

The Science of Functional Programming

A Tutorial, with Examples in Scala

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A *Transparent* copy of the source code for the book (LyX, LaTeX, graphics files, and build scripts), together with a full-color hyperlinked PDF file, is available at <https://github.com/winitzki/sofp>. The source code is also included as a “file attachment” `sofp-src.tar.bz2` within the PDF file `sofp.pdf`. To extract, run the command `'pdftk sofp.pdf unpack_files output .'` and then `'tar jxvf sofp-src.tar.bz2'`. See the file `README.md` for compilation instructions.

This pedagogical tutorial book presents the theoretical knowledge needed by practitioners of the functional programming paradigm. The main topics covered are: working with functional collections and recursion; the fundamental type and code constructions in functional programming; reasoning about types and code with the Curry-Howard correspondence; laws, structure theorems, and code derivation for functors, monads, and other important typeclasses; free type constructions; and recursive types. Detailed explanations are logically developed and accompanied by full derivations and worked examples tested in the Scala interpreter, as well as exercises. Readers should have a working knowledge of basic Scala; e.g. be able to write code that reads a small text file and prints the number of words in it. Readers should also know school-level mathematics; for example, should be able to simplify the expressions $\frac{1}{x-2} - \frac{1}{x+2}$ and $\frac{d}{dx} ((x+1)e^{-x})$.

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Preface

The goal of this book is to teach programmers how to reason mathematically about types and code, in a way that is directly relevant to software practice.

The material is presented here at medium to advanced level. It requires a certain amount of mathematical experience and is not suitable for people who are unfamiliar with school-level algebra, or unwilling to learn difficult concepts through prolonged mental concentration and effort.

The first part is introductory and may be suitable for beginners in programming. Starting from Chapter 5, the material becomes unsuitable for beginners.

The presentation in this book is self-contained, defining and explaining all required notations, concepts, and Scala language features from scratch. The emphasis is on clarity and understandability of all examples, mathematical notions, derivations, and code. To achieve a clearer presentation of the material, the book uses some non-standard notations (Appendix A) and terminology (Appendix B).

The vision of this book is to explain the mathematical principles that guide the practice of functional programming – that is, help people to write code. Therefore, all mathematical developments in this book are motivated and justified by practical programming issues and are accompanied by code examples that illustrate their usage. For instance, the equational laws for standard typeclasses (functor, applicative, monad, etc.) are first motivated heuristically with code examples. Only then a set of mathematical equations is derived and the laws are formulated as equations.

Each new concept or technique is clarified by solved examples and exercises. Answers to exercises are not provided, but it is verified that the exercises are doable and free of errors. More difficult examples and exercises are marked by an asterisk (*).

A software engineer needs to know only a few fragments of mathematical theory; namely, the fragments that answer questions arising in the practice of functional programming. So the theoretical material is kept to the minimum; *ars longa, vita brevis*. (Chapter 18 presents more discussion of the scope of the required theory.) Mathematical generalizations are not pursued beyond practical relevance or immediate pedagogical usefulness. This limits the scope of required mathematical knowledge to bare rudiments of category theory, type theory, and formal logic. For instance, the book does not mention “introduction/elimination rules”, “strong normalization”, “complete partial order domains”, “adjoint functors”, “pullbacks”, “topoi”, or “algebras”, because learning these concepts will not help a functional programmer write code. This book is also not an introduction to today’s research in the theory of programming languages. Instead, the focus is on practically useful material – including some rarely mentioned constructions, e.g. the “filterable functor” and “applicative contrafunctor” typeclasses.

All code examples in this book serve only for explanation and illustration, and are not optimized for performance or stack safety.

Formatting conventions used in this book

- Text in boldface indicates a new concept or term that is being defined. Text in italics is a logical emphasis. Example:

An **aggregation** is a function from a list of values to a *single* value.

- Sample Scala code is written inline using a small monospaced font, such as this: `val a = "xyz"`. Longer code examples are written in separate code blocks, which may also show the output from the Scala interpreter for certain lines of code:

```
val s = (1 to 10).toList

scala> s.product
res0: Int = 3628800
```

- In the introductory chapters, type expressions and code examples are written in the syntax of Scala. Starting from Chapters 4–5, the book introduces a new notation for types where e.g. the Scala type expression $((A, B)) \Rightarrow \text{Option}[A]$ is written as $A \times B \Rightarrow 1 + A$. Also, a new notation for code is introduced and developed in Chapters 5–6 for easier reasoning about equational laws. For example, the functor composition law is written in the code notation as

$$f^{\uparrow L} \circ g^{\uparrow L} = (f \circ g)^{\uparrow L} \quad ,$$

where L is a functor and $f^{A \Rightarrow B}$ and $g^{B \Rightarrow C}$ are arbitrary functions of the specified types. The symbol \circ denotes the forward composition of functions (Scala’s `andThen` method). Appendix A summarizes the conventions of the type and code notations.

- Derivations of laws are written in a two-column notation where the right column contains formulas in the code notation and the left column indicates the property or law used to derive the expression at right. A green underline in the *previous* expression shows the part rewritten using the indicated law:

$$\begin{aligned} \text{expect to equal } \text{pu}_M &: \text{pu}_M^{\uparrow \text{Id}} \circ \text{pu}_M \circ \text{ftn}_M \\ \text{lifting to the identity functor} &: = \text{pu}_M \circ \underline{\text{pu}_M \circ \text{ftn}_M} \\ \text{left identity law for } M &: = \text{pu}_M \quad . \end{aligned}$$

A green underline is sometimes also used at the *last* step of the derivation, to indicate the part of the expression that resulted from the most recent rewriting.

Part I

Beginner level

1 Mathematical formulas as code. I.

Nameless functions

1.1 Translating mathematics into code

1.1.1 First examples

We begin by writing Scala code for some computational tasks.

Factorial of 10 Find the product of integers from 1 to 10 (the **factorial** of 10).

First, we write a mathematical formula for the result:

$$\prod_{k=1}^{10} k \quad .$$

We can then write Scala code in a way that resembles this formula:

```
scala> (1 to 10).product
res0: Int = 3628800
```

The Scala interpreter indicates that the result is the value 3628800 of type `Int`. To define a name for this value, we use the “`val`” syntax:

```
scala> val fac10 = (1 to 10).product
fac10: Int = 3628800

scala> fac10 == 3628800
res1: Boolean = true
```

The code `(1 to 10).product` is an **expression**, which means that (1) the code can be evaluated (e.g. using the Scala interpreter) and yields a value, and (2) the code can be inserted as a part of a larger expression. For example, we could write

```
scala> 100 + (1 to 10).product + 100
res0: Int = 3629000
```

Factorial as a function Define a function that takes an integer n and computes the factorial of n .

A mathematical formula for this function can be written as

$$f(n) = \prod_{k=1}^n k \quad .$$

The corresponding Scala code is

```
def f(n: Int) = (1 to n).product
```

In Scala’s `def` syntax, we need to specify the type of a function’s argument; in this case, we write `n: Int`. In the usual mathematical notation, types of arguments are either not written at all, or written separately from the formula:

$$f(n) = \prod_{k=1}^n k, \quad \forall n \in \mathbb{N} \quad .$$

This indicates that n must be from the set of non-negative integers (denoted by \mathbb{N} in mathematics). This is similar to specifying the type `Int` in the Scala code. So, the argument's type in the code specifies the *domain* of a function.

Having defined the function f , we can now apply it to an integer argument:

```
scala> f(10)
res6: Int = 3628800
```

It is an error to apply f to a non-integer value, e.g. to a string:

```
scala> f("abc")
<console>:13: error: type mismatch;
 found   : String("abc")
 required: Int
    f("abc")
      ^
```

1.1.2 Nameless functions

The formula and the code, as written above, both involve *naming* the function as “ f ”. Sometimes a function does not really need a name, – for instance, if the function is used only once. “Nameless” mathematical functions are denoted using the symbol \mapsto (pronounced “mapped to”) like this:

$$x \mapsto (\text{some formula}) \quad .$$

So the mathematical notation for the nameless factorial function is

$$n \mapsto \prod_{k=1}^n k \quad .$$

This reads as “a function that maps n to the product of all k where k goes from 1 to n ”. The Scala expression implementing this mathematical formula is

```
(n: Int) => (1 to n).product
```

This expression shows Scala’s syntax for a **nameless** function. Here,

```
n: Int
```

is the function’s **argument**, while

```
(1 to n).product
```

is the function’s **body**. The arrow symbol `=>` separates the argument from the body.¹

Functions in Scala (whether named or nameless) are treated as values, which means that we can also define a Scala value as

```
scala> val fac = (n: Int) => (1 to n).product
fac: Int => Int = <function1>
```

We see that the value `fac` has the type `Int => Int`, which means that the function takes an integer (`Int`) argument and returns an integer result value. What is the value of the function `fac` *itself*? As we have just seen, the Scala interpreter prints `<function1>` as the “value” of `fac`. An alternative Scala interpreter² called `ammonite` prints something like this,

```
scala@ val fac = (n: Int) => (1 to n).product
fac: Int => Int = ammonite.$sess.cmd0$$$Lambda$1675/2107543287@1e44b638
```

¹In Scala, the two ASCII characters `=>` and the single Unicode character \Rightarrow have the same meaning. This book uses the symbol \Rightarrow for symbolic calculations and `=>` for Scala code. However, when doing calculations *by hand*, one could write \rightarrow instead of \Rightarrow since it is faster. Several programming languages, such as OCaml and Haskell, use the symbols `->` or the Unicode equivalent, \rightarrow , for the function arrow.

²<https://ammonite.io/>

This seems to indicate some identifying number, or perhaps a memory location.

One may imagine that a “function value” represents a block of compiled machine code, – code that will actually run and evaluate the function’s body when the function is applied to its argument.

Once defined, a function can be applied to an argument like this:

```
scala> fac(10)
res1: Int = 3628800
```

However, functions can be used without naming them. We can directly apply a nameless factorial function to an integer argument 10 instead of writing `fac(10)`:

```
scala> ((n: Int) => (1 to n).product)(10)
res2: Int = 3628800
```

One would not often write code like this because there is no advantage in creating a nameless function and then applying it right away to an argument. This is so because we can evaluate the expression

```
((n: Int) => (1 to n).product)(10)
```

by substituting 10 instead of `n` in the function body, which gives us

```
(1 to 10).product
```

If a nameless function uses the argument several times, for example

```
((n: Int) => n*n*n + n*n)(12345)
```

it is still better to substitute the argument and to eliminate the nameless function. We could write

```
12345*12345*12345 + 12345*12345
```

but, of course, it is better to avoid repeating the value 12345. To achieve that, we may define `n` as a value in an **expression block** like this:

```
scala> { val n = 12345; n*n*n + n*n }
res3: Int = 322687002
```

Defined in this way, the value `n` is visible only within the expression block. Outside the block, another value named `n` could be defined independently of this `n`. For this reason, the definition of `n` is called a **locally scoped** definition.

Nameless functions are convenient when they are themselves arguments of other functions, as we will see next.

Example: prime numbers Let us define a function that takes an integer argument n and determines whether n is a prime number.

A simple mathematical formula for this function can be written as

$$\text{is_prime}(n) = \forall k \in [2, n-1] : n \neq 0 \bmod k \quad . \quad (1.1)$$

This formula has two clearly separated parts: first, a range of integers from 2 to $n-1$, and second, a requirement that all these integers should satisfy a given condition, $n \neq 0 \bmod k$. Formula (1.1) is translated into Scala code as

```
def is_prime(n: Int) = (2 to n-1).forall(k => n % k != 0)
```

In this code, the two parts of the mathematical formula are implemented in a way that is closely similar to the mathematical notation, except for the arrow after k .

We can now apply the function `is_prime` to some integer values:

```
scala> is_prime(12)
res3: Boolean = false

scala> is_prime(13)
res4: Boolean = true
```

As we can see from the output above, the function `is_prime` returns a value of type `Boolean`. Therefore, the function `is_prime` has type `Int => Boolean`.

A function that returns a `Boolean` value is called a **predicate**.

In Scala, it is optional – but strongly recommended – to specify the return type of named functions. The required syntax looks like this,

```
def is_prime(n: Int): Boolean = (2 to n-1).forall(k => n % k != 0)
```

However, we do not need to specify the type `Int` for the argument `k` of the nameless function `k => n % k != 0`. The Scala compiler knows that `k` is going to iterate over the *integer* elements of the range `(2 to n-1)`, which effectively forces `k` to be of type `Int`.

1.1.3 Nameless functions and bound variables

The code for `is_prime` differs from the mathematical formula (1.1) in two ways.

One difference is that the interval $[2, n - 1]$ is in front of `forall`. Another is that the Scala code uses a nameless function (`k => n % k != 0`), while Eq. (1.1) does not seem to involve any functions.

To understand the first difference, we need to keep in mind that the Scala syntax such as `(2 to n-1).forall(k => ...)` means to apply a function called `forall` to *two* arguments: the first argument is the range `(2 to n-1)`, and the second argument is the nameless function (`k => ...`). In Scala, the ***infix*** syntax `x.f(z)`, or equivalently `x f z`, means that a function `f` is applied to its *two* arguments, `x` and `z`. In the ordinary mathematical notation, this would be $f(x, z)$. Infix notation is often easier to read and is widely used, e.g. when we write $x + y$ rather than something like *plus*(x, y).

A single-argument function could be also defined with infix notation, and then the syntax is `x.f`, as in the expression `(1 to n).product` we have seen before.

The infix methods `.product` and `.forall` are already provided in the Scala standard library, so it is natural to use them. If we want to avoid the infix syntax, we could define a function `for_all` with two arguments and write code like this,

```
for_all(2 to n-1, k => n % k != 0)
```

This would have brought the syntax somewhat closer to the formula (1.1).

However, there still remains the second difference: The symbol k is used as an *argument* of a nameless function (`k => n % k != 0`) in the Scala code, – while the mathematical formula

$$\forall k \in [2, n - 1] : n \neq 0 \bmod k \quad (1.2)$$

does not seem to use any functions but defines the symbol k that goes over the range $[2, n - 1]$. The variable k is then used for writing the predicate $n \neq 0 \bmod k$.

Let us investigate the role of k more closely. The mathematical variable k is actually defined *only inside* the expression “ $\forall k : \dots$ ” and makes no sense outside that expression. This becomes clear by looking at Eq. (1.1): The variable k is not present in the left-hand side and could not possibly be used there. The name “ k ” is defined only in the right-hand side, where it is first mentioned as the arbitrary element $k \in [2, n - 1]$ and then used in the sub-expression “ $\dots \bmod k$ ”.

So, the mathematical notation in Eq. (1.2) says two things: First, we use the name k for integers from 2 to $n - 1$. Second, for each of those k we evaluate the expression $n \neq 0 \bmod k$, which can be viewed as a certain given *function of* k that returns a `Boolean` value. Translating the mathematical notation into code, it is therefore natural to use the nameless function

$$k \mapsto n \neq 0 \bmod k$$

and to write Scala code applying this nameless function to each element of the range $[2, n - 1]$ and checking that all result values be `true`:

```
(2 to n-1).forall(k => n % k != 0)
```

Just as the mathematical notation defines the variable k only in the right-hand side of Eq. (1.1), the argument `k` of the nameless Scala function `k => n % k != 0` is defined only within that function’s body and cannot be used in any code outside the expression `n % k != 0`.

Variables that are defined only inside an expression and are invisible outside are called **bound variables**, or “variables bound in an expression”. Variables that are used in an expression but are defined outside it are called **free variables**, or “variables occurring free in an expression”. These concepts apply equally well to mathematical formulas and to Scala code. For example, in the mathematical expression $k \Rightarrow n \neq 0 \bmod k$ (which is a nameless function), the variable k is bound (it is defined only within that expression) but the variable n is free (it is defined outside that expression).

The main difference between free and bound variables is that bound variables can be *locally renamed* at will, unlike free variables. To see this, consider that we could rename k to z and write instead of Eq. (1.1) an equivalent definition

$$\text{is_prime}(n) = \forall z \in [2, n-1] : n \neq 0 \bmod z \quad ,$$

or in Scala code,

```
(2 to n-1).forall(z => n % z != 0)
```

The argument z in the nameless function $z \Rightarrow n \% z \neq 0$ may be renamed without changing the result of the entire program. No code outside that function needs to be changed after renaming z . But the value n is defined outside and cannot be renamed “locally” (i.e. only within the sub-expression). If we wanted to rename n in the sub-expression $z \Rightarrow n \% z \neq 0$, we would also need to change every place in the code that defines and uses n *outside* that expression, or else the program would become incorrect.

Mathematical formulas use bound variables in various constructions such as $\forall k : p(k)$, $\exists k : p(k)$, $\sum_{k=a}^b f(k)$, $\int_0^1 k^2 dk$, $\lim_{n \rightarrow \infty} f(n)$, and $\text{argmax}_k f(k)$. When translating mathematical expressions into code, we need to recognize the presence of bound variables, which the mathematical notation does not make quite so explicit. For each bound variable, we need to create a nameless function whose argument is that variable, e.g. $k \Rightarrow p(k)$ or $k \Rightarrow f(k)$ for the examples just shown. Only then will our code correctly reproduce the behavior of bound variables in mathematical expressions.

As an example, the mathematical formula $\forall k \in [1, n] : p(k)$ has a bound variable k and is translated into Scala code as

```
(1 to n).forall(k => p(k))
```

At this point we can apply a simplification trick to this code. The nameless function $k \Rightarrow p(k)$ does exactly the same thing as the (named) function p : It takes an argument, which we may call k , and returns $p(k)$. So, we can simplify the Scala code above to

```
(1 to n).forall(p)
```

The simplification of $x \Rightarrow f(x)$ to just f is always possible for functions f of a single argument.³

1.2 Aggregating data from sequences

Consider the task of counting how many even numbers there are in a given list L of integers. For example, the list $[5, 6, 7, 8, 9]$ contains *two* even numbers: 6 and 8.

A mathematical formula for this task can be written like this,

$$\begin{aligned} \text{count_even}(L) &= \sum_{k \in L} \text{is_even}(k) \quad , \\ \text{is_even}(k) &= \begin{cases} 1 & \text{if } k = 0 \bmod 2 \\ 0 & \text{otherwise} \end{cases} . \end{aligned}$$

Here we defined a helper function `is_even` in order to write more easily a formula for `count_even`. In mathematics, complicated formulas are often split into simpler parts by defining helper expressions.

³Certain features of Scala allow programmers to write code that looks like $f(x)$ but actually involves additional implicit or default arguments of the function f , or an implicit type conversion for its argument x . In those cases, replacing the code $x \Rightarrow f(x)$ by f will fail to compile. But these complications do not arise when working with simple functions.

We can write the Scala code similarly. We first define the helper function `is_even`; the Scala code can be written in a style quite similar to the mathematical formula:

```
def is_even(k: Int): Int = (k % 2) match {
  case 0 => 1 // First, check if it is zero.
  case _ => 0 // The underscore matches everything else.
}
```

For such a simple computation, we could also write shorter code using a nameless function,

```
val is_even = (k: Int) => if (k % 2 == 0) 1 else 0
```

Given this function, we now need to translate into Scala code the expression $\sum_{k \in L} \text{is_even}(k)$. We can represent the list L using the data type `List[Int]` from the Scala standard library.

To compute $\sum_{k \in L} \text{is_even}(k)$, we must apply the function `is_even` to each element of the list L , which will produce a list of some (integer) results, and then we will need to add all those results together. It is convenient to perform these two steps separately. This can be done with the functions `.map` and `.sum`, defined in the Scala standard library as infix methods for the data type `List`.

The method `.sum` is similar to `.product` and is defined for any `List` of numerical types (`Int`, `Float`, `Double`, etc.). It computes the sum of all numbers in the list:

```
scala> List(1, 2, 3).sum
res0: Int = 6
```

The method `.map` needs more explanation. This method takes a *function* as its second argument, applies that function to each element of the list, and puts all the results into a *new* list, which is then returned as the result value:

```
scala> List(1, 2, 3).map(x => x*x + 100*x)
res1: List[Int] = List(101, 204, 309)
```

In this example, the argument of `.map` is the nameless function $x \mapsto x^2 + 100x$. This function will be used repeatedly by `.map` to transform each integer from `List(1, 2, 3)`, creating a new list as a result.

It is equally possible to define the transforming function separately, give it a name, and then use it as the argument to `.map`:

```
scala> def func1(x: Int): Int = x*x + 100*x
func1: (x: Int)Int

scala> List(1, 2, 3).map(func1)
res2: List[Int] = List(101, 204, 309)
```

Usually, short functions are defined inline, and longer functions are given a name and defined separately.

An infix method, such as `.map`, can be also used with a “dotless” syntax:

```
scala> List(1, 2, 3) map func1
res3: List[Int] = List(101, 204, 309)
```

If the transforming function `func1` is used only once, and especially for a simple operation such as $x \mapsto x^2 + 100x$, it is easier to work with a nameless function.

We can now combine the methods `.map` and `.sum` to define `count_even`:

```
def count_even(s: List[Int]) = s.map(is_even).sum
```

This code can be also written using a nameless function instead of `is_even`:

```
def count_even(s: List[Int]): Int = s
  .map { k => if (k % 2 == 0) 1 else 0 }
  .sum
```

It is customary in Scala to use infix methods when chaining several operations. For instance `s.map(...).sum` means first apply `s.map(...)`, which returns a *new* list, and then apply `.sum` to that list. To make the code more readable, we put each of the chained methods on a new line.

To test this code, let us run it in the Scala interpreter. In order to let the interpreter work correctly with code entered line by line, the dot character needs to be at the *end* of the line. (In a compiled code, the dots can be at the beginning of the lines since the compiler reads the entire code at once.)

```
scala> def count_even(s: List[Int]): Int = s .
      map { k => if (k % 2 == 0) 1 else 0 } .
      sum
count_even: (s: List[Int])Int

scala> count_even(List(1,2,3,4,5))
res0: Int = 2

scala> count_even( List(1,2,3,4,5).map(x => x * 2) )
res1: Int = 5
```

Note that the Scala interpreter prints the types differently for functions defined using `def`. It prints `(s: List[Int])Int` for a function of type `List[Int] => Int`.

1.3 Filtering and truncating a sequence

In addition to the methods `.sum`, `.product`, `.map`, `.forall` that we have already seen, the Scala standard library defines many other useful methods. We will now take a look at using the methods `.max`, `.min`, `.exists`, `.size`, `.filter`, and `.takeWhile`.

The methods `.max`, `.min`, and `.size` are self-explanatory:

```
scala> List(10, 20, 30).max
res2: Int = 30

scala> List(10, 20, 30).min
res3: Int = 10

scala> List(10, 20, 30).size
res4: Int = 3
```

The methods `.forall`, `.exists`, `.filter`, and `.takeWhile` require a predicate as an argument. The `.forall` method returns `true` if and only if the predicate returns `true` for all values in the list; the `.exists` method returns `true` if and only if the predicate holds (returns `true`) for at least one value in the list. These methods can be written as mathematical formulas like this:

$$\begin{aligned}\text{forall}(S, p) &= \forall k \in S : p(k) = \text{true} \\ \text{exists}(S, p) &= \exists k \in S : p(k) = \text{true}\end{aligned}$$

However, there is no mathematical notation for operations such as “removing elements from a list”, so we will focus on the Scala syntax for these functions.

The `.filter` method returns a list that contains only the values for which the predicate returns `true`:

```
scala> List(1, 2, 3, 4, 5).filter(k => k % 3 != 0)
res5: List[Int] = List(1, 2, 4, 5)
```

The `.takeWhile` method truncates a given list, returning a new list with the initial portion of values from the original list for which predicate keeps being `true`:

```
scala> List(1, 2, 3, 4, 5).takeWhile(k => k % 3 != 0)
res6: List[Int] = List(1, 2)
```

In all these cases, the predicate’s argument `k` must be of the same type as the elements in the list. In the examples shown above, the elements are integers (i.e. the lists have type `List[Int]`), therefore `k` must be of type `Int`.

The methods `.max`, `.min`, `.sum`, and `.product` are defined on lists of *numeric types*, such as `Int`, `Double`, and `Long`. The other methods are defined on lists of all types.

Using these methods, we can solve many problems that involve transforming and aggregating data stored in lists (as well as in arrays, sets, or other similar data structures). A **transformation** is a function taking a list of values and returning another list of values; examples of transformation functions are `.filter` and `.map`. An **aggregation** is a function taking a list of values and returning a *single* value; examples of aggregation functions are `.max` and `.sum`.

Writing programs by chaining together various methods of transformation and aggregation is known as programming in the **map/reduce style**.

1.4 Solved examples

1.4.1 Aggregation

Example 1.4.1.1 Improve the code for `is_prime` by limiting the search to $k^2 \leq n$:

$$\text{is_prime}(n) = \forall k \in [2, n-1] \text{ such that } k^2 \leq n : n \neq 0 \bmod k \quad .$$

Solution: Use `.takeWhile` to truncate the initial list when $k^2 \leq n$ becomes false:

```
def is_prime(n: Int): Boolean =
  (2 to n-1)
    .takeWhile(k => k*k + 1 < n)
    .forall(k => n % k != 0)
```

Example 1.4.1.2 Compute $\prod_{k \in [1,10]} |\sin(k+2)|$.

Solution

```
(1 to 10)
  .map(k => math.abs(math.sin(k + 2)))
  .product
```

Example 1.4.1.3 Compute $\sum_{k \in [1,10]; \cos k > 0} \sqrt{\cos k}$.

Solution

```
(1 to 10)
  .filter(k => math.cos(k) > 0)
  .map(k => math.sqrt(math.cos(k)))
  .sum
```

It is safe to compute $\sqrt{\cos k}$, because we have first filtered the list by keeping only values k for which $\cos k > 0$. Let us check that this is so:

```
scala> (1 to 10).toList.filter(k => math.cos(k) > 0).map(x => math.cos(x))
res0: List[Double] = List(0.5403023058681398, 0.28366218546322625, 0.9601702866503661,
0.7539022543433046)
```

Example 1.4.1.4 Compute the average of a non-empty list of type `List[Double]`,

$$\text{average}(s) = \frac{1}{n} \sum_{i=0}^{n-1} s_i \quad .$$

Solution We need to divide the sum by the length of the list:

```
def average(s: List[Double]): Double = s.sum / s.size

scala> average(List(1.0, 2.0, 3.0))
res0: Double = 2.0
```

Example 1.4.1.5 Given n , compute the Wallis product⁴ truncated up to $\frac{2n}{2n+1}$:

$$\text{wallis}(n) = \frac{2}{1} \frac{2}{3} \frac{4}{5} \frac{6}{7} \dots \frac{2n}{2n+1}.$$

Solution Define the helper function `wallis_frac(i)` that computes the i^{th} fraction. The method `.toDouble` converts integers to `Double` numbers.

```
def wallis_frac(i: Int): Double = (2*i).toDouble/((2*i - 1)*(2*i)/(2*i + 1))

def wallis(n: Int) = (1 to n).map(wallis_frac).product

scala> math.cos(wallis(10000)) // Should be close to 0.
res0: Double = 3.9267453954401036E-5

scala> math.cos(wallis(100000)) // Should be even closer to 0.
res1: Double = 3.926966362362075E-6
```

The limit of the Wallis product is $\frac{\pi}{2}$, so the cosine of `wallis(n)` tends to zero in the limit of large n .

Example 1.4.1.6 Check numerically that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$: First, define a function of n that computes a partial sum of this series until $k = n$. Then compute the partial sum for a large value of n and compare with the limit value.

Solution

```
def euler_series(n: Int): Double = (1 to n).map(k => 1.0/k/k).sum

scala> euler_series(100000)
res0: Double = 1.6449240668982423

scala> val pi = 4*math.atan(1)
pi: Double = 3.141592653589793

scala> pi*pi/6
res1: Double = 1.6449340668482264
```

Example 1.4.1.7 Check numerically the infinite product formula

$$\prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2}\right) = \frac{\sin \pi x}{\pi x}.$$

Solution Compute this product up to $k = n$ for $x = 0.1$ and a large value of n , say $n = 10^5$, and compare with the right-hand side:

```
def sine_product(n: Int, x: Double): Double = (1 to n).map(k => 1.0 - x*x/k/k).product

scala> sine_product(n = 100000, x = 0.1) // Arguments may be named, for clarity.
res0: Double = 0.9836317414461351

scala> math.sin(pi*0.1)/pi/0.1
res1: Double = 0.9836316430834658
```

Example 1.4.1.8 Define a function p that takes a list of integers and a function $f: \text{Int} \Rightarrow \text{Int}$, and returns the largest value of $f(x)$ among all x in the list.

Solution

```
def p(s: List[Int], f: Int => Int): Int = s.map(f).max
```

Here is a test for this function:

```
scala> p(List(2, 3, 4, 5), x => 60 / x)
res0: Int = 30
```

⁴https://en.wikipedia.org/wiki/Wallis_product

Mathematical notation	Scala code
$x \mapsto \sqrt{x^2 + 1}$	<code>x => math.sqrt(x*x + 1)</code>
list $[1, 2, \dots, n]$	<code>(1 to n)</code>
list $[f(1), \dots, f(n)]$	<code>(1 to n).map(k => f(k))</code>
$\sum_{k=1}^n k^2$	<code>(1 to n).map(k => k*k).sum</code>
$\prod_{k=1}^n f(k)$	<code>(1 to n).map(f).product</code>
$\forall k$ such that $1 \leq k \leq n : p(k)$ holds	<code>(1 to n).forall(k => p(k))</code>
$\exists k, 1 \leq k \leq n$ such that $p(k)$ holds	<code>(1 to n).exists(k => p(k))</code>
$\sum_{k \in S \text{ such that } p(k) \text{ holds}} f(k)$	<code>s.filter(p).map(f).sum</code>

Table 1.1: Translating mathematics into code.

1.4.2 Transformation

Example 1.4.2.1 Given a list of lists, `s: List[List[Int]]`, select the inner lists of size at least 3. The result must be again of type `List[List[Int]]`.

Solution To “select the inner lists” means to compute a *new* list containing only the desired inner lists. We use `.filter` on the outer list `s`. The predicate for the filter is a function that takes an inner list and returns `true` if the size of that list is at least 3. Write the predicate as a nameless function, `t => t.size >= 3`, where `t` is of type `List[Int]`:

```
def f(s: List[List[Int]]): List[List[Int]] = s.filter(t => t.size >= 3)

scala> f(List( List(1,2), List(1,2,3), List(1,2,3,4) ))
res0: List[List[Int]] = List(List(1, 2, 3), List(1, 2, 3, 4))
```

The Scala compiler deduces the type of `t` from the code; no other type would work since we apply `.filter` to a *list of lists* of integers.

Example 1.4.2.2 Find all integers $k \in [1, 10]$ such that there are at least three different integers j , where $1 \leq j \leq k$, each j satisfying the condition $j^2 > 2k$.

Solution

```
scala> (1 to 10).toList.filter(k => (1 to k).filter(j => j*j > 2*k).size >= 3)
res0: List[Int] = List(6, 7, 8, 9, 10)
```

The argument of the outer `.filter` is a nameless function that also uses a `.filter`. The inner expression (shown at left) computes the list of j ’s that satisfy the condition $j^2 > 2k$, and then compares the size of that list with 3. In this way, we impose the requirement that there should be at least 3 values of j . We can see how the Scala code closely follows the mathematical formulation of the task.

1.5 Summary

Functional programs are mathematical formulas translated into code. Table 1.1 shows how to implement some often used mathematical constructions in Scala.

What problems can one solve with this knowledge?

- Compute mathematical expressions involving sums, products, and quantifiers, based on integer ranges, such as $\sum_{k=1}^n f(k)$ etc.

- Transform and aggregate data from lists using `.map`, `.filter`, `.sum`, and other methods from the Scala standard library.

What are examples of problems that are *not* solvable with these tools?

- Example 1: Compute the smallest $n \geq 1$ such that

$$f(f(f(\dots f(0)\dots)) > 1000 \quad ,$$

where the given function f is applied n times.

- Example 2: Given a list s of numbers, compute the list r of running averages:

$$r_n = \frac{1}{n} \sum_{k=0}^{n-1} s_k \quad .$$

- Example 3: Perform binary search over a sorted list of integers.

These computations involve *mathematical induction*, which we have not yet learned to translate into code in the general case.

Library functions we have seen so far, such as `.map` and `.filter`, implement a restricted class of iterative operations on lists: namely, operations that process each element of a given list independently and accumulate results. For instance, when computing `s.map(f)`, the number of function applications is given by the size of the initial list. However, Example 1 requires applying a function f repeatedly until a given condition holds – that is, repeating for an *initially unknown* number of times. So it is impossible to write an expression containing `.map`, `.filter`, `.takeWhile`, etc., that solves Example 1. We could write the solution of Example 1 as a formula by using mathematical induction, but we have not yet seen how to implement that in Scala code.

Example 2 can be formulated as a definition of a new list r by induction,

$$r_0 = s_0 \quad ; \quad r_i = s_i + r_{i-1} \text{ for } i = 1, 2, 3, \dots$$

However, operations such as `.map` and `.filter` cannot compute r_i depending on the value of r_{i-1} .

Example 3 defines the search result by induction: the list is split in half, and search is performed by inductive hypothesis in the half that contains the required value. This computation requires an initially unknown number of steps.

Chapter 2 explains how to implement these tasks by translating mathematical induction into code using recursion.

1.6 Exercises

1.6.1 Aggregation

Exercise 1.6.1.1 Machin's formula⁵ converges to π faster than Example 1.4.1.5:

$$\begin{aligned} \frac{\pi}{4} &= 4 \arctan \frac{1}{5} - \arctan \frac{1}{239} \quad , \\ \arctan \frac{1}{n} &= \frac{1}{n} - \frac{1}{3} \frac{1}{n^3} + \frac{1}{5} \frac{1}{n^5} - \dots = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} n^{-2k-1} \quad . \end{aligned}$$

Implement a function that computes the series for $\arctan \frac{1}{n}$ up to a given number of terms, and compute an approximation of π using this formula. Show that about 12 terms of the series are already sufficient for a full-precision `Double` approximation of π .

⁵<http://turner.faculty.swau.edu/mathematics/materialslibrary/pi/machin.html>

Exercise 1.6.1.2 Using the function `is_prime`, check numerically the Euler product formula⁶ for the Riemann zeta function $\zeta(4)$; it is known⁷ that $\zeta(4) = \frac{\pi^4}{90}$:

$$\prod_{k \geq 2; k \text{ is prime}} \frac{1}{1 - p^{-4}} = \frac{\pi^4}{90} \quad .$$

1.6.2 Transformation

Exercise 1.6.2.1 Define a function `add_20` of type `List[List[Int]] => List[List[Int]]` that adds 20 to every element of every inner list. A sample test:

```
scala> add_20( List( List(1), List(2, 3) ) )
res0: List[List[Int]] = List(List(21), List(22, 23))
```

Exercise 1.6.2.2 An integer n is called a “3-factor” if it is divisible by only three different integers j such that $2 \leq j < n$. Compute the set of all “3-factor” integers n among $n \in [1, \dots, 1000]$.

Exercise 1.6.2.3 Given a function `f: Int => Boolean`, an integer n is called a “3- f ” if there are only three different integers $j \in [1, \dots, n]$ such that `f(j)` returns `true`. Define a function that takes `f` as an argument and returns a sequence of all “3- f ” integers among $n \in [1, \dots, 1000]$. What is the type of that function? Implement Exercise 1.6.2.2 using that function.

Exercise 1.6.2.4 Define a function `see100` of type `List[List[Int]] => List[List[Int]]` that selects only those inner lists whose largest value is at least 100. Test with:

```
scala> see100( List( List(0, 1, 100), List(60, 80), List(1000) ) )
res0: List[List[Int]] = List(List(0, 1, 100), List(1000))
```

Exercise 1.6.2.5 Define a function of type `List[Double] => List[Double]` that “normalizes” the list: finds the element having the largest absolute value and, if that value is nonzero, divides all elements by that factor and returns a new list; otherwise returns the original list.

1.7 Discussion

1.7.1 Functional programming as a paradigm

Functional programming (FP) is a **paradigm** of programming, – that is, an approach that guides programmers to write code in specific ways, for a wide range of programming tasks.

The main principle of FP is to write code *as a mathematical expression or formula*. This approach allows programmers to derive code through logical reasoning rather than through guessing, – similarly to how books on mathematics reason about mathematical formulas and derive results systematically, without guessing or “debugging.” Similarly to mathematicians and scientists who reason about formulas, functional programmers can *reason about code* systematically and logically, based on rigorous principles. This is possible only because code is written as a mathematical formula.

Mathematical intuition is backed by the vast experience accumulated while working with data over millennia of human history. It took centuries to invent flexible and powerful notation such as $\sum_{k \in S} p(k)$ and to develop the corresponding rules of calculation. Functional programmers are fortunate to have these reasoning tools at their disposal.

As we have seen, the Scala code for certain computational tasks corresponds quite closely to mathematical formulas. (Scala conventions and syntax, of course, require programmers to spell out certain things that are omitted in the mathematical notation.) Just as in mathematics, large code expressions may be split into smaller expressions when needed. Expressions can be easily reused,

⁶https://en.wikipedia.org/wiki/Proof_of_the_Euler_product_formula_for_the_Riemann_zeta_function

⁷<https://tinyurl.com/yxey4tsd>

flexibly composed together, and written independently from each other. The FP community has developed a toolkit of functions (such as `.map`, `.filter`, etc.) that proved to be especially useful in real-life programming, although many of them are not standard in mathematical literature.

Mastering FP involves practicing to reason about programs as formulas, building up the specific kind of applied mathematical intuition, familiarizing oneself with concepts adapted to a programmer's needs, and learning how to translate the mathematics into code in various cases. The FP community has discovered a number of specific design patterns founded on mathematical principles but driven by practical necessities of programming (rather than by the needs of academic mathematics). This book explains the required mathematical design patterns in detail, developing them through intuition and examples of Scala code.

1.7.2 Functional programming languages

It is possible to apply the FP paradigm while writing code in any programming language. However, some languages lack certain features that make FP techniques much easier to use in practice. For example, in a language such as Python or Ruby, one can productively use only a limited number of FP idioms, such as the map/reduce operations. More advanced FP constructions are impractical in these languages because the required code becomes too hard to read and to write without errors, which negates the advantages of rigorous reasoning about functional programs.

Some programming languages, such as Haskell and OCaml, were designed specifically for advanced use in the FP paradigm. Other languages, such as ML, F#, Scala, Swift, Elm, and PureScript, have different design goals but still support enough FP features to be considered FP languages. This book uses Scala, but the same constructions may be implemented in other FP languages in a similar way. At the level of detail needed in this book, the differences between languages such as ML, OCaml, Haskell, F#, Scala, Swift, Elm, or PureScript do not play a significant role.

1.7.3 The mathematical meaning of “variables”

The usage of variables in functional programming is similar to how mathematical literature uses variables. In mathematics, **variables** are used first of all as *arguments* of functions; e.g. the formula

$$f(x) = x^2 + x$$

contains the variable x and defines a function f that takes x as its argument (to be definite, let us assume that x is an integer) and computes the value $x^2 + x$. The body of the function is the expression $x^2 + x$.

Mathematics has the convention that a variable, such as x , does not change its value within a formula. Indeed, there is no mathematical notation even to talk about “changing” the value of x *inside* the formula $x^2 + x$. It would be quite confusing if a mathematics textbook said “before adding the last x in the formula $x^2 + x$, we change that x by adding 4 to it”. If the “last x ” in $x^2 + x$ needs to have a 4 added to it, a mathematics textbook will just write the formula $x^2 + x + 4$.

Arguments of nameless functions are also immutable. Consider, for example,

$$f(n) = \sum_{k=0}^n (k^2 + k) \quad .$$

Here, n is the argument of the function f , while k is the argument of the nameless function $k \mapsto k^2 + k$. Neither n nor k can be “modified” in any sense within the expressions where they are used. The symbols k and n stand for some integer values, and these values are immutable. Indeed, it is meaningless to say that we “modified the integer 4”. In the same way, we cannot modify k .

So, a variable in mathematics remains constant *within the expression* where it is defined; in that expression, a variable is essentially a “named constant”. Of course, a function f can be applied to different values x , to compute a different result $f(x)$ each time. However, a given value of x will remain unmodified within the body of the function f while $f(x)$ is being computed.

Functional programming adopts this convention from mathematics: variables are immutable named constants. (Scala also has *mutable* variables, but we will not consider them in this book.)

In Scala, function arguments are immutable within the function body:

```
def f(x: Int) = x * x + x // Cannot modify 'x' here.
```

The *type* of each mathematical variable (such as integer, vector, etc.) is also fixed. Each variable is a value from a specific set (e.g. the set of all integers, the set of all vectors, etc.). Mathematical formulas such as $x^2 + x$ do not express any “checking” that x is indeed an integer and not, say, a vector, in the middle of evaluating $x^2 + x$. The types of all variables are checked in advance.

Functional programming adopts the same view: Each argument of each function must have a **type** that represents *the set of possible allowed values* for that function argument. The programming language’s compiler will automatically check the types of all arguments *before* the program runs. A program that calls functions on arguments of incorrect types will not compile.

The second usage of **variables** in mathematics is to denote expressions that will be reused. For example, one writes: let $z = \frac{x-y}{x+y}$ and now compute $\cos z + \cos 2z + \cos 3z$. Again, the variable z remains immutable, and its type remains fixed.

In Scala, this construction (defining an expression to be reused later) is written with the “**val**” syntax. Each variable defined using “**val**” is a named constant, and its type and value are fixed at the time of definition. Type annotations for “**val**”s are optional in Scala: for instance we could write

```
val x: Int = 123
```

or we could omit the type annotation `: Int` and write more concisely

```
val x = 123
```

because it is clear that this `x` is an integer. However, it is often helpful to write out types. If we do so, the compiler will check that the types match correctly and give an error message whenever wrong types are used. For example, a type error is detected when using a `String` instead of an `Int`:

```
scala> val x: Int = "123"
<console>:11: error: type mismatch;
 found   : String("123")
 required: Int
    val x: Int = "123"
              ^
```

1.7.4 Iteration without loops

Another distinctive feature of the FP paradigm is handling of iteration without writing loops.

Iterative computations are ubiquitous in mathematics. As an example, consider the formula for the standard deviation estimated from a sample,

$$\sigma(s) = \sqrt{\frac{1}{n-1} \sum_{i=1}^n \sum_{j=1}^n s_i s_j - \frac{1}{n(n-1)} \left(\sum_{i=1}^n s_i \right)^2} .$$

These expressions are computed by iterating over values of i and j . And yet, no mathematics textbook uses “loops” or says “now repeat this formula ten times”. Indeed, it would be pointless to evaluate a formula such as $x^2 + x$ ten times in a loop, or to “repeat” an equation such as

$$(x-1)(x^2+x+1) = x^3-1 .$$

Instead of loops, mathematicians write *expressions* such as $\sum_{i=1}^n s_i$, where symbols such as $\sum_{i=1}^n$ or $\prod_{i=1}^n$ denote iterative computations. Such computations are defined using mathematical induction. The functional programming paradigm has developed rich tools for translating mathematical induction into code. In this chapter, we have seen methods such as `.map`, `.filter`, and `.sum`, which

implement certain kinds of iterative computations. These and other operations can be combined in very flexible ways, which allows programmers to write iterative code without loops.

The programmer can avoid writing loops because the iteration is delegated to the library functions `.map`, `.filter`, `.sum`, and so on. It is the job of the library and the compiler to translate these functions into machine code. The machine code most likely *will* contain loops; but the functional programmer does not need to see that code or to reason about it.

1.7.5 Nameless functions in mathematical notation

Functions in mathematics are mappings from one set to another. A function does not necessarily *need* a name; the mapping just needs to be defined. However, nameless functions have not been widely used in the conventional mathematical notation. It turns out that nameless functions are quite important in functional programming because, in particular, they allow programmers to write code more concisely and use a straightforward, consistent syntax.

Nameless functions contain bound variables that are invisible outside the function's scope. This property is directly reflected by the prevailing mathematical conventions. Compare the formulas

$$f(x) = \int_0^x \frac{dx}{1+x} \quad ; \quad f(x) = \int_0^x \frac{dz}{1+z} \quad .$$

The mathematical convention is that one may rename the integration variable at will, and so these formulas define the same function f .

In programming, the only situation when a variable “may be renamed at will” is when the variable represents an argument of a function. It follows that the notations $\frac{dx}{1+x}$ and $\frac{dz}{1+z}$ correspond to a nameless function whose argument was renamed from x to z . In FP notation, this nameless function would be denoted as $z \Rightarrow \frac{1}{1+z}$, and the integral rewritten as code such as

```
integration(0, x, { z => 1.0 / (1 + z) } )
```

Now consider the traditional mathematical notations for summation, e.g.

$$\sum_{k=0}^x \frac{1}{1+k} \quad .$$

In sums, the bound variable k is introduced under the Σ symbol; but in integrals, the bound variable follows the special symbol “ d ”. This notational inconsistency could be removed if we were to use nameless functions explicitly, for example:

$$\begin{aligned} \sum_0^x \left(k \mapsto \frac{1}{1+k} \right) &\text{ instead of } \sum_{k=0}^x \frac{1}{1+k} \quad , \\ \int_0^x \left(z \mapsto \frac{1}{1+z} \right) &\text{ instead of } \int_0^x \frac{dz}{1+z} \quad . \end{aligned}$$

In this notation, the new summation symbol \sum_0^x does not mention the name “ k ” but takes a function as an argument. Similarly, the new integration symbol \int_0^x does not mention “ z ” and does not use the special symbol “ d ” but now takes a function as an argument. Written in this way, the operations of summation and integration become *functions* that take a function as argument. The above summation may be written in a consistent and straightforward manner as a Scala function:

```
summation(0, x, { y => 1.0 / (1 + y) } )
```

We could implement `summation(a,b,g)` as

```
def summation(a: Int, b: Int, g: Int => Double) = (a to b).map(g).sum

scala> summation(1, 10, x => math.sqrt(x))
res0: Double = 22.4682781862041
```


Numerical integration requires longer code, since the formulas are more complicated. For instance, Simpson's rule⁸ can be written as

$$\text{integration}(a, b, g) = \frac{\delta}{3} (g(a) + g(b) + 4s_1 + 2s_2) \quad ,$$

$$\text{where } n = 2 \left\lfloor \frac{b-a}{\varepsilon} \right\rfloor, \quad \delta_x = \frac{b-a}{n} \quad ,$$

$$s_1 = \sum_{i=1,3,\dots,n-1} g(a + i\delta_x) \quad , \quad s_2 = \sum_{i=2,4,\dots,n-2} g(a + i\delta_x) \quad .$$

A straightforward translation of this formula into Scala is

```
def integration(a: Double, b: Double, g: Double => Double, eps: Double) = {
  // First, we define some helper values and functions that replace
  // the definitions "where n = ..." in the mathematical formula.
  val n: Int = 2*((b - a)/eps).toInt
  val delta_x = (b - a) / n
  val g_i = (i: Int) => g(a + i*delta_x)
  val s1 = (1 to (n-1) by 2).map(g_i).sum
  val s2 = (2 to (n-2) by 2).map(g_i).sum
  // Now we write the expression for the final result.
  delta_x / 3 * (g(a) + g(b) + 4*s1 + 2*s2)
}

scala> integration(0, 5, x => x*x*x*x, eps = 0.01)    // Exact answer is 625.
res0: Double = 625.0000000004167

scala> integration(0, 7, x => x*x*x*x*x*x, eps = 0.01) // Exact answer is 117649.
res1: Double = 117649.00000014296
```

The entire code is one large *expression*, with a few sub-expressions defined for convenience as a few helper values and helper functions. In other words, this code is written in the FP paradigm.

1.7.6 Named and nameless expressions and their uses

It is a significant advantage if a programming language supports unnamed (or “nameless”) expressions. To see this, consider a familiar situation where we take the absence of names for granted.

In most programming languages today, we can directly write arithmetical expressions such as $(x+123)*y/(2+x)$. Here, x and y are variables with names. Note, however, that the entire expression does not need to have a name. Parts of that expression (such as $x+123$ or $2+x$) also do not have separate names. It would be quite inconvenient if we *needed* to assign a name separately to each sub-expression. The code for $(x+123)*y/(2+x)$ could then look like this:

```
r1 = 123
r2 = x + r1
r3 = r2 * y
r4 = 2
r5 = r4 + x
r6 = r3 / r5
return r6
```

This style of programming resembles assembly languages, where *every* sub-expression – that is, every step of every calculation, – must be named separately (and, in the assembly languages, assigned a memory address or a CPU register).

Programmers become more productive when their programming language supports nameless expressions. This is also common practice in mathematics; names are assigned when needed, but most expressions remain nameless.

It is similarly useful if data structures can be created without names. For instance, a **dictionary** (also called a “hashmap”) is created in Scala as

⁸https://en.wikipedia.org/wiki/Simpson%27s_rule

```
Map("a" -> 1, "b" -> 2, "c" -> 3)
```

This code is a nameless expression whose value is a dictionary. In programming languages that do not have such a construction, programmers have to write repetitive code that creates an initially empty dictionary and then fills it step by step with values:

```
// Scala code creating a dictionary:
Map("a" -> 1, "b" -> 2, "c" -> 3)

/* Shortest Java code for the same:
new HashMap<String, Integer>() {{
    put("a", 1);
    put("b", 2);
    put("c", 3);
}}; */
```

Nameless functions are useful for the same reason as nameless values of other types: they allow us to build larger programs from simpler parts in a uniform way.

1.7.7 Historical perspective on usage of nameless functions

Nameless functions were first used in 1936 in a theoretical programming language called “ λ -calculus”. In that language,⁹ all functions are nameless and have a single argument. The letter λ is a syntax separator denoting function arguments in nameless functions. For example, the nameless function $x \mapsto x + 1$ could be written as $\lambda x.add\ x\ 1$ in λ -calculus, if it had a function *add* for adding integers (which it does not).

In most programming languages that were in use until around 1990, all functions required names. But by 2015, most languages added support for nameless functions, because programming in the map/reduce style (which invites frequent use of nameless functions) turned out to be immensely productive. Table 1.2 shows the year when nameless functions were introduced in each language.

What this book calls a “nameless function” is also called anonymous function, function expression, function literal, closure, lambda function, lambda expression, or just a “lambda”.

Language	Year	Code for $k \mapsto k + 1$
λ -calculus	1936	$\lambda k.add\ k\ 1$
typed λ -calculus	1940	$\lambda k : int.add\ k\ 1$
LISP	1958	(lambda (k) (+ k 1))
Standard ML	1973	fn (k:int) => k + 1
Scheme	1975	(lambda (k) (+ k 1))
OCaml	1985	fun k -> k + 1
Haskell	1990	\ k -> k + 1
Oz	1991	fun {\$ K} K + 1
R	1993	function(k) k + 1
Python 1.0	1994	lambda k: k + 1
JavaScript	1995	function(k) { return k + 1; }
Mercury	1995	func(K) = K + 1
Ruby	1995	lambda { k k + 1 }
Lua 3.1	1998	function(k) return k + 1 end
Scala	2003	(k:Int) => k + 1
F#	2005	fun (k:int) -> k + 1
C# 3.0	2007	delegate(int k) { return k + 1; }
C++ 11	2011	[] (int k) { return k + 1; }
Go	2012	func(k int) { return k + 1 }
Kotlin	2012	{ k:Int -> k + 1 }
Swift	2014	{ (k:int) -> int in return k + 1 }
Java 8	2014	(int k) -> k + 1
Rust	2015	k:i32 k + 1

Table 1.2: Nameless functions in programming languages.

⁹Although called a “calculus,” it is a (drastically simplified) programming language. It has nothing to do with “calculus” as known in mathematics, such as differential or integral calculus. Also, the letter λ has no particular significance; it plays a purely syntactic role in the λ -calculus. Practitioners of functional programming usually do not need to study any λ -calculus. All practically relevant knowledge related to λ -calculus is explained in Chapter 4 of this book.

2 Mathematical formulas as code. II.

Mathematical induction

We will now study more flexible ways of working with data collections in the functional programming paradigm. The Scala standard library has methods for performing general iterative computations, that is, computations defined by induction. Translating mathematical induction into code is the focus of this chapter.

First, we need to become fluent in using tuple types with Scala collections.

2.1 Tuple types

2.1.1 Examples of using tuples

Many standard library methods in Scala work with tuple types. A simple example of a tuple is a *pair* of values, – such as, a pair of an integer and a string. The Scala syntax for this type of pair is

```
val a: (Int, String) = (123, "xyz")
```

The type expression `(Int, String)` denotes the type of this pair.

A **triple** is defined in Scala like this:

```
val b: (Boolean, Int, Int) = (true, 3, 4)
```

Pairs and triples are examples of tuples. A **tuple** can contain any number of values, which may be called **parts** of a tuple (they are also called **fields** of a tuple). The parts of a tuple can have different types, but the type of each part is fixed once and for all. Also, the number of parts in a tuple is fixed. It is a **type error** to use incorrect types in a tuple, or an incorrect number of parts of a tuple:

```
scala> val bad: (Int, String) = (1,2)
<console>:11: error: type mismatch;
 found   : Int(2)
 required: String
    val bad: (Int, String) = (1,2)
                                ^
scala> val bad: (Int, String) = (1,"a",3)
<console>:11: error: type mismatch;
 found   : (Int, String, Int)
 required: (Int, String)
    val bad: (Int, String) = (1,"a",3)
                                ^
```

Parts of a tuple can be accessed by number, starting from 1. The Scala syntax for **tuple accessor** methods looks like `._1`, for example:

```
scala> val a = (123, "xyz")
a: (Int, String) = (123,xyz)

scala> a._1
res0: Int = 123

scala> a._2
res1: String = xyz
```

It is a type error to access a tuple part that does not exist:

```
scala> a._0
<console>:13: error: value _0 is not a member of (Int, String)
  a._0
  ~

scala> a._5
<console>:13: error: value _5 is not a member of (Int, String)
  a._5
  ~
```

Type errors are detected at compile time, before any computations begin.

Tuples can be **nested**: any part of a tuple can be itself a tuple:

```
scala> val c: (Boolean, (String, Int), Boolean) = (true, ("abc", 3), false)
c: (Boolean, (String, Int), Boolean) = (true,(abc,3),false)

scala> c._1
res0: Boolean = true

scala> c._2
res1: (String, Int) = (abc,3)
```

To define functions whose arguments are tuples, we could use the tuple accessors. An example of such a function is

```
def f(p: (Boolean, Int), q: Int): Boolean = p._1 && (p._2 > q)
```

The first argument, p , of this function, has a tuple type. The function body uses accessor methods (`.1` and `.2`) to compute the result value. Note that the second part of the tuple p is of type `Int`, so it is valid to compare it with an integer q . It would be a type error to compare the *tuple* p with an *integer* using the expression $p > q$. It would be also a type error to apply the function f to an argument p that has a wrong type, e.g. the type `(Int, Int)` instead of `(Boolean, Int)`.

2.1.2 Pattern matching for tuples

Instead of using accessor methods when working with tuples, it is often convenient to use **pattern matching**. Pattern matching occurs in two situations in Scala:

- destructuring definition: `val pattern = ...`
- case expression: `case pattern => ...`

```
scala> val g = (1, 2, 3)
g: (Int, Int, Int) = (1,2,3)

scala> val (x, y, z) = g
x: Int = 1
y: Int = 2
z: Int = 3
```

An example of a **destructuring definition** is shown at left. The value g is a tuple of three integers. After defining g , we define the three variables x, y, z *at once* in a single `val` definition. We imagine that this definition “destructures” the data structure contained in g and decomposes it into three parts, then assigns the names x, y, z to these parts. The types of the new values are also assigned automatically.

In the example above, the left-hand side of the destructuring definition contains a tuple pattern (x, y, z) that looks like a tuple, except that its parts are names x, y, z that are so far *undefined*. These names are called **pattern variables**. The destructuring definition checks whether the structure of the value of g “matches” the three pattern variables. (If g does not contain a tuple with exactly three parts, the definition will fail.) This computation is called **pattern matching**.

Pattern matching is often used for working with tuples. The expression `{case (a, b, c) => ...}` called a **case expression** (shown at left) performs pattern matching on its argument. The pattern matching will

```
scala> (1, 2, 3) match { case (a, b, c) => a + b + c }
res0: Int = 6
```

“destructure” (i.e. decompose) a tuple and try to match it to the given pattern (a, b, c). In this pattern, a, b, c are as yet undefined new variables, – that is, they are pattern variables. If the pattern matching succeeds, the pattern variables a, b, c are assigned their values, and the function body can proceed to perform its computation. In this example, the pattern variables a, b, c will be assigned values 1, 2, and 3, and so the expression evaluates to 6.

Pattern matching is especially convenient for nested tuples. Here is an example where a nested tuple p is deconstructed by pattern matching:

```
def t1(p: (Int, (String, Int))): String = p match {
  case (x, (str, y)) => str + (x + y).toString
}

scala> t1((10, ("result is ", 2)))
res0: String = result is 12
```

The type structure of the argument (Int, (String, Int)) is visually repeated in the pattern (x, (str, y)), making it clear that x and y become integers and str becomes a string after pattern matching.

If we rewrite the code of t1 using the tuple accessor methods instead of pattern matching, the code will look like this:

```
def t2(p: (Int, (String, Int))): String = p._2._1 + (p._1 + p._2._2).toString
```

This code is shorter but harder to read. For example, it is not immediately clear that p._2._1 is a string. It is also harder to modify this code: Suppose we want to change the type of the tuple p to ((Int, String), Int). Then the new code is

```
def t3(p: ((Int, String), Int)): String = p._1._2 + (p._1._1 + p._2).toString
```

It takes time to verify, by going through every accessor method, that the function t3 computes the same expression as t2. In contrast, the code is changed easily when using the pattern matching expression instead of the accessor methods:

```
def t4(p: ((Int, String), Int)): String = p match {
  case ((x, str), y) => str + (x + y).toString
}
```

The only change in the function body, compared to t1, is in the pattern matcher. So it is visually clear that t4 computes the same expression as t1.

Sometimes we do not need some of the tuple parts in a pattern match. The following syntax is used to make this intention clear:

```
scala> val (x, _, _, z) = ("abc", 123, false, true)
x: String = abc
z: Boolean = true
```

The underscore symbol _ denotes the parts of the pattern that we want to ignore. The underscore will always match any value regardless of its type.

A feature of Scala is a short syntax for functions such as {case (x, y) => y} that extract elements from tuples. The shorter syntax is (t => t._2) or even shorter, (_. _2), as illustrated here:

```
scala> val p: ((Int, Int)) => Int = { case (x, y) => y }
p: ((Int, Int)) => Int = <function1>

scala> p((1, 2))
res0: Int = 2

scala> val q: ((Int, Int)) => Int = (t => t._2)
q: ((Int, Int)) => Int = <function1>

scala> q((1, 2))
res1: Int = 2

scala> Seq((1,10), (2,20), (3,30)).map(_. _2)
res2: Seq[Int] = List(10, 20, 30)
```

2.1.3 Using tuples with collections

Tuples can be combined with any other types without restrictions. For instance, we can define a tuple of functions,

```
val q: (Int => Int, Int => Int) = (x => x + 1, x => x - 1)
```

We can create a list of tuples,

```
val r: List[(String, Int)] = List(("apples", 3), ("oranges", 2), ("pears", 0))
```

We could define a tuple of lists of tuples of functions, or any other combination.

Here is an example of using the standard method `.map` to transform a list of tuples. The argument of `.map` must be a function taking a tuple as its argument. It is convenient to use pattern matching for writing such functions:

```
scala> val basket: List[(String, Int)] = List(("apples", 3), ("pears", 2), ("lemons", 0))
basket: List[(String, Int)] = List((apples,3), (pears,2), (lemons,0))
```

```
scala> basket.map { case (fruit, count) => count * 2 }
res0: List[Int] = List(6, 4, 0)
```

```
scala> basket.map { case (fruit, count) => count * 2 }.sum
res1: Int = 10
```

In this way, we can use the standard methods such as `.map`, `.filter`, `.max`, `.sum` to manipulate sequences of tuples. The names of the pattern variables “fruit”, “count” are chosen to help us remember the meaning of the parts of tuples.

We can easily transform a list of tuples into a list of values of a different type:

```
scala> basket.map { case (fruit, count) =>
  val isAcidic = (fruit == "lemons")
  (fruit, isAcidic)
}
res2: List[(String, Boolean)] = List((apples,false), (pears,false), (lemons,true))
```

In the Scala syntax, a nameless function written with braces `{ ... }` can define local values in its body. The return value of the function is the last expression written in the function body. In this example, the return value of the nameless function is the tuple `(fruit, isAcidic)`.

2.1.4 Treating dictionaries (Scala's Maps) as collections

In the Scala standard library, tuples are frequently used as types of intermediate values. For instance, tuples are used when iterating over dictionaries. The Scala type `Map[K,V]` represents a dictionary with keys of type `K` and values of type `V`. Here `K` and `V` are **type parameters**. Type parameters represent unknown types that will be chosen later, when working with values having specific types.

In order to create a dictionary with given keys and values, we can write

```
Map(("apples", 3), ("oranges", 2), ("pears", 0))
```

The same result is obtained by first creating a sequence of key/value *pairs* and then converting that sequence into a dictionary via the method `.toMap`:

```
List(("apples", 3), ("oranges", 2), ("pears", 0)).toMap
```

The same method works for other collection types such as `Seq`, `Vector`, `Stream`, and `Array`.

The Scala library defines a special infix syntax for pairs via the arrow symbol `->`. The expression `x -> y` is equivalent to the pair `(x, y)`:

```
scala> "apples" -> 3
res0: (String, Int) = (apples,3)
```

With this syntax, the code for creating a dictionary is easier to read:

```
Map("apples" -> 3, "oranges" -> 2, "pears" -> 0)
```

The method `.toSeq` converts a dictionary into a sequence of pairs:

```
scala> Map("apples" -> 3, "oranges" -> 2, "pears" -> 0).toSeq
res20: Seq[(String, Int)] = ArrayBuffer((apples,3), (oranges,2), (pears,0))
```

The `ArrayBuffer` is one of the many list-like data structures in the Scala library. All these data structures are subtypes of the common “sequence” type `Seq`. The methods defined in the Scala standard library sometimes return different implementations of the `Seq` type for reasons of performance.

The standard library has several useful methods that use tuple types, such as `.map` and `.filter` (with dictionaries), `.toMap`, `.zip`, and `.zipWithIndex`. The methods `.flatten`, `.flatMap`, `.groupBy`, and `.sliding` also work with most collection types, including dictionaries and sets. It is important to become familiar with these methods, because it will help writing code that uses sequences, sets, and dictionaries. Let us now look at these methods one by one.

The `.map` and `.toMap` methods Chapter 1 showed how the `.map` method works on sequences: the expression `xs.map(f)` applies a given function `f` to each element of the sequence `xs`, gathering the results in a new sequence. In this sense, we can say that the `.map` method “iterates over” sequences. The `.map` method works similarly on dictionaries, except that iterating over a dictionary of type `Map[K, V]` when applying `.map` looks like iterating over a sequence of *pairs*, `Seq[(K, V)]`. If `d: Map[K, V]` is a dictionary, the argument `f` of `d.map(f)` must be a function operating on tuples of type `(K, V)`. Typically, such functions are written using `case` expressions:

```
val m1 = Map("apples" -> 3, "pears" -> 2, "lemons" -> 0)

scala> m1.map { case (fruit, count) => count * 2 }
res0: Seq[Int] = ArrayBuffer(6, 4, 0)
```

If we want to transform a dictionary into another dictionary, we first create a sequence of pairs, transform it, and then convert it to a dictionary with the `.toMap` method:

```
scala> m1.map { case (fruit, count) => (fruit, count * 2) }.toMap
res1: Map[String,Int] = Map(apples -> 6, pears -> 4, lemons -> 0)
```

The `.filter` method works on dictionaries by iterating on key/value pairs. The filtering predicate must be a function of type `((K, V)) => Boolean`. For example:

```
scala> m1.filter { case (fruit, count) => count > 0 }.toMap
res2: Map[String,Int] = Map(apples -> 6, pears -> 4)
```

The `.zip` and `.zipWithIndex` methods The `.zip` method takes *two* sequences and produces a sequence of pairs, taking one element from each sequence:

```
scala> val s = List(1, 2, 3)
s: List[Int] = List(1, 2, 3)

scala> val t = List(true, false, true)
t: List[Boolean] = List(true, false, true)

scala> s.zip(t)
res3: List[(Int, Boolean)] = List((1,true), (2,false), (3,true))

scala> s zip t
res4: List[(Int, Boolean)] = List((1,true), (2,false), (3,true))
```

In the last line, the equivalent “dotless” infix syntax (`s zip t`) is shown to illustrate a syntax convention of Scala that we will sometimes use.

The `.zip` method works equally well on dictionaries: in that case, dictionaries are automatically converted to sequences of pairs before applying `.zip`.

The `.zipWithIndex` method transforms a sequence into a sequence of pairs, where the second part of the pair is the zero-based index:

```
scala> List("a", "b", "c").zipWithIndex
res5: List[(String, Int)] = List((a,0), (b,1), (c,2))
```


The `.flatten` method converts nested sequences to “flattened” ones:

```
scala> List(List(1, 2), List(2, 3), List(3, 4)).flatten
res6: List[Int] = List(1, 2, 2, 3, 3, 4)
```

The “flattening” operation computes the concatenation of the inner sequences. In Scala, sequences are concatenated using the operation `++`, e.g.:

```
scala> List(1, 2, 3) ++ List(4, 5, 6) ++ List(0)
res7: List[Int] = List(1, 2, 3, 4, 5, 6, 0)
```

So the `.flatten` method inserts the operation `++` between all the inner sequences.

Keep in mind that `.flatten` removes *only one* level of nesting, which is at the “outside” of the data structure. If applied to a `List[List[List[Int]]]`, the `.flatten` method returns a `List[List[Int]]`:

```
scala> List(List(List(1), List(2)), List(List(2), List(3))).flatten
res8: List[List[Int]] = List(List(1), List(2), List(2), List(3))
```

The `.flatMap` method is closely related to `.flatten` and can be seen as a shortcut, equivalent to first applying `.map` and then `.flatten`:

```
scala> List(1,2,3,4).map(n => (1 to n).toList)
res9: List[List[Int]] = List(List(1), List(1, 2), List(1, 2, 3), List(1, 2, 3, 4))

scala> List(1,2,3,4).map(n => (1 to n).toList).flatten
res10: List[Int] = List(1, 1, 2, 1, 2, 3, 1, 2, 3, 4)

scala> List(1,2,3,4).flatMap(n => (1 to n).toList)
res11: List[Int] = List(1, 1, 2, 1, 2, 3, 1, 2, 3, 4)
```

The `.flatMap` operation transforms a sequence by mapping each element to a potentially different number of new elements.

At first sight, it may be unclear why `.flatMap` is useful. (Should we perhaps combine `.filter` and `.flatten` into a `.flatMapFilter`, or combine `.zip` and `.flatten` into a `.flatMapZip`?) However, we will see later in this book that the use of `.flatMap`, which is related to “monads”, is one of the most versatile and powerful design patterns in functional programming. In this chapter, several examples and exercises will illustrate the use of `.flatMap` for working on sequences.

The `.groupBy` method rearranges a sequence into a dictionary where some elements of the original sequence are grouped together into subsequences. For example, given a sequence of words, we can group all words that start with the letter “y” into one subsequence, and all other words into another subsequence. This is accomplished by the following code,

```
scala> Seq("wombat", "xanthan", "yoghurt", "zebra").
  groupBy(s => if (s startsWith "y") 1 else 2)
res12: Map[Int,Seq[String]] = Map(1 -> List(yoghurt), 2 -> List(wombat, xanthan, zebra))
```

The argument of the `.groupBy` method is a *function* that computes a “key” out of each sequence element. The key can have an arbitrarily chosen type. (In the current example, that type is `Int`.) The result of `.groupBy` is a dictionary that maps each key to the sub-sequence of values that have that key. (In the current example, the type of the dictionary is therefore `Map[Int, Seq[String]]`.) The order of elements in the sub-sequences remains the same as in the original sequence.

As another example of using `.groupBy`, the following code will group together all numbers that have the same remainder after division by 3:

```
scala> List(1,2,3,4,5).groupBy(k => k % 3)
res13: Map[Int,List[Int]] = Map(2 -> List(2, 5), 1 -> List(1, 4), 0 -> List(3))
```

The `.sliding` method creates a sequence of sliding windows of a given width:

```
scala> (1 to 10).sliding(4).toList
res14: List[IndexedSeq[Int]] = List(Vector(1, 2, 3, 4), Vector(2, 3, 4, 5), Vector(3, 4, 5, 6),
  Vector(4, 5, 6, 7), Vector(5, 6, 7, 8), Vector(6, 7, 8, 9), Vector(7, 8, 9, 10))
```


Usually, this method is used together with an aggregation operation on the inner sequences. For example, the following code computes a sliding-window average with window width 50 over an array of 100 numbers:

```
scala> (1 to 100).map(x => math.cos(x)).sliding(50).
  map(_._sum / 50).take(5).toList
res15: List[Double] = List(-0.005153079196990285, -0.0011160413780774369, 0.003947079736951305,
  0.005381273944717851, 0.0018679497047270743)
```

The **.sortBy** method sorts a sequence according to a sorting key. The argument of **.sortBy** is a *function* that computes the sorting key from a sequence element. In this way, we can sort elements in an arbitrary way:

```
scala> Seq(1, 2, 3).sortBy(x => -x)
res0: Seq[Int] = List(3, 2, 1)

scala> Seq("xx", "z", "yyy").sortBy(word => word)           // Sort alphabetically.
res1: Seq[String] = List(xx, yyy, z)

scala> Seq("xx", "z", "yyy").sortBy(word => word.length) // Sort by word length.
res2: Seq[String] = List(z, xx, yyy)
```

Sorting by the elements themselves, as we have done here with **.sortBy(word => word)**, is only possible if the element's type has a well-defined ordering. For strings, this is the alphabetic ordering, and for integers, the standard arithmetic ordering. For such types, a convenience method **.sorted** is defined, and works equivalently to **.sortBy(x => x)**:

```
scala> Seq("xx", "z", "yyy").sorted
res3: Seq[String] = List(xx, yyy, z)
```

2.1.5 Solved examples: Tuples and collections

Example 2.1.5.1 For a given sequence x_i , compute the sequence of pairs $b_i = (\cos x_i, \sin x_i)$.

Hint: use **.map**, assume **xs:Seq[Double]**.

Solution We need to produce a sequence that has a pair of values corresponding to each element of the original sequence. This transformation is exactly what the **.map** method does. So the code is

```
xs.map { x => (math.cos(x), math.sin(x)) }
```

Example 2.1.5.2 Count how many times $\cos x_i > \sin x_i$ occurs in a sequence x_i .

Hint: use **.count**, assume **xs:Seq[Double]**.

Solution The method **.count** takes a predicate and returns the number of times the predicate was **true** while evaluated on the elements of the sequence:

```
xs.count { x => math.cos(x) > math.sin(x) }
```

We could also reuse the solution of Exercise 2.1.5.1 that computed the cosine and the sine values. The code would then become

```
xs.map { x => (math.cos(x), math.sin(x)) }
  .count { case (cosine, sine) => cosine > sine }
```

Example 2.1.5.3 For given sequences a_i and b_i , compute the sequence of differences $c_i = a_i - b_i$.

Hint: use **.zip**, **.map**, and assume **as** and **bs** are of type **Seq[Double]**.

Solution We can use **.zip** on **as** and **bs**, which gives a sequence of pairs,

```
as.zip(bs) : Seq[(Double, Double)]
```

We then compute the differences $a_i - b_i$ by applying **.map** to this sequence:

```
as.zip(bs).map { case (a, b) => a - b }
```

Example 2.1.5.4 In a given sequence p_i , count how many times $p_i > p_{i+1}$ occurs.

Hint: use `.zip` and `.tail`.

Solution Given `ps:Seq[Double]`, we can compute `ps.tail`. The result is a sequence that is 1 element shorter than `ps`, for example:

```
scala> val ps = Seq(1,2,3,4)
ps: Seq[Int] = List(1, 2, 3, 4)

scala> ps.tail
res0: Seq[Int] = List(2, 3, 4)
```

Taking a `.zip` of the two sequences `ps` and `ps.tail`, we get a sequence of pairs:

```
scala> ps.zip(ps.tail)
res1: Seq[(Int, Int)] = List((1,2), (2,3), (3,4))
```

Note that `ps.tail` is 1 element shorter than `ps`, and the resulting sequence of pairs is also 1 element shorter than `ps`. In other words, it is not necessary to truncate `ps` before computing `ps.zip(ps.tail)`. Now apply the `.count` method:

```
ps.zip(ps.tail).count { case (a, b) => a > b }
```

Example 2.1.5.5 For a given $k > 0$, compute the sequence $c_i = \max(b_{i-k}, \dots, b_{i+k})$.

Solution Applying the `.sliding` method to a list gives a list of nested lists:

```
scala> val bs = List(1,2,3,4,5)
bs: List[Int] = List(1, 2, 3, 4, 5)

scala> bs.sliding(3).toList
res0: List[List[Int]] = List(List(1, 2, 3), List(2, 3, 4), List(3, 4, 5))
```

For each b_i , we need to obtain a list of $2k + 1$ nearby elements $(b_{i-k}, \dots, b_{i+k})$. So we need to use `.sliding(2*k+1)` to obtain a window of the required size. Now we can compute the maximum of each of the nested lists by using the `.map` method on the outer list, with the `.max` method applied to the nested lists. So the argument of the `.map` method must be the function `nested => nested.max`:

```
bs.sliding(2 * k + 1).map(nested => nested.max)
```

In Scala, this code can be written more concisely using the syntax

```
bs.sliding(2 * k + 1).map(_.max)
```

because `_.max` means the nameless function `x => x.max`.

Example 2.1.5.6 Create a 10×10 multiplication table as a dictionary of type `Map[(Int, Int), Int]`. For example, a 3×3 multiplication table would be given by this dictionary,

```
Map( (1, 1) -> 1, (1, 2) -> 2, (1, 3) -> 3, (2, 1) -> 2,
      (2, 2) -> 4, (2, 3) -> 6, (3, 1) -> 3, (3, 2) -> 6, (3, 3) -> 9 )
```

Hint: use `.flatMap` and `.toMap`.

Solution We are required to make a dictionary that maps pairs of integers (x, y) to $x * y$. Begin by creating the list of *keys* for that dictionary, which must be a list of pairs (x, y) of the form `List((1,1), (1,2), ..., (2,1), (2,2), ...)`. We need to iterate over a sequence of values of x ; and for each x , we then need to iterate over another sequence to provide values for y . Try this computation:

```
scala> val s = List(1, 2, 3).map(x => List(1, 2, 3))
s: List[List[Int]] = List(List(1, 2, 3), List(1, 2, 3), List(1, 2, 3))
```

We would like to get `List((1,1), (1,2), (1,3))` etc., and so we use `.map` on the inner list with a nameless function `y => (1, y)` that converts a number into a tuple,

```
scala> List(1, 2, 3).map { y => (1, y) }
res0: List[(Int, Int)] = List((1,1), (1,2), (1,3))
```

The curly braces in `{y => (1, y)}` are only for clarity; we could also use parentheses and write `(y => (1, y))`.

Now, we need to have (x, y) instead of $(1, y)$ in the argument of `.map`, where x iterates over `List(1, 2, 3)` in the outside scope. Using this `.map` operation, we obtain

```
scala> val s = List(1, 2, 3).map(x => List(1, 2, 3).map { y => (x, y) })
s: List[List[(Int, Int)]] = List(List((1,1), (1,2), (1,3)), List((2,1), (2,2), (2,3)), List((3,1), (3,2), (3,3)))
```

This is almost what we need, except that the nested lists need to be concatenated into a single list. This is exactly what `.flatten` does:

```
scala> val s = List(1, 2, 3).map(x => List(1, 2, 3).map { y => (x, y) }).flatten
s: List[(Int, Int)] = List((1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3))
```

It is shorter to write `.flatMap(...)` instead of `.map(...).flatten`:

```
scala> val s = List(1, 2, 3).flatMap(x => List(1, 2, 3).map { y => (x, y) })
s: List[(Int, Int)] = List((1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3))
```

This is the list of keys for the required dictionary. The dictionary needs to map each *pair* of integers (x,y) to $x*y$. To create that dictionary, we will apply `.toMap` to a sequence of pairs $(\text{key}, \text{value})$, which in our case needs to be of the form of a nested tuple $((x,y), x*y)$. To achieve this, we use `.map` with a function that computes the product and creates these nested tuples:

```
scala> val s = List(1, 2, 3).flatMap(x => List(1, 2, 3).map { y => (x, y) }).
  map { case (x, y) => ((x, y), x * y) }
s: List[((Int, Int), Int)] = List(((1,1),1), ((1,2),2), ((1,3),3), ((2,1),2), ((2,2),4), ((2,3),6), ((3,1),3), ((3,2),6), ((3,3),9))
```

We can simplify this code if we notice that we are first mapping each y to a tuple (x,y) , and later map each tuple (x,y) to a nested tuple $((x,y), x*y)$. Instead, the entire computation can be done in the inner `.map` operation:

```
scala> val s = List(1, 2, 3).flatMap(x => List(1, 2, 3).map { y => ((x,y), x*y) })
s: List[((Int, Int), Int)] = List(((1,1),1), ((1,2),2), ((1,3),3), ((2,1),2), ((2,2),4), ((2,3),6), ((3,1),3), ((3,2),6), ((3,3),9))
```

It remains to convert this list of tuples to a dictionary with `.toMap`. Also, for better readability, we can use Scala's pair syntax `key -> value`, which is completely equivalent to writing the tuple $(\text{key}, \text{value})$. The final code is

```
(1 to 10).flatMap(x => (1 to 10).map { y => (x,y) -> x*y }).toMap
```

Example 2.1.5.7 For a given sequence x_i , compute the maximum of all of the numbers $x_i, x_i^2, \cos x_i, \sin x_i$. Hint: use `.flatMap`, `.max`.

Solution We will compute the required value if we take `.max` of a list containing all of the numbers. To do that, first map each element of the list `xs:Seq[Double]` into a sequence of three numbers:

```
scala> val xs = List(0.1, 0.5, 0.9) // An example list of some 'Double' numbers.
xs: List[Double] = List(0.1, 0.5, 0.9)

scala> xs.map { x => Seq(x, x*x, math.cos(x), math.sin(x)) }
res0: List[Seq[Double]] = List(List(0.1, 0.010000000000000002, 0.9950041652780258, 0.09983341664682815), List(0.5, 0.25, 0.8775825618903728, 0.479425538604203), List(0.9, 0.81, 0.6216099682706644, 0.7833269096274834))
```

This list is almost what we need, except we need to `.flatten` it:

```
scala> res0.flatten
res1: List[Double] = List(0.1, 0.010000000000000002, 0.9950041652780258, 0.09983341664682815, 0.5, 0.25, 0.8775825618903728, 0.479425538604203, 0.9, 0.81, 0.6216099682706644, 0.7833269096274834)
```

It remains to take the maximum of the resulting numbers:

```
scala> res1.max
res2: Double = 0.9950041652780258
```

The final code (starting from a given sequence `xs`) is

```
xs.flatMap { x => Seq(x, math.cos(x), math.sin(x)) }.max
```

Example 2.1.5.8 From a dictionary of type `Map[String, String]` mapping names to addresses, and assuming that the addresses do not repeat, compute a dictionary of type `Map[String, String]` mapping the addresses back to names.

Hint: use `.map` and `.toMap`.

Solution Keep in mind that iterating over a dictionary looks like iterating over a list of (key, value) pairs, and use `.map` to reverse each pair:

```
dict.map{ case (name, addr) => (addr, name) }.toMap
```

Example 2.1.5.9 Write the solution of Example 2.1.5.8 as a function with type parameters `Name` and `Addr` instead of the fixed type `String`.

Solution In Scala, the syntax for type parameters in a function definition is

```
def rev[Name, Addr](...) = ...
```

The type of the argument is `Map[Name, Addr]`, while the type of the result is `Map[Addr, Name]`. So we use the type parameters `Name` and `Addr` in the type signature of the function. The final code is

```
def rev[Name, Addr](dict: Map[Name, Addr]): Map[Addr, Name] =
  dict.map { case (name, addr) => (addr, name) }.toMap
```

The body of the function `rev` remains the same as in Example 2.1.5.8; only the type signature changed. This is because the procedure for reversing a dictionary works in the same way for dictionaries of any type. So the body of the function `rev` does not actually need to know the types of the keys and values in the dictionary. For this reason, it was easy for us to change the specific type `String` into type parameters in that function.

When the function `rev` is applied to a dictionary of a specific type, the Scala compiler will automatically set the type parameters `Name` and `Addr` that fit the required types of the dictionary's keys and values. For example, if we apply `rev` to a dictionary of type `Map[Boolean, Seq[String]]`, the type parameters will be set automatically as `Name = Boolean` and `Addr = Seq[String]`:

```
scala> val d = Map(true -> Seq("x", "y"), false -> Seq("z", "t"))
d: Map[Boolean, Seq[String]] = Map(true -> List(x, y), false -> List(z, t))

scala> rev(d)
res0: Map[Seq[String], Boolean] = Map(List(x, y) -> true, List(z, t) -> false)
```

Type parameters can be also set explicitly when using the function `rev`. If the type parameters are chosen incorrectly, the program will not compile:

```
scala> rev[Boolean, Seq[String]](d)
res1: Map[Seq[String], Boolean] = Map(List(x, y) -> true, List(z, t) -> false)

scala> rev[Int, Double](d)
<console>:14: error: type mismatch;
 found   : Map[Boolean,Seq[String]]
 required: Map[Int,Double]
    rev[Int, Double](d)
    ~
```

Example 2.1.5.10* Given a sequence `words:Seq[String]` of words, compute a sequence of type `Seq[(Seq[String], Int)]`, where each inner sequence contains all the words having the same length, paired with the integer value showing that length. So, the input `Seq("the", "food", "is", "good")` should produce the output

```
Seq( (Seq("is"), 2), (Seq("the"), 3), (Seq("food", "good"), 4) )
```

The resulting sequence must be ordered by increasing length of words.

Solution It is clear that we need to begin by grouping the words by length. The library method `.groupBy` takes a function that computes a grouping key from each element of a sequence. In our case, we need to group by word length, which is computed with the method `.length` if applied to a string. So the first step is to write

```
words.groupBy{ word => word.length }
```

or, more concisely, `words.groupBy(_.length)`. The result of this expression is a dictionary that maps each length to the list of words having that length:

```
scala> words.groupBy(_.length)
res0: scala.collection.immutable.Map[Int,Seq[String]] = Map(2 -> List(is), 4 -> List(food, good), 3
-> List(the))
```

This is close to what we need. If we convert this dictionary to a sequence, we will get a list of pairs

```
scala> words.groupBy(_.length).toSeq
res1: Seq[(Int, Seq[String])] = ArrayBuffer((2,List(is)), (4,List(food, good)), (3,List(the)))
```

It remains to swap the length and the list of words and to sort the result by increasing length. We can do this in any order: first sort, then swap; or first swap, then sort. The final code is

```
words
  .groupBy(_.length)
  .toSeq
  .sortBy { case (len, words) => len }
  .map { case (len, words) => (words, len) }
```

This can be written somewhat shorter if we use the code `_. _1` for selecting the first parts from pairs and `.swap` for swapping the two elements of a pair:

```
words.groupBy(_.length).toSeq.sortBy(_._1).map(_._swap)
```

However, the program may now be harder to read and to modify.

2.1.6 Reasoning about type parameters in collections

In Example 2.1.5.10 we have applied a chain of operations to a sequence. Let us add comments showing the type of the intermediate result after each operation:

```
words // Seq[String]
  .groupBy(_.length) // Map[Int, Seq[String]]
  .toSeq // Seq[(Int, Seq[String])]
  .sortBy { case (len, words) => len } // Seq[(Int, Seq[String])]
  .map { case (len, words) => (words, len) } // Seq[(Seq[String], Int)]
```

In computations like this, the Scala compiler verifies at each step that the operations are applied to values of the correct type.

For instance, `.sortBy` is defined for sequences but not for dictionaries, so it would be a type error to apply `.sortBy` to a dictionary without first converting it to a sequence using `.toSeq`. The type of the intermediate result after `.toSeq` is `Seq[(Int, Seq[String])]`, and the `.sortBy` operation is applied to that sequence. So the sequence element matched by `{ case (len, words) => len }` is a tuple `(Int, Seq[String])`, which means that the pattern variables `len` and `words` must have types `Int` and `Seq[String]` respectively. It would be a type error to use the sorting key function `{ case (len, words) => words }`: the sorting key can be an integer `len`, but not a string sequence `words` (because sorting by string sequences is not defined).

If we visualize how the type of the sequence should change at every step, we can more quickly understand how to implement the required task. Begin by writing down the intermediate types that would be needed during the computation:

```
words: Seq[String] // Need to group by word length.
Map[Int, Seq[String]] // Need to sort by word length; can't sort a dictionary!
// Need to convert this dictionary to a sequence:
```

```
Seq[ (Int, Seq[String]) ] // Now sort this! Sorting does not change the type.
// It remains to swap the parts of all tuples in the sequence:
Seq[ (Seq[String], Int) ] // We are done.
```

Having written down these types, we are better assured that the computation can be done correctly. Writing the code becomes straightforward, since we are guided by the already known types of the intermediate results:

```
words.groupBy(_.length).toSeq.sortBy(_._1).map(_._swap)
```

This example illustrates the main benefits of reasoning about types: it gives direct guidance about how to organize the computation, together with a greater assurance in the correctness of the code.

2.1.7 Exercises: Tuples and collections

Exercise 2.1.7.1 Find all pairs i, j within $(0, 1, \dots, 9)$ such that $i + 4 * j > i * j$.

Hint: use `.flatMap` and `.filter`.

Exercise 2.1.7.2 Same task as in Exercise 2.1.7.1, but for i, j, k and the condition $i + 4 * j + 9 * k > i * j * k$.

Exercise 2.1.7.3 Given two sequences $p: \text{Seq}[String]$ and $q: \text{Seq}[Boolean]$ of equal length, compute a $\text{Seq}[String]$ with those elements of p for which the corresponding element of q is `true`.

Hint: use `.zip`, `.map`, `.filter`.

Exercise 2.1.7.4 Convert a $\text{Seq}[Int]$ into a $\text{Seq}[(Int, Boolean)]$ where the `Boolean` value is `true` when the element is followed by a larger value. For example, the input sequence $\text{Seq}(1, 3, 2, 4)$ is to be converted into $\text{Seq}((1, true), (3, false), (2, true), (4, false))$. (The last element, 4, has no following element.)

Exercise 2.1.7.5 Given $p: \text{Seq}[String]$ and $q: \text{Seq}[Int]$ of equal length and assuming that values in q do not repeat, compute a $\text{Map}[Int, String]$ mapping numbers from q to the corresponding strings from p .

