Relations and Functions

→ In this, we extend the set theory Concepts of relation and function.

→ We shall study functions from a set-Theoretic approach that includes finite functions and introduce new counting ideas in the study.

Cartesian Products and Relations

Defn: For sets A, B the cartesian product, or cross product, of A and B is denoted by $A \times B$ and equals $\{(a, b) \mid a \in A, b \in B\}$

We say that the elements of $A \times B$ are ordered pairs. For $(a, b), (c, d) \in A \times B$, we have $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

If A, B are finite, it follows from the rule of product that $|A \times B| = |A| \cdot |B|$

Although, we generally will not have ~~|A×B|~~

$A \times B = B \times A$, we will have $|A \times B| = |B \times A|$

Here $A \subseteq U_1$ and $B \subseteq U_2$ and we may find that the universes are different. i.e $U_1 \neq U_2$. Also, even if $A, B \subseteq U$, it is not necessary that $A \times B \subseteq U$, So unlike the cases for union i.e intersections, here $P(U)$ is not necessarily closed under this binary Operation

[In General 'n' number of sets]

Example Let $A = \{2, 3, 4\}$

$B = \{4, 5\}$

Then a) $A \times B = \{(2,4)(2,5)(3,4)(3,5)(4,4)(4,5)\}$

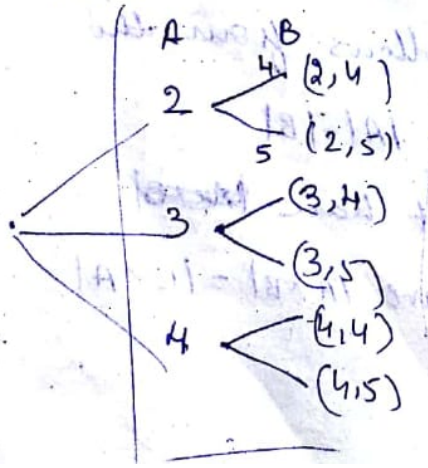
b) $B \times A = \{(4,2)(4,3)(4,4)(5,2)(5,3)(5,4)\}$

c) $B^2 = B \times B = \{(4,4), (4,5), (5,4), (5,5)\}$

d) $B^3 = B \times B \times B = \{(a,b,c) \mid a,b,c \in B\}$

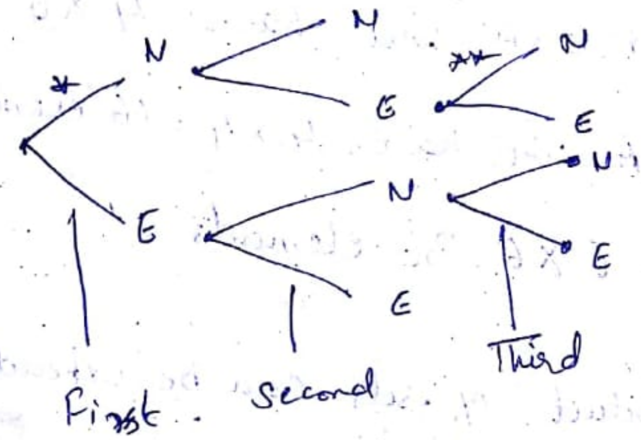
Eg. The Set $R \times R = \{(x,y) \mid x, y \in R\}$ is recognized as the real plane of Coordinate geometry and two dimensional Calculus.

$A \times B$



Eg At the Wimbledon Tennis Championships, women play at most three sets in a match. The winner is the first to win two sets. If we let N & E denote the two players, the tree diagram indicates the six ways in which this match can be won.

For eg. the starred line segment indicates that player E won the first set. The double starred edge indicates that player N has won the match by winning the first and third sets.



$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

$B \times A = \{(b, a) \mid b \in B \text{ and } a \in A\}$

$(a, b) \neq (b, a)$

$A = \{1, 0, -1\}$ and $B = \{2, 3\}$ then

$A \times B = \{(1,2) (1,3) (0,2) (0,3) (-1,2) (-1,3)\}$

$B \times A = \{(2,1) (2,0) (2-1) (3,1) (3,0) (3,-1)\}$

$A \times B \neq B \times A$

Eg For any set A Prove that

$$A \times \phi = \phi \times A = \phi$$

Suppose $A \times \phi \neq \phi$

Then, $A \times \phi$ has at least one element

(a,b) in it such that $a \in A$ & $b \in \phi$.

Now $b \in \phi$ means that ϕ is not the null set.

This is contradiction.

$\therefore A \times \phi = \phi$ Similarly $\phi \times A = \phi$.

Eg

For any non-empty sets A, B, C prove the following results:

(1) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(2) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

(3) $A \times (B \cup C) = (A \times B) \cap (A \times C)$

(4) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

(5) $A \times (B - C) = (A \times B) - (A \times C)$.

Soln: (1) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

$x, y \in A \times (B \cup C)$

$\Leftrightarrow \{\langle x,y \rangle \mid x \in A \wedge y \in B \cup C\}$

$\Leftrightarrow \{\langle x,y \rangle \mid x \in A \wedge (y \in B \vee y \in C)\}$

$\Leftrightarrow \{\langle x,y\rangle \mid (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)\}$

$\Leftrightarrow \{\langle x,y\rangle \mid x \in A \wedge y \in B\} \vee \{\langle x,y\rangle \mid x \in A \wedge y \in B\}$

$\Rightarrow \{\langle x,y\rangle \mid x \in A \wedge y \in C\}$

$\Rightarrow \{\langle x,y\rangle \mid x \in A \wedge y \in B\} \vee$ Distributes.

$\{\langle x,y\rangle\} x \in A \; y \in C\}$

$\Rightarrow \langle x,y\rangle \in \{A \times B \vee \langle x,y\rangle \in A \times C\}$

$\Rightarrow \langle x,y\rangle \in (A \times B) \cup (A \times C)$

OR - $\cup$
AND - $\times$

(1) $(x,y) \in \{A \times (B \cup C)\} \Leftrightarrow x \in A$ and $y \in (B \cup C)$

