BMS
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Module 2
Mathematical Induction
17CSL36

By,

Dr. Pushpa S. K
Professor
Dept. of Information Science & Engg.
BMS Institute of Technology & Mgmt. Bengaluru.

Mathematical Induction

Z : The set of integers = { 0,1,-1,2,-2,3,-3. . . . }

N : The set of nonnegative integers or natural numbers = { 0,1,2,3,. . . . }

Z^+ : The set of positive integers or natural numbers = { 1,2,3,. . . . } = { x € Z | x > 0 }

Q : The set of rational numbers = { a/b | a, b € Z, b ≠ 0 }

Q^+ : The set of positive rational numbers = { r € Q | r > 0 }

Q^* : The set of non zero rational numbers

R^* : The set of non zero real numbers

R : The set of real numbers

R^+ : The set of positive real numbers

Mathematical Induction

When x, y $\in$ Z , we know that x+y, xy, x-y $\in$ Z . Thus we say that the set Z is closed under +, * and -

2, 3 $\in$ Z , but 2/3 is not member of Z. So, the set Z of all integers is not closed under ÷.

To cope with this situation, we shall know somewhat restricted form of division for Z and shall concentrate on special elements of Z^+ called primes.

These primes are building blocks of the integers and they provide Fundamental Theorem of Arithmetic.

The well ordered principle

Given two integers x and y, we know that $x < y$ or $y < x$. Suppose we try to express the subset Z+ or Z, using the inequality symbols > and ≥.

We can define the set of positive elements of Z + as
$Z+ = \{x \in Z+ \mid x > 0\} = \{x \in Z+ \mid x \geq 1\}$.

When we do like wise for the rational and real numbers.

$Q+ = \{x \in Q+ \mid x > 0\}$ and $R+ = \{x \in R+ \mid x > 0\}$, but we cannot represent Q+ or R+ using ≥ as we did in Z+ .

The set Z+ is different from the sets Q+ and R +.

Because Q+ or R+ themselves do not contain least elements. (there is no small positive rational number or smallest positive real number)

If q is a positive rational number, then $0 \leq q / 2 < q$, would have the smaller positive rational number $q/2$. therefore $Z+ \subset Z$.

The Well Ordered Principle

Every nonempty subset of Z+ contains a smallest element. [It is often expressed by saying that Z+ is well ordered]

This distinguishes Z+ from Q+ and R+. But does it lead anywhere that is mathematically interesting ? Yes.

This leads to mathematical induction.

Induction

- The **principle of mathematical induction** is a useful tool for proving that a certain predicate is true for **all natural numbers**.

- It cannot be used to discover theorems, but only to prove them.

Induction

- If we have a propositional function P(n), and we want to prove that P(n) is true for any natural number n, we do the following:
- Show that P(0) is true.
  (basis step)
- Show that if P(n) then P(n + 1) for any $n \in N$.
  (inductive step)
- Then P(n) must be true for any $n \in N$.
  (conclusion)

Induction

- **Example1:**
- Show that $n < 2^n$ for all positive integers n.
- Let P(n) be the proposition "$n < 2^n$."
- 1. Show that P(1) is true.
  (basis step)
- P(1) is true, because $1 < 2^1 = 2$.

Induction

- 2. Show that if $P(n)$ is true, then $P(n + 1)$ is true. (inductive step)

Assume that $n < 2^n$ is true.

We need to show that $P(n + 1)$ is true, i.e.

$n + 1 < 2^{n+1}$

We start from $n < 2^n$:

$n + 1 < 2^n + 1 \leq 2^n + 2^n = 2^{n+1}$

Therefore, if $n < 2^n$ then $n + 1 < 2^{n+1}$

Induction

- Then P(n) must be true for any positive integer. (conclusion)

$n < 2^n$ is true for any positive integer.

End of proof.

Induction

- **Example 2:**
- $S(n) = 1 + 2 + \ldots + n = n\,(n + 1)/2$
- Show that $P(0)$ is true.
  (basis step)
- For $n = 0$ we get $0 = 0$. True.

Induction

- Example 3: Show that if $P(n)$ then $P(n + 1)$ for any $n \in N$. (inductive step)
- $1 + 2 + \ldots + n = n(n + 1)/2$
- $1 + 2 + \ldots + n + (n + 1) = n(n + 1)/2 + (n + 1)$
- $= (2n + 2 + n(n + 1))/2$
- $= (2n + 2 + n^2 + n)/2$
- $= (2 + 3n + n^2)/2$
- $= (n + 1)(n + 2)/2$
- $= (n + 1)((n + 1) + 1)/2$

Induction

- Then P(n) must be true for any $n \in N$. (conclusion)
- $1 + 2 + \ldots + n = n(n + 1)/2$ is true for all $n \in N$.
- End of proof.

Induction

The principle of mathematical induction:

- Show that P(0) is true.

(basis step)

- Show that if P(0) and P(1) and … and P(n), then P(n + 1) for any $n \in N$.

(inductive step)

- Then P(n) must be true for any $n \in N$.

(conclusion)

Example: Show that every integer greater than 1 can be written as the product of primes.

- Show that P(2) is true.

(basis step)

2 is the product of one prime: itself.

Induction

Show that if P(2) and P(3) and … and P(n), then P(n + 1) for any $n \in N$. (inductive step)

Two possible cases:

- If (n + 1) is **prime**, then obviously P(n + 1) is true.
- If (n + 1) is **composite**, it can be written as the product of two integers a and b such that

$2 \leq a \leq b < n + 1$.