Exercise 2.1.7.6 Write the solution of Exercise 2.1.7.5 as a function with type parameters P and Q instead of the fixed types `String` and `Int`. Test it with $P = Boolean$ and $Q = \text{Set}[Int]$.

Exercise 2.1.7.7 Given $p: \text{Seq}[String]$ and $q: \text{Seq}[Int]$ of equal length, compute a $\text{Seq}[String]$ that contains the strings from p ordered according to the corresponding numbers from q . For example, if $p = \text{Seq}("a", "b", "c")$ and $q = \text{Seq}(10, -1, 5)$ then the result must be $\text{Seq}("b", "c", "a")$.

Exercise 2.1.7.8 Write the solution of Exercise 2.1.7.7 as a function with type parameter s instead of the fixed type `String`. The required type signature and a sample test:

```
def reorder[S](p: Seq[S], q: Seq[Int]): Seq[S] = ???

scala> reorder(Seq(6.0, 2.0, 8.0, 4.0), Seq(20, 10, 40, 30))
res0: Seq[Double] = List(2.0, 6.0, 4.0, 8.0)
```

Exercise 2.1.7.9 Given a $\text{Seq}[(String, Int)]$ showing a list of purchased items (where item names may repeat), compute a $\text{Map}[String, Int]$ showing the total counts: e.g. for the input

```
Seq(("apple", 2), ("pear", 3), ("apple", 5))
```

the output must be

```
Map("apple" -> 7, "pear" -> 3)
```

Implement this computation as a function with type parameter s instead of `String`.

Hint: use `.groupBy`, `.map`, `.sum`.

Exercise 2.1.7.10 Given a $\text{Seq}[List[Int]]$, compute a new $\text{Seq}[List[Int]]$ where each inner list contains *three* largest elements from the initial inner list (or fewer than three if the initial inner list is shorter).

Hint: use `.map`, `.sortBy`, `.take`.

Exercise 2.1.7.11 (a) Given two sets `p:Set[Int]` and `q:Set[Int]`, compute a set of type `Set[(Int, Int)]` as the Cartesian product of the sets `p` and `q`; that is, the set of all pairs `(x, y)` where `x` is an element from the set `p` and `y` is an element from the set `q`.

(b) Implement this computation as a function with type parameters `I, J` instead of `Int`. The required type signature and a sample test:

```
def cartesian[I,J](p: Set[I], q: Set[J]): Set[(I, J)] = ???

scala> cartesian(Set("a", "b"), Set(10, 20))
res0: Set[(String, Int)] = Set((a,10), (a,20), (b,10), (b,20))
```

Hint: use `.flatMap` and `.map` on sets.

Exercise 2.1.7.12* Given a `Seq[Map[Person, Amount]]`, showing the amounts various people paid on each day, compute a `Map[Person, Seq[Amount]]`, showing the sequence of payments for each person. Assume that `Person` and `Amount` are type parameters. The required type signature and a sample test:

```
def payments[Person, Amount](data: Seq[Map[Person, Amount]]): Map[Person, Seq[Amount]] = ???

scala> payments(Seq(Map("Tarski" -> 10, "Church" -> 20), Map("Church" -> 100, "Gentzen" -> 40),
  Map("Tarski" -> 50)))
res0: Map[String, Seq[Int]] = Map(Gentzen -> List(40), Church -> List(20, 100), Tarski -> List(10, 50))
```

Hint: use `.flatMap`, `.groupBy`, `.mapValues` on dictionaries.

2.2 Converting a sequence into a single value

Until this point, we have been working with sequences using methods such as `.map` and `.zip`. These techniques are powerful but still insufficient for solving certain problems.

A simple computation that is impossible to do using `.map` is obtaining the sum of a sequence of numbers. The standard library method `.sum` already does this; but we cannot re-implement `.sum` ourselves by using `.map`, `.zip`, or `.filter`. These operations always compute *new sequences*, while we need to compute a single value (the sum of all elements) from a sequence.

We have seen a few library methods such as `.count`, `.length`, and `.max` that compute a single value from a sequence; but we still cannot implement `.sum` using these methods. What we need is a more general way of converting a sequence to a single value, such that we could ourselves implement `.sum`, `.count`, `.max`, and other similar computations.

Another task not solvable with `.map`, `.sum`, etc., is to compute a floating-point number from a given sequence of decimal digits (including a “dot” character):

```
def digitsToDouble(ds: Seq[Char]): Double = ???

scala> digitsToDouble(Seq('2', '0', '4', '.', '5'))
res0: Double = 204.5
```

Why is it impossible to implement this function using `.map`, `.sum`, and other methods we have seen so far? In fact, the same task for *integer* numbers (instead of floating-point numbers) can be implemented via `.length`, `.map`, `.sum`, and `.zip`:

```
def digitsToInt(ds: Seq[Int]): Int = {
  val n = ds.length
  // Compute a sequence of powers of 10, e.g. [1000, 100, 100, 1].
  val powers: Seq[Int] = (0 to n - 1).map(k => math.pow(10, n - 1 - k).toInt)
  // Sum the powers of 10 with coefficients from 'ds'.
  (ds zip powers).map { case (d, p) => d * p }.sum
}

scala> digitsToInt(Seq(2,4,0,5))
res0: Int = 2405
```


This task is doable because the required computation can be written as the formula

$$r = \sum_{k=0}^{n-1} d_k * 10^{n-1-k} \quad .$$

The sequence of powers of 10 can be computed separately and “zipped” with the sequence of digits d_k . However, for floating-point numbers, the sequence of powers of 10 depends on the position of the “dot” character. Methods such as `.map` or `.zip` cannot compute a sequence whose next elements depend on previous elements, and the dependence is described by some custom function.

2.2.1 Inductive definitions of aggregation functions

Mathematical induction is a general way of expressing the dependence of next values on previously computed values. To define a function from sequence to a single value (e.g. an aggregation function $f: \text{Seq}[\text{Int}] \Rightarrow \text{Int}$) via mathematical induction, we need to specify two computations:

- (The **base case** of the induction.) We need to specify what value the function f returns for an empty sequence, `Seq()`. If the function f is only defined for non-empty sequences, we need to specify what the function f returns for a one-element sequence such as `Seq(x)`, with any x .
- (The **inductive step**.) Assuming that the function f is already computed for some sequence xs (the **inductive assumption**), how to compute the function f for a sequence with one more element x ? The sequence with one more element is written as `xs ++ Seq(x)`. So, we need to specify how to compute $f(xs ++ Seq(x))$ assuming that $f(xs)$ is already known.

Once these two computations are specified, the function f is defined (and can in principle be computed) for an arbitrary input sequence. This is how induction works in mathematics, and it works in the same way in functional programming. With this approach, the inductive definition of the method `.sum` looks like this:

- The sum of an empty sequence is 0. That is, `Seq().sum == 0`.
- If the result is already known for a sequence `xs`, and we have a sequence that has one more element x , the new result is equal to `xs.sum + x`. In code, this is `(xs ++ Seq(x)).sum == xs.sum + x`.

The inductive definition of the function `digitsToInt` is:

- For an empty sequence of digits, `Seq()`, the result is 0. This is a convenient base case, even if we never call `digitsToInt` on an empty sequence.
- If `digitsToInt(xs)` is already known for a sequence `xs` of digits, and we have a sequence `xs ++ Seq(x)` with one more digit x , then

```
digitsToInt(xs ++ Seq(x)) = digitsToInt(xs) * 10 + x
```

Let us write inductive definitions for methods such as `.length`, `.max`, and `.count`:

- The length of a sequence:
 - for an empty sequence, `Seq().length == 0`
 - if `xs.length` is known then `(xs ++ Seq(x)).length == xs.length + 1`
- Maximum element of a sequence (undefined for empty sequences):
 - for a one-element sequence, `Seq(x).max == x`
 - if `xs.max` is known then `(xs ++ Seq(x)).max == math.max(xs.max, x)`
- Count the sequence elements satisfying a predicate p :

- for an empty sequence, `Seq().count(p) == 0`
- if `xs.count(p)` is known then `(xs ++ Seq(x)).count(p) == xs.count(p) + c`, where we set `c = 1` when `p(x) == true` and `c = 0` otherwise

There are two main ways of translating mathematical induction into code. The first way is to write a recursive function. The second way is to use a standard library function, such as `foldLeft` or `reduce`. Most often it is better to use the standard library functions, but sometimes the code is more transparent when using explicit recursion. So let us consider each of these ways in turn.

2.2.2 Implementing functions by recursion

A **recursive function** is any function that calls itself somewhere within its own body. The call to itself is the **recursive call**.

When the body of a recursive function is evaluated, it may repeatedly call itself with different arguments until the result value can be computed *without* any recursive calls. The last recursive call corresponds to the base case of the induction. It is an error if the base case is never reached, as in this example:

```
scala> def infiniteLoop(x: Int): Int = infiniteLoop(x+1)
infiniteLoop: (x: Int)Int

scala> infiniteLoop(2) // You will need to press Ctrl-C to stop this.
```

We translate mathematical induction into code by first writing a condition to decide whether we have the base case or the inductive step. As an example, let us define `.sum` by recursion. The base case returns 0, and the inductive step returns a value computed from the recursive call:

```
def sum(s: Seq[Int]): Int = if (s == Seq()) 0 else {
  val x = s.head // To split s = Seq(x) ++ xs, compute x
  val xs = s.tail // and xs.
  sum(prev) + next // Call sum(...) recursively.
}
```

In this example, we use the `if/else` expression to separate the base case from the inductive step. In the inductive step, we split the

given sequence `s` into a single-element sequence `Seq(x)`, the “head” of `s`, and the remainder (“tail”) sequence `xs`. So, we split `s` as `s = Seq(x) ++ xs` rather than as `s = xs ++ Seq(x)`.

For computing the sum of a numerical sequence, the order of summation does not matter. However, the order of operations *will* matter for many other computational tasks. We need to choose whether the inductive step should split the sequence as `s = Seq(x) ++ xs` or as `s = xs ++ Seq(x)`, according to the task at hand.

Consider the implementation of `digitsToInt` according to the inductive definition shown in the previous subsection:

```
def digitsToInt(s: Seq[Int]): Int = if (s == Seq()) 0 else {
  val x = s.last // To split s = xs ++ Seq(x), compute x
  val xs = s.take(s.length - 1) // and xs.
  digitsToInt(xs) * 10 + x // Call digitstoInt(...) recursively.
}
```

In this example, it is important to split the sequence into `s = xs ++ Seq(x)` in this order, and not in the order `Seq(x) ++ xs`. The reason is

that digits increase their numerical value from right to left, so we need to multiply the value of the *left* subsequence, `digitsToInt(xs)`, by 10, in order to compute the correct result.

These examples show how mathematical induction is converted into recursive code. This approach often works but has two technical problems. The first problem is that the code will fail due to the “stack overflow” when the input sequence `s` is long enough. In the next subsection, we will see how this problem is solved (at least in some cases) using “tail recursion”.

The second problem is that each inductively defined function repeats the code for checking the base case and the code for splitting the sequence `s` into the subsequence `xs` and the extra element `x`. This repeated common code can be put into a library function, and the Scala library provides such functions. We will look at using them in Section 2.2.4.

2.2.3 Tail recursion

The code of `lengthS` will fail for large enough sequences. To see why, consider an inductive definition of the `.length` method as a function `lengthS`:

```
def lengthS(s: Seq[Int]): Int =
  if (s == Seq()) 0
  else 1 + lengthS(s.tail)

scala> lengthS((1 to 1000).toList)
res0: Int = 1000

scala> val s = (1 to 100000).toList
s: List[Int] = List(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22,
  23, 24, 25, 26, 27, 28, 29, 30, 31, 32, ...)

scala> lengthS(s)
java.lang.StackOverflowError
  at .lengthS(<console>:12)
  at .lengthS(<console>:12)
  at .lengthS(<console>:12)
  at .lengthS(<console>:12)
  ...
```

The problem is not due to insufficient main memory: we *are* able to compute and hold in memory the entire sequence `s`. The problem is with the code of the function `lengthS`. This function calls itself *inside* an expression `1 + lengthS(...)`. So we can visualize how the computer evaluates this code:

```
lengthS(Seq(1, 2, ..., 100000))
= 1 + lengthS(Seq(2, ..., 100000))
= 1 + (1 + lengthS(Seq(3, ..., 100000)))
= ...
```

The function body of `lengthS` will evaluate the inductive step, that is, the “`else`” part of the “`if/else`”, about 100000 times. Each time, the sub-expression with nested computations `1+(1+(...))` will get larger.

This intermediate sub-expression needs to be held somewhere in memory, until at some point the function body goes into the base case and returns a value. When that happens, the entire intermediate sub-expression will contain about 100000 nested function calls still waiting to be evaluated. This sub-expression is held in a special area of memory called **stack memory**, where the not-yet-evaluated nested function calls are held in the order of their calls, as if on a “stack”. Due to the way computer memory is managed, the stack memory has a fixed size and cannot grow automatically. So, when the intermediate expression becomes large enough, it causes an overflow of the stack memory and crashes the program.

A way to avoid stack overflows is to use a trick called **tail recursion**. Using tail recursion means rewriting the code so that all recursive calls occur at the end positions (at the “tails”) of the function body. In other words, each recursive call must be *itself* the last computation in the function body, rather than placed inside other computations. Here is an example of tail-recursive code:

```
def lengthT(s: Seq[Int], res: Int): Int =
  if (s == Seq())
    res
  else
    lengthT(s.tail, 1 + res)
```

In this code, one of the branches of the `if/else` returns a fixed value without doing any recursive calls, while the other branch returns the result of recursive call to `lengthT(...)`. In the code of `lengthT`, recursive calls never occur within any sub-expressions.

It is not a problem that the recursive call to `lengthT` has some sub-expressions such as `1+res` as its arguments, because all these sub-expressions will be computed *before* `lengthT` is recursively called. The recursive call to `lengthT` is the *last* computation performed by this branch of the `if/else`. A tail-recursive function can have many `if/else` or `match/case` branches, with or without recursive calls; but all recursive calls must be always the last expressions returned.

The Scala compiler has a feature for checking automatically that a function’s code is tail-recursive: the `@tailrec` annotation. If a function with a `@tailrec` annotation is not tail-recursive, or is not recursive at all, the program will not compile.

```
@tailrec def lengthT(s: Seq[Int], res: Int): Int =
```

```
if (s == Seq()) res
else lengthT(s.tail, 1 + res)
```

Let us trace the evaluation of this function on an example:

```
lengthT(Seq(1,2,3), 0)
= lengthT(Seq(2, 3), 1 + 0) // = lengthT(Seq(2, 3), 1)
= lengthT(Seq(3), 1 + 1)    // = lengthT(Seq(3), 2)
= lengthT(Seq(), 1 + 2)     // = lengthT(Seq(), 3)
= 3
```

All sub-expressions such as $1 + 1$ and $1 + 2$ are computed *before* recursive calls to `lengthT`. Because of that, sub-expressions do not grow within the stack memory. This is the main benefit of tail recursion.

How did we rewrite the code of `lengths` to obtain the tail-recursive code of `lengthT`? An important difference between `lengths` and `lengthT` is the additional argument, `res`, called the **accumulator argument**. This argument is equal to an intermediate result of the computation. The next intermediate result ($1 + \text{res}$) is computed and passed on to the next recursive call via the accumulator argument. In the base case of the recursion, the function now returns the accumulated result, `res`, rather than `0`, because at that time the computation is finished.

Rewriting code by adding an accumulator argument to achieve tail recursion is called the **accumulator technique** or the “accumulator trick”.

One consequence of using the accumulator trick is that the function `lengthT` now always needs a value for the accumulator argument. However, our goal is to implement a function such as `length(s)` with just one argument, `s: Seq[Int]`. We can define `length(s) = lengthT(s, ???)` if we supply an initial accumulator value. The correct initial value for the accumulator is `0`, since in the base case (an empty sequence `s`) we need to return `0`.

So, a tail-recursive implementation of `lengthT` requires us to define *two* functions: the tail-recursive `lengthT` and an “adapter” function that will set the initial value of the accumulator argument. To emphasize that `lengthT` is a helper function, one could define it *inside* the adapter function:

```
def length[A](s: Seq[A]): Int = {
  @tailrec def lengthT(s: Seq[A], res: Int): Int = {
    if (s == Seq()) res
    else lengthT(s.tail, 1 + res)
  }
  lengthT(s, 0)
}
```

When `length` is implemented like that, users will not be able to call `lengthT` directly, because it is only visible within the body of the `length` function.

Another possibility in Scala is to use a **default value** for the `res` argument:

```
@tailrec def length[A](s: Seq[A], res: Int = 0): Int =
  if (s == Seq()) res
  else length(s.tail, 1 + res)
```

Giving a default value for a function argument is the same as defining *two* functions: one with that argument and one without. For example, the syntax

```
def f(x: Int, y: Boolean = false): Int = ... // Function body.
```

is equivalent to defining two functions (with the same name),

```
def f(x: Int, y: Boolean) = ... // Function body.
def f(x: Int): Int = f(x, false)
```

Using a default argument value, we can define the tail-recursive helper function and the adapter function at once, making the code shorter.

The accumulator trick works in a large number of cases, but it may be far from obvious how to introduce the accumulator argument, what its initial value must be, and how to define the induction step for the accumulator. In the example with the `lengthT` function, the accumulator trick works because of the following mathematical property of the expression being computed:

$$1 + (1 + (1 + (\dots + 1))) = (((1 + 1) + 1) + \dots) + 1 \quad .$$

This property is called the **associativity law** of addition. Due to this law, the computation can be rearranged so that additions associate to the left. In code, it means that intermediate expressions are computed immediately before making the recursive calls; this avoids the growth of the intermediate expressions.

Usually, the accumulator trick works because some associativity law is present. In that case, we are able to rearrange the order of recursive calls so that these calls always occur outside all other sub-expressions, – that is, in tail positions. However, not all computations obey a suitable associativity law. Even if a code rearrangement exists, it may not be immediately obvious how to find it.

As an example, consider a tail-recursive re-implementation of the function `digitsToInt` from the previous subsection, where the recursive call is within a sub-expression `digitsToInt(xs) * 10 + x`. To transform the code into a tail-recursive form, we need to rearrange the main computation,

$$r = d_{n-1} + 10 * (d_{n-2} + 10 * (d_{n-3} + 10 * (...d_0))) \quad ,$$

so that the operations group to the left. We can do this by rewriting r as

$$r = ((d_0 * 10 + d_1) * 10 + ...) * 10 + d_{n-1} \quad .$$

It follows that the digit sequence s must be split into the *leftmost* digit and the rest, $s = s.head ++ s.tail$. So, a tail-recursive implementation of the above formula is

```
@tailrec def fromDigits(s: Seq[Int], res: Int = 0): Int =
  // 'res' is the accumulator.
  if (s == Seq()) res
  else fromDigits(s.tail, 10 * res + s.head)
```

Despite a certain similarity between this code and the code of `digitsToInt` from the previous subsection, the implementation `fromDigits` cannot be directly derived from the inductive definition of `digitsToInt`.

One needs a separate proof that `fromDigits(s, 0)` computes the same result as `digitsToInt(s)`. The proof follows from the following property.

Statement 2.2.3.1 For any $xs: Seq[Int]$ and $r: Int$, we have

$$\text{fromDigits}(xs, r) = \text{digitsToInt}(xs) + r * \text{math.pow}(10, s.length)$$

Proof We prove this by induction. To shorten the proof, denote sequences by $[1, 2, 3]$ instead of `Seq(1, 2, 3)` and temporarily write $d(s)$ instead of `digitsToInt(s)` and $f(s, r)$ instead of `fromDigitsT(s, r)`. Then an inductive definition of $f(s, r)$ is

$$f([], r) = r \quad , \quad f([x] ++ s, r) = f(s, 10 * r + x) \quad . \quad (2.1)$$

Denoting the length of a sequence s by $|s|$, we reformulate Statement 2.2.3.1 as

$$f(s, r) = d(s) + r * 10^{|s|} \quad , \quad (2.2)$$

We prove Eq. (2.2) by induction. To prove the base case $s = []$, we have $f([], r) = r$ and $d([]) + r * 10^0 = r$ since $d([]) = 0$ and $|s| = 0$. The resulting equality $r = r$ proves the base case.

To prove the inductive step, we assume that Eq. (2.2) holds for a given sequence s ; then we need to prove that

$$f([x] ++ s, r) = d([x] ++ s) + r * 10^{|s|+1} \quad . \quad (2.3)$$

We will transform the left-hand side and the right-hand side separately, hoping to obtain the same expression. The left-hand side of Eq. (2.3):

$$\begin{aligned} & f([x] ++ s, r) \\ \text{use Eq. (2.1):} & \quad = f(s, 10 * r + x) \\ \text{use Eq. (2.2):} & \quad = d(s) + (10 * r + x) * 10^{|s|} \quad . \end{aligned}$$

The right-hand side of Eq. (2.3) contains $d([x] ++ s)$, which we somehow need to simplify. Assuming that $d(s)$ correctly calculates a number from its digits, we use a property of decimal notation: a digit x in front of n other digits has the value $x * 10^n$. This property can be formulated as an equation,

$$d([x] ++ s) = x * 10^{|s|} + d(s) \quad . \quad (2.4)$$

So, the right-hand side of Eq. (2.3) can be rewritten as

$$\begin{aligned}
 & d([x]++s) + r * 10^{|s|+1} \\
 \text{use Eq. (2.4) : } & = x * 10^{|s|} + d(s) + r * 10^{|s|+1} \\
 \text{factor out } 10^{|s|} : & = d(s) + (10 * r + x) * 10^{|s|} .
 \end{aligned}$$

We have successfully transformed both sides of Eq. (2.3) to the same expression.

We have not yet proved that the function d satisfies the property in Eq. (2.4). The proof uses induction and begins by writing the code of d in a short notation,

$$d([]) = 0 \quad , \quad d(s++[y]) = d(s) * 10 + y \quad . \quad (2.5)$$

The base case is Eq. (2.4) with $s = []$. It is proved by

$$x = d([]++[x]) = d([x]++) = x * 10^0 + d([]) = x \quad .$$

The induction step assumes Eq. (2.4) for a given x and a given sequence s , and needs to prove that for any y , the same property holds with $s++[y]$ instead of s :

$$d([x]++s++[y]) = x * 10^{|s|+1} + d(s++[y]) \quad . \quad (2.6)$$

The left-hand side of Eq. (2.6) is transformed into its right-hand side like this:

$$\begin{aligned}
 & d([x]++s++[y]) \\
 \text{use Eq. (2.5) : } & = d([x]++s) * 10 + y \\
 \text{use Eq. (2.4) : } & = (x * 10^{|s|} + d(s)) * 10 + y \\
 \text{expand parentheses : } & = x * 10^{|s|+1} + d(s) * 10 + y \\
 \text{use Eq. (2.5) : } & = x * 10^{|s|+1} + d(s++[y]) \quad .
 \end{aligned}$$

This demonstrates Eq. (2.6) and so concludes the proof.

2.2.4 Implementing general aggregation (foldLeft)

An **aggregation** converts a sequence of values into a single value. In general, the type of the result value may be different from the type of elements in the sequence. To describe this general situation, we introduce type parameters, A and B , so that the input sequence is of type $\text{Seq}[A]$ and the aggregated value is of type B . Then an inductive definition of any aggregation function $f: \text{Seq}[A] \Rightarrow B$ looks like this:

- (Base case.) For an empty sequence, $f(\text{Seq}()) = b_0$ where $b_0: B$ is a given value.
- (Induction step.) Assuming that $f(xs) = b$ is already computed, we define $f(xs ++ \text{Seq}(x)) = g(x, b)$ where g is a given function with type signature $g: (A, B) \Rightarrow B$.

The code implementing f is written using recursion:

```
def f[A, B](s: Seq[A]): B =
  if (s == Seq()) b0
  else g(s.last, f(s.take(s.length - 1)))
```

We can now refactor this code into a generic utility function, by making b_0 and g into parameters. A possible implementation is

```
def f[A, B](s: Seq[A], b: B, g: (A, B) => B): B =
  if (s == Seq()) b
  else g(s.last, f(s.take(s.length - 1), b, g))
```

expression will grow with the length of `s`, which is not acceptable. To rearrange the computation into a tail-recursive form, we need to start the base case at the innermost call `g(x, b)`, then compute `g(y, g(x, b))` and continue. In other words, we need to traverse the sequence starting from its *leftmost* element `x`, rather than starting from the right. So, instead of splitting the sequence `s` into `s.last ++ s.take(s.length - 1)` as we did in the code of `f`, we need to split `s` into `s.head ++ s.tail`. Let us also exchange the order of the arguments of `g`, in order to be more consistent with the way this code is implemented in the Scala library. The resulting code is tail-recursive:

```
@tailrec def leftFold[A, B](s: Seq[A], b: B, g: (B, A) => B): B =
  if (s == Seq()) b
  else leftFold(s.tail, g(b, s.head), g)
```

We call this function a “left fold” because it aggregates (or “folds”) the sequence starting from the leftmost element.

In this way, we have defined a general method of computing any inductively defined aggregation function on a sequence. The function `leftFold` implements the logic of aggregation defined via mathematical induction. Using `leftFold`, we can write concise implementations of methods such as `.sum`, `.max`, and many other aggregation functions. The method `leftFold` already contains all the code necessary to set up the base case and the induction step. The programmer just needs to specify the expressions for the initial value `b` and for the updater function `g`.

As a first example, let us use `leftFold` for implementing the `.sum` method:

```
def sum(s: Seq[Int]): Int = leftFold(s, 0, { (x, y) => x + y })
```

To understand in detail how `leftFold` works, let us trace the evaluation of this function when applied to `Seq(1, 2, 3)`:

```
sum(Seq(1, 2, 3)) == leftFold(Seq(1, 2, 3), 0, g)
                  // Here, g = { (x, y) => x + y }, so g(x, y) = x + y.
                  == leftFold(Seq(2, 3), g(0, 1), g)                // g(0, 1) = 1.
                  == leftFold(Seq(2, 3), 1, g)                    // Now expand the code of 'leftFold'.
                  == leftFold(Seq(3), g(1, 2), g)                 // g(1, 2) = 3; expand the code.
                  == leftFold(Seq(), g(3, 3), g)                  // g(3, 3) = 6; expand the code.
                  == 6
```

The second argument of `leftFold` is the accumulator argument. The initial value of the accumulator is specified when first calling `leftFold`. At each iteration, the new accumulator value is computed by calling the updater function `g`, which uses the previous accumulator value and the value of the next sequence element. To visualize the process of recursive evaluation, it is convenient to write a table showing the sequence elements and the accumulator values as they are updated:

Current element <code>x</code>	Old accumulator value	New accumulator value
1	0	1
2	1	3
3	3	6

We implemented `leftFold` only as an illustration. Scala’s library has a method called `.foldLeft` implementing the same logic using a slightly different type signature. To see this difference, compare the implementation of `sum` using our `leftFold` function and using the standard `.foldLeft` method:

```
def sum(s: Seq[Int]): Int = leftFold(s, 0, { (x, y) => x + y })

def sum(s: Seq[Int]): Int = s.foldLeft(0) { (x, y) => x + y }
```

The syntax of `.foldLeft` makes it more convenient to use a nameless function as the updater argument of `.foldLeft`, since curly braces separate that argument from others. We will use the standard `.foldLeft` method from now on.

In general, the type of the accumulator value can be different from the type of the sequence elements. An example is an implementation of `count`:

```
def count[A](s: Seq[A], p: A => Boolean): Int =
  s.foldLeft(0) { (x, y) => x + (if (p(y)) 1 else 0) }
```

The accumulator is of type `Int`, while the sequence elements can have an arbitrary type, parameterized by `A`. The `.foldLeft` method works in the same way for all types of accu-

mulators and all types of sequence elements.

The method `.foldLeft` is available in the Scala library for all collections, including dictionaries and sets. Since `.foldLeft` is tail-recursive, no stack overflows will occur even for very large sequences.

The Scala library contains several other methods similar to `.foldLeft`, such as `.foldRight` and `.reduce`. (However, `.foldRight` is not tail-recursive!)

2.2.5 Solved examples: using `foldLeft`

It is important to gain experience using the `.foldLeft` method.

Example 2.2.5.1 Use `.foldLeft` for implementing the `max` function for integer sequences. Return the special value `Int.MinValue` for empty sequences.

Solution Write an inductive formulation of the `max` function:

- (Base case.) For an empty sequence, return `Int.MinValue`.
- (Inductive step.) If `max` is already computed on a sequence `xs`, say `max(xs) = b`, the value of `max` on a sequence `xs ++ Seq(x)` is `math.max(b, x)`.

Now we can write the code:

```
def max(s: Seq[Int]): Int = s.foldLeft(Int.MinValue) { (b, x) => math.max(b, x) }
```

If we are sure that the function will never be called on empty sequences, we can implement `max` in a simpler way by using the `.reduce` method:

```
def max(s: Seq[Int]): Int = s.reduce { (x, y) => math.max(x, y) }
```

Example 2.2.5.2 Implement the `count` method on sequences of type `Seq[A]`.

Solution Using the inductive definition of the function `count` as shown in Section 2.2.1, we can write the code as

```
def count[A](s: Seq[A], p: A => Boolean): Int =
  s.foldLeft(0) { (b, x) => b + (if (p(x)) 1 else 0) }
```

Example 2.2.5.3 Implement the function `digitsToInt` using `.foldLeft`.

Solution The inductive definition of `digitsToInt` is directly translated into code:

```
def digitsToInt(d: Seq[Int]): Int = d.foldLeft(0) { (n, x) => n * 10 + x }
```

Example 2.2.5.4 For a given non-empty sequence `xs: Seq[Double]`, compute the minimum, the maximum, and the mean as a tuple $(x_{\min}, x_{\max}, x_{\text{mean}})$. The sequence should be traversed only once, i.e. the code must be `xs.foldLeft(...)`, using `.foldLeft` only once.

Solution Without the requirement of using a single traversal, we would write

```
(xs.min, xs.max, xs.sum / xs.length)
```

However, this code traverses `xs` at least three times, since each of the aggregations `xs.min`, `xs.max`, and `xs.sum` iterates over `xs`. We need to combine the four inductive definitions of `min`, `max`, `sum`, and `length` into a single inductive definition of some function. What is the type of that function's return value?

We need to accumulate intermediate values of *all four* numbers (min, max, sum, and length) in a tuple. So the required type of the accumulator is (Double, Double, Double, Double). To avoid repeating a long type expression, we can define a type alias for it, say, D4:

```
scala> type D4 = (Double, Double, Double, Double)
defined type alias D4
```

The updater function must update each of the four numbers according to the definitions of their inductive steps:

```
def update(p: D4, x: Double): D4 = p match {
  case (min, max, sum, length) =>
    (math.min(x, min), math.max(x, max), x + sum, length + 1)
}
```

Now we can write the code of the required function:

```
def f(xs: Seq[Double]): (Double, Double, Double) = {
  val init: D4 = (Double.PositiveInfinity, Double.NegativeInfinity, 0, 0)
  val (min, max, sum, length) = xs.foldLeft(init)(update)
  (min, max, sum/length)
}

scala> f(Seq(1.0, 1.5, 2.0, 2.5, 3.0))
res0: (Double, Double, Double) = (1.0,3.0,2.0)
```

Example 2.2.5.5* Implement the function `digitsToDouble` using `.foldLeft`. The argument is of type `Seq[Char]`. As a test, the expression `digitsToDouble(Seq('3','4','.', '2','5'))` must evaluate to 34.25. Assume that all input characters are either digits or a dot (so, negative numbers are not supported).

Solution The evaluation of a `.foldLeft` on a sequence of digits will visit the sequence from left to right. The updating function should work the same as in `digitsToInt` until a dot character is found. After that, we need to change the updating function. So, we need to remember whether a dot character has been seen. The only way for `.foldLeft` to “remember” any data is to hold that data in the accumulator value. We can choose the type of the accumulator according to our needs. So, for this task we can choose the accumulator to be a *tuple* that contains, for instance, the floating-point result constructed so far and a `Boolean` flag showing whether we have already seen the dot character.

To see what `digitsToDouble` must do, let us consider how the evaluation of `digitsToDouble(Seq('3', '4', '.', '2', '5'))` should go. We can write a table showing the intermediate result at each iteration. This will hopefully help us figure out what the accumulator and the updater function $g(\dots)$ must be:

Current digit c	Previous result n	New result $n' = g(n, c)$
'3'	0.0	3.0
'4'	3.0	34.0
','	34.0	34.0
'2'	34.0	34.2
'5'	34.2	34.25

While the dot character was not yet seen, the updater function multiplies the previous result by 10 and adds the current digit. After the dot character, the updater function must add to the previous result the current digit divided by a factor that represents increasing powers of 10. In other words, the update computation $n' = g(n, c)$ must be defined by these formulas:

- Before the dot character: $g(n, c) = n * 10 + c$.
- After the dot character: $g(n, c) = n + \frac{c}{f}$, where f is 10, 100, 1000, ..., for each new digit.

The updater function g has only two arguments: the current digit and the previous accumulator value. So, the changing factor f must be *part of* the accumulator value, and must be multiplied by

10 at each digit after the dot. If the factor f is not a part of the accumulator value, the function g will not have enough information for computing the next accumulator value correctly. So, the updater computation must be $n' = g(n, c, f)$, not $n' = g(n, c)$.

For this reason, we choose the accumulator type as a tuple (Double, Boolean, Double) where the first number is the result n computed so far, the Boolean flag indicates whether the dot was already seen, and the third number is f , that is, the power of 10 by which the current digit will be divided if the dot was already seen. Initially, the accumulator tuple will be equal to (0.0, false, 10.0). Then the updater function is implemented like this:

```
def update(acc: (Double, Boolean, Double), c: Char): (Double, Boolean, Double) =
  acc match { case (n, flag, factor) =>
    if (c == '.') (n, true, factor) // Set flag to 'true' after a dot character was seen.
    else {
      val digit = c - '0'
      if (flag) // This digit is after the dot. Update 'factor'.
        (n + digit/factor, flag, factor * 10)
      else // This digit is before the dot.
        (n * 10 + digit, flag, factor)
    }
  }
```

Now we can implement `digitsToDouble` as follows,

```
def digitsToDouble(d: Seq[Char]): Double = {
  val initAccumulator = (0.0, false, 10.0)
  val (n, _, _) =
    d.foldLeft(initAccumulator)(update)
  n
}

scala> digitsToDouble(Seq('3', '4', '.', '2', '5'))
res0: Double = 34.25
```

The result of calling `d.foldLeft` is a tuple (n, flag, factor), in which only the first part, `n`, is needed. In Scala's pattern matching expressions, the underscore symbol is used to denote the pattern variables whose values are not needed in the subsequent code. We could extract the first part using the accessor method `._1`, but the code will be more readable if we show all parts of the tuple by writing (n, _, _).

Example 2.2.5.6 Implement the `.map` method for sequences by using `.foldLeft`. The input sequence should be of type `Seq[A]` and the output sequence of type `Seq[B]`, where `A` and `B` are type parameters. The required type signature of the function and a sample test:

```
def map[A, B](xs: Seq[A])(f: A => B): Seq[B] = ???

scala> map(List(1, 2, 3)) { x => x * 10 }
res0: Seq[Int] = List(10, 20, 30)
```

Solution The required code should build a new sequence by applying the function f to each element. How can we build a new sequence using `.foldLeft`? The evaluation of `.foldLeft` consists of iterating over the input sequence and accumulating some result value, which is updated at each iteration. Since the result of a `.foldLeft` is always equal to the last computed accumulator value, it follows that the new sequence should be the accumulator value. So, we need to update the accumulator by appending the value $f(x)$, where x is the current element of the input sequence. We can append elements to sequences using the `:+` operation:

```
def map[A, B](xs: Seq[A])(f: A => B): Seq[B] =
  xs.foldLeft(Seq[B]()) { (acc, x) => acc :+ f(x) }
```

The operation `acc :+ f(x)` is equivalent to `acc ++ Seq(f(x))` but is shorter to write.

Example 2.2.5.7 Implement a function `toPairs` that converts a sequence of type `Seq[A]` to a sequence of pairs, `Seq[(A, A)]`, by putting together the adjacent elements pairwise. If the initial sequence has an odd number of elements, a given default value of type `A` is used to fill the last pair. The required type signature and an example test:

```
def toPairs[A](xs: Seq[A], default: A): Seq[(A, A)] = ???
```

```
scala> toPairs(Seq(1, 2, 3, 4, 5, 6), -1)
res0: Seq[(Int, Int)] = List((1,2), (3,4), (5,6))

scala> toPairs(Seq("a", "b", "c"), "<nothing>")
res1: Seq[(String, String)] = List((a,b), (c,<nothing>))
```

Solution We need to use `.foldLeft` to accumulate a sequence of pairs, and each pair needs two values. However, we iterate over values in the input sequence one by one. So, a new pair can be made only once every two iterations. The accumulator needs to hold the information about the current iteration being even or odd. For odd-numbered iterations, the accumulator also needs to store the previous element that is still waiting for its pair. Therefore, we choose the type of the accumulator to be a tuple `(Seq[(A, A)], Seq[A])`. The first sequence is the intermediate result, and the second sequence is the “remainder”: it holds the previous element for odd-numbered iterations and is empty for even-numbered iterations. Initially, the accumulator should be empty. A trace of the accumulator updates is shown in this table:

Current element x	Previous accumulator	Next accumulator
"a"	(Seq(), Seq())	(Seq(), Seq("a"))
"b"	(Seq(), Seq("a"))	(Seq(("a", "b")), Seq())
"c"	(Seq(("a", "b")), Seq())	(Seq(("a", "b")), Seq("c"))

Now it becomes clear how to implement the updater function.

```
type Acc = (Seq[(A, A)], Seq[A]) // Type alias, for brevity.
def updater(acc: Acc, x: A): Acc = acc match {
  case (result, Seq())    => (result, Seq(x))
  case (result, Seq(prev)) => (result ++ Seq((prev, x)), Seq())
}
```

We will call `.foldLeft` with this updater and then perform some post-processing to make sure we create the last pair in case the last iteration is odd-numbered, i.e. when the “remainder” is not empty after `.foldLeft` is finished. In this implementation, we use pattern matching to decide whether a sequence is empty:

```
def toPairs[A](xs: Seq[A], default: A): Seq[(A, A)] = {
  type Acc = (Seq[(A, A)], Seq[A]) // Type alias, for brevity.
  def init: Acc = (Seq(), Seq())
  def updater(acc: Acc, x: A): Acc = acc match {
    case (result, Seq())    => (result, Seq(x))
    case (result, Seq(prev)) => (result ++ Seq((prev, x)), Seq())
  }
  val (result, remainder) = xs.foldLeft(init)(updater)
  remainder match { // May need to append the last element to the result.
    case Seq()    => result
    case Seq(x)   => result ++ Seq((x, default))
  }
}
```

This code shows examples of partial functions that are applied safely. One of these partial functions is used in the expression

```
remainder match {
  case Seq()    => ...
  case Seq(a)   => ...
}
```

This code works when `remainder` is empty or has length 1 but would fail for longer sequences. Within the implementation of `toPairs`, the value of `remainder` will always be a sequence of length at most 1, so the code is safe.

2.2.6 Exercises: Using `foldLeft`

Exercise 2.2.6.1 Implement a function `fromPairs` that performs the inverse transformation to the `toPairs` function defined in Example 2.2.5.7. The required type signature and a sample test:

```
def fromPairs[A](xs: Seq[(A, A)]): Seq[A] = ???

scala> fromPairs(Seq((1, 2), (3, 4)))
res0: Seq[Int] = List(1, 2, 3, 4)
```

Hint: This can be done with `.foldLeft` or with `.flatMap`.

Exercise 2.2.6.2 Implement the `flatten` method for sequences by using `.foldLeft`. The required type signature and a sample test:

```
def flatten[A](xss: Seq[Seq[A]]): Seq[A] = ???

scala> flatten(Seq(Seq(1, 2, 3), Seq(), Seq(4)))
res0: Seq[Int] = List(1, 2, 3, 4)
```

Exercise 2.2.6.3 Use `.foldLeft` to implement the `zipWithIndex` method for sequences. The required type signature and a sample test:

```
def zipWithIndex[A](xs: Seq[A]): Seq[(A, Int)] = ???

scala> zipWithIndex(Seq("a", "b", "c", "d"))
res0: Seq[String] = List((a, 0), (b, 1), (c, 2), (d, 3))
```

Exercise 2.2.6.4 Use `.foldLeft` to implement a function `filterMap` that combines `.map` and `.filter` for sequences. The required type signature and a sample test:

```
def filterMap[A, B](xs: Seq[A])(pred: A => Boolean)(f: A => B): Seq[B] = ???

scala> filterMap(Seq(1, 2, 3, 4)) { x => x > 2 } { x => x * 10 }
res0: Seq[Int] = List(30, 40)
```

Exercise 2.2.6.5* Split a sequence into subsequences (“batches”) of length not larger than a given maximum length n . The required type signature and a sample test:

```
def byLength[A](xs: Seq[A], length: Int): Seq[Seq[A]] = ???

scala> byLength(Seq("a", "b", "c", "d"), 2)
res0: Seq[Seq[String]] = List(List(a, b), List(c, d))

scala> byLength(Seq(1, 2, 3, 4, 5, 6, 7), 3)
res1: Seq[Seq[Int]] = List(List(1, 2, 3), List(4, 5, 6), List(7))
```

Exercise 2.2.6.6* Split a sequence into batches by “weight” computed via a given function. The total weight of items in any batch should not be larger than a given maximum weight. The required type signature and a sample test:

```
def byWeight[A](xs: Seq[A], maxW: Double)(w: A => Double): Seq[Seq[A]] = ???

scala> byWeight((1 to 10).toList, 5.75){ x => math.sqrt(x) }
res0: Seq[Seq[Int]] = List(List(1, 2, 3), List(4, 5), List(6, 7), List(8), List(9), List(10))
```

Exercise 2.2.6.7* Use `.foldLeft` to implement a `groupBy` function. The type signature and a test:

```
def groupBy[A, K](xs: Seq[A])(by: A => K): Map[K, Seq[A]] = ???

scala> groupBy(Seq(1, 2, 3, 4, 5)){ x => x % 2 }
res0: Map[Int, Seq[Int]] = Map(1 -> List(1, 3, 5), 0 -> List(2, 4))
```

Hints: The accumulator should be of type `Map[K, Seq[A]]`. To work with dictionaries, you will need to use the methods `.getOrElse` and `.updated`. The method `.getOrElse` fetches a value from a dictionary by key, and returns the given default value if the dictionary does not contain that key:

```
scala> Map("a" -> 1, "b" -> 2).getOrElse("a", 300)
```

```
res0: Int = 1

scala> Map("a" -> 1, "b" -> 2).getOrElse("c", 300)
res1: Int = 300
```

The method `.updated` produces a new dictionary that contains a new value for the given key, whether or not that key already exists in the dictionary:

```
scala> Map("a" -> 1, "b" -> 2).updated("c", 300) // Key is new.
res0: Map[String,Int] = Map(a -> 1, b -> 2, c -> 300)

scala> Map("a" -> 1, "b" -> 2).updated("a", 400) // Key already exists.
res1: Map[String,Int] = Map(a -> 400, b -> 2)
```

2.3 Converting a single value into a sequence

An aggregation converts or “folds” a sequence into a single value; the opposite operation (“unfolding”) converts a single value into a sequence. An example of this task is to compute the sequence of decimal digits for a given integer:

```
def digitsOf(x: Int): Seq[Int] = ???

scala> digitsOf(2405)
res0: Seq[Int] = List(2, 4, 0, 5)
```

We cannot implement `digitsOf` using `.map`, `.zip`, or `.foldLeft`, because these methods work only if we *already have* a sequence; but the function `digitsOf` needs to create a new sequence. We could create a sequence via the expression `(1 to n)` if the required length of the sequence were

known in advance. However, the function `digitsOf` must produce a sequence whose length is determined by a condition that we cannot easily evaluate in advance.

A general “unfolding” operation requires us to build a sequence whose length is not determined in advance. This kind of sequence is called a **stream**. A stream is a sequence whose elements are computed only when necessary (unlike sequences such as a `List` or an `Array`, whose elements are all computed in advance and stored). The unfolding operation will keep computing the next element; this creates a stream. We can then apply `.takeWhile` to the stream, in order to stop it when a certain condition holds. Finally, if required, the truncated stream may be converted to a list or to another type of sequence. In this way, we can generate a sequence of initially unknown length according to any given requirements.

The Scala library has a general stream-producing function `Stream.iterate`. This function has two arguments, the initial value and a function that computes the next value from the previous one:

```
scala> Stream.iterate(2) { x => x + 10 }
res0: Stream[Int] = Stream(2, ?)
```

The stream is ready to start computing the next elements of the sequence (so far, only the first element, 2, has been computed). In order to see

the next elements, we need to stop the stream at a finite size and then convert the result to a list:

```
scala> Stream.iterate(2) { x => x + 10 }.take(6).toList
res1: List[Int] = List(2, 12, 22, 32, 42, 52)
```

If we try to evaluate `.toList` on a stream without first limiting its size via `.take` or `.takeWhile`, the program will keep producing more elements of the stream until it runs out of memory and crashes.

Streams are similar to sequences, and methods such as `.map`, `.filter`, `.flatMap` are also defined for streams. For instance, we can use the method `.drop` that skips a given number of initial elements:

```
scala> Seq(10, 20, 30, 40, 50).drop(3)
res2: Seq[Int] = List(40, 50)

scala> Stream.iterate(2) { x => x + 10 }.drop(3)
res3: Stream[Int] = Stream(32, ?)
```

This example shows that in order to evaluate `.drop(3)`, the stream had to compute its elements up to 32 (but the subsequent elements are still not computed).

To figure out the code for `digitsOf`, we first write this function as a mathematical formula. To compute the sequence of digits for, say, $n = 2405$, we need to divide n repeatedly by 10, getting a sequence n_k of intermediate numbers ($n_0 = 2405, n_1 = 240, \dots$) and the corresponding sequence of last digits, $n_k \bmod 10$ (in this example: 5, 0, ...). The sequence n_k is defined using mathematical induction:

- Base case: $n_0 = n$, where n is the given initial integer.
- Inductive step: $n_{k+1} = \left\lfloor \frac{n_k}{10} \right\rfloor$ for $k = 1, 2, \dots$

Here $\left\lfloor \frac{n_k}{10} \right\rfloor$ is the mathematical notation for the integer division by 10. Let us tabulate the evaluation of the sequence n_k for $n = 2405$:

$k =$	0	1	2	3	4	5	6
$n_k =$	2405	240	24	2	0	0	0
$n_k \bmod 10 =$	5	0	4	2	0	0	0

The numbers n_k will remain all zeros after $k = 4$. It is clear that the useful part of the sequence is before it becomes all zeros. In this example, the sequence n_k needs to be stopped at $k = 4$. The sequence of digits then becomes [5,0,4,2], and we need to

reverse it to obtain [2,4,0,5]. For reversing a sequence, the Scala library has the standard method `.reverse`. A complete implementation for `digitsOf` is thus

```
def digitsOf(n: Int): Seq[Int] =
  if (n == 0) Seq(0) else { // n == 0 is a special case.
    Stream.iterate(n) { nk => nk / 10 }
      .takeWhile { nk => nk != 0 }
      .map { nk => nk % 10 }
      .toList.reverse
  }
```

We can shorten the code by using the syntax such as `(_ % 10)` instead of `{ nk => nk % 10 }`,

```
def digitsOf(n: Int): Seq[Int] =
  if (n == 0) Seq(0) else { // n == 0 is a special case.
    Stream.iterate(n)(_ / 10)
      .takeWhile(_ != 0)
      .map(_ % 10)
      .toList.reverse
  }
```

The type signature of the method `Stream.iterate` can be written as

```
def iterate[A](init: A)(next: A => A): Stream[A]
```

and shows a close correspondence to a definition by mathematical induction. The base case is the first value, `init`, and the inductive step is a function, `next`, that computes the next element from the previous one. It is a general way of creating sequences whose length is not determined in advance.

2.4 Transforming a sequence into another sequence

We have seen methods such as `.map` and `.zip` that transform sequences into sequences. However, these methods cannot express a general transformation where the elements of the new sequence are defined by induction and depend on previous elements. An example of this kind is computing the partial sums of a given sequence x_i , say $b_k = \sum_{i=0}^{k-1} x_i$. This formula defines $b_0 = 0$, $b_1 = x_0$, $b_2 = x_0 + x_1$, $b_3 = x_0 + x_1 + x_2$, etc. A definition via mathematical induction may be written like this:

- (Base case.) $b_0 = 0$.
- (Induction step.) Given b_k , we define $b_{k+1} = b_k + x_k$ for $k = 0, 1, 2, \dots$

The Scala library method `.scanLeft` implements a general sequence-to-sequence transformation defined in this way. The code implementing the partial sums is

```
def partialSums(xs: Seq[Int]): Seq[Int] =
  xs.scanLeft(0){ (x, y) => x + y }

scala> partialSums(Seq(1, 2, 3, 4))
res0: Seq[Int] = List(0, 1, 3, 6, 10)
```

element of the first sequence and a previous element of the second sequence, and returns the next element of the second sequence. Note that the result of `.scanLeft` is one element longer than the original sequence, because the base case provides an initial value.

Until now, we have seen that `.foldLeft` is sufficient to re-implement almost every method that work on sequences, such as `.map`, `.filter`, or `.flatten`. Let us show, as an illustration, how to implement the method `.scanLeft` via `.foldLeft`. In the implementation, the accumulator contains the previous element of the second sequence together with a growing fragment of that sequence, which is updated as we iterate over the first sequence. The code is

```
1 def scanLeft[A, B](xs: Seq[A])(b0: B)(next: (B, A) => B)
2   : Seq[B] = {
3   val init: (B, Seq[B]) = (b0, Seq(b0))
4   val (_, result) = xs.foldLeft(init){ case ((b, seq), x) =>
5     val newB = next(b, x)
6     (newB, seq ++ Seq(newB))
7   }
8   result
9 }
```

ond is a value of type `A`. The pattern matching expression `{case ((b, seq), x) => ...}` appears to match a nested tuple. In reality, this expression matches the two arguments of the updater function and, at the same time, destructures the tuple argument as `(b, seq)`.

To implement the (nameless) updater function for `.foldLeft` in lines 4–7, we used the Scala feature that makes it easier to define functions with several arguments containing tuples. In our case, the updater function in `.foldLeft` has two arguments: the first is a tuple `(B, Seq[B])`, the second

2.5 Summary

We have seen a broad overview of translating mathematical induction into Scala code.

What problems can we solve now?

- Compute mathematical expressions involving arbitrary recursion.
- Use the accumulator trick to enforce tail recursion.
- Implement functions with type parameters.
- Use arbitrary inductive (i.e. recursive) formulas to:
 - convert sequences to single values (aggregation or “folding”);
 - create new sequences from single values (“unfolding”);
 - transform existing sequences into new sequences.

Definition by induction	Scala code example
$f([]) = b; f(s++[x]) = g(f(s), x)$	<code>f(xs) = xs.foldLeft(b)(g)</code>
$x_0 = b; x_{k+1} = g(x_k)$	<code>xs = Stream.iterate(b)(g)</code>
$y_0 = b; y_{k+1} = g(y_k, x_k)$	<code>ys = xs.scanLeft(b)(g)</code>

Table 2.1: Implementing mathematical induction.

values: “Starting from this value, we get that sequence, then transform it into this other sequence,” etc. This is a powerful way of working with sequences, dictionaries, and sets. Many kinds of programming errors (such as an incorrect array index) are avoided from the outset, and the code is shorter and easier to read than conventional code written using loops.

Table 2.1 shows Scala code implementing those tasks. Iterative calculations are implemented by translating mathematical induction directly into code. In the functional programming paradigm, the programmer does not need to write any loops or use array indices. Instead, the programmer reasons about sequences as mathematical

What tasks are not possible with these tools? We cannot implement a non-tail-recursive function without stack overflow (i.e. without unlimited growth of intermediate expressions). The accumulator trick does not always work! In some cases, it is impossible to implement tail recursion in a given recursive computation. An example of such a computation is the “merge-sort” algorithm where the function body must contain two recursive calls within a single expression. (It is impossible to rewrite *two* recursive calls as one.)

What if our recursive code cannot be transformed into tail-recursive code via the accumulator trick, but the recursion depth is so large that stack overflows occur? There exist special tricks (e.g. the “continuation-passing” or “trampolines”) that convert non-tail-recursive code into iterative code without stack overflows. Those advanced tricks are beyond the scope of this chapter.

2.5.1 Solved examples

Example 2.5.1.1 Compute the smallest n such that $f(f(f(\dots f(1)\dots)) \geq 1000$, where the function f is applied n times. Write this as a function taking f , 1, and 1000 as arguments. Test with $f(x) = 2x + 1$.

Solution We define a stream of values $[1, f(1), f(f(1)), \dots]$ and use `.takeWhile` to stop the stream when the given condition holds. The `.length` method then gives the length of the resulting sequence:

```
scala> Stream.iterate(1)(x => 2*x+1).takeWhile(x => x < 1000).toList
res0: List[Int] = List(1, 3, 7, 15, 31, 63, 127, 255, 511)

scala> Stream.iterate(1)(x => 2*x+1).takeWhile(x => x < 1000).length
res1: Int = 9
```

Example 2.5.1.2 (a) For a given `Stream[Int]`, compute the stream of the largest values seen so far.

(b) Compute the stream of k largest values seen so far (k is a given integer parameter).

Solution: We cannot use `.max` or sort the entire stream, since the length of the stream is not known in advance. So we need to use `.scanLeft`, which will build the output stream one element at a time.

(a) Maintain the largest value seen so far in the accumulator of the `.scanLeft`:

```
def maxSoFar(xs: Stream[Int]): Stream[Int] =
  xs.scanLeft(xs.head){ case (max, x) => math.max(max, x) }
    .drop(1)
```

We use `.drop(1)` to remove the initial value, `xs.head`, because it is not useful for our result but is always produced by `.scanLeft`.

To test this function, let us define a stream whose values go up and down:

```
val s = Stream.iterate(0)(x => 1 - 2*x)

scala> s.take(10).toList
res0: List[Int] = List(0, 1, -1, 3, -5, 11, -21, 43, -85, 171)

scala> maxSoFar(s).take(10).toList
res1: List[Int] = List(0, 1, 1, 3, 3, 11, 11, 43, 43, 171)
```

(b) We again use `.scanLeft`, where now the accumulator needs to keep the largest k values seen so far. There are two ways of maintaining this accumulator: First, to have a sequence of k values that we sort and truncate each time. Second, to use a specialized data structure such as a priority queue that automatically keeps values sorted and its length bounded. For the purposes of this tutorial, let us avoid using specialized data structures:

```
def maxKSoFar(xs: Stream[Int], k: Int): Stream[Seq[Int]] = {
  // The initial value of the accumulator is an empty Seq() of type Seq[Int].
  xs.scanLeft(Seq[Int]() { case (seq, x) =>
    // Sort in the descending order, and take the first k values.
    (seq :+ x).sorted.reverse.take(k)
  })
  .drop(1) // Skip the useless first value.
}
```



```
scala> maxKSoFar(s, 3).take(10).toList
res2: List[Seq[Int]] = List(List(0), List(1, 0), List(1, 0, -1), List(3, 1, 0), List(3, 1, 0),
    List(11, 3, 1), List(11, 3, 1), List(43, 11, 3), List(43, 11, 3), List(171, 43, 11))
```

Example 2.5.1.3 Find the last element of a non-empty sequence. (Hint: use `.reduce`.)

Solution This function is available in the Scala library as the standard method `.last` on sequences. Here we need to re-implement it using `.reduce`. Begin by writing an inductive definition:

- (Base case.) `last(Seq(x)) == x`.
- (Inductive step.) `last(Seq(x) ++ xs) == last(xs)` assuming `xs` is non-empty.

The `.reduce` method implements an inductive aggregation similarly to `.foldLeft`, except that for `.reduce` the base case is fixed – it always returns `x` for a 1-element sequence `Seq(x)`. This is exactly what we need here, so the inductive definition is directly translated into code, with the updater function $g(x, y) = y$:

```
def last[A](xs: Seq[A]): A = xs.reduce { case (x, y) => y }
```

Example 2.5.1.4 (a) Count the occurrences of each distinct word in a string:

```
def countWords(s: String): Map[String, Int] = ???

scala> countWords("a quick a quick a fox")
res0: Map[String, Int] = Map("a" -> 3, "quick" -> 2, "fox" -> 1)
```

(b) Count the occurrences of each distinct element in a sequence of type `Seq[A]`.

Solution (a) We split the string into an array of words via `s.split(" ")`, and apply a `.foldLeft` to that array, since the computation is a kind of aggregation over the array of words. The accumulator of the aggregation will be the dictionary of word counts for all the words seen so far:

```
def countWords(s: String): Map[String, Int] = {
  val init: Map[String, Int] = Map()
  s.split(" ").foldLeft(init) { (dict, word) =>
    val newCount = dict.getOrElse(word, 0) + 1
    dict.updated(word, newCount)
  }
}
```

An alternative, shorter implementation of the same function is

```
def countWords(s: String): Map[String, Int] = s.split(" ").groupBy(w => w).mapValues(_.length)
```

The `.groupBy` creates a dictionary in one function call rather than one entry at a time. But the resulting dictionary contains word lists instead of word counts, so we use `.mapValues`:

```
scala> "a a b b b c".split(" ").groupBy(w => w)
res0: Map[String,Array[String]] = Map(b -> Array(b, b, b), a -> Array(a, a), c -> Array(c))

scala> res0.mapValues(_.length)
res1: Map[String,Int] = Map(b -> 3, a -> 2, c -> 1)
```

(b) The main code of `countWords` does not depend on the fact that words are of type `String`. It will work in the same way for any other type of keys for the dictionary. So we keep the same code and define the type signature of the function to contain a type parameter `A` instead of `String`:

```
def countValues[A](xs: Seq[A]): Map[A, Int] =
  xs.foldLeft(Map[A, Int]()) { (dict, word) =>
    val newCount = dict.getOrElse(word, 0) + 1
    dict.updated(word, newCount)
  }

scala> countValues(Seq(100, 100, 200, 100, 200, 200, 100))
res0: Map[Int,Int] = Map(100 -> 4, 200 -> 3)
```


Example 2.5.1.5 (a) Implement the binary search algorithm¹ for a sorted sequence `xs: Seq[Int]` as a function returning the index of the requested value `goal` (assume that `xs` always contains `goal`):

```
@tailrec def binSearch(xs: Seq[Int], goal: Int): Int = ???

scala> binSearch(Seq(1, 3, 5, 7), 5)
res0: Int = 2
```

(b) Re-implement `binSearch` using `Stream.iterate` without writing explicitly recursive code.

Solution (a) The well-known binary search algorithm splits the array into two halves and may continue the search recursively in one of the halves. We need to write the solution as a tail-recursive function with an additional accumulator argument. So we expect that the code should look like this,

```
@tailrec def binSearch(xs: Seq[Int], goal: Int, acc: _ = ???): Int = {
  if (???are we done???) acc
  else {
    // Determine which half of the sequence contains 'goal'.
    // Then update the accumulator accordingly.
    val newAcc = ???
    binSearch(xs, goal, newAcc) // Tail-recursive call.
  }
}
```

We now decide the type and the initial value of the accumulator, and implement the updater for it.

The information required for the recursive call must show the segment of the sequence where the target number is present. That segment is defined by two indices i, j representing the left and the right bounds of the sub-sequence, such that the target element is x_n with $x_i \leq x_n < x_{j-1}$. It follows that the accumulator should be a pair of two integers (i, j) . The initial value of the accumulator is the pair $(0, N)$, where N is the length of the entire sequence. The search is finished when $i + 1 = j$. We now write the code, introducing for convenience *two* accumulator values representing (i, j) :

```
@tailrec def binSearch(xs: Seq[Int], goal: Int)(left: Int = 0, right: Int = xs.length): Int = {
  // Check whether 'goal' is at one of the boundaries.
  if (right - left <= 1 || xs(left) == goal) left
  else {
    val middle = (left + right) / 2
    // Determine which half of the array contains 'target'.
    // Update the accumulator accordingly.
    val (newLeft, newRight) =
      if (goal < xs(middle)) (left, middle)
      else (middle, right)
    binSearch(xs, goal)(newLeft, newRight) // Tail-recursive call.
  }
}

scala> binSearch(0 to 10, 3)() // Default accumulator values.
res0: Int = 3
```

Here we used a feature of Scala that allows us to set `xs.length` as a default value for the argument `right` of `binSearch`. This works because `right` is in a different **argument group** from `xs`. Default values in an argument group may depend on arguments in a *previous* argument group. However, the code

```
def binSearch(xs: Seq[Int], goal: Int, left: Int = 0, right: Int = xs.length)
```

will generate an error: the arguments in the same argument group cannot depend on each other. (The error will say `not found: value xs`.)

(b) We can visualize the binary search as a procedure that generates a sequence of progressively tighter bounds for the location of `goal`. The initial bounds are $(0, \text{xs.length})$, and the final bounds are $(k, k+1)$ for some k . We can generate the sequence of bounds using `Stream.iterate` and stop the sequence when the bounds become sufficiently tight. To make the use of `.takeWhile` more convenient, we add an extra sequence element where the bounds (k, k) are equal. The code becomes

¹https://en.wikipedia.org/wiki/Binary_search_algorithm

```
def binSearch(xs: Seq[Int], goal: Int): Int = {
  type Acc = (Int, Int)
  val init: Acc = (0, xs.length)
  val updater: Acc => Acc = { case (left, right) =>
    if (right - left <= 1) (left, left) // Extra element (k, k) in the stream.
    else if (xs(left) == goal) (left, left + 1)
    else {
      val middle = (left + right) / 2
      // Determine which half of the array contains 'target'.
      // Update the accumulator accordingly.
      if (goal < xs(middle)) (left, middle)
      else (middle, right)
    }
  }
  Stream.iterate(init)(updater)
    .takeWhile{ case (left, right) => right > left }
    .last._1 // Take the 'left' boundary from the last element.
}
```

This code is clearer because recursion is delegated to `Stream.iterate` and cleanly separated from the “business logic” (i.e. implementing the base case, the inductive step, and the post-processing).

Example 2.5.1.6 For a given positive $n: \text{Int}$, compute the sequence $[s_0, s_1, s_2, \dots]$ defined by $s_0 = SD(n)$ and $s_k = SD(s_{k-1})$ for $k > 0$, where $SD(x)$ is the sum of the decimal digits of the integer x , e.g. $SD(123) = 6$. Stop the sequence s_i when the numbers begin repeating. For example, $SD(99) = 18$, $SD(18) = 9$, $SD(9) = 9$. So, for $n = 99$, the sequence s_i must be computed as $[99, 18, 9]$.

Hint: use `Stream.iterate`; compute the digits in the reverse order since their sum will be the same.

Solution We need to implement a function `sdSeq` having the type signature

```
def sdSeq(n: Int): Seq[Int]
```

First we need to implement $SD(x)$. The sum of digits is obtained by almost the same code as in Section 2.3:

```
def SD(n: Int): Int = if (n == 0) 0 else
  Stream.iterate(n)(_ / 10).takeWhile(_ != 0).map(_ % 10).sum
```

Now we can try evaluating SD on some numbers to see its behavior:

```
scala> (1 to 15).toList.map(SD)
res0: List[Int] = List(1, 2, 3, 4, 5, 6, 7, 8, 9, 1, 2, 3, 4, 5, 6)
```

It is clear that $SD(n) < n$ as long as $n \geq 10$. So the sequence elements s_i will not repeat until they become smaller than 10, and then they will always repeat. This seems to be an easy way of stopping the sequence. Let us try that:

```
scala> Stream.iterate(99)(SD).takeWhile(x => x >= 10).toList
res1: List[Int] = List(99, 18)
```

We are missing the last element of the sequence, $SD(18) = 9$, because `.takeWhile` stops the sequence too early. In order to obtain the correct sequence, we need to compute one more element. To fix this, we can generate a stream of *pairs*:

```
scala> Stream.iterate((0, 99)){ case (prev, x) => (x, SD(x)) }.
  takeWhile{ case (prev, x) => prev >= 10 || x >= 10 }.toList
res2: List[(Int, Int)] = List((0,99), (99,18), (18,9))
```

This looks right; it remains to remove the first parts of the tuples:

```
def sdSeq(n: Int): Seq[Int] =
  Stream.iterate((0, n)){ case (prev, x) => (x, SD(x)) } // Stream[(Int, Int)]
    .takeWhile{ case (prev, x) => prev >= 10 || x >= 10 } // Stream[(Int, Int)]
    .map(_._2) // Stream[Int]
    .toList // List[Int]
```

```
scala> sdSeq(99)
res3: Seq[Int] = List(99, 18, 9)
```

Example 2.5.1.7 Implement a function `unfold` with the type signature

```
def unfold[A](init: A)(next: A => Option[A]): Stream[A]
```

The function should create a stream of values of type `A` with the initial value `init`. Next elements are computed from previous ones via the function `next` until it returns `None`. An example test:

```
scala> unfold(0) { x => if (x > 5) None else Some(x + 2) }
res0: Stream[Int] = Stream(0, ?)

scala> res0.toList
res1: List[Int] = List(0, 2, 4, 6)
```

Solution We can formulate the task as an inductive definition of a stream. If `next(init) == None`, the stream must stop at `init`. (This is the base case of the induction). Otherwise, `next(init) == Some(x)` yields a new value `x` and indicates that we need to continue to “unfold” the stream with `x` instead of `init`. (This is the induction step.) Streams can be created from individual values via the Scala standard library method `Stream.cons` that constructs a stream from a single value and a tail:

```
def unfold[A](init: A)(next: A => Option[A]): Stream[A] = next(init) match {
  case None      => Stream(init)           // A stream containing a single value 'init'.
  case Some(x)   => Stream.cons(init, unfold(x)(next)) // 'init' followed by the tail of stream.
}
```

Example 2.5.1.8 For a given stream $[s_0, s_1, s_2, \dots]$ of type `Stream[T]`, compute the “half-speed” stream $h = [s_0, s_0, s_1, s_1, s_2, s_2, \dots]$. The half-speed sequence h is defined as $h_{2k} = h_{2k+1} = s_k$ for $k = 0, 1, 2, \dots$

Solution We use `.map` to replace each element s_i by a sequence containing two copies of s_i . Let us try this on a sample sequence:

```
scala> Seq(1,2,3).map( x => Seq(x, x))
res0: Seq[Seq[Int]] = List(List(1, 1), List(2, 2), List(3, 3))
```

The result is almost what we need, except we need to `.flatten` the nested list:

```
scala> Seq(1,2,3).map( x => Seq(x, x)).flatten
res1: Seq[Int] = List(1, 1, 2, 2, 3, 3)
```

The composition of `.map` and `.flatten` is `.flatMap`, so the final code is

```
def halfSpeed[T](str: Stream[T]): Stream[T] = str.flatMap(x => Seq(x, x))

scala> halfSpeed(Seq(1,2,3).toStream)
res2: Stream[Int] = Stream(1, ?)

scala> halfSpeed(Seq(1,2,3).toStream).toList
res3: List[Int] = List(1, 1, 2, 2, 3, 3)
```

Example 2.5.1.9 (The loop detection problem.) Stop a given stream $[s_0, s_1, s_2, \dots]$ at a place k where the sequence repeats itself; that is, an element s_k equals some earlier element s_i with $i < k$.

Solution The trick is to create a half-speed sequence h_i out of s_i and then find an index $k > 0$ such that $h_k = s_k$. (The condition $k > 0$ is needed because we will always have $h_0 = s_0$.) If we find such an index k , it would mean that either $s_k = s_{k/2}$ or $s_k = s_{(k-1)/2}$; in either case, we will have found an element s_k that equals an earlier element.

As an example, for an input sequence $s = [1, 3, 5, 7, 9, 3, 5, 7, 9, \dots]$ we obtain the half-speed sequence $h = [1, 1, 3, 3, 5, 5, 7, 7, 9, 9, 3, 3, \dots]$. Looking for an index $k > 0$ such that $h_k = s_k$, we find that $s_7 = h_7 = 7$. The element s_7 indeed repeats an earlier element (although s_7 is not the first such repetition).

There are in principle two ways of finding an index $k > 0$ such that $h_k = s_k$: First, to iterate over a list of indices $k = 1, 2, \dots$ and evaluate the condition $h_k = s_k$ as a function of k . Second, to build a sequence of pairs (h_i, s_i) and use `.takeWhile` to stop at the required index. In the present case, we

cannot use the first way because we do not have a fixed set of indices to iterate over. Also, the condition $h_k = s_k$ cannot be directly evaluated as a function of k because s and h are streams that compute elements on demand, not lists whose elements are computed in advance and ready for use.

So the code must iterate over a stream of pairs (h_i, s_i) :

```
def stopRepeats[T](str: Stream[T]): Stream[T] = {
  val halfSpeed = str.flatMap(x => Seq(x, x))
  val result = halfSpeed.zip(str) // Stream[(T, T)]
  .drop(1) // Enforce the condition k > 0.
  .takeWhile { case (h, s) => h != s } // Stream[(T, T)]
  .map(_._2) // Stream[T]
  str.head +: result // Prepend the first element that was dropped.
}

scala> stopRepeats(Seq(1, 3, 5, 7, 9, 3, 5, 7, 9).toStream).toList
res0: List[Int] = List(1, 3, 5, 7, 9, 3, 5)
```

Example 2.5.1.10 Reverse each word in a string, but keep the order of words:

```
def revWords(s: String): String = ???

scala> revWords("A quick brown fox")
res0: String = A kciuq nworb xof
```

Solution The standard method `.split` converts a string into an array of words:

```
scala> "pa re ci vo mu".split(" ")
res0: Array[String] = Array(pa, re, ci, vo, mu)
```

Each word is reversed with `.reverse`; the resulting array is concatenated into a string with `.mkString`:

```
def revWords(s: String): String = s.split(" ").map(_.reverse).mkString(" ")
```

Example 2.5.1.11 Remove adjacent repeated characters from a string:

```
def noDups(s: String): String = ???

scala> noDups("abbcdeeeefddggggh")
res0: String = abcdefdgh
```

Solution A string is automatically converted into a sequence of characters when we use methods such as `.map` or `.zip` on it. So, we can use `s.zip(s.tail)` to get a sequence of pairs (s_k, s_{k+1}) where s_k is the k -th character of the string s . A `.filter` will then remove elements s_k for which $s_{k+1} = s_k$:

```
scala> val s = "abbcd"
s: String = abbcd

scala> s.zip(s.tail).filter { case (sk, skPlus1) => sk != skPlus1 }
res0: IndexedSeq[(Char, Char)] = Vector((a,b), (b,c), (c,d))
```

It remains to convert this sequence of pairs into the string `"abcd"`. One way of doing this is to project the sequence of pairs onto the second parts of the pairs,

```
scala> res0.map(_._2).mkString
res1: String = bcd
```

We just need to add the first character, `'a'`. The resulting code is

```
def noDups(s: String): String = if (s == "") "" else {
  val pairs = s.zip(s.tail).filter { case (x, y) => x != y }
  pairs.head._1 +: pairs.map(_._2).mkString
}
```

The method `+:` prepends an element to a sequence, so `x +: xs` is equivalent to `Seq(x) ++ xs`.

Example 2.5.1.12 For a given sequence of type `Seq[A]`, find the longest subsequence that does not contain any adjacent duplicate values.

```
def longestNoDups[A](xs: Seq[A]): Seq[A] = ???

scala> longestNoDups(Seq(1, 2, 2, 5, 4, 4, 4, 8, 2, 3, 3))
res0: Seq[Int] = List(4, 8, 2, 3)
```

Solution This is a dynamic programming² problem. Many such problems are solved with a single `.foldLeft`. The accumulator represents the current “state” of the dynamic programming solution, and the “state” is updated with each new element of the input sequence.

We first need to determine the type of the accumulator value, or the “state”. The task is to find the longest subsequence without adjacent duplicates. So the accumulator should represent the longest subsequence found so far, as well as any required extra information about other subsequences that might grow as we iterate over the elements of `xs`. What is that extra information in our case?

Imagine creating the set of *all* subsequences that have no adjacent duplicates. For the input sequence `[1, 2, 2, 5, 4, 4, 4, 8, 2, 3, 3]`, this set of all subsequences will be `{[1, 2], [2, 5, 4], [4, 8, 2, 3]}`. We can build this set incrementally in the accumulator value of a `.foldLeft`. To visualize how this set would be built, consider the partial result after seeing the first 8 elements of the input sequence, `[1, 2, 2, 5, 4, 4, 4, 8]`. The partial set of non-repeating subsequences is `{[1, 2], [2, 5, 4], [4, 8]}`. As we add another element, 2, we update the partial set to `{[1, 2], [2, 5, 4], [4, 8, 2]}`.

It is now clear that the subsequence `[1, 2]` has no chance of being the longest subsequence, since `[2, 5, 4]` is already longer. However, we do not yet know whether `[2, 5, 4]` or `[4, 8, 2]` is the winner, because the subsequence `[4, 8, 2]` could still grow and become the longest one (and it does become `[4, 8, 2, 3]` later). At this point, we need to keep both of these two subsequences in the accumulator, but we may already discard `[1, 2]`.

We have deduced that the accumulator needs to keep only *two* sequences: the first sequence is already terminated and will not grow, the second sequence ends with the current element and may yet grow. The initial value of the accumulator is empty. The first subsequence is discarded when it becomes shorter than the second. The code can be written now:

```
def longestNoDups[A](xs: Seq[A]): Seq[A] = {
  val init: (Seq[A], Seq[A]) = (Seq(), Seq())
  val (first, last) = xs.foldLeft(init) { case ((first, current), x) =>
    // If 'current' is empty, 'x' cannot be repeated.
    val xWasRepeated = current != Seq() && current.last == x
    val firstIsLongerThanCurrent = first.length > current.length
    // Compute the new pair '(first, current)'.
    // Keep 'first' only if it is longer; otherwise replace it by 'current'.
    val newFirst = if (firstIsLongerThanCurrent) first else current
    // Append 'x' to 'current' if 'x' is not repeated.
    val newCurrent = if (xWasRepeated) Seq(x) else current :+ x
    (newFirst, newCurrent)
  }
  // Return the longer of the two subsequences; prefer 'first'.
  if (first.length >= last.length) first else last
}
```

2.5.2 Exercises

Exercise 2.5.2.1 Compute the sum of squared digits of a given integer; e.g., `dsq(123) = 14` (see Example 2.5.1.6). Generalize the solution to take as an argument an function `f: Int => Int` replacing the squaring operation. The required type signature and a sample test:

```
def digitsMapSum(x: Int)(f: Int => Int): Int = ???

scala> digitsMap(123){ x => x * x }
```

²https://en.wikipedia.org/wiki/Dynamic_programming

```
res0: Int = 14

scala> digitsMap(123){ x => x * x * x }
res1: Int = 36
```

Exercise 2.5.2.2 Compute the **Collatz sequence** c_i as a stream defined by

$$c_0 = n \quad ; \quad c_{k+1} = \begin{cases} \frac{c_k}{2} & \text{if } c_k \text{ is even,} \\ 3c_k + 1 & \text{if } c_k \text{ is odd.} \end{cases}$$

Stop the stream when it reaches 1 (as one would expect³ it will).

Exercise 2.5.2.3 For a given integer n , compute the sum of cubed digits, then the sum of cubed digits of the result, etc.; stop the resulting sequence when it repeats itself, and so determine whether it ever reaches 1. (Use Exercise 2.5.2.1.)

```
def cubes(n: Int): Stream[Int] = ???

scala> cubes(123).take(10).toList
res0: List[Int] = List(123, 36, 243, 99, 1458, 702, 351, 153, 153, 153)

scala> cubes(2).take(10).toList
res1: List[Int] = List(2, 8, 512, 134, 92, 737, 713, 371, 371, 371)

scala> cubes(4).take(10).toList
res2: List[Int] = List(4, 64, 280, 520, 133, 55, 250, 133, 55, 250)

def cubesReach1(n: Int): Boolean = ???

scala> cubesReach1(10)
res3: Boolean = true

scala> cubesReach1(4)
res4: Boolean = false
```

Exercise 2.5.2.4 For a, b, c of type `Set[Int]`, compute the set of all sets of the form `Set(x, y, z)` where x is from a , y from b , and z from c . The required type signature and a sample test:

```
def prod3(a: Set[Int], b: Set[Int], c: Set[Int]): Set[Set[Int]] = ???

scala> prod3(Set(1,2), Set(3), Set(4,5))
res0: Set[Set[Int]] = Set(Set(1,3,4), Set(1,3,5), Set(2,3,4), Set(2,3,5))
```

Hint: use `.flatMap`.

Exercise 2.5.2.5* Same task as in Exercise 2.5.2.4 for a set of sets, i.e. given a `Set[Set[Int]]` instead of just three sets a, b, c . The required type signature and a sample test:

```
def prodSet(si: Set[Set[Int]]): Set[Set[Int]] = ???

scala> prodSet(Set(Set(1,2), Set(3), Set(4,5), Set(6)))
res0: Set[Set[Int]] = Set(Set(1,3,4,6), Set(1,3,5,6), Set(2,3,4,6), Set(2,3,5,6))
```

Hint: use `.foldLeft` and `.flatMap`.

Exercise 2.5.2.6* In a sorted array `xs: Array[Int]` where no values are repeated, find all pairs of values whose sum equals a given number n . Use tail recursion. A type signature and a sample test:

```
def pairs(goal: Int, xs: Array[Int]): Set[(Int, Int)] = ???

scala> pairs(10, Array(1, 2, 3, 4, 5, 6, 7, 8))()
res0: Set[(Int, Int)] = Set((2,8), (3,7), (4,6), (5,5))
```

³https://en.wikipedia.org/wiki/Collatz_conjecture

Exercise 2.5.2.7 Reverse a sentence's word order, but keep the words unchanged:

```
def revSentence(s: String): String = ???

scala> revSentence("A quick brown fox")
res0: String = "fox brown quick A"
```

Exercise 2.5.2.8 (a) Reverse an integer's digits (see Example 2.5.1.6) as shown:

```
def revDigits(n: Int): Int = ???

scala> revDigits(12345)
res0: Int = 54321
```

(b) A **palindrome integer** is an integer number n such that $\text{revDigits}(n) == n$. Write a predicate function of type $\text{Int} \Rightarrow \text{Boolean}$ that checks whether a given positive integer is a palindrome.

Exercise 2.5.2.9 Define a function $\text{findPalindrome}: \text{Long} \Rightarrow \text{Long}$ performing the following computation: First define $f(n) = \text{revDigits}(n) + n$ for a given integer n , where the function revDigits was defined in Exercise 2.5.2.8. If $f(n)$ is a palindrome integer, findPalindrome returns that integer. Otherwise, it keeps applying the same transformation and computes $f(n)$, $f(f(n))$, ..., until a palindrome integer is eventually found (this is mathematically guaranteed). A sample test:

```
scala> findPalindrome(10101)
res0: Long = 10101

scala> findPalindrome(123)
res0: Long = 444

scala> findPalindrome(83951)
res1: Long = 869363968
```

Exercise 2.5.2.10 Transform a given sequence $xs: \text{Seq}[\text{Int}]$ into a sequence $\text{Seq}[(\text{Int}, \text{Int})]$ of pairs that skip one neighbor. Implement this transformation as a function skip1 with a type parameter A instead of the type Int . The required type signature and a sample test:

```
def skip1[A](xs: Seq[A]): Seq[(A, A)] = ???

scala> skip1(List(1,2,3,4,5))
res0: List[Int] = List((1,3), (2,4), (3,5))
```

Exercise 2.5.2.11 (a) For a given integer interval $[n_1, n_2]$, find the largest integer $k \in [n_1, n_2]$ such that the decimal representation of k does *not* contain any of the digits 3, 5, or 7. **(b)** For a given integer interval $[n_1, n_2]$, find the integer $k \in [n_1, n_2]$ with the largest sum of decimal digits. **(c)** A positive integer n is called a **perfect number** if it is equal to the sum of its divisors (other integers k such that $k < n$ and n/k is an integer). For example, 6 is a perfect number because its divisors are 1, 2, and 3, and $1 + 2 + 3 = 6$, while 8 is not a perfect number because its divisors are 1, 2, and 4, and $1 + 2 + 4 = 7 \neq 8$. Write a function that determines whether a given number n is perfect. Determine all perfect numbers up to one million.

Exercise 2.5.2.12 Remove adjacent repeated elements from a sequence of type $\text{Seq}[A]$ when they are repeated more than k times. Repetitions up to k times should remain unchanged. The required type signature and a sample test:

```
def removeDups[A](s: Seq[A], k: Int): Seq[A] = ???

scala> removeDups(Seq(1, 1, 1, 1, 5, 2, 2, 5, 5, 5, 5, 5, 1), 3)
res0: Seq[Int] = List(1, 1, 1, 5, 2, 2, 5, 5, 5, 1)
```

Exercise 2.5.2.13 Implement a function unfold2 with the type signature

```
def unfold2[A,B](init: A)(next: A => Option[(A,B)]): Stream[B]
```


The function should create a stream of values of type `B` by repeatedly applying the given function `next` until it returns `None`. At each iteration, `next` should be applied to the value of type `A` returned by the previous call to `next`. An example test:

```
scala> unfold2(0) { x => if (x > 5) None else Some((x + 2, s"had $x")) }
res0: Stream[String] = Stream(had 0, ?)

scala> res0.toList
res1: List[String] = List(had 0, had 2, had 4)
```

Exercise 2.5.2.14* (a) Remove repeated elements (whether adjacent or not) from a sequence of type `Seq[A]`. (This re-implements the standard library's method `.distinct`.)

(b) For a sequence of type `Seq[A]`, remove all elements that are repeated (whether adjacent or not) more than k times:

```
def removeK[A](k: Int, xs: Seq[A]): Seq[A] = ???

scala> removeK(2, Seq("a", "b", "a", "b", "b", "c", "b", "a"))
res0: Seq[String] = List(a, b, a, b, c)
```

Exercise 2.5.2.15* For a given sequence `xs:Seq[Double]`, find a subsequence that has the largest sum of values. The sequence `xs` is not sorted, and its values may be positive or negative. The required type signature and a sample test:

```
def maxsub(xs: Seq[Double]): Seq[Double] = ???

scala> maxsub(Seq(1.0, -1.5, 2.0, 3.0, -0.5, 2.0, 1.0, -10.0, 2.0))
res0: Seq[Double] = List(2.0, 3.0, -0.5, 2.0, 1.0)
```

Hint: use dynamic programming and `.foldLeft`.

Exercise 2.5.2.16* Using tail recursion, find all common integers between two *sorted* sequences:

```
@tailrec def commonInt(xs: Seq[Int], ys: Seq[Int]): Seq[Int] = ???

scala> commonInt(Seq(1, 3, 5, 7), Seq(2, 3, 4, 6, 7, 8))
res0: Seq[Int] = List(3, 7)
```

2.6 Discussion

2.6.1 Total and partial functions

In Scala, functions can be total or partial. A **total** function will always compute a result value, while a **partial** function may fail to compute its result for certain values of its arguments.

A simple example of a partial function in Scala is the `.max` method: it only works for non-empty sequences. Trying to evaluate it on an empty sequence generates an error called an “exception”:

```
scala> Seq(1).tail
res0: Seq[Int] = List()
scala> res0.max
java.lang.UnsupportedOperationException: empty.max
  at scala.collection.TraversableOnce$class.max(TraversableOnce.scala:229)
  at scala.collection.AbstractTraversable.max(Traversable.scala:104)
  ... 32 elided
```

This kind of error may crash the entire program at run time. Unlike the type errors we saw before, which occur at compilation time (i.e. before the program can start), **run-time errors** occur while the program is running, and only when some partial function happens to get an incorrect input. The incorrect input may occur at any point after the program started running, which may crash the entire program in the middle of a long computation.

So, it seems clear that we should write code that does not generate such errors. For instance, it is safe to apply `.max` to a sequence if we know that it is non-empty.

Sometimes, a function that uses pattern matching turns out to be a partial function because its pattern matching code fails on certain input data.

If a pattern matching expression fails, the code will throw an exception and stop running. In functional programming, we usually want to avoid this situation because it makes it much harder to reason about program correctness. In most cases, programs can be written to avoid the possibility of match errors. An example of an unsafe pattern matching expression is

```
def h(p: (Int, Int)): Int = p match { case (x, 0) => x }

scala> h( (1,0) )
res0: Int = 1

scala> h( (1,2) )
scala.MatchError: (1,2) (of class scala.Tuple2$mcII$sp)
  at .h(<console>:12)
  ... 32 elided
```

Here the pattern contains a pattern variable `x` and a constant `0`. This pattern only matches tuples whose second part is equal to `0`. If the second argument is nonzero, a match error occurs and the program crashes. So, `h` is a partial function.

Pattern matching failures never happen if we match a tuple of correct size with a pattern such as `(x, y, z)`, because a pattern variable will always match a value. So, pattern matching with a pattern such as `(x, y, z)` is **infallible** (never fails at run time) when applied to a tuple with 3 elements.

Another way in which pattern matching can be made infallible is by including a pattern that matches everything:

```
p match {
  case (x, 0)   => ... // This only matches some tuples.
  case _       => ... // This matches everything.
}
```

If the first pattern `(x, 0)` fails to match the value `p`, the second pattern will be tried (and will always succeed). The `case` patterns in a `match` expression are tried in the order they are written. So, a `match` expression may be made infallible by adding a “match-all” underscore pattern.

2.6.2 Scope and shadowing of pattern matching variables

Pattern matching introduces **locally scoped** variables – that is, variables defined only on the right-hand side of the pattern match expression. As an example, consider this code:

```
def f(x: (Int, Int)): Int = x match { case (x, y) => x + y }

scala> f( (2,4) )
res0: Int = 6
```

The argument of `f` is the variable `x` of a tuple type `(Int, Int)`, but there is also a pattern variable `x` in the case expression. The pattern variable `x` matches the first part of the tuple and has type `Int`. Because variables are locally scoped, the pattern variable `x` is only defined within the expression `x + y`. The argument `x: (Int, Int)` is a completely different variable whose value has a different type.

The code works correctly but is confusing to read because of the name clash between the two quite different variables, both named `x`. Another negative consequence of the name clash is that the argument `x: (Int, Int)` is *invisible* within the case expression: if we write “`x`” in that expression, we will get the pattern variable `x: Int`. One says that the argument `x: (Int, Int)` has been **shadowed** by the pattern variable `x`.

The problem is easy to correct: we can give the pattern variable some other name. Since the pattern variable is locally scoped, it can be renamed within its scope without having to change any other code. A completely equivalent code is

```
def f(x: (Int, Int)): Int = x match { case (a, b) => a + b }

scala> f( (2,4) )
res0: Int = 6
```

2.6.3 Lazy values and sequences: iterators and streams

We have used streams to create sequences whose length is not known in advance. An example is a stream containing a sequence of increasing positive integers:

```
scala> val p = Stream.iterate(1)(_ + 1)
p: Stream[Int] = Stream(1, ?)
```

At this point, we have not defined a stopping condition for this stream. In some sense, streams are “infinite” sequences, although in practice a stream is always finite because computers cannot run infinitely long. Also, computers cannot store infinitely many values in memory.