$\Leftrightarrow x \in A$ and $\{y \in B$ OR $y \in C\}$

$\Leftrightarrow \{x \in A$ and $y \in B\}$ OR $\{x \in A$ and $y \in C\}$

$\Leftrightarrow (x,y) \in A \times B$ OR $(x,y) \in A \times C$

$\Leftrightarrow (x,y) \in \{(A \times B) \cup (A \times C)\}$

(2) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

$\Leftrightarrow x \in (A \cup B)$ And $y \in C$

$\Leftrightarrow \{x \in A$ OR $x \in B\}$ AND $y \in C$

$\Leftrightarrow \{x \in A$ AND $y \in C\}$ OR $\{x \in B$ And $y \in C\}$

$\Leftrightarrow (x,y) \in A \times C$ or $(x,y) \in B \times C$

$\Rightarrow (x,y) \in \{(A \times C) \cup (B \times C)\}$

Continue ----

Relations Continue

(3) $(x,y) \in A\times(B\cap C) \iff x\in A$ And $y\in B\cap C$

$\iff \{x\in A$ AND $(y\in B$ And $y\in C)\}$

$\iff (x\in A$ AND $y\in B)$ And $(x\in A$ and $y\in B)$

$\iff (x,y)\in A\times B$ AND $(x,y)\in A\times C)$

$\Rightarrow (x,y)\in A\times B$ AND $(x,y)\in A\times C$

$\Rightarrow (x,y)\in (A\times B)\cap(A\times C)$

(4) $(x,y)\in\{(A\cap B)\times C\} \iff x\in A\cap B$ and $y\in C$

$\iff \{x\in A$ and $x\in B\}$ and $y\in C$

$\iff \{x\in A$ and $y\in C\}$ and $\{x\in B$ and $y\in C\}$

$\iff (x,y)\in A\times C$ and $(x,y)\in B\times C\}$

$\iff \{(x,y)\in (A\times C)\cap(B\times C)\}$

$\Rightarrow$ ~~(x,y)~~

(5) $(x,y)\in\{A\times(B-C)\}$

$\Rightarrow x\in A$ and $y\in(B-C)$

$\Rightarrow x\in A$ and $y\in B$ and $y\notin C$

$\Rightarrow x\in A$ and $y\in B$ and $x\in A$ and $y\notin C$

$\Rightarrow (x,y)\in A\times B$ and $(x,y)\notin(A\times C)$

$\Rightarrow (x,y)\in\{A\times B\}-\{(x,y)\notin(A\times C)\}$

$\Rightarrow (x,y)\in(A\times B)-(A\times C)$

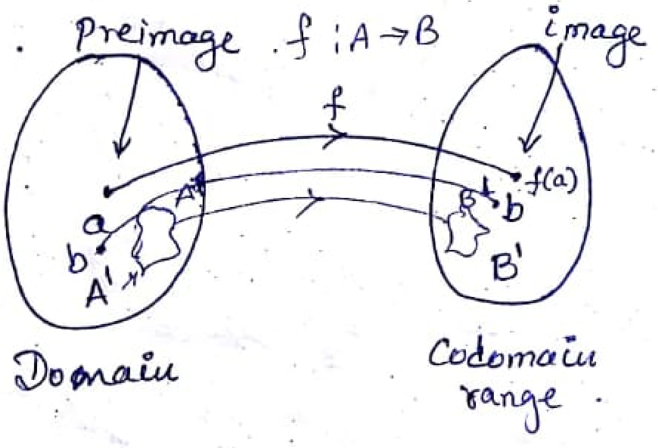
Functions

Let A & B be two non-empty sets.
Then a function f from A to B is a relation from A to B such that for each a in A there is a unique b in B such that $(a,b) \in f$.
Then we ~~write~~ $b = f(a)$.

Here, b is called <u>image</u> of a
& a is called <u>preimage</u> of b under f.

The element a is also called an argument of the function f,
and $b = f(a)$ is then called the value of the function f for the argument a

A function from A to B is denoted by $f: A \to B$.



Subsets of A' mapped to subset of B

$f(a)$ is the image of a
$f(b)$ is the image of b,
$f(B')$ is the image of A' under f

f:

The subset of B consisting of the images of all elements of A under f is called the <u>range</u> of f and is denoted by $f(A)$

Eg image $a \to 1$, $b \to 2$, $c \to 3$ ~~imaging~~ set $(1, 2)$ is the image of (a, b, c)

① Every a in A belongs to some pair $(a, b) \in f$, and if $(a, b_1) \in f$ and $(a, b_2) \in f$, then $b_1 = b_2$. This means that every element of A has an image in B (under f) and if an element a of A has two images in B, then the two images cannot be different.

② An element $b \in B$ need not have a preimage in A, under f

c is not having preimage

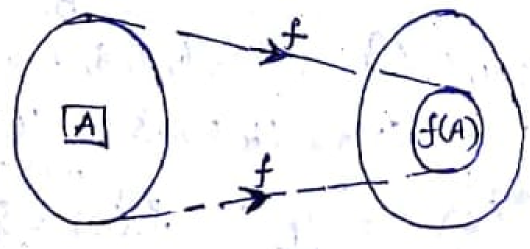
③ If an element $b \in B$ has a preimage $a \in A$ under f, the preimage need not be unique. In other words, two different elements of A can have the same image in B, under f.

④ The statement $(a, b) \in f$, afb and $b = f(a)$ are equivalent. ~~in the sense that~~

⑤ If g is a function from A to B (denoted by $g: A \to B$), then $f = g$ if and only if $f(a) = g(a)$ for every $a \in A$.

⑥ The range of $f: A \to B$ is given by

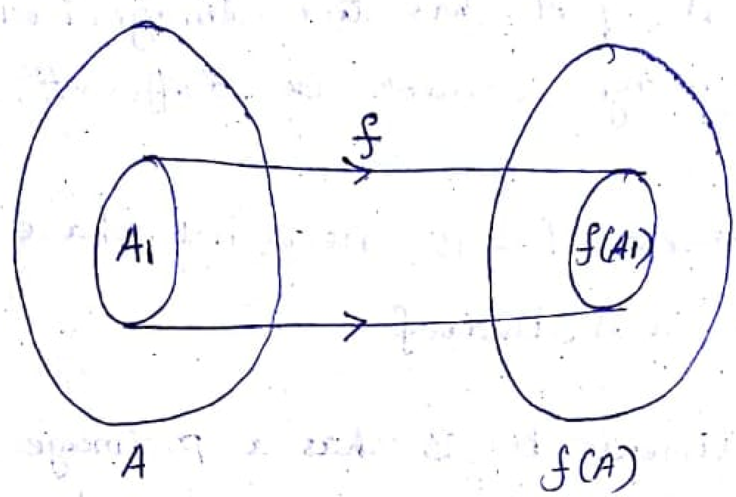
$$f(A) = \{f(x) \mid x \in A\}$$ So $f(A)$ is a subset of B.



