Exercise Sheet 1:

a) <u>Step 1: EigenValues:</u> $A\begin{pmatrix} 7 & 4 \\ 4 & 13 \end{pmatrix}$

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \begin{pmatrix} 7-\lambda & 4 \\ 4 & 13-\lambda \end{pmatrix} = (7-\lambda)(13-\lambda) - 16 = 0$$

$$91 - 7\lambda - 13\lambda + \lambda^2 - 16 = 0$$

$$= \lambda^2 - 20\lambda + 75 = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{20 \pm \sqrt{400 - 300}}{2}$$

$$= \frac{20 \pm 10}{2} \quad \begin{matrix} \to \lambda_1 = 15 \\ \to \lambda_2 = 5 \end{matrix}$$

<u>Step 2: EigenVectors:</u>

For $\lambda_1 = 15$, $(A - 15I)\vec{u} = 0$

$$\begin{pmatrix} -8 & 4 \\ 4 & -2 \end{pmatrix} \vec{u} = 0 \Longrightarrow -8x + 4y = 0$$

$$y = 2x$$

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $\lambda_2 = 5$, $(A - 5I)\vec{u} = 0 \Rightarrow$

$$\begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix} \vec{u} = 0 \Rightarrow 2x + 4y = 0 \Rightarrow y = -0.5x$$

$$\vec{u}_2 = \begin{pmatrix} 1 \\ -0.5 \end{pmatrix}$$

b) $\vec{x} = a\vec{u}_1 + b\vec{u}_2$

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} a + b \\ 2a - 0.5b \end{bmatrix}$$

$$3 = a + b \qquad b = 3 - a$$

$$1 = 2a - 0{,}5b$$

$$1 = 2a - 0.5(3 - a) \quad 1 = 2a - 1.5 + \frac{a}{2}$$

$a = 1$, $b = 2$

$\vec{x} = \vec{u}_1 + 2\vec{u}_2$ ✓

Task 2.

$x_1'' = \cos\alpha\, x + \sin\alpha\, y$

we want: $E[(x_1'')^2] = \iint_0^1 (\cos\alpha\, x + \sin\alpha\, y)^2 p(x,y)\,dx\,dy$

Where $p(x,y) = 1$ on $[0,1]^2$

$E[(x_1'')^2] = \cos^2\alpha\, E[x^2] + \sin^2\alpha\, E[y^2] + 2\cos\alpha\sin\alpha\, E[xy]$

$E[x^2] = \int_0^1 x^2 p(x)\,dx$ with $p(x) = 1 = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

By symmetry, $E[y^2] = \frac{1}{3}$

$E[xy] = \int_0^1\int_0^1 xy\,p(x)p(y)\,dx\,dy = \left(\int_0^1 x\,dx\right)\left(\int_0^1 y\,dy\right)$

$= \left[\frac{x^2}{2}\right]_0^1 \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$

$E[x_1''^2] = \cos^2\alpha\,\frac{1}{3} + \sin^2\alpha\,\frac{1}{3} + 2\cos\alpha\sin\alpha\,\frac{1}{4}$

$= \frac{1}{3}(\cos^2\alpha + \sin^2\alpha) + \frac{1}{4}\sin 2\alpha$

$= \frac{1}{3} + \frac{1}{4}\sin(2\alpha)$

$$\boxed{E[(x_1'')^2] = \frac{1}{3} + \frac{1}{4}\sin(2\alpha)}$$

Finding maximum: $\frac{d}{d\alpha}\left(\frac{1}{3} + \frac{1}{4}\sin 2\alpha\right) = \frac{1}{2}\cos 2\alpha \neq 0$

$\cos 2\alpha = 0$ $\quad 2\alpha = 90^\circ$ or 270°

$\alpha = 45^\circ, 135^\circ, 225^\circ, 315^\circ$

Max 2nd: $\sin 2\alpha = +1$ $\Rightarrow$ $\boxed{\alpha = 45^\circ \text{ or } 225^\circ}$