More precisely, streams are “partially computed” rather than “infinite”. The main difference between arrays and streams is that a stream’s elements are computed on demand and not all initially available, while an array’s elements are all computed in advance and are available immediately.

Generally, there are four possible ways a value could be available:

Availability	Explanation	Example Scala code
“eager”	computed in advance	<code>val z = f(123)</code>
“lazy”	computed upon first request	<code>lazy val z = f(123)</code>
“on-call”	computed each time it is requested	<code>def z = f(123)</code>
“never”	cannot be computed due to errors	<code>val (z, x) = "abc"</code>

A **lazy value** (declared as `lazy val` in Scala) is computed only when used in some other expression. Once computed, a lazy value stays in memory and will not be re-computed.

An “on-call” value is re-computed every time it is used. In Scala, this is the behavior of a `def` declaration.

Most collection types in Scala (such as `List`, `Array`, `Set`, and `Map`) are **eager**: all elements of an eager collection are already evaluated.

A stream is a **lazy collection**. Elements of a stream are computed when first needed; after that, they remain in memory and will not be computed again:

```
scala> val str = Stream.iterate(1)(_ + 1)
str: Stream[Int] = Stream(1, ?)

scala> str.take(10).toList
res0: List[Int] = List(1, 2, 3, 4, 5, 6, 7, 8, 9, 10)

scala> str
res1: Stream[Int] = Stream(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ?)
```

In many cases, it is not necessary to keep previous values of a sequence in memory. For example, consider the computation

```
scala> (1L to 10000000000L).sum
res0: Long = 500000000500000000
```

The computation just shown does *not* keep all the numbers in memory. The same computation fails if we use a list or a stream:

```
scala> (1L to 10000000000L).toStream.sum
java.lang.OutOfMemoryError: GC overhead limit exceeded
```

We do not actually need to keep a billion numbers in memory if we only want to compute their sum. Indeed, the computation just shown does *not* keep all the numbers in memory. The same computation fails if we use a list or a stream:

The code `(1L to 10000000000L).sum` works because `(1 to n)` produces a sequence whose

elements are computed whenever needed but do not remain in memory. This can be seen as a sequence with the “on-call” availability of elements. Sequences of this sort are called **iterators**:

```
scala> 1 to 5
res0: scala.collection.immutable.Range.Inclusive = Range(1, 2, 3, 4, 5)

scala> 1 until 5
res1: scala.collection.immutable.Range = Range(1, 2, 3, 4)
```

The types `Range` and `Range.Inclusive` are defined in the Scala standard library and are iterators. They behave as collections and support the usual methods (`.map`, `.filter`, etc.), but they do not store previously computed values in memory.

The `.view` method Eager collections such as `List` or `Array` can be converted to iterators by using the `.view` method. This is necessary when intermediate collections consume too much memory when fully evaluated. For example, consider the computation of Example 2.1.5.7 where we used `.flatMap` to replace each element of an initial sequence by three new numbers before computing `.max` of the resulting collection. If instead of three new numbers we wanted to compute *three million* new numbers each time, the intermediate collection created by `.flatMap` would require too much memory, and the computation would crash:

```
scala> (1 to 10).flatMap(x => 1 to 3000000).max
java.lang.OutOfMemoryError: GC overhead limit exceeded
```

for our computer’s memory. We can use `.view` to avoid this:

```
scala> (1 to 10).view.flatMap(x => 1 to 3000000).max
res0: Int = 3000000
```

tors behave similarly to other sequences. We may write programs in the map/reduce style, applying standard methods such as `.map`, `.filter`, etc., to streams and iterators. Mathematical reasoning about transforming a sequence is the same, whether the sequence is eager, lazy, or on-call.

The broken `Iterator` class The Scala library contains a class called `Iterator`, which has methods such as `Iterator.iterate` and other methods similar to `Stream`. However, `Iterator` actually not a correct iterator because it cannot be treated as a *value* in the mathematical sense:

```
scala> val iter = (1 until 10).toIterator
iter: Iterator[Int] = non-empty iterator

scala> iter.toList // Look at the elements of 'iter'.
res0: List[Int] = List(1, 2, 3, 4, 5, 6, 7, 8, 9)

scala> iter.toList // Look at those elements again...??
res1: List[Int] = List()

scala> iter
res2: Iterator[Int] = empty iterator
```

Evaluating the expression `iter.toList` two times produces a *different* result the second time. As we see, the value `iter` becomes “empty” after the first use.

This situation is impossible in mathematics: if x is some value, such as 100, and f is some function, such as $f(x) = \sqrt{x}$, then $f(x)$ will be the same, $f(100) = \sqrt{100} = 10$, no matter how many times we compute $f(x)$. For instance, we can compute $f(x) + f(x) = 20$ and obtain the correct result. We could also set $y = f(x)$ and compute $y + y = 20$, with the same result. This property is called **referential transparency** or **purity** of the function f .

When we set $x = 100$ and compute $f(x) + f(x)$, the number 100 does not “become empty” after the first use; its value remains the same. This behavior is called the **value semantics** of numbers. One says that integers “are values” in the mathematical sense. Alternatively, one says that numbers are **immutable**, i.e. cannot be changed. (What would it mean to “modify” the number 10?)

In programming, a type has value semantics if a given computation applied to it always gives

the same result. Usually, this means that the type contains immutable data, and the computation is referentially transparent. We can see that Scala's `Range` has value semantics and is immutable:

```
scala> val x = 1 until 10
x: scala.collection.immutable.Range = Range(1, 2, 3, 4, 5, 6, 7, 8, 9)

scala> x.toList
res0: List[Int] = List(1, 2, 3, 4, 5, 6, 7, 8, 9)

scala> x.toList
res1: List[Int] = List(1, 2, 3, 4, 5, 6, 7, 8, 9)
```

Collections such as `List`, `Map`, or `Stream` are immutable. Some elements of a `Stream` may not be evaluated yet, but this does not affect its value semantics:

```
scala> val str = (1 until 10).toStream
str: scala.collection.immutable.Stream[Int] = Stream(1, ?)

scala> str.toList
res0: List[Int] = List(1, 2, 3, 4, 5, 6, 7, 8, 9)

scala> str.toList
res1: List[Int] = List(1, 2, 3, 4, 5, 6, 7, 8, 9)
```

Iterators produced by applying `.view` also have value semantics:

```
scala> val v = (1 until 10).view
v: scala.collection.SeqView[Int,IndexedSeq[Int]] = SeqView(...)

scala> v.toList
res0: List[Int] = List(1, 2, 3, 4, 5, 6, 7, 8, 9)

scala> v.toList
res1: List[Int] = List(1, 2, 3, 4, 5, 6, 7, 8, 9)
```

Due to the lack of value semantics, programs written using `Iterator` may not obey the usual rules of mathematical reasoning. This makes it easy to write wrong code that looks correct.

To illustrate the problem, let us re-implement Example 2.5.1.9 by keeping the same code but using `Iterator` instead of `Stream`:

```
def stopRepeatsBad[T](iter: Iterator[T]): Iterator[T] = {
  val halfSpeed = iter.flatMap(x => Seq(x, x))
  halfSpeed.zip(iter) // Do not prepend the first element. It won't help.
    .drop(1)
    .takeWhile { case (h, s) => h != s }
    .map(_._2)
}

scala> stopRepeatsBad(Seq(1, 3, 5, 7, 9, 3, 5, 7, 9).toIterator).toList
res0: List[Int] = List(5, 9, 3, 7, 9)
```

The result `[5,9,3,7,9]` is incorrect, but not in an obvious way: the sequence *was* stopped at a repetition, as we expected, but some of the elements of the given sequence are missing (while other elements are present). It is difficult to debug a program when it produces *partially* correct numbers.

The error in this code occurs in the expression `halfSpeed.zip(iter)` due to the fact that `halfSpeed` was itself defined via `iter`. The result is that `iter` is *used twice* in this code, which leads to errors because `iter` is mutable and does not behave as a value. Creating an `Iterator` and using it twice in the same expression can give wrong results or even fail with an exception:

```
scala> val s = (1 until 10).toIterator
s: Iterator[Int] = non-empty iterator

scala> val t = s.zip(s).toList
java.util.NoSuchElementException: next on empty iterator
```

It is surprising and counter-intuitive that a variable cannot be used twice in some expression. Intuitively, we expect code such as `s.zip(s)` to work correctly even though the variable `s` is used twice. When we read the expression `s.zip(s)`, we imagine a given sequence `s` being “zipped” with itself. So we reason that `s.zip(s)` should produce a sequence of pairs. But Scala’s `Iterator` is mutable, which breaks the usual ways of mathematical reasoning about code.

An `Iterator` can be converted to a `Stream` using the `.toStream` method. This restores the value semantics, since streams are values:

```
scala> val iter = (1 until 10).toIterator
iter: Iterator[Int] = non-empty iterator

scala> val str = iter.toStream
str: Stream[Int] = Stream(1, ?)

scala> str.toList
res0: List[Int] = List(1, 2, 3, 4, 5, 6, 7, 8, 9)

scala> str.toList
res1: List[Int] = List(1, 2, 3, 4, 5, 6, 7, 8, 9)

scala> str.zip(str).toList
res2: List[(Int, Int)] = List((1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7), (8,8), (9,9))
```

Instead of `Iterator`, we can use `Stream` and `.view` when lazy or on-call collections are required.

3 The logic of types. I. Disjunctive types

Disjunctive types describe values that belong to a disjoint set of alternatives.

To see how Scala implements disjunctive types, we need to begin by looking at “case classes”.

3.1 Scala’s case classes

3.1.1 Tuple types with names

It is often helpful to use names for the different parts of a tuple. Suppose that some program represents the size and the color of socks with the tuple type `(Double, String)`. What if the same tuple type `(Double, String)` is used in another place in the program to mean the amount paid and the payee? A programmer could mix the two values by mistake, and it would be hard to find out why the program incorrectly computes, say, the total amount paid.

```
def totalAmountPaid(ps: Seq[(Double, String)]): Double = ps.map(_._1).sum
val x = (10.5, "white")      // Sock size and color.
val y = (25.0, "restaurant") // Payment amount and payee.

scala> totalAmountPaid(Seq(x, y)) // Nonsense.
res0: Double = 35.5
```

We would prevent this kind of mistake if we could use two *different* types, with names such as `MySock` and `Payment`, for the two kinds of data. There are three basic ways of defining a new named type in Scala: using a type alias, using a class (or “trait”), and using an opaque type.

Opaque types (hiding a type under a new name) is a feature of a future version of Scala 3; so we focus on type aliases and case classes.

A **type alias** is an alternative name for an existing (already defined) type. We could use type aliases in our example to add clarity to the code:

```
type MySockTuple = (Double, String)
type PaymentTuple = (Double, String)

scala> val s: MySockTuple = (10.5, "white")
s: MySockTuple = (10.5,white)

scala> val p: PaymentTuple = (25.0, "restaurant")
p: PaymentTuple = (25.0,restaurant)
```

But type aliases do not prevent mix-up errors:

```
scala> totalAmountPaid(Seq(s, p)) // Nonsense again.
res1: Double = 35.5
```

Scala’s **case classes** can be seen as “tuples with names”. A case class is equivalent to a tuple type that has a name chosen when we define the case class. Also, each part of the case class will have a separate name that we must choose. This is how to define case classes for the example with socks and payments:

```
case class MySock(size: Double, color: String)
case class Payment(amount: Double, name: String)

scala> val sock = MySock(10.5, "white")
sock: MySock = MySock(10.5,white)
```

```
scala> val paid = Payment(25.0, "restaurant")
paid: Payment = Payment(25.0,restaurant)
```

This code defines new types named `MySock` and `Payment`. Values of type `MySock` are written as `MySock(10.5, "white")`, which is similar to writing the tuple `(10.5, "white")` except for adding the name `MySock` in front of the tuple.

To access the parts of a case class, we use the part names:

```
scala> sock.size
res2: Double = 10.5

scala> paid.amount
res3: Double = 25.0
```

The mix-up error is now a type error detected by the compiler:

```
def totalAmountPaid(ps: Seq[Payment]): Double = ps.map(_.amount).sum

scala> totalAmountPaid(Seq(paid, paid))
res4: Double = 50.0

scala> totalAmountPaid(Seq(sock, paid))
<console>:19: error: type mismatch;
 found   : MySock
 required: Payment
    totalAmountPaid(Seq(sock, paid))
                        ^
```

A function whose argument is of type `MySock` cannot be applied to an argument of type `Payment`. Case classes with different names are *different types*, even if they contain the same parts.

Just as tuples can have any number of parts, case classes can have any number of parts, but the part names must be distinct, for example:

```
case class Person(firstName: String, lastName: String, age: Int)

scala> val noether = Person("Emmy", "Noether", 137)
einstein: Person = Person(Emmy,Noether,137)

scala> noether.firstName
res5: String = Emmy

scala> noether.age
res6: Int = 137
```

This data type carries the same information as a tuple `(String, String, Int)`. However, the declaration of a `case class` `Person` gives the programmer several features that make working with the tuple's data more convenient and less error-prone.

Some (or all) part names may be specified when creating a case class value:

```
scala> val poincaré = Person(firstName = "Henri", lastName = "Poincaré", 165)
poincaré: Person = Person(Henri,Poincaré,165)
```

It is a type error to use wrong types with a case class:

```
scala> val p = Person(140, "Einstein", "Albert")
<console>:13: error: type mismatch;
 found   : Int(140)
 required: String
    val p = Person(140, "Einstein", "Albert")
                        ^

<console>:13: error: type mismatch;
 found   : String("Albert")
 required: Int
    val p = Person(140, "Einstein", "Albert")
                        ^
```

This error is due to an incorrect order of parts when creating a case class value. However, parts can be specified in any order when using part names:

```
scala> val p = Person(age = 137, lastName = "Noether", firstName = "Emmy")
p: Person = Person(Emmy,Noether,137)
```

A part of a case class can have the type of another case class, creating a type similar to a nested tuple:

```
case class BagOfSocks(sock: MySock, count: Int)
val bag = BagOfSocks(MySock(10.5, "white"), 6)

scala> bag.sock.size
res7: Double = 10.5
```

3.1.2 Case classes with type parameters

Type classes can be defined with type parameters. As an example, consider a generalization of `MySock` where, in addition to the size and color, an “extended sock” holds another value. We could define several specialized case classes,

```
case class MySock_Int(size: Double, color: String, value: Int)
case class MySock_Boolean(size: Double, color: String, value: Boolean)
```

but it is better to define a single parameterized case class

```
case class MySockX[A](size: Double, color: String, value: A)
```

This case class can accommodate every type `A`. We may now create values of `MySockX` containing a value of any given type,

```
scala> val s = MySockX(10.5, "white", 123)
s: MySockX[Int] = MySockX(10.5,white,123)
```

Because 123 has type `Int`, the type parameter `A` in `MySockX[A]` was automatically set to the type `Int`.

Each time we create a value of type `MySockX`, a specific type will have to be used instead of the type parameter `A`. In other words, we can only create values of types `MySockX[Int]`, `MySockX[String]`, etc. If we want to be explicit, we may write

```
scala> val s = MySockX[String](10.5, "white", "last pair")
s: MySockX[String] = MySockX(10.5,white,last pair)
```

However, we can write code working with `MySockX[A]` **parametrically**, that is, keeping the type parameter `A` in the code. For example, a function that checks whether a sock of type `MySockX[A]` fits my foot can be written as

```
def fitsMe[A](sock: MySockX[A]): Boolean = (sock.size >= 10.5 && sock.size <= 11.0)
```

This function is defined for all types `A` at once, because its code works in the same way regardless of what `A` is. Scala will set the type parameter automatically when we apply `fitsMe` to an argument:

```
scala> fitsMe(MySockX(10.5, "blue", List(1,2,3))) // Type parameter A = List[Int].
res0: Boolean = true
```

This code forces the type parameter `A` to be `List[Int]`, and so we may omit the type parameter of `fitsMe`. When types become more complicated, it may be helpful to write out some type parameters. The compiler can detect a mismatch between the type parameter `A = List[Int]` used in the “sock” value and the type parameter `A = Int` in the function `fitsMe`:

```
scala> fitsMe[Int](MySockX(10.5, "blue", List(1,2,3)))
<console>:15: error: type mismatch;
 found   : List[Int]
 required: Int
    fitsMe[Int](MySockX(10.5, "blue", List(1,2,3)))
                                     ^
```


Case classes may have several type parameters, and the types of the parts may use these type parameters. Here is an artificial example of a case class using type parameters in different ways,

```
case class Complicated[A,B,C,D](x: (A, A), y: (B, Int) => A, z: C => C)
```

This case class contains parts of different types that use the type parameters `A`, `B`, `C` in tuples and functions. The type parameter `D` is not used at all; this is allowed (and occasionally useful).

A type with type parameters, such as `MySockX` or `Complicated`, is called a **type constructor**. A type constructor “constructs” a new type, such as `MySockX[Int]`, from a given type parameter `Int`. Values of type `MySockX` cannot be created without setting the type parameter. So, it is important to distinguish the type constructor, such as `MySockX`, from a type that can have values, such as `MySockX[Int]`.

3.1.3 Tuples with one part and with zero parts

Let us compare tuples and case classes more systematically.

Parts of a case class are accessed by name with a dot syntax, for example `sock.color`. Parts of a tuple are accessed with the accessors such as `x._1`. This syntax is the same as that for a case class whose parts have names `_1`, `_2`, etc. So, it appears that tuple parts *do* have names in Scala, although those names are always automatically chosen as `_1`, `_2`, etc. Tuple types are also automatically named in Scala as `Tuple2`, `Tuple3`, etc., and they are parameterized, since each part of the tuple may be of any chosen type. A tuple type expression such as `(Int, String)` is just a special syntax for the parameterized type `Tuple2[Int, String]`. One could define the tuple types as case classes like this,

```
case class Tuple2[A, B](_1: A, _2: B)
case class Tuple3[A, B, C](_1: A, _2: B, _3: C) // And so on with Tuple4, Tuple5...
```

if these types were not already defined in the Scala library.

Proceeding systematically, we ask whether tuple types can have just one part or even no parts. Indeed, Scala defines `Tuple1[A]` as a tuple with a single part. (This type is occasionally used in practice.)

The tuple with zero parts also exists and is called `Unit` (rather than “`Tuple0`”). The syntax for the value of the `Unit` type is the empty tuple, `()`. It is clear that there is *only one* value, `()`, of this type; this explains the name “unit”.

At first sight, the `Unit` type may appear to be completely useless: it is a tuple that contains *no data*. It turns out, however, that the `Unit` type is important in functional programming, and it is used as a type *guaranteed* to have only a single distinct value. This chapter will show some examples of using the `Unit` type.

Case classes may have one part or zero parts, similarly to the one-part and zero-part tuples:

```
case class B(z: Int) // Tuple with one part.
case class C()       // Tuple with no parts.
```

Scala has a special syntax for empty case classes:

```
case object C // Similar to ‘case class C()’.
```

There are two main differences between `case class C()` and `case object C`:

- A `case object` cannot have type parameters, while we may define, if needed, a `case class C[X, Y, Z]()` with type parameters `X`, `Y`, `Z`.
- A `case object` is allocated in memory only once, while new values of a `case class C()` will be allocated in memory each time `C()` is evaluated.

Other than that, `case class C()` and `case object C` have the same meaning: a named tuple with zero parts, which we may also view as a “named `Unit`” type. In this book, I will not use `case objects` because `case classes` are more general.

Let us summarize the correspondence between tuples and case classes:

Tuples	Case classes
<code>(123, "xyz"): Tuple2[Int, String]</code>	<code>case class A(x: Int, y: String)</code>
<code>(123,): Tuple1[Int]</code>	<code>case class B(z: Int)</code>
<code>(): Unit</code>	<code>case class C()</code>

3.1.4 Pattern matching for case classes

Scala performs pattern matching in two situations:

- destructuring definition: `val pattern = ...`
- case expression: `case pattern => ...`

Case classes can be used in both situations. A destructuring definition can be used in a function whose argument is of case class type `BagOfSocks`:

```
case class MySock(size: Double, color: String)
case class BagOfSocks(sock: MySock, count: Int)

def printBag(bag: BagOfSocks): String = {
  val BagOfSocks(MySock(size, color), count) = bag // Destructure the 'bag'.
  s"bag has $count $color socks of size $size"
}

val bag = BagOfSocks(MySock(10.5, "white"), 6)

scala> printBag(bag)
res0: String = bag has 6 white socks of size 10.5
```

A case expression will destructure a value and compute a result:

```
def fitsMe(bag: BagOfSocks): Boolean = bag match {
  case BagOfSocks(MySock(size, _), _) => (size >= 10.5 && size <= 11.0)
}
```

In the code of this function, we match the `bag` value against the pattern `BagOfSocks(MySock(size, _), _)`. This pattern will always match and will define `size` as a pattern variable of type `Double`.

The syntax for pattern matching expressions with case classes is similar to the syntax for pattern matching of tuples, except for the presence of the *names* of the case classes. For example, removing the case class names from the pattern

```
case BagOfSocks(MySock(size, _), _) => ...
```

we obtain the nested tuple pattern

```
case ((size, _), _) => ...
```

that could be used for values of type `((Double, String), Int)`. We see that within pattern matching expressions, case classes behave exactly as tuple types with added names.

Scala's "case classes" got their name from their use in case expressions. It is usually more convenient to use `match` / `case` expressions with case classes than to use destructuring.

3.2 Disjunctive types

3.2.1 Motivation and first examples

In many situations, it is useful to have several different shapes of data within the same type. As a first example, suppose we are looking for real roots of a quadratic equation $x^2 + bx + c = 0$. There are three cases: no real roots, one real root, and two real roots. It is convenient to have a type that

represents “the real roots of a quadratic equation”; call it `RootsOfQ`. Inside that type, we distinguish between the three cases, but outside it looks like a single type.

Another example is the binary search algorithm that looks for an integer x in a sorted array. Either the algorithm finds the index of x , or it determines that the array does not contain x . It is convenient if the algorithm could return a single value of a type (call it `SearchResult`) that represents *either* an index at which x is found, *or* the absence of an index.

More generally, we may have computations that *either* return a value *or* generate an error and fail to produce a result. It is then convenient to return a value of type (call it `Result`) that represents either a correct result or an error message.

In certain computer games, one has different types of “rooms”, each room having certain properties depending on its type. Some rooms are dangerous because of monsters, other rooms contain useful objects, certain rooms allow you to finish the game, and so on. We want to represent all the different kinds of rooms uniformly, as a type `Room`, so that a value of type `Room` automatically stores the correct properties in each case.

In all these situations, data comes in several mutually exclusive shapes. This data can be represented by a single type if that type is able to describe a mutually exclusive set of cases:

- `RootsOfQ` must be either the empty tuple `()`, or `Double`, or a tuple `(Double, Double)`
- `SearchResult` must be either `Int` or the empty tuple `()`
- `Result` must be either an `Int` value or a `String` message

We see that the empty tuple, also known as the `Unit` type, is natural to use in these situations. It is also helpful to assign names to each of the cases:

- `RootsOfQ` is “no roots” with value `()`, or “one root” with value `Double`, or “two roots” with value `(Double, Double)`
- `SearchResult` is “index” with value `Int`, or “not found” with value `()`
- `Result` is “value” of type `Int` or “error message” of type `String`

Scala’s case classes provide exactly what we need here – named tuples with zero, one, two and more parts, and so it is natural to use case classes instead of tuples:

- `RootsOfQ` is a value of type `case class NoRoots()`, or a value of type `case class OneRoot(x: Double)`, or a value of type `case class TwoRoots(x: Double, y: Double)`
- `SearchResult` is a value of type `case class Index(Int)` or a value of type `case class NotFound()`
- `Result` is a value of type `case class Value(x: Int)` or a value of type `case class Error(message: String)`

Our three examples are now described as types that select one case class out of a given set. It remains to see how Scala defines such types. For instance, the definition of `RootsOfQ` needs to indicate that the case classes `NoRoots`, `OneRoot`, and `TwoRoots` are exactly the three alternatives described by the type `RootsOfQ`. The Scala syntax for that definition looks like this:

```
sealed trait RootsOfQ
final case class NoRoots() extends RootsOfQ
final case class OneRoot(x: Double) extends RootsOfQ
final case class TwoRoots(x: Double, y: Double) extends RootsOfQ
```

In the definition of `SearchResult`, we have two cases:

```
sealed trait SearchResult
final case class Index(i: Int) extends SearchResult
final case class NotFound() extends SearchResult
```

The definition of the `Result` type is parameterized, so that we can describe results of any type (while error messages are always of type `String`):

```
sealed trait Result[A]
final case class Value[A](x: A) extends Result[A]
final case class Error[A](message: String) extends Result[A]
```

The “`sealed trait / final case class`” syntax defines a type that represents a choice of one case class from a fixed set of case classes. This kind of type is called a **disjunctive type** in this book.

3.2.2 Solved examples: Pattern matching for disjunctive types

Our first examples of disjunctive types are `RootsOfQ`, `SearchResult`, and `Result[A]` defined in the previous section. We will now look at the Scala syntax for creating values of disjunctive types and for using the created values.

Consider the disjunctive type `RootsOfQ` having three case classes (`NoRoots`, `OneRoot`, `TwoRoots`). The only way of creating a value of type `RootsOfQ` is to create a value of one of these case classes. This is done by writing expressions such as `NoRoots()`, `OneRoot(2.0)`, or `TwoRoots(1.0, -1.0)`. Scala will accept these expressions as having the type `RootsOfQ`:

```
scala> val x: RootsOfQ = OneRoot(2.0)
x: RootsOfQ = OneRoot(2.0)
```

Given a value `x:RootsOfQ`, how can we use it, say, as a function argument? The main tool for working with values of disjunctive types is pattern matching with `match / case` expressions. In Chapter 2, we used pattern matching to destructure tuples with syntax such as `{ case (x, y) => ... }`. To use `match / case` expressions with disjunctive types, we may have to write *more than one case* pattern in a `match` expression, because we need to match several possible cases of the disjunctive type:

```
def f(r: RootsOfQ): String = r match {
  case NoRoots()      => "no real roots"
  case OneRoot(r)     => s"one real root: $r"
  case TwoRoots(x, y) => s"real roots: ($x, $y)"
}

scala> f(x)
res0: String = "one real root: 2.0"
```

If we only need to recognize a specific case of a disjunctive type, we can match all other cases with an underscore:

```
scala> x match {
  case OneRoot(r) => s"one real root: $r"
  case _         => "have something else"
}
res1: String = one real root: 2.0
```

The `match / case` expression represents a choice over possible values of a given type. Note the similarity with this code:

```
def f(x: Int): Int = x match {
  case 0    => println(s"error: must be nonzero"); -1
  case 1    => println(s"error: must be greater than 1"); -1
  case _    => x
}
```

The values 0 and 1 are some possible values of type `Int`, just as `OneRoot(1.0)` is a possible value of type `RootsOfQ`. When used with disjunctive types, `match / case` expressions will usually contain a complete list of possibilities. If the list of cases is incomplete, the Scala compiler will print a warning:

```
scala> def g(x: RootsOfQ): String = x match {
  case OneRoot(r) => s"one real root: $r"
}
```

```
<console>:14: warning: match may not be exhaustive.
It would fail on the following inputs: NoRoots(), TwoRoots(_, _)
    def g(x: RootsOfQ): String = x match {
                                   ^
```

This code defines a *partial* function `g` that can be applied only to values of the form `OneRoot(...)` and will fail for other values.

Let us look at more examples of using the disjunctive types we just defined.

Example 3.2.2.1 Given a sequence of quadratic equations, compute a sequence containing their real roots as values of type `RootsOfQ`.

Solution Define a case class representing a quadratic equation $x^2 + bx + c = 0$:

```
case class QEqu(b: Double, c: Double)
```

The following function determines how many real roots an equation has:

```
def solve(quadraticEqu: QEqu): RootsOfQ = {
  val QEqu(b, c) = quadraticEqu // Destructure QEqu.
  val d = b * b / 4 - c
  if (d > 0) {
    val s = math.sqrt(d)
    TwoRoots(b / 2 - s, b / 2 + s)
  } else if (d == 0.0) OneRoot(b / 2)
  else NoRoots()
}
```

Test the `solve` function:

```
scala> solve(QEqu(1,1))
res1: RootsOfQ = NoRoots()

scala> solve(QEqu(1,-1))
res2: RootsOfQ = TwoRoots(-0.6180339887498949, 1.618033988749895)

scala> solve(QEqu(6,9))
res3: RootsOfQ = OneRoot(3.0)
```

We can now implement the function `findRoots`,

```
def findRoots(equs: Seq[QEqu]): Seq[RootsOfQ] = equs.map(solve)
```

If the function `solve` will not be used often, we may want to write it inline as a nameless function:

```
def findRoots(equs: Seq[QEqu]): Seq[RootsOfQ] = equs.map { case QEqu(b, c) =>
  (b * b / 4 - c) match {
    case d if d > 0 =>
      val s = math.sqrt(d)
      TwoRoots(b / 2 - s, b / 2 + s)
    case 0.0 => OneRoot(b / 2)
    case _ => NoRoots()
  }
}
```

This code depends on some features of Scala syntax. We can use the partial function `{ case QEqu(b, c) => ... }` directly as the argument of `.map` instead of defining this function separately. This avoids having to destructure `QEqu` at a separate step. The `if / else` expression is replaced by an “embedded” `if` within the `case` expression, which is easier to read. Test the final code:

```
scala> findRoots(Seq(QEqu(1,1), QEqu(2,1)))
res4: Seq[RootsOfQ] = List(NoRoots(), OneRoot(1.0))
```

Example 3.2.2.2 Given a sequence of values of type `RootsOfQ`, compute a sequence containing only the single roots. Example test:

```
def singleRoots(rs: Seq[RootsOfQ]): Seq[Double] = ???
```

```
scala> singleRoots(Seq(TwoRoots(-1, 1), OneRoot(3.0), OneRoot(1.0), NoRoots()))
res5: Seq[Double] = List(3.0, 1.0)
```

Solution We apply `.filter` and `.map` to the sequence of roots:

```
def singleRoots(rs: Seq[RootsOfQ]): Seq[Double] = rs.filter {
  case OneRoot(x) => true
  case _          => false
}.map { case OneRoot(x) => x }
```

In the `.map` operation, we need to cover only the one-root case because the other two possibilities have been “filtered out” by the preceding `.filter` operation.

Example 3.2.2.3 Implement binary search returning a `SearchResult`. We will modify the binary search implementation from Example 2.5.1.5(b) so that it returns a `NotFound` value when appropriate.

Solution The code from Example 2.5.1.5(b) will return *some* index even if the given number is not present in the array:

```
scala> binSearch(Array(1, 3, 5, 7), goal = 5)
res6: Int = 2

scala> binSearch(Array(1, 3, 5, 7), goal = 4)
res7: Int = 1
```

When the number is not present, the array’s element at the computed index will not be equal to `goal`. We should return `NotFound()` in that case. The new code can be written as a `match` / `case` expression for clarity:

```
def safeBinSearch(xs: Seq[Int], goal: Int): SearchResult =
  binSearch(xs, goal) match {
    case n if xs(n) == goal => Index(n)
    case _                  => NotFound()
  }
```

To test:

```
scala> safeBinSearch(Array(1, 3, 5, 7), 5)
res8: SearchResult = Index(2)

scala> safeBinSearch(Array(1, 3, 5, 7), 4)
res9: SearchResult = NotFound()
```

Example 3.2.2.4 Use the disjunctive type `Result[Int]` to implement “safe integer arithmetic”, where a division by zero or a square root of a negative number will give an error message. Define arithmetic operations directly for values of type `Result[Int]`. When errors occur, abandon further computations.

Solution Begin by implementing the square root:

```
def sqrt(r: Result[Int]): Result[Int] = r match {
  case Value(x) if x >= 0 => Value(math.sqrt(x).toInt)
  case Value(x)          => Error(s"error: sqrt($x)")
  case Error(m)          => Error(m) // Keep the error message.
}
```

The square root is computed only if we have the `Value(x)` case, and only if $x \geq 0$. If the argument `r` was already an `Error` case, we keep the error message and perform no further computations.

To implement the addition operation, we need a bit more work:

```
def add(rx: Result[Int], ry: Result[Int]): Result[Int] = (rx, ry) match {
  case (Value(x), Value(y)) => Value(x + y)
  case (Error(m), _)       => Error(m) // Keep the error message.
  case (_, Error(m))       => Error(m)
}
```

This code illustrates nested patterns that match the tuple `(rx, ry)` against various possibilities. In this way, the code is clearer than code written with nested `if / else` expressions.

Implementing the multiplication operation results in almost the same code:

```
def mul(rx: Result[Int], ry: Result[Int]): Result[Int] = (rx, ry) match {
  case (Value(x), Value(y)) => Value(x * y)
  case (Error(m), _)       => Error(m)
  case (_, Error(m))       => Error(m)
}
```

To avoid repetition, we may define a general function that “lifts” operations on integers to operations on `Result[Int]` types:

```
def do2(rx: Result[Int], ry: Result[Int])(op: (Int, Int) => Int): Result[Int] =
  (rx, ry) match {
    case (Value(x), Value(y)) => Value(op(x, y))
    case (Error(m), _)       => Error(m)
    case (_, Error(m))       => Error(m)
  }
```

Now we can easily “lift” any binary operation on integers to a binary operation on `Result[Int]`, assuming that the operation never generates an error:

```
def sub(rx: Result[Int], ry: Result[Int]): Result[Int] = do2(rx, ry){ (x, y) => x - y }
```

Custom code is still needed for operations that *may* generate errors:

```
def div(rx: Result[Int], ry: Result[Int]): Result[Int] = (rx, ry) match {
  case (Value(x), Value(y)) if y != 0 => Value(x / y)
  case (Value(x), Value(y))          => Error(s"error: $x / $y")
  case (Error(m), _)                 => Error(m)
  case (_, Error(m))                 => Error(m)
}
```

We can now test the new “safe arithmetic” on simple calculations:

```
scala> add(Value(1), Value(2))
res10: Result[Int] = Value(3)

scala> div(add(Value(1), Value(2)), Value(0))
res11: Result[Int] = Error(error: 3 / 0)
```

We see that indeed all further computations are abandoned once an error occurs. An error message shows only the immediate calculation that generated the error. For instance, the error message for $20 + 1/0$ never mentions 20:

```
scala> add(Value(20), div(Value(1), Value(0)))
res12: Result[Int] = Error(error: 1 / 0)

scala> add(sqrt(Value(-1)), Value(10))
res13: Result[Int] = Error(error: sqrt(-1))
```

3.2.3 Standard disjunctive types: Option, Either, Try

The Scala library defines the disjunctive types `Option`, `Either`, and `Try` because they are used often. We now look at each of them in turn.

The Option type is a disjunctive type with two cases: the empty tuple and a one-element tuple. The names of the two case classes are `None` and `Some`. If the `Option` type were not already defined in the standard library, one could define it with the code

```
sealed trait Option[T]
final case object None extends Option[Nothing]
final case class Some[T](t: T) extends Option[T]
```


This code is similar to the type `SearchResult` defined in Section 3.2.1, except that `Option` has a type parameter instead of a fixed type `Int`. Another difference is the use of a `case object` for the empty case instead of an empty case class, such as `None()`. Since Scala's `case objects` cannot have type parameters, the type parameter in the definition of `None` must be set to the special type `Nothing`, which is a type with *no* values (also called the **void type**).

An alternative (implemented in libraries such as `scalaz`) is to define the empty option value as

```
final case class None[T]() extends Option[T]
```

In that implementation, the empty option `None[T]()` has a type parameter.

Several consequences follow from the Scala library's decision to define `None` without a type parameter. One consequence is that `None` can be reused as a value of type `Option[A]` for any type `A`:

```
scala> val y: Option[Int] = None
y: Option[Int] = None

scala> val z: Option[String] = None
z: Option[String] = None
```

Typically, `Option` is used in situations where a value may be either present or missing, especially when a missing value is *not an error*. The missing-value case is represented by `None`, while `Some(x)` means that a value `x` is present.

Example 3.2.3.1 Suppose that information about subscribers to a certain online service must contain a name and an email address, but a telephone number is optional. To represent this information, we may define a case class like this,

```
case class Subscriber(name: String, email: String, phone: Option[Long])
```

What if we represent the missing telephone number by a special value such as `-1` and use the simpler type `Long` instead of `Option[Long]`? The disadvantage is that we would need to *remember* to check for the special value `-1` in all functions that take the telephone number as an argument. Looking at a function such as `sendSMS(phone: Long)` at a different place in the code, a programmer might forget that the telephone number is actually optional. In contrast, the type signature `sendSMS(phone: Option[Long])` unambiguously indicates that the telephone number might be missing and helps the programmer to remember to handle both cases.

Pattern-matching code involving `Option` can handle the two cases like this:

```
def getDigits(phone: Option[Long]): Option[Seq[Long]] = phone match {
  case None           => None           // Do nothing.
  case Some(number)   => Some(digitsOf(number))
}
```

Here we used the function `digitsOf` defined in Section 2.3.

At the two sides of `case None => None`, the value `None` has different types, namely `Option[Long]` and `Option[Seq[Long]]`. Since these types are declared in the type signature of the function `getDigits`, the Scala compiler is able to figure out the types of all expressions in the `match / case` construction. So, pattern-matching code can be written without explicit type annotations such as `(None: Option[Long])`.

If we now need to compute the number of digits, we can write

```
def numberOfDigits(phone: Option[Long]): Option[Long] = getDigits(phone) match {
  case None           => None           // Do nothing.
  case Some(digits)   => Some(digits.length)
}
```

These examples perform a computation when an `Option` value is non-empty, and leave it empty otherwise. This design pattern is used often. To avoid repeating the code, we can implement this design pattern as a function that takes the computation as an argument `f`:

```
def doComputation(x: Option[Long], f: Long => Long): Option[Long] = x match {
  case None           => None           // Do nothing.
  case Some(i)        => Some(f(i))
}
```



```
}

```

It is then natural to generalize this function to arbitrary types using type parameters instead of a fixed type `Long`. The resulting function is usually called `fmap`:

```
def fmap[A, B](f: A => B): Option[A] => Option[B] = {
  case None           => None           // Do nothing.
  case Some(a)        => Some(f(a))
}

scala> fmap(digitsOf)(Some(4096))
res0: Option[Seq[Long]] = Some(List(4, 0, 9, 6))

scala> fmap(digitsOf)(None)
res1: Option[Seq[Long]] = None

```

We say that the `fmap` operation **lifts** a given function of type `A => B` to the type `Option[A] => Option[B]`.

It is important to keep in mind that the code `case Some(a) => Some(f(a))` changes the type of the option value. On the left side of the arrow, the type is `Option[A]`, while on the right side it is `Option[B]`. The Scala compiler knows this from the given type signature of `fmap`, so an explicit type parameter, `Some[B](f(a))`, is not needed.

The Scala library implements an equivalent function as a method on the `Option` class, with the syntax `x.map(f)` rather than `fmap(f)(x)`. We can concisely rewrite the previous code using the standard library methods as

```
def getDigits(phone: Option[Long]): Option[Seq[Long]] = phone.map(digitsOf)
def numberOfDigits(phone: Option[Long]): Option[Long] = phone.map(digitsOf).map(_.length)

```

We see that the `.map` operation for the `Option` type is analogous to the `.map` operation for sequences.

The similarity between `Option[A]` and `Seq[A]` is clearer if we view `Option[A]` as a special kind of “sequence” whose length is restricted to be either 0 or 1. So, `Option[A]` can have all the operations of `Seq[A]` except operations such as `.concat` that may grow the sequence beyond length 1. The standard operations defined on `Option` include `.map`, `.filter`, `.forall`, `.exists`, `.flatMap`, and `.foldLeft`.

Example 3.2.3.2 Given a phone number as `Option[Long]`, extract the country code if it is present. (Assume that the country code is any digits in front of the 10-digit number; for the phone number 18004151212, the country code is 1.) The result must be again of type `Option[Long]`.

Solution If the phone number is a positive integer n , we may compute the country code simply as $n / 10000000000L$. However, if the result of that division is zero, we should return an empty `Option` (i.e. the value `None`) rather than 0. To implement this logic, we may begin by writing this code,

```
def countryCode(phone: Option[Long]): Option[Long] = phone match {
  case None      => None
  case Some(n)   =>
    val countryCode = n / 10000000000L
    if (countryCode != 0L) Some(countryCode) else None
}

```

We may notice that we have reimplemented the design pattern similar to `.map` in this code, namely “if `None` then return `None`, else do a computation”. So we may try to rewrite the code as

```
def countryCode(phone: Option[Long]): Option[Long] = phone.map { n =>
  val countryCode = n / 10000000000L
  if (countryCode != 0L) Some(countryCode) else None
} // Type error: the result is Option[Option[Long]], not Option[Long].

```

This code does not compile: we are returning an `Option[Long]` within a function lifted via `.map`, so the resulting type is `Option[Option[Long]]`. We may use `.flatten` to convert `Option[Option[Long]]` to the required type `Option[Long]`,

```
def countryCode(phone: Option[Long]): Option[Long] = phone.map { n =>
  val countryCode = n / 10000000000L
  if (countryCode != 0L) Some(countryCode) else None
}.flatten

```

```
}.flatten // Types are correct now.
```

Since the `.flatten` follows a `.map`, we can rewrite the code using `.flatMap`:

```
def countryCode(phone: Option[Long]): Option[Long] = phone.flatMap { n =>
  val countryCode = n / 10000000000L
  if (countryCode != 0L) Some(countryCode) else None
} // Types are correct now.
```

Another way of implementing this example is to notice the design pattern “if condition does not hold, return `None`, otherwise keep the value”. For an `Option` type, this is equivalent to the `.filter` operation (recall that `.filter` returns an empty sequence if the predicate never holds). The code is

```
def countryCode(phone: Option[Long]): Option[Long] = phone.map(_ / 10000000000L).filter(_ != 0L)
```

Test it:

```
scala> countryCode(Some(18004151212L))
res0: Option[Long] = Some(1)

scala> countryCode(Some(8004151212L))
res1: Option[Long] = None
```

Example 3.2.3.3 Add a new requirement to Example 3.2.3.2: if the country code is not present, we should return the default country code 1.

Solution This is an often used design pattern: “if empty, substitute a default value”. The Scala library has the method `.getOrElse` for this purpose:

```
scala> Some(100).getOrElse(1)
res2: Int = 100

scala> None.getOrElse(1)
res3: Int = 1
```

So we can implement the new requirement as

```
scala> countryCode(Some(8004151212L)).getOrElse(1L)
res4: Long = 1
```

Using Option with collections Many Scala library methods return an `Option` as a result. The main examples are `.find`, `.headOption`, and `.lift` for sequences, and `.get` for dictionaries.

The `.find` method returns the first element satisfying a predicate:

```
scala> (1 to 10).find(_ > 5)
res0: Option[Int] = Some(6)

scala> (1 to 10).find(_ > 10) // No element is > 10.
res1: Option[Int] = None
```

The `.lift` method returns the element of a sequence at a given index:

```
scala> (10 to 100).lift(0)
res2: Option[Int] = Some(10)

scala> (10 to 100).lift(1000) // No element at index 1000.
res3: Option[Int] = None
```

The `.headOption` method returns the first element of a sequence, unless the sequence is empty. This is equivalent to `.lift(0)`:

```
scala> Seq(1,2,3).headOption
res4: Option[Int] = Some(1)

scala> Seq(1,2,3).filter(_ > 10).headOption
res5: Option[Int] = None
```

Applying `.find(p)` computes the same result as `.filter(p).headOption`, but `.find(p)` may be faster.

The `.get` method for a dictionary returns the value if it exists for a given key, and returns `None` if the key is not in the dictionary:

```
scala> Map(10 -> "a", 20 -> "b").get(10)
res6: Option[String] = Some(a)

scala> Map(10 -> "a", 20 -> "b").get(30)
res7: Option[String] = None
```

The `.get` method provides safe by-key access to dictionaries, unlike the direct access method that may fail:

```
scala> Map(10 -> "a", 20 -> "b")(10)
res8: String = a

scala> Map(10 -> "a", 20 -> "b")(30)
java.util.NoSuchElementException: key not found: 30
    at scala.collection.MapLike$class.default(MapLike.scala:228)
    at scala.collection.AbstractMap.default(Map.scala:59)
    ... 32 elided
```

Similarly, `.lift` provides safe by-index access to collections, unlike the direct access that may fail:

```
scala> Seq(10,20,30)(0)
res9: Int = 10

scala> Seq(10,20,30)(5)
java.lang.IndexOutOfBoundsException: 5
    at scala.collection.LinearSeqOptimized$class.apply(LinearSeqOptimized.scala:65)
    at scala.collection.immutable.List.apply(List.scala:84)
    ... 32 elided
```

The Either type The standard disjunctive type `Either[A, B]` has two type parameters and is often used for computations that report errors. By convention, the *first* type (A) is the type of error, and the *second* type (B) is the type of the (non-error) result. The names of the two cases are `Left` and `Right`. A possible definition of `Either` may be written as

```
sealed trait Either[A, B]
final case class Left[A, B](value: A) extends Either[A, B]
final case class Right[A, B](value: B) extends Either[A, B]
```

By convention, a value `Left(x)` represents an error, and a value `Right(y)` represents a valid result.

As an example, the following function substitutes a default value and logs the error information:

```
def logError(x: Either[String, Int], default: Int): Int = x match {
  case Left(error) => println(s"Got error: $error"); default
  case Right(res)  => res
}
```

To test:

```
scala> logError(Right(123), -1)
res1: Int = 123

scala> logError(Left("bad result"), -1)
Got error: bad result
res2: Int = -1
```

Why use `Either` instead of `Option` for computations that may fail? A failing computation such as `"xyz".toInt` cannot return a result, and sometimes we might use `None` to indicate that a result is not available. However, when the result is a requirement for further calculations, we will usually need to know exactly *which* error prevented the result from being available. The `Either` type may provide detailed information about such errors, which `Option` cannot do.

The `Either` type generalizes the type `Result` defined in Section 3.2.1 to an arbitrary error type instead

of `String`. We have seen its usage in Example 3.2.2.4, where the design pattern was “if value is present, do a computation, otherwise keep the error”. This design pattern is implemented by the `.map` method on `Either`:

```
1 scala> Right(1).map(_ + 1)
2 res0: Either[Nothing, Int] = Right(2)
3
4 scala> Left[String, Int]("error").map(_ + 1)
5 res1: Either[String, Int] = Left("error")
```

The type `Nothing` was filled in by the Scala compiler because we did not specify the first type parameter of `Right` in line 1.

The methods `.filter`, `.flatMap`, `.fold`, and `.getOrElse` are also defined for `Either`, with the same convention that a `Left` value represents an error.¹

Exceptions and the Try type When computations fail for any reason, Scala generates an **exception** instead of returning a value. An exception means that the evaluation of some expression was stopped without returning a result.

As an example, exceptions are generated when the available memory is too small to store the resulting data (as we saw in Section 2.6.3), or if a stack overflow occurs during the computation (as we saw in Section 2.2.3). Exceptions may also occur due to programmer’s error: when a pattern matching operation fails, when a requested key does not exist in a dictionary, or when the `.head` operation is applied to an empty list.

Motivated by these examples, we may distinguish “planned” and “unplanned” exceptions.

A **planned** exception is generated by programmer’s code via the `throw` syntax:

```
scala> throw new Exception("This is a test... this is only a test.")
java.lang.Exception: This is a test... this is only a test.
```

The Scala library contains a `throw` operation in various places, such as in the code for applying the `.head` method to an empty sequence, as well as in other situations where exceptions are generated due to programmer’s errors. These exceptions are generated deliberately and in well-defined situations. Although these exceptions indicate errors, these errors are anticipated in advance and so may be handled by the programmer.

For example, many Java libraries will generate exceptions when function arguments have unexpected values, when a network operation takes too long or a network connection is unexpectedly broken, when a file is not found or cannot be read due to access permissions, and in many other situations. All these exceptions are “planned” because they are generated explicitly by library code such as `throw new FileNotFoundException(...)`. The programmer’s code is expected to catch these exceptions, to handle the error, and to continue the evaluation of the program.

An **unplanned** exception is generated by the Java runtime system when critical errors occur, such as an out-of-memory error. It is rare that a programmer writes `val y = f(x)` while *expecting* that an out-of-memory exception will sometimes occur at that point.² An unplanned exception indicates a serious and unforeseen problem with memory or another critically important resource, such as the operating system’s threads or file handles. Such problems usually cannot be fixed and will prevent the program from running any further. It is reasonable that the program evaluation should abruptly stop (or “crash” as programmers say) after such an error.

The use of planned exceptions assumes that the programmer will write code to handle each exception. This assumption makes it significantly harder to write programs correctly: it is hard to figure out and to keep in mind all the possible exceptions that a given library function may *throw* in its code and in the code of all other libraries on which it depends. Instead of using exceptions for indicating errors, Scala programmers can write functions that return a disjunctive type such as `Either`, describing both possibilities (a correct result or an error condition). Users of these functions will *have* to do pattern matching on the result values. This helps programmers to remember and to

¹These methods are available in Scala 2.12 or a later version.

²Just once in the author’s experience, an out-of-memory condition had to be anticipated in an Android app.

handle all relevant error situations.

However, programmers will often need to use Java or Scala libraries that `throw` exceptions. To help write code for these situations, the Scala library contains a helper function called `Try()` and a disjunctive type also called `Try`. The type `Try[A]` can be seen as similar to `Either[Throwable, A]`, where `Throwable` is the general type of all exceptions (i.e. values to which a `throw` operation can be applied). The two parts of the disjunctive type `Try[A]` are called `Failure` and `Success[A]` (instead of `Left[Throwable]` and `Right[A]` in the `Either` type). The function `Try(expr)` will catch all exceptions thrown while the expression `expr` is evaluated. If the evaluation of `expr` succeeds and returns a value `x:A`, the value of `Try(expr)` will be `Success(x)`. Otherwise it will be `Failure(t)`, where `t:Throwable` is the value associated with the generated exception. Here is an example of using `Try`:

```
import scala.util.{Try, Success, Failure}

scala> Try("xyz".toInt)
res0: Try[Int] = Failure(java.lang.NumberFormatException: For input string: "xyz")

scala> Try("0002".toInt)
res1: Try[Int] = Success(2)
```

The code `Try("xyz".toInt)` does not generate any exceptions and will not crash the program. Any computation that may throw an exception can be enclosed in a `Try()`, and the exception will be caught and encapsulated within the disjunctive type as a `Failure(...)` value.

The methods `.map`, `.filter`, `.flatMap`, `.foldLeft` are defined for the `Try` class similarly to the `Either` type. One additional feature of `Try` is to catch exceptions generated by the function arguments of `.map`, `.filter`, `.flatMap`, and other standard methods:

```
scala> val y = x.map(y => throw new Exception("ouch"))
y: Try[Int] = Failure(java.lang.Exception: ouch)

scala> val z = x.filter(y => throw new Exception("huh"))
z: Try[Int] = Failure(java.lang.Exception: huh)
```

In this example, the values `y` and `z` were computed *successfully* even though exceptions were thrown while the function arguments of `.map` and `.filter` were evaluated. Other code can use pattern

matching on the values `y` and `z` and examine those exceptions. However, it is important that these exceptions were caught and the program did not crash, so the other code is *able* to run.

While the standard types `Try` and `Either` will cover many use cases, programmers can also define custom disjunctive types in order to represent all the anticipated failures or errors in the business logic of a particular application. Representing all errors in the types helps assure that the program will not crash because of an exception that we forgot to handle or did not even know about.

3.3 Lists and trees: recursive disjunctive types

Consider this code defining a disjunctive type `NInt`:

```
sealed trait NInt
final case class N1(x: Int) extends NInt
final case class N2(n: NInt) extends NInt
```

The type `NInt` has two disjunctive parts, `N1` and `N2`. But the definition of the case class `N1` refers to the type `NInt` as if it were already defined.

A type whose definition uses that same type is called a **recursive type**. The type `NInt` is an example of a recursive disjunctive type.

We might imagine defining a disjunctive type `x` whose parts recursively refer to the same type `x` (and/or to each other) in complicated ways. What kind of data would be represented by such a type `x`, and in what situations would `x` be useful? In general, this question is not easy to answer. For instance, the simple definition

```
final case class Bad(x: Bad)
```

is useless: we can create a value of type `Bad` only if we already have a value of type `Bad`. This is an example of an infinite type recursion. We will never be able to create any values of type `Bad`, which

means that the type `Bad` is void (has no values, like the special type `Nothing`).

Chapter 14 studies recursive types in more detail. For now, we will look at the main examples of recursive disjunctive types that are *known* to be useful. These examples are lists and trees.

3.3.1 Lists

A list of values of type `A` is either empty, or has one value of type `A`, or two values of type `A`, etc. We can visualize the type `List[A]` as a disjunctive type defined by

```
sealed trait List[A]
final case class List0[A]()           extends List[A]
final case class List1[A](x: A)       extends List[A]
final case class List2[A](x1: A, x2: A) extends List[A]
??? // Need an infinitely long definition.
```

However, this definition is not practical: we cannot define a separate case class for *each* possible length. Instead, we define the type `List[A]` via mathematical induction on the length of the list:

- Base case: empty list, `case class List0[A]()`.
- Inductive step: given a list of a previously defined length, say `Listn-1`, define a new case class `Listn` describing a list with one more element of type `A`. So we could define `Listn = (Listn-1, A)`.

Let us try to write this inductive definition as code:

```
sealed trait ListI[A] // Inductive definition of a list.
final case class List0[A]()           extends ListI[A]
final case class List1[A](prev: List0[A], x: A) extends ListI[A]
final case class List2[A](prev: List1[A], x: A) extends ListI[A]
??? // Still need an infinitely long definition.
```

To avoid writing an infinitely long type definition, we need to use a trick. Notice that all definitions of `List1`, `List2`, etc., have a similar form (while `List0` is not similar). We can replace all the definitions `List1`, `List2`, etc., by a single definition if we use the type `ListI[A]` recursively inside the case class:

```
sealed trait ListI[A] // Inductive definition of a list.
final case class List0[A]()           extends ListI[A]
final case class ListN[A](prev: ListI[A], x: A) extends ListI[A]
```

The type definition has become recursive. For this trick to work, it is important to use `ListI[A]` and not `ListN[A]` inside the definition `ListN[A]`; or else we would have created an infinite type recursion similar to `case class Bad` shown previously.

Since we obtained the type definition of `ListI` via a trick, let us verify that the code actually defines the disjunctive type we wanted.

To create a value of type `ListI[A]`, we must use one of the two available case classes. Using the first case class, we may create a value `List0()`. Since this empty case class does not contain any values of type `A`, it effectively represents an empty list (the base case of the induction). Using the second case class, we may create a value `ListN(prev, x)` where `x` is of type `A` and `prev` is some previously constructed value of type `ListI[A]`. This represents the induction step, because the case class `ListN` is a named tuple containing `ListI[A]` and `A`. Now, the same consideration recursively applies to constructing the value `prev`, which must be either an empty list or a pair containing another list and an element of type `A`. The assumption that the value `prev: ListI[A]` is already constructed is equivalent to the inductive assumption that we already have a list of a previously defined length. So, we have verified that `ListI[A]` implements the inductive definition shown above.

Examples of values of type `ListI` are the empty list `List0()`, a one-element list `ListN(List0(), x)`, and a two-element list `ListN(ListN(List0(), x), y)`.

To illustrate writing pattern-matching code using this type, let us implement the method `headOption`:

```
@tailrec def headOption[A]: ListI[A] => Option[A] = {
  case List0() => None
```

```

case ListN(List0(), x)    => Some(x)
case ListN(prev, _)      => headOption(prev)
}

```

The Scala library already defines the type `List[A]` in an equivalent but different way: its case classes are named differently, and the second case class (with the special name `::`) places the value of type `A` *before* the previously constructed list,

```

sealed trait List[A]
final case object Nil extends List[Nothing]
final case class ::[A](head: A, tail: List[A]) extends List[A]

```

Because “operator-like” case class names, such as `::`, support the infix syntax, we may write `head :: tail` instead of `::(head, tail)`. Pattern matching with the standard `List` class looks like this:

```

def headOption[A]: List[A] => Option[A] = {
  case Nil          => None
  case head :: tail => Some(head)
}

```

Examples of values created using Scala’s standard `List` type are the empty list `Nil`, a one-element list `x :: Nil`, and a two-element list `x :: y :: Nil`. The same syntax `x :: y :: Nil` is used both for creating values of type `List` and for pattern-matching on such values.

The Scala library also defines the helper function `List()`, so that `List()` is the same as `Nil` and `List(1, 2, 3)` is the same as `1 :: 2 :: 3 :: Nil`. Lists are easier to use in the syntax `List(1, 2, 3)`. Pattern matching can also use that syntax when convenient:

```

val x: List[Int] = List(1, 2, 3)

x match {
  case List(a)          => ...
  case List(a, b, c)    => ...
  case _                => ...
}

```

3.3.2 Tail recursion with `List`

Because the `List` type is defined by induction, it is straightforward to implement iterative computations with the `List` type using recursion.

A first example is the `map` function. We use reasoning by induction in order to figure out the implementation of `map`. The required type signature is

```

def map[A, B](xs: List[A])(f: A => B): List[B] = ???

```

The base case is an empty list, and we return again an empty list:

```

def map[A, B](xs: List[A])(f: A => B): List[B] = xs match {
  case Nil => Nil
  ...
}

```

In the induction step, we have a pair `(head, tail)` in the case class `::`, with `head:A` and `tail:List[A]`. The pair can be pattern-matched with the syntax `head :: tail`. The `map` function should apply the argument `f` to the head value, which will give the first element of the resulting list. The remaining elements are computed by the induction assumption, i.e. by a recursive call to `map`:

```

def map[A, B](xs: List[A])(f: A => B): List[B] = xs match {
  case Nil          => Nil
  case head :: tail => f(head) :: map(tail)(f) // Not tail-recursive.
}

```

While this implementation is straightforward and concise, it is not tail-recursive. This will be a problem for large enough lists.

Instead of implementing the often-used methods such as `.map` or `.filter` one by one, let us implement `foldLeft` because most of the other methods can be expressed via `foldLeft`.

The required type signature is

```
def foldLeft[A, R](xs: List[A])(init: R)(f: (R, A) => R): R = ???
```

Reasoning by induction, we start with the base case `xs == Nil`, where the only possibility is to return the value `init`:

```
def foldLeft[A, R](xs: List[A])(init: R)(f: (R, A) => R): R = xs match {
  case Nil          => init
  ...
```

The induction step for `foldLeft` says that, given the values `head:A` and `tail:List[A]`, we need to apply the updater function to the previous accumulator value. That value is `init`. So we apply `foldLeft` recursively to the tail of the list once we have the updated accumulator value:

```
@tailrec def foldLeft[A, R](xs: List[A])(init: R)(f: (R, A) => R): R =
  xs match {
    case Nil          => init
    case head :: tail =>
      val newInit = f(init, head) // Update the accumulator.
      foldLeft(tail)(newInit)(f)  // Recursive call to 'foldLeft'.
  }
```

This implementation is tail-recursive because the recursive call to `foldLeft` is the last expression returned in a `case` branch.

Another example is a function for reversing a list. The Scala library defines the `.reverse` method for this task, but we will show an implementation using `foldLeft`. The updater function *prepends* an element to a previous list:

```
def reverse[A](xs: List[A]): List[A] =
  xs.foldLeft(Nil: List[A])((prev, x) => x :: prev)

scala> reverse(List(1, 2, 3))
res0: List[Int] = List(3, 2, 1)
```

Without the explicit type annotation `Nil:List[A]`, the Scala compiler will decide that `Nil` has type `List[Nothing]`, and the types will not match later in the code. In Scala, one often finds that the initial value for `.foldLeft` needs an explicit type annotation.

The `reverse` function can be used to obtain a tail-recursive implementation of `map` for `List`. The idea is to first use `foldLeft` to accumulate transformed elements:

```
scala> Seq(1, 2, 3).foldLeft(Nil:List[Int])((prev, x) => x*x :: prev)
res0: List[Int] = List(9, 4, 1)
```

The result is a reversed `.map(x => x*x)`, so we need to apply `.reverse`:

```
def map[A, B](xs: List[A])(f: A => B): List[B] =
  xs.foldLeft(Nil: List[B])((prev, x) => f(x) :: prev).reverse

scala> map(List(1, 2, 3))(x => x*x)
res2: List[Int] = List(1, 4, 9)
```

This achieves stack safety at the cost of traversing the list twice. (This implementation is shown only as an example. The Scala library implements `.map` for `List` using low-level tricks in order to achieve better performance.)

Example 3.3.2.1 A definition of the **non-empty list** is similar to `List` except that the empty-list case is replaced by a 1-element case:

```
sealed trait NEL[A]
final case class Last[A](head: A) extends NEL[A]
final case class More[A](head: A, tail: NEL[A]) extends NEL[A]
```


Values of a non-empty list look like this:

```
scala> val xs: NEL[Int] = More(1, More(2, Last(3))) // [1, 2, 3]
xs: NEL[Int] = More(1,More(2,Last(3)))

scala> val ys: NEL[String] = Last("abc") // One element, ["abc"].
ys: NEL[String] = Last(abc)
```

To create non-empty lists more easily, we implement a conversion function `toNEL` from an ordinary list. To guarantee that a non-empty list can be created, we give `toNEL` *two* arguments:

```
def toNEL[A](x: A, rest: List[A]): NEL[A] = rest match {
  case Nil      => Last(x)
  case y :: tail => More(x, toNEL(y, tail))
} // Not tail-recursive: 'toNEL()' is used inside 'More(...)'
```

To test:

```
scala> toNEL(1, List()) // Result = [1].
res0: NEL[Int] = Last(1)

scala> toNEL(1, List(2, 3)) // Result = [1, 2, 3].
res1: NEL[Int] = More(1,More(2,Last(3)))
```

The `head` method is safe for non-empty lists, unlike `.head` for an ordinary `List`:

```
def head[A]: NEL[A] => A = {
  case Last(x)      => x
  case More(x, _)   => x
}
```

We can also implement a tail-recursive `foldLeft` function for non-empty lists:

```
@tailrec def foldLeft[A, R](n: NEL[A])(init: R)(f: (R, A) => R): R = n match {
  case Last(x)      => f(init, x)
  case More(x, tail) => foldLeft(tail)(f(init, x))(f)
}

scala> foldLeft(More(1, More(2, Last(3))))(0)(_ + _)
res2: Int = 6
```

Example 3.3.2.2 Use `foldLeft` to implement a `reverse` function for the type `NEL`. The required type signature and a sample test:

```
def reverse[A]: NEL[A] => NEL[A] = ???

scala> reverse(toNEL(10, List(20, 30))) // Result must be [30, 20, 10].
res3: NEL[Int] = More(30,More(20,Last(10)))
```

Solution We will use `foldLeft` to build up the reversed list as the accumulator value. It remains to choose the initial value of the accumulator and the updater function. We have already seen the code for reversing the ordinary list via the `.foldLeft` method (Section 3.3.2),

```
def reverse[A](xs: List[A]): List[A] = xs.foldLeft(Nil: List[A])((prev, x) => x :: prev)
```

However, we cannot reuse the same code for non-empty lists by writing `More(x, prev)` instead of `x :: prev`, because the `foldLeft` operation works with non-empty lists differently. Since lists are always non-empty, the updater function is always applied to an initial value, and the code works incorrectly:

```
def reverse[A](xs: NEL[A]): NEL[A] =
  foldLeft(xs)(Last(head(xs)):NEL[A])((prev, x) => More(x, prev))

scala> reverse(toNEL(10, List(20, 30))) // Result = [30, 20, 10, 10].
res4: NEL[Int] = More(30,More(20,More(10,Last(10))))
```

The last element, 10, should not have been repeated. It was repeated because the initial accumulator value already contained the head element 10 of the original list. However, we cannot set the initial

accumulator value to an empty list, since a value of type `NEL[A]` must be non-empty. It seems that we need to handle the case of a one-element list separately. So we begin by matching on the argument of `reverse`, and apply `foldLeft` only when the list is longer than 1 element:

```
def reverse[A]: NEL[A] => NEL[A] = {
  case Last(x)      => Last(x)      // 'reverse' is trivial.
  case More(x, tail) =>             // Use foldLeft on 'tail'.
    foldLeft(tail)(Last(x):NEL[A])((prev,x) => More(x, prev))
}

scala> reverse(toNEL(10, List(20, 30))) // Result = [30, 20, 10].
res5: NEL[Int] = More(30,More(20,Last(10)))
```

Exercise 3.3.2.3 Implement a function `toList` that converts a non-empty list into an ordinary Scala List. The required type signature and a sample test:

```
def toList[A](nel: NEL[A]): List[A] = ???

scala> toList(More(1, More(2, Last(3)))) // This is [1, 2, 3].
res6: List[Int] = List(1, 2, 3)
```

Exercise 3.3.2.4 Implement a `map` function for the type `NEL`. Type signature and a sample test:

```
def map[A,B](xs: NEL[A])(f: A => B): NEL[B] = ???

scala> map[Int, Int](toNEL(10, List(20, 30)))(_ + 5) // Result = [15, 25, 35].
res7: NEL[Int] = More(15,More(25,Last(35)))
```

Exercise 3.3.2.5 Implement a function `concat` that concatenates two non-empty lists:

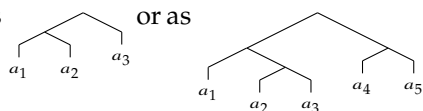
```
def concat[A](xs: NEL[A], ys: NEL[A]): NEL[A] = ???

scala> concat(More(1, More(2, Last(3))), More(4, Last(5))) // Result is [1, 2, 3, 4, 5].
res8: NEL[Int] = More(1,More(2,More(3,More(4,Last(5)))))
```

3.3.3 Binary trees

We will consider four kinds of trees defined as recursive disjunctive types: binary trees, rose trees, regular-shaped trees, and abstract syntax trees.

Examples of a **binary tree** with leaves of type `A` can be drawn as

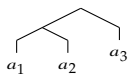


where a_i are some values of type `A`.

An inductive definition says that a binary tree is either a leaf with a value of type `A` or a branch containing *two* previously defined binary trees. Translating this definition into code, we get

```
sealed trait Tree2[A]
final case class Leaf[A](a: A) extends Tree2[A]
final case class Branch[A](x: Tree2[A], y: Tree2[A]) extends Tree2[A]
```

With this definition, the tree

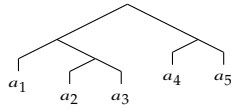


```
Branch(Branch(Leaf("a1"), Leaf("a2")), Leaf("a3"))
```

while the expression

```
Branch(Branch(Leaf("a1"), Branch(Leaf("a2"), Leaf("a3"))), Branch(Leaf("a4"), Leaf("a5")))
```

creates the tree



Recursive functions on trees are translated into concise code. For instance, the function `foldLeft` for trees of type `Tree2` is implemented as

```
def foldLeft[A, R](t: Tree2[A])(init: R)(f: (R, A) => R): R = t match {
  case Leaf(a)      => f(init, a)
  case Branch(t1, t2) =>
    val r1 = foldLeft(t1)(init)(f) // Fold the left branch.
    foldLeft(t2)(r1)(f) // Starting from 'r1', fold the right branch.
}
```

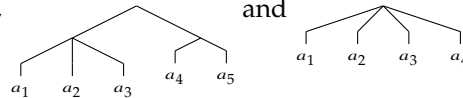
Note that this function *cannot* be made tail-recursive using the accumulator trick, because `foldLeft` needs to call itself twice in the `Branch` case. To test:

```
val t: Tree2[String] = Branch(Branch(Leaf("a1"), Leaf("a2")), Leaf("a3"))

scala> foldLeft(t)("")(_ + " " + _)
res0: String = " a1 a2 a3"
```

3.3.4 Rose trees

A **rose tree** is similar to the binary tree except the branches contain a non-empty list of trees. Because of that, a rose tree can fork into arbitrarily many branches at each node, rather than always into two branches as the binary tree does; for example,



A possible definition of a data type for the rose tree is

```
sealed trait TreeN[A]
final case class Leaf[A](a: A) extends TreeN[A]
final case class Branch[A](ts: NEL[TreeN[A]]) extends TreeN[A]
```

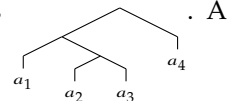
Exercise 3.3.4.1 Define the function `foldLeft` for a rose tree, using `foldLeft` for the type `NEL`. Type signature and a test:

```
def foldLeft[A, R](t: TreeN[A])(init: R)(f: (R, A) => R): R = ???

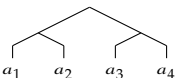
scala> foldLeft(Branch(More(Leaf(1), More(Leaf(2), Last(Leaf(3))))) (0) (_ + _))
res0: Int = 6
```

3.3.5 Regular-shaped trees

Binary trees and rose trees may choose to branch or not to branch at any given node, resulting in structures that may have different branching depths at different nodes, such as



regular-shaped tree always branches in the same way at every node until a chosen total depth, e.g.



, where all nodes at depth 0 and 1 always branch into two, while nodes at depth 2 do not branch. The branching number is fixed for a given type of a regular-shaped tree; in this example, the branching number is 2, so it is a regular-shaped *binary* tree.

How can we define a data type representing a regular-shaped binary tree? We need a tree that is either a single value, or a pair of values, or a pair of pairs, etc. Begin with the non-recursive (but, of course, impractical) definition

```
sealed trait RTree[A]
final case class Leaf[A](x: A) extends RTree[A]
final case class Branch1[A](xs: (A, A)) extends RTree[A]
final case class Branch2[A](xs: ((A, A), (A, A))) extends RTree[A]
??? // Need an infinitely long definition.
```

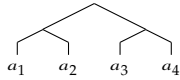
The case `Branch1` describes a regular-shaped tree with total depth 1, the case `Branch2` has total depth 2, and so on. Now, we cannot rewrite this definition as a recursive type because the case classes do not have the same structure. The non-trivial trick is to notice that each `Branchn` case class uses the previous case class's data structure *with the type parameter set to (A, A)* instead of `A`. So we can rewrite this definition as

```
sealed trait RTree[A]
final case class Leaf[A](x: A) extends RTree[A]
final case class Branch1[A](xs: Leaf[(A, A)]) extends RTree[A]
final case class Branch2[A](xs: Branch1[(A, A)]) extends RTree[A]
??? // Need an infinitely long definition.
```

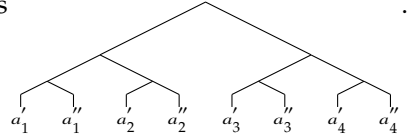
We can now apply the type recursion trick: replace the type `Branchn-1[(A, A)]` in the definition of `Branchn` by the type `RTree[(A, A)]`. This gives the type definition for a regular-shaped binary tree:

```
sealed trait RTree[A]
final case class Leaf[A](x: A) extends RTree[A]
final case class Branch[A](xs: RTree[(A, A)]) extends RTree[A]
```

Since we used some tricks to figure out the definition of `RTree[A]`, let us verify that this definition actually describes the recursive disjunctive type we wanted. The only way to create a structure of type `RTree[A]` is to create a `Leaf[A]` or a `Branch[A]`. A value of type `Leaf[A]` is a correct regularly-shaped tree. It remains to consider the case of `Branch[A]`. Creating a `Branch[A]` requires a previously created `RTree` with values of type `(A, A)` instead of `A`. By the inductive assumption, the previously created `RTree[A]` would have the correct shape. Now, it is clear that if we replace the type parameter `A` by the pair `(A, A)`, a regular-shaped tree such as



deeper, which can be drawn (replacing each a_i by a pair a'_i, a''_i) as



We see that `RTree[A]` is the correct definition of a regular-shaped binary tree.

Example 3.3.5.1 Define a (non-tail-recursive) `map` function for a regular-shaped binary tree. The required type signature and a test:

```
def map[A, B](t: RTree[A])(f: A => B): RTree[B] = ???

scala> map(Branch(Branch(Leaf(((1,2),(3,4))))))(_ * 10)
res0: RTree[Int] = Branch(Branch(Leaf(((10,20),(30,40)))))
```

Solution Begin by pattern-matching on the tree:

```
def map[A, B](t: RTree[A])(f: A => B): RTree[B] = t match {
  case Leaf(x)      => ???
  case Branch(xs)   => ???
}
```

In the base case, we have no choice but to return `Leaf(f(x))`.

```
def map[A, B](t: RTree[A])(f: A => B): RTree[B] = t match {
  case Leaf(x)      => Leaf(f(x))
  case Branch(xs)   => ???
}
```

In the inductive step, we are given a previous tree value `xs:RTree[(A, A)]`. It is clear that we need to apply `map` recursively to `xs`. Let us try:

```
def map[A, B](t: RTree[A])(f: A => B): RTree[B] = t match {
  case Leaf(x)      => Leaf(f(x))
  case Branch(xs)   => Branch(map(xs)(f))    // Type error!
}
```

Here, `map(xs)(f)` has an incorrect type of the function `f`. Since `xs` has type `RTree[(A, A)]`, the recursive call `map(xs)(f)` requires `f` to be of type `((A, A)) => (B, B)` instead of `A => B`.

So, we need to provide a function of the correct type instead of `f`. A function of type `((A, A)) => (B, B)` will be obtained out of `f: A => B` if we apply `f` to each part of the tuple `(A, A)`; the code for that function is `{ case (x, y) => (f(x), f(y)) }`. Therefore, we can implement `map` as

```
def map[A, B](t: RTree[A])(f: A => B): RTree[B] = t match {
  case Leaf(x)      => Leaf(f(x))
  case Branch(xs)   => Branch(map(xs){ case (x, y) => (f(x), f(y)) })
}
```

Exercise 3.3.5.2 Using tail recursion, compute the depth of a regular-shaped binary tree of type `RTree`. (An `RTree` of depth n has 2^n leaf values.) The required type signature and a test:

```
@tailrec def depth[A](t: RTree[A]): Int = ???

scala> depth(Branch(Branch(Leaf(((("a", "b"), ("c", "d"))))))
res2: Int = 2
```

Exercise 3.3.5.3* Define a tail-recursive function `foldLeft` for a regular-shaped binary tree. The required type signature and a test:

```
@tailrec def foldLeft[A, R](t: RTree[A])(init: R)(f: (R, A) => R): R = ???

scala> foldLeft(Branch(Branch(Leaf(((1,2),(3,4))))))(0)(_ + _)
res0: Int = 10

scala> foldLeft(Branch(Branch(Leaf(((("a", "b"), ("c", "d")))))))( "")(_ + _)
res1: String = abcd
```

3.3.6 Abstract syntax trees

Expressions in formal languages are represented by abstract syntax trees. An **abstract syntax tree** (or **AST** for short) is defined as either a leaf of one of the available leaf types, or a branch of one of the available branch types. All the available leaf and branch types must be specified as part of the definition of an AST. In other words, one must specify the data carried by leaves and branches, as well as the branching numbers.

To illustrate how ASTs are used, let us rewrite Example 3.2.2.4 via an AST. We view Example 3.2.2.4 as a small sub-language that deals with “safe integers” and supports the “safe arithmetic” operations `Sqrt`, `Add`, `Mul`, and `Div`. Example calculations in this sub-language are $\sqrt{16} * (1 + 2) = 12$; $20 + 1/0 = \text{error}$; and $10 + \sqrt{-1} = \text{error}$.

We can implement this sub-language in two stages. The first stage will create a data structure (an AST) that represents an unevaluated expression in the sub-language. The second stage will evaluate that AST to obtain either a number or an error message.

A straightforward way of defining a data structure for an AST is to use a disjunctive type whose cases describe all the possible operations of the sub-language. We will need one case class for each of `Sqrt`, `Add`, `Mul`, and `Div`. An additional operation, `Num`, will lift ordinary integers into “safe integers”. So, we define the disjunctive type for “arithmetic sub-language expressions” as

```
sealed trait Arith
final case class Num(x: Int) extends Arith
final case class Sqrt(x: Arith) extends Arith
final case class Add(x: Arith, y: Arith) extends Arith
```

```
final case class Mul(x: Arith, y: Arith) extends Arith
final case class Div(x: Arith, y: Arith) extends Arith
```

A value of type `Arith` is either a `Num(x)` for some integer `x`, or an `Add(x, y)` where `x` and `y` are previously defined `Arith` expressions, or another operation.

This type definition is similar to the binary tree type

```
sealed trait Tree
final case class Leaf(x: Int) extends Tree
final case class Branch(x: Tree, y: Tree) extends Tree
```

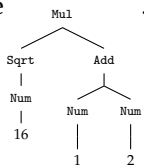
if we rename `Leaf` to `Num` and `Branch` to `Add`. However, the `Arith` type contains four different types of “branches”, some with branching number 1 and others with branching number 2.

This example illustrates the structure of an AST: it is a tree of a general shape, where leaves and branches are chosen from a specified set of allowed possibilities. In this example, we have a single allowed type of leaf (`Num`) and four allowed types of branches (`Sqrt`, `Add`, `Mul`, and `Div`).

This completes the first stage of implementing the sub-language of “safe arithmetic”. We may now use the disjunctive type `Arith` to create expressions in the sub-language. For example, $\sqrt{16} * (1 + 2)$ is represented by

```
scala> val x: Arith = Mul(Sqrt(Num(16)), Add(Num(1), Num(2)))
x: Arith = Mul(Sqrt(Num(16)), Add(Num(1), Num(2)))
```

We can visualize `x` as the abstract syntax tree



The expressions $20 + 1/0$ and $10 * \sqrt{-1}$ are represented by

```
scala> val y: Arith = Add(Num(20), Div(Num(1), Num(0)))
y: Arith = Add(Num(20), Div(Num(1), Num(0)))

scala> val z: Arith = Add(Num(10), Sqrt(Num(-1)))
z: Arith = Add(Num(10), Sqrt(Num(-1)))
```

As we see, the expressions `x`, `y`, and `z` *remain unevaluated*; each of them is a data structure that encodes a tree of operations of the sub-language. These operations will be evaluated at the second stage of implementing the sub-language.

To evaluate expressions in the “safe arithmetic”, we can implement a function with type signature `run: Arith => Either[String, Int]`. That function plays the role of an **interpreter** or “**runner**” for programs written in the sub-language. The runner will destructure the expression tree and execute all the operations, taking care of possible errors.

To implement `run`, we need to define required arithmetic operations on the type `Either[String, Int]`. For instance, we need to be able to add or multiply values of that type. Instead of custom code from Example 3.2.2.4, we can use the standard `.map` and `.flatMap` methods defined on `Either`. For example, addition and multiplication of two “safe integers” is implemented as

```
def add(x: Either[String, Int], y: Either[String, Int]):
  Either[String, Int] = x.flatMap { r1 => y.map(r2 => r1 + r2) }
def mul(x: Either[String, Int], y: Either[String, Int]):
  Either[String, Int] = x.flatMap { r1 => y.map(r2 => r1 * r2) }
```

while the “safe division” is

```
def div(x: Either[String, Int], y: Either[String, Int]):
  Either[String, Int] = x.flatMap { r1 => y.flatMap(r2 =>
    if (r2 == 0) Left(s"error: $r1 / $r2") else Right(r1 / r2) )
}
```

With this code, we can implement the runner as

```
def run: Arith => Either[String, Int] = {
  case Num(x)      => Right(x)
  case Sqrt(x)     => run(x).flatMap { r =>
    if (r < 0) Left(s"error: sqrt($r)") else Right(math.sqrt(r).toInt)
  }
  case Add(x, y)   => add(run(x), run(y))
  case Mul(x, y)   => mul(run(x), run(y))
  case Div(x, y)   => div(run(x), run(y))
}
```

Test it with the values x , y , z defined previously:

```
scala> run(x)
res0: Either[String, Int] = Right(12)

scala> run(y)
res1: Either[String, Int] = Left("error: 1 / 0")

scala> run(z)
res2: Either[String, Int] = Left("error: sqrt(-1)")
```

3.4 Summary

What problems can we solve now?

- Represent values from a disjoint domain by a custom-defined disjunctive type.
- Use disjunctive types instead of exceptions to indicate failures.
- Use standard disjunctive types `Option`, `Try`, `Either` and methods defined for them.
- Define recursive disjunctive types (e.g. lists and trees) and implement recursive functions that work with them.

The following examples and exercises illustrate these tasks.

3.4.1 Solved examples

Example 3.4.1.1 Define a disjunctive type `DayOfWeek` representing the seven days.

Solution Since there is no information other than the label on each day, we use empty case classes:

```
sealed trait DayOfWeek
final case class Sunday() extends DayOfWeek
final case class Monday() extends DayOfWeek
final case class Tuesday() extends DayOfWeek
final case class Wednesday() extends DayOfWeek
final case class Thursday() extends DayOfWeek
final case class Friday() extends DayOfWeek
final case class Saturday() extends DayOfWeek
```

This data type is analogous to an enumeration type in C or C++:

```
typedef enum { Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday } DayOfWeek;
```

Example 3.4.1.2 Modify `DayOfWeek` so that the values additionally represent a restaurant name and total amount for Fridays and a wake-up time on Saturdays.

Solution For the days where additional information is given, we use non-empty case classes:

```
sealed trait DayOfWeekX
final case class Sunday() extends DayOfWeekX
final case class Monday() extends DayOfWeekX
final case class Tuesday() extends DayOfWeekX
final case class Wednesday() extends DayOfWeekX
final case class Thursday() extends DayOfWeekX
final case class Friday(restaurant: String, amount: Int) extends DayOfWeekX
final case class Saturday(wakeUpAt: java.time.LocalDateTime) extends DayOfWeekX
```

This data type is no longer equivalent to an enumeration type.

Example 3.4.1.3 Define a disjunctive type that describes the real roots of the equation $ax^2 + bx + c = 0$, where a, b, c are arbitrary real numbers.

Solution Begin by solving the equation and enumerating all the possible cases. It may happen that $a = b = c = 0$, and then all x are roots. If $a = b = 0$ but $c \neq 0$, the equation is $c = 0$, which has no roots. If $a = 0$ but $b \neq 0$, the equation becomes $bx + c = 0$, having a single root. If $a \neq 0$ and $b^2 > 4ac$, we have two distinct real roots. If $a \neq 0$ and $b^2 = 4ac$, we have one real root. If $b^2 < 4ac$, we have no real roots. The resulting type definition can be written as

```
sealed trait RootsOfQ2
final case class AllRoots() extends RootsOfQ2
final case class ConstNoRoots() extends RootsOfQ2
final case class Linear(x: Double) extends RootsOfQ2
final case class NoRealRoots() extends RootsOfQ2
final case class OneRootQ(x: Double) extends RootsOfQ2
final case class TwoRootsQ(x: Double, y: Double) extends RootsOfQ2
```

This disjunctive type contains six parts, among which three parts are empty tuples and two parts are single-element tuples; but this is not a useless redundancy. We would lose information if we reused `Linear` for the two cases $a = 0, b \neq 0$ and $a \neq 0, b^2 = 4ac$, or if we reused `NoRoots()` for representing all three different no-roots cases.

Example 3.4.1.4 Define a function `rootAverage` that computes the average value of all real roots of a general quadratic equation, where the set of roots is represented by the type `RootsOfQ2` defined in Example 3.4.1.3. The required type signature is

```
val rootAverage: RootsOfQ2 => Option[Double] = ???
```

The function should return `None` if the average is undefined.

Solution The average is defined only in cases `Linear`, `OneRootQ`, and `TwoRootsQ`. In all other cases, we must return `None`. We implement this via pattern matching:

```
val rootAverage: RootsOfQ2 => Option[Double] = roots => roots match {
  case Linear(x)      => Some(x)
  case OneRootQ(x)    => Some(x)
  case TwoRootsQ(x, y) => Some((x + y) * 0.5)
  case _              => None
}
```

We do not need to enumerate all other cases since the underscore (`_`) matches everything that the previous cases did not match.

The often-used code pattern of the form `x => x match { case ... }` can be shortened to the nameless function syntax `{ case ... }`. The code then becomes

```
val rootAverage: RootsOfQ2 => Option[Double] = {
  case Linear(x)      => Some(x)
  case OneRootQ(x)    => Some(x)
  case TwoRootsQ(x, y) => Some((x + y) * 0.5)
  case _              => None
}
```

Test it:


```
scala> Seq(NoRealRoots(), OneRootQ(1.0), TwoRootsQ(1.0, 2.0), AllRoots()).map(rootAverage)
res0: Seq[Option[Double]] = List(None, Some(1.0), Some(1.5), None)
```

Example 3.4.1.5 Generate 100 quadratic equations $x^2 + bx + c = 0$ with random coefficients b, c (uniformly distributed between -1 and 1) and compute the mean of the largest real roots from all these equations.

Solution Use the type `QEqu` and the `solve` function from Example 3.2.2.1. A sequence of equations with random coefficients is created by applying the method `Seq.fill`:

```
def random(): Double = scala.util.Random.nextDouble() * 2 - 1
val coeffs: Seq[QEqu] = Seq.fill(100)(QEqu(random(), random()))
```

Now we can use the `solve` function to compute all roots:

```
val solutions: Seq[RootsOfQ] = coeffs.map(solve)
```

For each set of roots, compute the largest root:

```
scala> val largest: Seq[Option[Double]] = solutions.map {
  case OneRoot(x)      => Some(x)
  case TwoRoots(x, y) => Some(math.max(x, y))
  case _               => None
}
largest: Seq[Option[Double]] = List(None, Some(0.9346072365885472), Some(1.1356234869160806),
  Some(0.9453181931646322), Some(1.1595052441078866), None, Some(0.5762252742788) ...)
```

It remains to remove the `None` values and to compute the mean of the resulting sequence. The Scala library defines the `.flatten` method that removes `Nones` and transforms `Seq[Option[A]]` into `Seq[A]`:

```
scala> largest.flatten
res0: Seq[Double] = List(0.9346072365885472, 1.1356234869160806, 0.9453181931646322,
  1.1595052441078866, 0.5762252742788 ...)
```

Now compute the mean of the last sequence. Since the `.flatten` operation is preceded by `.map`, we can replace it by a `.flatMap`. The final code is

```
val largest = Seq.fill(100)(QEqu(random(), random()))
  .map(solve)
  .flatMap {
    case OneRoot(x)      => Some(x)
    case TwoRoots(x, y) => Some(math.max(x, y))
    case _               => None
  }

scala> largest.sum / largest.size
res1: Double = 0.7682649774589514
```

Example 3.4.1.6 Implement a function with type signature

```
def f1[A, B]: Option[Either[A, B]] => Either[A, Option[B]] = ???
```

The function should preserve information as much as possible.

Solution Begin by pattern matching on the argument:

```
1 def f1[A, B]: Option[Either[A, B]] => Either[A, Option[B]] = {
2   case None      => ???
3   case Some(eab: Either[A, B]) => ???
4 }
```

In line 3, we wrote the **type annotation** `: Either[A, B]` only for clarity; it is not required here because the Scala compiler can deduce the type

of the pattern variable `eab` from the fact that we are matching a value of type `Option[Either[A, B]]`.

