Category Theory for Programmers

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "g \circ f"'] & Y \arrow[d, "g"] \\
& Z
\end{tikzcd}
$$

Bartosz Milewski

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Preface

> For some time now I've been floating the idea of writing a book about category theory that would be targeted at programmers. Mind you, not computer scientists but programmers — engineers rather than scientists. I know this sounds crazy and I am properly scared. I can't deny that there is a huge gap between science and engineering because I have worked on both sides of the divide. But I've always felt a very strong compulsion to explain things. I have tremendous admiration for Richard Feynman who was the master of simple explanations. I know I'm no Feynman, but I will try my best. I'm starting by publishing this preface — which is supposed to motivate the reader to learn category theory — in hopes of starting a discussion and soliciting feedback.[1]

I WILL ATTEMPT, in the space of a few paragraphs, to convince you that this book is written for you, and whatever objections you might have to learning one of the most abstract branches of mathematics in your "copious spare time" are totally unfounded.

[1] You may also watch me teaching this material to a live audience.

My optimism is based on several observations. First, category theory is a treasure trove of extremely useful programming ideas. Haskell programmers have been tapping this resource for a long time, and the ideas are slowly percolating into other languages, but this process is too slow. We need to speed it up.

Second, there are many different kinds of math, and they appeal to different audiences. You might be allergic to calculus or algebra, but it doesn't mean you won't enjoy category theory. I would go as far as to argue that category theory is the kind of math that is particularly well suited for the minds of programmers. That's because category theory — rather than dealing with particulars — deals with structure. It deals with the kind of structure that makes programs composable.

Composition is at the very root of category theory — it's part of the definition of the category itself. And I will argue strongly that composition is the essence of programming. We've been composing things forever, long before some great engineer came up with the idea of a subroutine. Some time ago the principles of structural programming revolutionized programming because they made blocks of code composable. Then came object oriented programming, which is all about composing objects. Functional programming is not only about composing functions and algebraic data structures — it makes concurrency composable — something that's virtually impossible with other programming paradigms.

Third, I have a secret weapon, a butcher's knife, with which I will butcher math to make it more palatable to programmers. When you're a professional mathematician, you have to be very careful to get all your assumptions straight, qualify every statement properly, and construct all your proofs rigorously. This makes mathematical papers and books extremely hard to read for an outsider. I'm a physicist by training, and in physics we made amazing advances using informal reasoning. Mathematicians laughed at the Dirac delta function, which was made up on the spot by the great physicist P. A. M. Dirac to solve some differential equations. They stopped laughing when they discovered a completely new branch of calculus called distribution theory that formalized Dirac's insights.

Of course when using hand-waving arguments you run the risk of saying something blatantly wrong, so I will try to make sure that there is solid mathematical theory behind informal arguments in this book. I do have a worn-out copy of Saunders Mac Lane's *Category Theory for the Working Mathematician* on my nightstand.

Since this is category theory *for programmers* I will illustrate all major concepts using computer code. You are probably aware that functional languages are closer to math than the more popular imperative languages. They also offer more abstracting power. So a natural temptation would be to say: You must learn Haskell before the bounty of category theory becomes available to you. But that would imply that category theory has no application outside of functional programming and that's simply not true. So I will provide a lot of C++ examples. Granted, you'll have to overcome some ugly syntax, the patterns might not stand out from the background of verbosity, and you might be forced to do some copy and paste in lieu of higher abstraction, but that's just the lot of a C++ programmer.

But you're not off the hook as far as Haskell is concerned. You don't have to become a Haskell programmer, but you need it as a language for sketching and documenting ideas to be implemented in C++. That's exactly how I got started with Haskell. I found its terse syntax and powerful type system a great help in understanding and implementing C++ templates, data structures, and algorithms. But since I can't expect the

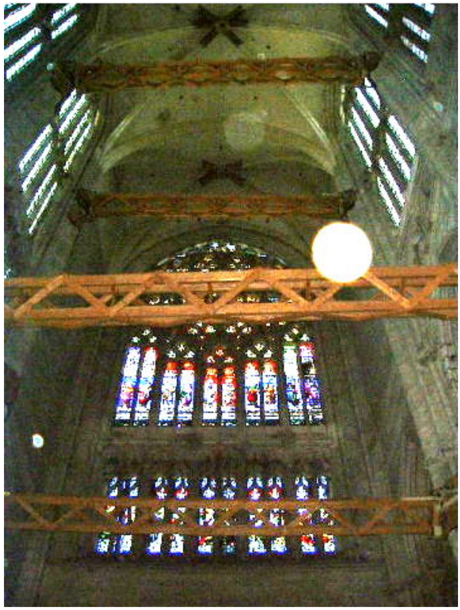
readers to already know Haskell, I will introduce it slowly and explain everything as I go.

If you're an experienced programmer, you might be asking yourself: I've been coding for so long without worrying about category theory or functional methods, so what's changed? Surely you can't help but notice that there's been a steady stream of new functional features invading imperative languages. Even Java, the bastion of object-oriented programming, let the lambdas in C++ has recently been evolving at a frantic pace — a new standard every few years — trying to catch up with the changing world. All this activity is in preparation for a disruptive change or, as we physicist call it, a phase transition. If you keep heating water, it will eventually start boiling. We are now in the position of a frog that must decide if it should continue swimming in increasingly hot water, or start looking for some alternatives.

One of the forces that are driving the big change is the multicore revolution. The prevailing programming paradigm, object oriented programming, doesn't buy you anything in the realm of concurrency and parallelism, and instead encourages dangerous and buggy design. Data hiding, the basic premise of object orientation, when combined with sharing and mutation, becomes a recipe for data races. The idea of combining a mutex with the data it protects is nice but, unfortunately, locks don't compose, and lock hiding makes deadlocks more likely and harder to debug.

But even in the absence of concurrency, the growing complexity of software systems is testing the limits of scalability of the imperative paradigm. To put it simply, side effects are getting out of hand. Granted, functions that have side effects are often convenient and easy to write. Their effects can in principle be encoded in their names and in the comments. A function called SetPassword or WriteFile is obviously mutating some state and generating side effects, and we are used to dealing with that. It's only when we start composing functions that have side effects on top of other functions that have side effects, and so on, that things start getting hairy. It's not that side effects are inherently bad — it's the fact that they are hidden from view that makes them impossible to manage at larger scales. Side effects don't scale, and imperative programming is all about side effects.

Changes in hardware and the growing complexity of software are forcing us to rethink the foundations of programming. Just like the builders of Europe's great gothic cathedrals we've been honing our craft to the limits of material and structure. There is an unfinished gothic cathedral in Beauvais, France, that stands witness to this deeply human struggle with limitations. It was intended to beat all previous records of height and lightness, but it suffered a series of collapses. Ad hoc measures like iron rods and wooden supports keep it from disintegrating, but obviously a lot of things went wrong. From a modern per-



Ad hoc measures preventing the Beauvais cathedral from collapsing.

spective, it's a miracle that so many gothic structures had been successfully completed without the help of modern material science, computer modelling, finite element analysis, and general math and physics. I hope future generations will be as admiring of the programming skills we've been displaying in building complex operating systems, web servers, and the internet infrastructure. And, frankly, they should, because we've done all this based on very flimsy theoretical foundations. We have to fix those foundations if we want to move forward.

Part I

Part One

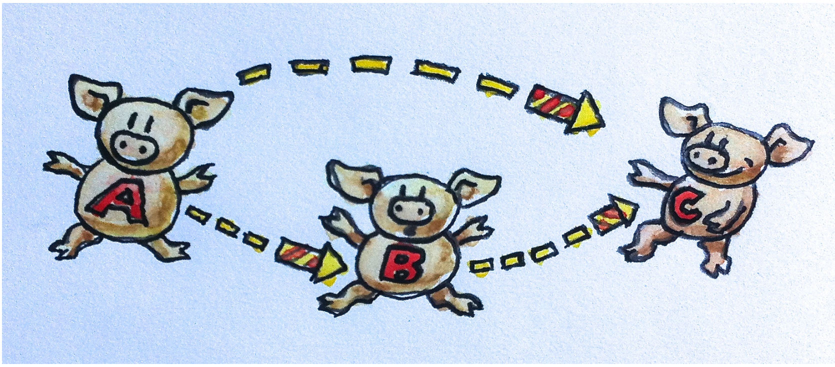
1

Category: The Essence of Composition

A CATEGORY is an embarrassingly simple concept. A category consists of *objects* and *arrows* that go between them. That's why categories are so easy to represent pictorially. An object can be drawn as a circle or a point, and an arrow… is an arrow. (Just for variety, I will occasionally draw objects as piggies and arrows as fireworks.) But the essence of a category is *composition*. Or, if you prefer, the essence of composition is a category. Arrows compose, so if you have an arrow from object A to object B, and another arrow from object B to object C, then there must be an arrow — their composition — that goes from A to C.

1.1 Arrows as Functions

Is this already too much abstract nonsense? Do not despair. Let's talk concretes. Think of arrows, which are also called *morphisms*, as functions. You have a function `f` that takes an argument of type A and



In a category, if there is an arrow going from A to B and an arrow going from B to C then there must also be a direct arrow from A to C that is their composition. This diagram is not a full category because it's missing identity morphisms (see later).

returns a B. You have another function `g` that takes a B and returns a C. You can compose them by passing the result of `f` to `g`. You have just defined a new function that takes an A and returns a C.

In math, such composition is denoted by a small circle between functions: $g \circ f$. Notice the right to left order of composition. For some people this is confusing. You may be familiar with the pipe notation in Unix, as in:

```
lsof | grep Chrome
```

or the chevron >> in F#, which both go from left to right. But in mathematics and in Haskell functions compose right to left. It helps if you read `g∘f` as "g *after* f."

Let's make this even more explicit by writing some C code. We have one function `f` that takes an argument of type `A` and returns a value of type `B`:

```
B f(A a);
```

and another:

```
C g(B b);
```

Their composition is:

```
C g_after_f(A a)
{
    return g(f(a));
}
```

Here, again, you see right-to-left composition: `g(f(a))`; this time in C.

I wish I could tell you that there is a template in the C++ Standard Library that takes two functions and returns their composition, but there isn't one. So let's try some Haskell for a change. Here's the declaration of a function from A to B:

```
f :: A -> B
```

Similarly:

```
g :: B -> C
```

Their composition is:

```
g . f
```

Once you see how simple things are in Haskell, the inability to express straightforward functional concepts in C++ is a little embarrassing. In fact, Haskell will let you use Unicode characters so you can write composition as:

```
g ∘ f
```

You can even use Unicode double colons and arrows:

```
f ∷ A → B
```

So here's the first Haskell lesson: Double colon means "has the type of…" A function type is created by inserting an arrow between two types. You compose two functions by inserting a period between them (or a Unicode circle).

1.2 Properties of Composition

There are two extremely important properties that the composition in any category must satisfy.

1. Composition is associative. If you have three morphisms, f, g, and h, that can be composed (that is, their objects match end-to-end), you don't need parentheses to compose them. In math notation this is expressed as:

   ```
   h∘(g∘f) = (h∘g)∘f = h∘g∘f
   ```

   In (pseudo) Haskell:

   ```
   f :: A -> B
   g :: B -> C
   h :: C -> D
   h . (g . f) == (h . g) . f == h . g . f
   ```

(I said "pseudo," because equality is not defined for functions.)

Associativity is pretty obvious when dealing with functions, but it may be not as obvious in other categories.

2. For every object A there is an arrow which is a unit of composition. This arrow loops from the object to itself. Being a unit of composition means that, when composed with any arrow that either starts at A or ends at A, respectively, it gives back the same arrow. The unit arrow for object A is called $\mathtt{id}_A$ (*identity* on A). In math notation, if `f` goes from A to B then

   $\mathtt{f} \circ \mathtt{id}_A = \mathtt{f}$

   and

   $\mathtt{id}_B \circ \mathtt{f} = \mathtt{f}$

When dealing with functions, the identity arrow is implemented as the identity function that just returns back its argument. The implementation is the same for every type, which means this function is universally polymorphic. In C++ we could define it as a template:

```
template<class T> T id(T x) { return x; }
```

Of course, in C++ nothing is that simple, because you have to take into account not only what you're passing but also how (that is, by value, by reference, by const reference, by move, and so on).

In Haskell, the identity function is part of the standard library (called Prelude). Here's its declaration and definition:

```
id :: a -> a
id x = x
```

As you can see, polymorphic functions in Haskell are a piece of cake. In the declaration, you just replace the type with a type variable. Here's the trick: names of concrete types always start with a capital letter, names of type variables start with a lowercase letter. So here `a` stands for all types.