(7) For $f: A \to B$, if $A_1 \subseteq A$ and $f(A_1)$ is defined by $f(A_1) = \{f(x) \mid x \in A_1\}$.

then $f(A_1) \subseteq f(A)$

[Here $f(A_1)$ is called the image of A_1 under f]



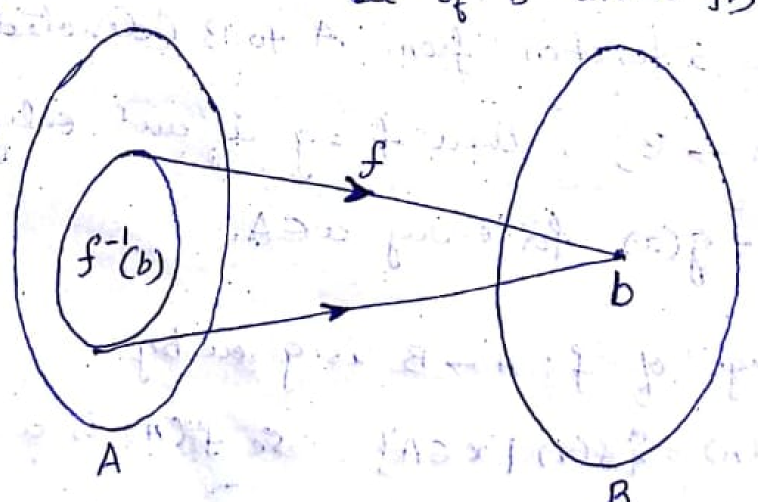
Eg: $A_1 : \{2, 3\}$

~~image~~ $f : f(x) = x^2 + 1$ image : $(5, 10)$

(8) For $f: A \to B$, if $b \in B$ and $f^{-1}(b)$ is defined by

$$f^{-1}(b) = \{x \in A \mid f(x) = b\}$$

then $f^{-1}(b) \subseteq A$ (Here $f^{-1}(b)$ is called the preimage set of b under f.)

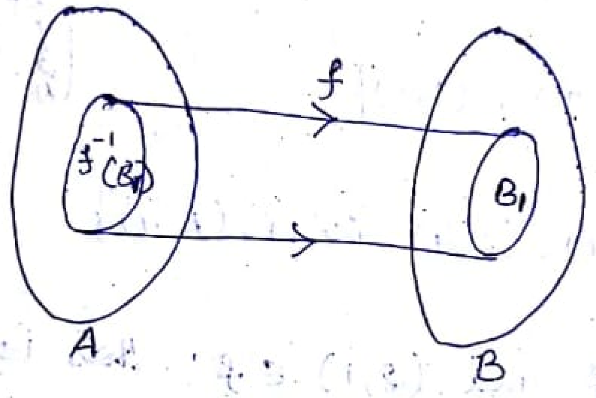


Eg $A = \{1, 2, 3, 4, 5, 6\}$ $B = \{6, 7, 8, 9, 10\}$

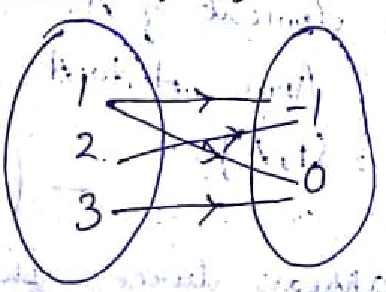
$f : \{(1,7)\ (2,7)\ (3,8)\ (4,6)\ (5,9)\ (6,9)\}$

determine $f^{-1}(6) = \underline{4}$ ($b = 6$) $f^{-1}(b) = \{1, 2, 3\}$ ($b = \{7, 8\}$)

(9) For $f: A \to B$, if $B_1 \subseteq B$ and $f^{-1}(B_1)$ is defined by $f^{-1}(B_1) = \{x \in A \mid f(x) \in B_1\}$.
then $f^{-1}(B_1) \subseteq A$. (Here $f^{-1}(B_1)$ is called the preimage of B_1 under f)

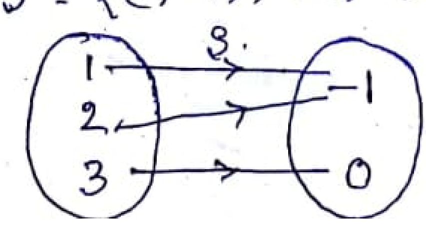


Eg1. Let $A = \{1, 2, 3\}$ and $B = \{-1, 0\}$ and R be a relation from A to B defined by
$R = \{(1,-1)\ (1,0)\ (2,-1)\ (3,0)\}$
Is R a function from A to B?



Since 1 is mapped to -1 & 0 it is not a function.

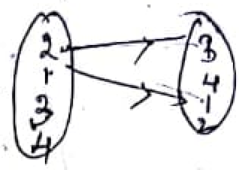
Eg2. Let $A = \{1, 2, 3\}$ and $B = \{-1, 0\}$ as in example-1, and S be a relation from A to B defined by $S = \{(1,-1), (2,-1)\ (3,0)\}$ Is S a function



Each element of A is related to a unique element of B.
$\therefore$ S is a function.

Eg3. Let $A = \{1, 2, 3, 4\}$ Determine whether or not the following relations on A are functions.

(1) $f = \{(2,3), (1,4), (2,1) (3,2) (4,4)\}$ not allowed.

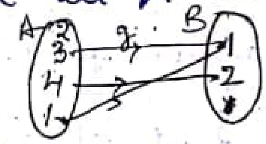


(2) $g = \{(3,1), (4,2), (1,1)\}$

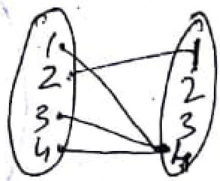
(3) $h = \{(2,1), (3,4), (1,4), (2,1), (4,4)\}$

Soln (1) $(2,3) \in f$ and $(2,1) \in f$; ~~this~~ ie 2 elements is related to two different elements 3 and 1 under f, Therefore f is not a function.

(2) 2 is not related to any element in A under g. Therefore, g is not a function.



(3) Under h, every element of A is related to a related to a unique element of A.
$\therefore$ h is a function from A to A
its range is $h(A) = \{1, 4\}$



In h, the term $(2,1)$ appears twice. This has no special significance; in a set, an element can appear any number of times.

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Range of a function.