In the scope of line 2, we need to return a value of type `Either[A, Option[B]]`. A value of that type must be either a `Left(x)` for some $x:A$, or a `Right(y)` for some $y:Option[B]$, where y must be either `None` or `Some(z)` with a $z:B$. However, in our case the code is of the form `case None => ???`, and we cannot produce any values $x:A$ or $z:B$ since A and B are arbitrary, unknown types. The only remaining

possibility is to return `Right(y)` with `y = None`, and so the code must be

```
...
case None => Right(None) // No other choice here.
```

In the next scope, we can perform pattern matching on the value `eab`:

```
...
case Some(eab: Either[A, B]) = eab match {
  case Left(a)    => ???
  case Right(b)   => ???
}
```

It remains to figure out what expressions to write in each case. In the case `Left(a) => ???`, we have a value of type `A`, and we need to compute a value of type `Either[A, Option[B]]`. We execute the same argument as before: The return value must be `Left(x)` for some `x:A`, or `Right(y)` for some `y:Option[B]`. At this point,

we have a value of type `A` but no values of type `B`. So we have two possibilities: to return `Left(a)` or to return `Right(None)`. If we decide to return `Left(a)`, the code is

```
1 def f1[A, B]: Option[Either[A, B]] => Either[A, Option[B]] = {
2   case None      => Right(None) // No other choice here.
3   case Some(eab) => eab match {
4     case Left(a) => Left(a)    // Could return Right(None) here.
5     case Right(b) => ???
6   }
7 }
```

Let us decide whether to return `Left(a)` or `Right(None)` in line 4. Both choices will satisfy the required return type `Either[A, Option[B]]`. However, if we return `Right(None)` in that line, we will ignore the given value `a:A`,

i.e. we will lose information. So we return `Left(a)` in line 4.

Reasoning similarly for line 5, we find that we may return `Right(None)` or `Right(Some(b))`. The first choice ignores the given value of `b:B`. To preserve information, we make the second choice:

```
1 def f1[A, B]: Option[Either[A, B]] => Either[A, Option[B]] = {
2   case None      => Right(None)
3   case Some(eab) => eab match {
4     case Left(a)  => Left(a)
5     case Right(b) => Right(Some(b))
6   }
7 }
```

We can now refactor this code into a somewhat more readable form by using nested patterns:

```
def f1[A, B]: Option[Either[A, B]] => Either[A, Option[B]] = {
  case None          => Right(None)
  case Some(Left(a)) => Left(a)
  case Some(Right(b)) => Right(Some(b))
}
```

Example 3.4.1.7 Implement a function with the type signature

```
def f2[A, B]: (Option[A], Option[B]) => Option[(A, B)] = ???
```

The function should preserve information as much as possible.

Solution Begin by pattern matching on the argument:

```
1 def f2[A, B]: (Option[A], Option[B]) => Option[(A, B)] = {
2   case (Some(a), Some(b)) => ???
3   ...
```

In line 2, we have values `a:A` and `b:B`, and we need to compute a value of type `Option[(A, B)]`. A value of that type is either `None` or `Some((x, y))`

where we would need to obtain values `x:A` and `y:B`. Since `A` and `B` are arbitrary types, we cannot produce new values `x` and `y` from scratch. The only way of obtaining `x:A` and `y:B` is to set `x = a` and `y = b`. So, our choices are to return `Some((a, b))` or `None`. We reject returning `None` since that would unnecessarily lose information. Thus, we continue writing code as

```
1 def f2[A, B]: (Option[A], Option[B]) => Option[(A, B)] = {
2   case (Some(a), Some(b)) => Some((a, b))
3   case (Some(a), None)   => ???
4   ...
```

In line 3, we have a value `a:A` but no values of type `B`. Since the type `B` is arbitrary, we cannot produce any values of type `B` to return a value of the form `Some((x, y))`. So, `None` is the only computable value of type `Option[(A, B)]` in line 3. We continue to write the code:

```
1 def f2[A, B]: (Option[A], Option[B]) => Option[(A, B)] = {
2   case (Some(a), Some(b)) => Some((a, b))
3   case (Some(a), None)   => None // No other choice here.
4   case (None, Some(b))   => ???
5   case (None, None)     => ???
6 }
```

In lines 4–5, we find that there is no choice other than returning `None`. So we can simplify the code:

```
def f2[A, B]: (Option[A], Option[B]) => Option[(A, B)] = {
  case (Some(a), Some(b)) => Some((a, b))
  case _                 => None // No other choice here.
}
```

3.4.2 Exercises

Exercise 3.4.2.1 Define a disjunctive type `CellState` representing the visual state of one cell in the *Minesweeper*³ game: A cell can be closed (showing nothing), or show a bomb, or be open and show the number of bombs in neighbor cells.

Exercise 3.4.2.2 Define a function from `Seq[Seq[CellState]]` to `Int`, counting the total number of cells with zero neighbor bombs shown.

Exercise 3.4.2.3 Define a disjunctive type `RootOfLinear` representing all possibilities for the solution of the equation $ax + b = 0$ for arbitrary real a, b . (The possibilities are: no roots; one root; all x are roots.) Implement the solution as a function `solve1` with type signature

```
def solve1: ((Double, Double)) => RootOfLinear = ???
```

Exercise 3.4.2.4 Given a `Seq[(Double, Double)]` containing pairs (a, b) of the coefficients of $ax + b = 0$, produce a `Seq[Double]` containing the roots of that equation when a unique root exists. Use the type `RootOfLinear` and the function `solve1` defined in Exercise 3.4.2.3.

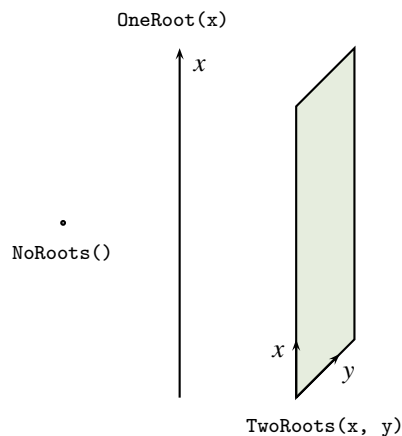
Exercise 3.4.2.5 The case class `Subscriber` was defined in Example 3.2.3.1. Given a `Seq[Subscriber]`, compute the sequence of email addresses for all subscribers that did *not* provide a phone number.

Exercise 3.4.2.6* In this exercise, a “procedure” is a function of type `Unit => Unit`; an example of a procedure is `{ () => println("hello") }`. Define a disjunctive type `Proc` for an abstract syntax tree representing three operations on procedures: 1) `Func[A](f)`, create a procedure from a function f of type `Unit => A`, where A is a type parameter. (Note that the type `Proc` does not have any type parameters.) 2) `Sequ(p1, p2)`, execute two procedures sequentially. 3) `Para(p1, p2)`, execute two procedures in parallel. Then implement a “runner” that converts a `Proc` into a `Future[Unit]`, running the computations either sequentially or in parallel as appropriate. Test with this code:

```
sealed trait Proc; final case class ??? // etc.
def runner: Proc => Future[Unit] = ???
val proc1: Proc = Func[_ => Thread.sleep(200); println("hello1")]
val proc2: Proc = Func[_ => Thread.sleep(400); println("hello2")]

scala> runner(Sequ(Para(proc2, proc1), proc2))
hello1
hello2
hello2
```

³[https://en.wikipedia.org/wiki/Minesweeper_\(video_game\)](https://en.wikipedia.org/wiki/Minesweeper_(video_game))

Figure 3.1: The disjoint domain represented by the `RootsOfQ` type.

Exercise 3.4.2.7 Implement functions that have a given type signature, preserving information as much as possible:

```
def f1[A, B]: Option[(A, B)] => (Option[A], Option[B]) = ???
def f2[A, B]: Either[A, B] => (Option[A], Option[B]) = ???
def f3[A,B,C]: Either[A, Either[B,C]] => Either[Either[A,B], C] = ???
```

Exercise 3.4.2.8 Define a parameterized type `EvenList[A]` representing a list of values of type `A` that is guaranteed to have an even number of elements (zero, two, four, etc.). Implement functions `foldLeft` and `map` for `EvenList`.

3.5 Discussion

3.5.1 Disjunctive types as mathematical sets

To understand the properties of disjunctive types from the mathematical point of view, consider a function whose argument is a disjunctive type, such as

```
def isDoubleRoot(r: RootsOfQ) = ...
```

The type of the argument `r:RootsOfQ` represents the mathematical domain of the function, that is, the set of admissible values of the argument `r`. What kind of domain is that? The set of real roots of a quadratic equation $x^2 + bx + c = 0$ can be empty, or it can contain a single real number x , or a pair of real numbers (x, y) . Geometrically, a number x is pictured as a point on a line (a one-dimensional space), and pair of numbers (x, y) is pictured as a point on a Cartesian plane (a two-dimensional space). The no-roots case corresponds to a zero-dimensional space, which is pictured as a single point (see Figure 3.1). The point, the line, and the plane do not intersect (have no common points); together, they form the domain of the possible roots. Such domains are called **disjoint**. So, the set of real roots of a quadratic equation $x^2 + bx + c = 0$ is a disjoint domain containing three parts.

In the mathematical notation, a one-dimensional real space is denoted by \mathbb{R} , a two-dimensional space by \mathbb{R}^2 , and a zero-dimensional space by \mathbb{R}^0 . At first, we may think that the mathematical representation of the type `RootsOfQ` is a union of the three sets, $\mathbb{R}^0 \cup \mathbb{R}^1 \cup \mathbb{R}^2$. But an ordinary union

of sets would not always work correctly because we need to distinguish the parts of the union unambiguously, even if some parts have the same type. For instance, the disjunctive type shown in Example 3.4.1.3 cannot be correctly represented by the mathematical union

$$\mathbb{R}^0 \cup \mathbb{R}^0 \cup \mathbb{R}^1 \cup \mathbb{R}^0 \cup \mathbb{R}^1 \cup \mathbb{R}^2$$

because $\mathbb{R}^0 \cup \mathbb{R}^0 = \mathbb{R}^0$ and $\mathbb{R}^1 \cup \mathbb{R}^1 = \mathbb{R}^1$, so

$$\mathbb{R}^0 \cup \mathbb{R}^0 \cup \mathbb{R}^1 \cup \mathbb{R}^0 \cup \mathbb{R}^1 \cup \mathbb{R}^2 = \mathbb{R}^0 \cup \mathbb{R}^1 \cup \mathbb{R}^2 \quad .$$

This representation has lost the distinction between e.g. `Linear(x)` and `OneRootQ(x)`.

In the Scala code, each part of a disjunctive type must be distinguished by a unique name such as `NoRoots`, `OneRoot`, and `TwoRoots`. To represent this mathematically, we can attach a distinct label to each part of the union. Labels are symbols without any special meaning, and we can just assume that labels are names of Scala case classes. Parts of the union are then represented by sets of pairs such as $(\text{OneRoot}, x)_{x \in \mathbb{R}^1}$. Then the domain `RootsOfQ` is expressed as

$$\text{RootsOfQ} = (\text{NoRoots}, u)_{u \in \mathbb{R}^0} \cup (\text{OneRoot}, x)_{x \in \mathbb{R}^1} \cup (\text{TwoRoots}, (x, y))_{(x, y) \in \mathbb{R}^2} \quad .$$

This is an ordinary union of mathematical sets, but each of the sets has a unique label, so no two values from different parts of the union could possibly be equal. This kind of labeled union is called a **disjoint union**. Each element of the disjoint union is a pair of the form $(\text{label}, \text{data})$, where the label uniquely identifies the part of the union, and the data can have any chosen type such as \mathbb{R}^1 . If we use disjoint unions, we cannot confuse different parts of the union even if their data have the same type, because labels are required to be distinct.

Disjoint unions are not often used in mathematics, but they are needed in software engineering because real-life data often belongs to disjoint domains.

Named Unit types At first sight, it may seem strange that the zero-dimensional space is represented by a set containing *one* point. Why should we not use an empty set (rather than a set with one point) to represent the case where the equation has no real roots? The reason is that we are required to represent not only the values of the roots but also the information *about* the existence of the roots. The case with no real roots needs to be represented by some *value* of type `RootsOfQ`. This value cannot be missing, which would happen if we used an empty set to represent the no-roots case. It is natural to use the named empty tuple `NoRoots()` to represent this case, just as we used a named 2-tuple `TwoRoots(x, y)` to represent the case of two roots.

Consider the value u used by the mathematical set $(\text{NoRoots}, u)_{u \in \mathbb{R}^0}$. Since \mathbb{R}^0 consists of a single point, there is only *one* possible value of u . Similarly, the `Unit` type in Scala has only one distinct value, written as `()`. A case class with no parts, such as `NoRoots`, has only one distinct value, written as `NoRoots()`. This Scala value is fully analogous to the mathematical notation $(\text{NoRoots}, u)_{u \in \mathbb{R}^0}$.

We see that case classes with no parts are quite similar to `Unit` except for an added name. For this reason, they can be viewed as “named unit” types.

3.5.2 Disjunctive types in other programming languages

Disjunctive types and the associated pattern matching turns out to be one of the defining features of functional programming languages. Languages that were not designed for functional programming do not support these features, while ML, OCaml, Haskell, F#, Scala, Swift, Elm, and PureScript support disjunctive types and pattern matching as part of the language design.

It is remarkable that the named tuple types (also called “structs” or “records”) are provided in almost every programming language, while disjunctive types are almost never present except in languages designed for the FP paradigm. (Ada and Pascal are the only languages that have disjunctive types without other FP features.⁴)

⁴[https://en.wikipedia.org/wiki/Comparison_of_programming_languages_\(basic_instructions\)#Other_types](https://en.wikipedia.org/wiki/Comparison_of_programming_languages_(basic_instructions)#Other_types)

The union types in C and C++ are not disjunctive types because it is not possible to determine which part of the union is represented by a given value. A `union` declaration in C looks like this,

```
union { int x; double y; long z; } di;
```

The problem is that we cannot determine whether a given value `di` represents an `int`, a `double`, or a `long`. This leads to errors that are hard to detect.

Programming languages of the C family (C, C++, Objective C, Java) support **enumeration** (`enum`) types, which are a limited form of disjunctive types. An `enum` type declaration in Java looks like this:

```
enum Color { RED, GREEN, BLUE; }
```

In Scala, this is equivalent to a disjunctive type containing three *empty* tuples,

```
sealed trait Color
final case class RED() extends Color
final case class GREEN() extends Color
final case class BLUE() extends Color
```

If `enum` types were “enriched” with extra data, so that the tuples could be non-empty, we would obtain the full functionality of disjunctive types. A definition of `RootsOfQ` could then look like this:

```
enum RootsOfQ {           // This is not valid in Java!
  NoRoots(), OneRoot(Double x), TwoRoots(Double x, Double y);
}
```

A future version of Scala 3 will have a shorter syntax for disjunctive types⁵ that indeed looks like an “enriched `enum`”:

```
enum RootsOfQ {
  case NoRoots
  case OneRoot(x: Double)
  case TwoRoots(x: Double, y: Double)
}
```

For comparison, the syntax for a disjunctive type equivalent to `RootsOfQ` in OCaml and Haskell is

```
(* OCaml *)
type RootsOfQ = NoRoots | OneRoot of float | TwoRoots of float*float

-- Haskell
data RootsOfQ = NoRoots | OneRoot Double | TwoRoots (Double, Double)
```

This is more concise than the Scala syntax. When reasoning about disjunctive types, it is inconvenient to write out long type definitions. Chapter 5 will define a mathematical notation designed for efficient reasoning about types.

3.5.3 Disjunctions and conjunctions in formal logic

In logic, a **proposition** is a logical formula that could be true or false. A **disjunction** of propositions A, B, C is denoted by $A \vee B \vee C$ and is true if and only if *at least one* of A, B, C is true. A **conjunction** of A, B, C is denoted by $A \wedge B \wedge C$ and is true if and only if *all* of the propositions A, B, C are true.

There is a similarity between a disjunctive data type and a logical disjunction of propositions. A value of the disjunctive data type `RootsOfQ` can be constructed only if we have one of the values `NoRoots()`, `OneRoot(x)`, or `TwoRoots(x, y)`. Let us now rewrite the previous sentence as a logical formula. Denote by $CH(A)$ the logical proposition “this Code \mathcal{H} has a value of type A ”, where “this code” refers to a particular function in a program. So, the proposition “the function *can* return a value of type `RootsOfQ`” is denoted by $CH(\text{RootsOfQ})$. We can then write the above sentence about `RootsOfQ` as the logical formula

$$CH(\text{RootsOfQ}) = CH(\text{NoRoots}) \vee CH(\text{OneRoot}) \vee CH(\text{TwoRoots}) \quad . \quad (3.1)$$

⁵<https://dotty.epfl.ch/docs/reference/enums/adts.html>

There is also a similarity between logical *conjunctions* and tuple types. Consider the named tuple (i.e. a case class) `TwoRoots(x: Double, y: Double)`. When can we have a value of type `TwoRoots`? Only if we have two values of type `Double`. Rewriting this sentence as a logical formula, we get

$$\mathcal{CH}(\text{TwoRoots}) = \mathcal{CH}(\text{Double}) \wedge \mathcal{CH}(\text{Double}) \quad .$$

Formal logic admits the simplification

$$\mathcal{CH}(\text{Double}) \wedge \mathcal{CH}(\text{Double}) = \mathcal{CH}(\text{Double}) \quad .$$

However, no such simplification will be available in the general case, e.g.

```
case class Data3(x: Int, y: String, z: Double)
```

For this type, we will have the formula

$$\mathcal{CH}(\text{Data3}) = \mathcal{CH}(\text{Int}) \wedge \mathcal{CH}(\text{String}) \wedge \mathcal{CH}(\text{Double}) \quad . \quad (3.2)$$

We find that tuples are related to logical conjunctions in the same way as disjunctive types are related to logical disjunctions. This is the main reason for choosing the name “disjunctive types”.⁶

The correspondence between disjunctions, conjunctions, and data types is explained in more detail in Chapter 5. For now, we note that the operations of conjunction and disjunction are not sufficient to produce all possible logical expressions. To obtain a complete logic, it is also necessary to have a logical negation $\neg A$ (“ A is not true”) or, equivalently, a logical implication $A \Rightarrow B$ (“if A is true then B is true”). It turns out that the logical implication $A \Rightarrow B$ is related to the function type `A => B`. In Chapter 4, we will study function types in depth.

⁶These types are also called “variants”, “sum types”, “co-product types”, and “tagged union types”.

Part II

Intermediate level

4 The logic of types. II. Higher-order functions

4.1 Functions that return functions

4.1.1 Motivation and first examples

Consider the task of preparing a logger function that prints messages with a configurable prefix.

A simple logger function can be a value of type `String => Unit`, such as

```
val logger: String => Unit = { message => println(s"INFO: $message") }

scala> logger("hello world")
INFO: hello world
```

This function prints any given message with the logging prefix `"INFO"`.

The standard library function `println(...)` always returns a `Unit` value after printing its arguments. As we already know, there is only a single value of type `Unit`, and that value is denoted by `()`. To see that `println` returns `Unit`, run this code:

```
scala> val x = println(123)
123
x: Unit = ()
```

The task is to make the logging prefix configurable. A simple solution is to implement a function `logWith` that takes a prefix as an argument and returns a new logger containing that prefix. Note that the function `logWith` returns a new *function*, i.e. a new value of type `String => Unit`:

```
def logWith(prefix: String): (String => Unit) = {
  message => println(s"$prefix: $message")
}
```

The body of `logWith` consists of a nameless function `message => println(...)`, which is a value of type `String => Unit`. This value will be returned when we evaluate `logWith("...")`.

We can now use `logWith` to create some logger functions:

```
scala> val info = logWith("INFO")
info: String => Unit = <function1>

scala> val warn = logWith("WARN")
warn: String => Unit = <function1>
```

The created logger functions are then usable as ordinary functions:

```
scala> info("hello")
INFO: hello

scala> warn("goodbye")
WARN: goodbye
```

The values `info` and `warn` can be used by any code that needs a logging function.

It is important that the prefix is “baked into” functions created by `logWith`. A logger such as `warn` will always print messages with the prefix `"WARN"`, and the prefix cannot be changed any more. This is so because the value `prefix` is treated as a local constant within the body of the nameless function computed and returned by `logWith`. For instance, the body of the function `warn` is equivalent to

```
{ val prefix = "WARN"; (message => s"$prefix: $message") }
```

Whenever a new function is created using `logWith(prefix)`, the (immutable) value of `prefix` is stored within the body of the newly created function. This is a general feature of nameless functions: the function's body keeps copies of all the outer-scope values it uses. One sometimes says that the function's body “closes over” those values; for this reason, nameless functions are also called “**closures**”. It would be clearer to say that nameless functions “capture” values from outer scopes.

As another example of the capture of values, consider this code:

```
val f: Int => Int = {
  val p = 10
  val q = 20
  x => p + q * x
}
```

The body of the function `f` is equivalent to `{ x => 10 + 20 * x }` because the values `p = 10` and `q = 20` were captured.

4.1.2 Curried and uncurried functions

Reasoning mathematically about the code

```
val info = logWith("INFO")
info("hello")
```

we would expect that `info` is *the same value* as `logWith("INFO")`, and so the code `info("hello")` should have the same effect as the code `logWith("INFO")("hello")`. This is indeed so:

```
scala> logWith("INFO")("hello")
INFO: hello
```

The syntax `logWith("INFO")("hello")` looks like the function `logWith` applied to *two* arguments. Yet, `logWith` was defined as a function with a single argument of type `String`. This is not a contradiction because `logWith("INFO")` returns a function that accepts an additional argument. So, both function applications `logWith("INFO")` and `logWith("INFO")("hello")` are valid. In this sense, we are allowed to apply `logWith` to one argument at a time.

A function that can be applied to arguments in this way is called a **curried** function.

While a curried function can be applied to one argument at a time, an **uncurried** function must be applied to all arguments at once, e.g.

```
def prefixLog(prefix: String, message: String): Unit = println(s"$prefix: $message")
```

The type of the curried function `logWith` is `String => (String => Unit)`. By Scala's syntax conventions, the function arrow (`=>`) groups to the *right*. So the parentheses in the type expression `String => (String => Unit)` are not needed; the function's type can be written as `String => String => Unit`.

The type `String => String => Unit` is different from `(String => String) => Unit`, which is the type of a function returning `Unit` and having a function of type `String => String` as its argument. When an argument's type is a function type, e.g. `String => String`, it must be enclosed in parentheses.

In general, a curried function takes an argument and returns another function that again takes an argument and returns another function, and so on, until finally a non-function type is returned. So, the type signature of a curried function generally looks like `A => B => C => ... => R => S`, where `A`, `B`, ..., `R` are the **curried arguments** and `S` is the “final” result type.

For example, in the type expression `A => B => C => D` the types `A`, `B`, `C` are the types of curried arguments, and `D` is the final result type. It takes time to get used to reading this kind of syntax.

In Scala, functions defined with multiple argument groups (enclosed in multiple pairs of parentheses) are curried functions. We have seen examples of curried functions before:

```
def map[A, B](xs: Seq[A])(f: A => B): Seq[B]
def fmap[A, B](f: A => B)(xs: Option[A]): Option[B]
```

```
def foldLeft[A, R](xs: Seq[A])(init: R)(update: (R, A) => R): R
```

The type signatures of these functions can be also written equivalently as

```
def map[A, B]: Seq[A] => (A => B) => Seq[B]
def fmap[A, B]: (A => B) => Option[A] => Option[B]
def foldLeft[A, R]: Seq[A] => R => ((R, A) => R) => R
```

Curried arguments of a function type, such as $(A \Rightarrow B)$, need parentheses.

To summarize, a curried function such as `logWith` can be defined in three equivalent ways in Scala:

```
1 def logWith1(prefix: String)(message: String): Unit = println(s"$prefix: $message")
2 def logWith2(prefix: String): String => Unit = { message => println(s"$prefix: $message") }
3 def logWith3: String => String => Unit = { prefix => message => println(s"$prefix: $message") }
```

We will sometimes enclose nameless functions in parentheses or curly braces to improve clarity.

Line 3 in the above code shows that the arrow symbol \Rightarrow groups to the right within the *code* of nameless functions: $x \Rightarrow y \Rightarrow \text{expr}$ means $\{x \Rightarrow \{y \Rightarrow \text{expr}\}\}$, a nameless function taking an argument x and returning a nameless function that takes an argument y and returns an expression expr . This syntax convention is adopted for two reasons. First, the code $x \Rightarrow y \Rightarrow z$ visually corresponds to the curried function's type signature $A \Rightarrow B \Rightarrow C$, which uses the same syntax convention. Second, the syntax $(x \Rightarrow y) \Rightarrow z$ could not work for a nameless function because $(x \Rightarrow y)$ is not a valid pattern expression for the function's argument; patterns cannot match arbitrary *functions* of type $A \Rightarrow B$. So $x \Rightarrow (y \Rightarrow z)$ is the only possible way of inserting parentheses into $x \Rightarrow y \Rightarrow z$.

Although the code $(x \Rightarrow y) \Rightarrow z$ is invalid, a type expression $(A \Rightarrow B) \Rightarrow C$ is valid. A nameless function of type $(A \Rightarrow B) \Rightarrow C$ is written as $f \Rightarrow \text{expr}$ where $f: A \Rightarrow B$ is its argument and expr its body.

4.1.3 Equivalence of curried and uncurried functions

We defined the curried function `logWith` in order to be able to create logger functions such as `info` and `warn`. However, some curried functions, such as `foldLeft`, are almost always applied to all possible arguments. A curried function applied to all its possible arguments is equivalent to an uncurried function that takes all those arguments at once. Let us look at this equivalence in more detail.

Consider a curried function with type signature $\text{Int} \Rightarrow \text{Int} \Rightarrow \text{Int}$. This function takes an integer and returns an (uncurried) function taking an integer and returning an integer. An example of such a curried function is

```
def f1(x: Int): Int => Int = { y => x - y }
```

The function takes an integer x and returns the expression $y \Rightarrow x - y$, which is a function of type $\text{Int} \Rightarrow \text{Int}$. The code of `f1` can be written equivalently as

```
val f1: Int => Int => Int = { x => y => x - y }
```

Let us compare the function `f1` with a function that takes its two arguments at once, e.g.

```
def f2(x: Int, y: Int): Int = x - y
```

The function `f2` has type signature $(\text{Int}, \text{Int}) \Rightarrow \text{Int}$.

The syntax for using the functions `f1` and `f2` is different:

```
scala> f1(20)(4)
res0: Int = 16

scala> f2(20, 4)
res1: Int = 16
```

The main difference between the usage of `f1` and `f2` is that `f2` must be applied *at once* to both arguments, while `f1` applied to just the first argument, 20. The result of evaluating `f1(20)` is a function that can be later applied to another argument:

```
scala> val r1 = f1(20)
```

```
r1: Int => Int = <function1>

scala> r1(4)
res2: Int = 16
```

Applying a curried function to some but not all possible arguments is called a **partial application**. Applying a curried function to all possible arguments is called a **saturated** application.

If we need to partially apply an *uncurried* function, we can use the underscore (_) symbol:

```
1 scala> val r2: Int => Int = f2(20, _)
2 r2: Int => Int = <function1>
3
4 scala> r2(4)
5 res3: Int = 16
```

(The type annotation `Int => Int` is required in line 1.) This code creates a function `r2` by partially applying `f2` to the first argument but not to the second. Other than that, `r2` is the same function as `r1` defined above; i.e. `r2` returns the same values for the same

arguments as `r1`. A more straightforward syntax for a partial application is

```
scala> val r3: Int => Int = { x => f2(20, x) }
r3: Int => Int = <function1>

scala> r3(4)
res4: Int = 16
```

We can see that a curried function, such as `f1`, is better adapted for partial application than `f2`, because the syntax is shorter. However, the functions `f1` and `f2` are **computationally equivalent** in the sense that given `f1` we can reconstruct `f2` and vice versa:

```
def f2new(x: Int, y: Int): Int = f1(x)(y)           // f2new is equal to f2
def f1new: Int => Int => Int = { x => y => f2(x, y) } // f1new is equal to f1
```

It is clear that the function `f1new` computes the same results as `f1`, and that the function `f2new` computes the same results as `f2`. The computational equivalence of the functions `f1` and `f2` is not *equality* – these functions are *different*; but one of them can be easily reconstructed from the other if necessary.

More generally, a curried function has a type signature of the form `A => B => C => ... => R => S`, where `A`, `B`, `C`, ..., `S` are some types. A function with this type signature is computationally equivalent to an uncurried function with type signature `(A,B,C,...,R) => S`. The uncurried function takes all arguments at once, while the curried function takes one argument at a time. Other than that, these two functions compute the same results given the same arguments.

We have seen how a curried function can be converted to an equivalent uncurried one, and vice versa. The Scala library defines the methods `curried` and `uncurried` that convert between these forms of functions. To convert between `f2` and `f1`:

```
scala> val f1c = (f2 _).curried
f1c: Int => (Int => Int) = <function1>

scala> val f2u = Function.uncurried(f1c)
f2u: (Int, Int) => Int = <function2>
```

The syntax `(f2 _)` is needed in Scala to convert methods to function values. Recall that Scala has two ways of defining a function: one as a method (defined using `def`), another as a function value (defined using `val`). The extra underscore will become unnecessary in Scala 3.

The methods `.curried` and `.uncurried` are easy to implement, as we will show in Section 4.2.1.

4.2 Fully parametric functions

We have seen that some functions are declared with type parameters, which are set only when the function is applied to specific arguments. Examples of such functions are the `map` and `filter` methods with type signatures

```
def map[A, B](xs: Seq[A])(f: A => B): Seq[B]
def filter[A](xs: Seq[A])(p: A => Boolean): Seq[A]
```

Such functions can be applied to arguments of different types without changing the function's code. It is clearly better to implement a single function with type parameters instead of writing several functions with repeated code only because we need to work with different types.

When we apply the function `map` as `map(xs)(f)` to a specific value `xs` of type, e.g., `Seq[Int]`, and a specific function `f` of type, say, `Int => String`, the Scala compiler will automatically set the type parameters `A = Int` and `B = String` in the definition of `map`. We may also set type parameters explicitly and write, for example, `map[Int, String](xs)(f)`. This syntax shows a certain similarity between type parameters such as `Int`, `String` and "value parameters" (arguments) `xs` and `f`. Setting type parameters (e.g. `map[Int, String]`) means setting type parameters equal to specific types (e.g. `A=Int, B=String`) in the type signature of the function, similarly to how setting value parameters means substituting specific values into the body of the function.

In the functions `map` and `filter` as just shown, some types are parameters while others are specific types, such as `Seq` and `Boolean`. It is sometimes possible to replace *all* types in the type signature of a function by type parameters.

A function is **fully parametric** if its arguments have types described by type parameters, and the code of the function treats all types as type parameters. In other words, fully parametric functions do not use any values of specific types, such as `Int` or `String`, in their type signature or in their body. A fully parametric function does not use any information about its argument types, other than assuming that the arguments' types correctly match the type signature.

What kind of functions are fully parametric? To build an intuition for that, let us compare these two functions having the same type signature:

```
def cos_sin(p: (Double, Double)): (Double, Double) = p match {
  case (x, y) =>
    val r = math.sqrt(x * x + y * y)
    (x / r, y / r) // Return cos and sin of the angle.
}

def swap(p: (Double, Double)): (Double, Double) = p match {
  case (x, y) => (y, x)
}
```

We can introduce type parameters into the type signature of `swap` to make it fully parametric, with no changes in the code of the function:

```
def swap[A, B](p: (A, B)): (B, A) = p match {
  case (x, y) => (y, x)
}
```

Generalizing `swap` to a fully parametric function is possible because the operation of swapping two parts of a tuple `(A, B)` works in the same way for all types `A, B`. The specialized version of

`swap` working on `(Double, Double)` can be obtained from the fully parametric version of `swap` if we set the type parameters as `A = Double, B = Double`.

In contrast, the function `cos_sin` performs a computation that is specific to the type `Double` and cannot be generalized to an arbitrary type parameter `A` instead of `Double`. So, `cos_sin` cannot be generalized to a fully parametric function.

The `swap` operation for pairs is already defined in the Scala library:

```
scala> (1, "abc").swap
res0: (String, Int) = (abc,1)
```

Other swapping functions can be defined for tuples with more elements, e.g.

```
def swap12[A,B,C]: ((A, B, C)) => (B, A, C) = {
  case (x, y, z) => (y, x, z)
}
```

The Scala syntax requires the double parentheses around tuple types of arguments but not around the tuple type of a function's result. So, the function `cos_sin` may be written as a value of type

```
val cos_sin: ((Double, Double)) => (Double, Double) = ...
```

4.2.1 Examples. Function composition

Further examples of fully parametric functions are the identity function, the constant function, the function composition methods, and the curry / uncurry conversions.

The identity function (available in the Scala library as `identity[T]`) is

```
def id[T]: T => T = (t => t)
```

The constant function (available in the Scala library as `Function.const`) takes an argument `c` and returns a new function that always returns `c`:

```
def const[C, X](c: C): X => C = (_ => c)
```

The syntax `_ => c` is used to emphasize that the function ignores its argument.

Function composition Consider two functions `f: Int => Double` and `g: Double => String`. We can apply `f` to an integer argument `x: Int` and get a result `f(x)` of type `Double`. We can then apply `g` to that result and obtain a `String` value `g(f(x))`. The transformation from the original integer `x: Int` to the final `String` value `g(f(x))` can be viewed as a new function of type `Int => String`. That new function is called the **forward composition** of the two functions `f` and `g`. In Scala, this operation is written as `f andThen g`:

```
val f: Int => Double = (x => 5.67 + x)
val g: Double => String = (x => f"Result x = ${x%3.2f}")

scala> val h = f andThen g
h: Int => String = <function1>

scala> h(40)
res36: String = Result x = 45.67
```

The Scala compiler derives the type of `h` automatically as `Int => String`.

The forward composition is denoted by \circ (pronounced “before”) and can be defined as

$$f \circ g \triangleq (x \Rightarrow g(f(x))) \quad . \quad (4.1)$$

The symbol \triangleq means “is defined as”.

We could write the forward composition as a fully parametric function,

```
def andThen[X, Y, Z](f: X => Y)(g: Y => Z): X => Z = { x => g(f(x)) }
```

This type signature requires the types of the function arguments to match in a certain way, or else the composition is undefined (and the code would produce a type error).

The method `andThen` is an example of a function that *both* returns a new function *and* takes other functions as arguments.

The **backward composition** of two functions `f` and `g` works in the opposite order: first `g` is applied and then `f` is applied to the result. Using the symbol \circ (pronounced “after”) for this operation, we can write

$$f \circ g \triangleq (x \Rightarrow f(g(x))) \quad . \quad (4.2)$$

In Scala, the backward composition is called `compose` and used as `f compose g`. This method may be implemented as a fully parametric function

```
def compose[X, Y, Z](f: Y => X)(g: Z => Y): Z => X = { z => f(g(z)) }
```

We have already seen the methods `curried` and `uncurried` defined by the Scala library. As an illustration, let us write our own code for converting curried functions to uncurried:

```
def uncurry[A, B, R](f: A => B => R): ((A, B)) => R = {
  case (a, b) => f(a)(b)
}
```

We conclude from these examples that fully parametric functions perform operations so general that they work in the same way for all types. Some arguments of fully parametric functions may

have complicated types such as $A \Rightarrow B \Rightarrow R$, which are type expressions made up from type parameters. But fully parametric functions do not use values of specific types such as `Int` or `String`.

The property of being fully parametric is also called **parametricity**. In some programming languages, functions with type parameters are called “generic”.

4.2.2 Laws of function composition

The operations of function composition, introduced in Section 4.2.1, have three important properties or “laws” that follow directly from the definitions. These laws are:

- The two identity laws: the composition of any function f with the identity function will give again the function f .
- The associativity law: the consecutive composition of three functions f, g, h does not depend on the order in which the pairs are composed.

These laws hold equally for the forward and the backward composition, since those are just syntactic variants of the same mathematical operation. Let us write these laws rigorously as mathematical equations and prove them.

Proofs in the forward notation The composition of the identity function with an arbitrary function f can be $\text{id} \circ f$ with the identity function to the left of f , or $f \circ \text{id}$ with the identity function to the right of f . In both cases, the result must be equal to the function f . The resulting two laws are

$$\begin{aligned} \text{left identity law of composition : } \text{id} \circ f &= f \quad , \\ \text{right identity law of composition : } f \circ \text{id} &= f \quad . \end{aligned}$$

To show that these laws always hold, we need to show that both sides of the laws, which are functions, give the same result when applied to an arbitrary value x . Let us first clarify how the type parameters must be set for the laws to have consistently matching types.

The laws must hold for an arbitrary function f . So we may assume that f has the type signature $A \Rightarrow B$, where A and B are arbitrary type parameters. Consider the left identity law. The function $(\text{id} \circ f)$ is, by definition (4.1), a function that takes an argument x , applies id to that x , and then applies f to the result:

$$\text{id} \circ f = (x \Rightarrow f(\text{id}(x))) \quad .$$

If f has type $A \Rightarrow B$, its argument must be of type A , or else the types will not match. Therefore, the identity function must have type $A \Rightarrow A$, and the argument x must have type A . With these choices of the type parameters, the function $(x \Rightarrow f(\text{id}(x)))$ will have type $A \Rightarrow B$, as it must since the right-hand side of the law is f . We add type annotations to the code as *superscripts*,

$$\text{id}^A \circ f^{A \Rightarrow B} = (x^A \Rightarrow f(\text{id}(x)))^{A \Rightarrow B} \quad .$$

In the Scala syntax, this formula may be written as

```
id[A] andThen (f: A => B) == { x: A => f(id(x)) }: A => B
```

It is quicker to write code in the mathematical notation than in the Scala syntax. We will follow the convention where type parameters are single uppercase letters, as is common in Scala code (although this convention is not enforced by the Scala compiler).

The colon symbol ($:$) in the superscript x^A means a type annotation, as in Scala code `x:A`. Superscripts without a colon, such as id^A , denote type parameters, as in Scala code `identity[A]`. Since the function `identity[A]` has type $A \Rightarrow A$, we can write id^A or equivalently (but more verbosely) $\text{id}^{A \Rightarrow A}$ to denote that function.

Since $\text{id}(x) = x$ by definition of the identity function, we find that

$$\text{id} \circ f = (x \Rightarrow f(\text{id}(x))) = (x \Rightarrow f(x)) = f \quad .$$

The last step works since $x \Rightarrow f(x)$ is a function taking an argument x and applying f to that argument; i.e. $x \Rightarrow f(x)$ is the same function as f .

Now consider the right identity law:

$$f \circ \text{id} = (x \Rightarrow \text{id}(f(x))) \quad .$$

To make the types match, assume that $f:A \Rightarrow B$. Then x must have type A , and the identity function must have type $B \Rightarrow B$. The result of $\text{id}(f(x))$ will also have type B . With these choices of type parameters, all types match:

$$f:A \Rightarrow B \circ \text{id}^B = (x:A \Rightarrow \text{id}(f(x)))^{A \Rightarrow B} \quad .$$

Since $\text{id}(f(x)) = f(x)$, we find that

$$f \circ \text{id} = (x \Rightarrow f(x)) = f \quad .$$

In this way, we have demonstrated that both identity laws hold.

The associativity law is written as an equation like this:

$$\text{associativity law of composition : } (f \circ g) \circ h = f \circ (g \circ h) \quad . \quad (4.3)$$

Let us verify that the types match here. The types of the functions f , g , and h must be such that all the function compositions exist. If f has type $A \Rightarrow B$ for some type parameters A and B , then the argument of g must be of type B ; so we can choose $g:B \Rightarrow C$, where C is another type parameter. The composition $f \circ g$ has type $A \Rightarrow C$, so h must have type $C \Rightarrow D$ for some type D . Assuming the types as $f:A \Rightarrow B$, $g:B \Rightarrow C$, and $h:C \Rightarrow D$, we find that the types in all the compositions $f \circ g$, $g \circ h$, $(f \circ g) \circ h$, and $f \circ (g \circ h)$ match. We can rewrite Eq. (4.3) with type annotations,

$$(f:A \Rightarrow B \circ g:B \Rightarrow C) \circ h:C \Rightarrow D = f:A \Rightarrow B \circ (g:B \Rightarrow C \circ h:C \Rightarrow D) \quad . \quad (4.4)$$

Having checked the types, we are ready to prove the associativity law. We note that both sides of the law (4.4) are functions of type $A \Rightarrow D$. To prove that two functions are equal means to prove that they always return the same results when applied to the same arguments. So we need to apply both sides of Eq. (4.4) to an arbitrary value $x:A$. Using the definition (4.1) of the forward composition,

$$(f \circ g)(x) = g(f(x)) \quad ,$$

we find

$$\begin{aligned} ((f \circ g) \circ h)(x) &= h((f \circ g)(x)) = h(g(f(x))) \quad , \\ (f \circ (g \circ h))(x) &= (g \circ h)(f(x)) = h(g(f(x))) \quad . \end{aligned}$$

Both sides of the law are now equal when applied to an arbitrary value x .

Because of the associativity law, we do not need parentheses when writing the expression $f \circ g \circ h$. The function $(f \circ g) \circ h$ is the same as $f \circ (g \circ h)$.

In the proof, we have omitted the type annotations since we already checked that the types match. Checking the types beforehand allows us to write shorter proofs.

Proofs in the backward notation This book uses the **forward notation** $f \circ g$ for writing function compositions, rather than the backward notation $g \circ f$. If necessary, all equations can be automatically converted from one notation to the other by reversing the order of compositions because

$$f \circ g = g \circ f$$

for any functions $f:A \Rightarrow B$ and $g:B \Rightarrow C$. Let us see how to prove the composition laws in the backward notation. We will just need to reverse the order of function compositions in the proofs above.

The left identity and right identity laws are

$$f \circ \text{id} = f \quad , \quad \text{id} \circ f = f \quad .$$

To match the types, we need to choose the type parameters as

$$f:A \Rightarrow B \circ \text{id}:A \Rightarrow A = f:A \Rightarrow B \quad , \quad \text{id}:B \Rightarrow B \circ f:A \Rightarrow B = f:A \Rightarrow B \quad .$$

We can apply both sides of the laws to an arbitrary value $x:A$. For the left identity law, we find from definition (4.2) that

$$f \circ \text{id} = (x \Rightarrow f(\text{id}(x))) = (x \Rightarrow f(x)) = f \quad .$$

Similarly for the right identity law,

$$\text{id} \circ f = (x \Rightarrow \text{id}(f(x))) = (x \Rightarrow f(x)) = f \quad .$$

The associativity law,

$$h \circ (g \circ f) = (h \circ g) \circ f \quad ,$$

is proved by applying both sides to an arbitrary value x of a suitable type:

$$\begin{aligned} (h \circ (g \circ f))(x) &= h((g \circ f)(x)) = h(g(f(x))) \quad , \\ ((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) = h(g(f(x))) \quad . \end{aligned}$$

The types are checked by assuming that f has the type $f:A \Rightarrow B$. The types in $g \circ f$ match only when $g:B \Rightarrow C$, and then $g \circ f$ is of type $A \Rightarrow C$. The type of h must be $h:C \Rightarrow D$ for the types in $h \circ (g \circ f)$ to match. We can write the associativity law with type annotations as

$$h:C \Rightarrow D \circ (g:B \Rightarrow C \circ f:A \Rightarrow B) = (h:C \Rightarrow D \circ g:B \Rightarrow C) \circ f:A \Rightarrow B \quad . \quad (4.5)$$

The associativity law allows us to omit parentheses in the expression $h \circ g \circ f$.

The length of calculations is the same in the forward and the backward notation. One difference is that types of function compositions are more visually clear in the forward notation: it is easier to check that types match in Eq. (4.4) than in Eq. (4.5).

4.2.3 Example: A function that violates parametricity

Fully parametric functions should not make any decisions based on the actual types of arguments. As an example of an *incorrect* implementation of a fully parametric function, consider the following “fake identity” function:

```
def fakeId[A]: A => A = { // Special code for A = Int:
  case x: Int    => (x - 1).asInstanceOf[A]
  case x         => x      // Common code for all other types A.
}
```

This function’s type signature is the same as that of `id[A]`, and its behavior is the same for all types A except for $A = \text{Int}$:

```
scala> fakeId("abc")
res0: String = abc

scala> fakeId(true)
res1: Boolean = true

scala> fakeId(0)
res2: Int = -1
```

While Scala allows us to write this kind of code, the resulting function does not appear to be useful. In any case, `fakeId` is not a fully parametric function.

The identity laws of composition will not hold if we use `fakeId[A]` instead of the correct function `id[A]`. For example, consider the composition of `fakeId` with a simple function `f_1` defined by

```
def f_1: Int => Int = { x => x + 1 }
```

The composition `(f_1 andThen fakeId)` will have type `Int => Int`. Since `f_1` has type `Int => Int`, Scala will automatically set the type parameter `A = Int` in `fakeId[A]`,

```
scala> def f_2 = f_1 andThen fakeId
f_2: Int => Int
```

The identity law says that $f_2 = f_1 \circ id = f_1$. But we can check that `f_1` and `f_2` are not the same:

```
scala> f_1(0)
res3: Int = 1

scala> f_2(0)
res4: Int = 0
```

It is important that we are able to detect a violation of parametricity by checking whether some equation holds, without need to examine the code of the function `fakeId`. In this book, we will always formulate the desired properties of functions through equations or “laws”. To prove the laws, we will need to perform symbolic calculations similar to the proofs in Section 4.2.2. These calculations are **symbolic** in the sense that we were manipulating symbols such as x , f , g , and h without substituting any specific values for these symbols but using only the general properties of functions. In the next section, we will get more experience with such calculations.

4.3 Symbolic calculations with nameless functions

4.3.1 Calculations with curried functions

In mathematics, functions are evaluated by substituting their argument values into their body. Nameless functions are evaluated in the same way. For example, applying the nameless function $x \Rightarrow x + 10$ to an integer 2, we substitute 2 instead of x in “ $x + 10$ ” and get “ $2 + 10$ ”, which we then evaluate to 12. The computation is written like this,

$$(x \Rightarrow x + 10)(2) = 2 + 10 = 12 \quad .$$

To run this computation in Scala, we need to add a type annotation:

```
scala> ((x: Int) => x + 10)(2)
res0: Int = 12
```

Curried function applications such as $f(x)(y)$ are rarely used in mathematics, so we need to gain some experience working with them.

Consider a curried nameless function being applied to arguments, such as $(x \Rightarrow y \Rightarrow x - y)(20)(4)$, and compute the result of this function application. Begin with the argument 20; applying a nameless function of the form $(x \Rightarrow \dots)$ to 20 means to substitute $x = 20$ into the body of the function. After that substitution, we obtain the expression $y \Rightarrow 20 - y$, which is again a nameless function. Applying that function to the remaining argument (4) means substituting $y = 4$ into the body of $y \Rightarrow 20 - y$. We get the expression $20 - 4$, which equals 16. Check the result with Scala:

```
scala> ((x: Int) => (y: Int) => x - y)(20)(4)
res1: Int = 16
```

Applying a curried function such as $x \Rightarrow y \Rightarrow z \Rightarrow \text{expr}(x, y, z)$ to three curried arguments 10, 20, and 30 means to substitute $x = 10$, $y = 20$, and $z = 30$ into the expression `expr`. In this way, we can easily apply a curried function to any number of curried arguments.

This calculation is helped by the convention that $f(g)(h)$ means first applying f to g and then applying the result to h . In other words, function application groups to the *left*: $f(g)(h) = (f(g))(h)$. It would be confusing if function application grouped to the right and $f(g)(h)$ meant first applying g to h and then applying f to the result. If *that* were the syntax convention, it would be harder to reason about applying a curried function to the arguments.

We see that the right grouping of the function arrow \Rightarrow is well adapted to the left grouping of function applications. All functional languages follow these syntactic conventions.

To make calculations shorter, we will write code in a mathematical notation rather than in the Scala syntax. Type annotations are written with a colon in the superscript, for example: $x^{\text{Int}} \Rightarrow x + 10$ instead of the code `((x: Int) => x + 10)`.

The symbolic evaluation of the Scala code `((x: Int) => (y: Int) => x - y)(20)(4)` can be written as

$$\begin{aligned} & (x^{\text{Int}} \Rightarrow y^{\text{Int}} \Rightarrow \underline{x} - y)(20)(4) \\ \text{apply function and substitute } x = 20 : & = (\underline{y}^{\text{Int}} \Rightarrow 20 - \underline{y})(4) \\ \text{apply function and substitute } y = 4 : & = 20 - 4 = 16 \end{aligned}$$

In the above step-by-step calculation, the colored underlines and comments at left are added for clarity. A colored underline indicates a sub-expression that is going to be rewritten at the next step.

Here we performed calculations by substituting an argument into a function at each step. A compiled Scala program is evaluated in a similar way at run time.

Nameless functions are *values* and so can be used as part of larger expressions, just as any other values. For instance, nameless functions can be arguments of other functions (nameless or not). Here is an example of applying a nameless function $f \Rightarrow f(2)$ to a nameless function $x \Rightarrow x + 4$:

$$\begin{aligned} & (\underline{f} \Rightarrow \underline{f(2)})(x \Rightarrow x + 4) \\ \text{substitute } f = (x \Rightarrow x + 4) : & = (\underline{x} \Rightarrow \underline{x} + 4)(2) \\ \text{substitute } x = 2 : & = 2 + 4 = 6 \end{aligned}$$

In the nameless function $f \Rightarrow f(2)$, the argument f has to be itself a function, otherwise the expression $f(2)$ would make no sense. The argument x of $f(x)$ must be an integer, or else we would not be able to compute $x + 4$. The result of computing $f(2)$ is 4, an integer. We conclude that f must have type $\text{Int} \Rightarrow \text{Int}$, or else the types will not match. To verify this result in Scala, we need to specify a type annotation for f :

```
scala> ((f: Int => Int) => f(2))(x => x + 4)
res2: Int = 6
```

No type annotation is needed for $x \Rightarrow x + 4$ since the Scala compiler already knows the type of f and can figure out that x in $x \Rightarrow x + 4$ must have type Int .

To summarize the syntax conventions for curried nameless functions:

- Function expressions group everything to the right: $x \Rightarrow y \Rightarrow z \Rightarrow e$ means $x \Rightarrow (y \Rightarrow (z \Rightarrow e))$.
- Function applications group everything to the left, so $f(x)(y)(z)$ means $((f(x))(y))(z)$.
- Function applications group stronger than infix operations, so $x + f(y)$ means $x + (f(y))$, as usual in mathematics.

Here are some more examples of performing function applications symbolically. Types are omitted for brevity; every non-function value is of type Int .

$$\begin{aligned} (x \Rightarrow x * 2)(10) &= 10 * 2 = 20 \quad . \\ (p \Rightarrow z \Rightarrow z * p)(t) &= (z \Rightarrow z * t) \quad . \\ (p \Rightarrow z \Rightarrow z * p)(t)(4) &= (z \Rightarrow z * t)(4) = 4 * t \quad . \end{aligned}$$

Some results of these computation are integer values such as 20; other results are nameless functions such as $z \Rightarrow z * t$. Verify this in Scala:

```
scala> ((x:Int) => x*2)(10)
res3: Int = 20

scala> ((p:Int) => (z:Int) => z*p)(10)
res4: Int => Int = <function1>

scala> ((p:Int) => (z:Int) => z*p)(10)(4)
res5: Int = 40
```

In the following examples, some arguments are themselves functions. Consider an expression that uses the nameless function $(g \Rightarrow g(2))$ as an argument:

$$(f \Rightarrow p \Rightarrow f(p))(g \Rightarrow g(2)) \quad (4.6)$$

$$\begin{aligned} \text{substitute } f = (g \Rightarrow g(2)) : &= p \Rightarrow (g \Rightarrow g(2))(p) \\ \text{substitute } g = p : &= p \Rightarrow p(2) \quad . \end{aligned} \quad (4.7)$$

The result of this expression is a function $p \Rightarrow p(2)$ that will apply *its* argument p to the value 2. A possible value for p is the function $x \Rightarrow x + 4$. So, let us apply Eq. (4.6) to that function:

$$\begin{aligned} & \underline{(f \Rightarrow p \Rightarrow f(p))(g \Rightarrow g(2))} (x \Rightarrow x + 4) \\ \text{use Eq. (4.7) :} &= \underline{(p \Rightarrow p(2))} (x \Rightarrow x + 4) \\ \text{substitute } p = (x \Rightarrow x + 4) : &= \underline{(x \Rightarrow x + 4)} (2) \\ \text{substitute } x = 2 : &= 2 + 4 = 6 \quad . \end{aligned}$$

To verify this calculation in Scala, we need to add appropriate type annotations for f and p . To figure out the types, we reason like this:

We know that the function $f \Rightarrow p \Rightarrow f(p)$ is being applied to the arguments $f = (g \Rightarrow g(2))$ and $p = (x \Rightarrow x + 4)$. So, the argument f in $f \Rightarrow p \Rightarrow f(p)$ must be a function that takes p as an argument.

The variable x in $x \Rightarrow x + 4$ must be of type `Int`, or else we cannot add x to 4. Thus, the type of the expression $x \Rightarrow x + 4$ is `Int => Int`, and so must be the type of the argument p . We write $p^{Int \Rightarrow Int}$.

Finally, we need to make sure that the types match in the function $f \Rightarrow p \Rightarrow f(p)$. Types match in $f(p)$ if the type of f 's argument is the same as the type of p , which is `Int => Int`. So f 's type must be $(Int \Rightarrow Int) \Rightarrow A$ for some type A . Since in our example $f = (g \Rightarrow g(2))$, types match only if g has type `Int => Int`. But then $g(2)$ has type `Int`, and so we must have $A = Int$. Thus, the type of f is $(Int \Rightarrow Int) \Rightarrow Int$. We know enough to write the Scala code now:

```
scala> ((f: (Int => Int) => Int) => p => f(p))(g => g(2))(x => x + 4)
res6: Int = 6
```

Type annotations for p , g , and x may be omitted because the Scala compiler will

figure out the missing types from the given type of f . However, it is never an error to specify more type annotations when writing code; it just makes the code longer.

4.3.2 Solved examples: Deriving a function's type from its code

Checking that the types match is an important part of the functional programming paradigm, both in the practice of writing code and in theoretical derivations of laws for various functions. For instance, in the derivations of the composition laws (Section 4.2.2), we were able to deduce the possible type parameters for f , g , and h in the expression $f \circ g \circ h$. This worked because the composition operation `andThen` (denoted by the symbol \circ) is fully parametric. Given a fully parametric function, it is often possible to derive the most general type signature that matches the body of that function. The same type-matching procedure may also help in converting a given function to a fully parametric form.

Let us look at some examples of doing this.

Example 4.3.2.1 The functions `const` and `id` were defined in Section 4.2.1. What is the value `const(id)` and what is its type? Determine the most general type parameters in the expression `const(id)`.

Solution We need to treat the functions `const` and `id` as values, since our goal is to apply `const` to `id`. Write the code of these functions in a short notation:

$$\begin{aligned}\text{const}^{C,X} &\triangleq (c{:}^C \Rightarrow _{:}^X \Rightarrow c) \quad , \\ \text{id}^A &\triangleq (a{:}^A \Rightarrow a) \quad .\end{aligned}$$

The types will match in the expression `const(id)` only if the argument of the function `const` has the same type as the type of `id`. Since “const” is a curried function, we need to look at its *first* curried argument, which is of type C . The type of `id` is $A \Rightarrow A$, where A is an arbitrary type so far. So, the type parameter C in $\text{const}^{C,X}$ must be equal to $A \Rightarrow A$:

$$C = A \Rightarrow A \quad .$$

The type parameter X in $\text{const}^{C,X}$ is not constrained, so we keep it as X . The result of applying `const` to `id` is of type $X \Rightarrow C$, which equals $X \Rightarrow A \Rightarrow A$. In this way, we find

$$\text{const}^{A \Rightarrow A, X}(\text{id}^A) : X \Rightarrow A \Rightarrow A \quad .$$

The types A and X can be arbitrary. The type $X \Rightarrow A \Rightarrow A$ is the most general type for the expression `const(id)` because we have not made any assumptions about the types except requiring that all functions must be always applied to arguments of the correct types.

To compute the value of `const(id)`, it remains to substitute the code of `const` and `id`. Since we already checked the types, we may omit all type annotations:

$$\begin{aligned}\text{const}(\text{id}) &= (\underline{c} \Rightarrow x \Rightarrow \underline{c})(\text{id}) \\ \text{definition of const :} &= (x \Rightarrow \underline{\text{id}}) \\ \text{apply function, substitute } c = \text{id} : &= (x \Rightarrow a \Rightarrow a) \quad .\end{aligned}$$

The function $(x \Rightarrow a \Rightarrow a)$ takes an argument $x{:}^X$ and returns the identity function $a{:}^A \Rightarrow a$. It is clear that the argument x is ignored by this function, so we can rewrite it equivalently as

$$\text{const}(\text{id}) = (_{:}^X \Rightarrow a{:}^A \Rightarrow a) \quad .$$

Example 4.3.2.2 Implement a function `twice` that takes a function `f: Int => Int` as its argument and returns a function that applies `f` twice. For example, if the function `f` is `{ x => x + 3 }`, the result of `twice(f)` should be equal to the function `x => x + 6`. After implementing the function `twice`, generalize it to a fully parametric function.

Solution According to the requirements, the function `twice` must return a new function of type `Int => Int`. So the type signature of `twice` is

```
def twice(f: Int => Int): Int => Int = ???
```

Since `twice(f)` must be a new function with an integer argument, we begin the code of `twice` by writing a new nameless function `{ (x: Int) => ... }`,

```
def twice(f: Int => Int): Int => Int = { (x: Int) => ??? }
```

The new function must apply `f` twice to its argument, that is, it must return `f(f(x))`. We can finish the implementation now:

```
def twice(f: Int => Int): Int => Int = { x => f(f(x)) }
```

The type annotation `(x: Int)` can be omitted. To test:

```
scala> val g = twice(x => x + 3)
g: Int => Int = <function1>

scala> g(10)
res0: Int = 16
```

To transform `twice` into a fully parametric function means replacing its type signature by a fully parameterized type signature while keeping the function body unchanged,

```
def twice[A, B, ...](f: ...): ... = { x => f(f(x)) }
```

To determine the type signature and the possible type parameters A, B, \dots , we need to determine the most general type that matches the function body. The function body is the expression $x \Rightarrow f(f(x))$. Assume that x has type A ; for types to match in the sub-expression $f(x)$, we need f to have type $A \Rightarrow B$ for some type B . The sub-expression $f(x)$ will then have type B . For types to match in $f(f(x))$, the argument of f must have type B ; but we already assumed $f:A \Rightarrow B$. This is consistent only if $A = B$. In this way, $x:A$ implies $f:A \Rightarrow A$, and the expression $x \Rightarrow f(f(x))$ has type $A \Rightarrow A$. We can now write the type signature of `twice`,

```
def twice[A](f: A => A): A => A = { x => f(f(x)) }
```

This fully parametric function has only one independent type parameter, A , and can be equivalently written in the code notation as

$$\text{twice}^A \triangleq f:A \Rightarrow A \Rightarrow x:A \Rightarrow f(f(x)) = (f:A \Rightarrow A \Rightarrow f \circ f) \quad . \quad (4.8)$$

The procedure of deriving the most general type for a given code is called **type inference**. In Example 4.3.2.2, the presence of the type parameter A and the type signature $(A \Rightarrow A) \Rightarrow A \Rightarrow A$ have been “inferred” from the code $f \Rightarrow x \Rightarrow f(f(x))$.

Example 4.3.2.3 Consider the fully parametric function `twice` defined in Example 4.3.2.2. What is the type of `twice(twice)`, and what computation does it perform? Test your answer on the expression `twice(twice[Int])(x => x+3)(10)`. What are the type parameters in that expression?

Solution Begin by figuring out the required type of `twice(twice)`. We introduce unknown type parameters as `twice[A](twice[B])`. The function `twice[A]` of type $(A \Rightarrow A) \Rightarrow A \Rightarrow A$ can be applied to the argument `twice[B]` only if `twice[B]` has type $A \Rightarrow A$. But `twice[B]` is of type $(B \Rightarrow B) \Rightarrow B \Rightarrow B$. Since the symbol \Rightarrow groups to the right, we have

$$(B \Rightarrow B) \Rightarrow B \Rightarrow B = (B \Rightarrow B) \Rightarrow (B \Rightarrow B) \quad .$$

This can match with $A \Rightarrow A$ only if we set $A = (B \Rightarrow B)$. So the most general type of `twice(twice)` is

$$\text{twice}^{B \Rightarrow B}(\text{twice}^B) : (B \Rightarrow B) \Rightarrow B \Rightarrow B \quad . \quad (4.9)$$

After checking that types match, we may omit types from further calculations.

Example 4.3.2.2 defined `twice` with the `def` syntax. To use `twice` as an argument in the expression `twice(twice)`, it is convenient to define `twice` as a value, `val twice = ...`. However, the function `twice` needs type parameters, and Scala 2 does not directly support `val` definitions with type parameters. Scala 3 will support type parameters appearing together with arguments in a nameless function:

```
val twice = [A] => (f: A => A) => (x: A) => f(f(x)) // Valid only in Scala 3.
```

Keeping this in mind, we will use the curried definition of `twice` given by Eq. (4.8), which shows that $\text{twice}(f) = f \circ f$. Substituting that into the expression `twice(twice)`, we find

$$\begin{aligned} \text{twice}(\text{twice}) &= \text{twice} \circ \text{twice} \\ \text{expand function composition} : &= f \Rightarrow \text{twice}(\text{twice}(f)) \quad . \\ \text{definition of } \text{twice}(f) : &= f \Rightarrow \text{twice}(f \circ f) \\ \text{definition of } \text{twice} : &= f \Rightarrow f \circ f \circ f \circ f \quad . \end{aligned}$$

So `twice(twice)` is a function that applies its (function-typed) argument *four* times.

The type parameters follow from Eq. (4.9) with $A = \text{Int}$ and can be written as `twiceInt⇒Int(twiceInt)`, or in Scala syntax, `twice[Int => Int](twice[Int])`. To test, we need to write at least one type parameter in the code, or else Scala cannot infer the types correctly:

```
scala> twice(twice[Int])(x => x + 3)(100) // _ + 3 + 3 + 3 + 3
res0: Int = 112

scala> twice[Int => Int](twice)(x => x + 3)(100)
res1: Int = 112
```

Example 4.3.2.4 Infer the type signature for the fully parametric function

```
def p[...]:... = { f => f(2) }
```

Can the types possibly match in the expression $p(p)$?

Solution In the nameless function $f \Rightarrow f(2)$, the argument f must be itself a function with an argument of type `Int`, otherwise the sub-expression $f(2)$ makes no sense. So, types will match if f has type $\text{Int} \Rightarrow \text{Int}$ or $\text{Int} \Rightarrow \text{String}$ or similar. The most general case is when f has type $\text{Int} \Rightarrow A$, where A is an arbitrary type (i.e. a type parameter). The type A will then be the (so far unknown) type of the value $f(2)$. Since nameless function $f \Rightarrow f(2)$ has an argument f of type $\text{Int} \Rightarrow A$ and the result of type A , we find that the type of p must be $(\text{Int} \Rightarrow A) \Rightarrow A$. With this type assignment, all types match. The type parameter A remains undetermined and is added to the type signature of the function p . The code is

```
def p[A]: (Int => A) => A = { f => f(2) }
```

To answer the question about the expression $p(p)$, we begin by writing that expression with new type parameters as $p[A](p[B])$. Then we try to choose A and B so that the types match in that expression. Does the type of $p[B]$, which is $(\text{Int} \Rightarrow B) \Rightarrow B$, match the type of the argument of $p[A]$, which is $\text{Int} \Rightarrow A$, with some choice of A and B ? A function type $P \Rightarrow Q$ matches $x \Rightarrow y$ only if $P = x$ and $Q = y$. So $(\text{Int} \Rightarrow B) \Rightarrow B$ can match $\text{Int} \Rightarrow A$ only if $\text{Int} \Rightarrow B$ matches Int and if $B = A$. But it is impossible for $\text{Int} \Rightarrow B$ to match Int , no matter how we choose B .

We conclude that types cannot be chosen consistently in $p[A](p[B])$. Such expressions contain a type error and are rejected by the Scala compiler. One also says that the expression $p(p)$ is **not well-typed**, or does not **typecheck**.

For any given code expression containing only function expressions, one can always find the most general type that makes all functions match their arguments, unless the expression does not typecheck. The Damas-Hindley-Milner algorithm¹ performs type inference (or determines that there is a type error) for a large class of expressions containing functions, tuples, and disjunctive types.

4.4 Summary

Table 4.1 shows the code notations introduced in this chapter.

What can we do with this chapter's techniques?

- Implement functions that return new functions and/or take functions as arguments.
- Simplify function applications symbolically.
- Infer the most general type for a given code expression.
- Convert functions to a fully parametric form when possible.

The following solved examples and exercises illustrate these techniques.

4.4.1 Solved examples

Example 4.4.1.1 Implement a function that applies a given function f repeatedly to an initial value x_0 , until a given function `cond` returns `true`:

¹https://en.wikipedia.org/wiki/Hindley%E2%80%93Milner_type_system#Algorithm_W

Code notation	Scala syntax	Comments
$x:A$	<code>x:A</code>	a value or an argument of type A
$f:A \Rightarrow B$	<code>f: A => B</code>	a function of type A => B
$x:\text{Int} \Rightarrow f(x)$	<code>{ x: Int => f(x)}</code>	nameless function with argument x
$f^{A,B} \triangleq \dots$	<code>def f[A, B] = ...</code>	define a function with type parameters
id^A , also $\text{id}:A \Rightarrow A$	<code>identity[A]</code>	the standard identity function
$A \Rightarrow B \Rightarrow C$	<code>A => B => C</code>	type of a curried function
$f \circ g$	<code>f andThen g</code>	forward composition of functions
$g \circ f$	<code>g compose f</code>	backward composition of functions

Table 4.1: Mathematical notation for symbolic computations with code.

```
def converge[X](f: X => X, x0: X, cond: X => Boolean): X = ???
```

Solution We create an iterator that keeps applying the function f , and use `.find` to stop the sequence when the condition first holds:

```
def converge[X](f: X => X, x0: X, cond: X => Boolean): X =
  Stream.iterate(x0)(f) // Type is Stream[X].
    .find(cond)         // Type is Option[X].
    .get                // Type is X.
```

The method `.get` is a partial function that can be applied only to non-empty `Option` values. But here it is safe to call `.get`, because the stream is unbounded and, if the condition `cond` never becomes `true`, the program will run out of memory (since `Stream.iterate` keeps all computed values in memory) or the user will run out of patience. So `.find(cond)` can never return an empty `Option` value. Of course, it is not satisfactory that the program crashes when the sequence does not converge. Exercise 4.4.2.7 will implement a safer version of this function by limiting the allowed number of iterations.

A tail-recursive implementation that works in constant memory is

```
@tailrec def converge[X](f: X => X, x0: X, cond: X => Boolean): X =
  if (cond(x0)) x0 else converge(f, f(x0), cond)
```

To test this code, compute an approximation to \sqrt{q} by Newton's method. The iteration function f is

$$f(x) = \frac{1}{2} \left(x + \frac{q}{x} \right) .$$

We iterate $f(x)$ starting with $x_0 = q/2$ until we obtain a given precision:

```
def approx_sqrt(q: Double, precision: Double): Double = {
  def cond(x: Double): Boolean = math.abs(x * x - q) <= precision
  def iterate_sqrt(x: Double): Double = 0.5 * (x + q / x)
  converge(iterate_sqrt, q / 2, cond)
}
```

Newton's method for \sqrt{q} is guaranteed to converge when $q \geq 0$. Test it:

```
scala> approx_sqrt(25, 1.0e-8)
res0: Double = 5.000000000016778
```

Example 4.4.1.2 Using both `def` and `val`, define a Scala function that takes an integer x and returns a function that adds x to its argument.

Solution Let us first write down the required type signature: the function must take an integer argument $x:\text{Int}$, and the return value must be a function of type `Int => Int`.


```
def add_x(x: Int): Int => Int = ???
```

We are required to return a function that adds x to its argument. Let us call that argument z , to avoid confusion with the x . So, we are required to return a function that we can write as $\{ z \Rightarrow z + x \}$. Since functions are values, we can directly return a new function by writing a nameless function expression:

```
def add_x(x: Int): Int => Int = { z => z + x }
```

To implement the same function as a `val`, we first convert the type signature of `add_x` to the equivalent type expression $\text{Int} \Rightarrow \text{Int} \Rightarrow \text{Int}$. Now we can write the Scala code of a function `add_x_v`:

```
val add_x_v: Int => Int => Int = { x => z => z + x }
```

The function `add_x_v` is equal to `add_x` except for using the `val` syntax instead of `def`. It is not necessary to specify the type of the arguments x and z because we already specified the type $\text{Int} \Rightarrow \text{Int} \Rightarrow \text{Int}$ for the value `add_x_v`.

Example 4.4.1.3 Using both `def` and `val`, implement a curried function `prime_f` that takes a function f and an integer x , and returns `true` when $f(x)$ is a prime number. Use the function `is_prime` defined in Section 1.1.2.

Solution First, determine the required type signature of `prime_f`. The value $f(x)$ must have type `Int`, or else we cannot check whether it is prime. So, f must have type $\text{Int} \Rightarrow \text{Int}$. Since `prime_f` should be a curried function, we need to put each argument into its own set of parentheses:

```
def prime_f(f: Int => Int)(x: Int): Boolean = ???
```

To implement `prime_f`, we need to return the result of `is_prime` applied to $f(x)$. A simple solution is

```
def prime_f(f: Int => Int)(x: Int): Boolean = is_prime(f(x))
```

To implement the same function as a `val`, rewrite its type signature as

```
val prime_f: (Int => Int) => Int => Boolean = ???
```

(The parentheses around `Int => Int` are mandatory since `Int => Int => Int => Boolean` would be a completely different type.) The implementation is

```
val prime_f: (Int => Int) => Int => Boolean = { f => x => is_prime(f(x)) }
```

The code `is_prime(f(x))` is a forward composition of the functions f and `is_prime`, so we can write

```
val prime_f: (Int => Int) => Int => Boolean = (f => f andThen is_prime)
```

A nameless function of the form `f => f.something` is equivalent to a shorter Scala syntax `(_.something)`. So we finally rewrite the code of `prime_f` as

```
val prime_f: (Int => Int) => Int => Boolean = (_ andThen is_prime)
```

Example 4.4.1.4 Implement a function `choice(x,p,f,g)` that takes a value x , a predicate p , and two functions f and g . The return value must be $f(x)$ if $p(x)$ returns `true`; otherwise the return value must be $g(x)$. Infer the most general type for this function.

Solution The code of this function must be

```
def choice[...](x,p,f,g) = if (p(x)) f(x) else g(x)
```

To infer the most general type for this code, begin by assuming that x has type A , where A is a type parameter. Then the predicate p must have type $A \Rightarrow \text{Boolean}$. Since p is an arbitrary predicate, the value $p(x)$ will be sometimes `true` and sometimes `false`. So, `choice(x,p,f,g)` will sometimes return $f(x)$ and sometimes $g(x)$. It follows that type A must be the argument type of both f and g , which means that the most general types so far are $f:A \Rightarrow B$ and $g:A \Rightarrow C$, yielding the type signature

$$\text{choice}(x:A, p:A \Rightarrow \text{Boolean}, f:A \Rightarrow B, g:A \Rightarrow C) \quad .$$

What could be the return type of `choice(x,p,f,g)`? If $p(x)$ returns `true`, the function `choice` returns $f(x)$, which is of type B . Otherwise, `choice` returns $g(x)$, which is of type C . However, the type signature of `choice` must be fixed in advance (at compile time) and cannot depend on the value $p(x)$ computed at run time. So, the types of $f(x)$ and of $g(x)$ must be the same, $B = C$. The type signature of `choice` will thus have only two type parameters, A and B :

```
def choice[A, B](x: A, p: A => Boolean, f: A => B, g: A => B): B =
  if (p(x)) f(x) else g(x)
```

Example 4.4.1.5 Infer the most general type for the fully parametric function

```
def q[...]: ... = { f => g => g(f) }
```

What types are inferred for the expressions $q(q)$ and $q(q(q))$?

Solution Begin by assuming $f^{:A}$ with a type parameter A . In the sub-expression $g \Rightarrow g(f)$, the curried argument g must itself be a function, because it is being applied to f as $g(f)$. So we assign types as $f^{:A} \Rightarrow g^{:A \Rightarrow B} \Rightarrow g(f)}$, where A and B are type parameters. Then the final returned value $g(f)$ has type B . Since there are no other constraints on the types, the types A and B remain arbitrary, so we add them to the type signature:

```
def q[A, B]: A => (A => B) => B = { f => g => g(f) }
```

To match types in the expression $q(q)$, we first assume arbitrary type parameters and write $q[A, B](q[C, D])$. We need to introduce new type parameters C, D because these type parameters will probably need to be set differently from A, B when we try to match the types in the expression $q(q)$.

The type of the first curried argument of $q[A, B]$, which is A , must match the entire type of $q[C, D]$, which is $C \Rightarrow (C \Rightarrow D) \Rightarrow D$. So we must set the type parameter A as

$$A = C \Rightarrow (C \Rightarrow D) \Rightarrow D \quad .$$

The type of $q(q)$ becomes

$$q^{A,B}(q^{C,D}) : ((C \Rightarrow (C \Rightarrow D) \Rightarrow D) \Rightarrow B) \Rightarrow B \quad , \\ \text{where } A = C \Rightarrow (C \Rightarrow D) \Rightarrow D \quad .$$

There are no other constraints on the type parameters B, C, D .

We use this result to infer the most general type for $q(q(q))$. We may denote $r \triangleq q(q)$ for brevity; then, as we just found, r has type $((C \Rightarrow (C \Rightarrow D) \Rightarrow D) \Rightarrow B) \Rightarrow B$. To infer types in the expression $q(r)$, we introduce new type parameters E, F and write $q[E, F](r)$. The type of the argument of $q[E, F]$ is E , and this must be the same as the type of r . This gives the constraint

$$E = ((C \Rightarrow (C \Rightarrow D) \Rightarrow D) \Rightarrow B) \Rightarrow B \quad .$$

Other than that, the type parameters are arbitrary. The type of the expression $q(q(q))$ is $(E \Rightarrow F) \Rightarrow F$. We conclude that the most general type of $q(q(q))$ is

$$q^{E,F}(q^{A,B}(q^{C,D})) : (((C \Rightarrow (C \Rightarrow D) \Rightarrow D) \Rightarrow B) \Rightarrow B) \Rightarrow F) \Rightarrow F \quad , \\ \text{where } A = C \Rightarrow (C \Rightarrow D) \Rightarrow D \\ \text{and } E = ((C \Rightarrow (C \Rightarrow D) \Rightarrow D) \Rightarrow B) \Rightarrow B \quad .$$

It is clear from this derivation that expressions such as $q(q(q(q)))$, $q(q(q(q(q))))$, etc., are well-typed.

Let us test these results in Scala, renaming the type parameters for clarity to A, B, C, D :

```
scala> def qq[A, B, C]: ((A => (A => B) => B) => C) => C = q(q)
qq: [A, B, C] => ((A => ((A => B) => B)) => C) => C

scala> def qqq[A, B, C, D]: (((A => (A => B) => B) => C) => C) => D = q(q(q))
qqq: [A, B, C, D] => (((A => ((A => B) => B)) => C) => C) => D => D
```

We did not need to write any type parameters within the expressions $q(q)$ and $q(q(q))$ because the full type signature was declared for each of these expressions. Since the Scala compiler did not print any error messages, we are assured that the types match correctly.

Example 4.4.1.6 Infer types in the code expression

$$(f \Rightarrow g \Rightarrow g(f)) (f \Rightarrow g \Rightarrow g(f)) (f \Rightarrow f(10))$$

and simplify the code by symbolic calculations.

Solution The given expression is a curried function $f \Rightarrow g \Rightarrow g(f)$ applied to two curried arguments. The plan is to consider each of these sub-expressions in turn, assigning types for them using type parameters, and then to figure out how to set the type parameters so that all types match.

Begin by renaming the shadowed variables (f and g) to remove shadowing:

$$(f \Rightarrow g \Rightarrow g(f)) (x \Rightarrow y \Rightarrow y(x)) (h \Rightarrow h(10)) \quad . \quad (4.10)$$

As we have seen in Example 4.4.1.5, the sub-expression $f \Rightarrow g \Rightarrow g(f)$ is typed as $f^{A \Rightarrow B} \Rightarrow g^{A \Rightarrow B} \Rightarrow g(f)$, where A and B are some type parameters. The sub-expression $x \Rightarrow y \Rightarrow y(x)$ is the same function as $f \Rightarrow g \Rightarrow g(f)$ but with possibly different type parameters, say, $x^{C \Rightarrow D} \Rightarrow y^{C \Rightarrow D} \Rightarrow y(x)$. The types A, B, C, D are so far unknown.

Finally, the variable h in the sub-expression $h \Rightarrow h(10)$ must have type $\text{Int} \Rightarrow E$, where E is another type parameter. So, the sub-expression $h \Rightarrow h(10)$ is a function of type $(\text{Int} \Rightarrow E) \Rightarrow E$.

The types must match in the entire expression (4.10):

$$(f^{A \Rightarrow B} \Rightarrow g^{A \Rightarrow B} \Rightarrow g(f))(x^{C \Rightarrow D} \Rightarrow y^{C \Rightarrow D} \Rightarrow y(x))(h^{\text{Int} \Rightarrow E} \Rightarrow h(10)) \quad . \quad (4.11)$$

It follows that f must have the same type as $x \Rightarrow y \Rightarrow y(x)$, while g must have the same type as $h \Rightarrow h(10)$. The type of g , which we know as $A \Rightarrow B$, will match the type of $h \Rightarrow h(10)$, which we know as $(\text{Int} \Rightarrow E) \Rightarrow E$, only if $A = (\text{Int} \Rightarrow E)$ and $B = E$. It follows that f has type $\text{Int} \Rightarrow E$. At the same time, the type of f must match the type of $x \Rightarrow y \Rightarrow y(x)$, which is $C \Rightarrow (C \Rightarrow D) \Rightarrow D$. This can work only if $C = \text{Int}$ and $E = (C \Rightarrow D) \Rightarrow D = (\text{Int} \Rightarrow D) \Rightarrow D$.

In this way, we have found all the relationships between the type parameters A, B, C, D, E in Eq. (4.11). The type D remains undetermined (i.e. arbitrary), while the type parameters A, B, C, E are expressed as

$$A = \text{Int} \Rightarrow (\text{Int} \Rightarrow D) \Rightarrow D \quad , \quad (4.12)$$

$$B = E = (\text{Int} \Rightarrow D) \Rightarrow D \quad , \quad (4.13)$$

$$C = \text{Int} \quad .$$

The entire expression in Eq. (4.11) is a saturated application of a curried function, and thus has the same type as the “final” result expression $g(f)$, which has type B . So, the entire expression in Eq. (4.11) has type $B = (\text{Int} \Rightarrow D) \Rightarrow D$.

Having established that types match, we can now omit the type annotations and rewrite the code:

$$\begin{aligned} & (f \Rightarrow g \Rightarrow \underline{g(f)})(x \Rightarrow y \Rightarrow y(x)) (h \Rightarrow h(10)) \\ \text{substitute } f, g : & = (h \Rightarrow \underline{h(10)})(x \Rightarrow y \Rightarrow y(x)) \\ \text{substitute } h : & = (x \Rightarrow y \Rightarrow \underline{y(x)})(10) \\ \text{substitute } x : & = y \Rightarrow y(10) \quad . \end{aligned}$$

The type of this expression is $(\text{Int} \Rightarrow D) \Rightarrow D$ with a type parameter D . Since the argument y is an arbitrary function, we cannot simplify $y(10)$ or $y \Rightarrow y(10)$ any further. We conclude that $y^{\text{Int} \Rightarrow D} \Rightarrow y(10)$ is the final simplified form of Eq. (4.10).

To test this, we first define the function $f \Rightarrow g \Rightarrow g(f)$ as in Example 4.4.1.5,

```
def q[A, B]: A => (A => B) => B = { f => g => g(f) }
```

We also define the function $h \Rightarrow h(10)$ with a general type $(\text{Int} \Rightarrow E) \Rightarrow E$,

```
def r[E]: (Int => E) => E = { h => h(10) }
```

To help Scala evaluate Eq. (4.11), we need to set the type parameters for the first q function as $q[A, B]$ where A and B are given by Eqs. (4.12)–(4.13):

```
scala> def s[D] = q[Int => (Int => D) => D, (Int => D) => D](q)(r)
s: [D] => (Int => D) => D
```

To verify that the function s^D indeed equals $y^{\text{Int} \Rightarrow D} \Rightarrow y(10)$, we apply s^D to some functions of type $\text{Int} \Rightarrow D$, say, for $D = \text{Boolean}$ or $D = \text{Int}$:

```
scala> s(_ > 0) // Set D = Boolean and evaluate (10 > 0).
res6: Boolean = true

scala> s(_ + 20) // Set D = Int and evaluate (10 + 20).
res7: Int = 30
```

4.4.2 Exercises

Exercise 4.4.2.1 For `id` and `const` as defined above, what are the types of `id(id)`, `id(id)(id)`, `id(id(id))`, `id(const)`, and `const(const)`? Simplify these code expressions by symbolic calculations.

Exercise 4.4.2.2 For the function `twice` from Example 4.3.2.2, infer the most general type for the function `twice(twice(twice)))`. What does that function do? Test your answer on an example.

Exercise 4.4.2.3 Define a function `thrice` similarly to `twice` except it should apply a given function 3 times. What does the function `thrice(thrice(thrice)))` do?

Exercise 4.4.2.4 Define a function `ence` similarly to `twice` except it should apply a given function n times, where n is an additional curried argument.

Exercise 4.4.2.5 Define a fully parametric function `flip(f)` that swaps arguments for any given function `f` having two arguments. To test:

```
def f(x: Int, y: Int) = x - y // Expect f(10, 2) == 8.
val g = flip(f) // Now expect g(2, 10) == 8.

scala> assert( f (10, 2) == 8 && g(2, 10) == 8 )
```

Exercise 4.4.2.6 Revise the function from Exercise 1.6.2.4, implementing it a curried function and replacing the hard-coded number 100 by a *curried* first argument. The type signature should become `Int => List[List[Int]] => List[List[Int]]`.

Exercise 4.4.2.7 Implement the function `converge` from Example 4.4.1.1 as a curried function, with an additional argument to set the maximum number of iterations, returning `Option[Double]` as the final result type. The new version of `converge` should return `None` if the convergence condition is not satisfied after the given maximum number of iterations. The type signature and an example test:

```
@tailrec def convergeN[X](cond: X => Boolean)(x0: X)(maxIter: Int)(f: X => X): Option[X] = ???

scala> convergeN[Int](_ < 0)(0)(10)(_ + 1) // This does not converge.
res0: Option[Int] = None

scala> convergeN[Double]{ x => math.abs(x * x - 25) < 1e-8 }(1.0)(10) { x => 0.5 * (x + 25 / x) }
res1: Option[Double] = Some(5.000000000053722)
```

Exercise 4.4.2.8 Write a function `curry2` converting an uncurried function of type $(\text{Int}, \text{Int}) \Rightarrow \text{Int}$ into an equivalent curried function of type $\text{Int} \Rightarrow \text{Int} \Rightarrow \text{Int}$.

Exercise 4.4.2.9 Apply the function $(x \Rightarrow _ \Rightarrow x)$ to the value $(z \Rightarrow z(q))$ where q is a given value of type Q . Infer types in these expressions.

Exercise 4.4.2.10 Infer types in the following expressions and test in Scala:

(a) $p \Rightarrow q \Rightarrow p(t \Rightarrow t(q))$.

(b) $p \Rightarrow q \Rightarrow q(x \Rightarrow x(p(q)))$.

Exercise 4.4.2.11 Show that the following expressions cannot be well-typed:

(a) $p \Rightarrow p(q \Rightarrow q(p))$.

(b) $p \Rightarrow q \Rightarrow q(x \Rightarrow p(q(x)))$.

Exercise 4.4.2.12 Infer types and simplify the following code expressions by symbolic calculations:

(a) $q \Rightarrow (x \Rightarrow y \Rightarrow z \Rightarrow x(z)(y(z)))(a \Rightarrow a)(b \Rightarrow b(q))$.

(b) $(f \Rightarrow g \Rightarrow h \Rightarrow f(g(h)))(x \Rightarrow x)$.

(c) $(x \Rightarrow y \Rightarrow x(y))(x \Rightarrow y \Rightarrow x)$.

(d) $(x \Rightarrow y \Rightarrow x(y))(x \Rightarrow y \Rightarrow y)$.

(e) $x \Rightarrow (f \Rightarrow y \Rightarrow f(y)(x))(z \Rightarrow _ \Rightarrow z)$.

(f) $z \Rightarrow (x \Rightarrow y \Rightarrow x)(x \Rightarrow x(z))(y \Rightarrow y(z))$.

4.5 Discussion

4.5.1 Higher-order functions

The **order** of a function is the number of function arrows (\Rightarrow) contained in the type signature of that function. If a function's type signature contains more than one arrow, the function is called a **higher-order** function. Higher-order functions take functions as arguments and/or return functions as result values.

The methods `andThen`, `compose`, `curried`, and `uncurried` are examples of higher-order functions that take other functions as arguments *and* return a new function.

The following example illustrates the concept of a function's order. Consider

```
def f1(x: Int): Int = x + 10
```

The function `f1` has type signature `Int => Int` and order 1, so it is *not* a higher-order function.

```
def f2(x: Int): Int => Int = (z => z + x)
```

The function `f2` has type signature `Int => Int => Int` and is a higher-order function of order 2.

```
def f3(g: Int => Int): Int = g(123)
```

The function `f3` has type signature `(Int => Int) => Int` and is a higher-order function of order 2.

Although `f2` is a higher-order function, its "higher-orderness" comes from the fact that the return value is of a function type. An equivalent computation can be performed by an uncurried function that is not higher-order:

```
scala> def f2u(x: Int, z: Int): Int = z + x
```

Unlike `f2`, the function `f3` *cannot* be converted to a non-higher-order function because `f3` has an argument of a function type. Converting to an uncurried form cannot eliminate an argument of a function type.

4.5.2 Name shadowing and the scope of bound variables

Bound variables are introduced in nameless functions whenever an argument is defined. For example, in the curried nameless function $x \Rightarrow y \Rightarrow x + y$, the bound variables are the curried arguments x and y . The variable y is only defined within the scope $(y \Rightarrow x + y)$ of the inner function; the variable x is defined within the entire scope of $x \Rightarrow y \Rightarrow x + y$.