Haskell function definitions consist of the name of the function followed by formal parameters — here just one, `x`. The body of the function follows the equal sign. This terseness is often shocking to newcomers but you will quickly see that it makes perfect sense. Function definition and function call are the bread and butter of functional programming so their syntax is reduced to the bare minimum. Not only are there no parentheses around the argument list but there are no commas between arguments (you'll see that later, when we define functions of multiple arguments).

The body of a function is always an expression — there are no statements in functions. The result of a function is this expression — here, just `x`.

This concludes our second Haskell lesson.

The identity conditions can be written (again, in pseudo-Haskell) as:

```
f . id == f
id . f == f
```

You might be asking yourself the question: Why would anyone bother with the identity function — a function that does nothing? Then again, why do we bother with the number zero? Zero is a symbol for nothing. Ancient Romans had a number system without a zero and they were able to build excellent roads and aqueducts, some of which survive to this day.

Neutral values like zero or `id` are extremely useful when working with symbolic variables. That's why Romans were not very good at algebra, whereas the Arabs and the Persians, who were familiar with the concept of zero, were. So the identity function becomes very handy as an argument to, or a return from, a higher-order function. Higher order functions are what make symbolic manipulation of functions possible. They are the algebra of functions.

To summarize: A category consists of objects and arrows (morphisms). Arrows can be composed, and the composition is associative. Every object has an identity arrow that serves as a unit under composition.

1.3 Composition is the Essence of Programming

Functional programmers have a peculiar way of approaching problems. They start by asking very Zen-like questions. For instance, when designing an interactive program, they would ask: What is interaction? When implementing Conway's Game of Life, they would probably ponder about the meaning of life. In this spirit, I'm going to ask: What is programming? At the most basic level, programming is about telling the computer what to do. "Take the contents of memory address x and add it to the contents of the register EAX." But even when we program in assembly, the instructions we give the computer are an expression of something more meaningful. We are solving a non-trivial problem (if it were trivial, we wouldn't need the help of the computer). And how do we solve problems? We decompose bigger problems into smaller problems. If the smaller problems are still too big, we decompose them further, and so on. Finally, we write code that solves all the small problems. And then comes the essence of programming: we compose those pieces of code to create solutions to larger problems. Decomposition wouldn't make sense if we weren't able to put the pieces back together.

This process of hierarchical decomposition and recomposition is not imposed on us by computers. It reflects the limitations of the human mind. Our brains can only deal with a small number of concepts at a time. One of the most cited papers in psychology, The Magical Number Seven, Plus or Minus Two, postulated that we can only keep 7 ± 2 "chunks" of information in our minds. The details of our understanding of the human short-term memory might be changing, but we know for sure that it's limited. The bottom line is that we are unable to deal with the soup of objects or the spaghetti of code. We need structure not because well-structured programs are pleasant to look at, but because otherwise our brains can't process them efficiently. We often describe some piece of code as elegant or beautiful, but what we really mean is that it's easy to process by our limited human minds. Elegant code creates chunks that are just the right size and come in just the right number for our mental digestive system to assimilate them.

So what are the right chunks for the composition of programs? Their surface area has to increase slower than their volume. (I like this analogy because of the intuition that the surface area of a geometric object grows with the square of its size — slower than the volume, which grows with the cube of its size.) The surface area is the information we need in order to compose chunks. The volume is the information we need in order to implement them. The idea is that, once a chunk is implemented, we can forget about the details of its implementation and concentrate on how it interacts with other chunks. In object-oriented programming, the surface is the class declaration of the object, or its abstract interface. In functional programming, it's the declaration of a function. (I'm simplifying things a bit, but that's the gist of it.)

Category theory is extreme in the sense that it actively discourages us from looking inside the objects. An object in category theory is an abstract nebulous entity. All you can ever know about it is how it relates to other object — how it connects with them using arrows. This is how internet search engines rank web sites by analyzing incoming and outgoing links (except when they cheat). In object-oriented programming, an idealized object is only visible through its abstract interface (pure surface, no volume), with methods playing the role of arrows. The moment you have to dig into the implementation of the object in order to understand how to compose it with other objects, you've lost the advantages of your programming paradigm.

1.4 Challenges

1. Implement, as best as you can, the identity function in your favorite language (or the second favorite, if your favorite language happens to be Haskell).
2. Implement the composition function in your favorite language. It takes two functions as arguments and returns a function that is their composition.
3. Write a program that tries to test that your composition function respects identity.
4. Is the world-wide web a category in any sense? Are links morphisms?
5. Is Facebook a category, with people as objects and friendships as morphisms?
6. When is a directed graph a category?

2

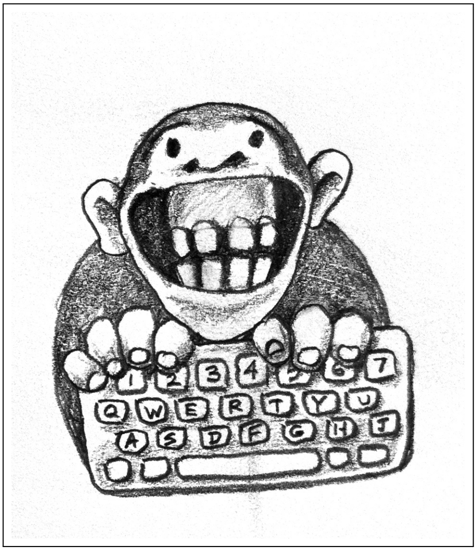
Types and Functions

THE CATEGORY OF TYPES AND FUNCTIONS plays an important role in programming, so let's talk about what types are and why we need them.

2.1 Who Needs Types?

There seems to be some controversy about the advantages of static vs. dynamic and strong vs. weak typing. Let me illustrate these choices with a thought experiment. Imagine millions of monkeys at computer keyboards happily hitting random keys, producing programs, compiling, and running them.

With machine language, any combination of bytes produced by monkeys would be accepted and run. But with higher level languages, we do appreciate the fact that a compiler is able to detect lexical and grammatical errors. Lots of monkeys will go without bananas, but the remaining programs will have a better chance of being useful. Type



checking provides yet another barrier against nonsensical programs. Moreover, whereas in a dynamically typed language, type mismatches would be discovered at runtime, in strongly typed statically checked languages type mismatches are discovered at compile time, eliminating lots of incorrect programs before they have a chance to run.

So the question is, do we want to make monkeys happy, or do we want to produce correct programs?

The usual goal in the typing monkeys thought experiment is the production of the complete works of Shakespeare. Having a spell checker and a grammar checker in the loop would drastically increase the odds. The analog of a type checker would go even further by making sure that, once Romeo is declared a human being, he doesn't sprout leaves or trap photons in his powerful gravitational field.

2.2 Types Are About Composability

Category theory is about composing arrows. But not any two arrows can be composed. The target object of one arrow must be the same as the source source object of the next arrow. In programming we pass the results on one function to another. The program will not work if the target function is not able to correctly interpret the data produced by the source function. The two ends must fit for the composition to work. The stronger the type system of the language, the better this match can be described and mechanically verified.

The only serious argument I hear against strong static type checking is that it might eliminate some programs that are semantically correct. In practice, this happens extremely rarely and, in any case, every language provides some kind of a backdoor to bypass the type system when that's really necessary. Even Haskell has `unsafeCoerce`. But such devices should be used judiciously. Franz Kafka's character, Gregor Samsa, breaks the type system when he metamorphoses into a giant bug, and we all know how it ends.

Another argument I hear a lot is that dealing with types imposes too much burden on the programmer. I could sympathize with this sentiment after having to write a few declarations of iterators in C++ myself, except that there is a technology called *type inference* that lets the compiler deduce most of the types from the context in which they are used. In C++, you can now declare a variable `auto` and let the compiler figure out its type.

In Haskell, except on rare occasions, type annotations are purely optional. Programmers tend to use them anyway, because they can tell a lot about the semantics of code, and they make compilation errors easier to understand. It's a common practice in Haskell to start a project by designing the types. Later, type annotations drive the implementation and become compiler-enforced comments.

Strong static typing is often used as an excuse for not testing the code. You may sometimes hear Haskell programmers saying, "If it compiles, it must be correct." Of course, there is no guarantee that a type-correct program is correct in the sense of producing the right output. The result of this cavalier attitude is that in several studies Haskell didn't come as strongly ahead of the pack in code quality as one would expect. It seems that, in the commercial setting, the pressure to fix bugs is applied only up to a certain quality level, which has everything to do with the economics of software development and the tolerance of the end user, and very little to do with the programming language or methodology. A better criterion would be to measure how many projects fall behind schedule or are delivered with drastically reduced functionality.

As for the argument that unit testing can replace strong typing, consider the common refactoring practice in strongly typed languages: changing the type of an argument of a particular function. In a strongly typed language, it's enough to modify the declaration of that function and then fix all the build breaks. In a weakly typed language, the fact that a function now expects different data cannot be propagated to call sites. Unit testing may catch some of the mismatches, but testing is almost always a probabilistic rather than a deterministic process. Testing is a poor substitute for proof.

2.3 What Are Types?

The simplest intuition for types is that they are sets of values. The type `Bool` (remember, concrete types start with a capital letter in Haskell) is a two-element set of `True` and `False`. Type `Char` is a set of all Unicode characters like `a` or `ą`.

Sets can be finite or infinite. The type of `String`, which is a synonym for a list of `Char`, is an example of an infinite set.

When we declare `x` to be an `Integer`:

```
x :: Integer
```

we are saying that it's an element of the set of integers. `Integer` in Haskell is an infinite set, and it can be used to do arbitrary precision arithmetic. There is also a finite-set `Int` that corresponds to machine type, just like the C++ `int`.

There are some subtleties that make this identification of types and sets tricky. There are problems with polymorphic functions that involve circular definitions, and with the fact that you can't have a set of all sets; but as I promised, I won't be a stickler for math. The great thing is that there is a category of sets, which is called **Set**, and we'll just work with it. In **Set**, objects are sets and morphisms (arrows) are functions.

Set is a very special category, because we can actually peek inside its objects and get a lot of intuitions from doing that. For instance, we know that an empty set has no elements. We know that there are special one-element sets. We know that functions map elements of one set to elements of another set. They can map two elements to one, but not one element to two. We know that an identity function maps each element of a set to itself, and so on. The plan is to gradually forget all this information and instead express all those notions in purely categorical terms, that is in terms of objects and arrows.

In the ideal world we would just say that Haskell types are sets and Haskell functions are mathematical functions between sets. There is just one little problem: A mathematical function does not execute any code — it just knows the answer. A Haskell function has to calculate the answer. It's not a problem if the answer can be obtained in a finite number of steps — however big that number might be. But there are some calculations that involve recursion, and those might never terminate. We can't just ban non-terminating functions from Haskell because distinguishing between terminating and non-terminating functions is undecidable — the famous halting problem. That's why computer scientists came up with a brilliant idea, or a major hack, depending on your point of view, to extend every type by one more special value called the *bottom* and denoted by `_|_`, or Unicode ⊥. This "value" corresponds to a non-terminating computation. So a function declared as:

```
f :: Bool -> Bool
```

may return `True`, `False`, or `_|_`; the latter meaning that it would never terminate.

Interestingly, once you accept the bottom as part of the type system, it is convenient to treat every runtime error as a bottom, and even allow functions to return the bottom explicitly. The latter is usually done using the expression `undefined`, as in:

```
f :: Bool -> Bool
f x = undefined
```

This definition type checks because `undefined` evaluates to bottom, which is a member of any type, including `Bool`. You can even write:

```
f :: Bool -> Bool
f = undefined
```

(without the `x`) because the bottom is also a member of the type `Bool->Bool`.

Functions that may return bottom are called partial, as opposed to total functions, which return valid results for every possible argument.

Because of the bottom, you'll see the category of Haskell types and functions referred to as **Hask** rather than **Set**. From the theoretical point of view, this is the source of never-ending complications, so at this point I will use my butcher's knife and terminate this line of reasoning. From the pragmatic point of view, it's okay to ignore non-terminating functions and bottoms, and treat **Hask** as bona fide **Set**.[1]

2.4 Why Do We Need a Mathematical Model?

As a programmer you are intimately familiar with the syntax and grammar of your programming language. These aspects of the language are usually described using formal notation at the very beginning of the language spec. But the meaning, or semantics, of the language is much harder to describe; it takes many more pages, is rarely formal enough, and almost never complete. Hence the never ending discussions among language lawyers, and a whole cottage industry of books dedicated to the exegesis of the finer points of language standards.

There are formal tools for describing the semantics of a language but, because of their complexity, they are mostly used with simplified academic languages, not real-life programming behemoths. One such tool called *operational semantics* describes the mechanics of program execution. It defines a formalized idealized interpreter. The semantics of

[1] Nils Anders Danielsson, John Hughes, Patrik Jansson, Jeremy Gibbons, Fast and Loose Reasoning is Morally Correct. This paper provides justification for ignoring bottoms in most contexts.

industrial languages, such as C++, is usually described using informal operational reasoning, often in terms of an "abstract machine."

