BMS
Institute of Technology and Management

Module – 2
Counting
17CSL36

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Fundamental Principal of Counting

Eg: 4 types of Piza
3 types of Burger

Select 1 item only ?

Rule of Sum

Select 1 Piza and 1 Burger

Rule of Product

Fundamental Principal of Counting

Study of discrete and combinatorial mathematics begins with two basic principles of counting.

The Rule of Sum and Product.

A college library has 40 text books on C++ and 50 text books dealing with Java. By the rule of sum, a student at his college can select among 40+50 = 90 text books in order to learn more about one or the other of these two subjects.

Solution: ? Rule of Sum

Fundamental Principal of Counting

Study of discrete and combinatorial mathematics begins with two basic principles of counting.

The Rule of Sum (OR)

A college library has 40 text books on C++ and 50 text books dealing with Java. By the rule of sum, a student at this college can select among 40+50 = 90 text books in order to learn more about one or the other of these two subjects.

Solution: ?

Fundamental Principal of Counting

The Rule of Product (AND)

A college library has 40 text books on C++ and 50 text books dealing with Java. By the rule of sum, a student at this college can select among 40+50 = 90 text books in order to learn more about one from C++ and one from Java of these two subjects.

Solution: ? Sum of Product

Counting

Example 1: Ravi goes to library to read a book. There are books of different courses, he goes to the shelf which has 7 Data Structures, 5 DMS books and 2 Jav. Ravi want to read precisely 1 book. How many choices does he have?

→ 7 Data Structures,
→ 5 DMS books and
→ 2 Java

Number of choices = 7 + 5+ 2 = 14 choices [Rule of Sum]

Ravi wants to chose exactly one book.

Counting

Example 2: On a website named ISE Learning channel, 8 projects and 4 seminar videos are uploaded. Ram wants to subscribe to 1 project and 1 seminar. How many choices does he have?

→ 8 Projects,
→ 4 Seminars
Ram wants 1 project and 1 seminar

Number of choices = $8 \times 4 = 32$ choices [Rule of Product]

Ram wants to subscribe to 1 project or 1 seminar. How many choices does he have?

Number of choices = $8 + 4 = 12$ choices [Rule of Sum]

Counting

Example 3: 5 men and 7 women contest in an election.

a) In how many ways can the people chose a leader?

b) Suppose people want two leaders a men and a women?

→ 5 Men,

→ 7 Women

a) Number of choices = 5 + 7 = 12 choices [Rule of Sum]

b) Number of choices = 5 × 7 = 35 choices [Rule of Product]

Counting

Example 4: Ram visits an ice-cream parlour to buy one, He sees that there are 4 cones, 5 ice-creams and 3 toppings. How many choices does he have?

→ 4 Cones
→ 5 Ice – cream flavours
→ 3 Seminars

Number of choices = $4 \times 5 \times 3 = 60$ choices [Rule of Product]

Rules of counting

If there are **n** choices for one event and **m** choices for another event, both cannot occur at the same time, then there are **n+m** choices for one event. ➔ Rule of Sum

If there are **n** choices for one event and **m** choices for another event, then there are n*m choices for both these events .
➔ Rule of Product

Permutation

In mathematics, permutation refers to the arrangement of all the members of a set in some order or sequence, while combination does not regard order as a parameter. It is just a way of selecting items from a set or collection.

Permutation Formula: A permutation is the choice of *r* things from a set of *n* things without replacement. Order matters in permutation.

$$nP_r = \frac{n!}{(n-r)!}$$

Combination Formula: A combination is the choice of *r* things from a set of *n* things without replacement. Order does not matter in combination.

$$nC_r = \frac{n!}{(n-r)!r!} = \frac{nPr}{r!}$$

Permutation Derivation

Number of permutations of n different things taking r at a time is = nPr

Let us assume that there are n basket balls and r vacant boxes and each of them can hold one basket ball. In how many ways can we fill r vacant boxes?