Another way of introducing bound variables in Scala is to write a `val` or a `def` within curly braces:

```
val x = {
  val y = 10          // Bound variable.
  y + y * y
} // Same as 'val x = 10 + 10 * 10'.
```

A bound variable is invisible outside the scope that defines it. So, it is easy to rename a bound variable: no outside code could possibly use it and depend on its value.

However, outside code may define a variable that (by chance) has the same name as a bound variable inside the scope. Consider this example from calculus: In the integral

$$f(x) = \int_0^x \frac{dx}{1+x} \quad ,$$

two bound variables named x are defined in two scopes: one in the scope of f , another in the scope of the nameless function $x \mapsto \frac{1}{1+x}$. The convention in mathematics is to treat these two x 's as two *completely different* variables that just happen to have the same name. In sub-expressions where both of these bound variables are visible, priority is given to the bound variable defined in the closest inner scope. The outer definition of x is then **shadowed**, i.e. hidden, by the definition of the inner x . For this reason, mathematicians expect that evaluating $f(10)$ will give

$$f(10) = \int_0^{10} \frac{dx}{1+x} = \log_e(11) \approx 2.398 \quad ,$$

rather than $\int_0^{10} \frac{dx}{1+10} = \frac{10}{11}$, because the outer definition $x = 10$ is shadowed within the expression $\frac{1}{1+x}$ by the definition of x in the local scope of $x \mapsto \frac{1}{1+x}$.

Since this is the standard mathematical convention, the same convention is adopted in functional programming. A variable defined in a function scope (i.e. a bound variable) is invisible outside that scope but will shadow any outside definitions of a variable with the same name.

Name shadowing is not advisable in practical programming, because it usually decreases the clarity of code and so invites errors. Consider the nameless function

$$x \Rightarrow x \Rightarrow x \quad ,$$

and let us decipher this confusing syntax. The symbol \Rightarrow groups to the right, so $x \Rightarrow x \Rightarrow x$ is the same as $x \Rightarrow (x \Rightarrow x)$. It is a function that takes x and returns $x \Rightarrow x$. Since the nameless function $(x \Rightarrow x)$ may be renamed to $(y \Rightarrow y)$ without changing its value, we can rewrite the code to

$$x \Rightarrow (y \Rightarrow y) \quad .$$

Having removed name shadowing, we can more easily understand this code and reason about it. For instance, it becomes clear that this function ignores its argument x and always returns the same value (the identity function $y \Rightarrow y$). So we can rewrite $(x \Rightarrow x \Rightarrow x)$ as $(_ \Rightarrow y \Rightarrow y)$, which is clearer.

4.5.3 Operator syntax for function applications

In mathematics, function applications are sometimes written without parentheses, for instance $\cos x$ or $\sin z$. Formulas such as $2 \sin x \cos x$ imply parentheses as $2 \cdot \sin(x) \cdot \cos(x)$. Functions such as $\cos x$ are viewed as “operators” that are applied to their arguments without parentheses, similar to the operators of summation $\sum_k f(k)$ and differentiation $\frac{d}{dx}f(x)$.

Many programming languages (such as ML, OCaml, F#, Haskell, Elm, PureScript) have adopted this “operator syntax”, making parentheses optional for function arguments so that $f\ x$ means the same as $f(x)$. Parentheses are still used where necessary to avoid ambiguity or for readability.²

The conventions for nameless functions in the operator syntax become:

²The operator syntax has a long history in programming. It is used in Unix shell commands, for example `cp file1 file2`. In LISP-like languages, function applications are enclosed in parentheses but the arguments are space-separated, for example `(f 10 20)`. Operator syntax is also used in some programming languages such as Tcl, Groovy, and CoffeeScript.

- Function expressions group to the right, so $x \Rightarrow y \Rightarrow z \Rightarrow e$ means $x \Rightarrow (y \Rightarrow (z \Rightarrow e))$.
- Function applications group to the left, so $f\ x\ y\ z$ means $((f\ x)y)z$.
- Function applications group stronger than infix operations, so $x + f\ y$ means $x + (f\ y)$, just as in mathematics “ $x + \cos y$ ” groups “ $\cos y$ ” stronger than the infix “ $+$ ” operation.

Thus, $x \Rightarrow y \Rightarrow a\ b\ c + p\ q$ means $x \Rightarrow (y \Rightarrow ((a\ b)\ c) + (p\ q))$. When this notation becomes hard to read correctly, one needs to add parentheses, e.g. to write $f(x \Rightarrow g\ h)$ instead of $f\ x \Rightarrow g\ h$.

This book will avoid using the “operator syntax” when reasoning about code. Scala does not support the parentheses-free operator syntax; parentheses are needed around each curried argument.

From the point of view of programming language theory, curried functions are “simpler” because they always have a *single* argument (and may return a function that will consume further arguments). From the point of view of programming practice, curried functions are often harder to read and to write.

In the operator syntax used e.g. in OCaml and Haskell, a curried function f is applied to curried arguments as, e.g., $f\ 20\ 4$. This departs further from the mathematical tradition and requires some getting used to. If the two arguments are more complicated than just 20 and 4, the resulting expression may become harder to read, compared with the syntax where commas are used to separate the arguments. (Consider, for instance, the expression $f\ (g\ 10)\ (h\ 20) + 30$.) To improve readability of code, programmers may prefer to first define short names for complicated expressions and then use these names as curried arguments.

In Scala, the choice of whether to use curried or uncurried function signatures is largely a matter of syntactic convenience. Most Scala code tends to be written with uncurried functions, while curried functions are used when they produce more easily readable code.

One of the syntactic features for curried functions in Scala is the ability to specify a curried argument using the curly brace syntax. Compare the two definitions of the function `summation` described in Section 1.7.5:

```
def summation1(a: Int, b: Int, g: Int => Int): Int = (a to b).map(g).sum

def summation2(a: Int, b: Int)(g: Int => Int): Int = (a to b).map(g).sum
```

To apply these functions to arguments, we need to use slightly different syntax:

```
scala> summation1(1, 10, { x => x*x*x + 2*x })
res0: Int = 3135

scala> summation2(1, 10) { x => x*x*x + 2*x }
res1: Int = 3135
```

The code that calls `summation2` is easier to read because the curried argument is syntactically separated from the rest of the code by curly braces. This is especially useful when the curried argument is itself a function with a complicated body, since Scala’s curly braces syntax allows function

bodies to contain their own local definitions (`val` or `def`).

Another feature of Scala is the “dotless” method syntax: for example, `xs map f` is equivalent to `xs.map(f)` and `f andThen g` is equivalent to `f.andThen(g)`. The “dotless” syntax is available only for infix methods, such as `.map`, defined on specific types such as `Seq`. In Scala 3, the “dotless” syntax will only work for methods having a special `@infix` annotation. Do not confuse Scala’s “dotless” method syntax with the operator syntax used in Haskell and some other languages.

4.5.4 Deriving a function’s code from its type signature

We have seen how the procedure of type inference derives the type signature from a function’s code. A well-known algorithm for type inference is the Damas-Hindley-Milner algorithm,³ with a Scala implementation available.⁴

³https://en.wikipedia.org/wiki/Hindley%E2%80%93Milner_type_system

⁴<http://dysphoria.net/2009/06/28/hindley-milner-type-inference-in-scala/>

It is remarkable that one can sometimes derive the function's *code* from the function's type signature. We will now look at some examples of this.

Consider a fully parametric function that performs partial applications for arbitrary other functions. A possible type signature is

```
def pa[A, B, C](x: A)(f: (A, B) => C): B => C = ???
```

The function `pa` substitutes a fixed argument value $x:A$ into another given function f .

How can we implement `pa`? Since `pa(x)(f)` must return a function of type $B \Rightarrow C$, we have no choice other than to begin writing a nameless function in the code,

```
def pa[A, B, C](x: A)(f: (A, B) => C): B => C = { y: B =>
  ??? // Need to compute a value of type C in this scope.
}
```

In the inner scope, we need to compute a value of type C , and we have values $x:A$, $y:B$, and $f: (A, B) \Rightarrow C$. How can we compute a value of type C ? If we knew that $C = \text{Int}$ when `pa(x)(f)` is applied, we could have simply selected a fixed integer value, say, `1`, as the value of type C . If we knew that $C = \text{String}$, we could have selected a fixed string, say, `"hello"`, as the value of type C . But a fully parametric function cannot use any knowledge of the types of its actual arguments.

So, a fully parametric function cannot produce a value of an arbitrary type C from scratch. The only way of producing a value of type C is by applying the function f to arguments of types A and B . Since the types A and B are arbitrary, we cannot obtain any values of these types other than $x:A$ and $y:B$. So, the only way of getting a value of type C is to compute $f(x, y)$. Thus, the body of `pa` must be

```
def pa[A, B, C](x: A)(f: (A, B) => C): B => C = { y => f(x, y) }
```

In this way, we have *unambiguously* derived the body of this function from its type signature, by assuming that the function must be fully parametric.

Another example is the operation of forward composition $f \circ g$ viewed as a fully parametric function with type signature

```
def before[A, B, C](f: A => B, g: B => C): A => C = ???
```

To implement `before`, we need to create a nameless function of type $A \Rightarrow C$,

```
def before[A, B, C](f: A => B, g: B => C): A => C = { x: A =>
  ??? // Need to compute a value of type C in this scope.
}
```

In the inner scope, we need to compute a value of type C from the values $x:A$, $f:A \Rightarrow B$, and $g:B \Rightarrow C$. Since the type C is arbitrary, the only way of obtaining a value of type C is by applying g to an argument of type B . In turn, the only way of obtaining a value of type B is to apply f to an argument of type A . Finally, we have only one value of type A , namely $x:A$. So, the only way of obtaining the required result is to compute $g(f(x))$.

We have unambiguously derived the body of the function from its type signature:

```
def before[A, B, C](f: A => B, g: B => C): A => C = { x => g(f(x)) }
```

In Chapter 5 and in Appendix C, we will see how a function's code can be derived from type signatures for a wide range of fully parametric functions.

5 The logic of types. III. The Curry-Howard correspondence

Fully parametric functions (introduced in Section 4.2) perform operations so general that their code does not depend on values of any specific data types such as `Int` or `String`. An example of a fully parametric function is

```
def before[A, B, C](f: A => B, g: B => C): A => C = { x => g(f(x)) }
```

We have also seen in Section 4.5.4 that for certain functions of this kind, the code can be derived unambiguously from the type signature.

There exists a mathematical theory (called the **Curry-Howard correspondence**) that provides precise conditions for the possibility of deriving a function's code from its type and a systematic derivation algorithm. Technical details about the algorithm are found in Appendix C. This chapter describes the main results and applications of this theory to functional programming.

5.1 Values computed by fully parametric functions

5.1.1 Motivation

Consider the Scala code of a fully parametric function,

```
def f[A, B, ...]: ... = {  
  val x: Either[A, B] = ... // Some expression here.  
  ... }
```

If this program compiles without type errors, it means that the types match and, in particular, that the function `f` is able to compute a value `x` of type `Either[A, B]`.

It is sometimes *impossible* to compute a value of a certain type within the body of a fully parametric function. For example, the fully parametric function `fmap` shown in Section 3.2.3.1 cannot compute any values of type `A`,

```
def fmap[A, B](f: A => B): Option[A] => Option[B] = {  
  val x: A = ??? // Cannot compute x here!  
  ... }
```

The reason is that a fully parametric function cannot compute values of type `A` from scratch without using previously given values of type `A` and without applying a function.

tion that returns values of type `A`. In `fmap`, no values of type `A` are given as arguments; the given function `f: A => B` returns values of type `B` and not `A`. The code of `fmap` must perform pattern matching on a value of type `Option[A]`:

```
def fmap[A, B](f: A => B): Option[A] => Option[B] = {  
  case None =>  
    val x: A = ??? // Cannot compute x here!  
    ...  
  case Some(a) =>  
    val x: A = a // Can compute x in this scope.  
    ... }
```

Since the case `None` has no values of type `A`, we are unable to compute a value `x` in that scope (as long as `fmap` remains a fully parametric function).

Being able to compute `x:A` “within the body of a function” means that, if needed, the function should be able to *return* `x` as a

result value. This requires computing `x` in all cases, not just within one part of the `match` expression.

The body of `fmap` also cannot compute any values of type `B`. Since no arguments of type `B` are given, the only way of obtaining a value of type `B` would be to apply the function `f: A => B` to *some* value of type `A`; but we just saw that the body of `fmap` cannot compute any values of type `A`.

Another example where one cannot compute a value of a certain type is in the following code:

```
def before[A, B, C](f: A => B, g: B => C): A => C = {
  // val h: C => A = ??? // Cannot compute h here!
  a => g(f(a)) // Can compute a value of type A => C.
}
```

The body of `before` may only use the arguments `f` and `g`. We can compute a value of type `A => C` by composing `f` and `g`, but it is impossible to compute a value `h` of type `C => A`.

`=> A`, no matter what code we try to write. The reason is that the body of `before` has no given values of type `A` and no functions that return values of type `A`, so a nameless function such as `{c:C => ???}` cannot compute its return value of type `A`. Since a fully parametric function cannot create values of an arbitrary type `A` from scratch, we see no possibility of computing `h` within the body of `before`.

Can we prove rigorously that a value of type `C => A` cannot be computed within the body of `before`? Or, perhaps, a clever trick *could* produce a value of that type? So far, we only saw informal arguments about whether values of certain types can be computed. To make the arguments rigorous, we need to translate statements such as “a fully parametric function `before` can compute a value of type `C => A`” into mathematical formulas, with rigorous rules for proving them true or false.

In Section 3.5.3, we denoted by $\mathcal{CH}(A)$ the proposition “the Code \mathcal{H} has a value of type A ”. By “the code” we now mean the body of a given fully parametric function. So, the notation $\mathcal{CH}(A)$ is not completely adequate because the validity of the proposition $\mathcal{CH}(A)$ depends not only on the choice of the type A but also on the place in the code fragment where the value of type A needs to be computed. What exactly is this additional dependency? In the above examples, we used the *types* of a function’s arguments when reasoning about getting a value of a given type A . Thus, a precise description of the proposition $\mathcal{CH}(A)$ is

$$\text{CH-proposition : } \text{a fully parametric function having arguments of types } X, Y, \dots, Z \text{ can compute a value of type } A \quad . \quad (5.1)$$

Here X, Y, \dots, Z, A may be either type parameters or more complicated type expressions such as $B \Rightarrow C$ or $(C \Rightarrow D) \Rightarrow E$, built from other type parameters.

If arguments of types X, Y, \dots, Z are given, it means we already have values of these types. So, the propositions $\mathcal{CH}(X), \mathcal{CH}(Y), \dots, \mathcal{CH}(Z)$ will be true. Thus, proposition (5.1) is equivalent to “ $\mathcal{CH}(A)$ assuming $\mathcal{CH}(X), \mathcal{CH}(Y), \dots, \mathcal{CH}(Z)$ ”. In mathematical logic, a statement of this form is called a **sequent** and is denoted by

$$\mathcal{CH}(X), \mathcal{CH}(Y), \dots, \mathcal{CH}(Z) \vdash \mathcal{CH}(A) \quad . \quad (5.2)$$

The assumptions $\mathcal{CH}(X), \mathcal{CH}(Y), \dots, \mathcal{CH}(Z)$ are called **premises** and the proposition $\mathcal{CH}(A)$ is called the **goal**. Showing rigorously the possibility of computing values in functions means proving sequents of the form (5.2). Our previous examples are denoted by the following sequents:

$$\begin{aligned} \text{fmap for Option : } & \mathcal{CH}(A \Rightarrow B) \vdash \mathcal{CH}(\text{Option}[A] \Rightarrow \text{Option}[B]) \\ \text{the function before : } & \mathcal{CH}(A \Rightarrow B), \mathcal{CH}(B \Rightarrow C) \vdash \mathcal{CH}(A \Rightarrow C) \\ \text{value of type } A \text{ within fmap : } & \mathcal{CH}(A \Rightarrow B), \mathcal{CH}(\text{Option}[A]) \vdash \mathcal{CH}(A) \\ \text{value of type } C \Rightarrow A \text{ within before : } & \mathcal{CH}(A \Rightarrow B), \mathcal{CH}(B \Rightarrow C) \vdash \mathcal{CH}(C \Rightarrow A) \end{aligned}$$

Calculations in formal logic are called **proofs**. So, in this section we gave informal arguments towards proving the first two sequents and disproving the last two. We will now develop tools for proving such sequents rigorously.

A proposition $\mathcal{CH}(A)$ may be true for one set of premises such as $\mathcal{CH}(X), \mathcal{CH}(Y), \dots, \mathcal{CH}(Z)$ but false for another. Here and in the following sections, we will be reasoning about \mathcal{CH} -propositions within the body of a *chosen* fully parametric function, i.e. with a fixed set of premises. We will then temporarily omit the premises and use the shorter notation $\mathcal{CH}(A)$.

5.1.2 Type notation and \mathcal{CH} -propositions for standard type constructions

In Section 3.5.3 we saw examples of reasoning about \mathcal{CH} -propositions for case classes and for disjunctive types. We will now extend this reasoning systematically to all type constructions that pro-

grams could use. A special **type notation** explained in this section will help us write type expressions more concisely. (See Appendix A for reference on the type notation.)

There are six **standard type constructions** supported by all functional languages: primitive types (including `Unit` type and the void type, called `Nothing` in Scala), product (tuple) types, co-product (disjunctive) types, function types, parameterized types, and recursive types. We will now derive the rules for writing CH -propositions for each of these type constructions (except recursive types).

1a) Rule for `Unit` type The `Unit` type has only a single value `()`, and this value (an “empty tuple”) can be *always* computed since it does not need any previous data:

```
def f[...]: ... = {
  ...
  val x: Unit = () // We can always compute a 'Unit' value.
  ...
}
```

So, the proposition $CH(\text{Unit})$ is always true. In the type notation, the `Unit` type is denoted by 1 .

Named unit types also have a single value that is always possible to compute. For example,

```
final case class N1()
```

defines a named unit type; we can compute

```
val x: N1 = N1()
```

So, the proposition $CH(N1)$ is always true. Named unit types are denoted by 1 , just as the `Unit` type itself.

1b) Rule for the void type The Scala type `Nothing` has no values, so the proposition $CH(\text{Nothing})$ is always false. The type `Nothing` is denoted by 0 in the type notation.

1c) Rule for primitive types For a specific primitive (or library-defined) type such as `Int` or `String`, the corresponding CH -proposition is *always true* because we could use any constant value, e.g.

```
def f[...]: ... {
  ...
  val x: String = "abc" // We can always compute a 'String' value.
  ...
}
```

So, the rule for primitive types is the same as the rule for the `Unit` type.

2) Rule for tuple types To compute a value of a tuple type (A, B) requires computing a value of type A and a value of type B . This is expressed by the logic formula $CH((A, B)) = CH(A) \wedge CH(B)$. A similar formula holds for case classes, as Eq. (3.2) shows. In the type notation, the tuple (A, B) is written as $A \times B$. Tuples and case classes with more than two parts are denoted similarly as $A \times B \times \dots \times C$. For example, the Scala definition

```
case class Person(firstName: String, lastName: String, age: Int)
```

is written in the type notation as $\text{String} \times \text{String} \times \text{Int}$. So, the rule for tuple types is

$$CH(A \times B \times \dots \times C) = CH(A) \wedge CH(B) \wedge \dots \wedge CH(C) \quad .$$

3) Rule for disjunctive types A disjunctive type may consist of several case classes. Having a value of a disjunctive type means to have a value of (at least) one of those case classes. An example of translating this relationship into a formula was shown by Eq. (3.1). For the standard disjunctive type `Either[A, B]`, we have the logical formula $CH(\text{Either}[A, B]) = CH(A) \vee CH(B)$. In the type notation, the Scala type `Either[A, B]` is written as $A + B$. A longer example: the Scala definition

```
sealed trait RootsOfQ
final case class NoRoots() extends RootsOfQ
final case class OneRoot(x: Double) extends RootsOfQ
final case class TwoRoots(x: Double, y: Double) extends RootsOfQ
```

is translated to the type notation as

$$\text{RootsOfQ} = 1 + \text{Double} + \text{Double} \times \text{Double} \quad .$$

The type notation is significantly shorter because it omits all case class names and part names from the type definitions. In this notation, the rule for disjunctive types is

$$CH(A + B + \dots + C) = CH(A) \vee CH(B) \vee \dots \vee CH(C) \quad .$$

4) Rule for function types Consider now a function type such as $A \Rightarrow B$. (This type is written in the type notation as $A \Rightarrow B$.) To compute a value of that type, we need to write code such as

```
val f: A => B = { (a: A) =>
  ??? // Compute a value of type B in this scope.
}
```

The inner scope of the function needs to compute a value of type B , and the given value $a:A$ may be used for that. So, $CH(A \Rightarrow B)$ is true if and only if we are able to compute a value of

type B when we are given a value of type A . To translate this statement into the language of logical propositions, we need to use the logical implication, $CH(A) \Rightarrow CH(B)$, which means that $CH(B)$ can be proved if $CH(A)$ already holds. So the rule for function types is

$$CH(A \Rightarrow B) = CH(A) \Rightarrow CH(B) \quad .$$

5) Rule for parameterized types Consider a function with type parameters, e.g.

```
def f[A, B]: A => (A => B) => B = { x => g => g(x) }
```

Being able to define the body of such a function is equivalent to being able to compute a value of type $A \Rightarrow (A \Rightarrow B) \Rightarrow B$ for *all* possible types A and B . In the notation of formal logic, this is written as

$$CH(\forall(A, B). A \Rightarrow (A \Rightarrow B) \Rightarrow B)$$

and is equivalent to

$$\forall(A, B). CH(A \Rightarrow (A \Rightarrow B) \Rightarrow B) \quad .$$

The code notation for the parameterized function f is

$$f^{A,B} : A \Rightarrow (A \Rightarrow B) \Rightarrow B \quad ,$$

and its type can be written as

$$\forall(A, B). A \Rightarrow (A \Rightarrow B) \Rightarrow B \quad .$$

The symbol \forall means “for all” and is known as the **universal quantifier** in logic.

In Scala, longer type expressions can be named and their names (called **type aliases**) can be used to make code shorter. Type aliases may also contain type parameters. Defining and using a type alias for the type of the function f looks like this,

```
type F[A, B] = A => (A => B) => B
def f[A, B]: F[A, B] = { x => g => g(x) }
```

This is written in the type notation as

$$F^{A,B} \triangleq A \Rightarrow (A \Rightarrow B) \Rightarrow B \quad ,$$

$$f^{A,B} : F^{A,B} \triangleq x^{A} \Rightarrow g^{A \Rightarrow B} \Rightarrow g(x) \quad ,$$

or equivalently (although somewhat less readably)

$$f : (\forall(A, B). F^{A,B}) \triangleq \forall(A, B). x^{A} \Rightarrow g^{A \Rightarrow B} \Rightarrow g(x) \quad .$$

In Scala 3, the function f can be written as a value via the syntax

```
val f: [A, B] => A => (A => B) => B = { // Valid only in Scala 3.
  [A, B] => (x: A) => (g: A => B) => g(x)
}
```

Type construction	Scala syntax	Type notation	\mathcal{CH} -proposition
type parameter	<code>[A]</code>	A	$\mathcal{CH}(A)$
product type (tuple)	<code>(A, B)</code>	$A \times B$	$\mathcal{CH}(A) \wedge \mathcal{CH}(B)$
disjunctive type	<code>Either[A, B]</code>	$A + B$	$\mathcal{CH}(A) \vee \mathcal{CH}(B)$
function type	<code>A => B</code>	$A \Rightarrow B$	$\mathcal{CH}(A) \Rightarrow \mathcal{CH}(B)$
unit or a “named unit” type	<code>Unit</code>	1	$\mathcal{CH}(1) = \text{True}$
primitive type	<code>Int, String, ...</code>	<code>Int, String, ...</code>	$\mathcal{CH}(\text{Int}) = \text{True}$
void type	<code>Nothing</code>	0	$\mathcal{CH}(0) = \text{False}$
value parameterized by type	<code>def f[A]: F[A]</code>	$f^A : F^A$	$\forall A. \mathcal{CH}(F^A)$
type with quantifier	<code>[A] => F[A]</code> (Scala 3)	$\forall A. F^A$	$\forall A. \mathcal{CH}(F^A)$

Table 5.1: The correspondence between type constructions and \mathcal{CH} -propositions.

This syntax corresponds more closely to the mathematical notation shown above.

So, the rule for parameterized types with the type notation F^A is

$$\mathcal{CH}(\forall A. F^A) = \forall A. \mathcal{CH}(F^A) .$$

Case classes and disjunctive types use *names* for the types and their parts. However, those names only add convenience for programmers and do not affect the computational properties of types. So, the type notation allows us to use nameless type expressions.

Table 5.1 summarizes the type notation and also shows how to translate it into logic formulas with propositions of the form $\mathcal{CH}(\dots)$.

The precedence of operators in the type notation is chosen to have fewer parentheses in the type expressions that are frequently used. Here are the rules of precedence:

- The type product operators (\times) group stronger than the disjunctive operators ($+$), so that type expressions such as $A + B \times C$ have the same operator precedence as in standard arithmetic. That is, $A + B \times C$ means $A + (B \times C)$. This convention makes type expressions easier to reason about (for people familiar with arithmetic).
- The function type arrows (\Rightarrow) group weaker than the operators $+$ and \times , so that often-used types such as $A \Rightarrow 1 + B$ (representing `A => Option[B]`) or $A \times B \Rightarrow C$ (representing `((A, B)) => c`) can be written without any parentheses. Type expressions such as $(A \Rightarrow B) \times C$ will require parentheses but are used less often.
- The type quantifiers group weaker than all other operators, so we can write types such as $\forall A. A \Rightarrow A \Rightarrow A$ without parentheses. Type quantifiers are most often placed outside a type expression. When this is not the case, parentheses are necessary, e.g. in the type expression $(\forall A. A \Rightarrow A) \Rightarrow 1$.

5.1.3 Solved examples: Type notation

From now on, we will prefer to write types in the type notation rather than in the Scala syntax. The type notation allows us to write nameless type expressions and, in particular, makes the structure of disjunctive types and their parts more transparent, compared with the Scala syntax. Names of types and parts of types are, of course, helpful for the programmer because they show the significance of data in the application. However, writing names for every part of every type is not helpful for

reasoning about the properties of types. Type notation makes reasoning about types easier, as we will see throughout this chapter. Once the programmer has finished deriving the necessary types and verifying their properties, the type expressions can be straightforwardly translated from the type notation into Scala code.

Let us get some experience converting between type notation and Scala code.

Example 5.1.3.1 Define a function `delta` taking an argument `x` and returning the pair `(x, x)`. Derive the most general type for this function. Write the type signature of `delta` in the type notation, and translate it into a \mathcal{CH} -proposition. Simplify the \mathcal{CH} -proposition if possible.

Solution Begin by writing the code of the function:

```
def delta(x: ...) = (x, x)
```

To derive the most general type for `delta`, first assume `x:A`, where `A` is a type parameter; then the tuple `(x, x)` has type `(A, A)`. We do not see any constraints on the type parameter `A`. So the type parameter represents an arbitrary type and needs to be added to the type signature of `delta`:

```
def delta[A](x: A): (A, A) = (x, x)
```

We find that the most general type of `delta` is `A => (A, A)`. We also note that there is only one way of implementing a fully parametric function with type signature `A => (A, A)`: the function must duplicate its given argument.

It is convenient to use the letter Δ for the function `delta`. In the type notation, the type signature of Δ is written as

$$\Delta^A : A \Rightarrow A \times A \quad .$$

So the proposition $\mathcal{CH}(\Delta)$ (meaning “the function Δ can be implemented”) is

$$\mathcal{CH}(\Delta) = \forall A. \mathcal{CH}(A \Rightarrow A \times A) \quad .$$

In the type expression `A => A × A`, the product symbol (\times) binds stronger than the function arrow (\Rightarrow), so the parentheses in `A => (A × A)` may be omitted.

Using the rules for transforming \mathcal{CH} -propositions, we rewrite

$$\begin{aligned} \mathcal{CH}(A \Rightarrow A \times A) \\ \text{rule for function types : } &= \mathcal{CH}(A) \Rightarrow \mathcal{CH}(A \times A) \\ \text{rule for tuple types : } &= \mathcal{CH}(A) \Rightarrow (\mathcal{CH}(A) \wedge \mathcal{CH}(A)) \quad . \end{aligned}$$

Thus the proposition $\mathcal{CH}(\Delta)$ is equivalent to

$$\mathcal{CH}(\Delta) = \forall A. \mathcal{CH}(A) \Rightarrow (\mathcal{CH}(A) \wedge \mathcal{CH}(A)) \quad .$$

Example 5.1.3.2 The standard disjunctive types `Either[A, B]` and `Option[A]` are written in the type notation as

$$\text{Either}^{A,B} \triangleq A + B \quad , \quad \text{Opt}^A \triangleq \mathbb{1} + A \quad .$$

The type `Either[A, B]` is written as `A + B` by definition of the disjunctive type operator (+). The type `Option[A]` has two disjoint cases, `None` and `Some[A]`. The case class `None` is a “named `Unit`” and is denoted by $\mathbb{1}$. The case class `Some[A]` contains a single value of type `A`. So, the type notation for `Option[A]` is $\mathbb{1} + A$.

Example 5.1.3.3 The Scala definition of the disjunctive type `UserAction`,

```
sealed trait UserAction
final case class SetName(first: String, last: String) extends UserAction
final case class SetEmail(email: String) extends UserAction
final case class SetUserId(id: Long) extends UserAction
```

is written in the type notation as

$$\text{UserAction} \triangleq \text{String} \times \text{String} + \text{String} + \text{Long} \quad . \quad (5.3)$$

The type operation \times groups stronger than $+$, as in arithmetic. To derive the type notation (5.3), we first drop all names from case classes and get three nameless tuples $(\text{String}, \text{String})$, (String) , and (Long) . Each of these tuples is then converted into a product using the operator \times , and all products are “summed” in the type notation using the operator $+$.

Example 5.1.3.4 The parameterized disjunctive type `Either3` is a generalization of `Either`:

```
sealed trait Either3[A, B, C]
final case class Left[A, B, C](x: A) extends Either3[A, B, C]
final case class Middle[A, B, C](x: B) extends Either3[A, B, C]
final case class Right[A, B, C](x: C) extends Either3[A, B, C]
```

This disjunctive type is written in the type notation as

$$\text{Either3}^{A,B,C} \triangleq A + B + C \quad .$$

Example 5.1.3.5 Define a Scala type constructor `F[A]` corresponding to the type notation

$$F^A \triangleq \mathbb{1} + \text{Int} \times A \times A + \text{Int} \times (\text{Int} \Rightarrow A) \quad .$$

Solution The formula for F^A defines a disjunctive type `F[A]` with three parts. To implement `F[A]` in Scala, we need to choose names for each of the disjoint parts, which will become case classes. For the purposes of this example, let us choose names `F1`, `F2`, and `F3`. Each of these case classes needs to have the same type parameter `A`. So we begin writing the code as

```
sealed trait F[A]
final case class F1[A](...) extends F[A]
final case class F2[A](...) extends F[A]
final case class F3[A](...) extends F[A]
```

Each of these case classes represents one part of the disjunctive type: `F1` represents $\mathbb{1}$, `F2` represents $\text{Int} \times A \times A$, and `F3` represents $\text{Int} \times (\text{Int} \Rightarrow A)$. To define these case classes, we need to name their parts. The final code is

```
sealed trait F[A]
final case class F1[A]() extends F[A] // Named unit type.
final case class F2[A](n: Int, x1: A, x2: A) extends F[A]
final case class F3[A](n: Int, f: Int => A) extends F[A]
```

The names `n`, `x1`, `x2`, and `f` are chosen purely for convenience.

Example 5.1.3.6 Write the type signature of the function

```
def fmap[A, B](f: A => B): Option[A] => Option[B]
```

in the type notation.

Solution This is a curried function, so we first rewrite the type signature as

```
def fmap[A, B]: (A => B) => Option[A] => Option[B]
```

The type notation for `Option[A]` is $\mathbb{1} + A$. Now we can write the type signature of `fmap` as

$$\begin{aligned} \text{fmap}^{A,B} : (A \Rightarrow B) \Rightarrow \mathbb{1} + A \Rightarrow \mathbb{1} + B \quad , \\ \text{or equivalently : } \text{fmap} : \forall(A, B). (A \Rightarrow B) \Rightarrow \mathbb{1} + A \Rightarrow \mathbb{1} + B \quad . \end{aligned}$$

We do not put parentheses around $\mathbb{1} + A$ and $\mathbb{1} + B$ because the function arrows (\Rightarrow) group weaker than the other type operations. Parentheses around $(A \Rightarrow B)$ are required.

We will usually prefer to write type parameters in superscripts rather than under type quantifiers. So, we will prefer to write $\text{id}^A = x^A \Rightarrow x$ rather than $\text{id} : (\forall A. A \Rightarrow A) = x^A \Rightarrow x$.

5.1.4 Exercises: Type notation

Exercise 5.1.4.1 Define a Scala disjunctive type $Q[T, A]$ corresponding to the type notation

$$Q^{T, A} \triangleq \mathbb{1} + T \times A + \text{Int} \times (T \Rightarrow T) + \text{String} \times A \quad .$$

Exercise 5.1.4.2 Rewrite `Either[(A, Int), Either[(A, Char), (A, Float)]]` in the type notation.

Exercise 5.1.4.3 Define a Scala type `OptE[A, B]` written in the type notation as $\text{OptE}^{A, B} \triangleq \mathbb{1} + A + B$.

Exercise 5.1.4.4 Write a Scala type signature for the fully parametric function

$$\text{flatMap}^{A, B} : \mathbb{1} + A \Rightarrow (A \Rightarrow \mathbb{1} + B) \Rightarrow \mathbb{1} + B$$

and implement this function, preserving information as much as possible.

5.2 The logic of \mathcal{CH} -propositions

5.2.1 Motivation and first examples

So far, we were able to convert statements such as “a fully parametric function can compute values of type A ” into logical propositions of the form $\mathcal{CH}(A)$ that we called \mathcal{CH} -propositions. The next step is to determine the proof rules suitable for reasoning about \mathcal{CH} -propositions.

Formal logic uses axioms and derivation rules for proving that certain formulas are true or false. A simple example of a true formula is “any proposition α is equivalent to itself”,

$$\forall \alpha. \alpha = \alpha \quad .$$

In logic, equivalence of propositions is usually understood as **implication** (\Rightarrow) in both directions: $\alpha = \beta$ means $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$. So, the above formula is the same as

$$\forall \alpha. \alpha \Rightarrow \alpha \quad .$$

If the proposition α is a \mathcal{CH} -proposition, $\alpha \triangleq \mathcal{CH}(A)$ for some type A , we obtain the formula

$$\forall A. \mathcal{CH}(A) \Rightarrow \mathcal{CH}(A) \quad . \tag{5.4}$$

We expect true \mathcal{CH} -propositions to correspond to types that *can* be computed in a fully parametric function. Let us see if this example fits our expectations. We can rewrite Eq. (5.4) as

$$\begin{aligned} \forall A. \underline{\mathcal{CH}(A) \Rightarrow \mathcal{CH}(A)} \\ \text{rule for function types : } &= \underline{\forall A. \mathcal{CH}(A \Rightarrow A)} \\ \text{rule for parameterized types : } &= \mathcal{CH}(\forall A. A \Rightarrow A) \quad . \end{aligned}$$

The last line shows the \mathcal{CH} -proposition that corresponds to the function type $\forall A. A \Rightarrow A$. Translating the type notation into a Scala type signature, we get

```
def f[A]: A => A
```

This type signature can be easily implemented,

```
def f[A]: A => A = { x => x }
```

So, in this example we see how we converted a true formula in logic into the type of a value `f` that can be implemented.

While the formula $\forall \alpha. \alpha = \alpha$ may be self-evident, the point of using formal logic is to have a set of axioms and proof rules that allow us to deduce *all* correct formulas systematically, without need for intuition or guessing. What axioms and proof rules are suitable for proving \mathcal{CH} -propositions?

A well-known set of logical rules is called Boolean logic. In that logic, each proposition is either *True* or *False*, and the implication operation (\Rightarrow) is defined by

$$(\alpha \Rightarrow \beta) \triangleq ((\neg\alpha) \vee \beta) \quad . \quad (5.5)$$

To verify a formula, substitute *True* or *False* into every variable and check if the formula has the value *True* in all possible cases. The result can be arranged into a truth table. The basic operations (disjunction, conjunction, negation, and implication) have the following truth tables:

α	β	$\alpha \vee \beta$	$\alpha \wedge \beta$	$\neg\alpha$	$\alpha \Rightarrow \beta$
<i>True</i>	<i>True</i>	<i>True</i>	<i>True</i>	<i>False</i>	<i>True</i>
<i>True</i>	<i>False</i>	<i>True</i>	<i>False</i>	<i>False</i>	<i>False</i>
<i>False</i>	<i>True</i>	<i>True</i>	<i>False</i>	<i>True</i>	<i>True</i>
<i>False</i>	<i>False</i>	<i>False</i>	<i>False</i>	<i>True</i>	<i>True</i>

So the formula $\alpha \Rightarrow \alpha$ has the value *True* whether α itself is *True* or *False*. This check is sufficient to show that $\forall\alpha. \alpha \Rightarrow \alpha$ is true in Boolean logic.

Here is the truth table for the formula $\forall(\alpha, \beta). (\alpha \wedge \beta) \Rightarrow \alpha$; that formula is true in Boolean logic since all values in the last column are *True*:

α	β	$\alpha \wedge \beta$	$(\alpha \wedge \beta) \Rightarrow \alpha$
<i>True</i>	<i>True</i>	<i>True</i>	<i>True</i>
<i>True</i>	<i>False</i>	<i>False</i>	<i>True</i>
<i>False</i>	<i>True</i>	<i>False</i>	<i>True</i>
<i>False</i>	<i>False</i>	<i>False</i>	<i>True</i>

The formula $\forall(\alpha, \beta). \alpha \Rightarrow (\alpha \wedge \beta)$ is not true in Boolean logic, which we can see from the following truth table (one value in the last column is *False*):

α	β	$\alpha \wedge \beta$	$\alpha \Rightarrow (\alpha \wedge \beta)$
<i>True</i>	<i>True</i>	<i>True</i>	<i>True</i>
<i>True</i>	<i>False</i>	<i>False</i>	<i>False</i>
<i>False</i>	<i>True</i>	<i>False</i>	<i>True</i>
<i>False</i>	<i>False</i>	<i>False</i>	<i>True</i>

Table 5.2 shows more examples of logical formulas that are true in Boolean logic. Each formula is first given in terms of \mathcal{CH} -propositions (we denoted $\alpha \triangleq \mathcal{CH}(A)$ and $\beta \triangleq \mathcal{CH}(B)$ for brevity) and then into a Scala type signature of a function that can be implemented.

Table 5.3 some examples of formulas that are *not true* in Boolean logic. Translated into type formulas and then into Scala, these formulas yield type signatures that *cannot* be implemented by fully parametric functions.

At first sight, it appears from these examples that whenever a logical formula is true in Boolean logic, the corresponding type signature can be implemented in code, and vice versa. However, this is *incorrect*: the rules of Boolean logic are not suitable for reasoning about types in a functional language. Below we will see some examples of formulas that are true in Boolean logic but yield unimplementable type signatures.

Logic formula	Type formula	Scala code
$\forall \alpha. \alpha \Rightarrow \alpha$	$\forall A. A \Rightarrow A$	<code>def id[A](x: A): A = x</code>
$\forall \alpha. \alpha \Rightarrow \text{True}$	$\forall A. A \Rightarrow \mathbb{1}$	<code>def toUnit[A](x: A): Unit = ()</code>
$\forall(\alpha, \beta). \alpha \Rightarrow (\alpha \vee \beta)$	$\forall(A, B). A \Rightarrow A + B$	<code>def toL[A, B](x: A): Either[A, B] = Left(x)</code>
$\forall(\alpha, \beta). (\alpha \wedge \beta) \Rightarrow \alpha$	$\forall(A, B). A \times B \Rightarrow A$	<code>def first[A, B](p: (A, B)): A = p._1</code>
$\forall(\alpha, \beta). \alpha \Rightarrow (\beta \Rightarrow \alpha)$	$\forall(A, B). A \Rightarrow (B \Rightarrow A)$	<code>def const[A, B](x: A): B => A = (_ => x)</code>

Table 5.2: Examples of logical formulas that are true theorems in Boolean logic.

Logic formula	Type formula	Scala type signature
$\forall \alpha. \text{True} \Rightarrow \alpha$	$\forall A. \mathbb{1} \Rightarrow A$	<code>def f[A](x: Unit): A</code>
$\forall(\alpha, \beta). (\alpha \vee \beta) \Rightarrow \alpha$	$\forall(A, B). A + B \Rightarrow A$	<code>def f[A, B](x: Either[A, B])</code>
$\forall(\alpha, \beta). \alpha \Rightarrow (\alpha \wedge \beta)$	$\forall(A, B). A \Rightarrow A \times B$	<code>def f[A, B](p: A): (A, B)</code>
$\forall(\alpha, \beta). (\alpha \Rightarrow \beta) \Rightarrow \alpha$	$\forall(A, B). (A \Rightarrow B) \Rightarrow A$	<code>def f[A, B](x: A => B): A</code>

Table 5.3: Examples of logical formulas that are *not* true in Boolean logic.

5.2.2 Example: Failure of Boolean logic for type reasoning

To see an explicit example of obtaining an incorrect result when using Boolean logic to reason about values computed by fully parametric functions, consider the following type,

$$\forall(A, B, C). (A \Rightarrow B + C) \Rightarrow (A \Rightarrow B) + (A \Rightarrow C) \quad , \quad (5.6)$$

which corresponds to the Scala type signature

```
def bad[A, B, C](g: A => Either[B, C]): Either[A => B, A => C] = ???
```

The function `bad` cannot be implemented as a fully parametric function. To see why, consider that the only available data is a function $g: A \Rightarrow B + C$, which returns values of type B or C depending (in some unknown way) on the input value of type A . The function `bad` must return either a function of type $A \Rightarrow B$ or a function of type $A \Rightarrow C$. How can the code of `bad` make this decision? The only input data is the function g that takes an argument of type A . We could imagine applying g to various arguments of type A and to see whether g returns a B or a C . However, the type A is arbitrary, and a fully parametric function cannot produce a value of type A in order to apply g to it. So the decision about whether to return $A \Rightarrow B$ or $A \Rightarrow C$ must be independent of the function g ; that decision must be hard-coded in the function `bad`.

Suppose we hard-coded the decision to return a function of type $A \Rightarrow B$. How can we create a function of type $A \Rightarrow B$ in the body of `bad`? Given a value $x: A$ of type A , we would need to compute some value of type B . Since the type B is arbitrary (it is a type parameter), we cannot produce a value of type B from scratch. The only potential source of values of type B is the given function g . The only way of using g is to apply it to $x: A$. However, for some x , the value $g(x)$ may have type `Right(c)` where c is of type C . In that case, we will have a value of type C , not B . So, in general, we cannot guarantee that we can always obtain a value of type B from a given value $x: A$. This means we cannot build a function of type $A \Rightarrow B$ out of the function g . Similarly, we cannot build a function of type $A \Rightarrow C$ out of g .

Whether we decide to return $A \Rightarrow B$ or $A \Rightarrow C$, we will not be able to return a value of the required type, as we just saw. We must conclude that we cannot implement `bad` as a fully parametric function.

We could try to switch between $A \Rightarrow B$ and $A \Rightarrow C$ depending on a given value of type A . This idea, however, means that we are working with a different type signature:

$$\forall(A, B, C). (A \Rightarrow B + C) \Rightarrow A \Rightarrow (A \Rightarrow B) + (A \Rightarrow C) \quad .$$

This type signature *can* be implemented, for instance, by this Scala code:

```
def q[A, B, C](g: A => Either[B, C]): A => Either[A=>B, A=>C] = { a =>
  g(a) match {
    case Left(b) => Left(_ => b)
    case Right(c) => Right(_ => c)
  }
}
```

But this is not the required type signature (5.6).

Now let us convert the type signature (5.6) into a \mathcal{CH} -proposition:

$$\begin{aligned} & \forall(\alpha, \beta, \gamma). (\alpha \Rightarrow (\beta \vee \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \vee (\alpha \Rightarrow \gamma)) \quad , \\ \text{where } & \alpha \triangleq \mathcal{CH}(A), \quad \beta \triangleq \mathcal{CH}(B), \quad \gamma \triangleq \mathcal{CH}(C) \quad . \end{aligned} \tag{5.7}$$

It turns out that this formula is true *in Boolean logic*. To prove this, we need to show that Eq. (5.7) is equal to *True* for any Boolean values of the variables α, β, γ . One way is to rewrite the expression (5.7) using the rules of Boolean logic, such as Eq. (5.5):

$$\begin{aligned} & \alpha \Rightarrow (\beta \vee \gamma) \\ \text{definition of } \Rightarrow \text{ via Eq. (5.5) : } & = (\neg \alpha) \vee \beta \vee \gamma \quad , \\ & (\alpha \Rightarrow \beta) \vee (\alpha \Rightarrow \gamma) \\ \text{definition of } \Rightarrow \text{ via Eq. (5.5) : } & = (\neg \alpha) \vee \beta \vee (\neg \alpha) \vee \gamma \\ \text{property } x \vee x = x \text{ in Boolean logic : } & = (\neg \alpha) \vee \beta \vee \gamma \quad , \end{aligned}$$

showing that $\alpha \Rightarrow (\beta \vee \gamma)$ is in fact *equal* to $(\alpha \Rightarrow \beta) \vee (\alpha \Rightarrow \gamma)$ in Boolean logic.

Let us also give a proof via truth-value reasoning. The only possibility for an implication $X \Rightarrow Y$ to be *False* is when $X = \text{True}$ and $Y = \text{False}$. So, Eq. (5.7) can be *False* only if $(\alpha \Rightarrow (\beta \vee \gamma)) = \text{True}$ and $(\alpha \Rightarrow \beta) \vee (\alpha \Rightarrow \gamma) = \text{False}$. A disjunction can be false only when both parts are false; so we must have both $(\alpha \Rightarrow \beta) = \text{False}$ and $(\alpha \Rightarrow \gamma) = \text{False}$. This is only possible if $\alpha = \text{True}$ and $\beta = \gamma = \text{False}$. But, with these value assignments, we find $(\alpha \Rightarrow (\beta \vee \gamma)) = \text{False}$ rather than *True* as we assumed. It follows that we cannot ever make Eq. (5.7) equal to *False*. This proves Eq. (5.7) to be true in Boolean logic.

5.2.3 The rules of proof for \mathcal{CH} -propositions

Section 5.2.2 shows that some true formulas in Boolean logic do not correspond to types of *implementable* fully parametric functions. However, we have also seen several other examples where Boolean logic does provide correct results: some true formulas correspond to implementable type signatures, while some false formulas correspond to non-implementable type signatures.

Instead of guessing whether the rules of Boolean logic are suitable, let us derive the suitable logical axioms and proof rules systematically.

The proposition $\mathcal{CH}(A)$ is true when a value of type A can be computed by a fully parametric function with a given type signature. To describe all possible ways of computing a value of type A , we need to enumerate all possible ways of writing code within a fully parametric function. The requirement of parametricity means that we are not allowed to use any specific types such as `Int` or `String`. We are only allowed to work with values of unknown types described by the given type parameters. We cannot use any concrete values such as `123` or `"hello"`, or any library functions that work with specific (non-parametric) types; however, we are permitted to use fully parametric types, such as `Either[A, B]` or `Option[A]`. The allowed eight code constructs are illustrated in this code fragment:

```

def f[A, B, ...](a: A, b: B)... = { // (A given type signature.)
  val x1: Unit = ()           // 1) Create a value of type Unit.
  val x2: A = a               // 2) Use a given argument.
  val x3 = { x: A => ... }     // 3) Create a function.
  val x4: D = x3(x2)          // 4) Use a function.
  val x5: (A, B) = (a, b)      // 5) Create a tuple.
  val x6: B = x5._2           // 6) Use a tuple.
  val x7: Option[A] = Some(x2) // 7) Create values of a disjunctive type.
  val x8 = x7 match { ... }    // 8) Use values of a disjunctive type.
}

```

A value of type X can be computed (i.e. $\mathcal{CH}(X)$ is true) if and only if we can create a sequence of computed values such as x_1, x_2, \dots , each being the result of one of these eight code constructs, ending with a value of type X . So, each of the eight code constructs should correspond to a logical rule for proving a \mathcal{CH} -proposition.

A set of axioms and proof rules defines a **formal logic**. So, we will now write the proof rules that will define *the* logic appropriate for reasoning about \mathcal{CH} -propositions.

Because each proof rule will be obtained from a specific code construct, any \mathcal{CH} -proposition such as $\mathcal{CH}(X)$ proved by applying a sequence of these rules will automatically correspond to a code fragment that combines the relevant code constructs to compute a value of type X . Conversely, any code computing a value of type X must be a combination of some of the eight code constructs, and that combination can be automatically translated into a sequence of applications of proof rules in the logic to produce a proof of the proposition $\mathcal{CH}(X)$.

Let us now write down the proof rules that follow from the eight code constructs. We will need to consider the full formulation (5.1) of \mathcal{CH} -propositions and write them as sequents such as Eq. (5.2). For brevity, we define $\alpha \triangleq \mathcal{CH}(A)$, $\beta \triangleq \mathcal{CH}(B)$, etc. It is also customary to use the letter Γ to denote a set of premises, such as $\mathcal{CH}(X), \mathcal{CH}(Y), \dots, \mathcal{CH}(Z)$ in Eq. (5.2). So, we can write a shorter formula $\Gamma \vdash \alpha$ instead of the sequent (5.2).

With these notations, we will now enumerate all the possible ways of proving that a \mathcal{CH} -proposition is true. We assume that the set of premises Γ is known.

1) Create a `Unit` value At any place in the code, we may write the expression `()` of type `Unit`. This expression corresponds to a proof of the proposition $\mathcal{CH}(\mathbb{1})$ with any set Γ of premises (even with an empty set of premises). So, the sequent $\Gamma \vdash \mathcal{CH}(\mathbb{1})$ is always true. The code corresponding to the proof of this sequent is an expression that creates a value of the `Unit` type:

$$\text{Proof}(\Gamma \vdash \mathcal{CH}(\mathbb{1})) = \mathbb{1} \quad ,$$

where we denoted by $\mathbb{1}$ the value `()`.

In formal logic, a sequent that is found to be always true, such as our $\Gamma \vdash \mathcal{CH}(\mathbb{1})$, is called an **axiom** and is written in the following notation,

$$\frac{}{\Gamma \vdash \mathcal{CH}(\mathbb{1})} \quad (\text{create unit}) \quad .$$

The “fraction with a label” represents a proof rule. The denominator of the “fraction” is the target sequent that we need to prove. The numerator of the “fraction” can have zero or more other sequents that need to be proved before the target sequent can be proved. In this case, the set of previous sequents is empty: the target sequent is an axiom and so requires no previous sequents for its proof. The label “create unit” is an arbitrary name used to refer to the rule.

2) Use a given argument At any place within the code of a fully parametric function, we may use one of the function’s arguments, say x^A . If some argument has type A , it means that $\alpha \triangleq \mathcal{CH}(A)$ belongs to the set of premises of the sequent we are trying to prove. To indicate this, we write the set of premises as “ Γ, α ”. The code construct `x:A` computes a value of type A , i.e. show that α is true, given these premises. This is expressed by the sequent $\Gamma, \alpha \vdash \alpha$. The proof of this sequent corresponds to an expression that returns one of the given arguments (which we here called x^A),

$$\text{Proof}(\Gamma, \alpha \vdash \alpha) = x^A \quad .$$

This sequent is an axiom since its proof requires no previous sequents, The formal logic notation for this axiom is

$$\frac{}{\Gamma, \alpha \vdash \alpha} \quad (\text{use arg}) \quad .$$

3) Create a function At any place in the code, we may compute a nameless function of type, say, $A \Rightarrow B$, by writing $(x:A) \Rightarrow \text{expr}$ as long as a value expr of type B can be computed in the inner scope of the function. The code for expr is also required to be fully parametric; it may use x and/or other values visible in that scope. So we now need to answer the question of whether a fully parametric function can compute a value of type B , given an argument of type A as well as all other arguments previously given to the parent function. This question is answered by a sequent whose premises contain one more proposition, $\mathcal{CH}(A)$, in addition to all previously available premises. Translating this into the language of \mathcal{CH} -propositions, we find that we will prove the sequent

$$\Gamma \vdash \mathcal{CH}(A \Rightarrow B) \quad = \quad \Gamma \vdash \mathcal{CH}(A) \Rightarrow \mathcal{CH}(B) \quad \triangleq \quad \Gamma \vdash \alpha \Rightarrow \beta$$

if we can prove the sequent $\Gamma, \mathcal{CH}(A) \vdash \mathcal{CH}(B) = \Gamma, \alpha \vdash \beta$. In the notation of formal logic, this is a **derivation rule** (rather than an axiom) and is written as

$$\frac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \alpha \Rightarrow \beta} \quad (\text{create function}) \quad .$$

The **turnstile** symbol, \vdash , groups weaker than other operators. So, we can write sequents such as $(\Gamma, \alpha) \vdash (\beta \Rightarrow \gamma)$ with fewer parentheses: $\Gamma, \alpha \vdash \beta \Rightarrow \gamma$.

What code corresponds to the “create function” rule? The proof of $\Gamma \vdash \alpha \Rightarrow \beta$ depends on a proof of another sequent. So, the corresponding code must be a *function* that takes a proof of the previous sequent as an argument and returns a proof of the new sequent. By the CH correspondence, a proof of a sequent corresponds to a code expression of the type given by the goal of the sequent; the expression may use arguments of types corresponding to the premises of the sequent. So, a proof of the sequent $\Gamma, \alpha \vdash \beta$ is an expression expr_B of type B that may use a given value of type A as well as any other arguments given previously. Then we can write the proof code for the sequent $\Gamma \vdash \alpha \Rightarrow \beta$ as the nameless function $(x:A) \Rightarrow \text{expr}_B$. This function has type $A \Rightarrow B$ and requires us to already have a suitable expr_B . This exactly corresponds to the proof rule “create function”. We may write the corresponding code as

$$\text{Proof}(\Gamma \vdash \mathcal{CH}(A) \Rightarrow \mathcal{CH}(B)) = x^A \Rightarrow \text{Proof}(\Gamma, x^A \vdash \mathcal{CH}(B)) \quad .$$

Here we wrote x^A instead of $\mathcal{CH}(A)$ since the value x^A is a proof of the proposition $\mathcal{CH}(A)$. We will see in Section 5.2.4 how premises such as Γ, x^A are implemented in code.

4) Use a function At any place in the code, we may apply an already defined function of type $A \Rightarrow B$ to an already computed value of type A . The result will be a value of type B . This corresponds to assuming $\mathcal{CH}(A \Rightarrow B)$ and $\mathcal{CH}(A)$, and then deriving $\mathcal{CH}(B)$. The formal logic notation for this proof rule is

$$\frac{\Gamma \vdash \alpha \quad \Gamma \vdash \alpha \Rightarrow \beta}{\Gamma \vdash \beta} \quad (\text{use function}) \quad .$$

The code corresponding to this proof rule takes previously computed values $x:A$ and $f:A \Rightarrow B$, and writes the expression $f(x)$. This can be written as a function application,

$$\text{Proof}(\Gamma \vdash \beta) = \text{Proof}(\Gamma \vdash \alpha \Rightarrow \beta) (\text{Proof}(\Gamma \vdash \alpha)) \quad .$$

5) Create a tuple If we have already computed some values $x:A$ and $y:B$, we may write the expression (x, y) and so compute a value of the tuple type (A, B) . The proof rule is

$$\frac{\Gamma \vdash \alpha \quad \Gamma \vdash \beta}{\Gamma \vdash \alpha \wedge \beta} \quad (\text{create tuple}) \quad .$$

We can write the corresponding code expression as

$$\text{Proof}(\Gamma \vdash \alpha \wedge \beta) = \text{Proof}(\Gamma \vdash \alpha) \times \text{Proof}(\Gamma \vdash \beta) \quad ,$$

where we write $a \times b$ to represent a pair of two values.

This rule describes creating a pair of values. A larger tuple, such as (w, x, y, z) , can be expressed via nested pairs, e.g. as $(w, (x, (y, z)))$. So it is sufficient to have a sequent rule for creating pairs; this rule can express the sequent rules for creating all other tuples, and we do not need to define separate rules for, say, $\Gamma \vdash \alpha \wedge \beta \wedge \gamma$.

6) Use a tuple If we already have a value $t : (A, B)$ of a tuple type $A \times B$, we can extract one of the parts of the tuple and obtain a value of type A or a value of type B . The code is `t._1` and `t._2` respectively, and the corresponding sequent proof rules are

$$\frac{\Gamma \vdash \alpha \wedge \beta}{\Gamma \vdash \alpha} \quad (\text{use tuple-1}) \qquad \frac{\Gamma \vdash \alpha \wedge \beta}{\Gamma \vdash \beta} \quad (\text{use tuple-2}) \quad .$$

The code can be written as

$$\begin{aligned} \text{Proof}(\Gamma \vdash \alpha) &= \nabla_1(\text{Proof}(\Gamma \vdash \alpha \wedge \beta)) \quad , \\ \text{Proof}(\Gamma \vdash \beta) &= \nabla_2(\text{Proof}(\Gamma \vdash \alpha \wedge \beta)) \quad , \end{aligned}$$

where we introduced the notation ∇_1 and ∇_2 to mean the Scala code `._1` and `._2`.

Since all tuples can be expressed through pairs, it is sufficient to have proof rules for pairs.

7) Create a disjunctive value The type `Either[A, B]` corresponding to the disjunction $\alpha \vee \beta$ can be used to define any other disjunctive type; e.g. a disjunctive type with three parts can be expressed as `Either[A, Either[B, C]]`. So it is sufficient to have proof rules for a disjunction of *two* propositions.

There are two ways of creating a value of the type `Either[A, B]`: the code expressions are `Left(x:A)` and `Right(y:B)`. The values $x:A$ or $y:B$ must have been computed previously (and correspond to previously proved sequents). So, the sequent proof rules are

$$\frac{\Gamma \vdash \alpha}{\Gamma \vdash \alpha \vee \beta} \quad (\text{create Left}) \qquad \frac{\Gamma \vdash \beta}{\Gamma \vdash \alpha \vee \beta} \quad (\text{create Right}) \quad .$$

The corresponding code can be written as

$$\begin{aligned} \text{Proof}(\Gamma \vdash \alpha \vee \beta) &= \text{Left}(\text{Proof}(\Gamma \vdash \alpha)) \quad , \\ \text{Proof}(\Gamma \vdash \alpha \vee \beta) &= \text{Right}(\text{Proof}(\Gamma \vdash \beta)) \quad . \end{aligned}$$

8) Use a disjunctive value The only way we may use a value of disjunctive type `Either[A, B]` is by pattern matching on it:

```
val result: C = (e: Either[A, B]) match {
  case Left(x:A)   => expr1(x)
  case Right(y:B)  => expr2(y)
}
```

Here, `expr1(x)` must be an expression of some type c , computed using $x:A$ and any previously available arguments (i.e. the premises Γ). Similarly, `expr2(y)` must be an expression of type c computed using $y:B$ and previous arguments. It is clear that `expr1(x)` represents a

proof of a sequent with an additional premise of type A , i.e. $\Gamma, \alpha \vdash \gamma$, where we denoted $\gamma \triangleq \mathcal{CH}(C)$. Similarly, `expr2(y)` is a proof of the sequent $\Gamma, \beta \vdash \gamma$. So, we can write the proof rule corresponding to the `match/case` expression as a rule with three previous sequents:

$$\frac{\Gamma \vdash \alpha \vee \beta \quad \Gamma, \alpha \vdash \gamma \quad \Gamma, \beta \vdash \gamma}{\Gamma \vdash \gamma} \quad (\text{use Either}) \quad .$$

The code can be written as

$$\text{Proof}(\Gamma \vdash \gamma) = \text{Proof}(\Gamma \vdash \alpha \vee \beta) \text{ match } \begin{cases} \text{have } x^A : & \text{Proof}(\Gamma, x^A \vdash \gamma) \\ \text{have } y^B : & \text{Proof}(\Gamma, y^B \vdash \gamma) \end{cases} \quad .$$

Table 5.4 summarizes the eight proof rules derived in this section. These proof rules define a logic known as the **intuitionistic propositional logic** or **constructive propositional logic**. We will call this logic “constructive” for short.

axioms :	$\frac{}{\Gamma \vdash \mathcal{CH}(\mathbb{1})}$ (create unit)	$\frac{}{\Gamma, \alpha \vdash \alpha}$ (use arg)
derivation rules :	$\frac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \alpha \Rightarrow \beta}$ (create function)	
	$\frac{\Gamma \vdash \alpha \quad \Gamma \vdash \alpha \Rightarrow \beta}{\Gamma \vdash \beta}$ (use function)	
	$\frac{\Gamma \vdash \alpha \quad \Gamma \vdash \beta}{\Gamma \vdash \alpha \wedge \beta}$ (create tuple)	
	$\frac{\Gamma \vdash \alpha \wedge \beta}{\Gamma \vdash \alpha}$ (use tuple-1)	$\frac{\Gamma \vdash \alpha \wedge \beta}{\Gamma \vdash \beta}$ (use tuple-2)
	$\frac{\Gamma \vdash \alpha}{\Gamma \vdash \alpha \vee \beta}$ (create Left)	$\frac{\Gamma \vdash \beta}{\Gamma \vdash \alpha \vee \beta}$ (create Right)
	$\frac{\Gamma \vdash \alpha \vee \beta \quad \Gamma, \alpha \vdash \gamma \quad \Gamma, \beta \vdash \gamma}{\Gamma \vdash \gamma}$ (use Either)	

Table 5.4: Proof rules for the constructive logic.

5.2.4 Example: Proving a \mathcal{CH} -proposition and deriving code

The task is to implement a fully parametric function

```
def f[A, B]: ((A => A) => B) => B = ???
```

Implementing this function is the same as being able to compute a value of type

$$F \triangleq \forall(A, B). ((A \Rightarrow A) \Rightarrow B) \Rightarrow B \quad .$$

Since the type parameters A and B are arbitrary, the body of the fully parametric function \mathfrak{f} cannot use any previously defined values of types A or B . So, the task is formulated as computing a value of type F with no previously defined values. This is written as the sequent $\Gamma \vdash \mathcal{CH}(F)$, where the set Γ of premises is empty, $\Gamma = \emptyset$. Rewriting this sequent using the rules of Table 5.1, we get

$$\forall(\alpha, \beta). \emptyset \vdash ((\alpha \Rightarrow \alpha) \Rightarrow \beta) \Rightarrow \beta \quad , \quad (5.8)$$

where we denoted $\alpha \triangleq \mathcal{CH}(A)$ and $\beta \triangleq \mathcal{CH}(B)$.

The next step is to prove the sequent (5.8) using the logic proof rules of Section 5.2.3. For brevity, we will omit the quantifier $\forall(\alpha, \beta)$ since it will be present in front of every sequent.

Begin by looking for a proof rule whose “denominator” has a sequent similar to Eq. (5.8), i.e. has an implication $(p \Rightarrow q)$ in the goal. We have only one rule that can prove a sequent of the form $\Gamma \vdash (p \Rightarrow q)$; this is the rule “create function”. That rule requires us to already have a proof of the sequent $(\Gamma, p) \vdash q$. So, we use this rule with $\Gamma = \emptyset$, $p = (\alpha \Rightarrow \alpha) \Rightarrow \beta$, and $q = \beta$:

$$\frac{(\alpha \Rightarrow \alpha) \Rightarrow \beta \vdash \beta}{\emptyset \vdash ((\alpha \Rightarrow \alpha) \Rightarrow \beta) \Rightarrow \beta} \quad .$$

We now need to prove the sequent $(\alpha \Rightarrow \alpha) \Rightarrow \beta \vdash \beta$, which we can write as $\Gamma_1 \vdash \beta$ where we defined $\Gamma_1 \triangleq [(\alpha \Rightarrow \alpha) \Rightarrow \beta]$ to be the set containing the single premise $(\alpha \Rightarrow \alpha) \Rightarrow \beta$.

There are no proof rules that derive a sequent with an explicit premise of the form of an implication $p \Rightarrow q$. However, we have a rule called “use function” that derives a sequent by assuming another sequent containing an implication. We would be able to use that rule,

$$\frac{\Gamma_1 \vdash \alpha \Rightarrow \alpha \quad \Gamma_1 \vdash (\alpha \Rightarrow \alpha) \Rightarrow \beta}{\Gamma_1 \vdash \beta} \quad ,$$

if we could prove the two sequents $\Gamma_1 \vdash \alpha \Rightarrow \alpha$ and $\Gamma_1 \vdash (\alpha \Rightarrow \alpha) \Rightarrow \beta$. To prove these sequents, note that the rule “create function” applies to $\Gamma_1 \vdash \alpha \Rightarrow \alpha$ as follows,

$$\frac{\Gamma_1, \alpha \vdash \alpha}{\Gamma_1 \vdash \alpha \Rightarrow \alpha} \quad .$$

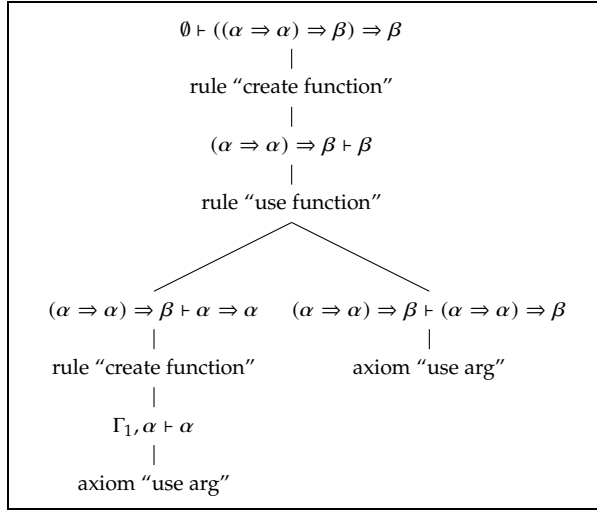


Figure 5.1: Proof tree for the sequent (5.8).

The sequent $\Gamma_1, \alpha \vdash \alpha$ is proved directly by the axiom “use arg”. The sequent $\Gamma_1 \vdash (\alpha \Rightarrow \alpha) \Rightarrow \beta$ is also proved by the axiom “use arg” because Γ_1 already contains $(\alpha \Rightarrow \alpha) \Rightarrow \beta$.

The proof of the sequent (5.8) is now complete and can be visualized as a tree (Figure 5.1). The next step is to derive the code from this proof.

To do that, we combine the code expressions that correspond to each of the proof rules we used. We need to retrace the proof backwards, starting from the leaves of the tree and going towards the root, and to assemble the Proof(...) code expressions one by one.

Begin with the left-most leaf “use arg”. This rule corresponds to the code $x^{:A}$,

$$\text{Proof}(\Gamma_1, \alpha \vdash \alpha) = x^{:A} \quad .$$

Here $x^{:A}$ must be a proof of the premise α in the sequent $\Gamma_1, \alpha \vdash \alpha$. So, we need to use the same $x^{:A}$ when we write the code for the previous rule, “create function”:

$$\text{Proof}(\Gamma_1 \vdash \alpha \Rightarrow \alpha) = (x^{:A} \Rightarrow \text{Proof}(\Gamma_1, \alpha \vdash \alpha)) = (x^{:A} \Rightarrow x) \quad .$$

The right-most leaf “use arg” corresponds to the code $f^{:(A \Rightarrow A) \Rightarrow B}$, where f is the premise contained in Γ_1 . So we can write

$$\text{Proof}(\Gamma_1 \vdash (\alpha \Rightarrow \alpha) \Rightarrow \beta) = f^{:(A \Rightarrow A) \Rightarrow B} \quad .$$

The previous rule, “use function”, combines the two preceding proofs:

$$\begin{aligned} & \text{Proof}((\alpha \Rightarrow \alpha) \Rightarrow \beta \vdash \beta) \\ &= \text{Proof}(\Gamma_1 \vdash (\alpha \Rightarrow \alpha) \Rightarrow \beta) (\text{Proof}(\Gamma_1 \vdash \alpha \Rightarrow \alpha)) \\ &= f^{:(A \Rightarrow A) \Rightarrow B} \Rightarrow (x^{:A} \Rightarrow x) \quad . \end{aligned}$$

Keep going backwards and find that the rule applied before “use function” is “create function”. We need to provide the same $f^{:(A \Rightarrow A) \Rightarrow B}$ as in the premise above, and so we obtain the code

$$\begin{aligned} & \text{Proof}(\emptyset \vdash ((\alpha \Rightarrow \alpha) \Rightarrow \beta) \Rightarrow \beta) \\ &= f^{:(A \Rightarrow A) \Rightarrow B} \Rightarrow \text{Proof}((\alpha \Rightarrow \alpha) \Rightarrow \beta \vdash \beta) \\ &= f^{:(A \Rightarrow A) \Rightarrow B} \Rightarrow f^{:(A \Rightarrow A) \Rightarrow B} \Rightarrow f^{:(A \Rightarrow A) \Rightarrow B} \Rightarrow (x^{:A} \Rightarrow x) \quad . \end{aligned}$$

This is the final code expression that implements the type $((A \Rightarrow A) \Rightarrow B) \Rightarrow B$. In this way, we have systematically derived the code from the type signature of a function. This function can be implemented in Scala as


```
def f[A, B]: ((A => A) => B) => B = { f => f(x => x) }
```

We found the proof tree in Figure 5.1 by guessing how to combine various proof rules. If we *somehow* find a proof tree for a sequent, we can prove the sequent and derive the corresponding code. However, it is not always obvious how to combine the proof rules for a given initial sequent. This is so because the rules of Table 5.7 do not provide an algorithm for finding a proof tree automatically for a given initial sequent. It turns out that such an algorithm exists (the “LJT algorithm”, see Appendix C). That algorithm can find proofs and derive code from type signatures containing tuples, disjunctive types, and function types (if the given type signature can be implemented).

The library `curryhoward`¹ implements the LJT algorithm. Here are some examples of using this library. We will run the `ammonite`² shell to load the library more easily.

Consider the type signature

$$\forall(A, B). (((A \Rightarrow B) \Rightarrow A) \Rightarrow A) \Rightarrow B \quad .$$

It is not immediately clear whether it is even possible to implement a function with this type signature. It turns out that it is possible, and the code can be derived automatically with help of the LJT algorithm. The library does this via the method `implement`:

```
@ import $ivy.`io.chymyst:curryhoward:0.3.7`, io.chymyst.ch._
@ def f[A, B]: (((A => B) => A) => A) => B = implement
defined function f
@ println(f.lambdaTerm.prettyPrint)
a => a (b => b (c => a (d => c)))
```

The code `a => a (b => b (c => a (d => c)))` was derived automatically for the function `f`. The function `f` was compiled and is ready to be used in any subsequent code.

A compile-time error occurs when trying to use a type signature that cannot be implemented as a fully parametric function:

```
@ def g[A, B]: ((A => B) => A) => A = implement
cmd3.sc:1: type ((A => B) => A) => A cannot be implemented
def g[A, B]: ((A => B) => A) => A = implement
~
Compilation Failed
```

The logical formula corresponding to this type signature is

$$\forall(\alpha, \beta). ((\alpha \Rightarrow \alpha) \Rightarrow \beta) \Rightarrow \beta \quad . \quad (5.9)$$

This formula is known as “Peirce’s law”.³ It is another example showing that the logic of types in functional programming languages is not Boolean. Peirce’s law is a true theorem in Boolean logic but does not hold in the constructive logic (i.e. it cannot be derived using the proof rules of Table 5.7). If we try to implement `g[A, B]` with the type signature shown above, we will fail to write fully parametric code for `g` that compiles without type errors. We will fail because no such code exists, – not because we are insufficiently clever. The LJT algorithm can *prove* that the given type signature cannot be implemented; the `curryhoward` library will then print an error message, and compilation will fail.

As another example, let us verify that the type signature from Section 5.2.2 cannot be implemented as a fully parametric function:

```
@ def bad[A, B, C](g: A => Either[B, C]): Either[A => B, A => C] = implement
cmd4.sc:1: type (A => Either[B, C]) => Either[A => B, A => C] cannot be implemented
def bad[A, B, C](g: A => Either[B, C]): Either[A => B, A => C] = implement
~
Compilation Failed
```

¹<https://github.com/Chymyst/curryhoward>

²<http://ammonite.io/#Ammonite-Shell>

³https://en.wikipedia.org/wiki/Peirce%27s_law

5.3 Solved examples: Equivalence of types

We found a correspondence between types, code, logical propositions, and proofs, which is known as the **Curry-Howard correspondence**. An example of the CH correspondence is that a proof of the logical proposition

$$\forall(\alpha, \beta). \alpha \Rightarrow (\beta \Rightarrow \alpha) \quad (5.10)$$

corresponds to the code of the function

```
def f[A, B]: A => (B => A) = { x => _ => x }
```

With the CH correspondence in mind, we may say that the function `f`'s code “is” the proof of the proposition (5.10). In this sense, the *existence* of the code `x => _ => x` with the type $A \Rightarrow (B \Rightarrow A)$ “is” a proof of the logical formula (5.10).

The Curry-Howard correspondence maps logic formulas such as $(\alpha \vee \beta) \wedge \gamma$ into type expressions such as $(A + B) \times C$. We have seen that types behave similarly to logic formulas in one respect: A logic formula is a true theorem when the corresponding type signature can be implemented as a fully parametric function, and vice versa.

It turns out that the similarity ends here. In other respects, type expressions behave as *arithmetic* expressions and not as logic formulas. For this reason, the type notation used in this book denotes disjunctive types by $A + B$ and tuples by $A \times B$, which is designed to remind us of arithmetic expressions (such as $1 + 2$ and 2×3) rather than of logic formulas (such as $A \vee B$ and $A \wedge B$).

The most important use of the type notation is for writing equations with types. Can we write type equations using the arithmetic intuition, such as

$$(A + B) \times C = A \times C + B \times C \quad ? \quad (5.11)$$

In this section, we will learn how to check whether one type expression is equivalent to another.

5.3.1 Logical identity does not correspond to type equivalence

The CH correspondence maps Eq. (5.11) into the logic formula

$$\forall(A, B, C). (A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C) \quad . \quad (5.12)$$

This formula is the well-known “distributive law”⁴ valid in Boolean logic as well as in the constructive logic. Since a logical equation $P = Q$ means $P \Rightarrow Q$ and $Q \Rightarrow P$, the distributive law (5.12) means that the two formulas hold,

$$\forall(A, B, C). (A \vee B) \wedge C \Rightarrow (A \wedge C) \vee (B \wedge C) \quad , \quad (5.13)$$

$$\forall(A, B, C). (A \wedge C) \vee (B \wedge C) \Rightarrow (A \vee B) \wedge C \quad . \quad (5.14)$$

The CH correspondence maps these logical formulas to fully parametric functions with types

```
def f1[A, B, C]: ((Either[A, B], C)) => Either[(A, C), (B, C)] = ???
def f2[A, B, C]: Either[(A, C), (B, C)] => (Either[A, B], C) = ???
```

In the type notation, these type signatures are written as

$$f_1 : \forall(A, B, C). (A + B) \times C \Rightarrow A \times C + B \times C \quad ,$$

$$f_2 : \forall(A, B, C). A \times C + B \times C \Rightarrow (A + B) \times C \quad .$$

Since the two logical formulas (5.13)–(5.14) are true theorems in constructive logic, we expect to be able to implement the functions `f1` and `f2`. It is not straightforward to guess how to combine the proof rules of Table 5.7 to obtain proofs of Eqs. (5.13)–(5.14). So, instead of deriving the implementations of `f1` and `f2` from the CH correspondence, we will write the Scala code directly.

To implement `f1`, we need to do pattern matching on the argument:

⁴https://en.wikipedia.org/wiki/Distributive_property#Rule_of_replacement

```
def f1[A, B, C]: ((Either[A, B], C)) => Either[(A, C), (B, C)] = {
  case (Left(a), c) => Left((a, c)) // No other choice here.
  case (Right(b), c) => Right((b, c)) // No other choice here.
}
```

In both cases, we have only one possible expression of the correct type.

Similarly, the implementation of `f2` leaves us no choices:

```
def f2[A, B, C]: Either[(A, C), (B, C)] => (Either[A, B], C) = {
  case Left((a, c)) => (Left(a), c) // No other choice here.
  case Right((b, c)) => (Right(b), c) // No other choice here.
}
```

The code of `f1` and `f2` never discards any given values; in other words, these functions appear to preserve information. We can formulate this property rig-

orously as a requirement that an arbitrary value $x: (Either[A, B], C)$ is mapped by `f1` to some value $y: Either[(A, C), (B, C)]$ and then mapped by `f2` back to *the same* value x . Similarly, any value y of type `Either[(A, C), (B, C)]` should be transformed by `f2` and then by `f1` back to the *same value* y .

Let us write these conditions as equations,

$$\forall x: (A+B) \times C. f_2(f_1(x)) = x \quad , \quad \forall y: A \times C + B \times C. f_1(f_2(y)) = y \quad .$$

If we show that these equations hold, it will follow that all the information in a value $x: (A+B) \times C$ is completely preserved inside the value $y \triangleq f_1(x)$; the original value x can be recovered as $x = f_2(y)$. Conversely, all the information in a value $y: A \times C + B \times C$ is preserved inside $x \triangleq f_2(y)$ and can be recovered by applying `f1`. Since the values $x: (A+B) \times C$ and $y: A \times C + B \times C$ are arbitrary, it will follow the *data types themselves*, $(A+B) \times C$ and $A \times C + B \times C$, carry equivalent information. Such types are called **equivalent** or **isomorphic**.

Generally, we say that types P and Q are **equivalent** or **isomorphic** (denoted $P \cong Q$) when there exist functions $f_1^{P \Rightarrow Q}$ and $f_2^{Q \Rightarrow P}$ that are inverses of each other. We can write these conditions using the notation $(f_1 \circ f_2)(x) \triangleq f_2(f_1(x))$ as

$$f_1 \circ f_2 = \text{id} \quad , \quad f_2 \circ f_1 = \text{id} \quad .$$

(We omit type annotations since we already checked that the types match. In Scala, the forward composition $f_1 \circ f_2$ is the function `f1 andThen f2`.) If these conditions hold, there is a one-to-one correspondence between the values of types P and Q . This is the same as to say that the data types P and Q “carry equivalent information”.

To verify that the Scala functions `f1` and `f2` defined above are inverses of each other, we first check if $f_1 \circ f_2 = \text{id}$. Applying $f_1 \circ f_2$ means to apply `f1` and then to apply `f2` to the result. Begin by applying `f1` to an arbitrary value $x: (A+B) \times C$. A value x of that type can be in only one of the two disjoint cases: a tuple `(Left(a), c)` or a tuple `(Right(b), c)`, for some values $a:A$, $b:B$, and $c:C$. The Scala code of `f1` maps these tuples to `Left((a, c))` and to `Right((b, c))` respectively; we can see this directly from the code of `f1`. We then apply `f2` to those values, which maps them back to a tuple `(Left(a), c)` or a tuple `(Right(b), c)` respectively, according to the code of `f2`. These tuples are exactly the value x we started with. So, applying $f_1 \circ f_2$ to an arbitrary $x: (A+B) \times C$ does not change the value x ; this is the same as to say that $f_1 \circ f_2 = \text{id}$.

To check whether $f_2 \circ f_1 = \text{id}$, we apply `f2` to an arbitrary value $y: A \times C + B \times C$, which must be one of the two disjoint cases, `Left((a, c))` or `Right((b, c))`. The code of `f2` maps these two cases into tuples `(Left(a), c)` and `(Right(b), c)` respectively. Then we apply `f1` and map these tuples back to `Left((a, c))` and `Right((b, c))` respectively. It follows that applying `f2` and then `f1` will always recover the initial value y . In other words, $f_2 \circ f_1 = \text{id}$.

By looking at the code of `f1` and `f2`, we can directly observe that these functions are inverses of each other: the tuple pattern `(Left(a), c)` is mapped to `Left((a, c))`, and the pattern `(Right(b), c)` to `Right((b, c))`, or vice versa. It is then visually clear that no information is lost and that the original values are restored by function compositions $f_1 \circ f_2$ or $f_2 \circ f_1$.

We find that the logical identity (5.12) leads to an equivalence of the corresponding types,

$$(A + B) \times C \cong A \times C + B \times C \quad . \quad (5.15)$$

To get Eq. (5.15) from Eq. (5.12), we just need to mentally replace the disjunction operations \vee by $+$ and the conjunctions \wedge by \times , as if we were converting to an arithmetic expression.

Consider another example of a logical identity: the associativity law for conjunction,

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma) \quad . \quad (5.16)$$

The corresponding types are $(A \times B) \times C$ and $A \times (B \times C)$; in Scala, $((A, B), C)$ and $(A, (B, C))$. We can define functions that convert between these types without information loss:

```
def f3[A, B, C]: (((A, B), C)) => (A, (B, C)) = { case ((a, b), c) => (a, (b, c)) }
def f4[A, B, C]: (A, (B, C)) => (((A, B), C)) = { case (a, (b, c)) => ((a, b), c) }
```

By applying these functions to arbitrary values of types $((A, B), C)$ and $(A, (B, C))$, it is easy to see that the functions `f3` and `f4` are inverses of each other. This is also directly visible in the code: the nested tuple pattern $((a, b), c)$ is mapped to the pattern $(a, (b, c))$ and back. So, the types $(A \times B) \times C$ and $A \times (B \times C)$ are equivalent, and we can write $A \times B \times C$ without parentheses.

Does a logical identity always correspond to an equivalence of types? This turns out to be *not* so. A simple example of a logical identity that does not correspond to a type equivalence is

$$True \vee \alpha = True \quad . \quad (5.17)$$

Since the CH correspondence maps the logical constant *True* into the unit type $\mathbb{1}$, the type equivalence corresponding to Eq. (5.17) is $\mathbb{1} + A \cong \mathbb{1}$. The type denoted by $\mathbb{1} + A$ means `Option[A]` in Scala, so the corresponding equivalence is `Option[A] ≅ Unit`. Intuitively, this type equivalence should not hold: an `Option[A]` may carry a value of type *A*, which cannot possibly be stored in a value of type `Unit`. We can verify this intuition rigorously by proving that any fully parametric functions with type signatures $g_1 : \mathbb{1} + A \Rightarrow \mathbb{1}$ and $g_2 : \mathbb{1} \Rightarrow \mathbb{1} + A$ will not satisfy $g_1 \circ g_2 = \text{id}$. To verify this, we note that $g_2 : \mathbb{1} \Rightarrow \mathbb{1} + A$ must have type signature

```
def g2[A]: Unit => Option[A] = ???
```

Such a function must always return `None`, since a fully parametric function cannot produce values of an arbitrary type *A* from scratch. Therefore, $g_1 \circ g_2$ is also a function that always returns `None`. The function $g_1 \circ g_2$ has type signature $\mathbb{1} + A \Rightarrow \mathbb{1} + A$ or, in Scala syntax, `Option[A] => Option[A]`, and is not equal to the identity function, because the identity function does not *always* return `None`.

Another example of a logical identity without a type equivalence is the distributive law

$$\forall(A, B, C). (A \wedge B) \vee C = (A \vee C) \wedge (B \vee C) \quad , \quad (5.18)$$

which is “dual” to the law (5.12), i.e. it is obtained from Eq. (5.12) by swapping all conjunctions (\wedge) with disjunctions (\vee). In logic, a dual formula to an identity is often also an identity. The CH correspondence maps Eq. (5.18) into the type equation

$$\forall(A, B, C). (A \times B) + C = (A + C) \times (B + C) \quad . \quad (5.19)$$

However, the types $A \times B + C$ and $(A + C) \times (B + C)$ are *not* equivalent. To see why, look at the possible code of the function $g_3 : (A + C) \times (B + C) \Rightarrow A \times B + C$:

```
1 def g3[A,B,C]: ((Either[A, C], Either[B, C])) => Either[(A, B), C] = {
2   case (Left(a), Left(b))    => Left((a, b)) // No other choice.
3   case (Left(a), Right(c))   => Right(c)      // No other choice.
4   case (Right(c), Left(b))   => Right(c)      // No other choice.
5   case (Right(c1), Right(c2)) => Right(c1)    // Must discard c1 or c2 here!
6 } // May return Right(c2) instead of Right(c1) in the last line.
```

In line 5, we have a choice of returning `Right(c1)` or `Right(c2)`. Whichever we choose, we will lose information because we will have discarded one of the given values `c1`, `c2`. After evaluating `g3`, we will not be able to restore *both* `c1` and `c2` – no matter what code we write for `g4`. So, the composition `g3 ∘ g4` cannot be equal to the identity function. The type equation (5.19) is incorrect.

We conclude that a logical identity $\mathcal{CH}(P) = \mathcal{CH}(Q)$ guarantees, via the CH correspondence, that we can implement *some* fully parametric functions of types $P \Rightarrow Q$ and $Q \Rightarrow P$. However, it is not guaranteed that these functions are inverses of each other, i.e. that the type conversions $P \Rightarrow Q$ or $Q \Rightarrow P$ have no information loss. So, the type equivalence $P \cong Q$ does not automatically follow from the logical identity $\mathcal{CH}(P) = \mathcal{CH}(Q)$.

The CH correspondence means that we can compute *some* value x^X of a given type X when the proposition $\mathcal{CH}(X)$ holds. However, the CH correspondence does not guarantee that the computed value x^X will satisfy any additional properties or laws.

5.3.2 Arithmetic identity corresponds to type equivalence

Looking at the examples of equivalent types, we notice that correct type equivalences correspond to *arithmetic* identities rather than *logical* identities. For instance, the logical identity in Eq. (5.12) leads to the type equivalence (5.15), which looks like a standard identity of arithmetic, such as

$$(1 + 10) \times 20 = 1 \times 20 + 10 \times 20 \quad .$$

The logical identity in Eq. (5.18), which does *not* yield a type equivalence, leads to an incorrect arithmetic equation 5.19, e.g. $(1 \times 10) + 20 \neq (1 + 20) \times (10 + 20)$. Similarly, the associativity law (5.16) leads to a type equivalence and to the arithmetic identity

$$(a \times b) \times c = a \times (b \times c) \quad ,$$

while the logical identity in Eq. (5.17), which does not yield a type equivalence, leads to an incorrect arithmetic statement $(\forall a. 1 + a = 1)$.

Table 5.5 summarizes these and other examples of logical identities, with the corresponding type equivalences. In all rows, quantifiers such as $\forall \alpha$ or $\forall(A, B)$ are implied when necessary.

Because we chose the type notation to be similar to the ordinary arithmetic notation, it is easy to translate a possible type equivalence into an arithmetic equation. In all cases, valid arithmetic identities correspond to type equivalences, and failures to obtain a type equivalence correspond to incorrect arithmetic identities. With regard to type equivalence, types such as $A + B$ and $A \times B$ behave similarly to arithmetic expressions such as $10 + 20$ and 10×20 and not similarly to logical formulas such as $\alpha \vee \beta$ and $\alpha \wedge \beta$.