The problem is that it's very hard to prove things about programs using operational semantics. To show a property of a program you essentially have to "run it" through the idealized interpreter.

It doesn't matter that programmers never perform formal proofs of correctness. We always "think" that we write correct programs. Nobody sits at the keyboard saying, "Oh, I'll just throw a few lines of code and see what happens." We think that the code we write will perform certain actions that will produce desired results. We are usually quite surprised when it doesn't. That means we do reason about programs we write, and we usually do it by running an interpreter in our heads. It's just really hard to keep track of all the variables. Computers are good at running programs — humans are not! If we were, we wouldn't need computers.

But there is an alternative. It's called *denotational semantics* and it's based on math. In denotational semantics every programing construct is given its mathematical interpretation. With that, if you want to prove a property of a program, you just prove a mathematical theorem. You might think that theorem proving is hard, but the fact is that we humans have been building up mathematical methods for thousands of years, so there is a wealth of accumulated knowledge to tap into. Also, as compared to the kind of theorems that professional mathematicians prove, the problems that we encounter in programming are usually quite simple, if not trivial.

Consider the definition of a factorial function in Haskell, which is a language quite amenable to denotational semantics:

```
fact n = product [1..n]
```

The expression `[1..n]` is a list of integers from `1` to `n`. The function `product` multiplies all elements of a list. That's just like a definition of factorial taken from a math text. Compare this with C:

```
int fact(int n) {
    int i;
    int result = 1;
    for (i = 2; i <= n; ++i)
        result *= i;
    return result;
}
```

Need I say more?

Okay, I'll be the first to admit that this was a cheap shot! A factorial function has an obvious mathematical denotation. An astute reader might ask: What's the mathematical model for reading a character from the keyboard or sending a packet across the network? For the longest time that would have been an awkward question leading to a rather convoluted explanation. It seemed like denotational semantics wasn't the best fit for a considerable number of important tasks that were essential for writing useful programs, and which could be easily tackled by operational semantics. The breakthrough came from category theory. Eugenio Moggi discovered that computational effect can be mapped to monads. This turned out to be an important observation that not only gave denotational semantics a new lease on life and made pure functional programs more usable, but also shed new light on traditional programming. I'll talk about monads later, when we develop more categorical tools.

One of the important advantages of having a mathematical model for programming is that it's possible to perform formal proofs of correctness of software. This might not seem so important when you're writing consumer software, but there are areas of programming where the price of failure may be exorbitant, or where human life is at stake. But even when writing web applications for the health system, you may appreciate the thought that functions and algorithms from the Haskell standard library come with proofs of correctness.

2.5 Pure and Dirty Functions

The things we call functions in C++ or any other imperative language, are not the same things mathematicians call functions. A mathematical function is just a mapping of values to values.

We can implement a mathematical function in a programming language: Such a function, given an input value will calculate the output value. A function to produce a square of a number will probably multiply the input value by itself. It will do it every time it's called, and it's guaranteed to produce the same output every time it's called with the same input. The square of a number doesn't change with the phases of the Moon.

Also, calculating the square of a number should not have a side effect of dispensing a tasty treat for your dog. A "function" that does that cannot be easily modelled as a mathematical function.

In programming languages, functions that always produce the same result given the same input and have no side effects are called *pure functions*. In a pure functional language like Haskell all functions are pure. Because of that, it's easier to give these languages denotational semantics and model them using category theory. As for other languages, it's always possible to restrict yourself to a pure subset, or reason about side effects separately. Later we'll see how monads let us model all kinds of effects using only pure functions. So we really don't lose anything by restricting ourselves to mathematical functions.

2.6 Examples of Types

Once you realize that types are sets, you can think of some rather exotic types. For instance, what's the type corresponding to an empty set? No, it's not C++ `void`, although this type *is* called `Void` in Haskell. It's a type that's not inhabited by any values. You can define a function that takes `Void`, but you can never call it. To call it, you would have to provide a value of the type `Void`, and there just aren't any. As for what this function can return, there are no restrictions whatsoever. It can return any type (although it never will, because it can't be called). In other words it's a function that's polymorphic in the return type. Haskellers have a name for it:

```
absurd :: Void -> a
```

(Remember, `a` is a type variable that can stand for any type.) The name is not coincidental. There is deeper interpretation of types and functions in terms of logic called the Curry-Howard isomorphism. The type `Void` represents falsity, and the type of the function `absurd` corresponds to the statement that from falsity follows anything, as in the Latin adage "ex falso sequitur quodlibet."

Next is the type that corresponds to a singleton set. It's a type that has only one possible value. This value just "is." You might not immediately recognise it as such, but that is the C++ `void`. Think of functions from and to this type. A function from `void` can always be called. If it's a pure function, it will always return the same result. Here's an example of such a function:

```
int f44() { return 44; }
```

You might think of this function as taking "nothing", but as we've just seen, a function that takes "nothing" can never be called because there is no value representing "nothing." So what does this function take? Conceptually, it takes a dummy value of which there is only one instance ever, so we don't have to mention it explicitly. In Haskell, however, there is a symbol for this value: an empty pair of parentheses, `()`. So, by a funny coincidence (or is it a coincidence?), the call to a function of void looks the same in C++ and in Haskell. Also, because of the Haskell's love of terseness, the same symbol `()` is used for the type, the constructor, and the only value corresponding to a singleton set. So here's this function in Haskell:

```
f44 :: () -> Integer
f44 () = 44
```

The first line declares that `f44` takes the type `()`, pronounced "unit," into the type `Integer`. The second line defines `f44` by pattern matching the only constructor for unit, namely `()`, and producing the number 44. You call this function by providing the unit value `()`:

```
f44 ()
```

Notice that every function of unit is equivalent to picking a single element from the target type (here, picking the `Integer` 44). In fact you could think of `f44` as a different representation for the number 44. This is an example of how we can replace explicit mention of elements of a set by talking about functions (arrows) instead. Functions from unit to any type A are in one-to-one correspondence with the elements of that set A.

What about functions with the `void` return type, or, in Haskell, with the unit return type? In C++ such functions are used for side effects, but we know that these are not real functions in the mathematical sense of the word. A pure function that returns unit does nothing: it discards its argument.

Mathematically, a function from a set A to a singleton set maps every element of A to the single element of that singleton set. For every A there is exactly one such function. Here's this function for `Integer`:

```
fInt :: Integer -> ()
fInt x = ()
```

You give it any integer, and it gives you back a unit. In the spirit of terseness, Haskell lets you use the wildcard pattern, the underscore, for an argument that is discarded. This way you don't have to invent a name for it. So the above can be rewritten as:

```
fInt :: Integer -> ()
fInt _ = ()
```

Notice that the implementation of this function not only doesn't depend on the value passed to it, but it doesn't even depend on the type of the argument.

Functions that can be implemented with the same formula for any type are called parametrically polymorphic. You can implement a whole family of such functions with one equation using a type parameter instead of a concrete type. What should we call a polymorphic function from any type to unit type? Of course we'll call it `unit`:

```
unit :: a -> ()
unit _ = ()
```

In C++ you would write this function as:

```
template<class T>
void unit(T) {}
```

Next in the typology of types is a two-element set. In C++ it's called `bool` and in Haskell, predictably, `Bool`. The difference is that in C++ `bool` is a built-in type, whereas in Haskell it can be defined as follows:

```
data Bool = True | False
```

(The way to read this definition is that `Bool` is either `True` or `False`.) In principle, one should also be able to define a Boolean type in C++ as an enumeration:

```
enum bool {
    true,
    false
};
```

but C++ `enum` is secretly an integer. The C++11 "`enum class`" could have been used instead, but then you would have to qualify its values with the class name, as in `bool::true` and `bool::false`, not to mention having to include the appropriate header in every file that uses it.

Pure functions from `Bool` just pick two values from the target type, one corresponding to `True` and another to `False`.

Functions to `Bool` are called *predicates*. For instance, the Haskell library `Data.Char` is full of predicates like `isAlpha` or `isDigit`. In C++ there is a similar library that defines, among others, `isalpha` and `isdigit`, but these return an `int` rather than a Boolean. The actual predicates are defined in `std::ctype` and have the form `ctype::is(alpha, c)`, `ctype::is(digit, c)`, etc.

2.7 Challenges

1. Define a higher-order function (or a function object) `memoize` in your favorite language. This function takes a pure function `f` as an argument and returns a function that behaves almost exactly like `f`, except that it only calls the original function once for every argument, stores the result internally, and subsequently returns this stored result every time it's called with the same argument. You can tell the memoized function from the original by watching its performance. For instance, try to memoize a function that takes a long time to evaluate. You'll have to wait for the result the first time you call it, but on subsequent calls, with the same argument, you should get the result immediately.
2. Try to memoize a function from your standard library that you normally use to produce random numbers. Does it work?
3. Most random number generators can be initialized with a seed. Implement a function that takes a seed, calls the random number generator with that seed, and returns the result. Memoize that function. Does it work?
4. Which of these C++ functions are pure? Try to memoize them and observe what happens when you call them multiple times: memoized and not.

    (a) The factorial function from the example in the text.

    (b) `std::getchar()`

    (c)
    ```
    bool f() {
        std::cout << "Hello!" << std::endl;
        return true;
    }
    ```

    (d) `int f(int x)`

```
{
    static int y = 0;
    y += x;
    return y;
}
```

5. How many different functions are there from `Bool` to `Bool`? Can you implement them all?
6. Draw a picture of a category whose only objects are the types `Void`, `()` (unit), and `Bool`; with arrows corresponding to all possible functions between these types. Label the arrows with the names of the functions.

3

Categories Great and Small

You can get real appreciation for categories by studying a variety of examples. Categories come in all shapes and sizes and often pop up in unexpected places. We'll start with something really simple.

3.1 No Objects

The most trivial category is one with zero objects and, consequently, zero morphisms. It's a very sad category by itself, but it may be important in the context of other categories, for instance, in the category of all categories (yes, there is one). If you think that an empty set makes sense, then why not an empty category?

3.2 Simple Graphs

You can build categories just by connecting objects with arrows. You can imagine starting with any directed graph and making it into a category by simply adding more arrows. First, add an identity arrow at each node. Then, for any two arrows such that the end of one coincides with the beginning of the other (in other words, any two *composable* arrows), add a new arrow to serve as their composition. Every time you add a new arrow, you have to also consider its composition with any other arrow (except for the identity arrows) and itself. You usually end up with infinitely many arrows, but that's okay.

Another way of looking at this process is that you're creating a category, which has an object for every node in the graph, and all possible *chains* of composable graph edges as morphisms. (You may even consider identity morphisms as special cases of chains of length zero.)

Such a category is called a *free category* generated by a given graph. It's an example of a free construction, a process of completing a given structure by extending it with a minimum number of items to satisfy its laws (here, the laws of a category). We'll see more examples of it in the future.

3.3 Orders

And now for something completely different! A category where a morphism is a relation between objects: the relation of being less than or equal. Let's check if it indeed is a category. Do we have identity morphisms? Every object is less than or equal to itself: check! Do we have composition? If `a <= b` and `b <= c` then `a <= c`: check! Is composition associative? Check! A set with a relation like this is called a *preorder*, so a preorder is indeed a category.

You can also have a stronger relation, that satisfies an additional condition that, if `a <= b` and `b <= a` then `a` must be the same as `b`. That's called a *partial order*.

Finally, you can impose the condition that any two objects are in a relation with each other, one way or another; and that gives you a *linear order* or *total order*.

Let's characterize these ordered sets as categories. A preorder is a category where there is at most one morphism going from any object a to any object b. Another name for such a category is "thin." A preorder is a thin category.

A set of morphisms from object a to object b in a category C is called a *hom-set* and is written as `C(a, b)` (or, sometimes, $\mathtt{Hom}_{\mathtt{C}}$`(a, b)`). So every hom-set in a preorder is either empty or a singleton. That includes the hom-set `C(a, a)`, the set of morphisms from a to a, which must be a singleton, containing only the identity, in any preorder. You may, however, have cycles in a preorder. Cycles are forbidden in a partial order.

It's very important to be able to recognize preorders, partial orders, and total orders because of sorting. Sorting algorithms, such as quicksort, bubble sort, merge sort, etc., can only work correctly on total orders. Partial orders can be sorted using topological sort.

3.4 Monoid as Set

Monoid is an embarrassingly simple but amazingly powerful concept. It's the concept behind basic arithmetics: Both addition and multiplication form a monoid. Monoids are ubiquitous in programming. They show up as strings, lists, foldable data structures, futures in concurrent programming, events in functional reactive programming, and so on.