By the **induction hypothesis**, both a and b can be written as the product of primes.
Therefore, $n + 1 = a \cdot b$ can be written as the product of primes.

Then P(n) must be true for any $n \in N$.
(conclusion)

End of proof.

We have shown that **every integer greater than 1** can be written as the product of primes.

Theorem

The principle of Mathematical Induction: Let S(n) denote an open mathematical statement that involves one or more occurrences of the variable n, which represents a positive integer.

a) If S(1) is true and
b) If whenever S(k) is true, then S(K+1) is true, then S(n) is true for all $n \in Z+$

Proof : Let S(n) be open statement satisfying conditions (a) & (b).
Let F = { t Є Z+ | S(t) is false}   // we need to prove that F ={Φ }

We assume that F ≠ {Φ }, then by the principle, F has a least element m.
Since S(1) is true, it says that m ≠ 1.
M > 1 and consequently m-1 Є Z+.

With $m-1 \neq F$, we have $S(m-1)$ true. So by Condition (b)

$S(m-1) + 1 = S(m)$ is true.
Contradicting $m \in F$.

This Contradiction arose from the assumption that $F \neq 0$.
Consequently , $F = 0$.

Example 4

For all n Є Z+, $\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 \ldots . +n(n+1)/2$

Proof:
For n=1 the open statement
S(n) = $\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 \ldots . +n(n+1)/2$

Basic :
S(1) =$\sum_{i=1}^{1} \frac{1(1+1)}{2}$ S(1) is true.
For n = k
S(k) forces us to s(k+1) = $\frac{k(k+1)}{2}$
$\sum_{i=1}^{k+1} i$ = (k+1)(k+1+1)/2
$\sum_{i=1}^{k+1} i$ = 1+2+3 ---- k + (k+1) = k(k+1)/2 + (k+1)

$\sum_{i=1}^{k+1} i$ = (k+1)(k+1+1)/2

$\sum_{i=1}^{k+1} i$ = k(k+1)/2 + (k+1) = k(k+1)/2 + 2(k+1)/2 = {(k+1)(k+1)}/2

Establishing inductive step of the theorem

S(n) = true for all n Є Z+

Problems

If n is a +ve integer prove that 1.2+2.3 + 3.4 + ---- + n(n+1) = [n(n+1)(n+2)] / 3 using mathematical induction.

Prove that for each n Є Z+, $\sum_{i=1}^{n} i \ * i = [n(n+1)(2n+1)]/6$

If n Є Z+, establish the validity of the open statement S(n) :
$\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 - - - n = \frac{[n*n+n+2]}{2}$

If n Є Z+, establish the validity of the open statement S(n) :
$\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 - - - n = \frac{[n*n+n+2]}{2}$

Assignment

1. By mathematical induction Prove that

$1^2 + 3^2 + 5^2 + \text{-----} + (2n-1)^2 = \dfrac{n(2n+1)(2n-1)}{3}$

2. By mathematical induction Prove that

$4 + 9 + 14 + 19 + \text{------} + 5n-1 = \dfrac{n[3+5n]}{2}$ for all $n \geq 4$

Example 5

Using mathematical induction show that 3 divides $n^3 - 7n + 3$ for all positive integer N.

S_n = $\mathbf{n^3 - 7n + 3}$ = 3. m // for some integer m Є N

Basic Step: [show that statement is true for n = 1]

S_n = $\mathbf{1^3 - 7.\,1 + 3}$ = -3 ---→ True

Inductive Hypothesis: (Assume n=k is true)

Here we know that S_n = $\mathbf{k^3 - 7k + 3}$ = 3. m is true for some integer m

Example 5

Inductive Step: [show that n=k is true n=k+1 is also true]

$S_{k+1} = (k+1)^3 - 7.(k+1) + 3 = (k+1)(k+1)(k+1) - 7(k+1) + 3$

$= K^3 + 3k^2 + 3k + 1 - 7k - 7 + 3$

$= (K^3 - 7k + 3) + 3(k^2 + k - 2)$

$= 3m + 3(k^2 + k - 2)$

$= 3(m + k^2 + k - 2)$

Now 3 divides $(K^3 - 7k + 3)$ and 3 divides $3(k^2 + k - 2)$

Hence , we find that 3 divides $(K^3 - 7k + 3) + 3(k^2 + k - 2)$

This implies that S_{k+1} is true.

Conclusion : This implies that 3 divides $S_{k+1} = (k+1)^3 - 7.(k+1) + 3$. Hence the statement holds good for n=1, n= k and n= k+1.

Example 6

Using mathematical induction prove that 7 divides $8^n - 1$ for all positive integer n.

$S_n = 8^n - 1 = 7.m$ // for some integer m Є N

Basic Step: [show that statement is true for n = 1]

$S_n = 8^1 - 1 = 7.1$ ---→ True

Inductive Hypothesis: (Assume n=k is true)

Here we know that $S_n = 8^k - 1 = 7.m$ is true for some integer m

Example 6

Inductive Step: [show that n=k is true n=k+1 is also true]

$S_{k+1} = 8^{k+1} – 1 = 8^{K}. 8^{1} - 1$

$8^{k} – 1 = 7m$

$8^{k} = 7m + 1$

Substitute $7m + 1$ for 8^{k}

$S_{k+1} = 8^{k+1} – 1 = 8^{K}. 8^{1} - 1$

$= S_{k+1} = 8^{k+1} – 1 = 8^{K}. 8^{1} - 1$

$= (7m+ 1). 8^{1} - 1$

$= 56m + 8-1$

$= 56m + 7$

$= 7(8m + 1)$

Hence 7 divides (8 m + 1)

Conclusion : This implies that 7 divides $8^{k+1} – 1$, Since the statement holds good for n=1, n= k and n= k+1

Example 7

Using mathematical in prove that 3 divides $n^3 + 2n$ for all positive integer n.