We already verified the first line and the last three lines of Table 5.5. Other identities are verified in a similar way. Let us begin with lines 3 and 4 of Table 5.5, which involve the proposition *False* and the corresponding void type \emptyset , defined in Scala as *Nothing*. Reasoning about the void type needs a special technique that we will now develop while verifying the type isomorphisms $\emptyset \times A \cong \emptyset$ and $\emptyset + A \cong A$.

Example 5.3.2.1 Verify the type equivalence $\emptyset \times A \cong \emptyset$.

Solution Recall that the type notation $\emptyset \times A$ represents the Scala tuple type $(\text{Nothing}, A)$. To demonstrate that the type $(\text{Nothing}, A)$ is equivalent to the type *Nothing*, we need to show that the type $(\text{Nothing}, A)$ has *no* values. Indeed, how could we create a value of type, say, $(\text{Nothing}, \text{Int})$? We would need to fill both parts of the tuple. We have values of type *Int*, but we can never get a value of type *Nothing*. So, regardless of the type A , it is impossible to create any values of type $(\text{Nothing}, A)$. In other words, the set of values of the type $(\text{Nothing}, A)$ is empty; but that is the definition of the type *Nothing*. The types $(\text{Nothing}, A)$ (denoted by $\emptyset \times A$) and *Nothing* (denoted by \emptyset) are both void and therefore equivalent.

Example 5.3.2.2 Verify the type equivalence $\emptyset + A \cong A$.

Logical identity	Type equivalence (if it holds)
$True \vee \alpha = True$	$1 + A \cong 1$
$True \wedge \alpha = \alpha$	$1 \times A \cong A$
$False \vee \alpha = \alpha$	$0 + A \cong A$
$False \wedge \alpha = False$	$0 \times A \cong 0$
$\alpha \vee \beta = \beta \vee \alpha$	$A + B \cong B + A$
$\alpha \wedge \beta = \beta \wedge \alpha$	$A \times B \cong B \times A$
$(\alpha \vee \beta) \vee \gamma = \alpha \vee (\beta \vee \gamma)$	$(A + B) + C \cong A + (B + C)$
$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$	$(A \times B) \times C \cong A \times (B \times C)$
$(\alpha \vee \beta) \wedge \gamma = (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$	$(A + B) \times C \cong A \times C + B \times C$
$(\alpha \wedge \beta) \vee \gamma = (\alpha \vee \gamma) \wedge (\beta \vee \gamma)$	$(A \times B) + C \cong (A + C) \times (B + C)$

Table 5.5: Logic identities with disjunction and conjunction, and the corresponding equivalences of types.

Solution Recall that the type notation $0 + A$ represents the Scala type `Either[Nothing, A]`. We need to show that any value of that type can be mapped without loss of information to a value of type `A`, and vice versa. This means implementing functions $f_1 : 0 + A \Rightarrow A$ and $f_2 : A \Rightarrow 0 + A$ such that $f_1 \circ f_2 = \text{id}$ and $f_2 \circ f_1 = \text{id}$.

The argument of f_1 is of type `Either[Nothing, A]`. How can we create a value of that type? Our only choices are to create a `Left(x)` with `x:Nothing`, or to create a `Right(y)` with `y:A`. However, we cannot create a value `x` of type `Nothing` because the type `Nothing` has *no* values; so we cannot create a `Left(x)`. The only remaining possibility is to create a `Right(y)` with some value `y` of type `A`. So, any values of type $0 + A$ must be of the form `Right(y)`, and we can extract that `y` to obtain a value of type `A`:

```
def f1[A]: Either[Nothing, A] => A = {
  case Right(y) => y
  // No need for case Left(x) => ... since no x can be given there.
}
```

For the same reason, there is only one implementation of the function f_2 ,

```
def f2[A]: A => Either[Nothing, A] = { y => Right(y) }
```

It is clear from the code that the functions f_1 and f_2 are inverses of each other.

We have just seen that a value of type $0 + A$ is always a `Right(y)` with some `y:A`. Similarly, a value of type $A + 0$ is always a `Left(x)` with some `x:A`. So, we will use the notation $A + 0$ and $0 + A$ to *denote* the `Left` and the `Right` parts of the disjunctive type `Either`. This notation agrees with the behavior of the Scala compiler, which will infer the types `Either[A, Nothing]` or `Either[Nothing, A]` for these parts:

```
def toLeft[A, B]: A => Either[A, B] = x => Left(x)
def toRight[A, B]: B => Either[A, B] = y => Right(y)

scala> toLeft(123)
res0: Either[Int, Nothing] = Left(123)

scala> toRight("abc")
res1: Either[Nothing, String] = Right("abc")
```

We can write the functions `toLeft` and `toRight` in a code notation as

$$\begin{aligned} \text{toLeft}^{A,B} &= x:A \Rightarrow x:A + 0:B, \\ \text{toRight}^{A,B} &= y:B \Rightarrow 0:A + y:B. \end{aligned}$$

In this notation, a value of the disjunctive type is shown without using Scala class

names such as `Either`, `Right`, and `Left`. This shortens the writing and speeds up code reasoning.

The type annotation $0:A$ is helpful to remind ourselves about the type parameter `A` used e.g. by the disjunctive value $0:A + y:B$ in the body of `toRight[A, B]`. Without this type annotation, $0 + y:B$ means

a value of type `Either[A, B]` where the parameter A is left unspecified and should be determined by matching the types of other expressions.

In the notation $0 + y^B$, we use the symbol 0 rather than an ordinary zero (0), to avoid suggesting that 0 is a value of type 0 . The void type 0 has no values, unlike the `Unit` type, 1 , which has a value denoted by 1 in the code notation.

Example 5.3.2.3 Verify the type equivalence $A \times 1 \cong A$.

Solution The corresponding Scala types are the tuple (A, Unit) and the type A . We need to implement functions $f_1 : \forall A. A \times 1 \Rightarrow A$ and $f_2 : \forall A. A \Rightarrow A \times 1$ and to demonstrate that they are inverses of each other. The Scala code for these functions is

```
def f1[A]: ((A, Unit)) => A = { case (a, ()) => a }
def f2[A]: A => (A, Unit) = { a => (a, ()) }
```

Let us first write a proof by reasoning directly with Scala code:

```
(f1 andThen f2)((a, ())) == f2(f1((a, ()))) == f2(a) == (a, ())
(f2 andThen f1)(a) == f1(f2(a)) == f1((a, ())) == a
```

Now let us write a proof in the code notation. The codes of f_1 and f_2 are

$$\begin{aligned} f_1 &= a^{A \times 1} \times 1 \Rightarrow a \quad , \\ f_2 &= a^{A \times 1} \Rightarrow a \times 1 \quad , \end{aligned}$$

where we denoted by 1 the value $()$ of the `Unit` type. We find

$$\begin{aligned} (f_1 \circ f_2)(a^{A \times 1} \times 1) &= f_2(f_1(a \times 1)) = f_2(a) = a \times 1 \quad , \\ (f_2 \circ f_1)(a^{A \times 1}) &= f_1(f_2(a)) = f_1(a \times 1) = a \quad . \end{aligned}$$

This shows that both compositions are identity functions. Another way of writing the proof is by computing the function compositions symbolically, without applying to a value $a^{A \times 1}$,

$$\begin{aligned} f_1 \circ f_2 &= (a \times 1 \Rightarrow a) \circ (a \Rightarrow a \times 1) = (a \times 1 \Rightarrow a \times 1) = \text{id}^{A \times 1} \quad , \\ f_2 \circ f_1 &= (a \Rightarrow a \times 1) \circ (a \times 1 \Rightarrow a) = (a \Rightarrow a) = \text{id}^A \quad . \end{aligned}$$

Example 5.3.2.4 Verify the type equivalence $A + B \cong B + A$.

Solution The corresponding Scala types are `Either[A, B]` and `Either[B, A]`. We use pattern matching to implement the functions required for the type equivalence:

```
def f1[A, B]: Either[A, B] => Either[B, A] = {
  case Left(a)    => Right(a) // No other choice here.
  case Right(b)   => Left(b)  // No other choice here.
}
def f2[A, B]: Either[B, A] => Either[A, B] = f1[B, A]
```

The functions f_1 and f_2 are implemented by code that can be derived unambiguously from the type signatures. For instance, the line `case Left(a) => ...` is required to return a value of type `Either[B, A]` by using only a given value $a:A$. The only way of doing that is by returning `Right(a)`.

A by using only a given value $a:A$. The only way of doing that is by returning `Right(a)`.

It is clear from the code that the functions f_1 and f_2 are inverses of each other. To verify that rigorously, we need show that f_1 andThen f_2 is equal to an identity function. The function f_1 andThen f_2 applies f_2 to the result of f_1 . The code of f_1 contains two `case ...` lines, each returning a result. So, we need to apply f_2 separately in each line. Evaluate the code symbolically:

```
(f1 andThen f2) == {
  case Left(a)    => f2(Right(a))
  case Right(b)   => f2(Left(b))
} == {
  case Left(a)    => Left(a)
  case Right(b)   => Right(b)
}
```

The result is a function of type `Either[A, B] => Either[A, B]` that does not change its argument; so it is equal to the identity function.

Let us now write the function `f1` in the code notation and perform the same derivation. We will also develop a useful notation for functions operating on disjunctive types.

The pattern matching construction in the Scala code of `f1` contains a pair of functions with types `A => Either[B, A]` and `B => Either[B, A]`. One of these functions is chosen depending on whether the argument of `f1` has type `A + 0` or `0 + B`. So, we may write the code of `f1` as

$$f_1 \triangleq x^{A+B} \Rightarrow \begin{cases} \text{if } x = a^{A} + 0^{B} & : 0^{B} + a^{A} \\ \text{if } x = 0^{A} + b^{B} & : b^{B} + 0^{A} \end{cases}$$

Since both the argument and the result of f_1 are disjunctive types with 2 parts each, it is convenient to write the code of f_1 as a 2×2 matrix that maps the input parts to the output parts:

```
def f1[A, B]: Either[A, B] => Either[B, A] = {
  case Left(a)    => Right(a)
  case Right(b)   => Left(b)
}
```

$$f_1 \triangleq \begin{array}{c|c|cc} & & B & A \\ \hline A & & 0 & a^{A} \Rightarrow a \\ \hline B & & b^{B} \Rightarrow b & 0 \end{array} .$$

The rows of the matrix correspond to the `case` rows in the Scala code; there is one row for each part of the disjunctive type of the argument. The columns of the matrix correspond to the parts of the disjunctive type of the result. The double line marks the input types of the functions.

The code of f_2 is written similarly; let us rename arguments for clarity:

```
def f2[A, B]: Either[B, A] => Either[A, B] = {
  case Left(b)    => Right(b)
  case Right(a)   => Left(a)
}
```

$$f_2 \triangleq \begin{array}{c|c|cc} & & A & B \\ \hline B & & 0 & y^{B} \Rightarrow y \\ \hline A & & x^{A} \Rightarrow x & 0 \end{array} .$$

The forward composition $f_1 \circ f_2$ is computed by the standard rules of row-by-column matrix multiplication.⁵ The 0 terms are omitted, and the remaining expressions are composed:

$$\begin{aligned} f_1 \circ f_2 &= \begin{array}{c|c|cc} & & B & A \\ \hline A & & 0 & a^{A} \Rightarrow a \\ \hline B & & b^{B} \Rightarrow b & 0 \end{array} \circ \begin{array}{c|c|cc} & & A & B \\ \hline B & & 0 & y^{B} \Rightarrow y \\ \hline A & & x^{A} \Rightarrow x & 0 \end{array} \\ \text{use matrix multiplication :} &= \begin{array}{c|c|cc} & & A & B \\ \hline A & & (a^{A} \Rightarrow a) \circ (x^{A} \Rightarrow x) & 0 \\ \hline B & & 0 & (b^{B} \Rightarrow b) \circ (y^{B} \Rightarrow y) \end{array} \\ \text{function composition :} &= \begin{array}{c|c|cc} & & A & B \\ \hline A & & \text{id} & 0 \\ \hline B & & 0 & \text{id} \end{array} = \text{id}^{A+B \Rightarrow A+B} . \end{aligned}$$

Several features of the matrix notation are helpful in such calculations. The parts of the code of f_1 are automatically composed with the corresponding parts of the code of f_2 . To check that the types match in the function composition, we just need to compare the types in the output row $\begin{array}{c|c} B & A \end{array}$ of

f_1 with the input column $\begin{array}{c} B \\ A \end{array}$ of f_2 . Once we verified that all types match, we may omit the type

⁵https://en.wikipedia.org/wiki/Matrix_multiplication

annotations and write the derivation as

$$\begin{aligned}
 f_1 \circ f_2 &= \left\| \begin{array}{cc} \mathbb{0} & a:A \Rightarrow a \\ b:B \Rightarrow b & \mathbb{0} \end{array} \right\| \circ \left\| \begin{array}{cc} \mathbb{0} & y:B \Rightarrow y \\ x:A \Rightarrow x & \mathbb{0} \end{array} \right\| \\
 \text{use matrix multiplication :} &= \left\| \begin{array}{cc} (a:A \Rightarrow a) \circ (x:A \Rightarrow x) & \mathbb{0} \\ \mathbb{0} & (b:B \Rightarrow b) \circ (y:B \Rightarrow y) \end{array} \right\| \\
 \text{function composition :} &= \left\| \begin{array}{cc} \text{id} & \mathbb{0} \\ \mathbb{0} & \text{id} \end{array} \right\| = \text{id} .
 \end{aligned}$$

The identity function is represented by the diagonal matrix $\left\| \begin{array}{cc} \text{id} & \mathbb{0} \\ \mathbb{0} & \text{id} \end{array} \right\|$, just as in standard matrix notation.

Exercise 5.3.2.5 Verify the type equivalence $A \times B \cong B \times A$.

Exercise 5.3.2.6 Verify the type equivalence $(A + B) + C \cong A + (B + C)$. The equivalences $(A + B) + C \cong A + (B + C)$ and $(A \times B) \times C \cong A \times (B \times C)$, which was verified in Section 5.3.1, permit us to write the type notation as $A + B + C$ and $A \times B \times C$ without the parentheses.

Exercise 5.3.2.7 Verify the type equivalence

$$(A + B) \times (A + B) = A \times A + 2 \times A \times B + B \times B ,$$

where 2 denotes the `Boolean` type.

5.3.3 Type cardinalities and type equivalence

To understand why type equivalences are related to arithmetic identities, consider the question of how many different values a given type can have.

Begin by counting the number of distinct values for simple types. For example, the `Unit` type has only one distinct value; the type `Nothing` has zero values; the `Boolean` type has two distinct values, `true` and `false`; and the type `Int` has 2^{32} distinct values.

It is more difficult to count the number of distinct values in a type such as `String`, which is equivalent to a list of unknown length, `List[Char]`. However, each computer's memory is limited, so there will exist a maximum length for values of type `String`, and so the total number of possible different strings will be finite (at least, for any given computer).

For a given type A , let us denote by $|A|$ the number of distinct values of type A . The number $|A|$ is called the **cardinality** of type A ; this is the same as the number of elements in the set of all values of type A . Since any computer's memory is finite, and since we may assume that we are already working with the largest possible computer, then there will be *finitely* many different values of a given type A that can exist in the computer. So, we may assume that $|A|$ is always a finite integer value. This assumption will simplify our reasoning. We will not actually need to compute the precise number of, say, all the different possible strings; it is sufficient to know that the set of all strings is finite, so that we can denote its cardinality by $|\text{String}|$.

The next step is to consider the cardinality of types such as $A \times B$ and $A + B$. If the types A and B have cardinalities $|A|$ and $|B|$, it follows that the set of all distinct pairs (a, b) has $|A| \times |B|$ elements. So the cardinality of the type $A \times B$ is equal to the (arithmetic) product of the cardinalities of A and B . The set of all pairs

$$\{(a, b) : a \in A, b \in B\}$$

is also known as the **Cartesian product** of sets A and B , and is denoted by $A \times B$. For this reason, the tuple type is also called the **product type**. Accordingly, the type notation adopts the symbol \times for the product type.

The set of all distinct values of the type $A + B$, i.e. of the Scala type `Either[A, B]`, is a disjoint union of the set of values of the form `Left(a)` and the set of values of the form `Right(b)`. It is clear that the cardinalities of these sets are equal to $|A|$ and $|B|$ respectively. So the cardinality of the type `Either[A, B]` is equal to $|A| + |B|$. For this reason, disjunctive types such as `Either[A, B]` are also called **sum types**, and the type notation adopts the symbol $+$ for these types.

We can write our conclusions as

$$\begin{aligned} |A \times B| &= |A| \times |B| \quad , \\ |A + B| &= |A| + |B| \quad . \end{aligned}$$

The type notation, $A \times B$ for the pairs and $A + B$ for the disjunctive types, translates directly into type cardinalities.

The last step is to notice that two types can be equivalent, $P \cong Q$, only if their cardinalities are equal, $|P| = |Q|$. When the cardinalities are not equal, $|P| \neq |Q|$, it will be impossible to have a one-to-one correspondence between the sets of values of type P and values of type Q . So it will be impossible to convert values from type P to type Q and back without loss of information.

We conclude that types are equivalent when a logical identity *and* an arithmetic identity hold.

The presence of both identities does not automatically guarantee a useful type equivalence. The fact that information in one type can be identically stored in another type does not necessarily mean that it is helpful to do so in a given application.

For example, the types `Option[Option[A]]` and `Either[Boolean, A]` are equivalent because both types contain $2 + |A|$ distinct values. The short notation for these types is $\mathbb{1} + \mathbb{1} + A$ and $2 + A$ respectively (the type `Boolean` is denoted by 2 since it has only two distinct values).

One could easily write code to convert between these types without loss of information:

```
def f1[A]: Option[Option[A]] => Either[Boolean, A] = {
  case None           => Left(false) // Or maybe Left(true)?
  case Some(None)     => Left(true)
  case Some(Some(x)) => Right(x)
}

def f2[A]: Either[Boolean, A] => Option[Option[A]] = {
  case Left(false) => None
  case Left(true)  => Some(None)
  case Right(x)   => Some(Some(x))
}
```

A sign of trouble is the presence of an arbitrary choice in this code. In `f1`, we could map `None` to `Left(false)` or to `Left(true)`, and adjust the rest of the code accordingly; the type equivalence would still hold. So, formally speaking, these types *are* equivalent, but there is no “natural” choice of the conversion functions `f1` and `f2` that will work correctly in all applications, because the meaning of these data types is application-dependent. This type equivalence is “accidental”.

Example 5.3.3.1 Are the types `Option[A]` and `Either[Unit, A]` equivalent? Check whether the corresponding logic identity and arithmetic identity hold.

Solution Begin by writing the given types in the type notation: `Option[A]` is written as $\mathbb{1} + A$, and `Either[Unit, A]` is written also as $\mathbb{1} + A$. This already indicates, by looking at the notation alone, that the types are equivalent. But let us verify explicitly that the type notation is not misleading here.

To establish type equivalence, we need to implement two fully parametric functions

```
def f1[A]: Option[A] => Either[Unit, A] = ???
def f2[A]: Either[Unit, A] => Option[A] = ???
```

such that $f_1 \circ f_2 = \text{id}$ and $f_2 \circ f_1 = \text{id}$. It is straightforward to implement `f1` and `f2`:

```
def f1[A]: Option[A] => Either[Unit, A] = {
  case None    => Left(())
  case Some(x) => Right(x)
}
```

```
def f2[A]: Either[Unit, A] => Option[A] = {
  case Left(())    => None
  case Right(x)    => Some(x)
}
```

The code clearly shows that f_1 and f_2 are inverses of each other; this verifies the type equivalence.

The logic identity is $True \vee A = True \vee A$ and holds trivially. It remains to check the arithmetic identity, which relates the number of distinct values of types `Option[A]` and `Either[Unit, A]`. Assume that the number of distinct values of type `A` is $|A|$. Any possible value of type `Option[A]` must be either `None` or `Some(x)`, where `x` is a value of type `A`. So the number of distinct values of type `Option[A]` is $1 + |A|$. All possible values of type `Either[Unit, A]` are of the form `Left(())` or `Right(x)`, where `x` is a value of type `A`. So the number of distinct values of type `Either[Unit, A]` is $1 + |A|$. We see that the arithmetic identity holds: the types `Option[A]` and `Either[Unit, A]` have equally many distinct values.

This example shows that the type notation is helpful for reasoning about type equivalences. The solution was found immediately when we wrote the type notation, $\mathbb{1} + A$, for the given types.

5.3.4 Type equivalence involving function types

Until now, we have looked at product types and disjunctive types. Let us now consider type constructions involving function types.

Consider two types `A` and `B`, whose cardinalities are known as $|A|$ and $|B|$. What is the cardinality of the set of all maps between given sets `A` and `B`? In other words, how many distinct values does the function type `A => B` have? A function $f: A \Rightarrow B$ needs to select a value of type `B` for each possible value of type `A`. Therefore, the number of different functions $f: A \Rightarrow B$ is $|B|^{|A|}$ (the arithmetic exponent, $|B|$ to the power $|A|$).

For the types `A = B = Int`, we have $|A| = |B| = 2^{32}$, and so the estimate will give

$$|A \Rightarrow B| = (2^{32})^{(2^{32})} = 2^{32 \times 2^{32}} = 2^{2^{37}} \approx 10^{4.1 \times 10^{10}}.$$

In fact, most of these functions will map integers to integers in a complicated (and practically useless) way and will be impossible to implement on a realistic computer because their code will be much longer than the available memory. So, the number of practically implementable functions of type `A => B` is often much smaller than $|B|^{|A|}$. Nevertheless, the estimate $|B|^{|A|}$ is useful since it shows the number of distinct functions that are possible in principle.

Let us now see what logic identities and arithmetic identities are available for type expressions involving function types. Table 5.6 lists the available identities and the corresponding type equivalences. (In the last column, we defined $a \triangleq |A|$, $b \triangleq |B|$, and $c \triangleq |C|$ for brevity.)

It is notable that no logic identity is available for the formula $\alpha \Rightarrow (\beta \vee \gamma)$, and correspondingly no type equivalence is available for the type expression `A => B + C` (although there is an identity for `A => B × C`). The presence of type expressions of the form `A => B + C` makes type reasoning more complicated because they cannot be transformed into equivalent formulas with simpler parts.

We will now prove some of the type identities in Table 5.6.

Example 5.3.4.1 Verify the type equivalence $\mathbb{1} \Rightarrow A \cong A$.

Solution Recall that the type notation $\mathbb{1} \Rightarrow A$ means the Scala function type `Unit => A`. There is only one value of type `Unit`, so the choice of a function of the type `Unit => A` is the same as the choice of a value of type `A`. Thus, the number of distinct values of the type $\mathbb{1} \Rightarrow A$ is $|A|$, and the arithmetic identity holds.

To verify the type equivalence explicitly, we need to implement two functions

```
def f1[A]: (Unit => A) => A = ???
def f2[A]: A => Unit => A = ???
```

The first function needs to produce a value of type `A`, given an argument of the function type `Unit => A`. The only possibility is to apply that function to the value of type `Unit`; we can always produce that value as `()`:

Logical identity (if holds)	Type equivalence	Arithmetic identity
$(True \Rightarrow \alpha) = \alpha$	$1 \Rightarrow A \cong A$	$a^1 = a$
$(False \Rightarrow \alpha) = True$	$0 \Rightarrow A \cong 1$	$a^0 = 1$
$(\alpha \Rightarrow True) = True$	$A \Rightarrow 1 \cong 1$	$1^a = 1$
$(\alpha \Rightarrow False) \neq False$	$A \Rightarrow 0 \neq 0$	$0^a \neq 0$
$(\alpha \vee \beta) \Rightarrow \gamma = (\alpha \Rightarrow \gamma) \wedge (\beta \Rightarrow \gamma)$	$A + B \Rightarrow C \cong (A \Rightarrow C) \times (B \Rightarrow C)$	$c^{a+b} = c^a \times c^b$
$(\alpha \wedge \beta) \Rightarrow \gamma = \alpha \Rightarrow (\beta \Rightarrow \gamma)$	$A \times B \Rightarrow C \cong A \Rightarrow B \Rightarrow C$	$c^{a \times b} = (c^b)^a$
$\alpha \Rightarrow (\beta \wedge \gamma) = (\alpha \Rightarrow \beta) \wedge (\alpha \Rightarrow \gamma)$	$A \Rightarrow B \times C \cong (A \Rightarrow B) \times (A \Rightarrow C)$	$(b \times c)^a = b^a \times c^a$

Table 5.6: Logical identities with implication, and the corresponding type equivalences and arithmetic identities.

```
def f1[A]: (Unit => A) => A = (h: Unit => A) => h()
```

Implementing `f2` is straightforward; we can just discard the `Unit` argument:

```
def f2[A]: A => Unit => A = (x: A) => _ => x
```

It remains to show that the functions `f1` and `f2` are inverses of each other. Let us perform the proof using Scala code and then using the code notation.

Writing Scala code, compute `f1(f2(x))` for an arbitrary `x:A`. Substituting the code, we get

```
f1(f2(x)) == f1(_ => x) == (_ => x)() == x
```

Now compute `f2(f1(h))` for arbitrary `h: Unit => A` in Scala code:

```
f2(f1(h)) == f2(h()) == { _ => h() }
```

How can we show that the function `{ _ => h() }` is equal to `h`? Whenever we apply equal functions to equal arguments, they return equal results. In our case, the argument of `h` is of type `Unit`, so we only need to verify that the result of applying `h` to the value `()` is the same as the result of applying `{ _ => h() }` to `()`. In other words, we need to apply both sides to an additional argument `()`:

```
f2(f1(h))() == { _ => h() } () == h()
```

This completes the proof.

For comparison, let us show the same proof in the code notation. The functions f_1 and f_2 are

$$\begin{aligned} f_1 &\triangleq h^{1 \Rightarrow A} \Rightarrow h(1) \quad , \\ f_2 &\triangleq x^{A \Rightarrow 1} \Rightarrow 1 \Rightarrow x \quad . \end{aligned}$$

Now write the function compositions in both directions:

$$\begin{aligned} f_1 \circ f_2 &= (h^{1 \Rightarrow A} \Rightarrow h(1)) \circ (x^{A \Rightarrow 1} \Rightarrow 1 \Rightarrow x) \\ \text{compute composition :} &= (h \Rightarrow 1 \Rightarrow h(1)) \\ \text{note that } 1 \Rightarrow h(1) \text{ is the same as } h : &= (h \Rightarrow h) = \text{id} \quad . \\ f_2 \circ f_1 &= (x^{A \Rightarrow 1} \Rightarrow 1 \Rightarrow x) \circ (h^{1 \Rightarrow A} \Rightarrow h(1)) \\ \text{compute composition :} &= x \Rightarrow (1 \Rightarrow x)(1) \\ \text{apply function :} &= (x \Rightarrow x) = \text{id} \quad . \end{aligned}$$

The type $1 \Rightarrow A$ is equivalent to the type A , but these types are not the same. The most important difference between these types is that a value of type A is available immediately, while a value of

type $\mathbb{1} \Rightarrow A$ is a function that still needs to be applied to an argument (of type $\mathbb{1}$) before a value of type A is obtained. The type $\mathbb{1} \Rightarrow A$ may represent an “on-call” value of type A ; that is, a value computed on demand every time. (See Section 2.6.3 for more details about “on-call” values.)

The void type $\mathbb{0}$ needs special reasoning.

Example 5.3.4.2 Verify the type equivalence $\mathbb{0} \Rightarrow A \cong \mathbb{1}$.

Solution What could be a function $f: \mathbb{0} \Rightarrow A$ from the type $\mathbb{0}$ to a type A ? Since there exist no values of type $\mathbb{0}$, the function f will never be applied to any arguments and so *does not need* to compute any actual values of type A . So, f is a function whose body may be empty; or at least it does not need to contain any expressions of type A . In Scala, such a function can be written as

```
def absurd[A]: Nothing => A = { ??? }
```

This code will compile without type errors. An equivalent code is

```
def absurd[A]: Nothing => A = { x => ??? }
```

The symbol `???` is defined in the Scala library and represents code that is “not implemented”. Trying to evaluate this symbol will produce an error:

```
scala> ???
scala.NotImplementedError: an implementation is missing
  scala.Predef$.qmark$qmark$qmark(Predef.scala:288)
```

Since the function `absurd` is impossible to apply to an argument, this error will never happen. So, we can pretend that the result value (which will never be computed or returned) has any required type, e.g. the type A .

Let us now verify that there exists *only one* function of type $\mathbb{0} \Rightarrow A$. Suppose there are two such functions, $f: \mathbb{0} \Rightarrow A$ and $g: \mathbb{0} \Rightarrow A$. Are f and g different functions? We would see that f and g are different only if we had a value x such that $f(x) \neq g(x)$, where x must have type $\mathbb{0}$. However, there are *no* values of type $\mathbb{0}$, and so we will never be able to find such x . It follows that any two functions f and g of type $\mathbb{0} \Rightarrow A$ are equal. In other words, there exists *only one* distinct value of type $\mathbb{0} \Rightarrow A$; i.e. the cardinality of the type $\mathbb{0} \Rightarrow A$ is 1. So, the type $\mathbb{0} \Rightarrow A$ is equivalent to the type $\mathbb{1}$.

Example 5.3.4.3 Show that $A \Rightarrow \mathbb{0} \not\cong \mathbb{0}$ and $A \Rightarrow \mathbb{0} \not\cong \mathbb{1}$.

Solution To prove that two types are *not* equivalent, it is sufficient to show that their type cardinalities are different. Let us determine the cardinality of the type $A \Rightarrow \mathbb{0}$, assuming that the cardinality of A is known. We note that a function of type, say, $\text{Int} \Rightarrow \mathbb{0}$ is impossible to implement. (If we had such a function $f: \text{Int} \Rightarrow \mathbb{0}$, we could evaluate, say, $x \triangleq f(123)$ and obtain a value x of type $\mathbb{0}$, which is impossible by definition of the type $\mathbb{0}$. It follows that $|\text{Int} \Rightarrow \mathbb{0}| = 0$. However, Example 5.3.4.2 shows that $\mathbb{0} \Rightarrow \mathbb{0}$ has cardinality 1. So, the cardinality $|A \Rightarrow \mathbb{0}| = 1$ if the type A is itself $\mathbb{0}$ but $|A \Rightarrow \mathbb{0}| = 0$ for all other types A . We conclude that the type $A \Rightarrow \mathbb{0}$ is not equivalent to $\mathbb{0}$ or $\mathbb{1}$ for all A ; it is equivalent to $\mathbb{0}$ only for non-void types A .

Example 5.3.4.4 Verify the type equivalence $A \Rightarrow \mathbb{1} \cong \mathbb{1}$.

Solution There is only one fully parametric function that returns $\mathbb{1}$:

```
def f[A]: A => Unit = { _ => () }
```

The function f cannot use its argument of type A since nothing is known about that type. So the code of f *must* discard its argument and return the fixed value `()` of type `Unit`. In the code notation, this function is written as

$$f: A \Rightarrow \mathbb{1} \triangleq (_ \Rightarrow 1) \quad .$$

We can show that there exist *only one* distinct function of type $A \Rightarrow \mathbb{1}$ (that is, the type $A \Rightarrow \mathbb{1}$ has cardinality 1). Assume that f and g are two such functions, and try to find a value $x: A$ such that $f(x) \neq g(x)$. We cannot find any such x because $f(x) = 1$ and $g(x) = 1$ for all x . So, any two functions f and g of type $A \Rightarrow \mathbb{1}$ must be equal to each other. Any type having cardinality 1 is equivalent to the `Unit` type, $\mathbb{1}$. So $A \Rightarrow \mathbb{1} \cong \mathbb{1}$.

Example 5.3.4.5 Verify the type equivalence

$$A + B \Rightarrow C \cong (A \Rightarrow C) \times (B \Rightarrow C) \quad .$$

Solution Begin by implementing two functions with type signatures

```
def f1[A,B,C]: (Either[A, B] => C) => (A => C, B => C) = ???
def f2[A,B,C]: ((A => C, B => C)) => Either[A, B] => C = ???
```

The code can be derived unambiguously from the type signatures. For the first function, we need to produce a pair of functions of type $(A \Rightarrow C, B \Rightarrow C)$. Can we produce the first part of that pair? Computing a function of type $A \Rightarrow C$ means that we need to produce a value of type C given an arbitrary value $a:A$. The available data is a function of type $\text{Either}[A, B] \Rightarrow C$ called, say, h . We can apply that function to $\text{Left}(a)$ and obtain a value of type C as required. So, a function of type $A \Rightarrow C$ is computed as $a \Rightarrow h(\text{Left}(a))$. Similarly, we produce a function of type $B \Rightarrow C$. The code is

```
def f1[A,B,C]: (Either[A, B] => C) => (A => C, B => C) =
  (h: Either[A, B] => C) => (a => h(Left(a)), b => h(Right(b)))
```

A code notation for this function is

$$f_1 : (A + B \Rightarrow C) \Rightarrow (A \Rightarrow C) \times (B \Rightarrow C) \quad ,$$

$$f_1 \triangleq h^{A+B \Rightarrow C} \Rightarrow (a^A \Rightarrow h(a + 0^B)) \times (b^B \Rightarrow h(0^A + b)) \quad .$$

For the function f_2 , we need to apply pattern matching to both curried arguments and then return a value of type C . This can be achieved in only one way:

```
def f2[A,B,C]: ((A => C, B => C)) => Either[A, B] => C = { case (f, g) =>
  {
    case Left(a)    => f(a)
    case Right(b)   => g(b)
  }
}
```

A code notation for this function can be written as

$$f_2 : (A \Rightarrow C) \times (B \Rightarrow C) \Rightarrow A + B \Rightarrow C \quad ,$$

$$f_2 \triangleq f^{A \Rightarrow C} \times g^{B \Rightarrow C} \Rightarrow \left| \begin{array}{c|c} & C \\ \hline A & a \Rightarrow f(a) \\ B & b \Rightarrow g(b) \end{array} \right| \quad .$$

The matrix in the last line has only one column because the result type, C , is not known to be a disjunctive type. We may simplify the functions, e.g. $a \Rightarrow f(a)$ into f , and write

$$f_2 \triangleq f^{A \Rightarrow C} \times g^{B \Rightarrow C} \Rightarrow f^{A \Rightarrow C} \times g^{B \Rightarrow C} \Rightarrow \left| \begin{array}{c|c} & C \\ \hline A & f \\ B & g \end{array} \right| \quad .$$

It remains to verify that $f_1 \circ f_2 = \text{id}$ and $f_2 \circ f_1 = \text{id}$. To compute the composition $f_1 \circ f_2$, we write (omitting types)

$$f_1 \circ f_2 = (h \Rightarrow (a \Rightarrow h(a + 0)) \times (b \Rightarrow h(0 + b))) \circ (f \times g \Rightarrow \left| \begin{array}{c} f \\ g \end{array} \right|)$$

$$\text{compute composition : } = h \Rightarrow \left| \begin{array}{c} a \Rightarrow h(a + 0) \\ b \Rightarrow h(0 + b) \end{array} \right| \quad .$$

To proceed, we need to simplify the expressions $h(a + \mathbb{0})$ and $h(\mathbb{0} + b)$. We rewrite the argument h (an arbitrary function of type $A + B \Rightarrow C$) in the matrix notation:

$$h \triangleq \left[\begin{array}{c|c} & C \\ \hline A & a \Rightarrow p(a) \\ B & b \Rightarrow q(b) \end{array} \right] = \left[\begin{array}{c|c} & C \\ \hline A & p \\ B & q \end{array} \right] ,$$

where $p:A \Rightarrow C$ and $q:B \Rightarrow C$ are new arbitrary functions. Since we already checked the types, we can omit all type annotations and express h as

$$h \triangleq \left[\begin{array}{c} p \\ q \end{array} \right] .$$

To evaluate expressions such as $h(a + \mathbb{0})$ and $h(\mathbb{0} + b)$, we need to use one of the rows of the column matrix h . The correct row will be selected *automatically* by the standard rules of matrix multiplication if we place a row vector to the left of the matrix, and if we use the convention of omitting any terms containing $\mathbb{0}$:

$$\begin{aligned} \left[a \quad \mathbb{0} \right] \triangleright \left[\begin{array}{c} p \\ q \end{array} \right] &= a \triangleright p , \\ \left[\mathbb{0} \quad b \right] \triangleright \left[\begin{array}{c} p \\ q \end{array} \right] &= b \triangleright q . \end{aligned}$$

Here we used the symbol \triangleright to separate an argument from a function when the argument is written to the *left* of the function. The symbol \triangleright (pronounced “pipe”) is defined by $x \triangleright f \triangleq f(x)$. In Scala, this operation is available as `x.pipe(f)` as of Scala 2.13.

We write values of disjunctive types, such as $a + \mathbb{0}$, as if they were row vectors:

$$h(a + \mathbb{0}) = (a + \mathbb{0}) \triangleright h = \left[a \quad \mathbb{0} \right] \triangleright h . \quad (5.20)$$

With these notations, we can compute further:

$$\begin{aligned} h(a + \mathbb{0}) &= \left[a \quad \mathbb{0} \right] \triangleright \left[\begin{array}{c} p \\ q \end{array} \right] = a \triangleright p = p(a) , \\ h(\mathbb{0} + b) &= \left[\mathbb{0} \quad b \right] \triangleright \left[\begin{array}{c} p \\ q \end{array} \right] = b \triangleright q = q(b) . \end{aligned}$$

Now we can complete the proof of $f_1 \circ f_2 = \text{id}$:

$$\begin{aligned} f_1 \circ f_2 &= h \Rightarrow \left[\begin{array}{c} a \Rightarrow h(a + \mathbb{0}) \\ b \Rightarrow h(\mathbb{0} + b) \end{array} \right] \\ \text{previous equations :} &= \left[\begin{array}{c} p \\ q \end{array} \right] \Rightarrow \left[\begin{array}{c} a \Rightarrow p(a) \\ b \Rightarrow q(b) \end{array} \right] \\ \text{simplify functions :} &= \left(\left[\begin{array}{c} p \\ q \end{array} \right] \Rightarrow \left[\begin{array}{c} p \\ q \end{array} \right] \right) = \text{id} . \end{aligned}$$

To prove that $f_2 \circ f_1 = \text{id}$, use the notation (5.20):

$$\begin{aligned}
 f_2 \circ f_1 &= \left(f \times g \Rightarrow \left\| \begin{array}{c} f \\ g \end{array} \right\| \right) \circ (h \Rightarrow (a \Rightarrow h(a + 0)) \times (b \Rightarrow h(0 + b))) \\
 \text{compute composition :} &= f \times g \Rightarrow (a \Rightarrow \left| \begin{array}{cc} a & 0 \end{array} \right| \triangleright \left\| \begin{array}{c} f \\ g \end{array} \right\|) \times (b \Rightarrow \left| \begin{array}{cc} 0 & b \end{array} \right| \triangleright \left\| \begin{array}{c} f \\ g \end{array} \right\|) \\
 \text{matrix notation :} &= f \times g \Rightarrow (a \Rightarrow \underline{a \triangleright f}) \times (b \Rightarrow \underline{b \triangleright g}) \\
 \text{definition of } \triangleright : &= f \times g \Rightarrow \underline{(a \Rightarrow f(a))} \times \underline{(b \Rightarrow g(b))} \\
 \text{simplify functions :} &= (f \times g \Rightarrow f \times g) = \text{id} \quad .
 \end{aligned}$$

In this way, we have proved that f_1 and f_2 are mutual inverses. The proofs appear long because we took time to motivate and introduce new notation for applying matrices to row vectors. Given this notation, the proof for $f_1 \circ f_2 = \text{id}$ can be written as

$$\begin{aligned}
 f_1 \circ f_2 &= (h \Rightarrow (a \Rightarrow (a + 0) \triangleright h) \times (b \Rightarrow (0 + b) \triangleright h)) \circ (f \times g \Rightarrow \left\| \begin{array}{c} f \\ g \end{array} \right\|) \\
 \text{compute composition :} &= h \Rightarrow \left\| \begin{array}{cc} a \Rightarrow \left| \begin{array}{cc} a & 0 \end{array} \right| \triangleright h \\ b \Rightarrow \left| \begin{array}{cc} 0 & b \end{array} \right| \triangleright h \end{array} \right\| = \left\| \begin{array}{c} p \\ q \end{array} \right\| \Rightarrow \left\| \begin{array}{cc} a \Rightarrow \left| \begin{array}{cc} a & 0 \end{array} \right| \triangleright \left\| \begin{array}{c} p \\ q \end{array} \right\| \\ b \Rightarrow \left| \begin{array}{cc} 0 & b \end{array} \right| \triangleright \left\| \begin{array}{c} p \\ q \end{array} \right\| \end{array} \right\| \\
 \text{matrix notation :} &= \left\| \begin{array}{c} p \\ q \end{array} \right\| \Rightarrow \left\| \begin{array}{cc} a \Rightarrow a \triangleright p \\ b \Rightarrow b \triangleright q \end{array} \right\| = \left(\left\| \begin{array}{c} p \\ q \end{array} \right\| \Rightarrow \left\| \begin{array}{c} p \\ q \end{array} \right\| \right) = \text{id} \quad .
 \end{aligned}$$

The code notation makes proofs shorter than they would be if we were manipulating code in Scala syntax. From now on, we will prefer to use the code notation in the proofs, keeping in mind that Scala code can be unambiguously recovered from the code notation, and vice versa.

Example 5.3.4.6 Verify the type equivalence

$$A \times B \Rightarrow C \cong A \Rightarrow B \Rightarrow C \quad .$$

Solution Begin by implementing the two functions

```
def f1[A,B,C]: (((A, B)) => C) => A => B => C = ???
def f2[A,B,C]: (A => B => C) => ((A, B)) => C = ???
```

The Scala code can be derived from the type signatures without ambiguity:

```
def f1[A,B,C]: (((A, B)) => C) => A => B => C = g => a => b => g((a, b))
def f2[A,B,C]: (A => B => C) => ((A, B)) => C = h => { case (a, b) => h(a)(b) }
```

Write these functions in the code notation:

$$\begin{aligned}
 f_1 &= g^{A \times B \Rightarrow C} \Rightarrow a^A \Rightarrow b^B \Rightarrow g(a \times b) \quad , \\
 f_2 &= h^{A \Rightarrow B \Rightarrow C} \Rightarrow (a \times b)^{A \times B} \Rightarrow h(a)(b) \quad .
 \end{aligned}$$

We denote by $(a \times b)^{A \times B}$ the argument of type (A, B) with pattern matching implied. This notation allows us to write shorter code notation involving tupled arguments.

Compute the function composition $f_1 \circ f_2$:

$$\begin{aligned}
 f_1 \circ f_2 &= (g \Rightarrow a \Rightarrow b \Rightarrow g(a \times b)) \circ (h \Rightarrow a \times b \Rightarrow h(a)(b)) \\
 \text{substitute } h &= a \Rightarrow b \Rightarrow g(a \times b) : &= g \Rightarrow a \times b \Rightarrow g(a \times b) \\
 \text{simplify function} : &= (g \Rightarrow g) = \text{id} \quad .
 \end{aligned}$$

Compute the function composition $f_2 \circ f_1$:

$$\begin{aligned}
 f_2 \circ f_1 &= (h \Rightarrow a \times b \Rightarrow h(a)(b)) \circ (g \Rightarrow a \Rightarrow b \Rightarrow g(a \times b)) \\
 \text{substitute } g &= a \times b \Rightarrow h(a)(b) : &= h \Rightarrow a \Rightarrow b \Rightarrow h(a)(b) \\
 \text{simplify function } b &\Rightarrow h(a)(b) : &= h \Rightarrow a \Rightarrow h(a) \\
 \text{simplify function } a &\Rightarrow h(a) \text{ to } h : &= (h \Rightarrow h) = \text{id} \quad .
 \end{aligned}$$

Exercise 5.3.4.7 Verify the type equivalence

$$(A \Rightarrow B \times C) \cong (A \Rightarrow B) \times (A \Rightarrow C) \quad .$$

5.4 Summary

What tasks can we perform now?

- Use the matrix notation and the pipe notation to write code that works on disjunctive types.
- Use the type notation (Table 5.1) for reasoning about types to:
 - Decide type equivalence using the rules in Tables 5.5–5.6.
 - Simplify type expressions before writing code.
- Convert a fully parametric type signature into a logical formula to:
 - Decide whether the type signature can be implemented in code.
 - If possible, derive the code using the CH correspondence.

What tasks cannot be performed with these tools?

- Automatically generate code for a recursive function. (The CH correspondence is based on propositional logic, which cannot describe recursion.)
- Automatically generate code satisfying a property (e.g. isomorphism). We may generate the code, but it is not guaranteed that properties will hold. The workaround is to verify the required properties manually, after deriving the code.
- Express complicated conditions (e.g. “array is sorted”) in a type signature. This can be done using **dependent types** (i.e. types that depend on run-time values in an arbitrary way) – an advanced technique that Scala does not fully support. Programming languages such as Coq, Agda, and Idris support full dependent types, but cannot automatically generate code from dependent type signatures.
- Generate code using type constructors with known properties (e.g. `.map`).

As an example of using type constructors with properties, consider this type signature:

```
def q[A]: Array[A] => (A => Option[B]) => Array[Option[B]]
```

Can we generate the code of this function from its type signature? We know that the Scala library defines a `.map` method on the `Array` type constructor, so the implementation of `q` is simple,

```
def q[A]: Array[A] => (A => Option[B]) => Array[Option[B]] = { arr => f => arr.map(f) }
```

However, it is hard to create an *algorithm* that can derive this implementation automatically from the type signature of `q` via the Curry-Howard correspondence. The algorithm would have to convert the type signature of `q` into the logical formula

$$CH(\text{Array}^A) \Rightarrow CH(A \Rightarrow \text{Opt}^B) \Rightarrow CH(\text{Array}^{\text{Opt}^B}) \quad . \quad (5.21)$$

To derive an implementation, the algorithm would need to use the available `.map` method for `Array`. That method has a type signature such as

$$\text{map} : \forall(A, B). \text{Array}^A \Rightarrow (A \Rightarrow B) \Rightarrow \text{Array}^B \quad .$$

To derive the CH -proposition (5.21), the algorithm will need to assume that the CH -proposition

$$CH(\forall(A, B). \text{Array}^A \Rightarrow (A \Rightarrow B) \Rightarrow \text{Array}^B) \quad (5.22)$$

already holds, i.e. that Eq. (5.22) is one of the premises of a sequent to be proved. Reasoning about propositions such as Eq. (5.22) requires **first-order logic** – a logic whose proof rules can handle quantified types such as $\forall(A, B)$ *inside* premises. However, first-order logic is **undecidable**: no algorithm can guarantee finding a proof or showing the absence of a proof in all cases.

The constructive propositional logic (with the rules listed in Section 5.2.3) is **decidable** and has an algorithm that either finds a proof or disproves any given formula. However, that logic cannot handle premises containing type quantifiers such as $\forall(A, B)$ *inside*, because all the available rules have the quantifiers implicitly placed *outside* the premises.

So, code for functions such as `q` can only be derived by trial and error, informed by intuition. This book will help functional programmers to acquire the necessary intuition and technique.

5.4.1 Solved examples

Example 5.4.1.1 Find the cardinality of the type `P = Option[Option[Boolean] => Boolean]`. Write `P` in the type notation and simplify to an equivalent type.

Solution Begin with the type `Option[Boolean]`, which can be either `None` or `Some(x)` with an `x:Boolean`. Since the type `Boolean` has 2 possible values, the type `Option[Boolean]` has 3 values:

$$|\text{Opt}^{\text{Bool}}| = |1 + \text{Bool}| = 1 + |\text{Bool}| = 3 \quad .$$

In the type notation, `Boolean` is denoted by the symbol `2`, and the type `Option[Boolean]` by `1 + 2`. So, the type notation `1 + 2` is consistent with the cardinality 3 of that type,

$$|1 + \text{Bool}| = |1 + 2| = 1 + 2 = 3 \quad .$$

The function type `Option[Boolean] => Boolean` is denoted by `1 + 2 => 2`. Its cardinality is computed as the arithmetic power

$$|\text{Opt}^{\text{Bool}} \Rightarrow \text{Bool}| = |1 + 2 \Rightarrow 2| = |2|^{|1+2|} = 2^3 = 8 \quad .$$

Finally, the we write `P` in the type notation as

$$P = 1 + (1 + 2 \Rightarrow 2)$$

and find

$$|P| = |1 + (1 + 2 \Rightarrow 2)| = 1 + |1 + 2 \Rightarrow 2| = 1 + 8 = 9 \quad .$$

Example 5.4.1.2 Implement a Scala type `P[A]` for the type notation

$$P^A \triangleq 1 + A + \text{Int} \times A + (\text{String} \Rightarrow A) \quad .$$

Solution To translate type notation into Scala code, begin by defining the disjunctive types as case classes (with names chosen for convenience). In this case, P^A is a disjunctive type with four parts, so we will need four case classes:

```
sealed trait P[A]
final case class P1[A](???) extends P[A]
final case class P2[A](???) extends P[A]
final case class P3[A](???) extends P[A]
final case class P4[A](???) extends P[A]
```

Each of the case classes represents one part of the disjunctive type. Now we write the contents for each of the case classes, in order to implement the data in each of the disjunctive parts:

```
sealed trait P[A]
final case class P1[A]() extends P[A]
final case class P2[A](x: A) extends P[A]
final case class P3[A](n: Int, x: A) extends P[A]
final case class P4[A](f: String => A) extends P[A]
```

Example 5.4.1.3 Find an equivalent disjunctive type for the type $P = (\text{Either}[A, B], \text{Either}[C, D])$.

Solution Begin by writing the given type in the type notation. The tuple becomes the product type, and `Either` becomes the disjunctive or “sum” type:

$$P \triangleq (A + B) \times (C + D) \quad .$$

We can use the usual rules of arithmetic to expand brackets in this type expression and to obtain an equivalent type:

$$P \cong A \times C + A \times D + B \times C + B \times D \quad .$$

This type is disjunctive, with 4 parts.

Example 5.4.1.4 Show that the following type equivalences do *not* hold: $A + A \not\cong A$ and $A \times A \not\cong A$, although the corresponding logical identities hold.

Solution Note that the arithmetic equalities do not hold, $A + A \neq A$ and $A \times A \neq A$. This already indicates that the types are not equivalent. To build further intuition, consider that a value of type $A + A$ (in Scala, `Either[A, A]`) is a `Left(a)` or a `Right(a)` for some $a:A$. In the code notation, it is either $a:A + 0$ or $0 + a:A$. So, a value of type $A + A$ contains a value of type A with an additional information about whether it is the first or the second part of the disjunctive type. We cannot represent all that information in a single value of type A .

A value of type $A \times A$ contains two (possibly different) values of type A , which cannot be represented by a single value of type A without loss of information.

However, the corresponding logical identities $\alpha \vee \alpha = \alpha$ and $\alpha \wedge \alpha = \alpha$ hold. To see that, we could derive the four formulas

$$\begin{aligned} \alpha \vee \alpha &\Rightarrow \alpha & , & & \alpha &\Rightarrow \alpha \vee \alpha & , \\ \alpha \wedge \alpha &\Rightarrow \alpha & , & & \alpha &\Rightarrow \alpha \wedge \alpha & , \end{aligned}$$

using the proof rules of Section 5.2.3. Alternatively, we may use the CH correspondence and show that the type signatures

$$\begin{aligned} \forall A. A + A &\Rightarrow A & , & & \forall A. A &\Rightarrow A + A & , \\ \forall A. A \times A &\Rightarrow A & , & & \forall A. A &\Rightarrow A \times A \end{aligned}$$

can be implemented via fully parametric functions. For a programmer, it is easier to write code than to guess the correct sequence of proof rules. For the first pair of type signatures, we find

```
def f1[A]: Either[A, A] => A = {
  case Left(a)   => a    // No other choice here.
  case Right(a)  => a    // No other choice here.
}
def f2[A]: A => Either[A, A] = { a => Left(a) } // Can be also Right(a).
```

The presence of an arbitrary choice, to return `Left(a)` or `Right(a)`, is a warning sign showing that additional information is required to create a value of type `Either[A, A]`. This is precisely the information present in the type $A + A$ but missing in the type A .

The code notation for these functions is

$$f_1 \triangleq \begin{array}{|c|} \hline A \\ \hline \begin{array}{|c|} \hline a \Rightarrow a \\ \hline a \Rightarrow a \end{array} \\ \hline \end{array} = \begin{array}{|c|} \hline A \\ \hline \begin{array}{|c|} \hline \text{id} \\ \hline \text{id} \end{array} \\ \hline \end{array} ,$$

$$f_2 \triangleq a : A \Rightarrow a + 0 : A = \begin{array}{|c|} \hline A \quad A \\ \hline \begin{array}{|c|} \hline a \Rightarrow a \quad 0 \\ \hline \end{array} \\ \hline \end{array} = \begin{array}{|c|} \hline A \quad A \\ \hline \begin{array}{|c|} \hline \text{id} \quad 0 \\ \hline \end{array} \\ \hline \end{array} .$$

The composition of these functions is not equal to identity:

$$f_1 \circ f_2 = \begin{array}{|c|} \hline \text{id} \\ \hline \text{id} \\ \hline \end{array} \circ \begin{array}{|c|} \hline \text{id} \quad 0 \\ \hline \end{array} = \begin{array}{|c|} \hline \text{id} \quad 0 \\ \hline \text{id} \quad 0 \\ \hline \end{array} \neq \text{id} = \begin{array}{|c|} \hline \text{id} \quad 0 \\ \hline 0 \quad \text{id} \\ \hline \end{array} .$$

For the second pair of type signatures, the code is

```
def f1[A]: ((A, A)) => A = { case (a1, a2) => a1 } // Can be also 'a2'.
def f2[A]: A => (A, A) = { x => (x, x) } // No other choice here.
```

It is clear that the first function loses information when it returns `a1` and discards `a2` (or vice versa).

The code notation for these functions is

$$f_1 \triangleq a_1^A \times a_2^A \Rightarrow a_1 ,$$

$$f_2 \triangleq a : A \Rightarrow a \times a .$$

The composition of these functions is not equal to identity:

$$f_1 \circ f_2 = (a_1 \times a_2 \Rightarrow a_1) \circ (a \Rightarrow a \times a)$$

$$= (a_1 \times a_2 \Rightarrow a_1 \times a_1) \neq \text{id} = (a_1 \times a_2 \Rightarrow a_1 \times a_2) .$$

We have implemented all four type signatures as fully parametric functions, which shows that the corresponding logical formulas are all true (i.e. can be derived using the proof rules). However, the functions cannot be inverses of each other. So, the type equivalences do not hold.

Example 5.4.1.5 Show that $((A \wedge B) \Rightarrow C) \neq (A \Rightarrow C) \vee (B \Rightarrow C)$ in the constructive logic, although the equality holds in Boolean logic. This is another example of the failure of Boolean logic to provide correct reasoning for types.

Solution Begin by rewriting the logical equality as two implications,

$$(A \wedge B \Rightarrow C) \Rightarrow (A \Rightarrow C) \vee (B \Rightarrow C) ,$$

$$((A \Rightarrow C) \vee (B \Rightarrow C)) \Rightarrow ((A \wedge B) \Rightarrow C) .$$

It is sufficient to show that one of these implications is incorrect. Rather than looking for a proof tree in the constructive logic (which would be difficult, since we would need to demonstrate that *no* proof tree exists), let us use the CH correspondence. So the task is to implement fully parametric functions with the type signatures

$$(A \times B \Rightarrow C) \Rightarrow (A \Rightarrow C) + (B \Rightarrow C) ,$$

$$(A \Rightarrow C) + (B \Rightarrow C) \Rightarrow A \times B \Rightarrow C .$$

For the first type signature, the Scala code is

```
def f1[A,B,C]: ((A, B)) => C => Either[A => C, B => C] = { k => ??? }
```

We are required to return either a `Left(g)` with $g: A \Rightarrow C$, or a `Right(h)` with $h: B \Rightarrow C$. The only given data is a function k of type $A \times B \Rightarrow C$, so the decision of whether to return a `Left` or a `Right` must be hard-coded in the function `f1` independently of k . Can we produce a function g of type $A \Rightarrow C$? Given a value of type A , we would need to return a value of type C . The only way to obtain a value of type C is by applying k to some arguments. But to apply k , we need a value of type B , which we do not have. So we cannot produce a $g: A \Rightarrow C$. Similarly, we cannot produce a function h of type $B \Rightarrow C$.

To repeat the same argument in the type notation: Obtaining a value of type $(A \Rightarrow C) + (B \Rightarrow C)$ means to compute either $g^{A \Rightarrow C} + 0$ or $0 + h^{B \Rightarrow C}$. This decision must be hard-coded since the only data is a function $k^{A \times B \Rightarrow C}$. We can compute a $g^{A \Rightarrow C}$ only by partially applying $k^{A \times B \Rightarrow C}$ to a value of type B . However, we cannot obtain any values of type B . Similarly, we cannot get an $h^{B \Rightarrow C}$.

The inverse type signature is implementable:

```
def f2[A,B,C]: Either[A=>C, B=>C] => ((A,B)) => C = {
  case Left(g)    => { case (a, b) => g(a) }
  case Right(h)   => { case (a, b) => h(b) }
}
```

$$f2 \triangleq \left| \begin{array}{c|c} & A \times B \Rightarrow C \\ \hline A \Rightarrow C & g^{A \Rightarrow C} \Rightarrow a \times b \Rightarrow g(a) \\ B \Rightarrow C & h^{B \Rightarrow C} \Rightarrow a \times b \Rightarrow h(b) \end{array} \right| .$$

Let us now show that the identity

$$((\alpha \wedge \beta) \Rightarrow \gamma) = ((\alpha \Rightarrow \gamma) \vee (\beta \Rightarrow \gamma)) \quad (5.23)$$

holds in Boolean logic. A straightforward calculation is to simplify the Boolean expression using Eq. (5.5), which only holds in Boolean logic (but not in the constructive logic). We find

$$\begin{aligned} \text{left-hand side of Eq. (5.23): } & (\alpha \wedge \beta) \Rightarrow \gamma \\ & \text{use Eq. (5.5): } = \neg(\alpha \wedge \beta) \vee \gamma \\ & \text{use de Morgan's law: } = \neg\alpha \vee \neg\beta \vee \gamma \\ \text{right-hand side of Eq. (5.23): } & (\alpha \Rightarrow \gamma) \vee (\beta \Rightarrow \gamma) \\ & \text{use Eq. (5.5): } = \neg\alpha \vee \gamma \vee \neg\beta \vee \gamma \\ & \text{use identity } \gamma \vee \gamma = \gamma: = \neg\alpha \vee \neg\beta \vee \gamma \end{aligned}$$

Both sides of Eq. (5.23) are equal to the same formula, $\neg\alpha \vee \neg\beta \vee \gamma$, so the identity holds.

This proof does not work in the constructive logic because neither the Boolean formula (5.5) nor the law of de Morgan,

$$\neg(\alpha \wedge \beta) = (\neg\alpha \vee \neg\beta) \quad ,$$

can be derived via the proof rules of the constructive logic.

Another way of proving the Boolean identity (5.23) is to enumerate all possible truth values for the variables α , β , and γ . The left-hand side, $(\alpha \wedge \beta) \Rightarrow \gamma$, can be *False* only if $\alpha \wedge \beta = \text{True}$ (that is, both α and β are *True*) and $\gamma = \text{False}$; for all other truth values of α , β , and γ , the formula $(\alpha \wedge \beta) \Rightarrow \gamma$ is *True*. Let us determine when the right-hand side, $(\alpha \Rightarrow \gamma) \vee (\beta \Rightarrow \gamma)$, can be *False*. This can happen only if both parts of the disjunction are *False*; that means $\alpha = \text{True}$, $\beta = \text{True}$, and $\gamma = \text{False}$. So, the two sides of the identity (5.23) are both *True* or both *False* with any choice of truth values of α , β , and γ . In Boolean logic, this is sufficient to prove the identity (5.23).

It is important to note that the proof rules of the constructive logic are not equivalent to checking whether some propositions are *True* or *False*. A general form of this statement was proved by K. Gödel in 1932.⁶ In this sense, constructive logic does not imply that every proposition is either *True* or *False*. This is not intuitive and requires getting used to.

The following example shows how to use the identities from Tables 5.5–5.6 to derive type equivalence for complicated type expressions, without need for proofs.

⁶See plato.stanford.edu/entries/intuitionistic-logic-development/

Example 5.4.1.6 Use known rules to verify the type equivalences:

(a) $A \times (A + 1) \times (A + 1 + 1) \cong A \times (1 + 1 + A \times (1 + 1 + 1 + A))$.

(b) $(1 + A + B) \Rightarrow 1 \times B \cong (B \Rightarrow B) \times (A \Rightarrow B) \times B$.

Solution (a) We can expand brackets in the type expression as in arithmetic,

$$\begin{aligned} A \times (A + 1) &\cong A \times A + A \times 1 \cong A \times A + A \\ A \times (A + 1) \times (A + 1 + 1) &\cong (A \times A + A) \times (A + 1 + 1) \\ &\cong A \times A \times A + A \times A + A \times A \times (1 + 1) + A \times (1 + 1) \\ &\cong A \times A \times A + A \times A \times (1 + 1 + 1) + A \times (1 + 1) \end{aligned}$$

The result looks like a polynomial in A , which we can now rearrange into the required form:

$$A \times A \times A + A \times A \times (1 + 1 + 1) + A \times (1 + 1) \cong A \times (1 + 1 + A \times (1 + 1 + 1 + A))$$

(b) Keep in mind that the conventions of the type notation make the function arrow (\Rightarrow) group weaker than other type operations. So, the type expression $(1 + A + B) \Rightarrow 1 \times B$ means a function from $1 + A + B$ to $1 \times B$.

Begin by using the rule $1 \times B \cong B$ to obtain $(1 + A + B) \Rightarrow B$. Now we use the rule

$$A + B \Rightarrow C \cong (A \Rightarrow C) \times (B \Rightarrow C)$$

and derive the equivalence

$$(1 + A + B) \Rightarrow B \cong (1 \Rightarrow B) \times (A \Rightarrow B) \times (B \Rightarrow B)$$

Finally, we note that $1 \Rightarrow B \cong B$ and that the type product is commutative, so we can rearrange the last type expression into the required form:

$$B \times (A \Rightarrow B) \times (B \Rightarrow B) \cong (B \Rightarrow B) \times (A \Rightarrow B) \times B$$

Example 5.4.1.7 Denote $\text{Read}^{E,T} \triangleq E \Rightarrow T$ and implement fully parametric functions with types $A \Rightarrow \text{Read}^{E,A}$ and $\text{Read}^{E,A} \Rightarrow (A \Rightarrow B) \Rightarrow \text{Read}^{E,B}$.

Solution Begin by defining a type alias for the type $\text{Read}^{E,T}$:

```
type Read[E, T] = E => T
```

The first type signature has only one implementation:

```
def p[E, A]: A => Read[E, A] = { x => _ => x }
```

We *must* discard the argument of type E ; we cannot use it for computing a value of type A given $x:A$.

The second type signature has three type parameters. It is the curried version of the function `map`:

```
def map[E, A, B]: Read[E, A] => (A => B) => Read[E, B] = ???
```

Expanding the type alias, we see that the two curried arguments are functions of types $E \Rightarrow A$ and $A \Rightarrow B$. The forward composition of these functions is a function of type $E \Rightarrow B$, or $\text{Read}^{E,B}$, which is exactly what we are required to return. So the code can be written as

```
def map[E, A, B]: (E => A) => (A => B) => E => B = { r => f => r andThen f }
```

If we did not notice this shortcut, we would reason differently: We are required to compute a value of type B given *three* curried arguments $r:E \Rightarrow A$, $f:A \Rightarrow B$, and $e:E$. Write this requirement as

$$\text{map} \triangleq r:E \Rightarrow A \Rightarrow f:A \Rightarrow B \Rightarrow e:E \Rightarrow ???^B,$$

The symbol $???^B$ is called a **typed hole**; it stands for a value that we are still figuring out how to compute, but whose type is already known. Typed holes are supported in Scala by an experimental compiler plugin.⁷ The plugin will print the known information about the typed hole.

⁷<https://github.com/cb372/scala-typed-holes>

To fill the typed hole $???^B$, we need a value of type B . Since no arguments have type B , the only way of getting a value of type B is to apply $f^{A \Rightarrow B}$ to some value of type A . So we write

$$\text{map} \triangleq r^{E \Rightarrow A} \Rightarrow f^{A \Rightarrow B} \Rightarrow e^E \Rightarrow f(???^A) \quad .$$

The only way of getting an A is to apply r to a value of type E ,

$$\text{map} \triangleq r^{E \Rightarrow A} \Rightarrow f^{A \Rightarrow B} \Rightarrow e^E \Rightarrow f(r(???^E)) \quad .$$

We have exactly one value of type E , namely e^E . So the code must be

$$\text{map}^{E, A, B} \triangleq r^{E \Rightarrow A} \Rightarrow f^{A \Rightarrow B} \Rightarrow e^E \Rightarrow f(r(e)) \quad .$$

Translate this to the Scala syntax:

```
def map[E, A, B]: (E => A) => (A => B) => E => B = { r => f => e => f(r(e)) }
```

We may now notice that the expression $e \Rightarrow f(r(e))$ is a function composition $r \circ f$ applied to e , and simplify the code accordingly.

Example 5.4.1.8 Show that the type signature `Read[A, T] => (A => B) => Read[B, T]` cannot be implemented as a fully parametric function.

Solution Expand the type signature and try to implement this function:

```
def m[A, B, T] : (A => T) => (A => B) => B => T = { r => f => b => ??? }
```

Given values $r^{A \Rightarrow T}$, $f^{A \Rightarrow B}$, and b^B , we need to compute a value of type T :

$$m = r^{A \Rightarrow T} \Rightarrow f^{A \Rightarrow B} \Rightarrow b^B \Rightarrow ???^T \quad .$$

The only way of getting a T is to apply r to some value of type A ,

$$m = r^{A \Rightarrow T} \Rightarrow f^{A \Rightarrow B} \Rightarrow b^B \Rightarrow r(???^A) \quad .$$

However, we do not have any values of type A . We have a function $f^{A \Rightarrow B}$ that *consumes* values of type A , and we cannot use f to produce any values of type A . So we seem to be unable to fill the typed hole $???^A$ and implement the function `m`.

In order to verify that `m` is unimplementable, we need to prove that the logical formula

$$\forall(\alpha, \beta, \tau). (\alpha \Rightarrow \tau) \Rightarrow (\alpha \Rightarrow \beta) \Rightarrow (\beta \Rightarrow \tau) \quad (5.24)$$

is not true in the constructive logic. We could use the `curryhoward` library for that:

```
@ def m[A, B, T] : (A => T) => (A => B) => B => T = implement
cmd1.sc:1: type (A => T) => (A => B) => B => T cannot be implemented
def m[A, B, T] : (A => T) => (A => B) => B => T = implement
      ^
```

Compilation Failed

Another way is to check whether this formula is true in Boolean logic. A formula that holds in constructive logic will always hold in Boolean logic, because all rules shown in Section 5.2.3 preserve Boolean truth values (see Section 5.5.4 for more details). It follows that any formula that fails to hold in Boolean logic will also not hold in constructive logic.

It is relatively easy to check whether a given Boolean formula is always equal to `True`. Simplifying Eq. (5.24) with the rules of Boolean logic, we find

$$\begin{aligned}
& (\alpha \Rightarrow \tau) \Rightarrow (\alpha \Rightarrow \beta) \Rightarrow (\beta \Rightarrow \tau) \\
\text{use Eq. (5.5):} & \quad = \neg(\underline{\alpha \Rightarrow \tau}) \vee \neg(\underline{\alpha \Rightarrow \beta}) \vee (\underline{\beta \Rightarrow \tau}) \\
\text{use Eq. (5.5):} & \quad = \underline{\neg(\neg\alpha \vee \tau)} \vee \underline{\neg(\neg\alpha \vee \beta)} \vee (\neg\beta \vee \tau) \\
\text{use de Morgan's law:} & \quad = (\alpha \wedge \neg\tau) \vee (\underline{\alpha \wedge \neg\beta}) \vee \neg\beta \vee \tau \\
\text{use identity } (p \wedge q) \vee q = q: & \quad = (\alpha \wedge \neg\tau) \vee \neg\beta \vee \tau \\
\text{use identity } (p \wedge \neg q) \vee q = p \vee q: & \quad = \alpha \vee \neg\beta \vee \tau \quad .
\end{aligned}$$

This formula is not identically *True*: it is *False* when $\alpha = \tau = \text{False}$ and $\beta = \text{True}$. So, Eq. (5.24) is not true in Boolean logic, and thus is not true in constructive logic. By the CH correspondence, we conclude that the type signature of map cannot be implemented by a fully parametric function.

Example 5.4.1.9 Define the type constructor $P^A \triangleq \mathbb{1} + A + A$ and implement map for it,

$$\text{map}^{A,B} : P^A \Rightarrow (A \Rightarrow B) \Rightarrow P^B \quad .$$

To check that map preserves information, verify the law $\text{map}(p)(x \Rightarrow x) == p$ for all $p : P[A]$.

Solution It is implied that map should be fully parametric and information-preserving. Begin by defining a Scala type for the notation $\mathbb{1} + A + A$:

```
sealed trait P[A]
final case class P1[A]() extends P[A]
final case class P2[A](x: A) extends P[A]
final case class P3[A](x: A) extends P[A]
```

Now implement the required type signature. Each time we find a choice in the implementation, we will choose to preserve information as much as possible.

```
def map[A, B]: P[A] => (A => B) => P[B] = p => f => p match {
  case P1() => P1()           // No other choice here.
  case P2(x) => ???
  case P3(x) => ???
}
```

In the case $P2(x)$, we are required to produce a value of type P^B from a value $x:A$ and a function $f:A \Rightarrow B$. Since P^B is a disjunctive type with three parts, we can produce a value of type P^B in three different ways: $P1()$, $P2(\dots)$, and $P3(\dots)$. If we return $P1()$, we will lose the information about the value x . If we return $P3(\dots)$, we will preserve the information about x but lose the information that the input value was a $P2$ rather than a $P3$. So we should return $P2(\dots)$ in that scope; in that way, we preserve the entire input information.

The value under $P2(\dots)$ must be of type B , and the only way of getting a value of type B is to apply f to x . So, we return $P2(f(x))$.

Similarly, in the case $P3(x)$, we should return $P3(f(x))$. The final code of map is

```
def map[A, B]: P[A] => (A => B) => P[B] = p => f => p match {
  case P1() => P1()           // No other choice here.
  case P2(x) => P2(f(x))      // Preserve information.
  case P3(x) => P3(f(x))      // Preserve information.
}
```

To verify the given law, we first write a matrix notation for map :

$$\text{map}^{A,B} \triangleq p : \mathbb{1} + A + A \Rightarrow f : A \Rightarrow B \Rightarrow p \triangleright \begin{array}{c|cc|cc} & & \mathbb{1} & B & B \\ \hline \mathbb{1} & \text{id} & 0 & 0 \\ A & 0 & f & 0 \\ A & 0 & 0 & f \end{array} .$$

The required law is written as an equation

$$\text{map}(p)(\text{id}) = p \quad .$$

Substituting the code notation for `map`, we verify the law:

$$\begin{aligned}
 &\text{expect to equal } p : \quad \text{map}(p)(\text{id}) \\
 &\text{apply map>() to arguments :} \quad = p \triangleright \left\| \begin{array}{ccc} \text{id} & 0 & 0 \\ 0 & \text{id} & 0 \\ 0 & 0 & \text{id} \end{array} \right\| \\
 &\text{identity in matrix notation :} \quad = p \triangleright \text{id} \\
 &\text{▷-notation :} \quad = \text{id}(p) = p \quad .
 \end{aligned}$$

Example 5.4.1.10 Implement `map` and `flatMap` for `Either [L, R]`, applied to the type parameter `L`.

Solution For a type constructor, say, P^A , the standard type signatures for `map` and `flatMap` are

$$\begin{aligned}
 \text{map} &: P^A \Rightarrow (A \Rightarrow B) \Rightarrow P^B \quad , \\
 \text{flatMap} &: P^A \Rightarrow (A \Rightarrow P^B) \Rightarrow P^B \quad .
 \end{aligned}$$

If a type constructor has more than one type parameter, e.g. $P^{A,S,T}$, one can define the functions `map` and `flatMap` applied to a chosen parameter. For example, when applied to the type parameter `A`, the type signatures are

$$\begin{aligned}
 \text{map} &: P^{A,S,T} \Rightarrow (A \Rightarrow B) \Rightarrow P^{B,S,T} \quad , \\
 \text{flatMap} &: P^{A,S,T} \Rightarrow (A \Rightarrow P^{B,S,T}) \Rightarrow P^{B,S,T} \quad .
 \end{aligned}$$

Being “applied to the type parameter `A`” means that the other type parameters S, T in $P^{A,S,T}$ remain fixed while the type parameter `A` is replaced by `B` in the type signatures of `map` and `flatMap`.

For the type `Either [L, R]` (i.e. $L + R$), we keep the type parameter `R` fixed while `L` is replaced by `M`. So we obtain the type signatures

$$\begin{aligned}
 \text{map} &: L + R \Rightarrow (L \Rightarrow M) \Rightarrow M + R \quad , \\
 \text{flatMap} &: L + R \Rightarrow (L \Rightarrow M + R) \Rightarrow M + R \quad .
 \end{aligned}$$

Implementing these functions is straightforward:

```

def map[L,M,R]: Either[L, R] => (L => M) => Either[M, R] = e => f => e match {
  case Left(x)    => Left(f(x))
  case Right(y)   => Right(y)
}
def flatMap[L,M,R]: Either[L, R] => (L => Either[M, R]) => Either[M, R] = e => f => e match {
  case Left(x)    => f(x)
  case Right(y)   => Right(y)
}

```

The code notation for these functions is

$$\begin{aligned}
 \text{map} &\triangleq e^{L+R} \Rightarrow f^{L \Rightarrow M} \Rightarrow e \triangleright \left\| \begin{array}{cc} & \begin{array}{cc} M & R \end{array} \\ \hline L & \begin{array}{cc} f & 0 \end{array} \\ R & \begin{array}{cc} 0 & \text{id} \end{array} \end{array} \right\| , \\
 \text{flatMap} &\triangleq e^{L+R} \Rightarrow f^{L \Rightarrow M+R} \Rightarrow e \triangleright \left\| \begin{array}{cc} & M + R \\ \hline L & f \\ R & y^{R \Rightarrow 0^M + y} \end{array} \right\| .
 \end{aligned}$$

Note that we cannot split f into the M and R columns since $f(x^{L \Rightarrow})$ could return either part of the disjunction $M + R$.

Example 5.4.1.11* Define a type $\text{State}^{S,A} \equiv S \Rightarrow A \times S$ and implement the functions:

- (a) $\text{pure}^{S,A} : A \Rightarrow \text{State}^{S,A}$.
- (b) $\text{map}^{S,A,B} : \text{State}^{S,A} \Rightarrow (A \Rightarrow B) \Rightarrow \text{State}^{S,B}$.
- (c) $\text{flatMap}^{S,A,B} : \text{State}^{S,A} \Rightarrow (A \Rightarrow \text{State}^{S,B}) \Rightarrow \text{State}^{S,B}$.

Solution It is assumed that all functions must be fully parametric and preserve as much information as possible. We define the type alias

```
type State[S, A] = S => (A, S)
```

- (a) The type signature is $A \Rightarrow S \Rightarrow A \times S$, and there is only one implementation,

```
def pure[S, A]: A => State[S, A] = a => s => (a, s)
```

In the code notation, this is written as

$$\text{pu}^{S,A} \triangleq a:A \Rightarrow s:S \Rightarrow a \times s \quad .$$

- (b) The explicit type signature is

$$\text{map}^{S,A,B} : (S \Rightarrow A \times S) \Rightarrow (A \Rightarrow B) \Rightarrow S \Rightarrow B \times S \quad .$$

Begin writing a Scala implementation:

```
def map[S, A, B]: State[S, A] => (A => B) => State[S, B] = { t => f => s => ??? }
```

We need to compute a value of $B \times S$ from the curried arguments $t:S \Rightarrow A \times S$, $f:A \Rightarrow B$, and $s:S$. We begin writing the code of `map` as a typed hole,

$$\text{map} \triangleq t:S \Rightarrow A \times S \Rightarrow f:A \Rightarrow B \Rightarrow s:S \Rightarrow ???^B \times ???^S \quad .$$

The only way of getting a value of type B is by applying f to a value of type A :

$$\text{map} \triangleq t:S \Rightarrow A \times S \Rightarrow f:A \Rightarrow B \Rightarrow s:S \Rightarrow f(???^A) \times ???^S \quad .$$

To fill the typed hole, we need a value of type A . The only possibility of obtaining a value of type A is by applying t to a value of type S ; we already have such a value, $s:S$. Computing $t(s)$ yields a pair of type $A \times S$, from which we may take the first part (of type A) to fill the typed hole $???^A$. The second part of the pair is a value of type S that we may use to fill the second typed hole, $???^S$. So the Scala code is

```
1 def map[S, A, B]: State[S, A] => (A => B) => State[S, B] = {
2   t => f => s =>
3     val (a, s2) = t(s)
4     (f(a), s2)    // We could also return '(f(a), s)' here.
5 }
```

($f(a)$, s) in line 4, we will have discarded the computed value $s2$, which is a loss of information.

To write the code notation for `map`, we need to destructure the pair that $t(s)$ returns. We can write explicit destructuring code like this:

$$\text{map} \triangleq t:S \Rightarrow A \times S \Rightarrow f:A \Rightarrow B \Rightarrow s:S \Rightarrow (a:A \times s_2:S \Rightarrow f(a) \times s_2)(t(s)) \quad .$$

If we temporarily denote by q the destructuring function

$$q \triangleq (a:A \times s_2:S \Rightarrow f(a) \times s_2) \quad ,$$

we will notice that the expression $s \Rightarrow q(t(s))$ is a function composition applied to s . So, we rewrite $s \Rightarrow q(t(s))$ as the composition $t \circ q$ and obtain shorter code,

$$\text{map} \triangleq t:S \Rightarrow A \times S \Rightarrow f:A \Rightarrow B \Rightarrow t \circ q (a:A \times s:S \Rightarrow f(a) \times s) \quad .$$

Why not return the original value s in the tuple $B \times S$, instead of the new value $s2$? The reason is that we would like to preserve information as much as possible. If we return

Shorter formulas are often easier to reason about in derivations (although not necessarily easier to read when converted to program code).

(c) The required type signature is

$$\text{flatMap}^{S,A,B} : (S \Rightarrow A \times S) \Rightarrow (A \Rightarrow S \Rightarrow B \times S) \Rightarrow S \Rightarrow B \times S \quad .$$

We perform t reasoning with typed holes:

$$\text{flatMap} \triangleq t^{S \Rightarrow A \times S} \Rightarrow f^{A \Rightarrow S \Rightarrow B \times S} \Rightarrow s^S \Rightarrow ???^{B \times S} \quad .$$

To fill $???^B$, we need to apply f to some arguments, since f is the only function that returns any values of type B . A saturated application of f will yield a value of type $B \times S$, which we can return without change:

$$\text{flatMap} \triangleq t^{S \Rightarrow A \times S} \Rightarrow f^{A \Rightarrow S \Rightarrow B \times S} \Rightarrow s^S \Rightarrow f(???^A)(???^S) \quad .$$

To fill the new typed holes, we need to apply t to an argument of type S . We have only one given value s^S of type S , so we must compute $t(s)$ and destructure it:

$$\text{flatMap} \triangleq t^{S \Rightarrow A \times S} \Rightarrow f^{A \Rightarrow S \Rightarrow B \times S} \Rightarrow s^S \Rightarrow (a \times s_2 \Rightarrow f(a)(s_2)) (t(s)) \quad .$$

Translating this notation into Scala code, we obtain

```
def flatMap[S,A,B]: State[S, A] => (A => State[S, B]) => State[S, B] = {
  t => f => s =>
    val (a, s2) = t(s)
    f(a)(s2)           // We could also return 'f(a)(s)' here.
}
```

As before, in order to preserve information, we choose not to discard the computed value s_2 .

The code notation for `flatMap` can be simplified to

$$\text{flatMap} \triangleq t^{S \Rightarrow A \times S} \Rightarrow f^{A \Rightarrow S \Rightarrow B \times S} \Rightarrow t \circ (a \times s \Rightarrow f(a)(s)) \quad .$$

5.4.2 Exercises

Exercise 5.4.2.1 Find the cardinality of the Scala type `Option[Boolean] => Option[Boolean]`. Show that this type is equivalent to `Option[Boolean] => Boolean`, and argue that the equivalence is accidental and not “natural”.

Exercise 5.4.2.2 Verify the type equivalences $A + A \cong 2 \times A$ and $A \times A \cong 2 \Rightarrow A$, where 2 denotes the Boolean type.

Exercise 5.4.2.3 Show that $A \Rightarrow (B \vee C) \neq (A \Rightarrow B) \wedge (A \Rightarrow C)$ in logic.

Exercise 5.4.2.4 Use known rules to verify the type equivalences:

- (a) $(A + B) \times (A \Rightarrow B) \cong A \times (A \Rightarrow B) + (\mathbb{1} + A \Rightarrow B)$.
- (b) $(A \times (\mathbb{1} + A) \Rightarrow B) \cong (A \Rightarrow B) \times (A \Rightarrow A \Rightarrow B)$.
- (c) $A \Rightarrow (\mathbb{1} + B) \Rightarrow C \times D \cong (A \Rightarrow C) \times (A \Rightarrow D) \times (A \Rightarrow B \Rightarrow C) \times (A \Rightarrow B \Rightarrow D)$.

Exercise 5.4.2.5 Write the type notation for `Either[(A, Int), Either[(A, Char), (A, Float)]]`. Transform this type into an equivalent type of the form $A \times (\dots)$.

Exercise 5.4.2.6 Define a type $\text{OptE}^{T,A} \triangleq \mathbb{1} + T + A$ and implement information-preserving `map` and `flatMap` for it, applied to the type parameter A . Get the same result using the equivalent type $(\mathbb{1} + A) + T$, i.e. `Either[Option[A], T]`. The required type signatures are

$$\begin{aligned} \text{map}^{A,B,T} : \text{OptE}^{T,A} &\Rightarrow (A \Rightarrow B) \Rightarrow \text{OptE}^{T,B} \quad , \\ \text{flatMap}^{A,B,T} : \text{OptE}^{T,A} &\Rightarrow (A \Rightarrow \text{OptE}^{T,B}) \Rightarrow \text{OptE}^{T,B} \quad . \end{aligned}$$

Exercise 5.4.2.7 Implement the `map` function for $P[A]$ (see Example 5.4.1.2). The required type signature is $P^A \Rightarrow (A \Rightarrow B) \Rightarrow P^B$.

Exercise 5.4.2.8 For the type constructor $Q^{T,A}$ defined in Exercise 5.1.4.1, define the `map` function with the type signature

$$\text{map}^{T,A,B} : Q^{T,A} \Rightarrow (A \Rightarrow B) \Rightarrow Q^{T,B} .$$

The implementation should preserve information as much as possible.

Exercise 5.4.2.9 Define a recursive type constructor Tr_3 as $\text{Tr}_3^A \triangleq \mathbb{1} + A \times A \times A \times \text{Tr}_3^A$ and implement the `map` function for it, with the standard type signature

$$\text{map}^{A,B} : \text{Tr}_3^A \Rightarrow (A \Rightarrow B) \Rightarrow \text{Tr}_3^B .$$

Exercise 5.4.2.10 Implement fully parametric functions with the following types:

- (a) $A + Z \Rightarrow (A \Rightarrow B) \Rightarrow B + Z$.
- (b) $A + Z \Rightarrow B + Z \Rightarrow (A \Rightarrow B \Rightarrow C) \Rightarrow C + Z$.
- (c) $\text{flatMap}^{E,A,B} : \text{Read}^{E,A} \Rightarrow (A \Rightarrow \text{Read}^{E,B}) \Rightarrow \text{Read}^{E,B}$.
- (d) $\text{State}^{S,A} \Rightarrow (S \times A \Rightarrow S \times B) \Rightarrow \text{State}^{S,B}$.

Exercise 5.4.2.11* Denote $\text{Cont}^{R,T} \triangleq (T \Rightarrow R) \Rightarrow R$ and implement the functions:

- (a) $\text{map}^{R,T,U} : \text{Cont}^{R,T} \Rightarrow (T \Rightarrow U) \Rightarrow \text{Cont}^{R,U}$.
- (b) $\text{flatMap}^{R,T,U} : \text{Cont}^{R,T} \Rightarrow (T \Rightarrow \text{Cont}^{R,U}) \Rightarrow \text{Cont}^{R,U}$.

Exercise 5.4.2.12* Denote $\text{Select}^{Z,T} \triangleq (T \Rightarrow Z) \Rightarrow T$ and implement the functions:

- (a) $\text{map}^{Z,A,B} : \text{Select}^{Z,A} \Rightarrow (A \Rightarrow B) \Rightarrow \text{Select}^{Z,B}$.
- (b) $\text{flatMap}^{Z,A,B} : \text{Select}^{Z,A} \Rightarrow (A \Rightarrow \text{Select}^{Z,B}) \Rightarrow \text{Select}^{Z,B}$.

5.5 Discussion

5.5.1 Using the Curry-Howard correspondence for writing code

This chapter shows how the CH correspondence performs two practically important tasks of type-level reasoning: checking whether a type signature can be implemented as a fully parametric function, and determining whether two types are equivalent. The first task is accomplished by mapping type expressions into formulas in the constructive logic and by applying the proof rules of that logic. The second task is accomplished by mapping type expressions into *arithmetic* formulas and applying the ordinary rules of arithmetic.

Fully parametric functions can be often derived from their type signatures alone. It is useful for a programmer to know that certain type signatures, such as

```
def f[A, B]: A => (A => B) => B
```

have only one possible implementation, while other type signatures, such as

```
def g[A, B]: A => (B => A) => B
def h[A, B]: ((A => B) => A) => A
```

cannot be implemented as fully parametric functions.

Although tools such as the `curryhoward` library could perform the code derivation, in most cases it is more beneficial if a programmer is able to derive an implementation by hand, or to see quickly that an implementation is impossible. Exercises in this chapter build up the required technique. The type notation introduced in this book is designed to help programmers to recognize patterns in type expressions and to reason about them more easily.

Throughout this chapter, we required all functions to be fully parametric. The reason is that the CH correspondence becomes informative only with parameterized types and with fully parametric

functions. For concrete types, e.g. `Int`, one can always produce *some* value even with no previous data, so the proposition $\mathcal{CH}(\text{Int})$ is always true within any code.