Traditionally, a monoid is defined as a set with a binary operation. All that's required from this operation is that it's associative, and that there is one special element that behaves like a unit with respect to it.

For instance, natural numbers with zero form a monoid under addition. Associativity means that:

```
(a + b) + c = a + (b + c)
```

(In other words, we can skip parentheses when adding numbers.)

The neutral element is zero, because:

```
0 + a = a
```

and

```
a + 0 = a
```

The second equation is redundant, because addition is commutative (`a + b = b + a`), but commutativity is not part of the definition of a monoid. For instance, string concatenation is not commutative and yet it forms a monoid. The neutral element for string concatenation, by the way, is an empty string, which can be attached to either side of a string without changing it.

In Haskell we can define a type class for monoids — a type for which there is a neutral element called `mempty` and a binary operation called `mappend`:

```
class Monoid m where
    mempty  :: m
    mappend :: m -> m -> m
```

The type signature for a two-argument function, `m->m->m`, might look strange at first, but it will make perfect sense after we talk about currying. You may interpret a signature with multiple arrows in two basic ways: as a function of multiple arguments, with the rightmost type being the return type; or as a function of one argument (the leftmost one), returning a function. The latter interpretation may be emphasized by adding parentheses (which are redundant, because the arrow is right-associative), as in: `m->(m->m)`. We'll come back to this interpretation in a moment.

Notice that, in Haskell, there is no way to express the monoidal properties of `mempty` and `mappend` (i.e., the fact that `mempty` is neutral and that `mappend` is associative). It's the responsibility of the programmer to make sure they are satisfied.

Haskell classes are not as intrusive as C++ classes. When you're defining a new type, you don't have to specify its class up front. You are free to procrastinate and declare a given type to be an instance of some class much later. As an example, let's declare `String` to be a monoid by providing the implementation of `mempty` and `mappend` (this is, in fact, done for you in the standard Prelude):

```
instance Monoid String where
    mempty = ""
    mappend = (++)
```

Here, we have reused the list concatenation operator `(++)`, because a `String` is just a list of characters.

A word about Haskell syntax: Any infix operator can be turned into a two-argument function by surrounding it with parentheses. Given two strings, you can concatenate them by inserting `++` between them:

```
"Hello " ++ "world!"
```

or by passing them as two arguments to the parenthesized `(++)`:

```
(++) "Hello " "world!"
```

Notice that arguments to a function are not separated by commas or surrounded by parentheses. (This is probably the hardest thing to get used to when learning Haskell.)

It's worth emphasizing that Haskell lets you express equality of functions, as in:

```
mappend = (++)
```

Conceptually, this is different than expressing the equality of values produced by functions, as in:

```
mappend s1 s2 = (++) s1 s2
```

The former translates into equality of morphisms in the category **Hask** (or **Set**, if we ignore bottoms, which is the name for never-ending calculations). Such equations are not only more succinct, but can often be generalized to other categories. The latter is called *extensional* equality, and states the fact that for any two input strings, the outputs of `mappend` and `(++)` are the same. Since the values of arguments are sometimes called *points* (as in: the value of f at point x), this is called point-wise equality. Function equality without specifying the arguments is described as *point-free*. (Incidentally, point-free equations often involve composition of functions, which is symbolized by a point, so this might be a little confusing to the beginner.)

The closest one can get to declaring a monoid in C++ would be to use the (proposed) syntax for concepts.

```
template<class T>
  T mempty = delete;

template<class T>
  T mappend(T, T) = delete;

template<class M>
  concept bool Monoid = requires (M m) {
    { mempty<M> } -> M;
    { mappend(m, m); } -> M;
  };
```

The first definition uses a value template (also proposed). A polymorphic value is a family of values — a different value for every type.

The keyword `delete` means that there is no default value defined: It will have to be specified on a case-by-case basis. Similarly, there is no default for `mappend`.

The concept `Monoid` is a predicate (hence the `bool` type) that tests whether there exist appropriate definitions of `mempty` and `mappend` for a given type `M`.

An instantiation of the Monoid concept can be accomplished by providing appropriate specializations and overloads:

```
template<>
std::string mempty<std::string> = {""};

std::string mappend(std::string s1, std::string s2) {
    return s1 + s2;
}
```

3.5 Monoid as Category

That was the “familiar” definition of the monoid in terms of elements of a set. But as you know, in category theory we try to get away from sets and their elements, and instead talk about objects and morphisms. So let’s change our perspective a bit and think of the application of the binary operator as “moving” or “shifting” things around the set.

For instance, there is the operation of adding 5 to every natural number. It maps 0 to 5, 1 to 6, 2 to 7, and so on. That’s a function defined on the set of natural numbers. That’s good: we have a function and a set. In general, for any number n there is a function of adding n — the “adder” of n.

How do adders compose? The composition of the function that adds 5 with the function that adds 7 is a function that adds 12. So the composition of adders can be made equivalent to the rules of addition. That’s good too: we can replace addition with function composition.

But wait, there’s more: There is also the adder for the neutral element, zero. Adding zero doesn’t move things around, so it’s the identity function in the set of natural numbers.

Instead of giving you the traditional rules of addition, I could as well give you the rules of composing adders, without any loss of information. Notice that the composition of adders is associative, because the composition of functions is associative; and we have the zero adder corresponding to the identity function.

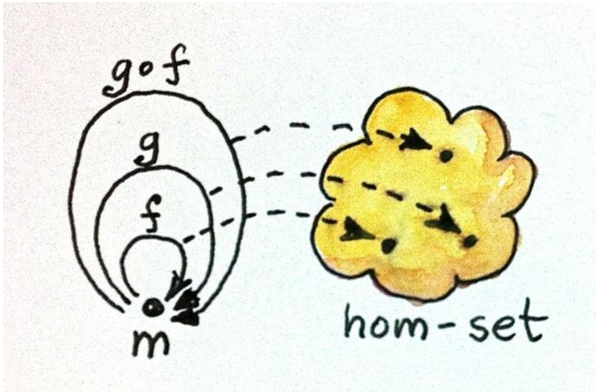
An astute reader might have noticed that the mapping from integers to adders follows from the second interpretation of the type signature of `mappend` as `m->(m->m)`. It tells us that `mappend` maps an element of a monoid set to a function acting on that set.

Now I want you to forget that you are dealing with the set of natural

numbers and just think of it as a single object, a blob with a bunch of morphisms — the adders. A monoid is a single object category. In fact the name monoid comes from Greek *mono*, which means single. Every monoid can be described as a single object category with a set of morphisms that follow appropriate rules of composition.

String concatenation is an interesting case, because we have a choice of defining right appenders and left appenders (or *prependers*, if you will). The composition tables of the two models are a mirror reverse of each other. You can easily convince yourself that appending "bar" after "foo" corresponds to prepending "foo" after prepending "bar".

You might ask the question whether every categorical monoid — a one-object category — defines a unique set-with-binary-operator monoid.



Monoid hom-set seen as morphisms and as points in a set.

It turns out that we can always extract a set from a single-object category. This set is the set of morphisms — the adders in our example. In other words, we have the hom-set M(m, m) of the single object m in the category M. We can easily define a binary operator in this set: The monoidal product of two set-elements is the element corresponding to the composition of the corresponding morphisms. If you give me two elements of M(m, m) corresponding to f and g, their product will correspond to the composition g   f. The composition always exists, because the source and the target for these morphisms are the same object. And it's associative by the rules of category. The identity morphism is the neutral element of this product. So we can always recover a set monoid from a category monoid. For all intents and purposes they are one and the same.

There is just one little nit for mathematicians to pick: morphisms don't have to form a set. In the world of categories there are things larger than sets. A category in which morphisms between any two objects form a set is called locally small. As promised, I will be mostly ignoring such subtleties, but I thought I should mention them for the record.

A lot of interesting phenomena in category theory have their root in the fact that elements of a hom-set can be seen both as morphisms, which follow the rules of composition, and as points in a set. Here, composition of morphisms in M translates into monoidal product in the set `M(m, m)`.

3.6 Acknowledgments

I'd like to thank Andrew Sutton for rewriting my C++ monoid concept code according to his and Bjarne Stroustrup's latest proposal.

3.7 Challenges

1. Generate a free category from:
   (a) A graph with one node and no edges
   (b) A graph with one node and one (directed) edge (hint: this edge can be composed with itself)
   (c) A graph with two nodes and a single arrow between them
   (d) A graph with a single node and 26 arrows marked with the letters of the alphabet: a, b, c ... z.
2. What kind of order is this?
   (a) A set of sets with the inclusion relation: A is included in B if every element of A is also an element of B.
   (b) C++ types with the following subtyping relation: T1 is a subtype of T2 if a pointer to T1 can be passed to a function

that expects a pointer to T2 without triggering a compilation error.

3. Considering that Bool is a set of two values True and False, show that it forms two (set-theoretical) monoids with respect to, respectively, operator `&&` (AND) and `||` (OR).
4. Represent the Bool monoid with the AND operator as a category: List the morphisms and their rules of composition.
5. Represent addition modulo 3 as a monoid category.

4

Kleisli Categories

You've seen how to model types and pure functions as a category. I also mentioned that there is a way to model side effects, or non-pure functions, in category theory. Let's have a look at one such example: functions that log or trace their execution. Something that, in an imperative language, would likely be implemented by mutating some global state, as in:

```
string logger;

bool negate(bool b) {
    logger += "Not so! ";
    return !b;
}
```

You know that this is not a pure function, because its memoized version would fail to produce a log. This function has *side effects*.

In modern programming, we try to stay away from global mutable state as much as possible — if only because of the complications of concurrency. And you would never put code like this in a library.

Fortunately for us, it's possible to make this function pure. You just have to pass the log explicitly, in and out. Let's do that by adding a string argument, and pairing regular output with a string that contains the updated log:

```
pair<bool, string> negate(bool b, string logger) {
    return make_pair(!b, logger + "Not so! ");
}
```

This function is pure, it has no side effects, it returns the same pair every time it's called with the same arguments, and it can be memoized if necessary. However, considering the cumulative nature of the log, you'd have to memoize all possible histories that can lead to a given call. There would be a separate memo entry for:

```
negate(true, "It was the best of times. ");
```

and

```
negate(true, "It was the worst of times. ");
```

and so on.

It's also not a very good interface for a library function. The callers are free to ignore the string in the return type, so that's not a huge burden; but they are forced to pass a string as input, which might be inconvenient.

Is there a way to do the same thing less intrusively? Is there a way to separate concerns? In this simple example, the main purpose of the function `negate` is to turn one Boolean into another. The logging is secondary. Granted, the message that is logged is specific to the function, but the task of aggregating the messages into one continuous log is a separate concern. We still want the function to produce a string, but we'd like to unburden it from producing a log. So here's the compromise solution:

```
pair<bool, string> negate(bool b) {
    return make_pair(!b, "Not so! ");
}
```

The idea is that the log will be aggregated *between* function calls.

To see how this can be done, let's switch to a slightly more realistic example. We have one function from string to string that turns lower case characters to upper case:

```
string toUpper(string s) {
    string result;
    int (*toupperp)(int) = &toupper; // toupper is overloaded
    transform(begin(s), end(s), back_inserter(result), toupperp);
    return result;
}
```

and another that splits a string into a vector of strings, breaking it on whitespace boundaries:

```
vector<string> toWords(string s) {
    return words(s);
}
```

The actual work is done in the auxiliary function `words`:

```
vector<string> words(string s) {
    vector<string> result{""};
    for (auto i = begin(s); i != end(s); ++i)
    {
        if (isspace(*i))
            result.push_back("");
        else
            result.back() += *i;
    }
    return result;
}
```

We want to modify the functions `toUpper` and `toWords` so that they piggyback a message string on top of their regular return values.

We will "embellish" the return values of these functions. Let's do it in a generic way by defining a template `Writer` that encapsulates a pair whose first component is a value of arbitrary type `A` and the second component is a string:

```
template<class A>
using Writer = pair<A, string>;
```

Here are the embellished functions:

```
Writer<string> toUpper(string s) {
```

```
    string result;
    int (*toupperp)(int) = &toupper;
    transform(begin(s), end(s), back_inserter(result), toupperp);
    return make_pair(result, "toUpper ");
}

Writer<vector<string>> toWords(string s) {
    return make_pair(words(s), "toWords ");
}
```

We want to compose these two functions into another embellished function that uppercases a string and splits it into words, all the while producing a log of those actions. Here's how we may do it:

```
Writer<vector<string>> process(string s) {
    auto p1 = toUpper(s);
    auto p2 = toWords(p1.first);
    return make_pair(p2.first, p1.second + p2.second);
}
```

We have accomplished our goal: The aggregation of the log is no longer the concern of the individual functions. They produce their own messages, which are then, externally, concatenated into a larger log.