$Sn = n^3 + 2n = 3. m$ // for some integer m Є N

Basic Step: [show that statement is true for $n = 1$]

$Sn = 1^3 + 2.1 = 1 + 2 = 3. 1 = 3. m$ ---→ True

Inductive Hypothesis: (Assume n=k is true)

Here we know that $Sn = k^3 + 2.k = 1 + 2 = 3. m$ ---→ True for some integer m.

Example 7

Inductive Step: [show that n=k is true n=k+1 is also true]

$S_{k+1} = (k+1)^3 + 2.(k+1) = (k+1)(k+1)(k+1) + 2k+2$

$= K^3 + 3k^2 + 3k+1+2k+2$

$= K^3 + 3k^2 + 5k+3$

$= K^3 + 3k^2 + 3k+2k+3$

$= K^3 + 2k + 3k^2 + 3k+3$

$= 3m + 3k^2 + 3k+2k+3$

$= 3(m + k^2 + k+1)$

Hence 3 divides S_{k+1}

Conclusion : This implies that 3 divides $S_n = n^3 + 2n$, Since the statement holds good for n=1, n= k and n= k+1

Assignment

1. Prove that $9^n - 2^n$ is divisible by 7 for all $n \in N$

2. Prove that for any positive integer n 7 divides $3^{2n+1} + 2^{n+2}$

3. Prove by mathematical induction that for every positive integer 8 divides $5^n + 2 . 3^{n-1} + 1$

Example 8

Using mathematical induction prove that $3^n < (n+1)!$ for all positive integer $n \geq 4$.

$S_n = 3^n < (n+1)!$ For all **$n \geq 4$**

Basic Step: [show that statement is true for $n = 4$]

$S_n = 3^4 < (4+1)! = 81 < 120$ ---→ True

Inductive Hypothesis: (Assume n=k is true)

Here we know that $S_k = 3^k < (k+1)!$ for some k **≥ 4 .**

Example 8

Inductive Step: [show that n=k is true n=k+1 is also true]

$3^{k+1} = 3^k \cdot 3^1$

$< (k+1)! \cdot 3$ **[by assumption]**

$< (k+2)\,(k+1)!$

$< (k+2)!$

$3^{k+1} < [\,(k+1)+1]!$

the inequality $k \geq 4$

This implies **$4 \leq k$**

$3 < k$

$3 < k+2$

$3(k+1)! < (k+2)(k+1)!$

Hence True

Conclusion : Since the statement is true for n=4 and true for n=k follows that n is true for n= k+1.

Example 9

Using mathematical induction prove that $2^n \geq 1 + n$ for all positive integer $n \geq 1$.

$S_n = 2^n \geq (1+n)$ For all $\mathbf{n \geq 1}$

Basic Step: [show that statement is true for n = 1]

$S_n = 2^1 \geq (1+1) = 2 \geq 2$ ---→ True

Inductive Hypothesis: (Assume n=k is true)

Here we know that $S_k = 2^k \geq (1+k)$ for some k in N .

Example 9

Inductive Step: [show that n=k is true n=k+1 is also true]

$2^{k+1} = 2^k \cdot 2^1$

$\geq (1+k) \cdot \mathbf{2}$ [by assumption]

$\geq 2 + 2k$

$\geq 2 + k$

$\geq 1+1+k$

$\geq 1+(1+k)$

$2^{k+1} \geq \mathbf{1+(k+1)}$

the inequality $k \geq 1$

This implies $2k \geq \mathbf{k}$

Hence its True.

Conclusion : Since the statement is true for n=1 and true for n=k follows that n is true for n= k+1.

Assignment

1. Prove that $2^n \geq n^2$ for n = {4,5,6,----}

2. Prove that $2^n \geq 2n$ for $n \geq 1$

3. Prove that $4n < (n^2 - 7)$ for all positive integer $n \geq 6$

4. By mathematical induction Prove that $n! \geq 2^{n-1}$ for all integers $n \geq 1$

Example 10

Harmonic numbers H1, H2, H3 ---- where

H1 = 1, H2 = 1 $+\frac{1}{2}$ **-------**

Hn = 1 $+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$ **+ ------ + +** $\frac{1}{n}$

Prove that $S_n = \sum_{j=1}^{n} H_j = (n+1)H_n - n$ For every positive integer n>=1.

$S_n = \sum_{j=1}^{n} H_j = (n+1)H_n - n$ For every positive integer n,

Basic Step: [show that statement is true for n = 1]

$S_n = \sum_{j=1}^{1} H_j = H_1 = 1 = LHS$

$= (1 + 1)\, H_1 - 1$

$= 1$ RHS ---→ True

Inductive Hypothesis: (Assume n=k is true)

Here we know that $S_k = \sum_{j=1}^{k} H_j = (k+1)H_k - k$ for some k in N .