Consider the function $(x:\text{Int}) \Rightarrow x + 1$. Its type signature, $\text{Int} \Rightarrow \text{Int}$, is insufficient to specify the code of the function, because there are many different functions with the same type signature, such as $x \Rightarrow x - 1$, $x \Rightarrow x * 2$, etc. So, deriving code from the type signature $\text{Int} \Rightarrow \text{Int}$ is not a meaningful task. Only a fully parametric type signature, such as $A \Rightarrow (A \Rightarrow B) \Rightarrow B$, gives enough information for possibly deriving the function's code. If we permit functions that are not fully parametric, we will not be able to reason about implementability of type signatures or about code derivation.

Information about the implementability of type signatures is given by logical formulas involving \mathcal{CH} -propositions. The validity of \mathcal{CH} -propositions means that we can compute *some* values of given types, but it does not give any information about the properties of those values, such as whether they satisfy any additional laws. This is why type equivalence is not determined by an equivalence of logical formulas.

It is useful for programmers to be able to reason about types and transform type expressions to equivalent simpler types before starting to write code. We have shown that a type equivalence corresponds to *each* standard arithmetic identity such as $(a + b) + c = a + (b + c)$, $(a \times b) \times c = a \times (b \times c)$, $1 \times a = a$, $(a + b) \times c = a \times c + b \times c$, etc. So, we are allowed to transform and simplify types as if they were arithmetic expressions, e.g. to rewrite

$$(A + B) \times C + D \cong D + A \times C + B \times C \quad .$$

The type notation makes this reasoning more intuitive (for people familiar with arithmetic).

These results apply to all type expressions built up using product types, disjunctive types (also called “sum” types because they correspond to arithmetic sums), and function types (also called “exponential” types because they correspond to arithmetic exponentials). Type expressions that contain only products and sum types may be called **polynomial**. Type expressions that also contain function types may be called **exponential-polynomial**.⁸ The set of exponential-polynomial types covers almost all data types and design patterns used in functional programming, and so this book focuses on these types.

There are no type constructions corresponding to subtraction or division, so equations such as

$$(1 - t) \times (1 + t) = 1 - t \times t \quad \text{or} \quad \frac{t + t \times t}{t} = 1 + t$$

do not directly yield any type equivalences. However, consider this well-known formula,

$$\frac{1}{1 - t} = 1 + t + t^2 + t^3 + \dots + t^n + \dots \quad .$$

At first sight, this formula appears to involve subtraction, division, and an infinite series, and thus cannot be directly translated into a type equivalence. However, the formula can be rewritten as

$$\frac{1}{1 - t} \triangleq L(t) = 1 + t + t^2 + t^3 + \dots + t^n \times L(t) \quad , \quad (5.25)$$

which is finite and only contains additions and multiplications. So, Eq. (5.25) can be translated into a type equivalence:

$$L^A \cong 1 + A + A \times A + A \times A \times A + \dots + \underbrace{A \times \dots \times A}_n \times L^A \quad . \quad (5.26)$$

This type formula (with $n = 1$) is equivalent to a recursive definition of the `List` type,

$$\text{List}^A \triangleq 1 + A \times \text{List}^A \quad .$$

The type equivalence (5.26) suggests that we may view the recursive type `List` as an “infinite disjunction” describing lists of zero, one, etc. elements.

⁸Polynomial types are often called “algebraic”, but we will use more specific terms “polynomial” and “exponential-polynomial”.

5.5.2 Implications for designing new programming languages

The functional programming paradigm assumes that programmers will use the six standard type constructions (Section 5.1.2) and the eight standard code constructions (Section 5.2.3). These constructions are foundational in the sense that they are used to express all design patterns of functional programming. A language that does not directly support some of these constructions cannot be considered a functional programming language.

A remarkable consequence of the CH correspondence is that the type system of any programming language (functional or not) is mapped into a *certain logic*, i.e. a system of logical operations and proof rules. A logical operation will correspond to each of the type constructions available in the programming language; a proof rule will correspond to each of the available code constructions. Functional programming languages that support all the standard type and code constructions – for instance, OCaml, Haskell, F#, Scala, Swift, etc., – will be mapped into the constructive logic with all standard logical operations available (*True*, *False*, disjunction, conjunction, and implication). Languages such as C, C++, Java, C# are mapped into logics that do not have the disjunction operation or the constants *True* and *False*. In other words, these languages are mapped into *incomplete* logics where some theorems will not be provable. (If $CH(A)$ is true but not provable, a value of type A is not directly computable by programs, although it could have been.) Languages such as Python, JavaScript, Ruby, Clojure have no type checking and so are mapped to *inconsistent* logics where any proposition can be derived – even propositions normally considered *False*.

Incompleteness of the logic of types will make a programming language unable to express certain computations, e.g. directly handle data that belongs to a disjoint domain. Inconsistency of a logic means that the logic may derive *False* from *True*. The CH correspondence will map such derivations to code that appears to compute a certain value although that value is not actually available. In practice, such code *crashes* (although all types match!) because the computed value has a wrong type, is “null”, or is a pointer to an invalid memory location.

None of these errors will happen in a programming language whose logic of types is complete and consistent, provided that types are checked at compile time.

So, the CH correspondence gives a mathematically justified procedure for designing type systems in new programming languages. The procedure has the following steps:

- Choose a complete formal logic that is free of inconsistencies.
- For each logical operation, provide a type construction in the language.
- For each proof rule, provide a code construction in the language.

Mathematicians have studied different logics (e.g. modal logic, temporal logic, or linear logic). Compared with the constructive logic, these other logics have some additional operations. (For instance, modal logic adds the operations “necessarily” and “possibly”, and temporal logic adds the operation “until”.) For each logic, mathematicians have determined the minimal complete sets of operations, axioms, and proof rules that do not lead to inconsistency. Programming language designers can choose a logic and translate it into a minimal programming language where the code is guaranteed *not to crash* as long as types match. This mathematical guarantee (known as **type safety**) is a powerful help for programmers since it automatically prevents a large set of programming errors. So, programmers will benefit if their programming language is designed using the CH correspondence.

Practically useful programming languages will, of course, introduce many more features than the minimal, mathematically necessary constructions derived from the chosen logic. Programmers will still benefit from type safety as long as the program stays within the mathematically consistent subset of the language. For Scala, a “safe” subset is identified by the `scalazzi` project.⁹

At the present time, it is not fully understood whether it would be better for a programming language to be based not on constructive logic but on, say, linear logic, temporal logic, or some other logic that adds more operations to the constructive logic. Practical experience suggests that at least

⁹<https://github.com/scalaz/scalazzi>

the operations of the constructive logic should be available. So, it appears that the six type constructions and the eight code constructions will remain available in all future languages of functional programming.

5.5.3 Uses of the void type

Scala's void type (`Nothing`) corresponds to the logical constant *False*. The practical uses of *False* are quite limited. One use case is for a branch of a `match` / `case` expression that does not return a value because it throws an exception. Such branches are considered formally to return a value of type `Nothing`, which can then be mapped to a value of any other type (through the function `absurd[A]: Nothing => A`, see Example 5.3.4.2).

To see how this trick is used, consider this code defining a value `x`,

```
val x: Double = if (t >= 0.0) math.sqrt(t) else { throw new Exception("error") }
```

The `else` branch does not return a value, but `x` is declared to be of type `Double`. For this code to type-check, both branches must return values of the same type. So, the compiler needs to pretend that the `else` branch also returns a value of type `Double`. The compiler first assigns the type `Nothing` to the expression `throw ...` and then implicitly uses the function `absurd: Nothing => Double` to convert that type to `Double`. In this way, types will match in the definition of the value `x`.

We will not use exceptions in this book. The functional programming paradigm avoids exceptions because their presence significantly complicates reasoning about code.

So far, none of our examples involved the logical **negation** operation. It is defined as

$$\neg A \triangleq A \Rightarrow \text{False} \quad ,$$

and its practical use is as limited as that of *False* and the void type. However, logical negation plays an important role in Boolean logic, which we will discuss next.

5.5.4 Relationship between Boolean logic and constructive logic

We have seen that some true theorems of Boolean logic are not true in constructive logic. For example, the Boolean identities $\neg(\neg\alpha) = \alpha$ and $(\alpha \Rightarrow \beta) = (\neg\alpha \vee \beta)$ do not hold in the constructive logic. However, any theorem of constructive logic is also a theorem of Boolean logic. The reason is that all eight rules of constructive logic (Section 5.2.3) are also true in Boolean logic.

To verify that a formula is true in Boolean logic, we only need to check that the value of the formula is *True* for all possible truth values (*True* or *False*) of its variables. A sequent such as $\alpha, \beta \vdash \gamma$ is true in Boolean logic if and only if $\gamma = \text{True}$ under the assumption that $\alpha = \beta = \text{True}$. So, the sequent $\alpha, \beta \vdash \gamma$ is translated into the Boolean formula

$$\alpha, \beta \vdash \gamma = ((\alpha \wedge \beta) \Rightarrow \gamma) = (\neg\alpha \vee \neg\beta \vee \gamma) \quad .$$

Table 5.7 translates all proof rules of Section 5.2.3 into Boolean formulas. The first two lines are axioms, while the subsequent lines are Boolean theorems that can be verified by calculation.

To simplify the calculations, note that all terms in the formulas contain the operation $(\neg\Gamma \vee \dots)$ corresponding to the context Γ . Now, if Γ is *False*, the entire formula becomes automatically *True*, and there is nothing else to check. So, it remains to verify the formula in case $\Gamma = \text{True}$, and then we can simply omit all instances of $\neg\Gamma$ in the formulas. Let us show the Boolean derivations for the rules “use function” and “use Either”; other formulas are checked in a similar way.

$$\begin{aligned} \text{formula “use function” : } & (\alpha \wedge (\alpha \Rightarrow \beta)) \Rightarrow \beta \\ \text{use Eq. (5.5) : } & = \neg(\alpha \wedge (\neg\alpha \vee \beta)) \vee \beta \\ \text{de Morgan’s laws : } & = \neg\alpha \vee (\alpha \wedge \neg\beta) \vee \beta \\ \text{identity } p \vee (\neg p \wedge q) = p \vee q \text{ with } p = \neg\alpha \text{ and } q = \beta : & = \neg\alpha \vee \neg\beta \vee \beta \\ \text{axiom “use arg” : } & = \text{True} \quad . \end{aligned}$$

Constructive logic	Boolean logic
$\frac{}{\Gamma \vdash \mathcal{H}(1)}$ (create unit)	$\neg \Gamma \vee \text{True} = \text{True}$
$\frac{}{\Gamma, \alpha \vdash \alpha}$ (use arg)	$\neg \Gamma \vee \neg \alpha \vee \alpha = \text{True}$
$\frac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \alpha \Rightarrow \beta}$ (create function)	$(\neg \Gamma \vee \neg \alpha \vee \beta) = (\neg \Gamma \vee (\alpha \Rightarrow \beta))$
$\frac{\Gamma \vdash \alpha \quad \Gamma \vdash \alpha \Rightarrow \beta}{\Gamma \vdash \beta}$ (use function)	$((\neg \Gamma \vee \alpha) \wedge (\neg \Gamma \vee (\alpha \Rightarrow \beta))) \Rightarrow (\neg \Gamma \vee \beta)$
$\frac{\Gamma \vdash \alpha \quad \Gamma \vdash \beta}{\Gamma \vdash \alpha \wedge \beta}$ (create tuple)	$(\neg \Gamma \vee \alpha) \wedge (\neg \Gamma \vee \beta) = (\neg \Gamma \vee (\alpha \wedge \beta))$
$\frac{\Gamma \vdash \alpha \wedge \beta}{\Gamma \vdash \alpha}$ (use tuple-1)	$(\neg \Gamma \vee (\alpha \wedge \beta)) \Rightarrow (\neg \Gamma \vee \alpha)$
$\frac{\Gamma \vdash \alpha}{\Gamma \vdash \alpha \vee \beta}$ (create Left)	$(\neg \Gamma \vee \alpha) \Rightarrow (\neg \Gamma \vee (\alpha \vee \beta))$
$\frac{\Gamma \vdash \alpha \vee \beta \quad \Gamma, \alpha \vdash \gamma \quad \Gamma, \beta \vdash \gamma}{\Gamma \vdash \gamma}$ (use Either)	$((\neg \Gamma \vee \alpha \vee \beta) \wedge (\neg \Gamma \vee \neg \alpha \vee \gamma) \wedge (\neg \Gamma \vee \neg \beta \vee \gamma)) \Rightarrow (\neg \Gamma \vee \gamma)$

Table 5.7: Proof rules of constructive logic are true also in the Boolean logic.

$$\begin{aligned}
\text{formula "use Either"} : & ((\alpha \vee \beta) \wedge (\alpha \Rightarrow \gamma) \wedge (\beta \Rightarrow \gamma)) \Rightarrow \gamma \\
\text{use Eq. (5.5)} : & = \neg((\alpha \vee \beta) \wedge (\neg \alpha \vee \gamma) \wedge (\neg \beta \vee \gamma)) \vee \gamma \\
\text{de Morgan's laws} : & = (\neg \alpha \wedge \neg \beta) \vee \underline{(\alpha \wedge \neg \gamma)} \vee \underline{(\beta \wedge \neg \gamma)} \vee \gamma \\
\text{identity } p \vee (\neg p \wedge q) = p \vee q : & = \underline{(\neg \alpha \wedge \neg \beta)} \vee \alpha \vee \beta \vee \gamma \\
\text{identity } p \vee (\neg p \wedge q) = p \vee q : & = \underline{\neg \alpha \vee \alpha} \vee \beta \vee \gamma \\
\text{axiom "use arg"} : & = \text{True} .
\end{aligned}$$

Since each proof rule of the constructive logic is translated into a true formula in Boolean logic, it follows that a proof tree in the constructive logic will be translated into a tree of Boolean formulas that have value *True* for each axiom or proof rule. The result is that any constructive proof for a sequent such as $\emptyset \vdash f(\alpha, \beta, \gamma)$ is translated into a chain of Boolean implications that look like this,

$$\text{True} = (\dots) \Rightarrow (\dots) \Rightarrow \dots \Rightarrow f(\alpha, \beta, \gamma) .$$

Since $(\text{True} \Rightarrow \alpha) = \alpha$, this chain proves the Boolean formula $f(\alpha, \beta, \gamma)$.

For example, the proof tree shown in Figure 5.1 is translated into

$$\begin{aligned}
\text{axiom "use arg"} : & \text{True} = (\neg((\alpha \Rightarrow \alpha) \Rightarrow \beta) \vee \neg \alpha \vee \alpha) \\
\text{rule "create function"} : & \Rightarrow \neg((\alpha \Rightarrow \alpha) \Rightarrow \beta) \vee (\alpha \Rightarrow \alpha) . \\
\text{axiom "use arg"} : & \text{True} = \neg((\alpha \Rightarrow \alpha) \Rightarrow \beta) \vee ((\alpha \Rightarrow \alpha) \Rightarrow \beta) . \\
\text{rule "use function"} : & \text{True} \Rightarrow (\neg((\alpha \Rightarrow \alpha) \Rightarrow \beta) \vee \beta) \\
\text{rule "create function"} : & \Rightarrow (((\alpha \Rightarrow \alpha) \Rightarrow \beta) \Rightarrow \beta) .
\end{aligned}$$

It is easier to check Boolean truth than to find a proof tree in constructive logic (or to establish that no proof tree exists). So, if we find that a formula is not true in Boolean logic, we know it is also not true in constructive logic. This gives us a quick way of proving that some type signatures are not implementable as fully parametric functions. In addition to formulas shown in Table 5.3 (Section 5.2.1), further examples of formulas that are not true in Boolean logic are

$$\begin{aligned}
& \forall A. A \quad , \\
& \forall (A, B). A \Rightarrow B \quad , \\
& \forall (A, B). (A \Rightarrow B) \Rightarrow B \quad .
\end{aligned}$$

Table 5.7 uses the Boolean identity $(\alpha \Rightarrow \beta) = (\neg \alpha \vee \beta)$, which does not hold in the constructive logic, to translate the constructive axiom "use arg" into the Boolean axiom $\neg \alpha \vee \alpha = \text{True}$. The

formula $\neg\alpha \vee \alpha = \text{True}$ is known as the **law of excluded middle**,¹⁰ and is equivalent to saying that any proposition α is either true or false. It is remarkable that the constructive logic *does not have* the law of excluded middle; it is neither an axiom nor a derived theorem of constructive logic.

To see why, consider what it would mean for $\neg\alpha \vee \alpha = \text{True}$ to hold in the constructive logic. The negation operation, $\neg\alpha$, is defined as the implication $\alpha \Rightarrow \text{False}$. So, the logical formula $\forall\alpha. \neg\alpha \vee \alpha$ corresponds to the type $\forall A. (A \Rightarrow \mathbb{0}) + A$. Can we compute a value of this type in a fully parametric function? We would need to compute either a value of type $A \Rightarrow \mathbb{0}$ or a value of type A ; this decision needs to be made in advance independently of A , because the code of a fully parametric function must operate in the same way for all types. Should we decide to return A or $A \Rightarrow \mathbb{0}$? We certainly cannot compute a value of type A from scratch, since A is a type parameter. As we have seen in Example 5.3.4.3, a value of type $A \Rightarrow \mathbb{0}$ exists if the type A is itself $\mathbb{0}$; but we do not know that, and a fully parametric function needs to have the same code for all types A . Since there are no values of type $\mathbb{0}$, and the type parameter A could be, say, `Int`, we cannot compute a value of type $A \Rightarrow \mathbb{0}$.

Example 5.3.4.3 showed that the type $A \Rightarrow \mathbb{0}$ is equivalent to $\mathbb{0}$ if A is not itself void ($A \neq \mathbb{0}$), and to $\mathbb{1}$ otherwise. Surely, any type A is either void or not void. So, why exactly is it impossible to implement a value of the type $(A \Rightarrow \mathbb{0}) + A$? We could say that if A is void then $(A \Rightarrow \mathbb{0}) \cong \mathbb{1}$ is not void, and so one of the types in the disjunction $(A \Rightarrow \mathbb{0}) \vee A$ should be non-void (i.e. have values).

However, this reasoning is incorrect. A fully parametric functions' code must work in the same general way for all types; the code cannot decide what to do depending on a specific type. So, it is insufficient to show that a value “should exist”; the real requirement is to compute a value of type $(A \Rightarrow \mathbb{0}) + A$ in fully parametric code that works the same way for all types A . But, as we have seen, this is impossible.

In Boolean logic, it is sufficient to prove that a value “should exist” (or that the non-existence of a value is contradictory in some way). However, any practically useful program needs to “construct” (i.e. compute) actual values and return them. The “constructive” logic got its name from this requirement. So, it is the constructive logic (not the Boolean logic) that provides correct reasoning about the types of values computable by functional programs.

¹⁰https://en.wikipedia.org/wiki/Law_of_excluded_middle

6 Functors, contrafunctors, and profunctors

Type constructors such as `Seq[A]` or `Array[A]` are data structures that hold or “wrap” zero or more values of a given type `A`. These data structures are fully parametric: they work in the same way for every type `A`. Working with parametric “data wrappers” or “data containers” turns out to be a powerful design pattern of functional programming. To fully realize its benefits, we will formalize the concept of “data wrapping” through a set of mathematical laws. We will then extend that design pattern to all data types for which the laws hold.

6.1 Practical use

6.1.1 Motivation: Type constructors that wrap data

How to formalize the idea of “wrapped data”? An intuitive view is that the data is “still there”, i.e. we should be able to manipulate the data held within the wrapper. In functional programming, to “manipulate” means to apply functions to data. So, if an integer value 123 is “wrapped”, we should be able somehow to apply a function such as `{x => x * 2}` and obtain a “wrapped” value 246.

Let us look at some often used type constructors defined in the Scala standard library, such as `Seq[A]`, `Try[A]`, and `Future[A]`. We notice the common features:

- There are some methods for creating a data structure that wraps zero or more values of a given type. For example, the Scala code `List.fill(10)(0)` creates a list of ten zeros of type `Int`.
- There are some methods for reading the wrapped values, if they exist. For example, the `List` class has the method `.headOption` that returns a non-empty option when the first element exists.
- There are some methods for manipulating the wrapped values while *keeping* them wrapped. For example, `List(10, 20, 30).map(_ + 5)` evaluates to `List(15, 25, 35)`.

The data types `Seq[A]`, `Try[A]`, and `Future[A]` express quite different kinds of “wrapping”. The data structure implementing `Seq[A]` can hold a variable number of values of type `A`. The data structure `Try[A]` holds either a successfully computed value of type `A` or a failure. The data structure `Future[A]` implements a computation that has been scheduled to run but may not have finished yet, and may return a value of type `A` (or a failure) when finished at a later time.

Since the meaning of the “wrappers” `Seq`, `Try`, and `Future` is quite different, the methods for creating and reading the wrapped values have different type signatures for each “wrapper”. However, the method `.map` is similar in all three examples. We can say generally that the `.map` method will apply a given function $f:A \Rightarrow B$ to the data of type `A` held inside the wrapper, and the new data (of type `B`) will remain within a wrapper of the same type:

```
val a = List(x,y,z).map(f) // Result is List(f(x), f(y), f(z)).
val b = Try(x).map(f)      // Result is Try(f(x)).
val c = Future(x).map(f)   // Result is Future(f(x)).
```

This motivates us to use the `map` function as the requirement for the “wrapping” functionality: A type constructor `Wrap[A]` is a “wrapper” if there exists a function `map` with the type signature

```
def map[A, B]: Wrap[A] => (A => B) => Wrap[B]
```

We can see that `Seq`, `Try`, and `Future` are “wrappers” because they have a suitable `.map` method. This chapter focuses on the properties of `.map` that are common to *all* “wrapper” types. We will ignore

all other features – reading data out of the wrapper, inserting or deleting data, waiting until data becomes available etc., – implemented by different methods specific to each wrapper type.

6.1.2 Example: Option and the identity law

As another example of a “data wrapper”, consider the type constructor `Option[A]`, which is written in the type notation as

$$\text{Opt}^A \triangleq \mathbb{1} + A \quad .$$

The type signature of its `map` function is

$$\text{map}^{A,B} : \mathbb{1} + A \Rightarrow (A \Rightarrow B) \Rightarrow \mathbb{1} + B \quad .$$

This function produces a new `Option[B]` value, possibly holding transformed data. We will now use this example to develop intuition about manipulating data in a wrapper.

Two possible implementations of `map` fit the type signature:

```
def mapX[A, B](oa: Option[A])(f: A => B): Option[B] = None
def mapY[A, B](oa: Option[A])(f: A => B): Option[B] = oa match {
  case None      => None
  case Some(x)   => Some(f(x))
}
```

The implementation `mapX` loses information since it ignores all input and always returns a constant value `None`. The implementation `mapY` preserves information and is clearly a more useful version of `map`.

How can we formulate this property of `map` in a rigorous way? The trick is to write the expression `map(oa)(f)` and to choose the argument $f:A \Rightarrow B$ to be the identity function $\text{id}:A \Rightarrow A$ (setting `map`’s type parameters to be equal, $A = B$, so that the types match). Using an identity function to transform the data wrapped in a given value of type `Option[A]` should not change that value. To check if this is so, substitute the identity function instead of `f` into `mapY` and get

```
mapY[A, A](oa: Option[A])(identity[A]: A => A): Option[A]
== oa match {
  case None      => None           // No change.
  case Some(x)   => Some(x)       // No change.
} == oa
```

The result is always equal to `oa`. We can write that fact as an equation,

$$\forall x:A. \text{map}(x)(\text{id}) = x \quad .$$

This equation is called the **identity law** of `map`. The identity law is a formal way of expressing the information-preserving property of the `map` function. The implementation `mapX` violates the identity law since it always returns `None` and so `mapX(oa)(id) == None` and not equal to `oa` for arbitrary values of `oa`. A data wrapper should not unexpectedly lose information when we manipulate the wrapped data. So, the correct implementation of `map` is `mapY`. The code notation for `map` is

$$\text{map}^{A,B} \triangleq p:\mathbb{1}+A \Rightarrow f:A \Rightarrow B \Rightarrow p \triangleright \begin{array}{c|c|c|c} & & \mathbb{1} & B \\ \hline & \mathbb{1} & \text{id} & 0 \\ \hline A & 0 & 0 & f \end{array} \quad .$$

When writing code, it is convenient to use the `.map` method defined by the Scala library. However, when reasoning about the properties of `map`, it turns out to be more convenient to flip the order of the curried arguments and to use the equivalent function, called `fmap`, with the type signature

$$\text{fmap}^{A,B} : (A \Rightarrow B) \Rightarrow \mathbb{1} + A \Rightarrow \mathbb{1} + B \quad .$$

The Scala implementation of `fmap` is shorter than that of `map`:

```
def fmap[A, B](f: A => B): Option[A] => Option[B] = {
  case None    => None
  case Some(x) => Some(f(x))
}
```

The code notation for this function is

$$\text{fmap}(f^{A \Rightarrow B}) \triangleq \begin{array}{c|c|c} & 1 & B \\ \hline 1 & \text{id} & 0 \\ \hline A & 0 & f \end{array} . \quad (6.1)$$

The identity law also looks simpler if expressed in terms of `fmap`, namely `fmap(id) = id`. Here we omit the type parameters *A* and *B*, which must be both equal.

Note that the type signature of `fmap` looks like a transformation from functions of type `A => B` to functions of type `Option[A] => Option[B]`. This transformation is called **lifting** because it “lifts” a function $f^{A \Rightarrow B}$ operating on simple values into a function operating on `Option`-wrapped values.

So, the identity law can be formulated as “a lifted identity function is also an identity function”. If we lift an identity function (or a composition of functions that equals an identity), we expect the wrapped data not to change. The identity law expresses this expectation in a mathematical equation.

6.1.3 Motivation for the composition law

The main feature of a “data wrapper” or “data container” is to allow us to manipulate the data inside it by applying functions to that data. The corresponding Scala code is `p.map(f)`, where *p* is a value of a “wrapper” type. It is natural to expect that lifted functions behave in the same way as the “unlifted” ones. For example, suppose we need to increment a counter *c* of type `Option[Int]`. The `Option` type means that the counter may be empty or non-empty; if it is non-empty, we increment the integer value wrapped inside the `Option` using the incrementing function

$$\text{incr} \triangleq x^{\text{Int}} \Rightarrow x + 1 .$$

In order to apply this function to the counter *c*, we need to lift it. The Scala code is

```
def incr: Int => Int = x => x + 1
val c: Option[Int] = Some(0)

scala> c.map(incr)
res0: Option[Int] = Some(1)
```

If we apply the lifted function twice, we expect that the counter will be incremented twice:

```
scala> c.map(incr).map(incr)
res1: Option[Int] = Some(2)
```

This result is the same as when applying a lifted function $x \Rightarrow x + 2$:

```
scala> c.map(x => x + 2)
res2: Option[Int] = Some(2)
```

It would be confusing and counter-intuitive if `c.map(x => x + 2)` did not give the same result as `c.map(incr).map(incr)`.

We can formulate this property more generally: liftings should preserve function composition for arbitrary functions $f^{A \Rightarrow B}$ and $g^{B \Rightarrow C}$. This is written as

$$\text{fmap}(f^{A \Rightarrow B} ; g^{B \Rightarrow C}) = \text{fmap}(f^{A \Rightarrow B}) ; \text{fmap}(g^{B \Rightarrow C}) .$$

This equation is called the **composition law**.

Let us formally verify the composition law for the `Option` type. To practice the code derivations, we will perform the calculations both by writing Scala code fragments and by writing expressions in the code notation.

The Scala code for the function `fmap` was given in Section 6.1.2. To evaluate `fmap(f ; g)`, we apply `fmap(f andThen g)`, where $f: A \Rightarrow B$ and $g: B \Rightarrow C$ are arbitrary functions, to an arbitrary value `oa: Option[A]`. In Scala code, it is more convenient to use the infix method `.map` and write `oa.map(f)` instead of the equivalent expression `fmap(f)(oa)`:

```
fmap(f andThen g)(oa) == oa.map(f andThen g) == oa match {
  case None      => None
  case Some(x)   => (f andThen g)(x)
}
```

Since $(f \text{ andThen } g)(x) == g(f(x))$, we rewrite the result as

```
oa.map(f andThen g) == oa match {
  case None      => None
  case Some(x)   => g(f(x))
}
```

Now we consider the right-hand side of the law, $\text{fmap}(f) \circ \text{fmap}(g)$, and write the Scala expressions:

```
oa.map(f).map(g) == (oa match {
  case None      => None
  case Some(x)   => f(x)
}).map(g) == (oa match {
  case None      => None
  case Some(x)   => f(x)
}).match {
  case None      => None
  case Some(y)   => g(y)
} == oa match {
  case None      => None
  case Some(x)   => g(f(x))
}
```

So we find that the two sides of the law have identical code.

The derivation is much shorter in the matrix notation; we use Eq. (6.1) as the definition of fmap and omit the types:

$$\begin{aligned} \text{fmap}(f) \circ \text{fmap}(g) &= \begin{vmatrix} \text{id} & 0 \\ 0 & f \end{vmatrix} \circ \begin{vmatrix} \text{id} & 0 \\ 0 & g \end{vmatrix} \\ \text{matrix composition :} &= \begin{vmatrix} \text{id} \circ \text{id} & 0 \\ 0 & f \circ g \end{vmatrix} = \begin{vmatrix} \text{id} & 0 \\ 0 & f \circ g \end{vmatrix} \\ \text{definition of fmap :} &= \text{fmap}(f \circ g) \end{aligned}$$

These calculations prove that the `map` method of the `Option` type satisfies the composition law. If the composition law did not hold, we would not be able to understand how `map` manipulates data within the `Option` wrapper. Looking at the Scala code example above, we expect `c.map(incr).map(incr)` to increment the data wrapped by `c` two times. If the result of `c.map(incr).map(incr)` were not `Some(2)` but, say, `Some(1)` or `None`, our ordinary intuitions about data transformations would be incorrect. Violations of the composition law prevent us from understanding the code via mathematical reasoning about transformation of data values.

The composition law is a rigorous formulation of the requirement that wrapped data should be transformed (by lifted functions) in the same way as ordinary data. For example, the following associativity property holds for lifted functions:

Statement 6.1.3.1 For arbitrary functions $f:A \Rightarrow B$, $g:B \Rightarrow C$, and $h:C \Rightarrow D$, we have

$$\text{fmap}(f) \circ \text{fmap}(g \circ h) = \text{fmap}(f \circ g) \circ \text{fmap}(h)$$

Proof The left-hand side is rewritten as

$$\begin{aligned} & \text{fmap}(f) \circ \text{fmap}(g \circ h) \\ \text{composition law for } (g \circ h) : &= \text{fmap}(f) \circ (\text{fmap}(g) \circ \text{fmap}(h)) \\ \text{associativity law (4.3) :} &= (\text{fmap}(f) \circ \text{fmap}(g)) \circ \text{fmap}(h) \\ \text{composition law for } (f \circ g) : &= \text{fmap}(f \circ g) \circ \text{fmap}(h) \end{aligned}$$

which now equals the right-hand side. This proves the statement.

6.1.4 Functors: definition and examples

Separating the functionality of “data wrapper” from any other features of a data type, we obtain:

- A data type with a type parameter, e.g. `L[A]`. We will use the notation L^\bullet (in Scala, `L[_]`) for the type constructor itself, and also when the name of the type parameter is not needed.

- A fully parametric function `fmap` with type signature

$$\text{fmap}_L : (A \Rightarrow B) \Rightarrow L^A \Rightarrow L^B \quad ,$$

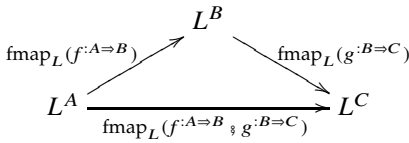
- satisfying the laws

$$\text{identity law for } L : \text{fmap}_L(\text{id}^{A \Rightarrow A}) = \text{id}^{L^A \Rightarrow L^A} \quad , \quad (6.2)$$

$$\text{composition law for } L : \text{fmap}_L(f^{A \Rightarrow B} \circ g^{B \Rightarrow C}) = \text{fmap}_L(f^{A \Rightarrow B}) \circ \text{fmap}_L(g^{B \Rightarrow C}) \quad . \quad (6.3)$$

A type constructor L^\bullet with these properties is called a **functor**. The laws (6.2)–(6.3) are the functor laws of identity and composition.

When a law involves function compositions, it is helpful to draw a type diagram to clarify how the functions transform various types involved in the law. A **type diagram** is a directed graph whose vertices are types and edges are functions mapping one type to another. Function composition corresponds to following a path in the diagram. A type diagram for the composition law (6.3) is shown at left. The diagram contains two paths from L^A to L^C ; by Eq. (6.3), both paths must give the same result. Mathematicians call such diagrams **commutative**.



Type diagrams are easier to read when using the *forward* composition $(f ; g)$ because the order of edges is the same as the order of functions in the composition. To see this, compare Eq. (6.3) and the type diagram above with the same law written using the backward composition,

$$\text{fmap}_L(g^{B \Rightarrow C} \circ f^{A \Rightarrow B}) = \text{fmap}_L(g^{B \Rightarrow C}) \circ \text{fmap}_L(f^{A \Rightarrow B}) \quad .$$

The function `map` is computationally equivalent to `fmap` and can be defined through `fmap` by

$$\begin{aligned} \text{map}_L : L^A \Rightarrow (A \Rightarrow B) \Rightarrow L^B \quad , \\ \text{map}_L(x^{L^A})(f^{A \Rightarrow B}) = \text{fmap}_L(f^{A \Rightarrow B})(x^{L^A}) \quad . \end{aligned}$$

Each of the type constructors `Option`, `Seq`, `Try`, and `Future` has its own definition of `map`; but the functor laws remain the same. We use the subscript L when writing `mapL` and `fmapL`, in order to indicate clearly the type constructor those functions work with.

We will now look at some examples of type constructors that are functors.

Standard data structures Many type constructors defined in the Scala library have a `.map` method, and almost all of them are functors. The most often used functors are:

- The standard disjunctive types `Option`, `Try`, and `Either[A, B]` (where, by default, transformations apply to the type parameter `B`).
- The linear sequence `Seq` and its various derived classes such as `List`, `Range`, `Vector`, `IndexedSeq`, and `Stream`.
- The “task-like” constructors: `Future` and its alternatives: `Task` (provided by the `monix` library), `Async` and `Concurrent` (provided by the `cats-effect` library), `ZIO` (provided by the `zio` library).
- Dictionaries: `Map[K, V]` with respect to the type parameter `V`. The method is called `.mapValues` instead of `.map`: it transforms the values in the dictionary (leaving the keys unchanged).

Application-specific, custom type constructors defined by the programmer, such as case classes with type parameters, are often functors. Their structure is simple and helps build intuition for functors, so let us now consider some examples of case classes that are functors. In this book, they are called polynomial functors.

Polynomial functors Type constructors built with primitive types, type parameters, products, and disjunctions (or “sums”) are often used to represent application-specific data. Consider the code

```
case class Counted[A](n: Int, a: A) {
  def map[B](f: A => B): Counted[B] = Counted(n, f(a))
}
```

The data type `Counted[A]` may be used to describe n repetitions of a given value $a:A$. The code already defines the infix method `.map` for the `Counted` class, which can be used like this,

```
scala> Counted(10, "abc").map(s => "prefix " + s)
res0: Counted[String] = Counted(10,prefix abc)
```

It is often more convenient to implement `.map` as an infix method rather than as a function such as

```
def map[A, B](c: Counted[A])(f: A => B): Counted[B] = c match {
  case Counted(n, a) => Counted(n, f(a))
}
```

The type notation for `Counted` is

$$\text{Counted}^A \triangleq \text{Int} \times A \quad ,$$

showing that `Counted[_]` is a polynomial type constructor. The existence of a `map` method suggests that `Counted[_]` is a functor. We still need to check that the functor laws hold for it.

Example 6.1.4.1 Verify that the above implementation of `map` for `Counted` satisfies the functor laws.

Solution The implementation of `map` is fully parametric since it does not perform any type-specific operations; it uses the value $n:\text{Int}$ as if `Int` were a type parameter. It remains to check that the laws hold. We will first verify the laws using the Scala syntax and then using the code notation.

The identity law means that for all $n:\text{Int}$ and $a:A$ we must have

```
Counted(n, a).map(identity) == Counted(n, a)
```

To verify this, we substitute the code of `.map` and find

```
Counted(n, a).map(identity) == Counted(n, identity(a)) == Counted(n, a)
```

The composition law means that for all $n:\text{Int}$, $a:A$, $f: A \Rightarrow B$, and $g: B \Rightarrow C$, we must have

```
Counted(n, a).map(f).map(g) == Counted(n, a).map(f andThen g)
```

Substitute the Scala code of `.map` into the left-hand side:

```
Counted(n, a).map(f).map(g) == Counted(n, f(a)).map(g) == Counted(n, g(f(a)))
```

The right-hand side can be transformed to the same expression:

```
Counted(n, a).map(f andThen g) == Counted(n, (f andThen g)(a)) == Counted(n, g(f(a)))
```

Let us now write a proof in the code notation, formulating the laws via the `fmap` method:

$$\text{fmap}_{\text{Counted}}(f:A \Rightarrow B) \triangleq (n:\text{Int} \times a:A \Rightarrow n \times f(a)) \quad .$$

To verify the identity law, we write

$$\begin{aligned} \text{expect to equal id} : & \text{fmap}_{\text{Counted}}(\text{id}) \\ \text{definition of fmap}_{\text{Counted}} : & = (n \times a \Rightarrow n \times \text{id}(a)) \\ \text{definition of id} : & = (n \times a \Rightarrow n \times a) = \text{id} \quad . \end{aligned}$$

To verify the composition law,

$$\begin{aligned} \text{expect to equal fmap}_{\text{Counted}}(f \circ g) : & \text{fmap}_{\text{Counted}}(f) \circ \text{fmap}_{\text{Counted}}(g) \\ \text{definition of fmap}_{\text{Counted}} : & = (n \times a \Rightarrow n \times f(a)) \circ (n \times b \Rightarrow n \times g(b)) \\ \text{compute composition} : & = n \times a \Rightarrow n \times \underline{g(f(a))} \\ \text{definition of } (f \circ g) : & = (n \times a \Rightarrow n \times (f \circ g)(a)) = \text{fmap}_{\text{Counted}}(f \circ g) \quad . \end{aligned}$$

We will prove later that all polynomial type constructors have a definition of `map` that satisfies the functor laws. It will be clear that we defined `map` for `Counted` correctly here.

What could be an *incorrect* implementation of `map`? Suppose that `n` counts the number of times `map` is applied:

```
def map_bad[A, B](c: Counted[A])(f: A => B): Counted[B] = c match {
  case Counted(n, a) => Counted(n + 1, f(a))
}
```

This implementation may appear reasonable. However, it violates both functor laws; for instance,

```
Counter(n, a) != map_bad(Counter(n, a))(identity) == Counter(n + 1, a)
```

The failure of functor laws leads to surprising behavior because a code refactoring changes the result:

```
Counter(n, a).map(incr).map(incr) != Counter(n, a).map(x => x + 2)
```

Let us look at some other simple examples of polynomial type constructors.

Example 6.1.4.2 Implement the `fmap` function for the type constructor

```
case class Vec3[A](x: A, y: A, z: A)
```

Solution Begin by implementing a fully parametric function:

```
def fmap[A, B](f: A => B): Vec3[A] => Vec3[B] = {
  case Vec3(x, y, z) => Vec3(f(x), f(y), f(z))
}
```

Since all three values `f(x)`, `f(y)`, `f(z)` have type `B`, the code of `fmap` would still satisfy the required type signature by returning, say, `Vec3(f(z), f(x), f(x))` or some other combination of these values. However, that implementation would not preserve information about the values `x`, `y`, `z` and about the ordering of these values in the original data `Vec3(x, y, z)`. For this reason, we choose the implementation of `fmap` shown above.

The type notation for the type constructor `Vec3[_]` is

$$\text{Vec}_3^A \triangleq A \times A \times A \quad ,$$

and the code notation for `fmap` is

$$\text{fmap}_{\text{Vec}_3}(f:A \Rightarrow B) \triangleq x:A \times y:A \times z:A \Rightarrow f(x) \times f(y) \times f(z) \quad .$$

Example 6.1.4.3 Implement the `fmap` function for the type constructor

$$\text{QueryResult}^A \triangleq \text{String} + \text{String} \times \text{Long} \times A \quad .$$

Solution Begin by implementing the type constructor in Scala,

```
sealed trait QueryResult[A]
case class Error[A](message: String) extends QueryResult[A]
case class Success[A](name: String, time: Long, data: A) extends QueryResult[A]
```

Now implement a fully parametric, information-preserving function with the type signature of `fmap` for this type constructor:

```
def fmap[A, B](f: A => B): QueryResult[A] => QueryResult[B] = {
  case Error(message) => Error(message)
  case Success(name, time, data) => Success(name, time, f(data))
}
```

As in the previous example, we treat specific types (`Long`, `String`) as if they were type parameters. In this way, we obtain a correct implementation of `fmap` that satisfies the functor laws.

Recursive polynomial functors Recursive disjunctive type constructors shown in Section 3.3, such as lists and trees, are functors. Their `fmap` methods are recursive functions; they usually *cannot* be directly implemented with tail recursion.

Example Define a list of *odd* length as a recursive type LO^* ,

$$\begin{aligned} LO^A &\triangleq A + A \times A \times LO^A \\ &\cong A + A \times A \times A + A \times A \times A \times A + \dots \end{aligned} \quad (6.4)$$

and implement `fmap` for it.

Solution The Scala definition of the type constructor `LO[_]` is

```
sealed trait LO[A]
final case class LO1[A](x: A) extends LO[A]
final case class LO2[A](x: A, y: A, tail: LO[A]) extends LO[A]
```

We can implement `fmap` as a recursive function:

```
def fmap[A, B](f: A => B): LO[A] => LO[B] = {
  case LO1(x) => LO1[B](f(x))
  case LO2(x, y, tail) => LO2[B](f(x), f(y), fmap(f)(tail))
}
```

This code for `fmap` is not tail-recursive because `fmap` is called inside the case class constructor `LO2`.

The type constructor LO^* is a **recursive polynomial functor** because it is defined by a recursive type equation (6.4) that uses only polynomial type operations (sums and products) in its right-hand side. For the same reason, lists and trees are recursive polynomial functors.

6.1.5 Functor block expressions

Computations with wrapped values often require a chain of `.map` methods, e.g.

```
scala> val result = Map(1 -> "one", 2 -> "two", 3 -> "three").
  map { case (i, name) => (i * i, name) }. // Compute i * i.
  map { case (x, name) => (x, s"$name * $name") }. // Compute product message.
  map { case (x, product) => s"$product is $x" } // Compute final message.
result: Seq[String] = List(one * one is 1, two * two is 4, three * three is 9)
```

Such code can be rewritten equivalently in the **functor block** syntax:

```
val result = for {
  (i, name) <- Map(1 -> "one", 2 -> "two", 3 -> "three") // For each (i, name)...
  x = i * i // define 'x' by computing i * i...
  product = s"$name * $name" // define 'product'...
} yield s"$product is $x" // and put these expressions into the 'result' sequence.
result: Seq[String] = List(one * one is 1, two * two is 4, three * three is 9)
```

Written in this way, the computations are easier to understand for two main reasons:

- There is less code to read and to write; no `.map` or `case` and fewer curly braces.
- Values such as `name` and `x` need to be kept in tuples and passed from one `.map` function to another, but any line in a functor block can directly reuse all values defined in previous lines.

The functor block is an important idiom in functional programming because it replaces a chain of `.map` methods (as well as `.filter` and `.flatMap` methods, as we will see in later chapters) by a visually clearer sequence of definitions and expressions. Scala defines a functor block via the keywords `for` and `yield`. We will see many examples of functor blocks throughout this book. In this chapter, we only consider functor blocks that are equivalent to a chain of `.map` operations on a functor value $p:L[A]$. These functor blocks can be recognized because they contain *only one* left arrow (in the first line). Here is how to replace a chain of `.map` operations by a functor block:

```
p.map(x => f(x)).map(y => g(y)).map(z => h(z)) == for {
  x <- p // The first line must contain a left arrow.
  y = f(x) // Some computation involving 'x'.
  z = g(y) // Another computation, uses 'y'.
} yield h(z) // The 'yield' may use 'x', 'y', 'z', and any other defined variables.
```

Translating functor blocks back into a chain of `.map` operations is straightforward except for one complication: if later lines in the functor block make use of variables defined in previous lines, the `.map` operations may need to create some intermediate tuples that are not present in the functor block syntax. Consider the code

```
val result: L[B] = for {
  x <- p           // The first line must contain a left arrow.
  y = f(x)         // Some computation involving 'x'.
  z = g(x, y)      // Another computation, uses 'x' and 'y'.
  ...
} yield q(x, y, z) // The 'yield' may use 'x', 'y', 'z', and any other defined variables.
```

The above functor block code assumes that `q(x, y, z)` has type `B`, and is equivalent to

```
val result: L[B] = p
  .map { x => (x, f(x)) } // Create a tuple because we need to keep 'x'.
  .map { case (x, y) => (x, y, g(x, y)) } // Need to keep 'x' and 'y'.
  ...
  .map { case (x, y, z) => q(x, y, z) } // Need to keep 'x', 'y', 'z'.
```

This code creates intermediate tuples only because the values `x`, `y`, `z` need to be used in later calculations. The functor block code is easier to read, write, and modify.

If desired, functor blocks may be written in a single line, by using semicolons to separate the individual steps:

```
scala> for { x <- List(1, 2, 3); y = x * x; z = y + 2 } yield z
res0: List[Int] = List(3, 6, 11)
```

A confusing feature of the `for` / `yield` syntax is that, at first sight, functor blocks (such as the code shown at left) appear to compute (or to “yield”) the value `expr(x)`. However, this is not so. As the above examples show, if `p` is a sequence then the functor block also computes a *sequence*. In general, the result value of a functor block is a “wrapped” value, where the type of the “wrapper” is determined by the first line of the functor block. The first line must have a left arrow followed by an expression of a functor type, i.e. of type `L[A]` for some functor `L[_]`. The result type will be `L[B]` where `B` is the type of the expression after the `yield` keyword.

For instance, the first line of the following functor block contains an `Option` type constructor, so the value of the entire expression will also be wrapped in an `Option` type constructor:

```
scala> for {
  x <- Some(123)
  y = (x - 3) / 10
} yield { if (y > 0) s"Have $y" else "Error" }
res1: Option[String] = Some(Have 12)
```

In this code, the `yield` keyword is followed by an expression of type `String`. So, the result of the entire functor block is of type `Option[String]`. (The expression following `yield` can be a large block containing new `vals` and/or other `for` / `yield` functor blocks if needed.)

Functor blocks can be used with any functor that has an infix `.map` method, not only with library-defined type constructors such as `Seq` or `Option`. Here are some examples of defining the `.map` methods and using functor blocks with disjunctive types.

The type constructor `QueryResult[_]` may define the `.map` method on the trait itself and split its implementation between the case classes like this:

```
sealed trait QueryResult[A] {
  def map[B](f: A => B): QueryResult[B] // No implementation here.
}
case class Error[A](message: String) extends QueryResult[A] {
  def map[B](f: A => B): QueryResult[B] = Error(message)
}
case class Success[A](name: String, time: Long, data: A) extends QueryResult[A] {
  def map[B](f: A => B): QueryResult[B] = Success(name, time, f(data))
}
```

After these definitions, we can use `QueryResult` in functor blocks:

```
val q: QueryResult[Int] = Success("addresses", 123456L, 10)
scala> val result = for {
  x <- q
  y = x + 2
} yield s"$y addresses instead of $x"
result: QueryResult[String] = Success(addresses,123456,12 addresses instead of 10)
```

As another example, let us define the `.map` method on the `L0` trait (a recursive disjunctive type):

```
sealed trait L0[A] {
  def map[B](f: A => B): L0[B]
}
final case class L01[A](x: A) extends L0[A] {
  def map[B](f: A => B): L0[B] = L01[B](f(x))
}
final case class L02[A](x: A, y: A, tail: L0[A]) extends L0[A] {
  def map[B](f: A => B): L0[B] = L02[B](f(x), f(y), tail.map(f))
}
```

After these definitions, we may use values of type `L0[_]` in functor blocks:

```
scala> val result = for {
  x <- L02("a", "quick", L02("brown", "fox", L01("jumped")))
  y = x.capitalize
  z = y + "/"
} yield (z, z.length)
result: L0[(String, Int)] = L02((A/,2),(Quick/,6),L02((Brown/,6),(Fox/,4),L01((Jumped/,7))))
```

Functor blocks and functor laws There is an important connection between the functor laws and the properties of code in functor blocks. Consider the following code,

```
def f(x: Int) = x * x    // Some computations.
def g(x: Int) = x - 1    // More computations.

scala> for {
  y <- List(10, 20, 30)
  x = y
  z = f(x)    // Perform computations.
} yield g(z)
res0: List[Int] = List(99, 399, 899)
```

The code says that `x = y`, so it appears reasonable to eliminate `y` and simplify this code into

```
scala> for {
  x <- List(10, 20, 30)    // Eliminated 'y' from the code.
  z = f(x)    // Perform computations.
} yield g(z)
res1: List[Int] = List(99, 399, 899)
```

Another example of refactoring that appears reasonable is to combine transformations:

```
scala> for {
  x <- List(10, 20, 30)
  y = x + 1
  z = f(y)    // Perform computations.
} yield g(z)
res2: List[Int] = List(120, 440, 960)
```

The code says that `y = x + 1`, so we may want to replace `f(y)` by `f(x + 1)`:

```
scala> for {
  x <- List(10, 20, 30)
  z = f(x + 1)    // Eliminated 'y' from the code.
} yield g(z)
res3: List[Int] = List(120, 440, 960)
```

Looking at these code changes, we expect that the computed results will remain the same. Indeed,

Functor block syntax	Chains of <code>.map</code> methods
<pre>for { // Code fragment 1a. y <- List(10, 20, 30) x = y z = f(x) } yield g(z)</pre>	<pre>List(10, 20, 30) // Code fragment 1b. .map(y => y) .map(x => f(x)) .map(z => g(z))</pre>
<pre>for { // Code fragment 2a. x <- List(10, 20, 30) z = f(x) } yield g(z)</pre>	<pre>List(10, 20, 30) // Code fragment 2b. .map(x => f(x)) .map(z => g(z))</pre>
<pre>for { // Code fragment 3a. x <- List(10, 20, 30) y = x + 1 z = f(y) } yield g(z)</pre>	<pre>List(10, 20, 30) // Code fragment 3b. .map(x => x + 1) .map(y => f(y)) .map(z => g(z))</pre>
<pre>for { // Code fragment 4a. x <- List(10, 20, 30) z = f(x + 1) } yield g(z)</pre>	<pre>List(10, 20, 30) // Code fragment 4b. .map(x => f(x + 1)) .map(z => g(z))</pre>

Table 6.1: Example translations of functor blocks into `.map` methods.

when the code directly states that $x = y$, it would be confusing and counter-intuitive if the result value changed after replacing y by x . When the code says that $y = x + 1$, ordinary mathematical reasoning suggests that $f(y)$ can be replaced by $f(x + 1)$ without affecting the results.

To see the connection with the functor laws, we translate the functor block syntax into `.map` expressions; the resulting code fragments are summarized in Table 6.1.

We find that code fragments 1b and 2b are equal if `.map(y => y)` does not modify the list to which it applies. This holds if the `.map` method obeys the functor identity law, `p.map(identity) == p`, for all `p` of the appropriate type. We also find that code fragments 3b and 4b are equal if we can replace `.map(x => x + 1).map(f)` by `.map(x => f(x + 1))`. This replacement is justified as long as the `.map` method obeys the functor composition law,

```
p.map(h).map(f) == p.map(x => f(h(x)))
```

for all `p` and functions `h` and `f` of appropriate types.

Functor laws guarantee that we can correctly understand and modify code written in functor blocks, reasoning about transformations of values as we do in mathematics.

6.1.6 Examples of non-functors

What properties of a data type make it a functor? To build an intuition, it is helpful to see examples of data types that are *not* functors.

There are several possibilities for a type constructor to fail being a functor:

- A `map` function's type signature cannot be implemented at all.
- A `map` function can be implemented but cannot satisfy the functor laws.
- A given `map` function is incorrect (does not satisfy the laws), although the error could be fixed: a different implementation of `map` satisfies the laws.
- A given `map[A, B]` function satisfies the laws for most types `A` and `B`, but violates the laws for certain specially chosen types.

We will now look at examples illustrating these possibilities.

Cannot implement map 's type signature Consider the type constructor C^\bullet defined by

$$C^A \triangleq A \Rightarrow \text{Int} \quad .$$

Scala code for this type notation can be

```
final case class C[A](r: A => Int)
```

The data type `C[A]` does not wrap data of type `A`; instead, it is a function that *consumes* data of type `A`. One cannot implement a fully parametric `map` function with the required type signature

$$\text{map}^{A,B} : (A \Rightarrow \text{Int}) \Rightarrow (A \Rightarrow B) \Rightarrow (B \Rightarrow \text{Int}) \quad .$$

To see this, recall that a fully parametric function needs to treat all types as type parameters, including the primitive type `Int`. So the code

```
def map[A, B]: C[A] => (A => B) => C[B] = { r => f => C(_ => 123) }
```

satisfies the type signature of `map` but is not fully parametric because it returns a specific value `123` of type `Int`, which is not allowed. Replacing the type `Int` by a new type parameter `N`, we obtain the type signature

$$\text{map}^{A,B,N} : (A \Rightarrow N) \Rightarrow (A \Rightarrow B) \Rightarrow B \Rightarrow N \quad .$$

We have seen in Example 5.4.1.8 that this type signature is not implementable. So, the type constructor `C` is not a functor.

Another important example of type constructors where the `map`'s type signature cannot be implemented are certain kinds of **generalized algebraic data types** (GADTs). In this book, they are called **unfunctors**. An unfunctor is a type constructor that has specially constructed values when its type parameter is set only to certain specific types. An example of an unfunctor defined in Scala is

```
sealed trait ServerAction[R]
final case class GetResult[R](r: String => R) extends ServerAction[R]
final case class StoreId(x: Long, y: String) extends ServerAction[Boolean]
final case class StoreName(name: String) extends ServerAction[Unit]
```

We see that some parts of the disjunctive type `ServerAction[R]` do not carry the type parameter `R` but instead set `R` to a specific type, such as `R=Boolean` or `R=Unit`. As a consequence, the case class `StoreName` has no type parameters and can only represent values of type `ServerAction[Unit]` but not, say, `ServerAction[Int]`. To implement a fully parametric function with type signature

```
def map[A, B]: ServerAction[A] => (A => B) => ServerAction[B]
```

we are required to support any choice of the type parameters `A` and `B`. For example, with `A=Unit`, we must be able to transform a `StoreName("abc")` from its fixed type `ServerAction[Unit]` to a value of type `ServerAction[B]` with a given type parameter `B`. However, the only way of creating a value of type `ServerAction[B]` with an arbitrary type `B` is to use the `GetResult[B]` case class, which requires us to have a function of type `String => B`. It is impossible for us to produce such a function because the type `B` is unknown and no values of type `B` are given.

We are prevented from implementing `map` because some type parameters are already set in the definition of `ServerAction[R]`. One can say that the unfunctor `ServerAction[_]` fails to be fully parametric in its type definition. This behavior of unfunctors is typical, and unfunctors are only used in situations where the lack of type parametricity does not lead to problems (see Chapter 12).

Cannot implement a lawful map An example of a non-functor of the second kind is

$$Q^A \triangleq (A \Rightarrow \text{Int}) \times A \quad .$$

Scala code for this type constructor is

```
final case class Q[A](q: A => Int, a: A)
```

A fully parametric `map` function with the correct type signature *can* be implemented (and there is only one such implementation):

$$\text{map}^{A,B} \triangleq q^{A \Rightarrow \text{Int}} \times a^A \Rightarrow f^{A \Rightarrow B} \Rightarrow (_ \Rightarrow q(a))^{B \Rightarrow \text{Int}} \times f(a) \quad .$$

The corresponding Scala code is

```
def map[A, B]: Q[A] => (A => B) => Q[B] = { qa => f =>
  Q[B](_ => qa.q(qa.a), f(qa.a))
}
```

This `map` function is fully parametric (since it treats the type `Int` as a type parameter) and has the right type signature, but the functor laws do not hold. To see the violation of the identity law, we consider an arbitrary value $q^{A \Rightarrow \text{Int}} \times a^A$ and compute:

$$\begin{aligned} \text{expect to equal } q \times a : \quad & \text{map}(q \times a)(\text{id}) \\ \text{definition of map} : \quad & = (_ \Rightarrow q(a)) \times \underline{\text{id}(a)} \\ \text{definition of id} : \quad & = (_ \Rightarrow q(a)) \times a \\ \text{function expansion, } q = (x \Rightarrow q(x)) : \quad & \neq q \times a = (x \Rightarrow q(x)) \times a \quad . \end{aligned}$$

The law must hold for arbitrary functions $q^{A \Rightarrow \text{Int}}$, but the function $(_ \Rightarrow q(a))$ always returns the same value $q(a)$, and thus is not equal to the original function q . Thus, the result of evaluating the expression $\text{map}(q \times a)(\text{id})$ is not always equal to the original value $q \times a$.

Since this `map` function is the only available implementation of the required type signature, we conclude that Q^\bullet is not a functor (we cannot implement `map` that satisfies the laws).

Mistakes in implementing `map` Non-functors of the third kind are type constructors with an incorrectly implemented `map`. An example is a type constructor $P^A \triangleq A \times A$ with the `map` function

$$\text{map} \triangleq x^A \times y^A \Rightarrow f^{A \Rightarrow B} \Rightarrow f(y) \times f(x) \quad .$$

Here is the Scala code corresponding to this code notation:

```
def map[A, B](p: (A, A))(f: A => B): (B, B) = p match {
  case (x, y) => (f(y), f(x))
}
```

This code swaps the values in the pair (x, y) and thus does not preserve information about the ordering of these values. We find that the functor identity law does not hold:

$$\begin{aligned} \text{expect to equal } x \times y : \quad & \text{map}(x^A \times y^A)(\text{id}^A) \\ \text{definition of map} : \quad & = \underline{\text{id}(y)} \times \underline{\text{id}(x)} \\ \text{definition of id} : \quad & = y \times x \neq x \times y \quad . \end{aligned}$$

We should not have swapped the values in the pair. The correct implementation of `map`,

$$\text{map} \triangleq x^A \times y^A \Rightarrow f^{A \Rightarrow B} \Rightarrow f(x) \times f(y) \quad ,$$

preserves information and satisfies the functor laws.

Example 6.1.4.2 shows the type constructor Vec_3^\bullet with an incorrect implementation of `map` that reorders some parts of a tuple and duplicates other parts. The correct implementation preserves the order of parts and does not duplicate or omit any parts.

Another case of an incorrect implementation is the following `map` function for `Option[_]`:

```
def map_bad[A, B]: Option[A] => (A => B) => Option[B] = { _ => _ => None }
```

This function always returns `None`, losing information and violating the identity law. However, we have already seen that `Option[_]` has a different implementation of `map` that satisfies the functor laws.

Similarly, one could define `map` for the `List[_]` type constructor to always return an empty list:

```
def map_bad[A, B]: List[A] => (A => B) => List[B] = { _ => _ => List() }
```

This implementation loses information and violates the functor laws. Of course, the Scala library provides a correct implementation of `map` for `List[_]`.

Example 6.1.4.1 is another situation where an incorrectly implemented `map` violates functor laws.

Functor laws will also be violated when `map` is not fully parametric. For instance, consider an implementation of `fmap[A, B](f)` that checks whether the two type parameters *A* and *B* are equal to each other *as types*, and if so, applies the function argument *f* twice. We need to use special features of Scala for comparing two type parameters as types:

```
import scala.reflect.runtime.universe._
def getType[T: TypeTag]: Type = weakTypeOf[T]
def equalTypes[A: TypeTag, B: TypeTag]: Boolean = getType[A] == getType[B]

def fmap_bad[A: TypeTag, B: TypeTag](f: A => B)(oa: Option[A]): Option[B] = oa match {
  case None => None
  case Some(x) => // If A = B, compute f(f(x)), else compute f(x).
    val z: B = if (equalTypes[A, B]) f(f(x).asInstanceOf[A]) else f(x)
    Some(z)
}
```

Testing shows that this function works as designed:

```
scala> fmap_bad[Int, String](_ + " a")(Some(123)) // Appends " a" once.
res0: Option[String] = Some(123 a)

scala> fmap_bad[String, String](_ + " a")(Some("123")) // Appends " a" twice.
res1: Option[String] = Some(123 a a)
```

The function `fmap_bad[A, B]` satisfies the identity law but violates the composition law when *A=B*:

```
scala> fmap_bad[String, String](_ + " b")(Some("123 a a"))
res2: Option[String] = Some(123 a a b b)

scala> fmap_bad[String, String](_ + " a b")(Some("123"))
res3: Option[String] = Some(123 a b a b)
```

In all these examples, we can implement a different `map` function that satisfies the functor laws. Strictly speaking, a functor is a type constructor *together with* a `map` function. So, it is not precise to say that a type constructor such as `Vec3[_]` is by itself a functor: being a functor depends on the given `map` function. To be precise, we need to add a condition that a correct implementation of `map` *can* be found. In this sense, we say that e.g. the type constructors `Option[_]` and `List[_]` “are” functors.

Laws hold for some types but not for others The Scala standard library contains `map` methods for the type constructors `Set` (transforming the values in a set) and `Map` (transforming both the keys and values in a dictionary). However, `Set[K]` and `Map[K, V]` fail to be lawful functors with respect to the type parameter *K*. The reason for this failure is complicated. A value of type `Set[K]` represents a set of zero or more values of type *K*, and it is enforced that all values in the set are distinct. So, the correct functionality of `Set` implies that we are able to compare two values of type *K* for equality. A standard way of comparing values for equality is the `equals` method defined in the Scala library:

```
scala> List(1, 2, 3).equals(List(1, 2, 3))
res0: Boolean = true

scala> List(1, 2, 3).equals(List(1, 2, 3, 4))
res1: Boolean = false
```

However, the `.equals` operation must obey the mathematical laws of reflexivity and transitivity that may or may not hold for the `.equals` method of a given type *K*. In most practical applications, the specific type *K* (such as `String` or `Int`) will have a mathematically correct `.equals` method.

Here is an artificial example¹ of a type whose `.equals` method violates the required properties:

```
final case class Weird(n: Int) {
  override def equals(x: Any): Boolean = x match {
    case w: Weird => n == 1 && w.n == 1 // Only equal if n == 1.
    case _        => false
  }
}
```

This implementation of `equals` is mathematically invalid: it violates the reflexivity law ($\forall x. x = x$) because values of the form `Weird(n)` with $n \neq 1$ are never equal:

```
scala> Weird(2) equals Weird(2)
res2: Boolean = false
```

As a result, values of type `Set[Weird]` will violate the composition law of functors if we create a `Set[Weird]` containing several values of `Weird(2)`, for instance. The implementation of `.map` on `Set` fails to detect that the five values `Weird(2)` are equal:

```
val f: Int => Weird = { a => Weird(a % 2) }
val g: Weird => Weird = { w => if(w.n == 1) Weird(2) else w }
val ws: Set[Int] = (1 to 10).toSet

scala> ws.map(f andThen g).toList.map(_._n)
res3: List[Int] = List(0, 0, 0, 0, 0, 2, 2, 2, 2, 2)

scala> ws.map(f).map(g).toList.map(_._n)
res4: List[Int] = List(0, 0, 0, 0, 0, 2)
```

The functor laws for a type constructor L^\bullet do not assume that the types A, B used in the function

$$\text{fmap}_L : (A \Rightarrow B) \Rightarrow L^A \Rightarrow L^B$$

should have a mathematically valid definition of the `.equals` method (or of any other operation). The `map` operation of a functor L^\bullet must be **lawful**, i.e. must satisfy the functor laws (6.2)–(6.3) for all types A, B . The functor laws must hold even if a type A 's implementations of certain operations violate some other laws. For this reason, `Set[_]` cannot be considered a functor in a rigorous sense.

The `map` method for dictionaries shows similar behavior because the “keys” of a dictionary must be distinct and are compared using the `.equals` method. In other words, a dictionary `Map[K, V]` behaves as a set with respect to the type κ of the dictionary's “keys” and thus is not a rigorous functor with respect to the type parameter κ .

The Scala library still provides the `.map`, `.flatMap`, and other methods for sets `Set[K]` and dictionaries `Map[K, V]` because in most applications the type κ will have a correctly defined `.equals` method, and so it is likely that we will not see any violations of the functor laws.

6.1.7 Contrafunctors

As we have seen in Section 6.1.6, the type constructor C^\bullet defined by

$$C^A \triangleq A \Rightarrow \text{Int}$$

is not a functor because it is impossible to implement the type signature of `map` for it,

$$\text{map}^{A,B} : (A \Rightarrow \text{Int}) \Rightarrow (A \Rightarrow B) \Rightarrow B \Rightarrow \text{Int} \quad .$$

Consider the code

$$\text{map}(c^{A \Rightarrow \text{Int}})(f^{A \Rightarrow B})(b^B) = ???^{?} \text{Int} \quad .$$

Since $c^{A \Rightarrow \text{Int}}$ consumes (rather than wraps) values of type A , no values of type A are given, and so we cannot apply the function $f^{A \Rightarrow B}$. However, it is possible to apply a function of type $B \Rightarrow A$ since

¹The following example was given in a comment by Paweł Szulc at <https://gist.github.com/tpolecat/7401433>

a value of type B is given as one of the curried arguments, b^B . So, we can implement a function called `contramap` with a different type signature where the function type is $B \Rightarrow A$ instead of $A \Rightarrow B$:

$$\text{contramap}^{A,B} : (A \Rightarrow \text{Int}) \Rightarrow (B \Rightarrow A) \Rightarrow B \Rightarrow \text{Int} \quad .$$

The implementation of this function is written in the code notation as

$$\text{contramap} \triangleq c^{A \Rightarrow \text{Int}} \Rightarrow f^{B \Rightarrow A} \Rightarrow (f \circ c)^{B \Rightarrow \text{Int}} \quad ,$$

and the corresponding Scala code is

```
def contramap[A, B](ca: C[A])(f: B => A): C[B] = { f andThen ca }
```

Flipping the order of the curried arguments in `contramap`, we define `contrafmap` as

$$\begin{aligned} \text{contrafmap}^{A,B} : (B \Rightarrow A) \Rightarrow C^A \Rightarrow C^B \quad , \\ \text{contrafmap} \triangleq f^{B \Rightarrow A} \Rightarrow c^{A \Rightarrow \text{Int}} \Rightarrow (f \circ c)^{B \Rightarrow \text{Int}} \quad . \end{aligned} \quad (6.5)$$

The type signature of `contrafmap` has the form of a “reverse lifting”: functions of type $B \Rightarrow A$ are lifted into the type $C[A] \Rightarrow C[B]$. The Scala code for `contrafmap` is

```
def contrafmap[A, B](f: B => A): C[A] => C[B] = { ca => f andThen ca }
```

We can check that this `contrafmap` satisfies two laws analogous to the functor laws:

$$\begin{aligned} \text{identity law : } & \text{contrafmap}^{A,A}(\text{id}^{A \Rightarrow A}) = \text{id}^{C^A \Rightarrow C^A} \quad , \\ \text{composition law : } & \text{contrafmap}^{A,B}(f^{B \Rightarrow A}) \circ \text{contrafmap}^{B,C}(g^{C \Rightarrow B}) = \text{contrafmap}^{A,C}(g \circ f) \quad . \end{aligned}$$

Since the function argument $f^{B \Rightarrow A}$ has the reverse order of types, the composition law reverses the order of composition ($g \circ f$) on one side; in this way, all types match. To verify the identity law:

$$\begin{aligned} \text{expect to equal id : } & \text{contrafmap}(\text{id}) \\ \text{use Eq. (6.5) : } & = c \Rightarrow (\text{id} \circ c) \\ \text{definition of id : } & = (c \Rightarrow c) = \text{id} \quad . \end{aligned}$$

To verify the composition law:

$$\begin{aligned} \text{expect to equal contrafmap}(g \circ f) : & \text{contrafmap}(f) \circ \text{contrafmap}(g) \\ \text{use Eq. (6.5) : } & = (c \Rightarrow (f \circ c)) \circ (c \Rightarrow (g \circ c)) \\ \text{rename } c \text{ to } d \text{ for clarity : } & = (c \Rightarrow (f \circ c)) \circ (d \Rightarrow (g \circ d)) \\ \text{compute composition : } & = (c \Rightarrow g \circ f \circ c) \\ \text{use Eq. (6.5) : } & = \text{contrafmap}(g \circ f) \quad . \end{aligned}$$

A type constructor with a fully parametric `contramap` is called a **contrafunctor** if the identity and the composition laws are satisfied.

Example 6.1.7.1 Show that the type constructor

$$D^A \triangleq A \Rightarrow A \Rightarrow \text{Int}$$

is a contrafunctor.

Solution The required type signature for `contramap` is

```
def contramap[A, B](d: A => A => Int)(f: B => A): B => B => Int = ???
```

We begin implementing `contramap` by writing code with a typed hole:

$$\text{contramap}^{A,B} \triangleq d:A \Rightarrow A \Rightarrow \text{Int} \Rightarrow f:B \Rightarrow A \Rightarrow b_1^B \Rightarrow b_2^B \Rightarrow ???:\text{Int} \quad .$$

To fill the typed hole, we need to compute a value of type `Int`. The only possibility is to apply d to two curried arguments of type A . We have two curried arguments of type B . So we apply $f:B \Rightarrow A$ to those arguments, obtaining two values of type A . To avoid information loss, we need to preserve the order of the curried arguments. So the resulting expression is

$$\text{contramap}^{A,B} \triangleq d:A \Rightarrow A \Rightarrow \text{Int} \Rightarrow f:B \Rightarrow A \Rightarrow b_1^B \Rightarrow b_2^B \Rightarrow d(f(b_1))(f(b_2)) \quad .$$

The corresponding Scala code is

```
def contramap[A, B](d: A => A => Int)(f: B => A): B => B => Int = {
  b1 => b2 => d(f(b1))(f(b2))
}
```

To verify the laws, it is easier to use the equivalent `contrafmap` defined by

$$\text{contrafmap}^{A,B}(f:B \Rightarrow A) \triangleq d:A \Rightarrow A \Rightarrow \text{Int} \Rightarrow b_1^B \Rightarrow b_2^B \Rightarrow d(f(b_1))(f(b_2)) \quad . \quad (6.6)$$

To verify the identity law:

$$\begin{aligned} \text{expect to equal id : } & \text{contrafmap}(\text{id}) \\ \text{use Eq. (6.6) : } & = d \Rightarrow b_1 \Rightarrow b_2 \Rightarrow \underline{d(\text{id}(b_1))(\text{id}(b_2))} \\ \text{definition of id : } & = d \Rightarrow \underline{b_1 \Rightarrow b_2 \Rightarrow d(b_1)(b_2)} \\ \text{simplify curried function : } & = (d \Rightarrow d) = \text{id} \quad . \end{aligned}$$

To verify the composition law, we rewrite its left-hand side into the right-hand side:

$$\begin{aligned} & \text{contrafmap}(f) \circ \text{contrafmap}(g) \\ \text{use Eq. (6.6) : } & = (d \Rightarrow b_1 \Rightarrow b_2 \Rightarrow d(f(b_1))(f(b_2))) \circ (\underline{d} \Rightarrow b_1 \Rightarrow b_2 \Rightarrow \underline{d}(g(b_1))(g(b_2))) \\ \text{rename } d \text{ to } e : & = (d \Rightarrow b_1 \Rightarrow b_2 \Rightarrow d(f(b_1))(f(b_2))) \circ (e \Rightarrow b_1 \Rightarrow b_2 \Rightarrow e(g(b_1))(g(b_2))) \\ \text{compute composition : } & = d \Rightarrow b_1 \Rightarrow b_2 \Rightarrow d(f(g(b_1)))(f(g(b_2))) \\ \text{use Eq. (6.6) : } & = \text{contrafmap}(b \Rightarrow f(g(b))) \\ \text{definition of } (g \circ f) : & = \text{contrafmap}(g \circ f) \quad . \end{aligned}$$

The type C^A represents a function that consumes a value of type A to produce an integer; the type D^A represents a curried function consuming *two* values of type A . These examples suggest the heuristic view that contrafunctors “consume” data while functors “wrap” data. By looking at the position of a given type parameter in a type expression such as $A \times \text{Int}$ or $A \Rightarrow A \Rightarrow \text{Int}$, we can see whether the type parameter is “consumed” or “wrapped”: A type parameter to the left of a function arrow is being “consumed”; a type parameter to the right of a function arrow (or used without a function arrow) is being “wrapped”. We will make this intuition precise in Section 6.2.

Type constructors that are not contrafunctors A type constructor that both consumes *and* wraps data is neither a functor nor a contrafunctor. An example of such a type constructor is

$$N^A \triangleq (A \Rightarrow \text{Int}) \times (\mathbb{1} + A) \quad .$$

We can implement neither `map` nor `contramap` for N^\bullet . Intuitively, the type parameter A is used both to the left of a function arrow (being “consumed”) and outside of a function (being “wrapped”).

Unfunctors (type constructors that violate parametricity) also cannot be contrafunctors because the required type signature for `contramap` cannot be implemented by a fully parametric function. To show that `ServerAction[_]` cannot be a contrafunctor, we can straightforwardly adapt the reasoning used in Section 6.1.6 for showing that `ServerAction[_]` cannot be a functor.

6.1.8 Subtyping, covariance, and contravariance

A type P is called a **subtype** of a type Q if there exists a designated **type conversion** function of type $P \Rightarrow Q$ that the compiler will automatically use whenever necessary to match types. For instance, applying a function of type $Q \Rightarrow Z$ to a value of type P is ordinarily a type error,

```
val h: Q => Z = ???
val p: P = ???
h(p) // Type error: the argument of h must be of type Q, not P.
```

However, this code will work when P is a subtype of Q because the compiler will automatically use the type conversion $P \Rightarrow Q$ before applying the function h .

Different programming languages define subtyping differently because they make different choices of the type conversion functions and of types P, Q to which type conversions apply. Most frequently, the language designers choose the type conversion functions to be *identity* functions that merely reassign the types. Let us look at some examples of such type conversion functions.

Within the focus of this book, the main example of subtyping is with disjunctive types. Consider this definition,

```
sealed trait AtMostTwo
final case class Zero() extends AtMostTwo
final case class One(x: Int) extends AtMostTwo
final case class Two(x: Int, y: Int) extends AtMostTwo
```

The corresponding type notation can be written as

$$\text{AtMostTwo} \triangleq \mathbb{1} + \text{Int} + \text{Int} \times \text{Int} \quad .$$

Each of the case classes (`Zero`, `One`, and `Two`) defines a type that is a subtype of `AtMostTwo`. To see that, we need to implement type conversion functions from each of the three case classes to `AtMostTwo`. The required functions reassign the types but perform no transformations on the data:

```
def f0: Zero => AtMostTwo = { case Zero() => Zero() }
def f1: One => AtMostTwo = { case One(x) => One(x) }
def f2: Two => AtMostTwo = { case Two(x, y) => Two(x, y) }
```

The implementation of these type conversion functions looks like the code of *identity* functions. In the matrix notation, we can write

$$\begin{array}{l} f_0 \triangleq \begin{array}{c|ccc} & \text{Zero} & \text{One} & \text{Two} \\ \hline \text{Zero} & \text{id} & 0 & 0 \end{array} , \quad f_0(\mathbb{1}^{\text{Zero}}) \triangleq \mathbb{1} + 0^{\text{One}} + 0^{\text{Two}} , \\ f_1 \triangleq \begin{array}{c|ccc} & \text{Zero} & \text{One} & \text{Two} \\ \hline \text{One} & 0 & \text{id} & 0 \end{array} , \quad f_1(x^{\text{Int}}) \triangleq 0^{\text{Zero}} + x^{\text{One}} + 0^{\text{Two}} , \\ f_2 \triangleq \begin{array}{c|ccc} & \text{Zero} & \text{One} & \text{Two} \\ \hline \text{Two} & 0 & 0 & \text{id} \end{array} , \quad f_2(x^{\text{Int}} \times y^{\text{Int}}) \triangleq 0^{\text{Zero}} + 0^{\text{One}} + (x \times y)^{\text{Two}} . \end{array}$$

This notation emphasizes that the code consists of identity functions with reassigned types.

Another example is a subtyping relation between function types. Consider the types

```
type P = (AtMostTwo => Int)
type Q = (Two => Int)
```

We can convert a function f of type P into a function g of type Q because f includes all the information necessary to define g . The Scala code for the type conversion is

```
def p2q(f: P): Q = { t: Two => f(t) }
```

This is written in the code notation as

$$\text{p2q}(f:\text{AtMostTwo} \Rightarrow \text{Int}) \triangleq t:\text{Two} \Rightarrow f(t) \quad .$$

Note that $t:\text{Two} \Rightarrow f(t)$ is the same function as f , except applied to a subtype `Two` of `AtMostTwo`. So, the implementation of `p2q` is an identity function with reassigned types.

In these cases, it is useful if the compiler could insert the appropriate conversion functions automatically whenever necessary. Any function that consumes an argument of type Q could be then automatically applicable to an argument of type P . The compiler could also remove the identity functions from the code, since they do not perform any data transformations. In this way, code involving subtypes becomes more concise with no decrease in performance.

To achieve this, we need to declare to the Scala compiler that certain types have a subtyping relation involving an identity function as the type conversion. This can be done in one of three ways depending on the situation at hand:

1. Declaring a class that `extends` another class (as we have just seen).
2. Declaring type parameters with a “variance annotation” such as `L[+A]` or `L[-B]`.
3. Declaring type parameters with a “subtyping annotation” (`A <: B`).

Subtyping for disjunctive types A function with argument of type `AtMostTwo` can be applied to a value of type `Two` with no additional code written by the programmer:

```
def head: AtMostTwo => Option[Int] = {
  case Zero()      => None
  case One(x)       => Some(x)
  case Two(x, y)    => Some(x)
}

scala> head(Two(10, 20))
res0: Option[Int] = Some(10)
```

We may imagine that the compiler automatically used the type conversion function `f2` shown above to convert a value of the type `Two` into a value of the type `AtMostTwo`. Since the code of `f2` is equivalent to an identity function, the type conversion does not change any data and only reassigns the types of the given values. So the compiler does not need to insert any additional code. Type conversion does not lead to any performance cost here.

Subtyping for type constructors If a type constructor L^A is a functor, we can use its `fmapL` method to lift a type conversion function $f : P \Rightarrow Q$ into

$$\text{fmap}_L(f) : L^P \Rightarrow L^Q \quad ,$$

which gives a type conversion function from L^P to L^Q . This gives a subtyping relation between the types L^P and L^Q because the code of the lifted function `fmapL(f)` is an identity function, due to functor L ’s identity law, `fmapL(id) = id`.

If a type constructor C^A is a contrafunctor, a type conversion function $f:P \Rightarrow Q$ is lifted to

$$\text{contrafmap}_C(f) : C^Q \Rightarrow C^P \quad ,$$

showing that C^Q is a subtype of C^P . The identity law of the contrafunctor C ,

$$\text{contrafmap}_C(\text{id}) = \text{id} \quad ,$$

shows that the lifted conversion function is an identity function with reassigned types.

A type constructor F is called **covariant** if F^A is a subtype of F^B whenever A is a subtype of B . A **contravariant** type constructor C has the subtype relation in the opposite direction: C^B is a subtype of C^A . So, we have established that all functors are covariant type constructors, and all contrafunctors are contravariant type constructors.²

The Scala compiler does not automatically determine whether a given type constructor `F[A]` is covariant with respect to a given type parameter `A`. To indicate the covariance property, the programmer needs to use a **variance annotation**, which looks like `F[+A]`, on the relevant type parameters. For

²The name “contrafunctor” was chosen in this book as a shortened form of “contravariant functor”.

example, the type constructor `Counted[A]` defined in Section 6.1.4 is a functor and so is covariant in its type parameter `A`. If we use the variance annotation `Counted[+A]` in the definition, Scala will automatically consider the type `Counted[Two]` as a subtype of `Counted[AtMostTwo]`. So we may now apply a function to a value of type `Counted[Two]` as if it had type `Counted[AtMostTwo]`:

```
final case class Counted[+A](n: Int, a: A)

def total(c: Counted[AtMostTwo]): Int = c match {
  case Counted(n, Zero())      => 0
  case Counted(n, One(_))      => n
  case Counted(n, Two(_, _))   => n * 2
}

scala> total(Counted(2, Two(10, 20)))
res1: Int = 4
```

The contravariance property for contrafunctors can be annotated using the syntax `F[-A]`.

A given type constructor may have several type parameters and may be covariant with respect to some of them and contravariant with respect to others. As we have seen, the position of a type parameter in a type expression indicates whether the value is “wrapped” (used in a **covariant position**) or “consumed” (used in a **contravariant position**). Covariant positions are to the right of function arrows, or outside function arrows; contravariant positions are to the left of a function arrow. The next examples confirm this intuition, which will be made rigorous in Section 6.2.

6.1.9 Solved examples: functors and contrafunctors

Example 6.1.9.1 Consider this implementation of `map` for the type constructor `Option[_]`:

```
def map[A, B](oa: Option[A])(f: A => B): Option[B] = oa match {
  case None      => None
  case Some(x: Int) => Some(f((x+1).asInstanceOf[A]))
  case Some(x)    => Some(f(x))
}
```

This code performs a non-standard computation if the type parameter `A` is set to `Int`. Show that this implementation of `map` violates the functor laws.

Solution If the type parameter `A` is not `Int`, or if the argument `oa` is `None`, the given code is the same as the standard (correct) implementation of `map` for `Option`. The function does something non-standard when e.g. `oa == Some(123)`. Substitute this value of `oa` into the identity law, `map(oa)(identity) == oa`, and compute symbolically (using Scala syntax)

```
map(oa)(identity) == Some(identity((123+1).asInstanceOf[Int])) == Some(124) != oa
```

This shows a violation of the functor identity law.

Example 6.1.9.2 Define case classes and implement `fmap` for the given type constructors:

- (a) $\text{Data}^A \triangleq \text{String} + A \times \text{Int} + A \times A \times A$.
- (b) $\text{Data}^A \triangleq \mathbb{1} + A \times (\text{Int} \times \text{String} + A)$.
- (c) $\text{Data}^A \triangleq (\text{String} \Rightarrow \text{Int} \Rightarrow A) \times A + (\text{Bool} \Rightarrow \text{Double} \Rightarrow A) \times A$.

Solution (a) Begin by defining a case class for each part of the disjunctive type:

```
sealed trait Data[A]
final case class Message[A](message: String) extends Data[A]
final case class Have1[A](x: A, n: Int) extends Data[A]
final case class Have3[A](x: A, y: A, z: A) extends Data[A]
```

The names `Message`, `Have1`, `Have3`, `n`, `x`, `y`, `z` are chosen arbitrarily.

The function `fmap` must have the type signature

$$\text{fmap}^{A,B} : f^{A \Rightarrow B} \Rightarrow \text{Data}^A \Rightarrow \text{Data}^B \quad .$$

To implement `fmap` correctly, we need to transform each part of the disjunctive type `Data[A]` into the corresponding part of `Data[B]` without loss of information. To clarify where the transformation $f^{A \Rightarrow B}$

need to be applied, let us write the type notation for Data^A and Data^B side by side:

$$\begin{aligned}\text{Data}^A &\triangleq \text{String} + A \times \text{Int} + A \times A \times A \quad , \\ \text{Data}^B &\triangleq \text{String} + B \times \text{Int} + B \times B \times B \quad .\end{aligned}$$

Now it is clear that we need to apply f to each value of type A present in Data^A , preserving the order of values. The Scala code is

```
def fmap[A, B](f: A => B): Data[A] => Data[B] = {
  case Message(message) => Message(message)
  case Have1(x, n)      => Have1(f(x), n)
  case Have3(x, y, z)   => Have3(f(x), f(y), f(z))
}
```

(b) It is convenient to define the disjunctive type $\text{Int} \times \text{String} + A$ separately as P^A :

```
sealed trait P[A]
final case class Message[A](code: Int, message: String) extends P[A]
final case class Value[A](x: A) extends P[A]
```

Now we notice that the type expression $(\mathbb{1} + \dots)$ can be encoded via the standard `Option` type. So, the Scala code for Data^A is

```
final case class Data[A](d: Option[(A, P[A])])
```

To help us implement `fmap` correctly, we write the type expressions

$$\begin{aligned}\text{Data}^A &\triangleq \mathbb{1} + A \times (\text{Int} \times \text{String} + A) \quad , \\ \text{Data}^B &\triangleq \mathbb{1} + B \times (\text{Int} \times \text{String} + B) \quad ,\end{aligned}$$

and transform Data^A into Data^B by applying $f^{A \Rightarrow B}$ at the correct places:

```
def fmap[A, B](f: A => B): Data[A] => Data[B] = {
  case Data(None) => Data(None)
  case Data(Some((x, Message(code, message)))) => Data(Some((f(x), Message(code, message))))
  case Data(Some((x, Value(y)))) => Data(Some((f(x), Value(f(y)))))
}
```

When deeply nested patterns become hard to read, we may handle the nested structure separately:

```
def fmap[A, B](f: A => B): Data[A] => Data[B] = {
  case Data(None) => Data(None)
  case Data(Some((x, p))) =>
    val newP: P[B] = p match {
      case Message(code, message) => Message(code, message)
      case Value(x) => Value(f(x))
    }
    Data(Some((f(x), newP)))
}
```

(c) Since the type structures $(\text{String} \Rightarrow \text{Int} \Rightarrow A) \times A$ and $(\text{Bool} \Rightarrow \text{Double} \Rightarrow A) \times A$ have a similar pattern, let us define a parameterized type

$$Q^{X,Y,A} \triangleq (X \Rightarrow Y \Rightarrow A) \times A \quad ,$$

and express the given type expression as

$$\text{Data}^A \triangleq Q^{\text{String}, \text{Int}, A} + Q^{\text{Bool}, \text{Double}, A} \quad .$$

It is then convenient to define `Data[A]` using the standard disjunctive type `Either`:

```
type Q[X, Y, A] = (X => Y => A, A)
type Data[A] = Either[Q[String, Int, A], Q[Boolean, Double, A]]
```

To make the code clearer, we will implement `fmap` separately for Q^\bullet and $Data^\bullet$.