Now imagine a whole program written in this style. It's a nightmare of repetitive, error-prone code. But we are programmers. We know how to deal with repetitive code: we abstract it! This is, however, not your run of the mill abstraction — we have to abstract *function composition* itself. But composition is the essence of category theory, so before we write more code, let's analyze the problem from the categorical point of view.

4.1 The Writer Category

The idea of embellishing the return types of a bunch of functions in order to piggyback some additional functionality turns out to be very fruitful. We'll see many more examples of it. The starting point is our regular category of types and functions. We'll leave the types as objects, but redefine our morphisms to be the embellished functions.

For instance, suppose that we want to embellish the function `isEven` that goes from `int` to `bool`. We turn it into a morphism that is represented by an embellished function. The important point is that this morphism is still considered an arrow between the objects `int` and `bool`, even though the embellished function returns a pair:

```
pair<bool, string> isEven(int n) {
    return make_pair(n % 2 == 0, "isEven ");
}
```

By the laws of a category, we should be able to compose this morphism with another morphism that goes from the object `bool` to whatever. In particular, we should be able to compose it with our earlier `negate`:

```
pair<bool, string> negate(bool b) {
    return make_pair(!b, "Not so! ");
}
```

Obviously, we cannot compose these two morphisms the same way we compose regular functions, because of the input/output mismatch. Their composition should look more like this:

```
pair<bool, string> isOdd(int n) {
    pair<bool, string> p1 = isEven(n);
```

```
    pair<bool, string> p2 = negate(p1.first);
    return make_pair(p2.first, p1.second + p2.second);
}
```

So here's the recipe for the composition of two morphisms in this new category we are constructing:

1. Execute the embellished function corresponding to the first morphism
2. Extract the first component of the result pair and pass it to the embellished function corresponding to the second morphism
3. Concatenate the second component (the string) of of the first result and the second component (the string) of the second result
4. Return a new pair combining the first component of the final result with the concatenated string.

If we want to abstract this composition as a higher order function in C++, we have to use a template parameterized by three types corresponding to three objects in our category. It should take two embellished functions that are composable according to our rules, and return a third embellished function:

```
template<class A, class B, class C>
function<Writer<C>(A)> compose(function<Writer<B>(A)> m1,
                               function<Writer<C>(B)> m2)
{
    return [m1, m2](A x) {
        auto p1 = m1(x);
        auto p2 = m2(p1.first);
        return make_pair(p2.first, p1.second + p2.second);
```

```
    };
}
```

Now we can go back to our earlier example and implement the composition of `toUpper` and `toWords` using this new template:

```
Writer<vector<string>> process(string s) {
    return compose<string, string, vector<string>>(toUpper,
     ↪  toWords)(s);
}
```

There is still a lot of noise with the passing of types to the `compose` template. This can be avoided as long as you have a C++14-compliant compiler that supports generalized lambda functions with return type deduction (credit for this code goes to Eric Niebler):

```
auto const compose = [](auto m1, auto m2) {
    return [m1, m2](auto x) {
        auto p1 = m1(x);
        auto p2 = m2(p1.first);
        return make_pair(p2.first, p1.second + p2.second);
    };
};
```

In this new definition, the implementation of `process` simplifies to:

```
Writer<vector<string>> process(string s) {
    return compose(toUpper, toWords)(s);
}
```

But we are not finished yet. We have defined composition in our new category, but what are the identity morphisms? These are not our regular identity functions! They have to be morphisms from type A back to type A, which means they are embellished functions of the form:

```
Writer<A> identity(A);
```

They have to behave like units with respect to composition. If you look at our definition of composition, you'll see that an identity morphism should pass its argument without change, and only contribute an empty string to the log:

```
template<class A> Writer<A> identity(A x) {
    return make_pair(x, "");
}
```

You can easily convince yourself that the category we have just defined is indeed a legitimate category. In particular, our composition is trivially associative. If you follow what's happening with the first component of each pair, it's just a regular function composition, which is associative. The second components are being concatenated, and concatenation is also associative.

An astute reader may notice that it would be easy to generalize this construction to any monoid, not just the string monoid. We would use `mappend` inside `compose` and `mempty` inside `identity` (in place of `+` and `""`). There really is no reason to limit ourselves to logging just strings. A good library writer should be able to identify the bare minimum of constraints that make the library work — here the logging library's only requirement is that the log have monoidal properties.

4.2 Writer in Haskell

The same thing in Haskell is a little more terse, and we also get a lot more help from the compiler. Let's start by defining the `Writer` type:

```
type Writer a = (a, String)
```

Here I'm just defining a type alias, an equivalent of a `typedef` (or `using`) in C++. The type `Writer` is parameterized by a type variable `a` and is equivalent to a pair of `a` and `String`. The syntax for pairs is minimal: just two items in parentheses, separated by a comma.

Our morphisms are functions from an arbitrary type to some `Writer` type:

```
a -> Writer b
```

We'll declare the composition as a funny infix operator, sometimes called the "fish":

```
(>=>) :: (a -> Writer b) -> (b -> Writer c) -> (a -> Writer c)
```

It's a function of two arguments, each being a function on its own, and returning a function. The first argument is of the type `(a->Writer b)`, the second is `(b->Writer c)`, and the result is `(a->Writer c)`.

Here's the definition of this infix operator — the two arguments `m1` and `m2` appearing on either side of the fishy symbol:

```
m1 >=> m2 = \x ->
    let (y, s1) = m1 x
        (z, s2) = m2 y
    in (z, s1 ++ s2)
```

The result is a lambda function of one argument `x`. The lambda is written as a backslash — think of it as the Greek letter λ with an amputated leg.

The `let` expression lets you declare auxiliary variables. Here the result of the call to `m1` is pattern matched to a pair of variables `(y, s1)`; and the result of the call to `m2`, with the argument `y` from the first pattern, is matched to `(z, s2)`.

It is common in Haskell to pattern match pairs, rather than use accessors, as we did in C++. Other than that there is a pretty straightforward correspondence between the two implementations.

The overall value of the `let` expression is specified in its `in` clause: here it's a pair whose first component is `z` and the second component is the concatenation of two strings, `s1++s2`.

I will also define the identity morphism for our category, but for reasons that will become clear much later, I will call it `return`.

```
return :: a -> Writer a
return x = (x, "")
```

For completeness, let's have the Haskell versions of the embellished functions `upCase` and `toWords`:

```
upCase :: String -> Writer String
upCase s = (map toUpper s, "upCase ")

toWords :: String -> Writer [String]
toWords s = (words s, "toWords ")
```

The function `map` corresponds to the C++ `transform`. It applies the character function `toUpper` to the string `s`. The auxiliary function `words` is defined in the standard Prelude library.

Finally, the composition of the two functions is accomplished with the help of the fish operator:

```
process :: String -> Writer [String]
process = upCase >=> toWords
```

4.3 Kleisli Categories

You might have guessed that I haven't invented this category on the spot. It's an example of the so called Kleisli category — a category based on a monad. We are not ready to discuss monads yet, but I wanted to give you a taste of what they can do. For our limited purposes, a Kleisli category has, as objects, the types of the underlying programming language. Morphisms from type A to type B are functions that go from A to a type derived from B using the particular embellishment. Each Kleisli category defines its own way of composing such morphisms, as well as the identity morphisms with respect to that composition. (Later we'll see that the imprecise term "embellishment" corresponds to the notion of an endofunctor in a category.)

The particular monad that I used as the basis of the category in this post is called the *writer monad* and it's used for logging or tracing the execution of functions. It's also an example of a more general mechanism for embedding effects in pure computations. You've seen previously that we could model programming-language types and functions in the category of sets (disregarding bottoms, as usual). Here we have extended this model to a slightly different category, a category where morphisms are represented by embellished functions, and their composition does more than just pass the output of one function to the input of another. We have one more degree of freedom to play with: the composition itself. It turns out that this is exactly the degree of freedom which makes it possible to give simple denotational semantics to programs that in imperative languages are traditionally implemented using side effects.

4.4 Challenge

A function that is not defined for all possible values of its argument is called a partial function. It's not really a function in the mathematical sense, so it doesn't fit the standard categorical mold. It can, however, be represented by a function that returns an embellished type `optional`:

```
template<class A> class optional {
    bool _isValid;
    A _value;
public:
    optional()    : _isValid(false) {}
    optional(A v) : _isValid(true), _value(v) {}
    bool isValid() const { return _isValid; }
    A value() const { return _value; }
};
```

As an example, here's the implementation of the embellished function `safe_root`:

```
optional<double> safe_root(double x) {
    if (x >= 0) return optional<double>{sqrt(x)};
    else return optional<double>{};
}
```

Here's the challenge:

1. Construct the Kleisli category for partial functions (define composition and identity).

2. Implement the embellished function `safe_reciprocal` that returns a valid reciprocal of its argument, if it's different from zero.
3. Compose `safe_root` and `safe_reciprocal` to implement `safe_root_reciprocal` that calculates `sqrt(1/x)` whenever possible.

4.5 Acknowledgments

I'm grateful to Eric Niebler for reading the draft and providing the clever implementation of `compose` that uses advanced features of C++14 to drive type inference. I was able to cut the whole section of old fashioned template magic that did the same thing using type traits. Good riddance! I'm also grateful to Gershom Bazerman for useful comments that helped me clarify some important points.

5

Products and Coproducts

The Ancient Greek playwright Euripides once said: "Every man is like the company he is wont to keep." We are defined by our relationships. Nowhere is this more true than in category theory. If we want to single out a particular object in a category, we can only do this by describing its pattern of relationships with other objects (and itself). These relationships are defined by morphisms.

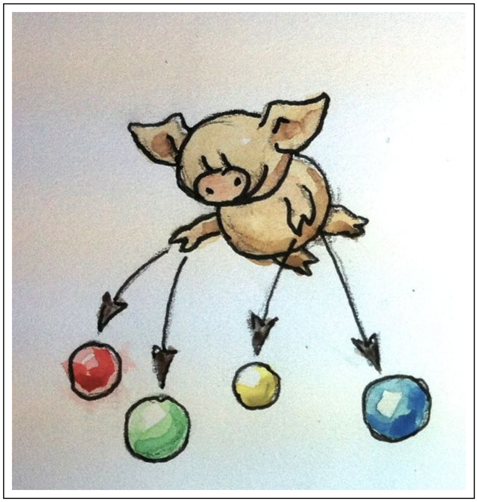
There is a common construction in category theory called the *universal construction* for defining objects in terms of their relationships. One way of doing this is to pick a pattern, a particular shape constructed from objects and morphisms, and look for all its occurrences in the category. If it's a common enough pattern, and the category is large, chances are you'll have lots and lots of hits. The trick is to establish some kind of ranking among those hits, and pick what could be considered the best fit.

This process is reminiscent of the way we do web searches. A query is like a pattern. A very general query will give you large *recall*: lots of hits. Some may be relevant, others not. To eliminate irrelevant hits, you refine your query. That increases its *precision*. Finally, the search engine will rank the hits and, hopefully, the one result that you're interested in will be at the top of the list.

5.1 Initial Object

The simplest shape is a single object. Obviously, there are as many instances of this shape as there are objects in a given category. That's a lot to choose from. We need to establish some kind of ranking and try to find the object that tops this hierarchy. The only means at our disposal are morphisms. If you think of morphisms as arrows, then it's possible that there is an overall net flow of arrows from one end of the category to another. This is true in ordered categories, for instance in partial orders. We could generalize that notion of object precedence by saying that object a is "more initial" than object b if there is an arrow (a morphism) going from a to b. We would then define *the* initial object as one that has arrows going to all other objects. Obviously there is no guarantee that such an object exists, and that's okay. A bigger problem is that there may be too many such objects: The recall is good, but precision is lacking. The solution is to take a hint from ordered categories — they allow at most one arrow between any two objects: there is only one way of being less-than or equal-to another object. Which leads us to this definition of the initial object:

> The **initial object** is the object that has one and only one morphism going to any object in the category.



However, even that doesn't guarantee the uniqueness of the initial object (if one exists). But it guarantees the next best thing: uniqueness *up to isomorphism*. Isomorphisms are very important in category theory, so I'll talk about them shortly. For now, let's just agree that uniqueness up to isomorphism justifies the use of "the" in the definition of the initial object.

Here are some examples: The initial object in a partially ordered set (often called a *poset*) is its least element. Some posets don't have an initial object — like the set of all integers, positive and negative, with less-than-or-equal relation for morphisms.

In the category of sets and functions, the initial object is the empty set. Remember, an empty set corresponds to the Haskell type `Void` (there is no corresponding type in C++) and the unique polymorphic function from `Void` to any other type is called `absurd`:

```
absurd :: Void -> a
```

It's this family of morphisms that makes `Void` the initial object in the category of types.