No. of ways first box can be filled: n

No. of ways second box can be filled: $(n - 1)$

No. of ways third box can be filled: $(n - 2)$

No. of ways fourth box can be filled: $(n - 3)$

No. of ways rth box can be filled: $(n - (r - 1))$

Permutation Derivation

Therefore, no. of ways of filling in r boxes in succession can be given by:

$$n\,(n-1)\,(n-2)\,(\text{n-3})\,.\,.\,.\,(n-(r-1))$$

This can be written as:

$$n\,(n-1)\,(n-2)\,\ldots\,(n-r+1)$$

The no. of permutations of n different basket balls taken r at a time, where $0 < r \le n$ and the balls do not repeat is:

$$n\,(\text{n}-1)\,(\text{n}-2)\,(\text{n}-3)\,.\,.\,.\,(\text{n}-r+1).$$

$$\Rightarrow \quad \text{nPr} = \text{n}\,(\text{n}-1)\,(\text{n}-2)(\text{n}-3).\,.\,.(\text{n}-\text{r}+1)$$

Multiplying numerator and denominator by $(n-r)\,(n-r-1)\,.\,.\,.\,3 \times 2 \times 1$, we get

$$nP_r = \frac{[n(n-1)(n-2)(n-3)\ldots(n-r+1)(n-r)(n-r-1)..3\times2\times1]}{(n-r)(n-r-1)..3\times2\times1} = \frac{n!}{(n-r)!}$$

Permutation

Example 5: 5 girls ABCDE want to take a picture but the constraint is at a time only 3 friends are allowed to take photo.

In how many ways can 5 girls take a picture ensuring the picture comprises of 3 people

Total ways to line up r objects out of n objects, where order is important. ➔ ${}^{n}P_{r} = \frac{n!}{(n-r)!}$

${}^{n}P_{r} = \frac{n!}{(n-r)!}$

$= \frac{5!}{(5-3)!} = 5*4*3 = 60$ ways.

Permutation

Example 6: How many 3 letter words with / without meaning can be formed from LOGARITHMS . If repetition is not allowed.

We have 10 letters, pick 3 digits letter words. Picking 3 letter out of 10 letter ➔ $^{n}P_{r} = \frac{n!}{(n-r)!}$

$^{n}P_{r} = \frac{n!}{(n-r)!}$

$= \frac{10!}{(7)!} =$ 720 ways.

Permutation

Example 7: In how many ways the letters of the word LEADER be arranged?

We have 6 letters, there are 2 E's repeated

Number of ways to arrange = $\frac{6!}{2!}$ = 360 ways.

Example 8: A company has 10 members on its board of directors. In how many ways can they elect a president, a secretory , and a vice president ?

We have 10 members, 3 members to be selected

Number of ways to arrange ${}^{10}P_3 = \frac{10!}{(10-3)!}$ =

Permutation

In how many ways can the word HOLIDAY be arranged such that the letter 'I' will always come to the left of the letter 'L' ?

We have 7 letters word ➔ HOLIDAY , all letters are distinct 7!
Ways of arranging letters of HOLODAY
Constraint : Number of ways in which letter 'I' comes to the left of 'L' i.e 'LI'.

This happens in exactly half the arrangements.
Number of ways to arrange = $\frac{7!}{2}$ = ?

Permutation

Example 11: In how many ways can MATHEMATICS be arranged so that the vowels come together?

Vowels → A E A I (Let us consider it has 1 Unit)
Consonants → M T H M T C S + (AEAI) → 8 letters
Constraint : vowels come together.

Number of ways of arranging M T H M T C S + (AEAI) = $\frac{8!}{2!\,.2!}$

Number of ways of arranging AEAI = $\frac{4!}{2!}$

Total Number of ways of arranging MATHEMATICS with vowels together $= \frac{8!}{2!\,.2!} \cdot \frac{4!}{2!} = ?$

Permutation

Example 12: Find the number of permutations of the word CLIMATE such that the vowels occur in odd places.