Find the range of the real valued functions $(x^2-4)/x-2$ and $\log|4-x^2|$?

$$\frac{x^2-4}{x-2} = \frac{f(x)}{g(x)}, \ g(x) \neq 0$$

$x-2 \neq 0$

$x \neq 2$

Domain $= R - \{2\}$

$$f(x) = \frac{x^2-4}{x-2}$$

$$= \frac{x^2-2^2}{x-2} = \frac{(\cancel{x-2})(x+2)}{(\cancel{x-2})}$$

$f(x) = (x+2)$

$y = (x+2)$

If $x \neq 2$ then

$y = 2+2$

$y \neq 4$

$\therefore$ Range $= R - \{4\}$

$\rightarrow \log|4-x^2|, \ f(x), \ f(x) \neq 0$

$4 - x^2 \neq 0$

$x^2 \neq 4$

$x \neq \pm 2$

Domain $= R - \{-2, 2\}$

Let $y = f(x)$

$y = \log_e|4-x^2|$

$e^y = 4 - x^2$

$\log_e^e = 1$

$e^y \neq 0$

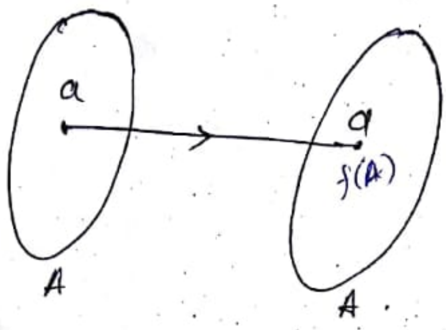
$x \in R \quad e^y > 0$ Range $= R$

Types of functions

Identity function (I_A)

A function $f: A \to A$ such that $f(a) = a$ for every $a \in A$ is called the identity function or identity mapping on A.

ie $f(A) = A$

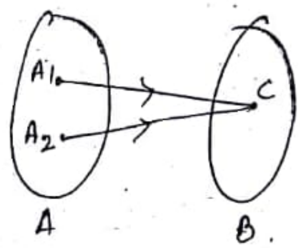


Constant function

A function $f: A \to B$ such that $f(a) = c$ for every $a \in A$, where c is a fixed element of B is called a constant function

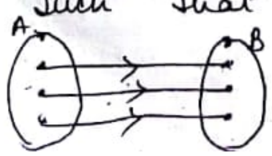
ie $f(A_1) = c$

$f(A_2) = c$



Onto function (Surjective function).

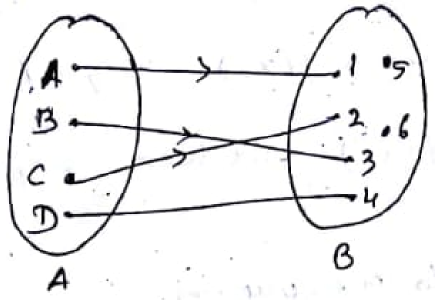
A function $f: A \to B$ is said to be an onto function if for every element b of B there is an element a of A such that $f(a) = b$.



One to One function (injective function)

A function $f : A \to B$ is said to be a one to one function if different elements of A have different images in B under f. i.e

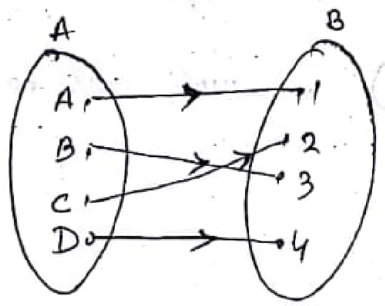
Note: function for which each element of the set A is mapped to a different element of the set B are said to be one-to-one



$a_1, a_2 \in A$ with $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$

One to one Correspondence

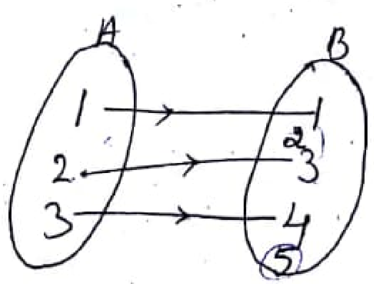
A function which is both one-to-one and onto is called a one-to-one Correspondence or bijective function.



Eg: Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$. Find ~~the~~ whether the following functions from A to B are
(a) one-to-one (b) Onto.

(i) $f : \{(1,1), (2,3), (3,4)\}$ ii) $g : \{(1,1), (2,2), (3,3)$

Soln (i)



Every element of A has a Unique element in B and no two elements of A have the same image in B.
$\therefore$ f is one-to-one.

Note: 2, 5 of B has no preimage in A.
$\therefore$ it is not onto.

Eg: The function $f : R \to R$ and $g : R \to R$ are defined by $f(x) = 3x+7$ for all $x \in R$ and $g(x) = x(x^3-1)$ for all $x \in R$. Verify that 'f' is one-to-one but g is not.

Soln: For any $x_1, x_2 \in R$, we have

$$f(x_1) = 3x_1 + 7 \quad f(x_2) = 3x_2 + 7$$

ie if $f(x_1) = f(x_2)$ we have $3x_1 + 7 = 3x_2 + 7$

$$\therefore x_1 = x_2.$$

$\therefore f$ is an one-to-one function.

~~Next~~, ~~we note that~~

$$g(x) = x(x^3-1)$$

$$g(0) = 0(0-1) = 0 \quad x_1 = 0$$

$$g(1) = 1(1-1) = 0 \quad x_2 = 1$$

$\therefore g(x_1) \neq g(x_2)$ $\therefore$ not one to one

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~~Pigeonhole~~ ~~Principle~~ Properties of function

Theorem 1: Let $X \to Y$ be a function and A and B be arbitrary nonempty subsets of X. Then:

(1) If $A \subseteq B$, then $f(A) \subseteq f(B)$

(2) $f(A \cup B) = f(A) \cup f(B)$

(3) $f(A \cap B) \subseteq f(A) \cap f(B)$, and the equality holds if f is one to one.

Proof: Let $y \in Y$ then.

(1) $y \in f(A) \Rightarrow y = f(x)$ for some $x \in A$

$\Rightarrow y = f(x)$ for some $x \in B$, because $A \subseteq B$

$\Rightarrow y \in f(B)$

(2) $y \in f(A \cup B) \Rightarrow y = f(x)$ for some $x \in A \cup B$.

$\Rightarrow y = f(x)$ for $x \in A$ or $x \in B$

$\Rightarrow y \in f(A)$ or $y \in f(B)$

$\Rightarrow y \in \{f(A) \cup f(B)\}$

$\therefore f(A \cup B) \subseteq f(A) \cup f(B)$ ——— (1)

Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$

$\therefore f(A) \subseteq f(A \cup B)$ and $f(B) \subseteq f(A \cup B)$

$\therefore f(A) \cup f(B) \subseteq f(A \cup B)$ ——— (2)

Result (1) & (2) $\Rightarrow$ that $f(A) \cup f(B) = f(A \cup B)$.