Example 10

Inductive Step: [show that n=k is true n=k+1 is also true]

$S_{k+1} = \sum_{j=1}^{k+1} H_j \quad = \sum_{j=1}^{k} H_j + H_{k+1}$

$= \{(k+1)H_k - k\} + H_{k+1}$

$= (k+1)\{H_{k+1} - \frac{1}{k+1}\} - k + H_{k+1}$ ➔

$= (k+1)H_{k+1} - 1 - k + H_{k+1}$

$= (k+2)H_{k+1} - 1 - k$

$= (k+2)H_{k+1} - (k+1)$ ➔ **S_{k+1} is true.**

$H_{k+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{------} + \frac{1}{k}$

$H_{k+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{------} \quad \frac{1}{k} + \frac{1}{k+1}$

$H_{k+1} = H_k + \frac{1}{k+1}$

$H_k = H_{k+1} - \frac{1}{k+1}$

Example 10

Conclusion:

Hence by mathematical induction, it follows that S(n) = $\sum_{j=1}^{n} H_j = (n+1)H_n - n$ is true for all positive integers n ≥ 1. This proves the required result.

Example 11

Let Hn be defined as Harmonic sequence, prove that

$S_n = \sum_{j=1}^{n} jH_j = \frac{(n+1)n}{2} H_{n+1} - \frac{(n+1)n}{4}$ For every positive integer n ≥ 1.

$S_n = \sum_{j=1}^{n} jH_j = \frac{(n+1)n}{2} H_{n+1} - \frac{(n+1)n}{4}$ For every positive integer n ≥ 1.

Basic Step: [show that statement is true for n = 1]

$\sum_{j=1}^{1} 1.1 = 1$ (LHS)

$H_1 = \frac{2.1}{2} H_2 - \frac{2.1}{4} = 1$ (RHS) ➔ True

Inductive Hypothesis: (Assume n=k is true)

Here we know that $S_k = \sum_{j=1}^{k} jH_j = \frac{(k+1)k}{2} H_{k+1} - \frac{(k+1)k}{4}$ for some k in n ≥ 1.

Example 11

Inductive Step: [show that n=k is true n=k+1 is also true]

$S_{k+1} = \sum_{j=1}^{k+1} jH_j = \sum_{j=1}^{k} jH_j + (k+1)H_{k+1}$ ➔ $\sum_{j=1}^{k} jH_j = \frac{(k+1)k}{2}H_{k+1} - \frac{(k+1)k}{4}$

$= [\frac{(k+1)k}{2}H_{k+1} - \frac{(k+1)k}{4}] + (k+1)H_{k+1}$

$= (k+1)\,(\frac{k}{2}H_{k+1} - \frac{k}{4} + H_{k+1})$

$= (k+1)\,[(\frac{k}{2}+1)H_{k+1} - \frac{k}{4}]$ ➔

$= (k+1)\,[\left(\frac{k}{2}+1\right)\left\{H_{k+2} - \frac{1}{k+2}\right\} - \frac{k}{4}]$

$= (k+1)\,[\{\frac{k+2}{2}.H_{k+2} - \frac{k+2}{2}.\frac{1}{k+2}\} - \frac{k}{4}]$

$\boldsymbol{H_{k+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{------} + \frac{1}{k}}$

$\boldsymbol{H_{k+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{------}\ \frac{1}{k} + \frac{1}{k+1}}$

$\boldsymbol{H_{k+1} = H_k + \frac{1}{k+1}}$

$\boldsymbol{H_k = H_{k+1} - \frac{1}{k+1}}$

$\boldsymbol{H_{k+1} = H_{k+2} - \frac{1}{k+2}}$

Example 11

$$= (k+1)\left[\frac{k+2}{2} H_{k+2} - \frac{1}{2} - \frac{k}{4}\right]$$

$$= (k+1)\left[\frac{(k+2)}{2} H_{k+2} - \frac{(k+2)}{4}\right]$$

$$= \frac{(k+2)(k+1)}{2} H_{k+2} - \frac{(k+2)(k+1)}{4}$$

➔ **S_{k+1} is true.**

Conclusion:

Hence by mathematical induction, it follows that $S(n) = \sum_{j=1}^{n} jH_j = \frac{(n+1)n}{2} H_{n+1} - \frac{(n+1)n}{4}$ is true for all positive integers n ≥ 1. This proves the required result.

Example 12

Prove by mathematical induction that, for any positive integer n, the number $11^{n+2} + 12^{2n+1}$ is divisible by 133.

$A_n = 11^{n+2} + 12^{2n+1}$ For every positive integer n.

Basic Step: [show that statement is true for n = 1]

$A_n = 11^{1+2} + 12^{2.1+1}$ = 1331 +1728 = 3059 = 23 ×133 , so that 133 divides 3059 ➔ Hence True.

Inductive Hypothesis: (Assume n=k is true)

Here we know that $A_k = 11^{k+2} + 12^{2k+1}$ for some $k \geq 1$.

Example 12

Inductive Step: [show that n=k is true n=k+1 is also true]

$A_{k+1} = 11^{k+3} + 12^{2(k+1)+1}$

$A_{k+1} = (11^{k+2} \times 11) + (12^{2k+1} \times 12^2)$

$A_{k+1} = (11^{k+2} \times 11) + (12^{2k+1} \times 144)$

$A_{k+1} = (11^{k+2} \times 11) + (12^{2k+1} \times (11 + 133))$

$A_{k+1} = (11^{k+2} + 12^{2k+1}) \times 11 + (12^{2k+1} \times 133)$

$A_{k+1} = (A_k \times 11) + (12^{2k+1} \times 133)$

$\Rightarrow$ A_{k+1} is true.

Hence its True.

Conclusion: Hence A_{k+1} is divisible by 133 when $\mathbf{A_n = 11^{n+2} + 12^{2n+1}}$ is divisible by 133.

Recursive Method

For example a_0, a_1, a_2, a_3 ----,{ a_n}.

- The explicit method
- Recursive method

In explicit method, the general term of the sequence is explicitly mentioned.