5.2 Terminal Object

Let's continue with the single-object pattern, but let's change the way we rank the objects. We'll say that object a is "more terminal" than object b if there is a morphism going from b to a (notice the reversal of direction). We'll be looking for an object that's more terminal than any other object in the category. Again, we will insist on uniqueness:

> The **terminal object** is the object with one and only one morphism coming to it from any object in the category.

And again, the terminal object is unique, up to isomorphism, which I will show shortly. But first let's look at some examples. In a poset, the terminal object, if it exists, is the biggest object. In the category of sets, the terminal object is a singleton. We've already talked about singletons — they correspond to the `void` type in C++ and the unit type `()` in Haskell. It's a type that has only one value — implicit in C++ and explicit in Haskell, denoted by `()`. We've also established that there is one and only one pure function from any type to the unit type:

```
unit :: a -> ()
unit _ = ()
```

so all the conditions for the terminal object are satisfied.

Notice that in this example the uniqueness condition is crucial, because there are other sets (actually, all of them, except for the empty set) that have incoming morphisms from every set. For instance, there is a Boolean-valued function (a predicate) defined for every type:

```
yes :: a -> Bool
yes _ = True
```

But `Bool` is not a terminal object. There is at least one more `Bool`-valued function from every type:

```
no :: a -> Bool
no _ = False
```

Insisting on uniqueness gives us just the right precision to narrow down the definition of the terminal object to just one type.

5.3 Duality

You can't help but to notice the symmetry between the way we defined the initial object and the terminal object. The only difference between the two was the direction of morphisms. It turns out that for any category C we can define the *opposite category* C^{op} just by reversing all the arrows. The opposite category automatically satisfies all the requirements of a category, as long as we simultaneously redefine composition. If original morphisms `f::a->b` and `g::b->c` composed to `h::a->c` with `h=g∘f`, then the reversed morphisms f^{op}`::b->a` and g^{op}`::c->b` will compose to h^{op}`::c->a` with $h^{op} = f^{op} \circ g^{op}$. And reversing the identity arrows is a (pun alert!) no-op.

Duality is a very important property of categories because it doubles the productivity of every mathematician working in category theory. For every construction you come up with, there is its opposite; and for every theorem you prove, you get one for free. The constructions in the opposite category are often prefixed with "co", so you have products and coproducts, monads and comonads, cones and cocones, limits and colimits, and so on. There are no cocomonads though, because reversing the arrows twice gets us back to the original state.

It follows then that a terminal object is the initial object in the opposite category.

5.4 Isomorphisms

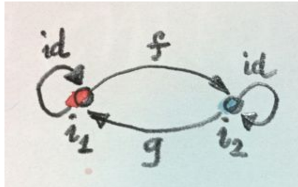
As programmers, we are well aware that defining equality is a nontrivial task. What does it mean for two objects to be equal? Do they have to occupy the same location in memory (pointer equality)? Or is it enough that the values of all their components are equal? Are two complex numbers equal if one is expressed as the real and imaginary part, and the other as modulus and angle? You'd think that mathematicians would have figured out the meaning of equality, but they haven't. They have the same problem of multiple competing definitions for equality. There is the propositional equality, intensional equality, extensional equality, and equality as a path in homotopy type theory. And then there are the weaker notions of isomorphism, and even weaker of equivalence.

The intuition is that isomorphic objects look the same — they have the same shape. It means that every part of one object corresponds to some part of another object in a one-to-one mapping. As far as our instruments can tell, the two objects are a perfect copy of each other. Mathematically it means that there is a mapping from object a to object b, and there is a mapping from object b back to object a, and they are the inverse of each other. In category theory we replace mappings with morphisms. An isomorphism is an invertible morphism; or a pair of morphisms, one being the inverse of the other.

We understand the inverse in terms of composition and identity: Morphism g is the inverse of morphism f if their composition is the identity morphism. These are actually two equations because there are two ways of composing two morphisms:

```
f . g = id
g . f = id
```

When I said that the initial (terminal) object was unique up to isomorphism, I meant that any two initial (terminal) objects are isomorphic. That's actually easy to see. Let's suppose that we have two initial objects i_1 and i_2. Since i_1 is initial, there is a unique morphism f from i_1 to i_2. By the same token, since i_2 is initial, there is a unique morphism g from i_2 to i_1. What's the composition of these two morphisms?



All morphisms in this diagram are unique.

The composition $g \circ f$ must be a morphism from i_1 to i_1. But i_1 is initial so there can only be one morphism going from i_1 to i_1. Since we are in a category, we know that there is an identity morphism from i_1 to i_1, and since there is room for only one, that must be it. Therefore $g \circ f$ is equal to identity. Similarly, $f \circ g$ must be equal to identity, because there can be only one morphism from i_2 back to i_2. This proves that f and g must be the inverse of each other. Therefore any two initial objects are isomorphic.

Notice that in this proof we used the uniqueness of the morphism from the initial object to itself. Without that we couldn't prove the "up to isomorphism" part. But why do we need the uniqueness of f and g? Because not only is the initial object unique up to isomorphism, it is unique up to *unique* isomorphism. In principle, there could be more than one isomorphism between two objects, but that's not the case here. This "uniqueness up to unique isomorphism" is the important property of all universal constructions.

5.5 Products

The next universal construction is that of a product. We know what a cartesian product of two sets is: it's a set of pairs. But what's the pattern that connects the product set with its constituent sets? If we can figure that out, we'll be able to generalize it to other categories.

All we can say is that there are two functions, the projections, from the product to each of the constituents. In Haskell, these two functions are called `fst` and `snd` and they pick, respectively, the first and the second component of a pair:

```haskell
fst :: (a, b) -> a
fst (x, y) = x

snd :: (a, b) -> b
snd (x, y) = y
```

Here, the functions are defined by pattern matching their arguments: the pattern that matches any pair is `(x, y)`, and it extracts its components into variables `x` and `y`.

These definitions can be simplified even further with the use of wildcards:

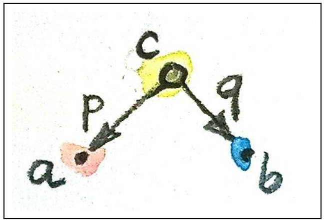
```haskell
fst (x, _) = x
snd (_, y) = y
```

In C++, we would use template functions, for instance:

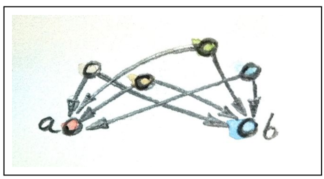
```cpp
template<class A, class B> A
fst(pair<A, B> const & p) {
    return p.first;
}
```

Equipped with this seemingly very limited knowledge, let's try to define a pattern of objects and morphisms in the category of sets that will lead us to the construction of a product of two sets, a and b. This pattern consists of an object c and two morphisms p and q connecting it to a and b, respectively:

```
p :: c -> a
q :: c -> b
```



All *c*s that fit this pattern will be considered candidates for the product. There may be lots of them.



For instance, let's pick, as our constituents, two Haskell types, `Int` and `Bool`, and get a sampling of candidates for their product.

Here's one: `Int`. Can `Int` be considered a candidate for the product of `Int` and `Bool`? Yes, it can — and here are its projections:

```
p :: Int -> Int
p x = x

q :: Int -> Bool
q _ = True
```

That's pretty lame, but it matches the criteria.

Here's another one: `(Int, Int, Bool)`. It's a tuple of three elements, or a triple. Here are two morphisms that make it a legitimate candidate (we are using pattern matching on triples):

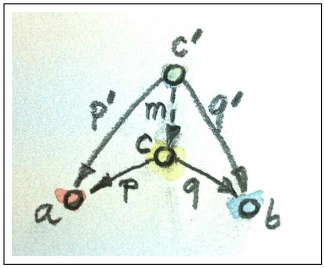
```
p :: (Int, Int, Bool) -> Int
p (x, _, _) = x

q :: (Int, Int, Bool) -> Bool
q (_, _, b) = b
```

You may have noticed that while our first candidate was too small — it only covered the `Int` dimension of the product; the second was too big — it spuriously duplicated the `Int` dimension.

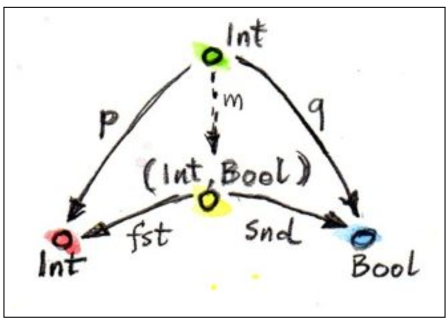
But we haven't explored yet the other part of the universal construction: the ranking. We want to be able to compare two instances of our pattern. We want to compare one candidate object *c* and its two projections *p* and *q* with another candidate object *c'* and its two projections *p'* and *q'*. We would like to say that *c* is "better" than *c'* if there is a morphism *m* from *c'* to *c* — but that's too weak. We also want its projections to be "better," or "more universal," than the projections of *c'*. What it means is that the projections *p'* and *q'* can be reconstructed from *p* and *q* using *m*:

```
p' = p . m
q' = q . m
```



Another way of looking at these equation is that *m factorizes p'* and *q'*. Just pretend that these equations are in natural numbers, and the dot is multiplication: *m* is a common factor shared by *p'* and *q'*.

Just to build some intuitions, let me show you that the pair `(Int, Bool)` with the two canonical projections, `fst` and `snd` is indeed *better* than the two candidates I presented before.



The mapping `m` for the first candidate is:

```
m :: Int -> (Int, Bool)
m x = (x, True)
```

Indeed, the two projections, `p` and `q` can be reconstructed as:

```
p x = fst (m x) = x
q x = snd (m x) = True
```

The `m` for the second example is similarly uniquely determined:

```
m (x, _, b) = (x, b)
```

We were able to show that `(Int, Bool)` is better than either of the two candidates. Let's see why the opposite is not true. Could we find some `m'` that would help us reconstruct `fst` and `snd` from `p` and `q`?

```
fst = p . m'
snd = q . m'
```

In our first example, `q` always returned `True` and we know that there are pairs whose second component is `False`. We can't reconstruct `snd` from `q`.

The second example is different: we retain enough information after running either `p` or `q`, but there is more than one way to factorize `fst` and `snd`. Because both `p` and `q` ignore the second component of the triple, our `m’` can put anything in it. We can have:

```
m' (x, b) = (x, x, b)
```

or

```
m' (x, b) = (x, 42, b)
```

and so on.

Putting it all together, given any type `c` with two projections `p` and `q`, there is a unique `m` from `c` to the cartesian product `(a, b)` that factorizes them. In fact, it just combines `p` and `q` into a pair.

```
m :: c -> (a, b)
m x = (p x, q x)
```

That makes the cartesian product `(a, b)` our best match, which means that this universal construction works in the category of sets. It picks the product of any two sets.

Now let's forget about sets and define a product of two objects in any category using the same universal construction. Such product doesn't always exist, but when it does, it is unique up to a unique isomorphism.

> A **product** of two objects *a* and *b* is the object *c* equipped with two projections such that for any other object *c'* equipped with two projections there is a unique morphism *m* from *c'* to *c* that factorizes those projections.

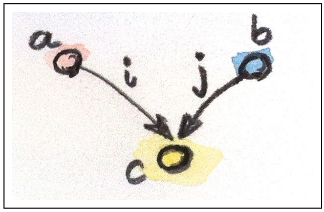
A (higher order) function that produces the factorizing function `m` from two candidates is sometimes called the *factorizer*. In our case, it would be the function:

```
factorizer :: (c -> a) -> (c -> b) -> (c -> (a, b))
factorizer p q = \x -> (p x, q x)
```

5.6 Coproduct

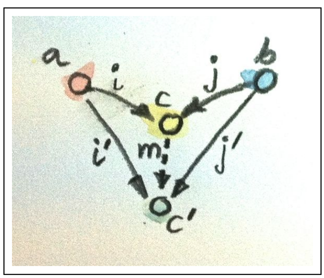
Like every construction in category theory, the product has a dual, which is called the coproduct. When we reverse the arrows in the product pattern, we end up with an object *c* equipped with two *injections*, `i` and `j`: morphisms from *a* and *b* to *c*.

```
i :: a -> c
j :: b -> c
```



The ranking is also inverted: object *c* is “better” than object *c’* that is equipped with the injections *i’* and *j’* if there is a morphism *m* from *c* to *c’* that factorizes the injections:

```
i' = m . i
j' = m . j
```



The “best” such object, one with a unique morphism connecting it to any other pattern, is called a coproduct and, if it exists, is unique up to unique isomorphism.