Word → C L I M A T E (No repetition)

Constraint : vowels should be in odd places

We have vowels: I A E

Consonants: C L M T

Total : 7 letters

Permutation

7 slots

1	2	3	4	5	6	7
O	E	O	E	O	E	O

Odd places = 4 Even places = 3

4 ➔ Consonants , 3 ➔ Vowels

Number of ways 3 vowels can occur in 4 slots $= {}^{4}P_{3}$

$$= \frac{4!}{(4-3)!} = 24 \text{ ways}$$

Number of ways 4 consonants can occur in 4 slots $= {}^{4}P_{4}$

$$= \frac{4!}{0!} = \frac{4!}{1} = 24 \text{ ways}$$

Total number of permutations $= 24 \times 24 = 576$ ways

Combinations Derivation

Hence, the total number of permutations of *n* different things taken *r* at a time is nCr ×r! On the other hand, it is nPr.

$$nP_r = nC_r \times r!$$

$$nC_r = \frac{nP_r}{r!} = \frac{n!}{(n-r)!r!}$$

Combinations

In how many ways can 4 people take a photo with 2 of them in it, without worrying about the order?

$^4C_2 = \frac{^4P_2}{2!} =$

In a football match championship, there are 21 matches. If each team plays one match with each other team, find the number of teams ?

$^nC_2 = 21 = \frac{n(n-)}{2} = \quad 21 = n(n-1) = 42 \;,\; n = 7$

Combinations

Form a committee of 7 members from 8 women and 9 men?

1. No condition
2. Form Committee such that 6 are women
3. Form committee such that atleast 6 are women
4. Form Committee such that atleast 1 is a women

1. We have 8 women and 9 men = 17

$$^{17}C_7 = \frac{17!}{7!\ 10!} = ?$$

2. Given 6 are women

   6 W and 1 M

   $^{8}C_6 \times {}^{9}C_1 = ?$

Combinations

3. Given atleast 6 are women

6 W AND 1 M OR 7 W AND 0 M

$({}^{8}C_{6} \times {}^{9}C_{1}) + ({}^{8}C_{7} \times {}^{9}C_{0})$

4. Given atleast 1 is a women

1 W OR 2 W OR 3 W OR 4 W OR 5 W OR 6 W OR 7 W

If 0 W OR 1 W OR 2 W OR 3 W OR 4 W OR 5 W OR 6 W OR 7 W

This is similar to Q1, but remove 0 W

$({}^{17}C_{7} - {}^{9}C_{7}) =$

Note: Atleast 1 $\equiv$ Total - none

Combinations

Form a committee of 7 members from 8 women and 9 men. Show that 4 are women ?

4. Given atleast 4 are women

0 W OR 1 W OR 2 W OR 3 W | 4 W OR 5 W OR 6 W OR 7 W

~~0 W OR 1 W OR 2 W OR 3 W~~ | 4 W OR 5 W OR 6 W OR 7 W

$$({}^{8}C_{4} \times {}^{9}C_{3}) + ({}^{8}C_{5} \times {}^{9}C_{2}) + ({}^{8}C_{6} \times {}^{9}C_{1}) + ({}^{8}C_{7}) = ?$$

Combinations

Form a group of 16 players, we want to select a team of 11. Out of theses 16 one is sachin and another is rahul ?

4. Given atleast 4 are women

0 W OR 1 W OR 2 W OR 3 W | 4 W OR 5 W OR 6 W OR 7 W

~~0 W OR 1 W OR 2 W OR 3 W~~ | 4 W OR 5 W OR 6 W OR 7 W

$$({}^{8}C_{4} \times {}^{9}C_{3}) + ({}^{8}C_{5} \times {}^{9}C_{2}) + ({}^{8}C_{6} \times {}^{9}C_{1}) + ({}^{8}C_{7})=?$$

Combinations

5 Scholarships to be distributed among 20 students
Arrangement or selection.

Distribute 5 distinct scholarship among 20 students.
Arrangement
Permutation

Distributing 5 identical scholarships among 20 students.
Selection
Combination

Combinations

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Select any # of objects from n distinct objects .

0 OR 1 OR 2 OR 3 OR 4 OR 5 ---------- OR n

$^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \text{------} = 2^n$

Let, a room has 5 bulbs, in the # ways in which bulbs are lighted

$^5C_0 + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = 2^5$

$\cancel{^5C_0 +} {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = (2^5 - 1)$

(not accepted) at least 1 is selected

Note: Since each object has two options either select or not select

Combinations

Select any # of objects from n distinct objects .