In recursive method, first few terms are explicitly indicated and general term is indicated through formula / rule.

Eg: $a_1 = 2$ and $a_n = a_{n-1} + 2$ for $n >= 2$

- Example: Let a (n) (iterative form)
- $a(n) = 1^2 + 2^2 + 3^2 + \ldots + n^2$
- Rewrite a(n) (Recursive form)
- $a(n) = 1^2 + 2^2 + 3^2 + \ldots (n-1)^2 + n^2$

  $a(n) = a(n-1) + n^2$

$a(n) = a(n-1) + n^2 \quad , a(1) = 1$

$a(5) = a(4) + 5^2 \quad = 30 + 25 = 55$

$a(4) = a(3) + 4^2 = 14 + 16 = 30$

$a(3) = a(2) + 3^2 = 5 + 9 = 14$

$a(2) = a(1) + 2^2 = 1 + 4 = 5$

$a(1) = 1$ (base case)

Example 13

A sequence $\{a_n\}$ is defined recursively by $a_1 = 4$, $a_n = a_{n-1} + n$ for $n \geq 2$. Find a_n in implicit form

We Know that $a_n = a_{n-1} + n$

Apply back substitution $a_n = a_{n-1} + n$

$$a_n = (a_{n-2} + n-1) + n$$

$$a_n = (a_{n-3} + n-2) + n-1 + n$$

$$a_n = (a_{n-3} + n-3) + n-2 + n-1 + n$$

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$$= (a_{n-n-1} + 2 + 3 + 4 + 5 + 6 + 7 \text{ ---} n-1 + n)$$

$$= (a_1 - 1\ ) + (1 + 2 + 3 + 5 \text{ -----} + n)$$

Example 13

$= (a_1 - 1\ ) + (1 + 2 + 3 + 5 \text{-----} + n)$

Using a_1=4 and the standard result

$a_n = (4 - 1\ ) + (n(n+1)/2)$ ➔

$a_n = 3 + \frac{n(n+1)}{2}$

$1+2+3+4+5 + \text{-----} + n = \frac{n(n+1)}{2}$

Explicit formula

Example 14

Find an explicit definition of the sequence defined recursively by $a_1 = 7$, $a_n = 2a_{n-1} + 1$ for $n \geq 2$.

$a_n = 2a_{n-1} + 1$ Apply back substitution

$a_n = 2\{2a_{n-2} + 1\} + 1$

$a_n = 2[2\{2a_{n-3} + 1\} + 1] + 1$

……

……

$= 2^{n-1} (a_{n-(n-1)} + 2^{n-2} + 2^{n-3} + ----- + 2^2 + 2 + 1)$

$= 2^{n-1} a_1 + (1 + 2 + 2^2 + 2^3 + ----- + 2^{n-3} + 2^{n-2})$

Using $a_1 = 7$ and the standard result

$1 + a + a^2 + ------- + a^{n-1} = \frac{a^n - 1}{a - 1}$ for $a > 1$

$a_n = 7 . 2^{n-1} + (2^{n-1} - 1)$

$= (8 . 2^{n-1}) - 1$ ➔ explicit definition.

Example 14

$= (a_1 - 1) + (1 + 2 + 3 + 5 \text{-----} + n)$

Using $a_1=4$ and the standard result

$a_n = (4 - 1) + (n(n+1)/2)$

$a_n = 3 + \dfrac{n(n+1)}{2}$

$1+2+3+4+5 + \text{-----} + n = \dfrac{n(n+1)}{2}$

Explicit formula

Applications of Fibonacci Nos.

The first 15 Fibonacci numbers F_n for $n = 0, 1, 2, \ldots, 14$ are:

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}
0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610

The sequence can also be extended to negative index n :

$$F_{n-2} = F_n - F_{n-1}$$

which yields the sequence of "negafibonacci" numbers satisfying:

$$F_{-n} = (-1)^{n+1} F_n$$

Thus the bidirectional sequence is:

F_{-7}	F_{-6}	F_{-5}	F_{-4}	F_{-3}	F_{-2}	F_{-1}	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7
13	−8	5	−3	2	−1	1	0	1	1	2	3	5	8	13

Applications of Fibonacci Nos

ISE Dept.
Transform Here

Figure 5 The Fibonacci spiral appeared in some kind of aloes

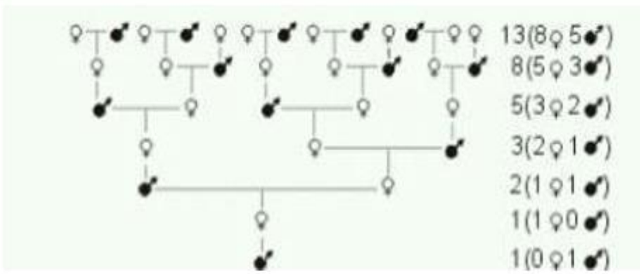
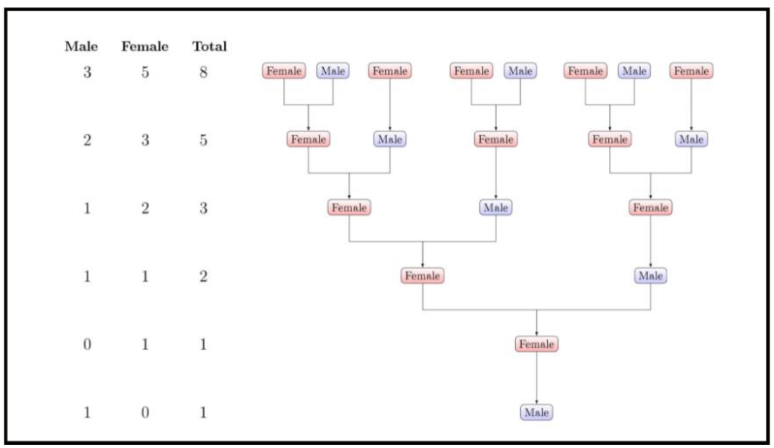
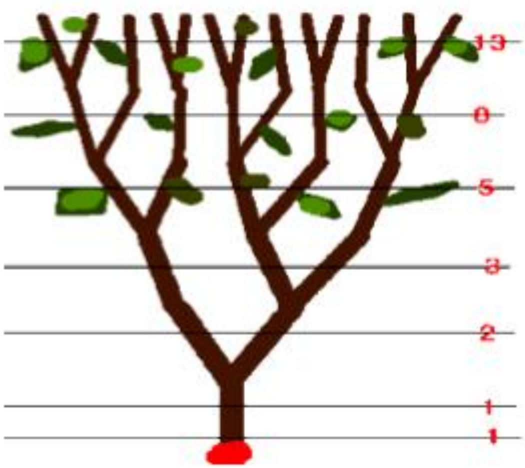