(3) $y \in f(A \cap B) \Rightarrow y = f(x)$ for some $x \in A \cap B$

$\Rightarrow y = f(x)$ for $x \in A$ and $x \in B$

$\Rightarrow y = f(x)$ for $x \in A$ and $y = f(x)$ for $x \in B$

$\Rightarrow y \in f(A)$ and $y \in f(B)$

$\Rightarrow y \in \{f(A) \cap f(B)\}$

$\therefore f(A \cap B) \subseteq f(A) \cap f(B)$

$y \in \{f(A) \cap f(B)\} \Rightarrow y \in f(A)$ and $y \in f(B)$

$\Rightarrow y = f(x_1)$ for some $x_1 \in A$, and $y = f(x_2)$ for some $x_2 \in B$.

$\Rightarrow y = f(x_1) = f(x_2)$

$\Rightarrow \cancel{x_1 \neq x_2}$ $x_1 = x_2$ if f is one-to-one

$\Rightarrow y = f(x_1)$ for some $\underbrace{x_1 \in A \text{ and } x_1 \in B}$

$\Rightarrow y = f(x_1)$ for some $x_1 \in A \cap B$

$\Rightarrow y \in f(A \cap B)$

Thus, if f is one-to-one then $\{f(A) \cap f(B)\} \subseteq f(A \cap B)$

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Theorem 2: Let A and B be finite sets and f be a function from A to B. Then the following are true:

(i.) If f is one-to-one, then $|A| \leq |B|$

(2) If f is onto, then $|B| \leq |A|$

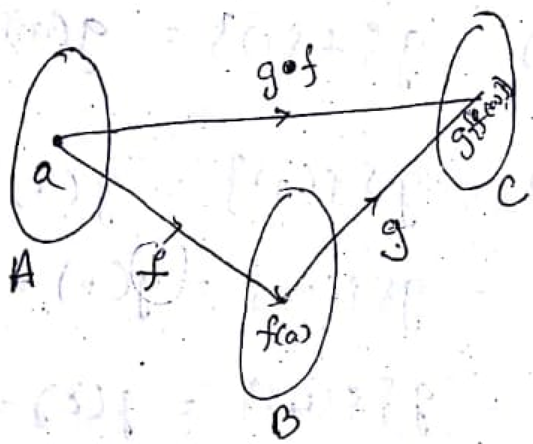
(3) If f is a one-to-one correspondence, then $|A| = |B|$

(4) If $|A| > |B|$, then at least two different elements of A have the same image under f.

Theorem 3: Suppose A and B are finite sets having the same number of elements and f is a function from A to B. Then f is one-to-one if and only if f is onto.

Composition of functions (∘)

Consider three non-empty sets A, B, C and the functions $f: A \to B$ and $g: B \to C$. The composition (or product) of these two functions is defined as the function $g \circ f : A \to C$ with $(g \circ f)(a) = g\{f\{a\}\}$ for all $a \in A$.



$f: A \to B$
$g: B \to C$.
Same

For a function $f: A \to A$

$f^0 : I_A$

$f(x) = x$

$f^1 = f$

$f^2 = f \circ f$

$f^3 = f \circ f \circ f$

$f^{n+1} = f^n \cdot f$

$f: A \to A$

$f \circ f \to f^2$

$f \circ f^2 \to f^3$

$n \geq 2$

$f \cdot f^{n-1} = f^n$

is a recursive function

Problem Let $A = \{1, 2, 3, 4\}$ $B = \{a, b, c\}$ and $C = \{w, x, y, z\}$ with $f : A \to B$ and $g : B \to C$ given by

$f : \{(1,a)\ (2,a)\ (3,b)\ (4,c)\}$

and $g : \{(a,x), (b,y), (c,z)\}$ Find $g \circ f$

Soln

By the defn

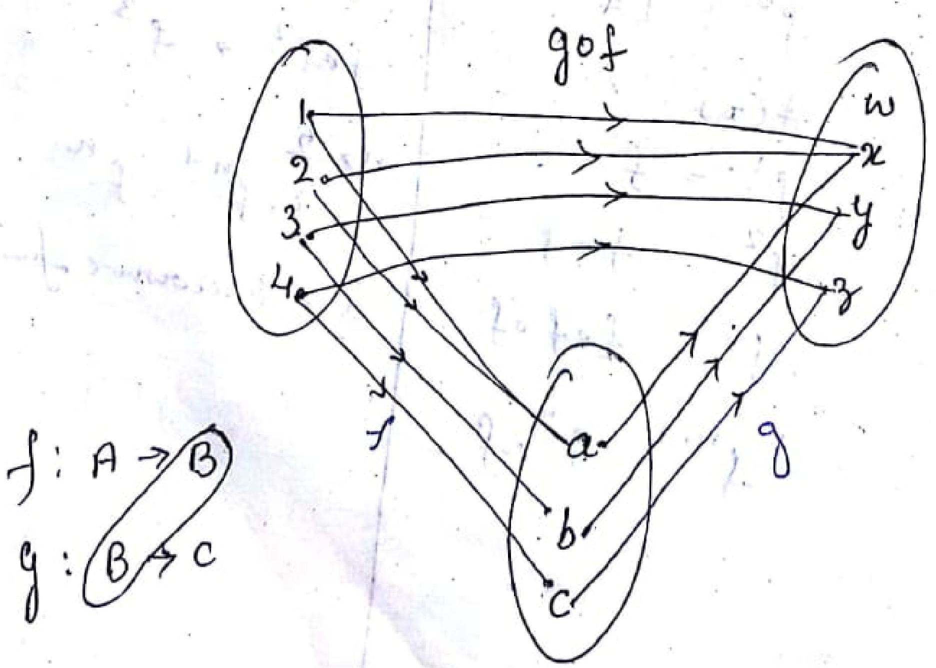
$$g \circ f(1) = g\{f(1)\} = g(a) = x.$$

$$g \circ f(2) = g\{f(2)\} = g(a) = x$$

$$g \circ f(3) = g\{f(3)\} = g(b) = y$$

$$g \circ f(4) = g\{f(4)\} = g(c) = z$$

Thus $g \circ f = \{(1,x), (2,x)\ (3,y)\ (4,z)\}$.



Eg2 Consider the function f and g defined by $f(x) = x^3$ and $g(x) = x^2+1$, $\forall x \in R$

Find gof, fog, f^2 and g^2

Soln Here both f and g are defined on R.

$\therefore$ All of the functions $g\,of$, fog, $f^2 = fof$ and $g^2 = gog$ are defined on R., we find that

$$(gof)(x) = g\{f(x)\} = g(x^3) = (x^3)^2+1$$
$$= x^6+1.$$

$$(fog)(x) = f\{g(x)\} = f(x^2+1) = (x^2+1)^3$$

$$f^2(x) = fof(x) = f\{f(x)\} = f(x^3) = (x^3)^3 = x^9$$
$$g^2(x) = gog(x) = g\{g(x)\} = g(x^2+1) = (x^2+1)^2+1$$

The above expressions define the functions gof, fog, f^2 and g^2 respectively.

