-k}\right)(2-k)^4$$

$$= \frac{1}{2!}\left(2^4 - 2\times 1^4\right) = 7$$

$$S(4,3) = \frac{1}{3!}\sum_{k=0}^{3}(-1)^k\left({}^3C_{3-k}\right)(3-k)^4$$

$$= \frac{1}{6}\left\{3^4 - 3\times 2^4 + 3\times 1^4\right\} = 6$$

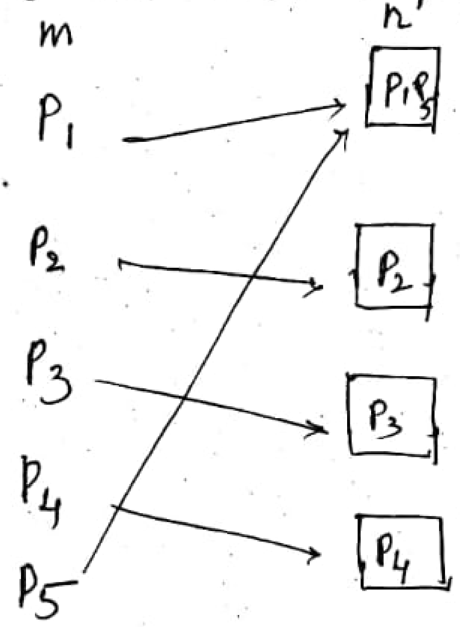
$$\therefore P(4) = 1 + 7 + 6 = \underline{\underline{14}}.$$

The Pigeonhole Problem

If m pigeons occupy n pigeonholes and if $m>n$, then two or more pigeons occupy the same pigeonhole.

(or)

If m pigeons occupy n pigeon holes, where $m>n$, then at least one pigeonhole must contain two or more pigeons in it.



This stmt is known as Pigeon hole principle.

i.e. If m pigeons occupy n pigeonholes, then at least one pigeonhole must contain $(p+1)$ or more pigeons, where $p = \lfloor (m-1)/n \rfloor$

Proof :- by Contradiction method.

Assume that the conclusion part of the Principle is not true. Then, no pigeonhole contains $(p+1)$ or more pigeons.

This means that every pigeonhole contains p or less number of pigeons.

Then,

Total number of pigeons $\le np = n \times \lfloor (m-1)/n \rfloor$

$$\cancel{n}\left(\frac{m-1}{\cancel{n}}\right)$$

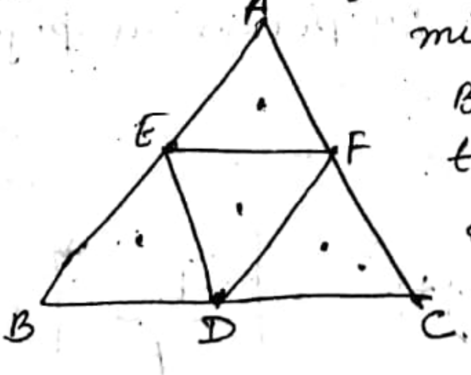
$$= m-1$$

This contradicts, because the total number of pigeon's is m. Hence our assumption is wrong, and the principle is true.

Problem

ABC is an equilateral triangle whose sides are of length 1 cm each. If we select 5 points inside the triangle, prove that at least two of these points are such that the distance between them is less than $1/2$ cm.

Soln Consider the triangle DEF formed by the mid points of the sides BC, CA & AB of the triangle as shown in fig.



Treating each of these four portions as a pigeonhole and five points chosen inside the triangle as pigeons, atleast one portion must contain 2 or more points. The distance b/n such points is less than 1/2 cm.

Eg If 5 colours are used to paint 26 doors, P.T at least 6 doors will have the same colour.

Soln Treating 26 doors as pigeons and 5 colours as pigeonholes,

$m = 26$
$n = 5$

$$\therefore p = \left\lfloor \frac{26-1}{5} \right\rfloor + 1 = \underline{6}.$$

$$5 + 1 = \underline{6}.$$

Eg Prove that in any set of 29 persons at least five persons must have been born on the same day of the week.

Soln Treating the seven days of a week as 7 pigeonholes and 29 persons as pigeons.

$m = 29$
$n = 7$

$$\text{ie } p = \left(\frac{m-1}{n}\right) + 1 = \left\lfloor \frac{29-1}{7} \right\rfloor + 1 =$$

$$4 + 1 = 5$$

ie at least 5 of any 29 persons must have been born on the same day of the week.

Eg How many persons must be chosen in order that at least five of them will have birth days in the same calendar month?

Soln Persons = m = ?

$n = 12$

Since the number of months over which the birthdays are distributed is 12,
The least no of persons who have BD in the same month is 5

ie $\left\lfloor \frac{m-1}{12} \right\rfloor + 1 = 5$

$m - 1 + 12 = 12 \times 5$

$m = \underline{\underline{49}}$

ie no of persons is $\underline{\underline{49}}$.

Eg. Find the least number of ways of choosing three different numbers from 1 to 10 so that all choices have the same sum.

Soln

From the numbers from 1 to 10, we can choose three different numbers in $({}^{10}C_3) = 120$ ways.

Smallest possible sum $= 1+2+3 = 6$ $\quad \frac{1 + 27}{27C_6}$

largest possible sum $= 8+9+10 = 27$

Including both (6 & 27) it is 22 number.

Now $m = 120$

$n = 22$

$$\therefore \left\lfloor \frac{120-1}{22} \right\rfloor + 1 = \lfloor 6.4 \rfloor \approx 6$$