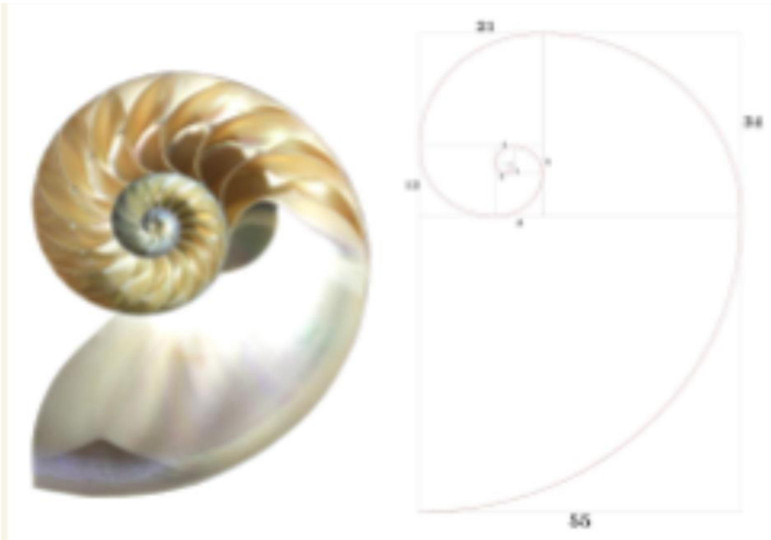
Figure 10 The family tree of a male bee [6]



Male Bee Genealogy



Applications of Fibonacci Nos



shell and Equiangular Spiral

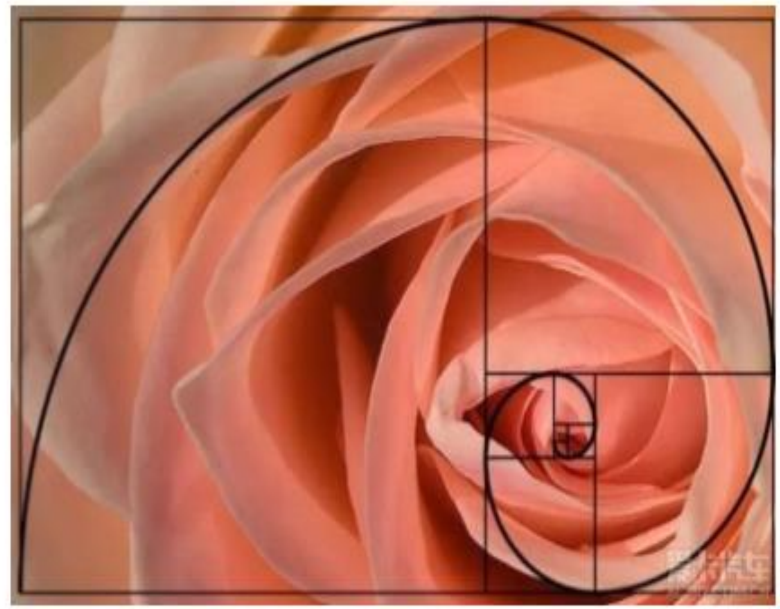
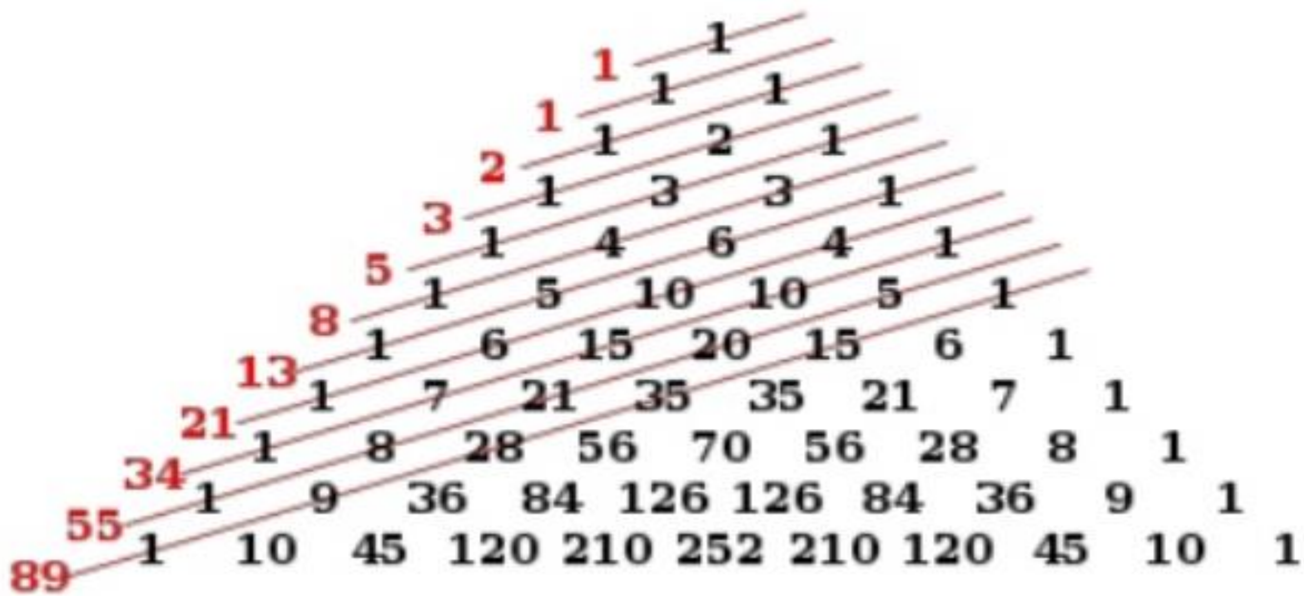