0 OR 1 OR 2 OR 3 OR 4 OR 5 ---------- OR n

$^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 +$ ------ $= 2^n$

Let, a room has 5 bulbs, in the # ways in which bulbs are lighted

$^5C_0 + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = 2^5$

$\cancel{^5C_0} + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = (2^5 - 1)$

(not accepted) at least 1 is selected

Note: Since each object has two options either select or not select

Combinations

Assume we have three distinct objects select any # of objects from n distinct objects

There are 2^3 Ways $2^3 – 1 = 7$ ways (At least one is selected)

Select any # of objects from n identical objects
Assume we have 5 identical books on JAVA in a library.

0 OR 1 OR 2 OR 3 OR 4 OR 5
~~0~~ OR 1 OR 2 OR 3 OR 4 OR 5 = (n + 1) ways

Atleast one object = n+1-1 = n ways
(each book has only one way)

Combinations

Assume we have 5 roses, each of them is different color distinct and 6 white lily (identical), in # of ways can we purchase S. T atleast 1 rose and atleast 1 lily.

$$(2^5 - 1) \times 6$$

Combination with Repetition

n different objects, objects can be repeated.

The number of r – combinations $= (n+r-1)\,C_r = {}^{n}C_r = {}^{n}C_{n-r}$

The number of solutions for x1 + x2 + x3 = 20 where

(i) $x1 \geq 0, x2 \geq 0\ x3 \geq 0$

   (i) X1 + x2+ x3 ------Xn = r

   (ii) n = 3, r = 20, n different objects, objects can be repeated.

The number of r – combinations $= (n+r-1)\,C_r$

$(3+20-1)\,C_{20} = {}^{22}C_{20} = {}^{22}C_2$

Combination with Repetition

How many ways 10 roses, 14 sunflowers, 15 daffodils can be distributed among 3 girls?

The number of solutions for x1 + x2 + x3 = 20 where

(i) $x1 \geq 0, x2 \geq 0\ x3 \geq 0$

  (i) x1 + x2+ x3 = r

  (ii) n = 3, r = 20 N different objects, objects can be repeated.

The number of r – combinations = (n+r-1) C_r

(20+3-1) C_{20}

Combination with Repetition

An ice cream vendor sells 3 flavors

1. Vanilla
2. Chocolate
3. Mango

Assume 10 kids visit the shop.

In how many ways can he sell 10 ice creams with 3 flavours?

Properties of Binomial Theorem

Properties of Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} (1)^{n-k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \ldots\ldots + \binom{n}{n} x^0 y^n$$

$$= x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots\ldots\ldots + y^n$$

r^{th} term $= \binom{n}{r-1} x^{n-r+1} y^{r-1}$

3^{rd} term $= \binom{n}{2} x^{n-2} y^{2}$

$$\boxed{T_r = \binom{n}{r-1} x^{n-r+1} y^{r-1}}$$

Properties of Binomial Theorem

Middle Term in the expansion of $(x+y)^n$:

n is even, $\left(\frac{n+2}{2}\right)^{th}$ term is the middle term.

n is odd,

$$(x+y)^3 = x^3 + 3x^2y + 3y^2x + y^3$$

$\left(\frac{n+1}{2}\right)^{th}$ term & $\left(\frac{n+1}{2}+1\right)^{th}$ term are middle terms.

The largest coeffecient in the expansion of $(x+y)^n \longrightarrow$ Coeffecient of middle term

1) Expand $(a+b)^6$

$$= \binom{6}{0}a^6b^0 + \binom{6}{1}a^5b + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}ab^5 + \binom{6}{6}a^0b^6$$

$$= 1a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5$$

2) Expand $(1.04)^4$.

$$(1.04)^4 = (1+0.04)^4$$

$$= \binom{4}{0}(0.04)^0 + \binom{4}{1}(0.04) + \binom{4}{2}(0.04)^2 + \binom{4}{3}(0.04)^3 + \binom{4}{4}(0.04)^4$$

$$= 1 + 4(0.04) + 6(0.0016) + 4(0.000064) + 1(0.00000256)$$

$$= 1.16985856$$

$$(1.04)^4 \simeq \boxed{1.17}$$

3) Find the 4^{th} term in the expansion of $\left(\frac{x^3}{2} - \frac{2}{x^2}\right)^9$.