> A **coproduct** of two objects *a* and *b* is the object *c* equipped with two injections such that for any other object *c'* equipped with two injections there is a unique morphism *m* from *c* to *c'* that factorizes those injections.

In the category of sets, the coproduct is the *disjoint union* of two sets. An element of the disjoint union of *a* and *b* is either an element of *a* or an element of *b*. If the two sets overlap, the disjoint union contains two copies of the common part. You can think of an element of a disjoint union as being tagged with an identifier that specifies its origin.

For a programmer, it's easier to understand a coproduct in terms of types: it's a tagged union of two types. C++ supports unions, but they are not tagged. It means that in your program you have to somehow keep track which member of the union is valid. To create a tagged union, you have to define a tag — an enumeration — and combine it with the union. For instance, a tagged union of an `int` and a `char const *` could be implemented as:

```
struct Contact {
    enum { isPhone, isEmail } tag;
    union { int phoneNum; char const * emailAddr; };
};
```

The two injections can either be implemented as constructors or as functions. For instance, here's the first injection as a function `PhoneNum`:

```
Contact PhoneNum(int n) {
    Contact c;
    c.tag = isPhone;
    c.phoneNum = n;
```

```
    return c;
}
```

It injects an integer into `Contact`.

A tagged union is also called a *variant*, and there is a very general implementation of a variant in the boost library, `boost::variant`.

In Haskell, you can combine any data types into a tagged union by separating data constructors with a vertical bar. The `Contact` example translates into the declaration:

```
data Contact = PhoneNum Int | EmailAddr String
```

Here, `PhoneNum` and `EmailAddr` serve both as constructors (injections), and as tags for pattern matching (more about this later). For instance, this is how you would construct a contact using a phone number:

```
helpdesk :: Contact;
helpdesk = PhoneNum 2222222
```

Unlike the canonical implementation of the product that is built into Haskell as the primitive pair, the canonical implementation of the coproduct is a data type called `Either`, which is defined in the standard Prelude as:

```
Either a b = Left a | Right b
```

It is parameterized by two types, `a` and `b` and has two constructors: `Left` that takes a value of type `a`, and `Right` that takes a value of type `b`.

Just as we've defined the factorizer for a product, we can define one for the coproduct. Given a candidate type `c` and two candidate injections `i` and `j`, the factorizer for `Either` produces the factoring function:

```haskell
factorizer :: (a -> c) -> (b -> c) -> Either a b -> c
factorizer i j (Left a)  = i a
factorizer i j (Right b) = j b
```

5.7 Asymmetry

We've seen two set of dual definitions: The definition of a terminal object can be obtained from the definition of the initial object by reversing the direction of arrows; in a similar way, the definition of the coproduct can be obtained from that of the product. Yet in the category of sets the initial object is very different from the final object, and coproduct is very different from product. We'll see later that product behaves like multiplication, with the terminal object playing the role of one; whereas coproduct behaves more like the sum, with the initial object playing the role of zero. In particular, for finite sets, the size of the product is the product of the sizes of individual sets, and the size of the coproduct is the sum of the sizes.

This shows that the category of sets is not symmetric with respect to the inversion of arrows.

Notice that while the empty set has a unique morphism to any set (the `absurd` function), it has no morphisms coming back. The singleton set has a unique morphism coming to it from any set, but it *also* has outgoing morphisms to every set (except for the empty one). As we've seen before, these outgoing morphisms from the terminal object play a very important role of picking elements of other sets (the empty set has no elements, so there's nothing to pick).

It's the relationship of the singleton set to the product that sets it apart from the coproduct. Consider using the singleton set, represented by the unit type `()`, as yet another — vastly inferior — candidate for the product pattern. Equip it with two projections `p` and `q`: functions from the singleton to each of the constituent sets. Each selects a concrete element from either set. Because the product is universal, there is also a (unique) morphism `m` from our candidate, the singleton, to the product. This morphism selects an element from the product set — it selects a concrete pair. It also factorizes the two projections:

```
p = fst . m
q = snd . m
```

When acting on the singleton value `()`, the only element of the singleton set, these two equations become:

```
p () = fst (m ())
q () = snd (m ())
```

Since `m ()` is the element of the product picked by `m`, these equations tell use that the element picked by `p` from the first set, `p ()`, is the first component of the pair picked by `m`. Similarly, `q ()` is equal to the second component. This is in total agreement with our understanding that elements of the product are pairs of elements from the constituent sets.

There is no such simple interpretation of the coproduct. We could try the singleton set as a candidate for a coproduct, in an attempt to extract the elements from it, but there we would have two injections going into it rather than two projections coming out of it. They'd tell us nothing about their sources (in fact, we've seen that they ignore the input parameter). Neither would the unique morphism from the coproduct to our singleton. The category of sets just looks very different when seen from the direction of the initial object than it does when seen from the terminal end.

This is not an intrinsic property of sets, it's a property of functions, which we use as morphisms in **Set**. Functions are, in general, asymmetric. Let me explain.

A function must be defined for every element of its domain set (in programming, we call it a *total* function), but it doesn't have to cover the whole codomain. We've seen some extreme cases of it: functions from a singleton set — functions that select just a single element in the codomain. (Actually, functions from an empty set are the real extremes.) When the size of the domain is much smaller than the size of the codomain, we often think of such functions as embedding the domain in the codomain. For instance, we can think of a function from a singleton set as embedding its single element in the codomain. I call them *embedding* functions, but mathematicians prefer to give a name to the opposite: functions that tightly fill their codomains are called *surjective* or *onto*.

The other source of asymmetry is that functions are allowed to map many elements of the domain set into one element of the codomain. They can collapse them. The extreme case are functions that map whole sets into a singleton. You've seen the polymorphic `unit` function that does just that. The collapsing can only be compounded by composition. A composition of two collapsing functions is even more collapsing than the individual functions. Mathematicians have a name for non-collapsing functions: they call them *injective* or *one-to-one*

Of course there are some functions that are neither embedding nor collapsing. They are called *bijections* and they are truly symmetric, because they are invertible. In the category of sets, an isomorphism is the same as a bijection.

5.8 Challenges

1. Show that the terminal object is unique up to unique isomorphism.
2. What is a product of two objects in a poset? Hint: Use the universal construction.
3. What is a coproduct of two objects in a poset?
4. Implement the equivalent of Haskell `Either` as a generic type in your favorite language (other than Haskell).
5. Show that `Either` is a "better" coproduct than `int` equipped with two injections:

   ```
   int i(int n) { return n; }
   int j(bool b) { return b? 0: 1; }
   ```

   Hint: Define a function

   ```
   int m(Either const & e);
   ```

   that factorizes `i` and `j`.
6. Continuing the previous problem: How would you argue that `int` with the two injections `i` and `j` cannot be "better" than `Either`?
7. Still continuing: What about these injections?

   ```
   int i(int n) {
       if (n < 0) return n;
       return n + 2;
   }

   int j(bool b) { return b? 0: 1; }
   ```

8. Come up with an inferior candidate for a coproduct of `int` and `bool` that cannot be better than `Either` because it allows multiple acceptable morphisms from it to `Either`.

5.9 Bibliography

1. The Catsters, Products and Coproducts video.

5.10 Acknowledments

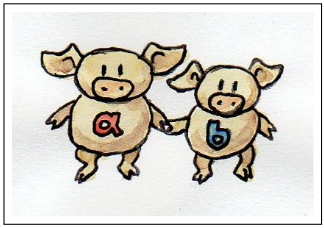
I'm grateful to Gershom Bazerman for reviewing this post before publication and for stimulating discussions.

6

Simple Algebraic Data Types

We've seen two basic ways of combining types: using a product and a coproduct. It turns out that a lot of data structures in everyday programming can be built using just these two mechanisms. This fact has important practical consequences. Many properties of data structures are composable. For instance, if you know how to compare values of basic types for equality, and you know how to generalize these comparisons to product and coproduct types, you can automate the derivation of equality operators for composite types. In Haskell you can automatically derive equality, comparison, conversion to and from string, and more, for a large subset of composite types.

Let's have a closer look at product and sum types as they appear in programming.



6.1 Product Types

The canonical implementation of a product of two types in a programming language is a pair. In Haskell, a pair is a primitive type constructor; in C++ it's a relatively complex template defined in the Standard Library.

Pairs are not strictly commutative: a pair `(Int, Bool)` cannot be substituted for a pair `(Bool, Int)`, even though they carry the same information. They are, however, commutative up to isomorphism — the isomorphism being given by the `swap` function (which is its own inverse):

```
swap :: (a, b) -> (b, a)
swap (x, y) = (y, x)
```

You can think of the two pairs as simply using a different format for storing the same data. It's just like big endian vs. little endian.

You can combine an arbitrary number of types into a product by nesting pairs inside pairs, but there is an easier way: nested pairs are equivalent to tuples. It's the consequence of the fact that different ways of nesting pairs are isomorphic. If you want to combine three types in a product, `a`, `b`, and `c`, in this order, you can do it in two ways:

```
((a, b), c)
```

or

```
(a, (b, c))
```

These types are different — you can't pass one to a function that expects the other — but their elements are in one-to-one correspondence. There is a function that maps one to another:

```
alpha :: ((a, b), c) -> (a, (b, c))
alpha ((x, y), z) = (x, (y, z))
```

and this function is invertible:

```
alpha_inv :: (a, (b, c)) -> ((a, b), c)
alpha_inv (x, (y, z)) = ((x, y), z)
```

so it's an isomorphism. These are just different ways of repackaging the same data.

You can interpret the creation of a product type as a binary operation on types. From that perspective, the above isomorphism looks very much like the associativity law we've seen in monoids:

```
(a * b) * c = a * (b * c)
```

Except that, in the monoid case, the two ways of composing products were equal, whereas here they are only equal "up to isomorphism."

If we can live with isomorphisms, and don't insist on strict equality, we can go even further and show that the unit type, `()`, is the unit of the product the same way 1 is the unit of multiplication. Indeed, the pairing of a value of some type a with a unit doesn't add any information. The type:

```
(a, ())
```

is isomorphic to `a`. Here's the isomorphism:

```
rho :: (a, ()) -> a
rho (x, ()) = x

rho_inv :: a -> (a, ())
rho_inv x = (x, ())
```

These observations can be formalized by saying that **Set** (the category of sets) is a *monoidal category*. It's a category that's also a monoid, in the sense that you can multiply objects (here, take their cartesian product). I'll talk more about monoidal categories, and give the full definition in the future.

There is a more general way of defining product types in Haskell — especially, as we'll see soon, when they are combined with sum types. It uses named constructors with multiple arguments. A pair, for instance, can be defined alternatively as:

```
data Pair a b = P a b
```

Here, `Pair a b` is the name of the type paremeterized by two other types, `a` and `b`; and `P` is the name of the data constructor. You define a pair type by passing two types to the `Pair` type constructor. You construct a pair value by passing two values of appropriate types to the constructor `P`. For instance, let's define a value `stmt` as a pair of `String` and `Bool`:

```
stmt :: Pair String Bool
stmt = P "This statements is" False
```

The first line is the type declaration. It uses the type constructor `Pair`, with `String` and `Bool` replacing `a` and the `b` in the generic definition of `Pair`. The second line defines the actual value by passing a concrete string and a concrete Boolean to the data constructor `P`. Type constructors are used to construct types; data constructors, to construct values.

Since the name spaces for type and data constructors are separate in Haskell, you will often see the same name used for both, as in:

```
data Pair a b = Pair a b
```

And if you squint hard enough, you may even view the built-in pair type as a variation on this kind of declaration, where the name `Pair` is replaced with the binary operator `(,)`. In fact you can use `(,)` just like any other named constructor and create pairs using prefix notation:

```
stmt = (,) "This statement is" False
```

Similarly, you can use `(,,)` to create triples, and so on.

Instead of using generic pairs or tuples, you can also define specific named product types, as in:

```
data Stmt = Stmt String Bool
```

which is just a product of `String` and `Bool`, but it's given its own name and constructor. The advantage of this style of declaration is that you may define many types that have the same content but different meaning and functionality, and which cannot be substituted for each other.

Programming with tuples and multi-argument constructors can get messy and error prone — keeping track of which component represents what. It's often preferable to give names to components. A product type with named fields is called a record in Haskell, and a `struct` in C.

