Homework 1

Problem 1

1.

Since $u \to v \equiv \neg u \vee v$, then:

$$u \vee v \equiv (\neg u) \to v$$

Furthermore,

$$\begin{aligned} \neg(u \to \neg v) &\equiv \neg(\neg u \vee \neg v) \\ &\equiv u \wedge v \end{aligned}$$

Thus, all logical statements that can be represented with $\{\neg, \wedge, \vee\}$ can be represented with $\{\to, \neg\}$. Since $\{\neg, \wedge, \vee\}$ is complete, we've reached the conclusion that $\{\to, \neg\}$ is also complete.

2.

$\{\wedge, \vee\}$ is not complete.

`CLAIM` :

if f is a logical statement composed of $\{\wedge, \vee, \mathbb{T}, \mathbb{F}, p\}$ only, where p is arbitrary proposition, then the f can only be equivlaent to:

$$\begin{aligned} f &\equiv p \\ f &\equiv \mathbb{F} \\ f &\equiv \mathbb{T} \end{aligned}$$

`Base Case`: Statement hold trivially for the cases where $f \in \{p, \mathbb{T}, \mathbb{F}\}$。

`Induction Step`: Suppose f_1, f_2 are logical statements composed of $\{\wedge, \vee, \mathbb{T}, \mathbb{F}, p\}$ only, and they can only be equivalent to p, $\mathbb{T}$, or $\mathbb{F}$, then, by cases analysis, and Idempotent Law of $\wedge$ and $\vee$, we have:

	$f_1 = p$	$f_1 = \mathbb{T}$	$f_1 = \mathbb{F}$
$f_2 = p$	$f_1 \wedge f_2 = p \wedge p = p$ $f_1 \vee f_2 = p \vee p = p$	$f_1 \wedge f_2 = \mathbb{T} \wedge p = p$ $f_1 \vee f_2 = \mathbb{T} \vee p = \mathbb{T}$	$f_1 \wedge f_2 = \mathbb{F} \wedge p = \mathbb{F}$ $f_1 \vee f_2 = \mathbb{F} \vee p = p$
$f_2 = \mathbb{T}$		$f_1 \wedge f_2 = \mathbb{T} \wedge \mathbb{T} = \mathbb{T}$ $f_1 \vee f_2 = \mathbb{T} \vee \mathbb{T} = \mathbb{T}$	$f_1 \wedge f_2 = \mathbb{F} \wedge \mathbb{T} = \mathbb{F}$ $f_1 \vee f_2 = \mathbb{F} \vee \mathbb{T} = \mathbb{T}$
$f_2 = \mathbb{F}$			$f_1 \wedge f_2 = \mathbb{F} \wedge \mathbb{F} = \mathbb{F}$ $f_1 \vee f_2 = \mathbb{F} \vee \mathbb{F} = \mathbb{F}$

Other cases left blanked also hold, since commutative law holds for $\wedge$ and $\vee$.

By `CLAIM` above, there's no logical statement f composed of $\{\wedge, \vee, \mathbb{T}, \mathbb{F}, p\}$ only such that $f \equiv \neg p$, since it contradicts to `CLAIM`.

Problem 2

1.

Prove that $[\neg q \wedge (p \to q)] \to \neg p$ is tautology

$$\begin{aligned}
[\neg q \wedge (p \to q)] \to \neg p &\equiv [\neg q \wedge (\neg p \vee q)] \to \neg p && \text{(implication law)} \\
&\equiv \neg[\neg q \wedge (\neg p \vee q)] \vee \neg p && \text{(implication law)} \\
&\equiv [\neg(\neg q) \vee \neg(\neg p \vee q)] \vee \neg p && \text{(De Morgan's Law)} \\
&\equiv [q \vee \neg(\neg p \vee q)] \vee \neg p && \text{(Double Negation Law)} \\
&\equiv [\neg(\neg p \vee q) \vee q] \vee \neg p && \text{(Commutative Law)} \\
&\equiv \neg(\neg p \vee q) \vee (q \vee \neg p) && \text{(Associative Law)} \\
&\equiv \neg(\neg p \vee q) \vee (\neg p \vee q) && \text{(Commutative Law)} \\
&\equiv \mathbb{T} && \text{Negation Law}
\end{aligned}$$

2.

Prove that $(\neg p \wedge (q \wedge r)) \vee ((\neg p \wedge q) \wedge \neg r)$ is equivalent to $q \wedge \neg p$

$$\begin{aligned}
\text{原題} &\equiv ((\neg p \wedge q) \wedge r) \vee ((\neg p \wedge q) \wedge \neg r) && \text{(Associative Law)} \\
&\equiv (\neg p \wedge q) \vee (r \wedge \neg r) && \text{(Distributive Law)} \\
&\equiv (\neg p \wedge q) \vee \mathbb{F} && \text{(Negation Law)} \\
&\equiv (\neg p \wedge q) && \text{(Identity Law)} \\
&\equiv (q \wedge \neg p) && \text{(Comutative Law)}
\end{aligned}$$

Problem 3

1.