Ans:

$$T_r = \binom{n}{r-1} a^{n-r+1} b^{r-1}$$

$$r = 4$$

$$T_4 = \binom{9}{3} a^6 b^3 \qquad a = \frac{x^3}{2} \quad b = \frac{-2}{x^2}$$

$$T_4 = \frac{9!}{3!\,6!}\left(\frac{x^3}{2}\right)^6\left(\frac{-2}{x^2}\right)^3 = 84\,\frac{x^{18}}{2^6} \times \frac{-8}{x^6}$$

$$T_4 = \boxed{\frac{-21}{2}x^3}$$

4) Determine if the expansion of $\underline{\left(x^2 - \frac{2}{x}\right)^{18}}$ will contain a term containing x^{10}

Ans:

$$T_r = \binom{n}{r-1} a^{n-r+1} b^{r-1}$$

$$= \binom{18}{r-1}(x^2)^{18-r+1}\left(\frac{-2}{x}\right)^{r-1} = \binom{18}{r-1}(x)^{38-2r}\frac{(-2)^{r-1}}{x^{r-1}}$$

$$= \binom{18}{r-1} x^{39-3r}(-2)^{r-1}$$

$$\boxed{}\, x^{10} = x^{39-3r}$$

$$39 - 3r = 10 \qquad r = \cancel{\frac{29}{3}}$$

The expansion does not contain the term x^{10}

Multinomial Coefficient

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Multinomial theorem

Sports club of a school has 36 girls. They want to form 4 volleyball teams of 9 girls each. In how many ways can they do this?

36 girls, 4 teams – T_1, T_2, T_3, T_4

9 girls in each team

T_1 : 9 girls to be chosen out of 36 $\longrightarrow \binom{36}{9}$

T_2 : 9 girls to be chosen out of 27 $\longrightarrow \binom{27}{9}$

T_3 : 9 girls to be chosen out of 18 $\longrightarrow \binom{18}{9}$

T_4 : 9 girls to be chosen out of 9 $\longrightarrow \binom{9}{9}$

MORE VIDEOS

Multinomial Coefficient

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$$\binom{36}{9} \times \binom{27}{9} \times \binom{18}{9} \times \binom{9}{9} \longrightarrow \text{Number of ways in which 4 teams will be formed.}$$

$$\frac{36!}{9!\,\cancel{27!}} \times \frac{\cancel{27!}}{9!\,\cancel{18!}} \times \frac{\cancel{18!}}{9!\,\cancel{9!}} \times \frac{\cancel{9!}}{9!\,0!} = \frac{36!}{9!\,9!\,9!\,9!}$$

Multinomial Coefficient

Number of ways to choose :

n_1 objects from n objects, n_2 objects from $(n-n_1)$ objects, n_3 objects from $(n-n_1-n_2)$ objects, ..., n_k objects from $(n-n_1-n_2-\cdots\cdots-n_{k-1})$ objects.

$$\binom{n}{n_1} \times \binom{n-n_1}{n_2} \times \binom{n-n_1-n_2}{n_3} \times \cdots \times \binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!\,n_2!\cdots\cdots n_k!} \longrightarrow \binom{n}{n_1, n_2, n_3, \ldots, n_k}$$

Multinomial Coefficient

$$\frac{n!}{n_1!\,n_2!\,\cdots\cdots\,n_k!}$$

coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots\cdots x_k^{n_k}$ in the expansion of

$$(x_1 + x_2 + \cdots\cdots + x_k)^n$$

$$k = 2,\ (x_1 + x_2)^n$$

Multinomial Coefficient

7) What is the co-efficient of x^2yz in the expansion of $(x+y+z)^4$?

Ans: The coefficient of $x_1^{n_1} y^{n_2} z^{n_3}$ is —

$$\frac{n!}{n_1!\,n_2!\,n_3!}$$

Coeffecient of x^2yz = $\frac{4!}{2!\,1!\,1!}$

Multinomial coefficients

8) What is co-efficient of $x_1^2 x_3 x_4^3 x_5$ in the expansion of $(x_1+x_2+x_3+x_4+x_5)^7$?

<u>Ans</u>: The co-efficient of $x_1^2 x_3 x_4^3 x_5$ $= \dfrac{7!}{2!\,1!\,3!\,1!}$

$$= \frac{7!}{2!\,3!} = \boxed{420}$$

Thank You