$\therefore$ Composition of functions is not Commutative.

Problem Let I_A and I_B denote the Identity funs on sets A & B respectively. For any function $f: A \to B$ P.T $foI_A = f = I_B of$.

Soln $a \in A$,

$$(f o I_A)\,a = f\{I_A(a)\} = f(a)$$
$$(I_B of)\,a = I_B\{f(a)\} = f(a)$$

$$\boxed{\therefore f o I_A = f \quad \text{and} \quad I_B o f = f.}$$

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Eg 4 Let f and g be functions from R to R defined by $f(x) = ax + b$ and $g(x) = 1 - x + x^2$

If $(g \circ f)(x) = 9x^2 - 9x + 3$, determine a, b.

Soln

$$9x^2 - 9x + 3 = (g \circ f)(x)$$
$$= g\{f(x)\} = g\{ax + b\}$$
$$= 1 - (ax + b) + (ax + b)^2$$
$$= 1 - a^2x^2 + 2ax \cdot b - (ax + b) + b^2$$
$$= a^2x^2 + (2ab - a)x + (1 - b + b^2)$$

Comparing the ~~co-efficients~~ Corresponding co-efficients, we get

$9 = a^2$, $9 = a - 2ab$ or $-9 = (2ab - a)$, $3 = 1 - b + b^2$

$a^2 = 9$

$a = \pm 3$

$a = 3$

$9 = a - 2ab$

$9 = 3 - 2 \cdot 3 \cdot b$

$9 = 3 - 6b$

$6 = -6b$

$b = -1$

$2 = -b + b^2$

$2 = -(-1) + (-1)$

$2 = 1 + 1$

$a = -3$

$b = 2$

Theorem 1 :- Let $f : A \to B$ and $g : B \to C$ be any two functions. Then the following are true

(1) If f and g are one-to-one, so is $g \circ f$

(2) If $g \cdot o f$ is one-to-one, then f is one-to-one

(3) If f and g are onto, so is $g \circ f$.

(4) If $g \circ f$ is onto, then g is onto.

Proof :- $g \circ f : A \to C$ $\begin{cases} f : A \to B \\ g : B \to C \end{cases}$

(1) Take any $a_1, a_2 \in A$. we find that

$(g \circ f)(a_1) = (g \circ f)(a_2)$

$\Rightarrow g\{f(a_1)\} = g\{f(a_2)\}$

$\Rightarrow f(a_1) = f(a_2)$ [$\because$ g is also one-to-one]

$\Rightarrow a_1 = a_2$ [because f is one-to-one]

$\therefore g \circ f$ is one-to-one.

(2) Take any $a_1, a_2 \in A$ Then, $f(a_1)\, f(a_2) \in B$

and $f(a_1) = f(a_2) = g\{f(a_1)\} = g(f(a_2))$,

[because g is a function from B]

$\Longrightarrow g \circ f(a_1) = g \circ f(a_2)$

$\Rightarrow a_1 = a_2$ [because $g \circ f$ is one to one]

This shows that f : is one-to one.

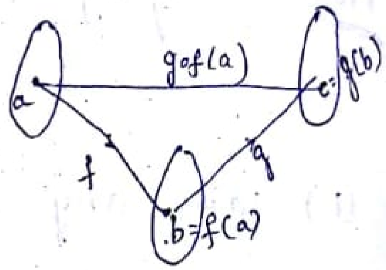
(3) Take any $c \in C$. Since g is onto, there is some $b \in B$ ST $g(b) = c$. Since $b \in B$ and f is onto, there is some $a \in A$ ST $f(a) = b$.

Consequently

$$(g \circ f)(a) = g(f(a)) = g(b) = c$$

Thus, for any $c \in C$, there is some $a \in A$ such that $f(a) = b$ consequently.

$$(g \circ f)(a) = g(f(a))$$
$$= g(b)$$
$$= c.$$



Thus, for any $c \in C$ there is some $a \in A$ such that $(g \circ f)(a) = c$.

$\therefore g \circ f$ is onto.

(4) Take any $c \in C$. Since $g \circ f : A \to C$ is onto,

There is some $a \in A$ ST $g(f(a)) = c$

Since $f(a) \in B$, This means that, given any $c \in C$, there is an element $f(a)$ in B such that $g\{f(a)\} = c$

$\therefore g$ is onto

Theorem 2

Let $f : A \to B$, $g : B \to C$ and $h : C \to D$ be three functions. Then

$$(h \circ g) \circ f = h \circ (g \circ f)$$

Proof: We first note that both $(h \circ g) \circ f$ & $h \circ (g \circ f)$ are functions from A to D.

For any $x \in A$, we have

$$[(h \circ g) \circ f]\, x = (h \circ g)[f(x)]$$
$$= (h \circ g)\, y. \qquad y = f(x).$$
$$= h[g(y)]$$
$$= h(z) \qquad \text{where } z = g(y) \quad \text{--- (1)}$$

and

$$[h \circ (g \circ f)\, x] = h[(g \circ f)(x)]$$
$$= h[g\{f(x)\}]$$
$$= h[g(y)]$$
$$= h(z) \quad \text{--- (2)}$$

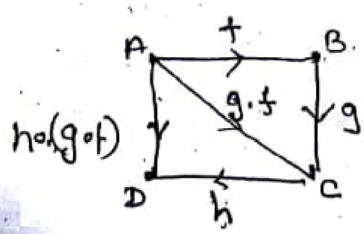
Result (i) and (ii) S.T

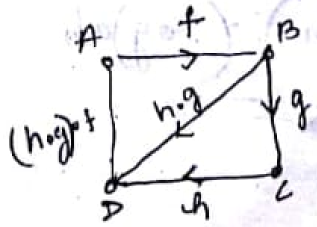
$$[(h \circ g) \circ f]\, x = [h \circ (g \circ f)](x) \quad \text{for every } x \in A$$

$\therefore (h \circ g) \circ f = h \circ (g \circ f)$; and the theorem is proved.

$\therefore$ Composition of functions is associative.