Define:

$$\begin{aligned} P(x) &: \text{student } x \text{ has a person computer} \\ E(x) &: \text{student } x \text{ is in EE} \\ C(x) &: \text{student } x \text{ is in CS} \\ c &: \text{Paul} \end{aligned}$$

Then:

Steps	**Reason**	**No**
$\forall x(\neg P(x) \to \neg E(x))$	Premise	1
$\forall x(C(x) \to P(x))$	Premise	2
$E(c) \lor C(c)$	Premise	3
$\neg P(c) \to \neg E(c)$	1, Universal Instantiation	4
$C(c) \to P(c)$	2, Universal Instantiation	5
$\neg\neg(P(c)) \lor \neg E(c)$	4, Implication Law	6
$P(c) \lor \neg E(c)$	6. Double Negation Law	7
$\neg C(c) \lor P(c)$	5, Implication Law	8
$C(c) \lor E(c)$	3, Commutative Law	9
$P(c) \lor E(c)$	8, 9 Resolution	10
$\neg E(c) \lor P(c)$	7, Commutative	11
$E(c) \lor P(c)$	10, Commutative	12
$P(c) \lor P(c)$	11, 12, Resolution	13
$P(c)$	13, Idempotent. QED.	14

2.

$$
\begin{aligned}
P &: \text{superman is able to prevent evil} \\
Q &: \text{superman does prevent evil} \\
E &: \text{superman exists} \\
I &: \text{superman is impotent} \\
L &: \text{It is impossible to learn logic}
\end{aligned}
$$

then:

No	Steps	Reason
1	$P \to Q$	Premise
2	$\neg P \to I$	Premise
3	$E \to \neg I$	Premise
4	$E \to \neg P$	Premise
5	$\neg E \vee \neg I$	3, Implication Law
6	$\neg I \vee \neg E$	5, Commutative Law
7	$I \to \neg E$	6, Implication Law
8	$E \to I$	4, 2, Hypothetical Syllogism
9	$E \to \neg E$	8, 7, Hypothetical Syllogism
10	$\neg E \vee \neg E$	9, Implication Law
11	$\neg E$	10, Idempotent Law
12	$\neg E \vee L$	11, Addition
13	$E \to L$	12, Implication Law. QED

3.

$$
\begin{aligned}
T(x) &: x \text{ lives in Taipei} \\
D(x) &: x \text{ lives within 100 km to the ocean} \\
F(x) &: x \text{ never eats seafood}
\end{aligned}
$$

Then:

No	Steps	Reason
1	$\forall x(T(x) \to D(x))$	Premise
2	$\exists y(T(y) \land F(y))$	Premise
3	$T(c) \land F(c)$	2, Existential Instantiation
4	$T(c) \to D(c)$	1, Universal Instantiation
5	$T(c)$	3, Simplification
6	$D(c)$	4, 5, Modus Ponens
7	$D(c) \land T(c)$	5, 6, Conjunction
8	$\exists z.\, D(z) \land T(z)$	7, Existential Generalization. QED

4.

$$\begin{aligned} D(x) &: x \text{ in discrete mathematic class} \\ S(x) &: x \text{ knows calculus} \\ C(x) &: x \text{ knows C++} \\ p &: \text{Peter} \end{aligned}$$

then

No	Step	Reason
1	$\forall x(D(x) \to S(x))$	Premise
2	$\forall x(\neg C(x) \to \neg D(x))$	Premise
3	$S(p) \to \neg C(p)$	Premise
4	$D(p) \to S(p)$	1, Universal Instantiation
5	$\neg C(p) \to \neg D(p)$	2, Universal Instantiation

6	$D(p) \to \neg C(p)$	4, 3 Hypothetical Syllogism
7	$D(p) \to \neg D(p)$	6, 5 Hypothetical Syllogism
8	$\neg D(p) \lor \neg D(p)$	7, Implication Law
7	$\neg D(p)$	8, Idempotent Law. QED.

Problem 4

1.

$$\begin{aligned} \neg((p \lor q) \land (r \lor s)) &\equiv \neg((r \lor s) \land (p \lor q)) && \text{(Commutative Law)} \\ &\equiv \neg(r \lor s) \lor \neg(p \lor q) && \text{(De Morgan's Law)} \\ &\equiv (\neg r \land \neg s) \lor \neg(p \lor q) && \text{(De Morgan's Law)} \\ &\equiv (\neg r \land \neg s) \lor (\neg p \land \neg q) && \text{(De Morgan's Law)} \\ &\equiv (\neg s \land \neg r) \lor (\neg p \land \neg q) && \text{(Commutative Law)} \end{aligned}$$

2.

$$\begin{aligned} \overline{(A \cup B) \cap (C \cup D)} &\equiv \overline{(A \cup B)} \cup \overline{(C \cup D)} && \text{(De Morgan's Law)} \\ &\equiv (\bar{A} \cap \bar{B}) \cup \overline{(C \cup D)} && \text{(De Morgan's Law)} \\ &\equiv (\bar{A} \cap \bar{B}) \cup (\bar{C} \cap \bar{D}) && \text{(De Morgan's Law)} \\ &\equiv (\bar{A} \cap \bar{B}) \cup (\bar{D} \cap \bar{C}) && \text{(Commutative Law)} \\ &\equiv (\bar{D} \cap \bar{C}) \cup (\bar{A} \cap \bar{B}) && \text{(Commutative Law)} \end{aligned}$$

Problem 5

$$\begin{aligned} 2^{S_1} &= \{\varnothing, \{\{1,2\}\}\} \\ 2^{S_2} &= \{\varnothing\} \end{aligned}$$

Collaborators & Reference

我是外系邊緣人，所以沒有跟人一起討論作業，也沒有 Reference。