6.2 Records

Let's have a look at a simple example. We want to describe chemical elements by combining two strings, name and symbol; and an integer, the atomic number; into one data structure. We can use a tuple `(String, String, Int)` and remember which component represents what. We would extract components by pattern matching, as in this function that checks if the symbol of the element is the prefix of its name (as in **He** being the prefix of **Helium**):

```
startsWithSymbol :: (String, String, Int) -> Bool
startsWithSymbol (name, symbol, _) = isPrefixOf symbol name
```

This code is error prone, and is hard to read and maintain. It's much better to define a record:

```
data Element = Element { name :: String
                       , symbol :: String
                       , atomicNumber :: Int }
```

The two representations are isomorphic, as witnessed by these two conversion functions, which are the inverse of each other:

```
tupleToElem :: (String, String, Int) -> Element
tupleToElem (n, s, a) = Element { name = n
                                , symbol = s
                                , atomicNumber = a }

elemToTuple :: Element -> (String, String, Int)
elemToTuple e = (name e, symbol e, atomicNumber e)
```

Notice that the names of record fields also serve as functions to access these fields. For instance, `atomicNumber e` retrieves the `atomicNumber` field from `e`. We use `atomicNumber` as a function of the type:

```
atomicNumber :: Element -> Int
```

With the record syntax for `Element`, our function `startsWithSymbol` becomes more readable:

```
startsWithSymbol :: Element -> Bool
startsWithSymbol e = isPrefixOf (symbol e) (name e)
```

We could even use the Haskell trick of turning the function `isPrefixOf` into an infix operator by surrounding it with backquotes, and make it read almost like a sentence:

```
startsWithSymbol e = symbol e `isPrefixOf` name e
```

The parentheses could be omitted in this case, because an infix operator has lower precedence than a function call.

6.3 Sum Types

Just as the product in the category of sets gives rise to product types, the coproduct gives rise to sum types. The canonical implementation of a sum type in Haskell is:

```
data Either a b = Left a | Right b
```

And like pairs, `Either`s are commutative (up to isomorphism), can be nested, and the nesting order is irrelevant (up to isomorphism). So we can, for instance, define a sum equivalent of a triple:

```
data OneOfThree a b c = Sinistral a | Medial b | Dextral c
```

and so on.

It turns out that **Set** is also a (symmetric) monoidal category with respect to coproduct. The role of the binary operation is played by the disjoint sum, and the role of the unit element is played by the initial object. In terms of types, we have `Either` as the monoidal operator and `Void`, the uninhabited type, as its neutral element. You can think of `Either` as plus, and `Void` as zero. Indeed, adding `Void` to a sum type doesn't change its content. For instance:

```
Either a Void
```

is isomorphic to `a`. That's because there is no way to construct a `Right` version of this type — there isn't a value of type `Void`. The only inhabitants of `Either a Void` are constructed using the `Left` constructors and they simply encapsulate a value of type `a`. So, symbolically, `a + 0 = a`.

Sum types are pretty common in Haskell, but their C++ equivalents, unions or variants, are much less common. There are several reasons for that.

First of all, the simplest sum types are just enumerations and are implemented using `enum` in C++. The equivalent of the Haskell sum type:

```
data Color = Red | Green | Blue
```

is the C++:

```
enum { Red, Green, Blue };
```

An even simpler sum type:

```
data Bool = True | False
```

is the primitive `bool` in C++.

Simple sum types that encode the presence or absence of a value are variously implemented in C++ using special tricks and "impossible" values, like empty strings, negative numbers, null pointers, etc. This kind of optionality, if deliberate, is expressed in Haskell using the `Maybe` type:

```
data Maybe a = Nothing | Just a
```

The `Maybe` type is a sum of two types. You can see this if you separate the two constructors into individual types. The first one would look like this:

```
data NothingType = Nothing
```

It's an enumeration with one value called `Nothing`. In other words, it's a singleton, which is equivalent to the unit type `()`. The second part:

```
data JustType a = Just a
```

is just an encapsulation of the type `a`. We could have encoded `Maybe` as:

```
data Maybe a = Either () a
```

More complex sum types are often faked in C++ using pointers. A pointer can be either null, or point to a value of specific type. For instance, a Haskell list type, which can be defined as a (recursive) sum type:

```
List a = Nil | Cons a (List a)
```

can be translated to C++ using the null pointer trick to implement the empty list:

```
template<class A>
class List {
    Node<A> * _head;
public:
    List() : _head(nullptr) {} // Nil
    List(A a, List<A> l)       // Cons
      : _head(new Node<A>(a, l))
    {}
};
```

Notice that the two Haskell constructors `Nil` and `Cons` are translated into two overloaded `List` constructors with analogous arguments (none, for `Nil`; and a value and a list for `Cons`). The `List` class doesn't need a tag to distinguish between the two components of the sum type. Instead it uses the special `nullptr` value for `_head` to encode `Nil`.

The main difference, though, between Haskell and C++ types is that Haskell data structures are immutable. If you create an object using one particular constructor, the object will forever remember which constructor was used and what arguments were passed to it. So a `Maybe` object that was created as `Just "energy"` will never turn into `Nothing`. Similarly, an empty list will forever be empty, and a list of three elements will always have the same three elements.

It's this immutability that makes construction reversible. Given an object, you can always disassemble it down to parts that were used in its construction. This deconstruction is done with pattern matching and it reuses constructors as patterns. Constructor arguments, if any, are replaced with variables (or other patterns).

The `List` data type has two constructors, so the deconstruction of an arbitrary `List` uses two patterns corresponding to those constructors. One matches the empty `Nil` list, and the other a `Cons`-constructed list. For instance, here's the definition of a simple function on `Lists`:

```
maybeTail :: List a -> Maybe (List a)
maybeTail Nil = Nothing
maybeTail (Cons _ t) = Just t
```

The first part of the definition of `maybeTail` uses the `Nil` constructor as pattern and returns `Nothing`. The second part uses the `Cons` constructor as pattern. It replaces the first constructor argument with a wildcard, because we are not interested in it. The second argument to `Cons` is bound to the variable `t` (I will call these things variables even though, strictly speaking, they never vary: once bound to an expression, a variable never changes). The return value is `Just t`. Now, depending on how your `List` was created, it will match one of the clauses. If it was created using `Cons`, the two arguments that were passed to it will be retrieved (and the first discarded).

Even more elaborate sum types are implemented in C++ using polymorphic class hierarchies. A family of classes with a common ancestor may be understood as one variant type, in which the vtable serves as a hidden tag. What in Haskell would be done by pattern matching on the constructor, and by calling specialized code, in C++ is accomplished by dispatching a call to a virtual function based on the vtable pointer.

You will rarely see `union` used as a sum type in C++ because of severe limitations on what can go into a union. You can't even put a `std::string` into a union because it has a copy constructor.

6.4 Algebra of Types

Taken separately, product and sum types can be used to define a variety of useful data structures, but the real strength comes from combining the two. Once again we are invoking the power of composition.

Let's summarize what we've discovered so far. We've seen two commutative monoidal structures underlying the type system: We have the sum types with `Void` as the neutral element, and the product types with the unit type, `()`, as the neutral element. We'd like to think of them as analogous to addition and multiplication. In this analogy, `Void` would correspond to zero, and unit, `()`, to one.

Let's see how far we can stretch this analogy. For instance, does multiplication by zero give zero? In other words, is a product type with one component being `Void` isomorphic to `Void`? For example, is it possible to create a pair of, say `Int` and `Void`?

To create a pair you need two values. Although you can easily come up with an integer, there is no value of type `Void`. Therefore, for any type `a`, the type `(a, Void)` is uninhabited — has no values — and is therefore equivalent to `Void`. In other words, `a*0 = 0`.

Another thing that links addition and multiplication is the distributive property:

```
a * (b + c) = a * b + a * c
```

Does it also hold for product and sum types? Yes, it does — up to isomorphisms, as usual. The left hand side corresponds to the type:

```
(a, Either b c)
```

and the right hand side corresponds to the type:

```
Either (a, b) (a, c)
```

Here's the function that converts them one way:

```
prodToSum :: (a, Either b c) -> Either (a, b) (a, c)
prodToSum (x, e) =
    case e of
      Left  y -> Left  (x, y)
      Right z -> Right (x, z)
```

and here's one that goes the other way:

```
sumToProd :: Either (a, b) (a, c) -> (a, Either b c)
sumToProd e =
    case e of
      Left  (x, y) -> (x, Left  y)
      Right (x, z) -> (x, Right z)
```

The `case of` statement is used for pattern matching inside functions. Each pattern is followed by an arrow and the expression to be evaluated when the pattern matches. For instance, if you call `prodToSum` with the value:

```
prod1 :: (Int, Either String Float)
prod1 = (2, Left "Hi!")
```

the `e` in `case e of` will be equal to `Left "Hi!"`. It will match the pattern `Left y`, substituting `"Hi!"` for `y`. Since the `x` has already been matched to `2`, the result of the `case of` clause, and the whole function, will be `Left (2, "Hi!")`, as expected.

I'm not going to prove that these two functions are the inverse of each other, but if you think about it, they must be! They are just trivially re-packing the contents of the two data structures. It's the same data, only different format.

Mathematicians have a name for such two intertwined monoids: it's called a *semiring*. It's not a full *ring*, because we can't define subtraction of types. That's why a semiring is sometimes called a *rig*, which is a pun on "ring without an *n*" (negative). But barring that, we can get a lot of mileage from translating statements about, say, natural numbers, which form a rig, to statements about types. Here's a translation table with some entries of interest:

Numbers	Types
0	`Void`
1	`()`
a + b	`Either a b = Left a | Right b`
a * b	`(a, b)` or `Pair a b = Pair a b`
2 = 1 + 1	`data Bool = True | False`
1 + a	`data Maybe = Nothing | Just a`

The list type is quite interesting, because it's defined as a solution to an equation. The type we are defining appears on both sides of the equation:

```
List a = Nil | Cons a (List a)
```

If we do our usual substitutions, and also replace `List a` with `x`, we get the equation:

```
x = 1 + a * x
```

We can't solve it using traditional algebraic methods because we can't subtract or divide types. But we can try a series of substitutions, where we keep replacing `x` on the right hand side with `(1 + a*x)`, and use the distributive property. This leads to the following series:

```
x = 1 + a*x
x = 1 + a*(1 + a*x) = 1 + a + a*a*x
x = 1 + a + a*a*(1 + a*x) = 1 + a + a*a + a*a*a*x
...
x = 1 + a + a*a + a*a*a + a*a*a*a...
```

We end up with an infinite sum of products (tuples), which can be interpreted as: A list is either empty, `1`; or a singleton, `a`; or a pair, `a*a`; or a triple, `a*a*a`; etc... Well, that's exactly what a list is — a string of `a`s!

There's much more to lists than that, and we'll come back to them and other recursive data structures after we learn about functors and fixed points.

Solving equations with symbolic variables — that's algebra! It's what gives these types their name: algebraic data types.

Finally, I should mention one very important interpretation of the algebra of types. Notice that a product of two types `a` and `b` must contain both a value of type `a` *and* a value of type `b`, which means both types must be inhabited. A sum of two types, on the other hand, contains either a value of type `a` *or* a value of type `b`, so it's enough if one of them is inhabited. Logical *and* and *or* also form a semiring, and it too can be mapped into type theory:

Logic	Types
false	`Void`

Logic	Types
true	`()`
a || b	`Either a b = Left a | Right b`
a && b	`(a, b)`

This analogy goes deeper, and is the basis of the Curry-Howard isomorphism between logic and type theory. We'll come back to it when we talk about function types.

6.5 Challenges

1. Show the isomorphism between `Maybe a` and `Either () a`.
2. Here's a sum type defined in Haskell:

   ```
   data Shape = Circle Float
              | Rect Float Float
   ```

   When we want to define a function like `area` that acts on a `Shape`, we do it by pattern matching on the two constructors:

   ```
   area :: Shape -> Float
   area (Circle r) = pi * r * r
   area (Rect d h) = d * h
   ```

   Implement `Shape` in C++ or Java as an interface and create two classes: `Circle` and `Rect`. Implement `area` as a virtual function.
3. Continuing with the previous example: We can easily add a new function `circ` that calculates the circumference of a `Shape`. We can do it without touching the definition of `Shape`:

```
circ :: Shape -> Float
circ (Circle r) = 2.0 * pi * r
circ (Rect d h) = 2.0 * (d + h)
```

Add `circ` to your C++ or Java implementation. What parts of the original code did you have to touch?

4. Continuing further: Add a new shape, `Square`, to `Shape` and make all the necessary updates. What code did you have to touch in Haskell vs. C++ or Java? (Even if you're not a Haskell programmer, the modifications should be pretty obvious.)
5. Show that `a + a = 2 * a` holds for types (up to isomorphism). Remember that `2` corresponds to `Bool`, according to our translation table.

6.6 Acknowledgments

Thanks go to Gershom Bazerman for reviewing this post and helpful comments.

Index

> Any inaccuracies in this index may be explained by the fact that it has been prepared with the help of a computer.
>
> —Donald E. Knuth, *Fundamental Algorithms* (Volume 1 of *The Art of Computer Programming*)

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THIS BOOK is ...

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