Figure 6 Picture of rose

Applications of Fibonacci Nos

The Fibonacci numbers occur in the sums of "shallow" diagonals in Pascal's triangle:



Lucas Numbers

INTRODUCTION

Similar to the Fibonacci numbers, each Lucas number is defined to be the sum of its two immediate previous terms. The first two Lucas numbers are $L_0 = 2$ and $L_1 = 1$ as opposed to the first two Fibonacci numbers $F_0 = 0$ and $F_1 = 1$. Though closely related in definition, Lucas and Fibonacci numbers exhibit distinct properties.

2, 1, 3, 4 , 7, 11, 18, 29, 47, 76, 123, …

Lucas Numbers

RELATIONSHIP TO FIBONACCI NUMBERS

The Lucas numbers are related to the Fibonacci numbers by the identities :

- $L_n = F_{n-1} + F_{n+1} = F_n + 2F_{n-1}$
- $L_{m+n} = L_{m+1}F_n + L_m F_{n-1}$
- $L_n^2 = 5F_n^2 + 4(-1)^n$, and thus as n approaches $+\infty$, the ratio $\frac{L_n}{F_n}$ approaches $\sqrt{5}$.
- $F_{2n} = L_n F_n$
- $F_{n+k} + (-1)^k F_{n-k} = L_k F_n$
- $F_n = \frac{L_{n-1} + L_{n+1}}{5}$

Example 15

The Fibonacci numbers are defined recursively by
$F_0 = 0, F_1 = 1, F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$
prove that $\sum_{i=0}^{n} {F_i}^2 = F_n \times F_{n+1}$ for all positive integers n.

Basic Step: [show that statement is true for n = 1]
$\sum_{i=0}^{1} {F_i}^2 = {F_0}^2 \times {F_1}^2 = 0 + 1 = 1 = 1 . 1 = F_1 \times F_2 = 1$ ➔ Hence True.

Inductive Hypothesis: (Assume $n=k \geq 1$ is true)
Here we know that $\sum_{i=0}^{k} {F_i}^2 = F_k \times F_{k+1}$ for some k **≥ 1.**

Example 15

Inductive Step: [show that n=k is true n=k+1 is also true]

$\sum_{i=0}^{k+1} F_i^{\,2} = \sum_{i=0}^{k} F_i^{\,2} + F^2_{k+1}$ ➔

Because $\sum_{i=0}^{k} F_i^{\,2} = F_k \times F_{k+1}$ by assumption.

$= (F_k \times F_{k+1}) + F^2_{k+1}$

$= (F_k \times F_{k+1}) + (F_{k+1} \times F_{k+1})$

$= F_{k+1} \times (F_k + F_{k+1})$ ➔

Because $F_n = F_{n-1} + F_{n-2}$

$F_k = (F_{k-1} + F_{k-2})$

$F_{k+1} = (F_k + F_{k-1})$

$\mathbf{F_{k+2} = (F_{k+1} + F_k)}$

$= (F_{k+1} \times F_{k+2})$

Hence its true for n= k+1.

Conclusion: Hence $\sum_{i=0}^{n} F_i^{\,2} = F_n \times F_{n+1}$ is true for all integers n ≥ 1.

Example 16

The Fibonacci numbers are defined recursively by

$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

prove that $\sum_{i=1}^{n} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{n+2}}{2^n}$

for all positive integers n.

Basic Step: [show that statement is true for n = 1]

$\frac{F_0}{2} = 1 - \frac{F_3}{2} = 1 - \frac{2}{2} = 0 \rightarrow$ Hence True.