$f : A \to B \quad g : B \to C \quad h : C \to D.$





Let f, g, h be functions from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $f(x) = x-1$, $g(x) = 3x$.

$$h(x) = \begin{cases} 0 & \text{if } x \text{ is even} \\ 1 & \text{if } x \text{ is odd} \end{cases}$$

Determine $f\circ(g\circ h)(x)$ and $((f\circ g)\circ h)(x)$ and verify that $f\circ(g\circ h) = (f\circ g)\circ h$

Soln

$$(g\circ h)(x) = g\{h(x)\} = 3h(x)$$

$$\therefore f\circ(g\circ h)(x) = f\{(g\circ h)(x)\}$$

$$= f\{3h(x)\} = 3h(x) - 1$$

$$= \begin{cases} -1 & \text{if } x \text{ is even} \\ 2 & \text{if } x \text{ is odd.} \end{cases}$$

On the other hand, $(f\circ g)(x) = f\{g(x)\}$

$$= g(x) - 1$$

$$= 3x - 1$$

$$\therefore [(f\circ g)\circ h](x) = (f\circ g)\{hx\}$$

$$= 3x - 1$$

$$= 3h(x) - 1$$

$$= \begin{cases} -1 & \text{if } x \text{ is even} \\ 2 & \text{if } x \text{ is odd} \end{cases}$$

$$\therefore f\circ(g\circ h) = (f\circ g)\circ h$$

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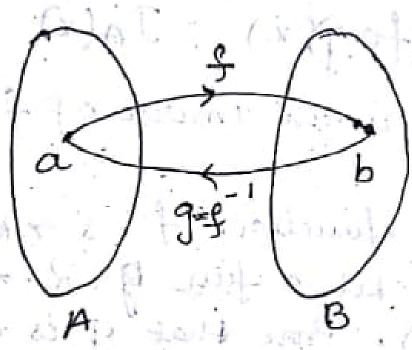
Invertible functions

A function $f: A \to B$ is said to be invertible if there exists a function $g: B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$ where I_A is the identity function on A and I_B is the identity function on B.

then g, is called an inverse of f & we write

$g = f^{-1}$

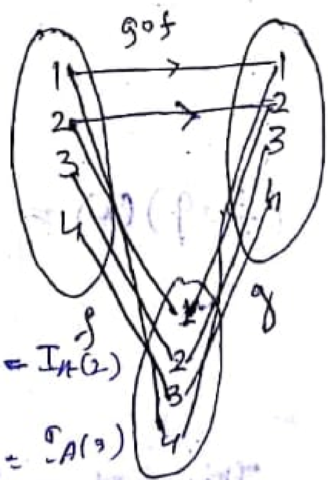


Eg - let $A = \{1, 2, 3, 4\}$ and f & g be functions from A to A given by

$f : \{(1,4)\ (2,1)\ (3,2)\ (4,3)\}$ $g : \{(1,2), (2,3), (3,4), (4,1)\}$

P.T f and g are inverses of each other

soln



$g \circ f(1) = g\{f(1)\}$

$= g(4)$

$= 1 = I_A(1)$

$g \circ f(2) = g\{f(2)\} = g(1) = 2 = I_A(2)$

$g \circ f(3) = g\{f(3)\} = g(2) = 3 = I_A(3)$

$g \circ f(4) = g\{f(4)\} = g(3) = 4 = I_A(4)$

$f \circ g\ (1) = f\{g(1)\} = f(2) = 1 = I_A(1)$

$f \circ g(2) = f\{g(2)\} = f(3) = 2$

$(f \circ g)(3) = f\{g(3)\} = f(4) = 3$

$(f \circ g)(4) = f\{g(4)\} = f(1) = 4$

$\because \forall x \in A,\ (g \circ f)(x) = I_A(x)$

and $(f \circ g)(x) = I_A(x)$.

$\therefore$ g is an inverse of f, & f is an inverse of g.

Eg2: Consider the function $f: R \to R$ defined by $f(x) = 2x + 5$, Let a fun $g: R \to R$ be defined by $g(x) = \frac{1}{2}(x-5)$. Prove that g is an inverse of f.

$\frac{1}{2}x - 5$

Soln: Let $x \in R$

$(g \circ f)(x) = g\{f(x)\} = g(2x+5)$

$= 2 \cdot \frac{1}{2}x - 5 + 5$

$= \frac{1}{2}(2x + \cancel{5} - \cancel{5})$

$= \frac{1}{\cancel{2}}(\cancel{2}x)$ $\quad \frac{1}{2}2x - 5 +$

$= x$

$= I_R(x)$

$(f \circ g)(x) = f\{g(x)\} = f\left(\frac{1}{2}(x-5)\right) = 2\{\frac{1}{2}(x-5)\} + 5$

$= \frac{1}{2}(2x \cancel{+5} \cancel{-5}) = 2\frac{1}{2}x - \frac{1}{2}5 +$

$= \frac{1}{\cancel{2}} \cdot \cancel{2}x = x = I_R(x)$

There show that $I_R(x) = I_R(x)$

$\therefore f$ is an inverse of g.

Theorem 1: If a function $f: A \to B$ is invertible then it has a unique inverse. Further, if $f(a) = b$ then $f^{-1}(b) = a$

Proof: Suppose $f: A \to B$ is invertible and it has g and h as inverses. Then g and h are functions from B to A such that

$$g \circ f = I_A \quad h \circ f = I_A$$

$$f \circ g = I_B \quad f \circ h = I_B$$

then we find that

$$h = h \circ I_B$$

$$= h \circ (f \circ g)$$

$$= (h \circ f) \circ g$$

$$= I_A \circ g$$

$$= g$$

This proves that h & g are not different.

Thus f has a unique inverse (when it is invertible)

Now, Suppose that $f(a) = b$. Then, if g is the inverse of f, we have

$$a = I_A(a)$$

$$= (g \circ f)(a)$$

$$= g\{f(a)\}$$

$$= g(b)$$

$$= a$$

$f(a) = b$

$g(b) = a$

Since $g = f^{-1}$ this proves that

$$f^{-1}(b) = a$$

1. If f is invertible, the statement $f(a) = b$ and $a = f^{-1}(b)$ are equivalent.

2. If $f = \{(a,b) / a \in A, b \in B\}$ is invertible, then $f^{-1} = \{(b,a) \mid b \in B, a \in A\}$ and conversely

3. If f is invertible then f^{-1} is invertible, and $(f^{-1})^{-1} = f$.

Theorem 2

✓ A function $f: A \to B$ is invertible if and only if it is one-to-one and onto.

Proof: First suppose that f is invertible.

Then there exists a unique function $g: B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$.

Take any $a_1, a_2 \in A$ then

$$f(a_1) = f(a_2) \Rightarrow g\{f(a_1)\} = g\{f(a_2)\}.$$
$$\Rightarrow (g \circ f)(a_1) = (g \circ f)(a_2)$$
$$\Rightarrow I_A a_1 = I_A a_2$$
$$\Rightarrow a_1 = a_2$$

This proves that f is one-to-one.

Next let $b \in B$. Then $g(b) \in A$

$$b = I_B(b) = (f \circ g)(b) = f\{g(b)\}.$$

Thus b is the image of an element $g(b) \in A$ under f.

$\therefore$ f is onto as well.

Conversely, Suppose that f is one-to-one & onto.
Then for each $b \in B$ There is a unique $a \in A$
Such that $b = f(a)$

Now, Consider the function $g : B \to A$

defined by $g(b) = a$

$$\text{then } (g \circ f)(a) = g\{f(a)\} = g(b) = a = I_A(a)$$

$$f \circ g(b) = f\{g(b)\} = f(a) = b = I_B(b).$$

$f(a) = b$
$g(b) = a$
$f(a) = g(b) = a$
$g(b) = a$.

$\therefore$ f is invertible with g as the inverse

Proof

Sol. Suppose $f: A \to B$ is one to one.
Since A & B are finite sets with $|A| = |B|$, it follows that f is onto, Consequently f is invertible.
Conversely, Suppose f is invertible, then f is one-to-one & onto.

Refer properties of functions Theorem 3

Theorem 4 :- If $f: A \to B$ and $g: B \to C$ are invertible functions, then $g \circ f: A \to C$ is an invertible function and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Proof : Since f and g are invertible functions, they are both one-to-one and onto.
Consequently $g \circ f$ is both one-to-one & onto.
Therefore, $g \circ f$ is invertible.
Now, the inverse f^{-1} of f is a function from B to A and the inverse g^{-1} of g is a function from C to B
Therefore if $h = f^{-1} \circ g^{-1}$ then h is a function from C to A
we find that

$$(g \circ f) \circ h = (g \circ f) \circ (f^{-1} \circ g^{-1})$$
$$= g \circ (f \circ f^{-1}) \circ g^{-1}$$
$$= g \circ I_B \circ g^{-1}$$
$$= (g \circ g^{-1}) = I_C$$

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and $h \circ (g \circ f) = (f^{-1} \circ g^{-1}) \circ (g \circ f)$

$$= f^{-1} \circ (g^{-1} \circ g) \circ f$$

$$= f^{-1} \circ I_B \circ f$$

$$= f^{-1} \circ f$$

$$= I_A.$$

i.e. h is the inverse of $g \circ f$: $h = (g \circ f)^{-1}$

$$(g \circ f)^{-1} = h = f^{-1} \circ g^{-1}$$

Problem

Consider the function $f: R \to R$ defined by $f(x) = x^2$ $\forall x \in R$, Is f invertible?

Soln $a \in R$

$$f(a) = a^2$$

$$f(-a) = (-a)^2 = a^2$$

thus both a and $-a$ have the same image a^2 under f. $\therefore f$ is not one-to-one

$\therefore f$ is not invertible.

Problem

Let $A = \{x \mid x$ is real and $x \geq -1\}$ and $B = \{x \mid x$ is real and $x \geq 0\}$ Consider the function $f: A \to B$ defined by $f(a) = \sqrt{a+1}$, $\forall a \in A$. ST f is invertible and determine f^{-1}.

Soln: Let us first check that f is one-to-one and onto.

Take any $a_1, a_2 \in A$. Then $f(a_1) = \sqrt{a_1+1}$ and $f(a_2) = \sqrt{a_2+1}$ and $f(a_1) = f(a_2)$

$$\Rightarrow \sqrt{a_1+1} = \sqrt{a_2+1}$$

$a_1 + 1 = a_2 + 1 \Rightarrow a_1 = a_2$ Hence f is one-to-one

Take any $b \in B$. Then $b = f(a)$ holds if $b = \sqrt{a+1}$ or

$b^2 = a+1$

$a = b^2 - 1 \quad b \geq 0$

$b^2 - 1 \geq -1$

$\therefore$ thus every $b \in B$ has $a = b^2 - 1$ as a preimage in A under f.

Hence f is onto as well.

This proves that f is invertible. The inverse of f is given by $f^{-1}(b) = b^2 - 1$ for $\forall\ b \in B$.

Eg 3 Let $A = B = R$, the set of all real numbers, and the functions $f: A \to B$ and $g: B \to A$ be defined by $f(x) = 2x^3 - 1$, $\forall x \in A$, $g(y) = \left\{\frac{1}{2}(y+1)\right\}^{1/3}$, $\forall y \in B$

S.T each of f and g is the inverse of the other.

Soln Let $x \in A$

$$(g \circ f)(x) = g(f(x)) \qquad y = f(x)$$

$$= g(y) \qquad g(y) = \left\{\frac{1}{2}(y+1)\right\}^{1/3}$$

$$= \left\{\frac{1}{2}(y+1)\right\}^{1/3}$$

$$= \left\{\frac{1}{2}\left(2x^3 - 1 + 1\right)\right\}^{1/3} \qquad f(x) = 2x^3 - 1$$

$$y = f(x) = 2x^3 - 1$$

$$= x.$$

$$\left[\frac{1}{\cancel{2}} \cdot \cancel{2} x^3\right]^{1/3} = x^{3 \cdot \frac{1}{3}} = x$$

Thus $g \circ f = I_A$

Let $y \in B$

$$f \circ g(y) = f(g(y)) = f\left[\left\{\frac{1}{2}(y+1)\right\}^{1/3}\right]$$

$$= 2\left[\left\{\frac{1}{2}(y+1)\right\}^{1/3}\right]^3 - 1$$

$$= 2\left[\frac{1}{2}(y+1)\right] - 1 = y$$

$x = \frac{1}{2}(y+1) - 1$

$\cancel{4} + \cancel{x} - \cancel{y}$

$= y.$

Thus, $f \circ g = I_B$

Accordingly, each of f and g is an invertible function, and further more each is the inverse of the other.

Q4. Let $A = B = C = R$ and $f : A \to B$ and $g : B \to C$ be defined by

$f(a) = 2a + 1$, $g(b) = \frac{1}{3}b$, $\forall a \in A$, $\forall b \in B$.

Complete $g \circ f$ and show that $g \circ f$ is invertible. What is $(g \circ f)^{-1}$?

Soln.
$$(g \circ f)(a) = g(f(a))$$
$$= g(2a+1)$$
$$= \frac{1}{3}(2a+1)$$

thus $g \circ f : A \to C$ is defined by $(g \circ f)(a) = \frac{1}{3}(2a+1)$

We check that 'f' is invertible with $f^{-1}(b) = \frac{1}{2}(b-1)$ and g is invertible with $g^{-1}(c) = 3c$.

$\therefore$ $g \circ f$ is invertible, and its inverse is given by

$$(g \circ f)^{-1}(c) = f^{-1} \circ g^{-1}(c)$$
$$= f^{-1}\{g^{-1}(c)\}$$
$$= f^{-1}(3c)$$
$$= \frac{1}{2}(3c - 1)$$

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