Inductive Hypothesis: (Assume n=k ≥ 1 is true)

Here we know that $\sum_{i=1}^{k} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{k+2}}{2^k}$ for some k **≥ 1.**

Example 16

Inductive Step: [show that n=k is true n=k+1 is also true]

$$\sum_{i=1}^{k+1} \frac{F_{i-1}}{2^i} = \sum_{i=1}^{k} \frac{F_{i-1}}{2^i} + \frac{F_k}{2^{k+1}}$$

➔ We know that, $\sum_{i=1}^{k} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{k+2}}{2^k}$

$$= \left(1 - \frac{F_{k+2}}{2^k}\right) + \frac{F_k}{2^{k+1}}$$

$$= 1 - \left(\frac{F_{k+2}}{2^k} - \frac{F_k}{2^{k+1}}\right)$$

➔ Takeing LCM , $\frac{1}{2^{k+1}} (2\, F_{k+2} - F_k)$

$$= 1 - \frac{1}{2^{k+1}} (2\, F_{k+2} - F_k)$$

Example 16

$= 1 - \frac{1}{2^{k+1}} \{(F_{k+2} - F_k) + F_{k+2}\} \rightarrow$

$= 1 - \frac{1}{2^{k+1}} \{F_{k+2} - F_k + F_{k+1} + F_k\}$

$= 1 - \frac{1}{2^{k+1}} \{F_{k+1} + F_{k+2}\}$

$= 1 - \frac{1}{2^{k+1}} F_{k+3}$

Because $F_n = F_{n-1} + F_{n-2}$
$F_k = (F_{k-1} + F_{k-2})$
$F_{k+1} = (F_k + F_{k-1})$
$\mathbf{F_{k+2} = (F_{k+1} + F_k)}$

Hence its true for n= k+1.

Conclusion: Hence $\sum_{i=1}^{k+1} \frac{F_{i-1}}{2^i} = \sum_{i=1}^{k} \frac{F_{i-1}}{2^i} + \frac{F_k}{2^{k+1}}$ is true for all integers n ≥ 1.

Example 17

The Lucas numbers are defined recursively by
$L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.
Evaluate L_2 to L_{10}

Using the definition $L_n = L_{n-1} + L_{n-2}$ and $L_0 = 2, L_1 = 1$
We get $L_2 = L_1 + L_0 = 1+2 = 3$
$L_3 = L_2 + L_1 = 3 + 1 = 4$
$L_4 = L_3 + L_2 = 4 + 3 = 7$
….. 11, 18, 29, 47, 76,123…. So on.

Example 17

If $F_{i's}$ are the Fibonacci numbers and $L_{i's}$ are the Lucas numbers, prove that $L_n = F_{n-1} + F_{n+1}$ for all positive integers n.

We know that $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2$ and $F_n = F_{n-1} + F_{n-2}$
$L_0 = 2, L_1 = 1, L_2 = 3\ L_3 = 4$ and $L_n = L_{n-1} + L_{n-2}$

Basic Step: [show that statement is true for n = 1, 2]

$L_n = F_{n-1} + F_{n+1}$

$L_1 = 1$ (LHS)

$F_{n-1} + F_{n+1} = 0 + 1 = F_0 + F_2 = 1$ (RHS) ➔ Hence True.

Inductive Hypothesis: (Assume $n=k \geq 2$ is true)

Here we know that $L_k = F_{k-1} + F_{k+1}$ for some $k \geq 2$.

Example 17

Inductive Step: [show that n=k is true n=k+1 is also true]

$L_{k+1} = L_k + L_{k-1}$ ➔

By the definition $L_n = L_{n-1} + L_{n-2}$

$= (F_{k-1} + F_{k+1}) + (F_{k-2} + F_k)$

By the definition

$= (F_{k-1} + F_{k-2}) + (F_{k+1} + F_k)$

$= F_k + F_{k+2}$ ➔

By the definition of F_n

Hence its true for n= k+1.

Conclusion: Hence $L_n = F_{n-1} + F_{n+1}$ is true for all positive integers n

Example 18

If $F_{i's}$ are the Fibonacci numbers and $L_{i's}$ are the Lucas numbers, prove that $L_{n+4} - L_n = 5F_{n+2}$ for all positive integers n >= 0.

We know that $F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2$ and
$L_0 = 2, L_1 = 1, L_4 = 7\ L_5 = 11$

Basic Step: [show that statement is true for n = 0, 1]

$L_{n+4} - L_n = 5F_{n+2}$

When n =0,

$L_4 - L_0 = 7 – 2 = 5$(LHS)

$5F_{n+2} = 5\ F_2 = 5.\ 1 = 5$ (RHS) ➔ Hence True.

Inductive Hypothesis: (Assume n=k ≥ 2 is true)

Here we know that $L_{k+4} - L_k = 5F_{k+2}$ for some k **≥ 1.**

Example 18

Inductive Step: [shows that n=k is true n=k+1 is also true]

Here we know that $L_{k+4} - L_k = 5F_{k+2}$ for some $k \geq$ **1.**

$L_{k+5} - L_{k+1} = (L_{k+4} + L_{k+3}) - (L_k + L_{k-1})$, using the definition of Li's

$= (L_{k+4} - L_k) + (L(_{k+3} - L_{k-1})$

$= (L_{k+4} - L_k) + (L(_{k-1)+4} - L_{k-1})$ using the above formula

$= 5F_{k+2} + 5F_{k+1}$

$= 5F_{k+3}$

Hence true for n = k+1

Conclusion: Hence $L_{n+4} - L_n = 5F_{n+2}$ is true for all positive integers n >= 0.

Assignment

1. A sequence $\{C_n\}$ is defined recursively by $C_n = 3C_{n-1} - 2C_{n-2}$ for all $n \geq 3$, with $C_1 = 5$ and $C_2 = 3$ as the initial conditions. Show that $C_n = -2^n + 7$.

2. A sequence $\{a_n\}$ is defined recursively by a0 = 1, a1 =1,a2 =1, and $a_n = a_{n-1} + a_{n-3}$ for all $n \geq 3$. Prove that $a_{n+2} \geq \left(\sqrt{2}\right)^n$ for all $n \geq 0$.

Thank You