To derive the code of `fmap` for Q^\bullet , we begin with the type signature

$$\text{fmap}_Q^{A,B} : (A \Rightarrow B) \Rightarrow (X \Rightarrow Y \Rightarrow A) \times A \Rightarrow (X \Rightarrow Y \Rightarrow B) \times B$$

and start writing the code using typed holes,

$$\text{fmap}_Q(f^{A \Rightarrow B}) \triangleq g^{X \Rightarrow Y \Rightarrow A} \times a^{A \Rightarrow ???^{X \Rightarrow Y \Rightarrow B}} \times ???^{B} \quad .$$

The typed hole $???^B$ is filled by $f(a)$. To fill the remaining type hole, we write

$$\begin{aligned} & ???^{X \Rightarrow Y \Rightarrow B} \\ &= x^{X \Rightarrow} y^{Y \Rightarrow} \Rightarrow \underline{???^B} \\ &= x^{X \Rightarrow} y^{Y \Rightarrow} \Rightarrow f(???^A) \quad . \end{aligned}$$

It would be wrong to fill the typed hole $???^A$ by a^A because a value of type $X \Rightarrow Y \Rightarrow B$ should be computed using the given data $g^{X \Rightarrow Y \Rightarrow A}$ of type $X \Rightarrow Y \Rightarrow A$. So we write

$$???^{X \Rightarrow Y \Rightarrow B} = x^{X \Rightarrow} y^{Y \Rightarrow} \Rightarrow f(g(x)(y)) \quad .$$

The corresponding Scala code is

```
def fmap_Q[A, B, X, Y](f: A => B): Q[X, Y, A] => Q[X, Y, B] = {
  case (g, a) => (x => y => f(g(x)(y)), f(a))
  // Could also write the code as
  // case (g, a) => (x => g(x) andThen f, f(a))
}
```

Finally, we can write the code for `fmapData`:

```
def fmap_Data[A, B](f: A => B): Data[A] => Data[B] = {
  case Left(q)    => Left(fmap_Q(f)(q))
  case Right(q)   => Right(fmap_Q(f)(q))
}
```

The Scala compiler will automatically infer the type parameters required by `fmap_Q` and check that all types match. With all inferred types written out, the code above would be

```
def fmap_Data[A, B](f: A => B): Data[A] => Data[B] = {
  case Left(q: Q[String, Int, A])    =>
    Left[Q[String, Int, B]](fmap_Q[A, B, String, Int](f)(q))
  case Right(q: Q[Boolean, Double, A]) =>
    Right[Q[Boolean, Double, B]](fmap_Q[A, B, Boolean, Double](f)(q))
}
```

When types become complicated, it may help to write out some of the type parameters in the code.

Example 6.1.9.3 Decide which of these types are functors or contrafunctors, and implement `fmap` or `contrafmap` as appropriate:

(a) $Data^A \triangleq (A \Rightarrow Int) + (A \Rightarrow A \Rightarrow String)$.

(b) $Data^{A,B} \triangleq (A + B) \times ((A \Rightarrow Int) \Rightarrow B)$.

Solution (a) The type constructor $Data^A$ uses its type parameter A always as an argument of some functions, i.e. to the left of function arrows:

```
type Data[A] = Either[A => Int, A => A => String]
```

So, $Data^A$ consumes values of type A , and we expect that $Data^A$ is a contrafunctor. Indeed, we can implement `contrafmap`:

```
def contrafmap[A, B](f: B => A): Data[A] => Data[B] = {
  case Left(a2Int)    => Left(b => a2Int(f(b)))
  case Right(a2a2String) => Right(b1 => b2 => a2a2String(f(b1))(f(b2)))
}
```

(b) The type constructor $\text{Data}^{A,B}$ has *two* type parameters, and so we need to answer the question separately for each of them. Write the Scala type definition as

```
type Data[A, B] = (Either[A, B], (A => Int) => B)
```

Begin with the type parameter A , and notice that a value of type $\text{Data}^{A,B}$ possibly contains a value of type A within `Either[A, B]`. In other words, A is “wrapped”, i.e. it is in a covariant position within the first part of the tuple. It remains to check the second part of the tuple, which is a higher-order function of type $(A \Rightarrow \text{Int}) \Rightarrow B$. This function consumes a function of type $A \Rightarrow \text{Int}$, which in turn consumes a value of type A . Consumers of A are contravariant in A , but it turns out that a “consumer of a consumer of A ” is covariant in A . So we will be able to implement `fmap` that applies to the type parameter A of $\text{Data}^{A,B}$. Renaming the type parameter B to Z for clarity, we write the type signature for `fmap` like this,

$$\text{fmap}^{A,C,Z} : (A \Rightarrow C) \Rightarrow (A + Z) \times ((A \Rightarrow \text{Int}) \Rightarrow Z) \Rightarrow (C + Z) \times ((C \Rightarrow \text{Int}) \Rightarrow Z) \quad .$$

We need to transform each part of the tuple separately. Transforming $A + Z$ into $C + Z$ is straightforward via the function

$$\begin{array}{c|c|c} & C & Z \\ \hline A & f & 0 \\ \hline Z & 0 & \text{id} \end{array} \quad .$$

The corresponding Scala code is

```
{
  case Left(x)    => Left(f(x))
  case Right(z)   => Right(z)
}
```

To derive code transforming $(A \Rightarrow \text{Int}) \Rightarrow Z$ into $(C \Rightarrow \text{Int}) \Rightarrow Z$, we use typed holes:

$$\begin{aligned} f:A \Rightarrow C &\Rightarrow g:(A \Rightarrow \text{Int}) \Rightarrow Z \Rightarrow \underline{???:(C \Rightarrow \text{Int}) \Rightarrow Z} \\ \text{nameless function :} &= f:A \Rightarrow C \Rightarrow g:(A \Rightarrow \text{Int}) \Rightarrow Z \Rightarrow p:C \Rightarrow \text{Int} \Rightarrow \underline{??? : Z} \\ \text{get a } Z \text{ by applying } g : &= f \Rightarrow g \Rightarrow p \Rightarrow g(\underline{??? : A \Rightarrow \text{Int}}) \\ \text{nameless function :} &= f \Rightarrow g \Rightarrow p \Rightarrow g(a:A \Rightarrow \underline{??? : \text{Int}}) \\ \text{get an } \text{Int} \text{ by applying } p : &= f \Rightarrow g \Rightarrow p \Rightarrow g(a \Rightarrow p(\underline{??? : C})) \\ \text{get a } C \text{ by applying } f : &= f \Rightarrow g \Rightarrow p \Rightarrow g(a \Rightarrow p(f(\underline{??? : A}))) \\ \text{use argument } a:A : &= f \Rightarrow g \Rightarrow p \Rightarrow g(a \Rightarrow p(f(a))) \quad . \end{aligned}$$

In the resulting Scala code for `fmap`, we write out some types for clarity:

```
def fmapA[A, Z, C](f: A => C): Data[A, Z] => Data[C, Z] = {
  case (e: Either[A, Z], g: ((A => Int) => Z)) =>
    val newE: Either[C, Z] = e match {
      case Left(x)    => Left(f(x))
      case Right(z)   => Right(z)
    }
    val newG: (C => Int) => Z = { p => g(a => p(f(a))) }
    (newE, newG) // This has type Data[C, Z].
}
```

This suggests that $\text{Data}^{A,Z}$ is covariant with respect to the type parameter A . The results of Section 6.2 will show rigorously that the functor laws hold for this implementation of `fmap`.

The analysis is simpler for the type parameter B because it is always used in covariant positions, i.e. never to the left of a function arrow. So we expect $\text{Data}^{A,B}$ to be a functor with respect to B . Implementing the corresponding `fmap` is straightforward:


```
def fmapB[Z, B, C](f: B => C): Data[Z, A] => Data[Z, B] = {
  case (e: Either[Z, B], g: ((Z => Int) => B)) =>
    val newE: Either[Z, B] = e match {
      case Left(x)    => Left(f(x))
      case Right(z)   => Right(z)
    }
    val newG: (C => Int) => Z = { p => g(a => p(f(a))) }
    (newE, newG) // This has type Data[C, Z].
}
```

The code indicates that $\text{Data}^{A,B}$ is a functor with respect to both A and B .

Example 6.1.9.4 Rewrite the following code in the type notation; identify covariant and contravariant type usages; verify with the Scala compiler that the variance annotations are correct:

```
sealed trait Coi[A, B]
final case class Pa[A, B](b: (A, B), c: B => Int) extends Coi[A, B]
final case class Re[A, B](d: A, e: B, c: Int) extends Coi[A, B]
final case class Ci[A, B](f: String => A, g: B => A) extends Coi[A, B]
```

Solution The type notation puts together all parts of the disjunctive type:

$$\text{Coi}^{A,B} \triangleq A \times B \times (B \Rightarrow \text{Int}) + A \times B \times \text{Int} + (\text{String} \Rightarrow A) \times (B \Rightarrow A) \quad .$$

Now find which types are wrapped and which are consumed in this type expression. We find that the type parameter A is wrapped and never consumed, but B is both wrapped and consumed (in $B \Rightarrow A$). So, the type constructor coi is covariant in A but neither covariant nor contravariant in B . We can check this by compiling the corresponding Scala code with variance annotations:

```
sealed trait Coi[+A, B]
case class Pa[+A, B](b: (A, B), c: B => Int) extends Coi[A, B]
case class Re[+A, B](d: A, e: B, c: Int) extends Coi[A, B]
case class Ci[+A, B](f: String => A, g: B => A) extends Coi[A, B]
```

We could also replace the fixed types `Int` and `String` by type parameters \mathbb{N} and s . A similar analysis shows that \mathbb{N} is in covariant positions while s is in a contravariant position. We can then check that Scala accepts the following type definition with variance annotations:

```
sealed trait Coi2[+A, B, +N, -S]
case class Pa2[+A, B, +N, -S](b: (A, B), c: B => N) extends Coi[A, B, N, S]
case class Re2[+A, B, +N, -S](d: A, e: B, c: N) extends Coi[A, B, N, S]
case class Ci2[+A, B, +N, -S](f: S => A, g: B => A) extends Coi[A, B, N, S]
```

6.1.10 Exercises: functors and contrafunctors

Exercise 6.1.10.1 An implementation of `fmap` for the type constructor `Either[A, A]` is given as

```
def fmap[A, B](f: A => B): Either[A, A] => Either[B, B] = {
  case Left(a: Int)    => Right(f(a))
  case Left(a)         => Left(f(a))
  case Right(a)        => Right(f(a))
}
```

Show that this implementation of `fmap` violates the functor laws. Implement `fmap` correctly for this type constructor.

Exercise 6.1.10.2 Define these type constructors in Scala, decide whether they are covariant or contravariant, and implement `fmap` or `contrafmap` as appropriate:

- $\text{Data}^A \triangleq (\mathbb{1} + A) \times (\mathbb{1} + A) \times \text{String} \quad .$
- $\text{Data}^A \triangleq (A \Rightarrow \text{Bool}) \Rightarrow (A \times (\text{Int} + A)) \quad .$
- $\text{Data}^{A,B} \triangleq (A \Rightarrow \text{Bool}) \times ((A + B) \Rightarrow \text{Int}) \quad .$
- $\text{Data}^A \triangleq (\mathbb{1} + (A \Rightarrow \text{Bool})) \Rightarrow (\mathbb{1} + (A \Rightarrow \text{Int})) \Rightarrow \text{Int} \quad .$

(e) $\text{Data}^B \triangleq (B + (\text{Int} \Rightarrow B)) \times (B + (\text{String} \Rightarrow B))$.

Exercise 6.1.10.3 Rewrite the following code in the type notation; identify covariant and contravariant type usages; add variance annotations and verify that the resulting code compiles:

```
sealed trait Result[A,B]
final case class P[A,B](a: A, b: B, c: Int) extends Result[A,B]
final case class Q[A,B](d: Int => A, e: Int => B) extends Result[A,B]
final case class R[A,B](f: A => A, g: A => B) extends Result[A,B]
```

6.2 Laws and structure

A type constructor is a functor if it admits a lawful `map` function. How can we recognize quickly that a given type constructor is a functor or perhaps a contrafunctor? For example, consider the type constructor $Z^{A,R}$ defined by

$$Z^{A,R} \triangleq ((A \Rightarrow A \Rightarrow R) \Rightarrow R) \times A + (1 + R \Rightarrow A + \text{Int}) + A \times A \times \text{Int} \times \text{Int} \quad . \quad (6.7)$$

Is $Z^{A,R}$ a functor with respect to A , or perhaps with respect to R ? To answer these questions, we will systematically build up various type expressions for which the functor or contrafunctor laws hold.

6.2.1 Reformulations of laws

We begin by introducing a more convenient notation for the functor laws. The laws (6.2)–(6.3) were defined in terms of the function `fmap`. When written in terms of the curried function `map`, the structure of the laws becomes less clear:

$$\begin{aligned} \text{map}_L(x : L^A)(\text{id} : A \Rightarrow A) &= x \quad , \\ \text{map}_L(x : L^A)(f : A \Rightarrow B ; g : B \Rightarrow C) &= \text{map}_L(\text{map}_L(x)(f))(g) \quad . \end{aligned}$$

However, the laws again look clearer when using the infix method `.map`:

```
x.map(identity) == x
x.map(f).map(g) == x.map(f andThen g)
```

To take advantage of this syntax, we can use the pipe notation where $x \triangleright \text{fmap}(f)$ means `x.map(f)`, and write the functor laws as

$$\begin{aligned} x \triangleright \text{fmap}_L(\text{id}) &= x \quad , \\ x \triangleright \text{fmap}_L(f) \triangleright \text{fmap}_L(g) &= x \triangleright \text{fmap}_L(f ; g) \quad . \end{aligned}$$

In later chapters of this book, we will find that the `.map` methods (equivalently, the `fmap` function) are used so often in different contexts that the notation $\text{fmap}_L(f)$ becomes too verbose. To make code expressions visually easy to manipulate, we need a shorter notation. At the same time, it is important to mark clearly the relevant type constructor L . Dropping the symbol L can lead to errors, since it will be sometimes unclear what type constructors are involved in an expression such as `x.map(f).map(g)` and whether we are justified in replacing that expression with `x.map(f andThen g)`.

For these reasons, we introduce the superscript notation \uparrow^L (pronounced “lifted to L ”) defined, for any function f , by

$$(f : A \Rightarrow B)^{\uparrow^L} : L^A \Rightarrow L^B \quad , \quad f^{\uparrow^L} \triangleq \text{fmap}_L(f) \quad .$$

Now we can write

$$\begin{aligned} x \triangleright \text{fmap}_L(f) &= x \triangleright f^{\uparrow^L} = f^{\uparrow^L}(x) \quad , \\ \text{map}_L(x)(f) &= f^{\uparrow^L}(x) \quad . \end{aligned}$$

In this notation, the identity and composition laws for a functor L are especially easy to use:

$$\text{id}^{\uparrow^L} = \text{id} \quad , \quad (f ; g)^{\uparrow^L} = f^{\uparrow^L} ; g^{\uparrow^L} \quad .$$

Applying a composition of lifted functions to a value looks like this,

$$x \triangleright (f \circ g)^{\uparrow L} = x \triangleright f^{\uparrow L} \circ g^{\uparrow L} = x \triangleright f^{\uparrow L} \triangleright g^{\uparrow L} \quad .$$

This equation directly represents the Scala code syntax

```
x.map(f andThen g) == x.map(f).map(g)
```

since the piping symbol (\triangleright) groups weaker than the composition symbol (\circ).

Written in the *backward* notation ($f \circ g$), the functor composition law is

$$(g \circ f)^{\uparrow L} = g^{\uparrow L} \circ f^{\uparrow L} \quad .$$

The analogous notation for a contrafunctor C^\bullet is

$$f^{\downarrow C} \triangleq \text{contrafmap}_C(f) \quad .$$

The contrafunctor laws are then written as

$$\text{id}^{\downarrow C} = \text{id} \quad , \quad (f \circ g)^{\downarrow C} = g^{\downarrow C} \circ f^{\downarrow C} \quad , \quad (g \circ f)^{\downarrow C} = f^{\downarrow C} \circ g^{\downarrow C} \quad .$$

We will mostly use the forward composition $f \circ g$ in this book, keeping in mind that one can straightforwardly and mechanically translate between forward and backward notations via

$$f \circ g = g \circ f \quad , \quad x \triangleright f = f(x) \quad .$$

6.2.2 Bifunctors

A type constructor can be a functor with respect to several type parameters. A **bifunctor** is a type constructor with *two* type parameters that satisfies the functor laws with respect to both type parameters at once.

As an example, consider the type constructor F defined by

$$F^{A,B} \triangleq A \times B \times B \quad .$$

If we fix the type parameter B but let the parameter A vary, we get a type constructor that we can denote as $F^{\bullet,B}$. We see that the type constructor $F^{\bullet,B}$ is a functor, with the corresponding `fmap` function

$$\text{fmap}_{F^{\bullet,B}}(f^{A \Rightarrow C}) \triangleq a^{:A} \times b_1^{:B} \times b_2^{:B} \Rightarrow f(a) \times b_1 \times b_2 \quad .$$

Instead of saying that $F^{\bullet,B}$ is a functor, we can also say more verbosely that $F^{A,B}$ is a functor with respect to A .

If we now fix the type parameter A , we find that the type constructor $F^{A,\bullet}$ is a functor, with the `fmap` function

$$\text{fmap}_{F^{A,\bullet}}(g^{B \Rightarrow D}) \triangleq a^{:A} \times b_1^{:B} \times b_2^{:B} \Rightarrow a \times g(b_1) \times g(b_2) \quad .$$

Since the bifunctor $F^{\bullet,\bullet}$ is a functor with respect to each type parameter separately, we can transform a value of type $F^{A,B}$ to a value of type $F^{C,D}$ by applying the two `fmap` functions one after another. It is convenient to denote this transformation by a single operation called `bimap` that uses two functions $f^{A \Rightarrow C}$ and $g^{B \Rightarrow D}$ as arguments:

$$\begin{aligned} \text{bimap}_F(f^{A \Rightarrow C})(g^{B \Rightarrow D}) &: F^{A,B} \Rightarrow F^{C,D} \quad , \\ \text{bimap}_F(f^{A \Rightarrow C})(g^{B \Rightarrow D}) &\triangleq \text{fmap}_{F^{\bullet,B}}(f^{A \Rightarrow C}) \circ \text{fmap}_{F^{A,\bullet}}(g^{B \Rightarrow D}) \quad . \end{aligned} \tag{6.8}$$

In the condensed notation, this is written as

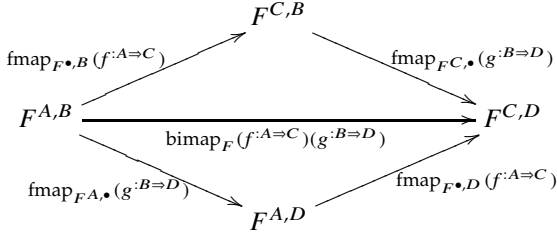
$$\text{bimap}_F(f^{A \Rightarrow C})(g^{B \Rightarrow D}) \triangleq f^{\uparrow F^{\bullet,B}} \circ g^{\uparrow F^{A,\bullet}} \quad ,$$

but in this case the longer notation in Eq. (6.8) is easier to reason about.

What if we apply the two fmap functions in the opposite order? Since these functions work with different type parameters, it is reasonable to expect that the transformation $F^{A,B} \Rightarrow F^{C,D}$ should be independent of the order of application:

$$\text{fmap}_{F^{\bullet,B}}(f:A \Rightarrow C) \circ \text{fmap}_{F^{C,\bullet}}(g:B \Rightarrow D) = \text{fmap}_{F^{A,\bullet}}(g:B \Rightarrow D) \circ \text{fmap}_{F^{\bullet,D}}(f:A \Rightarrow C) \quad (6.9)$$

This equation is illustrated by the type diagram below.



Different paths in this diagram give the same results if they arrive at the same vertex (as mathematicians say, “the diagram commutes”). In this way, the diagram illustrates at once the commutativity law (6.9) and the definition (6.8) of bimap_F .

Let us verify the commutativity law for the bifunctor $F^{A,B} \triangleq A \times A \times B$:

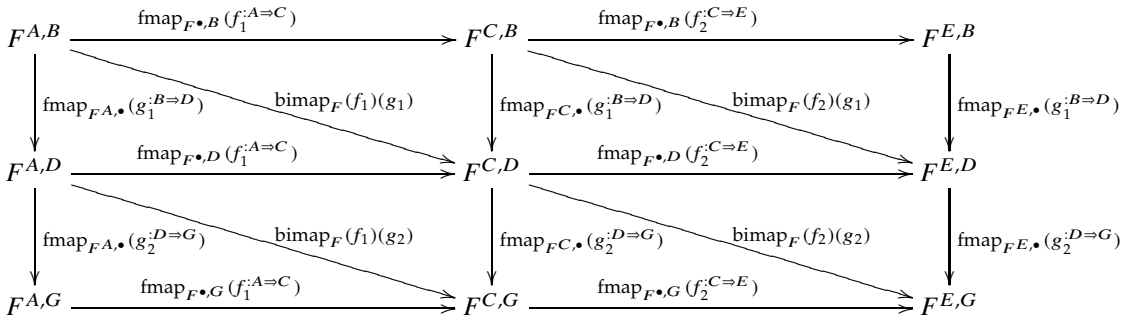
$$\begin{aligned} \text{left-hand side : } & \text{fmap}_{F^{\bullet,B}}(f:A \Rightarrow C) \circ \text{fmap}_{F^{C,\bullet}}(g:B \Rightarrow D) \\ \text{definitions of fmap : } & = (a^:A \times b_1^:B \times b_2^:B \Rightarrow f(a) \times b_1 \times b_2) \circ (c^:C \times b_1^:B \times b_2^:B \Rightarrow c \times g(b_1) \times g(b_2)) \\ \text{compute composition : } & = a^:A \times b_1^:B \times b_2^:B \Rightarrow f(a) \times g(b_1) \times g(b_2) \quad , \\ \text{right-hand side : } & \text{fmap}_{F^{A,\bullet}}(g:B \Rightarrow D) \circ \text{fmap}_{F^{\bullet,D}}(f:A \Rightarrow C) \\ \text{definitions of fmap : } & = (a^:A \times b_1^:B \times b_2^:B \Rightarrow a \times g(b_1) \times g(b_2)) \circ (a^:A \times d_1^:D \times d_2^:D \Rightarrow f(a) \times d_1 \times d_2) \\ \text{compute composition : } & = a^:A \times b_1^:B \times b_2^:B \Rightarrow f(a) \times g(b_1) \times g(b_2) \quad . \end{aligned}$$

Both sides of the law are equal.

The commutativity law (6.9) leads to the composition law for bimap ,

$$\text{bimap}_F(f_1:A \Rightarrow C)(g_1:B \Rightarrow D) \circ \text{bimap}_F(f_2:C \Rightarrow E)(g_2:D \Rightarrow G) = \text{bimap}_F(f_1 \circ f_2)(g_1 \circ g_2) \quad (6.10)$$

The following type diagram shows the relationships between various bimap and fmap functions:



To derive the composition law from Eq. (6.9), write

$$\begin{aligned} & \text{bimap}_F(f_1)(g_1) \circ \text{bimap}_F(f_2)(g_2) \\ \text{use Eq. (6.8) : } & = \text{fmap}_{F^{\bullet,B}}(f_1) \circ \text{fmap}_{F^{C,\bullet}}(g_1) \circ \text{fmap}_{F^{\bullet,D}}(f_2) \circ \text{fmap}_{F^{E,\bullet}}(g_2) \\ \text{commutativity law (6.9) : } & = \text{fmap}_{F^{\bullet,B}}(f_1) \circ \text{fmap}_{F^{\bullet,B}}(f_2) \circ \text{fmap}_{F^{E,\bullet}}(g_1) \circ \text{fmap}_{F^{E,\bullet}}(g_2) \\ \text{composition laws : } & = \text{fmap}_{F^{\bullet,B}}(f_1 \circ f_2) \circ \text{fmap}_{F^{E,\bullet}}(g_1 \circ g_2) \\ \text{use Eq. (6.8) : } & = \text{bimap}_F(f_1 \circ f_2)(g_1 \circ g_2) \quad . \end{aligned}$$

Conversely, we can derive Eq. (6.9) from the composition law (6.10). We write the composition law with specially chosen functions:

$$\text{bimap}_F(f:A \Rightarrow C)(g:B \Rightarrow D) = \text{bimap}_F(\text{id}:A \Rightarrow A)(g:B \Rightarrow D) \circ \text{bimap}_F(f:A \Rightarrow C)(\text{id}:D \Rightarrow D) \quad (6.11)$$

Using Eq. (6.8), we find

$$\begin{aligned} \text{expect } \text{fmap}_{F^A, \bullet}(g) \circ \text{fmap}_{F^{\bullet}, D}(f) &: \text{fmap}_{F^{\bullet}, B}(f:A \Rightarrow C) \circ \text{fmap}_{FC, \bullet}(g:B \Rightarrow D) \\ \text{use Eq. (6.8)} &: = \text{bimap}_F(f:A \Rightarrow C)(g:B \Rightarrow D) \\ \text{use Eq. (6.11)} &: = \text{bimap}_F(\text{id}:A \Rightarrow A)(g:B \Rightarrow D) \circ \text{bimap}_F(f:A \Rightarrow C)(\text{id}:D \Rightarrow D) \\ \text{use Eq. (6.8)} &: = \text{fmap}_{F^{\bullet}, B}(\text{id}) \circ \text{fmap}_{F^A, \bullet}(g) \circ \text{fmap}_{F^{\bullet}, D}(f) \circ \text{fmap}_{FC, \bullet}(\text{id}) \\ \text{identity laws for } F &: = \text{fmap}_{F^A, \bullet}(g) \circ \text{fmap}_{F^{\bullet}, D}(f) \quad . \end{aligned}$$

The identity law for `bimap` holds as well,

$$\begin{aligned} \text{expect to equal id} &: \text{bimap}_F(\text{id}:A \Rightarrow A)(\text{id}:B \Rightarrow B) \\ \text{use Eq. (6.8)} &: = \text{fmap}_{F^{\bullet}, B}(\text{id}) \circ \text{fmap}_{FC, \bullet}(\text{id}) \\ \text{identity laws for } F &: = \text{id} \circ \text{id} = \text{id} \quad . \end{aligned}$$

If $F^{A,B}$ is known to be a functor separately with respect to A and B , will the commutativity law (6.9) always hold? The calculation for the example $F^{A,B} \triangleq A \times B \times B$ shows that the two `fmap` functions commute because they work on different parts of the data structure $F^{A,B}$. This turns out³ to be true in general: the commutativity law follows from the parametricity of the `fmap` functions. Because of this, we do not need to verify the `bimap` laws as long as $F^{\bullet,B}$ and $F^{A,\bullet}$ are lawful functors.

Type constructors with more than two type parameters have similar properties. It is sufficient to check the functor laws with respect to each type parameter separately.

In general, a type constructor may be a functor with respect to some type parameters and a contrafunctor with respect to others. For brevity, we will talk about “covariant” and “contravariant” type parameters in those cases.

6.2.3 Functor constructions

What type expressions will give us a functor? Functional programming languages support the six standard type constructions (see Section 5.1.2). We can check all the available type constructions for obeying the functor laws. The results are summarized in Table 6.2.

In each of these constructions, the `fmap` function for a new functor is defined either from scratch or by using the known `fmap` functions for previously defined type constructors. We will now show the code for these constructions and prove their validity. We will use the code notation for brevity, occasionally showing the translation into the Scala syntax.

Statement 6.2.3.1 The type constructor $\text{Id}^A \triangleq A$ is a lawful functor (the **identity functor**).

Proof The `fmap` function is defined by

$$\begin{aligned} \text{fmap}_{\text{Id}} : (A \Rightarrow B) &\Rightarrow \text{Id}^A \Rightarrow \text{Id}^B \cong (A \Rightarrow B) \Rightarrow A \Rightarrow B \quad , \\ \text{fmap}_{\text{Id}} &\triangleq (f:A \Rightarrow B \Rightarrow f) = \text{id}^{(A \Rightarrow B) \Rightarrow A \Rightarrow B} \quad . \end{aligned}$$

The identity function is the only fully parametric implementation of the type signature $(A \Rightarrow B) \Rightarrow A \Rightarrow B$. Since the code of `fmap` is the identity function, the laws are satisfied automatically:

$$\begin{aligned} \text{identity law} &: \text{fmap}_{\text{Id}}(\text{id}) = \text{id}(\text{id}) = \text{id} \quad , \\ \text{composition law} &: \text{fmap}_{\text{Id}}(f \circ g) = f \circ g = \text{fmap}_{\text{Id}}(f) \circ \text{fmap}_{\text{Id}}(g) \quad . \end{aligned}$$

³The proof is beyond the scope of this chapter; see <https://byorgey.wordpress.com/2018/03/30/>

Construction	Type notation	Comment
type parameter	$L^A \triangleq A$	the identity functor
product type	$L^A \triangleq P^A \times Q^A$	the functor product; P and Q must be functors
disjunctive type	$L^A \triangleq P^A + Q^A$	the functor co-product; P and Q must be functors
function type	$L^A \triangleq C^A \Rightarrow P^A$	the exponential functor; P is a functor and C a contrafunctor
primitive type	$L^A \triangleq Z$	the constant functor; Z is a fixed type
type constructor	$L^A \triangleq P Q^A$	functor composition; P and Q are both functors or both contrafunctors
recursive type	$L^A \triangleq S^A, L^A$	recursive functor; $S^{A,B}$ must be a functor w.r.t. both A and B

Table 6.2: Functor constructions defining a functor L^A .

Statement 6.2.3.2 The type constructor $\text{Const}^{Z,A} \triangleq Z$ is a lawful functor (a **constant functor**) with respect to the type parameter A .

Proof The fmap function is defined by

$$\begin{aligned} \text{fmap}_{\text{Const}} : (A \Rightarrow B) &\Rightarrow \text{Const}^{Z,A} \Rightarrow \text{Const}^{Z,B} \cong (A \Rightarrow B) \Rightarrow Z \Rightarrow Z \quad , \\ \text{fmap}_{\text{Const}}(f^{A \Rightarrow B}) &\triangleq (z^{Z \Rightarrow Z} \Rightarrow z) = \text{id}^{Z \Rightarrow Z} \quad . \end{aligned}$$

It is a constant function that ignores f and returns the identity $\text{id}^{Z \Rightarrow Z}$. The laws are satisfied:

$$\begin{aligned} \text{identity law} : \quad &\text{fmap}_{\text{Const}}(\text{id}) = \text{id} \quad , \\ \text{composition law} : \quad &\text{fmap}_{\text{Const}}(f \circ g) = \text{id} = \text{fmap}_{\text{Const}}(f) \circ \text{fmap}_{\text{Const}}(g) = \text{id} \circ \text{id} \quad . \end{aligned}$$

The corresponding Scala code is

```
type Const[Z, A] = Z
def fmap[A, B](f: A => B): Const[Z, A] => Const[Z, B] = identity[Z]
```

The identity functor Id^\bullet and the constant functor $\text{Const}^{Z,\bullet}$ are not often used: their fmap implementations are identity functions, and so they rarely provide useful functionality.

Functor product We have seen that type constructors with product types, such as $L^A \triangleq A \times A \times A$, are functors. The next construction (the **functor product**) explains why.

Statement 6.2.3.3 If L^\bullet and M^\bullet are two functors then the product $P^A \triangleq L^A \times M^A$ is also a functor.

Proof The fmap function for P is defined by

```
def fmap[A, B](f: A => B): (L[A], M[A]) => (L[B], M[B]) = {
  case (la, ma) => (la.map(f), ma.map(f))
}
```

The corresponding code notation is

$$f^{\uparrow P} \triangleq l^{L^A} \times m^{M^A} \Rightarrow f^{\uparrow L}(l) \times f^{\uparrow M}(m) \quad .$$

Writing this code using the pipe (\triangleright) operation makes it somewhat closer to the Scala syntax:

$$(l^{L^A} \times m^{M^A}) \triangleright f^{\uparrow P} \triangleq (l \triangleright f^{\uparrow L}) \times (m \triangleright f^{\uparrow M}) \quad . \quad (6.12)$$

An alternative notation uses the **function product** symbol \boxtimes defined by

$$\begin{aligned} p^{A \Rightarrow B} \boxtimes q^{C \Rightarrow D} : A \times C &\Rightarrow B \times D \quad , \\ p \boxtimes q &\triangleq a \times c \Rightarrow p(a) \times q(c) \quad , \\ (a \times c) \triangleright (p \boxtimes q) &= (a \triangleright p) \times (c \triangleright q) \quad . \end{aligned}$$

In this notation, the lifting for P is defined more concisely:

$$f^{\uparrow P} \triangleq f^{\uparrow L} \boxtimes f^{\uparrow M} \quad .$$

We need to verify the identity law and the composition law.

To verify the identity law for P , pipe an arbitrary value of type $L^A \times M^A$ into both sides of the law:

$$\begin{aligned}
 \text{expect to equal } l \times m : & \quad (l^{L^A} \times m^{M^A}) \triangleright \text{id}^{\uparrow P} \\
 \text{definition of } f^{\uparrow P} : & \quad = (l \triangleright \underline{\text{id}^{\uparrow L}}) \times (m \triangleright \underline{\text{id}^{\uparrow M}}) \\
 \text{identity laws for } L, M : & \quad = (l \triangleright \text{id}) \times (m \triangleright \text{id}) \\
 \text{definition of id} : & \quad = l \times m \quad .
 \end{aligned}$$

To verify the composition law for P , we need to show that

$$f^{\uparrow P} \circ g^{\uparrow P} = (f \circ g)^{\uparrow P} \quad .$$

Apply both sides of this equation to an arbitrary value of type $L^A \times M^A$:

$$\begin{aligned}
 \text{expect to equal } (l \times m) \triangleright (f \circ g)^{\uparrow P} : & \quad (l^{L^A} \times m^{M^A}) \triangleright f^{\uparrow P} \circ g^{\uparrow P} \\
 \triangleright \text{notation} : & \quad = \underline{(l^{L^A} \times m^{M^A}) \triangleright f^{\uparrow P}} \triangleright g^{\uparrow P} \\
 \text{use Eq. (6.12)} : & \quad = \underline{((l \triangleright f^{\uparrow L}) \times (m \triangleright f^{\uparrow M})) \triangleright g^{\uparrow P}} \\
 \text{use Eq. (6.12)} : & \quad = (l \triangleright f^{\uparrow L} \triangleright g^{\uparrow L}) \times (m \triangleright f^{\uparrow M} \triangleright g^{\uparrow M}) \\
 \triangleright \text{notation} : & \quad = (l \triangleright \underline{f^{\uparrow L} \circ g^{\uparrow L}}) \times (m \triangleright \underline{f^{\uparrow M} \circ g^{\uparrow M}}) \\
 \text{composition laws for } L, M : & \quad = (l \triangleright (f \circ g)^{\uparrow L}) \times (m \triangleright (f \circ g)^{\uparrow M}) \\
 \text{use Eq. (6.12)} : & \quad = (l \times m) \triangleright (f \circ g)^{\uparrow P} \quad .
 \end{aligned}$$

The calculations are shorter if we use the function product notation:

$$\begin{aligned}
 \text{expect to equal } (f \circ g)^{\uparrow P} : & \quad f^{\uparrow P} \circ g^{\uparrow P} = (f^{\uparrow L} \boxtimes f^{\uparrow M}) \circ (g^{\uparrow L} \boxtimes g^{\uparrow M}) \\
 \text{composition of functions under } \boxtimes : & \quad = \underline{(f^{\uparrow L} \circ g^{\uparrow L}) \boxtimes (f^{\uparrow M} \circ g^{\uparrow M})} \\
 \text{composition laws for } L, M : & \quad = (f \circ g)^{\uparrow L} \boxtimes (f \circ g)^{\uparrow M} = (f \circ g)^{\uparrow P} \quad .
 \end{aligned}$$

For comparison, the same derivation using the Scala code syntax looks like this,

```
(( l, m )).map(f).map(g) == (( l.map(f), m.map(f) )).map(g)
== (( l.map(f).map(g), m.map(f).map(g) ))
== (( l.map(f andThen g), m.map(f andThen g) ))
```

assuming that the `.map` method is defined on pairs by Eq. (6.12),

```
(( l, m )).map(f) == (( l.map(f), m.map(f) ))
```

The proof written in the Scala syntax does not show the type constructors whose `.map` methods are used in each expression. For instance, it is not indicated that the two `.map` methods used in the expression `m.map(f).map(g)` belong to the *same* type constructor M and thus obey M 's composition law. The code notation shows this more concisely and more clearly, helping us in reasoning:

$$m \triangleright f^{\uparrow M} \triangleright g^{\uparrow M} = m \triangleright f^{\uparrow M} \circ g^{\uparrow M} = m \triangleright (f \circ g)^{\uparrow M} \quad .$$

By the conventions of the pipe notation, we have

$$(x \triangleright f) \triangleright g = x \triangleright f \triangleright g = x \triangleright f \circ g = x \triangleright (f \circ g) = (f \circ g)(x) = g(f(x)) \quad .$$

We will often use this notation in code derivations.

Statement 6.2.3.4 If P^A and Q^A are functors then $L^A \triangleq P^A + Q^A$ is a functor, with `fmap` defined by

```
def fmap[A, B](f: A => B): Either[P[A], Q[A]] => Either[P[B], Q[B]] = {
  case Left(pa)    => Left(fmap_P(f)(pa))    // Use fmap for P.
  case Right(qa)   => Right(fmap_Q(f)(qa))   // Use fmap for Q.
}
```

The functor L^\bullet is the **functor co-product** of P^\bullet and Q^\bullet . The code notation for the `fmap` function is

$$\text{fmap}_L(f^{A \Rightarrow B}) = f^{\uparrow L} \triangleq \begin{vmatrix} & \parallel & P^B & Q^B \\ P^A & f^{\uparrow P} & \mathbb{0} & \\ Q^A & \mathbb{0} & f^{\uparrow Q} & \end{vmatrix}.$$

Here we assume that lawful `fmap` functions are given for the functors P and Q .

Proof Omitting the type annotations, we write the code of $\text{fmap}_L(f)$ as

$$\text{fmap}_L(f) = f^{\uparrow L} = \begin{vmatrix} f^{\uparrow P} & \mathbb{0} \\ \mathbb{0} & f^{\uparrow Q} \end{vmatrix}. \quad (6.13)$$

To verify the identity law, use Eq. (6.13) and the identity laws for P and Q :

$$\begin{aligned} \text{expect to equal id : } \text{id}^{\uparrow L} &= \begin{vmatrix} \text{id}^{\uparrow P} & \mathbb{0} \\ \mathbb{0} & \text{id}^{\uparrow Q} \end{vmatrix} = \begin{vmatrix} \text{id} & \mathbb{0} \\ \mathbb{0} & \text{id} \end{vmatrix} \\ \text{identity function in matrix notation : } &= \text{id} \end{aligned}$$

To verify the composition law:

$$\begin{aligned} \text{expect to equal } (f \circ g)^{\uparrow L} : f^{\uparrow L} \circ g^{\uparrow L} &= \begin{vmatrix} f^{\uparrow P} & \mathbb{0} \\ \mathbb{0} & f^{\uparrow Q} \end{vmatrix} \circ \begin{vmatrix} g^{\uparrow P} & \mathbb{0} \\ \mathbb{0} & g^{\uparrow Q} \end{vmatrix} \\ \text{matrix composition : } &= \begin{vmatrix} \underline{f^{\uparrow P} \circ g^{\uparrow P}} & \mathbb{0} \\ \mathbb{0} & \underline{f^{\uparrow Q} \circ g^{\uparrow Q}} \end{vmatrix} \\ \text{composition laws for } P, Q : &= \begin{vmatrix} (f \circ g)^{\uparrow P} & \mathbb{0} \\ \mathbb{0} & (f \circ g)^{\uparrow Q} \end{vmatrix} = (f \circ g)^{\uparrow L} \end{aligned}$$

The last two statements show that any type constructor built up only using primitive types, type parameters, products and co-products, such as $L^A \triangleq \mathbb{1} + (\text{String} + A) \times A \times \text{Int} + A$, is a functor. Functors of this kind are called **polynomial functors** because they are analogous to ordinary arithmetic polynomial functions of a variable A . The type notation with its symbols $(+)$, (\times) makes this analogy visually clear.

Implementing `fmap` for a polynomial functor is straightforward: `fmap` replaces each occurrence of the a value of type A by the corresponding value of type B , leaving constant types unchanged and keeping the order of products and disjunctions. Previously, our implementations of `fmap` for various type constructors (such as shown in Example 6.1.9.2) were guided by the idea of preserving information. Statements 6.2.3.3–6.2.3.4 explain why those implementations of the `fmap` are correct (i.e. obey the functor laws).

The next construction shows when a function type is a functor: the argument of the function must be a contrafunctor.

Statement 6.2.3.5 If C is a contrafunctor and P is a functor then $L^A \triangleq C^A \Rightarrow P^A$ is a functor, called an **exponential functor**, with fmap defined by

$$\begin{aligned} \text{fmap}_L^{A,B}(f:A \Rightarrow B) &: (C^A \Rightarrow P^A) \Rightarrow C^B \Rightarrow P^B, \\ \text{fmap}_L(f:A \Rightarrow B) &= f^{\uparrow L} \triangleq h:C^A \Rightarrow P^A \Rightarrow f^{\downarrow C} \circ h \circ f^{\uparrow P}. \end{aligned} \quad (6.14)$$

The corresponding Scala code is

```
def fmap_L[A, B](f: A => B)(h: C[A] => P[A]): C[B] => P[B] = {
  contrafmap_C(f) andThen h andThen fmap_P(f)
}
```

A type diagram for fmap_L can be drawn as

$$\begin{array}{ccc} & C^A & \xrightarrow{h} P^A \\ \text{contrafmap}_C(f:A \Rightarrow B) \nearrow & & \searrow \text{fmap}_P(f:A \Rightarrow B) \\ C^B & \xrightarrow{\text{fmap}_L(f:A \Rightarrow B)(h:C^A \Rightarrow P^A)} & P^B \end{array}$$

Proof Since the types are already checked, we can use Eq. (6.14) without type annotations,

$$h \triangleright f^{\uparrow L} = f^{\downarrow C} \circ h \circ f^{\uparrow P}. \quad (6.15)$$

To verify the identity law for L , show that $\text{id}^{\uparrow L}(h) = h$:

$$\begin{aligned} \text{expect to equal } h &: h \triangleright \text{id}^{\uparrow L} \\ \text{definition (6.15) of } \uparrow^L &: = \text{id}^{\downarrow C} \circ h \circ \text{id}^{\uparrow P} \\ \text{identity laws for } C \text{ and } P &: = \text{id} \circ h \circ \text{id} \\ \text{definition of id} &: = h. \end{aligned}$$

To verify the composition law for L :

$$\begin{aligned} \text{expect to equal } h \triangleright f^{\uparrow L} \circ g^{\uparrow L} &: h \triangleright (f \circ g)^{\uparrow L} \\ \text{definition (6.15) of } \uparrow^L &: = (f \circ g)^{\downarrow C} \circ h \circ (f \circ g)^{\uparrow P} \\ \text{composition laws for } C \text{ and } P &: = g^{\downarrow C} \circ \underline{f^{\downarrow C} \circ h \circ f^{\uparrow P}} \circ g^{\uparrow P} \\ \text{definition (6.15) of } \uparrow^L &: = \underline{g^{\downarrow C} \circ (h \triangleright f^{\uparrow L})} \circ g^{\uparrow P} \\ \text{definition (6.15) of } \uparrow^L &: = (h \triangleright f^{\uparrow L}) \circ g^{\uparrow L} = h \triangleright f^{\uparrow L} \circ g^{\uparrow L}. \end{aligned}$$

It is important for this proof that C is a contrafunctor and so the order of lifted function compositions is reversed, $(f \circ g)^{\downarrow C} = g^{\downarrow C} \circ f^{\downarrow C}$. If C were a functor, the proof would not work: we would have obtained $f^{\downarrow C} \circ g^{\downarrow C}$ instead of $g^{\downarrow C} \circ f^{\downarrow C}$, and we would not be able to group $f^{\downarrow C} \circ h \circ f^{\uparrow P}$ together (the order of composition cannot be permuted for arbitrary functions f, g).

Examples of functors obtained via the exponential functor construction are $L^A \triangleq Z \Rightarrow A$ (with the contrafunctor C^A chosen as the constant contrafunctor Z , where Z is a fixed type) and $L^A \triangleq (A \Rightarrow Z) \Rightarrow A$ (with the contrafunctor $C^A \triangleq A \Rightarrow Z$). Statement 6.2.3.5 generalizes those examples to arbitrary contrafunctors C^A used as arguments of function types.

In a similar way, one can prove that $P^A \Rightarrow C^A$ is a contrafunctor (Exercise 6.3.1.2). Together with the results of Statements 6.2.3.3–6.2.3.5, this establishes the rules of reasoning about covariance and contravariance of type parameters in arbitrary type expressions. Every function arrow flips the variance from covariant to contravariant and back. For instance, the identity functor $L^A \triangleq A$ is covariant in A , while $A \Rightarrow Z$ is contravariant in A , and $(A \Rightarrow Z) \Rightarrow Z$ is again covariant in A . As we

have seen, $A \Rightarrow A \Rightarrow Z$ is contravariant in A , so any number of curried arrows count as one in this consideration (and, in any case, $A \Rightarrow A \Rightarrow Z \cong A \times A \Rightarrow Z$). Products and disjunctions do not change variance, so $(A \Rightarrow Z_1) \times (A \Rightarrow Z_2) + (A \Rightarrow Z_3)$ is still contravariant in A . We will see more examples of such reasoning below (Section 6.2.5).

The remaining constructions set a type parameter to another type constructor. The **functor composition** P^{Q^A} , written in Scala as `P[Q[A]]`, is analogous to a function composition such as $f(g(x))$ except for type constructors. Viewed in this way, type constructors are **type-level functions** (i.e. maps on the set of types). For this reason, functor composition can be also written as $P \circ Q$, similar to the function composition $f \circ g$.

An example of functor composition in Scala is `List[Option[A]]`. Since both `List` and `Option` have a `.map` method, we may write code such as

```
val p: List[Option[Int]] = List(Some(1), None, Some(2), None, Some(3))

scala> p.map(_.map(x => x + 10))
res0: List[Option[Int]] = List(Some(11), None, Some(12), None, Some(13))
```

The code `p.map(_.map(f))` lifts an $f: A \Rightarrow B$ into a function of type `List[Option[A]] => List[Option[B]]`. In this way, we may perform the `.map` operation on the composite data type `List[Option[_]]`.

The next statement shows that this trick always produces a lawful `.map` function. In other words, the composition of functors is always a functor:

Statement 6.2.3.6 If P^A and Q^A are functors then $L^A \triangleq P^{Q^A}$ is also a functor, with `fmap` defined by

```
def fmap_L[A, B](f: A => B): P[Q[A]] => P[Q[B]] = fmap_P(fmap_Q(f))
```

Here we assumed that the functions `fmapP` and `fmapQ` are known and satisfy the functor laws.

In the code notation, `fmapL` is written equivalently as

$$\begin{aligned} \text{fmap}_L : f: A \Rightarrow B &\Rightarrow P^{Q^A} \Rightarrow P^{Q^B} \quad , \\ \text{fmap}_L(f) &\triangleq \text{fmap}_P(\text{fmap}_Q(f)) \quad , \quad \text{fmap}_L \triangleq \text{fmap}_Q \circ \text{fmap}_P \quad , \\ \text{in a shorter notation : } f^{\uparrow L} &\triangleq (f^{\uparrow Q})^{\uparrow P} \triangleq f^{\uparrow Q \uparrow P} \quad . \end{aligned}$$

Proof To verify the identity law for L , use the identity laws for P and Q :

$$\text{id}^{\uparrow L} = (\text{id}^{\uparrow Q})^{\uparrow P} = \text{id}^{\uparrow P} = \text{id} \quad .$$

To verify the composition law for L , use the composition laws for P and Q :

$$(f \circ g)^{\uparrow L} = ((f \circ g)^{\uparrow Q})^{\uparrow P} = (f^{\uparrow Q} \circ g^{\uparrow Q})^{\uparrow P} = f^{\uparrow Q \uparrow P} \circ g^{\uparrow Q \uparrow P} \quad .$$

Finally, we consider recursive data types such as lists and trees (Section 3.3). It is helpful to use the type notation for reasoning about those types. The list type,

```
sealed trait List[A]
final case class Empty() extends List[A]
final case class Head[A](head: A, tail: List[A]) extends List[A]
```

is written as

$$\text{List}^A \triangleq \mathbb{1} + A \times \text{List}^A \quad .$$

The binary tree type,

```
sealed trait Tree2[A]
final case class Leaf[A](a: A) extends Tree2[A]
final case class Branch[A](x: Tree2[A], y: Tree2[A]) extends Tree2[A]
```

Description	Type definition	Bifunctor $S^{A,R}$
list	$L^A \triangleq \mathbb{1} + A \times L^A$	$S^{A,R} \triangleq \mathbb{1} + A \times R$
non-empty list	$NEL^A \triangleq A + A \times NEL^A$	$S^{A,R} \triangleq A + A \times R$
list of odd length	$L^A \triangleq A + A \times A \times L^A$	$S^{A,R} \triangleq A + A \times A \times R$
binary tree	$L^A \triangleq A + L^A \times L^A$	$S^{A,R} \triangleq A + R \times R$
rose tree	$L^A \triangleq A + NEL^{L^A}$	$S^{A,R} \triangleq A + NEL^R$
regular-shaped tree	$L^A \triangleq A + L^{A \times A}$	not possible
abstract syntax tree	$L^A \triangleq P^A + Q^{L^A}$	$S^{A,R} = P^A + Q^R$

Table 6.3: Recursive disjunctive types defined using type equations.

is defined by

$$\text{Tree}_2^A \triangleq A + \text{Tree}_2^A \times \text{Tree}_2^A \quad .$$

These definitions of recursive types look like “type equations”. We can generalize these examples to a recursive definition

$$L^A \triangleq S^{A,L^A} \quad , \quad (6.16)$$

where $S^{A,R}$ is a suitably chosen type constructor with two type parameters A, R . If the type constructor $S^{\bullet,\bullet}$ is given, the Scala code defining L^\bullet can be written as

```
type S[A, R] = ... // Must be defined previously as type alias, class, or trait.
final case class L[A](x: S[A, L[A]])
```

We must use a case class to define L because Scala does not support recursive type aliases:

```
scala> type L[A] = Either[A, L[A]]
<console>:14: error: illegal cyclic reference involving type L
      type L[A] = Either[A, L[A]]
                        ^

scala> final case class L[A](x: Either[A, L[A]])
defined class L
```

Table 6.3 summarizes our previous examples of recursive types and shows the relevant choices of $S^{A,R}$, which turns out to be always a bifunctor. For abstract syntax trees, the functors P^\bullet and Q^\bullet must be given; they specify the available shapes of leaves and branches respectively.

We will now prove that Eq. (6.16) always defines a functor.

Statement 6.2.3.7 If $S^{A,B}$ is a bifunctor (a functor with respect to both type parameters A and B) then the recursively defined type constructor L^A is a functor,

$$L^A \triangleq S^{A,L^A} \quad .$$

The `fmap` method for L is a recursive function implemented as

$$\text{fmap}_L(f^{A \Rightarrow B}) \triangleq \text{bimap}_S(f)(\text{fmap}_L(f)) \quad .$$

The corresponding Scala code is

```
final case class L[A](x: S[A, L[A]]) // The type constructor S[_, _] must be defined previously.

def bimap_S[A, B, C, D](f: A => C)(g: B => D): S[A, B] => S[C, D] = ??? // Must be defined.

def fmap_L[A, B](f: A => B): L[A] => L[B] = { case L(x) =>
  val newX: S[B, L[B]] = bimap_S(f)(fmap_L(f))(x) // Recursive call to contrafmap(f).
```

```
L(newX)          // Need to wrap the value of type S[B, L[B]] into the type constructor L.
}
```

Proof This is the first time we prove a property of a recursive function (fmap_L). Since the implementation of fmap_L is recursive, its code contains some calls to itself. When we are proving some property of fmap_L , we may *assume* in the proof that the property already holds for all recursive calls of fmap_L . As long as recursion is not infinite, this kind of proof will be equivalent to a proof by induction: The base case corresponds to the final non-recursive evaluation within the code of fmap_L , and the inductive assumption states that the recursive calls to fmap_L already satisfy the property we are trying to prove.

For clarity, we add an overline to recursive calls in the code:

$$\text{fmap}_L(f) \triangleq \text{bimap}_S(f)(\overline{\text{fmap}_L(f)}) \quad .$$

To prove the identity law:

```
expect to equal id :  fmap_L(id)
definition of fmap_L :  = bimap_S(id)(fmap_L(id))
inductive assumption :  = bimap_S(id)(id)
identity law for S :    = id    .
```

To prove the composition law:

```
expect to equal fmap_L(f ; g) :  fmap_L(f) ; fmap_L(g)
definition of fmap_L :            = bimap_S(f)(fmap_L(f)) ; bimap_S(g)(fmap_L(g))
composition law for S :          = bimap_S(f ; g)(fmap_L(f) ; fmap_L(g))
inductive assumption :           = bimap_S(f ; g)(fmap_L(f ; g))
definition of fmap_L :            = fmap_L(f ; g)    .
```

For the regular-shaped binary tree, the construction (6.16) is insufficient: no bifunctor S^{A,L^A} can replace the type argument A in L^A to obtain $L^{A \times A}$. To see that, consider that S^{A,L^A} is an application of a type-level function $S^{\bullet,\bullet}$ to its two type parameters, which are set to A and L^A . In Scala syntax, S^{A,L^A} is written as $\text{s}[A, L[A]]$. No matter how we define the type constructor S , the resulting type expression $\text{s}[A, L[A]]$ will always use the type constructor L as $L[A]$ and not as $L[(A, A)]$.

To describe regular-shaped trees, we need to generalize the construction by adding another arbitrary functor, P^\bullet , in the type argument of L^\bullet :

$$L^A \triangleq S^{A, L^{P^A}} \quad . \tag{6.17}$$

Regular-shaped trees are defined by Eq. (6.17) with $S^{A,R} \triangleq A + R$ and $P^A \triangleq A \times A$. The Scala code for these definitions is

```
type S[A, R] = Either[A, R]
type P[A] = (A, A)
final case class L[A](s: S[A, L[P[A]]]) // Equivalently, case class L[A](s: Either[A, L[(A, A)]])
```

Different choices of P will define regular-shaped trees with different kinds of branching.

6.2.4 Contrafunctor constructions

In the previous section, we have found constructions that create new functors. *Mutatis mutandis*, similar constructions work for contrafunctors. A systematic checking of all possible type constructions yields Table 6.4. One difference with respect to Table 6.2 is that the identity type constructor,

Construction	Type notation	Comment
tuple	$C^A \triangleq P^A \times Q^A$	the product contrafunctor; P and Q must be contrafunctors
disjunctive type	$C^A \triangleq P^A + Q^A$	the co-product contrafunctor; P and Q must be contrafunctors
function type	$C^A \triangleq L^A \Rightarrow H^A$	the exponential contrafunctor; L is a functor and H a contrafunctor
primitive type	$C^A \triangleq Z$	the constant contrafunctor; Z is a fixed type
type constructor	$C^A \triangleq P Q^A$	the composition; P is a functor and Q a contrafunctor (or vice versa)
type recursion	$C^A \triangleq S^{A,C^A}$	$S^{A,B}$ must be a contrafunctor w.r.t. A and functor w.r.t. B

Table 6.4: Contrafunctor constructions defining a contrafunctor C^A .

$L^A \triangleq A$, is not present because it is a functor and not a contrafunctor. However, the constant type constructor, $L^A \triangleq Z$, is a functor and a contrafunctor at the same time.

Let us now prove the validity of some of these constructions.

Statement 6.2.4.1 If Z is any fixed type, the constant type constructor $C^A \triangleq Z$ is a contrafunctor (the **constant contrafunctor**) whose `contrafmap` returns an identity function of type $Z \Rightarrow Z$:

```
type Const[Z, A] = Z
def contrafmap[Z, A, B](f: B => A): Const[Z, A] => Const[Z, B] = identity[Z]
```

Proof All laws hold because `contrafmap` returns an identity function:

identity law : `contrafmap (id) = id` ,

composition law : `contrafmap (f) ; contrafmap (g) = id ; id = id = contrafmap (g ; f)` .

Statement 6.2.4.2 If P^A is a functor and Q^A is a contrafunctor then $L^A \triangleq P Q^A$ is a contrafunctor with `contrafmap` defined by

```
def contrafmap[A, B](f: B => A): P[Q[A]] => P[Q[B]] = fmap_P(contrafmap_Q(f))
```

where lawful implementations of `fmapP` and `contrafmapQ` are assumed to be given.

Proof Convert the Scala implementation of `contrafmapL` into the code notation:

$$\text{contrafmap}_L(f^{B \Rightarrow A}) \triangleq \text{fmap}_P(\text{contrafmap}_Q(f)) \quad .$$

It is easier to reason about this function if we rewrite it as

$$f^{\downarrow L} \triangleq (f^{\downarrow Q})^{\uparrow P} \quad .$$

The contrafunctor laws for L are then proved like this:

identity law : $\text{id}^{\downarrow L} = (\text{id}^{\downarrow Q})^{\uparrow P} = \text{id}^{\uparrow P} = \text{id}$.

composition law : $f^{\downarrow L} ; g^{\downarrow L} = (f^{\downarrow Q})^{\uparrow P} ; (g^{\downarrow Q})^{\uparrow P}$

use P 's composition law : $= \underline{(f^{\downarrow Q} ; g^{\downarrow Q})}^{\uparrow P} = \underline{((g ; f)^{\downarrow Q})}^{\uparrow P} = (g ; f)^{\downarrow L}$.

Finally, the recursive construction works for contrafunctors, except that the type constructor $S^{A,R}$ must be a contrafunctor in A (but still a functor in R). An example of such a type constructor is

$$S^{A,R} \triangleq (A \Rightarrow \text{Int}) + R \times R \quad . \quad (6.18)$$

The type constructor $S^{\bullet,\bullet}$ is not a bifunctor because it is contravariant in its first type parameter; so we cannot define a `bimap` function for it. However, we can define an analogous function called `xmap`, with the type signature

```
def xmap[A, B, Q, R](f: B => A)(g: Q => R): S[A, Q] => S[B, R]
```

$$\begin{aligned} \text{xmap}_S &: (B \Rightarrow A) \Rightarrow (Q \Rightarrow R) \Rightarrow S^{A,Q} \Rightarrow S^{B,R} , \\ \text{xmap}_S(f^{B \Rightarrow A})(g^{Q \Rightarrow R}) &\triangleq \text{fmap}_{S^{A,\bullet}}(g) \circ \text{contrafmap}_{S^{\bullet,R}}(f) . \end{aligned}$$

The function `xmap` obeys the laws of identity and composition:

$$\text{identity law : } \text{xmap}_S(\text{id})(\text{id}) = \text{id} , \quad (6.19)$$

$$\text{composition law : } \text{xmap}_S(f_1)(g_1) \circ \text{xmap}_S(f_2)(g_2) = \text{xmap}_S(f_2 \circ f_1)(g_1 \circ g_2) . \quad (6.20)$$

These laws are similar to the identity and composition laws for bifunctors (Section 6.2.2), except for inverting the order of the composition ($f_2 \circ f_1$). The laws hold automatically whenever the functor and contrafunctor methods for S ($\text{fmap}_{S^{A,\bullet}}$ and $\text{contrafmap}_{S^{\bullet,R}}$) are fully parametric. We omit the details since they are quite similar to what we saw in Section 6.2.2 for bifunctors.

If we define a type constructor L^\bullet using the recursive equation

$$L^A \triangleq S^{A,L^A} \triangleq (A \Rightarrow \text{Int}) + L^A \times L^A ,$$

we obtain a contrafunctor in the shape of a binary tree whose leaves are functions of type $A \Rightarrow \text{Int}$. The next statement shows that recursive type equations of this kind always define contrafunctors.

Statement 6.2.4.3 If $S^{A,R}$ is a contrafunctor with respect to A and a functor with respect to R then the recursively defined type constructor C^A is a contrafunctor,

$$C^A \triangleq S^{A,C^A} .$$

Given the functions $\text{contrafmap}_{S^{\bullet,R}}$ and $\text{fmap}_{S^{A,\bullet}}$ for S , we implement contrafmap_C as

$$\begin{aligned} \text{contrafmap}_C(f^{B \Rightarrow A}) &: C^A \Rightarrow C^B \cong S^{A,C^A} \Rightarrow S^{B,C^B} , \\ \text{contrafmap}_C(f^{B \Rightarrow A}) &\triangleq \text{xmap}_S(f)(\overline{\text{contrafmap}_C(f)}) . \end{aligned}$$

The corresponding Scala code can be written as

```
final case class C[A](x: S[A, C[A]]) // The type constructor S[_, _] must be defined previously.

def xmap_S[A,B,Q,R](f: B => A)(g: Q => R): S[A, Q] => S[B, R] = ??? // Must be defined.

def contrafmap[A, B](f: B => A): C[A] => C[B] = { case C(x) =>
  val sbcb: S[B, C[B]] = xmap_S(f)(contrafmap(f))(x) // Recursive call to contrafmap(f).
  C(sbcb) // Need to wrap the value of type S[B, C[B]] into the type constructor C.
}
```

Proof The code of `contrafmap` is recursive, and the recursive call is marked by an overline:

$$\text{contrafmap}_C(f) \triangleq f^{\downarrow C} \triangleq \text{xmap}_S(f)(\overline{\text{contrafmap}_C(f)}) .$$

To verify the identity law:

$$\begin{aligned} \text{expect to equal id : } \text{contrafmap}_C(\text{id}) &= \text{xmap}_S(\text{id})(\overline{\text{contrafmap}_C(\text{id})}) \\ \text{inductive assumption : } &= \text{xmap}_S(\text{id})(\text{id}) \\ \text{identity law for xmap}_S : &= \text{id} . \end{aligned}$$

To verify the composition law:

$$\begin{aligned} \text{expect to equal } (g^{\downarrow C} \circ f^{\downarrow C}) : & (f^{D \Rightarrow B} \circ g^{B \Rightarrow A})^{\downarrow C} = \text{xmap}_S(f \circ g)(\overline{\text{contrafmap}_C(f \circ g)}) \\ \text{inductive assumption : } &= \text{xmap}_S(f \circ g)(\overline{\text{contrafmap}_C(g) \circ \text{contrafmap}_C(f)}) \\ \text{composition laws for xmap}_S : &= \text{xmap}_S(g)(\overline{\text{contrafmap}_C(g)}) \circ \text{xmap}_S(f)(\overline{\text{contrafmap}_C(f)}) \\ \text{definition of } \downarrow C : &= g^{\downarrow C} \circ f^{\downarrow C} . \end{aligned}$$

6.2.5 Solved examples: How to recognize functors and contrafunctors

Sections 6.2.3 and 6.2.4 describe how functors and contrafunctors are built from other type expressions. We can see from Tables 6.2 and 6.4 that *every* one of the six type constructions (Section 5.1.2) gives either a new functor or a new contrafunctor. So, we expect to be able to decide, for *every* type expression built from primitive types, type parameters, product types, sum types, function (exponential) types, and type parameter substitutions, whether it is a functor or a contrafunctor. The decision algorithm is based on the results shown in Tables 6.2 and 6.4:

- Primitive types $\mathbb{1}$, Int , String , etc. can be viewed both as constant functors and as constant contrafunctors (since they do not contain type parameters).
- Polynomial type expressions (not containing any function arrows) are always functors with respect to every type parameter. Equivalently, we may say that all polynomial type constructors are covariant in all their type parameters. For example, the type expression $A \times B + (A + \mathbb{1} + B) \times A \times C$ is covariant in each of the type parameters A, B, C .
- Type parameters to the right of a function arrow are in a covariant position. For example, $\text{Int} \Rightarrow A$ is covariant in A .
- Each time a type parameter is placed to the left of an *uncurried* function arrow \Rightarrow , the variance is reversed: covariant becomes contravariant and vice versa. For example,

this is covariant in A : $\mathbb{1} + A \times A$,
 this is contravariant in A : $(\mathbb{1} + A \times A) \Rightarrow \text{Int}$,
 this is covariant in A : $((\mathbb{1} + A \times A) \Rightarrow \text{Int}) \Rightarrow \text{Int}$,
 this is contravariant in A : $((((\mathbb{1} + A \times A) \Rightarrow \text{Int}) \Rightarrow \text{Int}) \Rightarrow \text{Int}) \Rightarrow \text{Int}$.

- Repeated curried function arrows work as one arrow: $A \Rightarrow \text{Int}$ is contravariant in A , and $A \Rightarrow A \Rightarrow A \Rightarrow \text{Int}$ is still contravariant in A . This is so because the type $A \Rightarrow A \Rightarrow A \Rightarrow \text{Int}$ is equivalent to $A \times A \times A \Rightarrow \text{Int}$, which is of the form $F^A \Rightarrow \text{Int}$ with a type constructor $F^A \triangleq A \times A \times A$. Exercise 6.3.1.1 will show that $F^A \Rightarrow \text{Int}$ is contravariant in A .
- Nested type constructors combine their variances: e.g. if we know that F^A is contravariant in A then $F^{A \Rightarrow \text{Int}}$ is covariant in A , while $F^{A \times A \times A}$ is contravariant in A .

For any exponential-polynomial type expression, such as Eq. (6.7),

$$Z^{A,R} \triangleq ((A \Rightarrow A \Rightarrow R) \Rightarrow R) \times A + (\mathbb{1} + R \Rightarrow A + \text{Int}) + A \times A \times \text{Int} \times \text{Int} ,$$

we can mark the position of each type parameter as either covariant (+) or contravariant (−), according to the number of nested function arrows:

$$((\underset{+}{(A \Rightarrow \underset{+}{A} \Rightarrow \underset{-}{R})} \Rightarrow \underset{+}{R}) \times \underset{+}{A} + (\mathbb{1} + \underset{-}{R} \Rightarrow \underset{+}{A} + \underset{+}{\text{Int}}) + \underset{+}{A} \times \underset{+}{A} \times \underset{+}{\text{Int}} \times \underset{+}{\text{Int}}) .$$

We find that A is always in covariant positions, while R is sometimes in covariant and sometimes in contravariant positions. So, we expect that $Z^{A,R}$ is a functor with respect to A , but not a functor (nor a contrafunctor) with respect to R .

To show that $Z^{A,R}$ is indeed a functor in the parameter A , we need to implement a suitable `map` method and verify that the functor laws hold. To do that from scratch, we could use the techniques explained in this and the previous chapters: starting from the type signature

$$\text{map}_Z : Z^{A,R} \Rightarrow (A \Rightarrow B) \Rightarrow Z^{B,R} ,$$

we could derive a fully parametric, information-preserving implementation of `map`. We could then look for equational proofs of the identity and composition laws for that `map` function. This would require a lot of work for a complicated type constructor such as $Z^{A,R}$.

However, that work can be avoided if we find a way of building up $Z^{A,R}$ step by step via the known functor and contrafunctor constructions. Each step automatically provides both a fragment of the code of `map` and a proof that the functor laws hold up to that step. In this way, we will avoid the need to look for an implementation of `map` and proofs of laws for each new functor and contrafunctor. The next examples illustrate the procedure for a simpler type constructor.

Example 6.2.5.1 Rewrite this Scala definition in the type notation and decide whether it is covariant or contravariant with respect to each type parameter:

```
final case class G[A, Z](p: Either[Int, A], q: Option[Z => Int => Z => (Int, A)])
```

Solution The type notation is

$$G^{A,Z} \triangleq (\text{Int} + A) \times (\mathbb{1} + (Z \Rightarrow \text{Int} \Rightarrow Z \Rightarrow \text{Int} \times A)) \quad .$$

Mark the covariant and the contravariant positions in this type expression:

$$(\text{Int} + \underset{+}{A}) \times (\mathbb{1} + (\underset{-}{Z} \Rightarrow \text{Int} \Rightarrow \underset{-}{Z} \Rightarrow \text{Int} \times \underset{+}{A})) \quad .$$

All Z positions in the sub-expression $Z \Rightarrow \text{Int} \Rightarrow Z \Rightarrow \text{Int} \times A$ are contravariant since the function arrows are curried rather than nested. We see that A is always in covariant positions (+) while Z is always in contravariant positions (-). It follows that $G^{A,Z}$ is covariant in A and contravariant in Z .

Example 6.2.5.2 Use known functor constructions to implement the `map` method with respect to A for the type `G[A, Z]` from Example 6.2.5.1.

Solution We need to build $G^{A,Z}$ via step-by-step constructions that start from primitive types and type parameters. At the top level of its type expression, $G^{A,Z}$ is a product type, so we begin by using the “product functor” construction (Statement 6.2.3.3),

$$G^{A,Z} \cong G_1^A \times G_2^{A,Z} \quad ,$$

$$\text{where } G_1^A \triangleq \text{Int} + A \quad \text{and} \quad G_2^{A,Z} \triangleq \mathbb{1} + (Z \Rightarrow \text{Int} \Rightarrow Z \Rightarrow \text{Int} \times A) \quad .$$

We continue with G_1^A , which is a co-product of `Int` (a constant functor) and A (the identity functor). The constant functor and the identity functor have lawful `map` implementations that are already known (Statements 6.2.3.1–6.2.3.2). Now, the “co-product functor” construction (Statement 6.2.3.3) produces a `map` implementation for G_1^A together with a proof that it satisfies the functor laws:

$$\text{fmap}_{G_1}(f^{A \Rightarrow B}) = f^{\uparrow G_1} \triangleq \begin{array}{c|c|c} & & \text{Int} \quad B \\ \hline & \text{Int} & \text{id} \quad \mathbb{0} \\ \hline A & & \mathbb{0} \quad f \end{array} \quad .$$

Turning our attention to $G_2^{A,Z}$, we find that it is a disjunctive type containing a curried function type that ultimately returns the product type $\text{Int} \times A$. This tells us to use the “co-product functor”, the “exponential functor”, and the “product functor” constructions. Write down the functor constructions needed at each step as we decompose $G_2^{A,Z}$:

$$G_2^{A,Z} \triangleq \mathbb{1} + (Z \Rightarrow \text{Int} \Rightarrow Z \Rightarrow \text{Int} \times A) \quad .$$

$$\text{co-product functor : } G_2^{A,Z} \cong \mathbb{1} + G_3^{A,Z} \quad \text{where } G_3^{A,Z} \triangleq Z \Rightarrow \text{Int} \Rightarrow Z \Rightarrow \text{Int} \times A \quad .$$

$$\text{exponential functor : } G_3^{A,Z} \cong Z \Rightarrow G_4^{A,Z} \quad \text{where } G_4^{A,Z} \triangleq \text{Int} \Rightarrow Z \Rightarrow \text{Int} \times A \quad .$$

$$\text{exponential functor : } G_4^{A,Z} \cong \text{Int} \Rightarrow G_5^{A,Z} \quad \text{where } G_5^{A,Z} \triangleq Z \Rightarrow \text{Int} \times A \quad .$$

$$\text{exponential functor : } G_5^{A,Z} \cong Z \Rightarrow G_6^A \quad \text{where } G_6^A \triangleq \text{Int} \times A \quad .$$

$$\text{product functor : } G_6^A \cong \text{Int} \times A \cong \text{Const}^{\text{Int},A} \times \text{Id}^A \quad .$$

Each of the type constructors G_1, \dots, G_6 is a functor in A because all of the functor constructions preserve functor laws. Therefore, $G^{A,Z}$ is a functor in A .

It remains to derive the code for the `fmap` method of G . Each of the functor constructions combines the `fmap` implementations from previously defined functors into a new `map` implementation, so we just need to combine the code fragments in the order of constructions. For brevity, we will use the notations $f^{\uparrow L} \triangleq \text{fmap}_L(f)$ and $x \triangleright f^{\uparrow L}$ instead of the Scala code `x.map(f)` throughout the derivations:

$$\begin{aligned}
 \text{product functor : } G^{A,Z} &\cong G_1^A \times G_2^{A,Z} \quad , \quad (g_1 \times g_2) \triangleright f^{\uparrow G} = (g_1 \triangleright f^{\uparrow G_1}) \times (g_2 \triangleright f^{\uparrow G_2}) \quad . \\
 \text{co-product functor : } G_1^A &\triangleq \text{Int} + A \quad , \quad f^{\uparrow G_1} = \begin{vmatrix} \text{id} & 0 \\ 0 & f \end{vmatrix} \quad . \\
 \text{co-product functor : } G_2^{A,Z} &\triangleq \mathbb{1} + G_3^{A,Z} \quad , \quad f^{\uparrow G_2} = \begin{vmatrix} \text{id} & 0 \\ 0 & f^{\uparrow G_3} \end{vmatrix} \quad . \\
 \text{exponential functor : } G_3^{A,Z} &\triangleq Z \Rightarrow G_4^{A,Z} \quad , \quad g_3 \triangleright f^{\uparrow G_3} = g_3 \circ f^{\uparrow G_4} = (z : Z \Rightarrow g_3(z) \triangleright f^{\uparrow G_4}) \quad .
 \end{aligned}$$

Applying the exponential functor construction three times, we finally obtain

$$\begin{aligned}
 G_3^{A,Z} &\triangleq Z \Rightarrow \text{Int} \Rightarrow Z \Rightarrow G_6^A \quad , \quad g_3 \triangleright f^{\uparrow G_3} = z_1^Z \Rightarrow n^{\text{Int}} \Rightarrow z_2^Z \Rightarrow g_3(z_1)(n)(z_2) \triangleright f^{\uparrow G_6} \quad . \\
 G_6^A &\triangleq \text{Int} \times A \quad , \quad (i \times a) \triangleright f^{\uparrow G_6} = i \times f(a) \quad .
 \end{aligned}$$

We can now write the corresponding Scala code for `fmapG`:

```
def fmap_G[A, B, Z](f: A => B): G[A, Z] => G[B, Z] = { case G(p, q) =>
  val newP: Either[Int, B] = p.map(f)           // Use the standard .map method for Either[Int, A].
  val newQ: Option[Z => Int => Z => (Int, B)] = q.map { // Use the .map method for Option[_].
    (g3: Z => Int => Z => (Int, A)) =>
      z1 => n => z2 => // The code of .map for G_3.
        val (i, a) = g3(z1)(n)(z2)
        (i, f(a))      // The code of .map for G_6.
      }
    G(newP, newQ)      // The code of .map for G_1.
  }
}
```

6.3 Summary

What tasks can we perform with the techniques of this chapter?

- Quickly decide if a given type constructor is a functor, a contrafunctor, or neither.
- Implement a `fmap` or a `contrafmap` function that satisfies the appropriate laws.
- Use constructions to derive the correct code of `fmap` or `contrafmap` without trial and error.
- Use functor blocks to manipulate data wrapped in functors with more readable code.

6.3.1 Exercises: Functor and contrafunctor constructions

Exercise 6.3.1.1 If H^A is a contrafunctor and L^A is a functor, show that $C \triangleq L^A \Rightarrow H^A$ is a contrafunctor with the `contrafmap` method defined by the following code:

```
def contrafmap[A, B](f: B => A)(c: L[A] => H[A]): L[B] => H[B] = {
  lb: L[B] => contrafmap_H(f)(c(fmap_L(f)(lb)))
}
```

Here, `contrafmap_H` and `fmap_L` are the methods already defined for H and L .

Exercise 6.3.1.2 In each case, implement the required `fmap` or `contrafmap` function for the new type constructor L and prove that the appropriate laws hold. Write the implementations both in Scala and in the code notation. Assume that the given type constructors F and G already satisfy the appropriate laws.

- (a) $L^A \triangleq F^A \times G^A$ is a contrafunctor if F^A and G^A are contrafunctors.
- (b) $L^A \triangleq F^A + G^A$ is a contrafunctor if F^A and G^A are contrafunctor.
- (c) $L^A \triangleq F^{G^A}$ is a functor when both F and G are contrafunctors.
- (d) $L^A \triangleq F^{G^A}$ is a contrafunctor when F is a contrafunctor and G is a functor.

Exercise 6.3.1.3 Show that the type constructor L defined by Eq. (6.17) is a functor for any given bifunctor S and functor P .

Exercise 6.3.1.4 Show that $L^A \triangleq F^A \Rightarrow G^A$ is, in general, neither a functor nor a contrafunctor when both F^A and G^A are functors or both are contrafunctors (an example of suitable F^A and G^A is sufficient).

Exercise 6.3.1.5 For each of the Scala type constructors defined below, formulate the definition in the type notation and decide whether the type constructors are functors, contrafunctors, or neither.

```
type F[A] = Int => (Option[A], Either[A, Int], Either[A, A])
type G[A] = ((Int, A)) => Either[Int, A]
type H[A] = Either[A, (A, Option[A])] => Int => Int
```

Exercise 6.3.1.6 Using the known constructions, determine which of the following are functors or contrafunctors (or neither) and implement `fmap` or `contrafmap` if appropriate. Answer this question with respect to each type parameter separately.

- (a) $F^A \triangleq \text{Int} \times A \times A + (\text{String} \Rightarrow A) \times A$.
- (b) $G^{A,B} \triangleq (A \Rightarrow \text{Int} \Rightarrow \mathbb{1} + B) + (A \Rightarrow \mathbb{1} + A \Rightarrow \text{Int})$.
- (c) $H^{A,B,C} \triangleq (A \Rightarrow A \Rightarrow B \Rightarrow C) \times C + (B \Rightarrow A)$.
- (d) $P^{A,B} \triangleq (((A \Rightarrow B) \Rightarrow A) \Rightarrow B) \Rightarrow A$.

Exercise 6.3.1.7 Show that the recursive type constructor L^\bullet defined by

$$L^A \triangleq \mathbb{1} + A + L^A$$

is a functor, and implement a `map` or `fmap` function for L in Scala.

6.4 Discussion

6.4.1 Profunctors

We have seen that some type constructors are neither functors nor contrafunctors because their type parameters appear both in covariant and contravariant positions. An example of such a type constructor is

$$P^A \triangleq A + (A \Rightarrow \text{Int}) \quad .$$

It is not possible to define either a `map` or a `contrafmap` function for P : the required type signatures cannot be implemented. However, we *can* implement a function called `xmap`, with the type signature

$$\text{xmap}_P : (B \Rightarrow A) \Rightarrow (A \Rightarrow B) \Rightarrow P^A \Rightarrow P^B \quad .$$

To see why, let us temporarily rename the contravariant occurrence of A to Z and define a new type constructor \tilde{P} by

$$\tilde{P}^{Z,A} \triangleq A + (Z \Rightarrow \text{Int}) \quad .$$

The original type constructor P^A is expressed as $P^A = \tilde{P}^{A,A}$. Now, $\tilde{P}^{Z,A}$ is covariant in A and contravariant in Z . We can implement `xmapP` as a composition of `fmap` with respect to A and `contrafmap`

with respect to Z , similarly to what we saw in the proof of Statement 6.2.4.3. The function $\text{xmap}_{\tilde{P}}$ will satisfy the identity and composition laws (6.19)–(6.20). Setting the type parameter $Z = A$, we will obtain the xmap_P function for P . The identity and composition laws for xmap_P will hold, since the laws of $\tilde{P}^{Z,A}$ hold for all type parameters:

P's identity law : $\text{xmap}_P(\text{id}^{A \Rightarrow A})(\text{id}^{A \Rightarrow A}) = \text{id}$,

P's composition law : $\text{xmap}_P(f_1^{B \Rightarrow A})(g_1^{A \Rightarrow B}) \circ \text{xmap}_P(f_2^{C \Rightarrow B})(g_2^{B \Rightarrow C}) = \text{xmap}_P(f_2 \circ f_1)(g_1 \circ g_2)$.

A type constructor P^A with these properties ($P^A = \tilde{P}^{A,A}$ where $\tilde{P}^{Z,A}$ has a lawful $\text{xmap}_{\tilde{P}}$) is called a **profunctor**.

Consider an exponential-polynomial type constructor P^A , no matter how complicated, such as

$$P^A \triangleq (1 + A \times A \Rightarrow A) \times A \Rightarrow 1 + (A \Rightarrow A + \text{Int}) \quad .$$

Each copy of a type parameter will occur either in covariant or in a contravariant position because no other possibility is available in exponential-polynomial type expressions. So, we can always rename all contravariant occurrences of the type parameter A to “ Z ” and so obtain a new type constructor $\tilde{P}^{Z,A}$, which will be covariant in A and contravariant in Z . Since $\tilde{P}^{A,Z}$ is a functor in A and a contrafunctor in Z , we will be able to define a function $\text{xmap}_{\tilde{P}}$ satisfying the identity and composition laws. Setting $Z = A$, we will obtain a lawful xmap_P , which makes P a profunctor. Thus, *every* exponential-polynomial type constructor is a profunctor.

An unfunctor, such as the disjunctive type `ServerAction [R]` shown in Section 6.1.6, cannot be made into a profunctor. The type signature of xmap cannot be implemented for `ServerAction [R]` because it is not a fully parametric type constructor (and so is not exponential-polynomial).

Profunctors are not often used in practical coding. We will occasionally see profunctors in later chapters.

6.4.2 Subtyping with injective or surjective conversion functions

In some cases, P is a subtype of Q when the set of values of P is a *subset* of values of Q . In other words, the conversion function $P \Rightarrow Q$ is injective and embeds all information from a value of type P into a value of type Q . This kind of subtyping works for parts of disjunctive types, such as `Some [A] <: Option [A]` (in the type notation, $0 + A <: 1 + A$). The set of all values of type `Some [A]` is a subset of the set of values of type `Option [A]`, and the conversion function is injective because it is an identity function, $0 + x^A \Rightarrow 0 + x$, that merely reassigns types.

However, subtyping does not necessarily imply that the conversion function is injective. An example of a subtyping relation with a *surjective* conversion function is between the function types $P \triangleq 1 + A \Rightarrow \text{Int}$ and $Q \triangleq 0 + A \Rightarrow \text{Int}$ (in Scala, `P = Option [A] => Int` and `Q = Some [A] => Int`). We have $P <: Q$ because $P \cong C^{1+A}$ and $Q \cong C^{0+A}$, where $C^X \triangleq X \Rightarrow \text{Int}$ is a contrafunctor. The conversion function $P \Rightarrow Q$ is an identity function that reassigns types,

```
def p2q[A](p: Option[A] => Int): Some[A] => Int = { x: Some[A] => p(x) }
```

In the code notation, $p \Rightarrow x \Rightarrow p(x)$ is easily seen to be the same as $p \Rightarrow p$.

Nevertheless, it is not true that all information from a value of type P is preserved in a value of type Q : the type P describes functions that also accept `None` as an argument, while functions of type Q do not. So, there is strictly more information in the type P than in Q . The conversion function $p2q : P \Rightarrow Q$ is surjective.

We have now seen examples of either injective or surjective type conversions. Suppose $P_1 <: Q_1$ and $P_2 <: Q_2$, and consider the product types $P_1 \times P_2$ and $Q_1 \times Q_2$. Since the product type is one of the functor constructions, the product $A \times B$ is covariant in both type parameters. It follows that $P_1 \times P_2 <: Q_1 \times Q_2$. If $r_1 : P_1 \Rightarrow Q_1$ is injective but $r_2 : P_2 \Rightarrow Q_2$ is surjective, the function product $r_1 \boxtimes r_2 : P_1 \times P_2 \Rightarrow Q_1 \times Q_2$ is neither injective nor surjective. So, type conversion functions are not necessarily injective or surjective; they can also be anything “in between”.

An important property of functor liftings is that they preserve injectivity and surjectivity: if a function $f:A \Rightarrow B$ is injective, it is lifted to an injective function $f^{\uparrow L} : L^A \Rightarrow L^B$; and similarly for surjective functions f . Let us prove this property for injective functions; the proof for surjective functions is quite similar.

Statement 6.4.2.1 If L^A is a lawful functor and $f:A \Rightarrow B$ is an injective function then $\text{fmap}_L(f)$ is also an injective function of type $L^A \Rightarrow L^B$.

Proof We begin by noting that an injective function $f:A \Rightarrow B$ must somehow embed all information from a value of type A into a value of type B . The **image** of f (the subset of all values of type B that can be obtained as $f(a)$ for some $a:A$) thus contains a distinct value of type B for each distinct value of type A . So, there exists a function that maps any b from the image of f back to a value $a:A$ it came from; call that function $g:B \Rightarrow A$. The function g must satisfy

$$\forall a:A. g(f(a)) = a \quad ;$$

equivalently written as

$$g \circ f = \text{id} \quad .$$

It is important that g is a partial function. The function g is partial because it is defined only for a subset of values of type B , which are in the image of f . Despite the equation $g \circ f = \text{id}$, the function g is not an inverse for f . An inverse function for f must be a *total* (not a partial) function h satisfying $h \circ f = \text{id}$ and $f \circ h = \text{id}$. The function g is called a **left inverse** for f because $f \circ g \neq \text{id}$, since $f \circ g$ is only a partial function.

The fact that f has a left inverse is *equivalent* to the assumption that f is injective. Indeed, if any function f has a left inverse g , we can show that f is injective. Assume some x and y such that $f(x) = f(y)$; we will prove that f is injective if we show that $x = y$. Applying g to both sides of $f(x) = f(y)$, we get

$$x = g(f(x)) = g(f(y)) = y \quad .$$

Now we apply this trick to functions lifted into the functor L . To prove that $\text{fmap}_L(f)$ is injective, we need to show that it has a left inverse. We can lift both sides of the equation $g \circ f = \text{id}$ to get

$$\begin{aligned} & \text{fmap}_L(g) \circ \text{fmap}_L(f) \\ \text{composition law for } L : &= \text{fmap}_L(g \circ f) \\ \text{use } g \circ f = \text{id} : &= \text{fmap}_L(\text{id}) \\ \text{identity law for } L : &= \text{id} \quad . \end{aligned}$$

It follows that $\text{fmap}_L(g) \circ \text{fmap}_L(f) = \text{id}$, i.e. $\text{fmap}_L(g)$ is a left inverse for $\text{fmap}_L(f)$. Since $\text{fmap}_L(f)$ has a left inverse, it is injective.

18 “Applied functional type theory”: A proposal

What exactly is the extent of “theory” that a practicing functional programmer should know in order to be effective at writing code in the functional paradigm? This question is not yet resolved. This book presents a coherent body of theoretical knowledge that, in the author’s view, fits the description of “practicable functional programming theory”. This body of knowledge could be understood as a separate branch of computer science called **applied functional type theory** (AFTT). This is the area of theoretical computer science that serves the needs of functional programmers working as software engineers.

It is for these practitioners, rather than for academic researchers, that this book sets out to examine the functional programming inventions over the last 30 years, – such as the “functional pearls” papers¹ – and to determine the scope of theoretical material that has demonstrated its pragmatic usefulness and thus belongs to AFTT, as opposed to material that is purely academic and may be tentatively omitted. This book is a first step towards formulating AFTT.

In this book, code is written in Scala because the author is fluent in that language. However, most of this material will work equally well in Haskell, OCaml, and other FP languages. This is so because the science of functional programming, called AFTT, is not a set of tricks specific to Scala or Haskell. An advanced user of any other functional programming language will have to face the same questions and struggle with the same practical issues.

18.1 AFTT is not covered by computer science curricula

Traditional courses of theoretical computer science (algorithms and data structures, complexity theory, distributed systems, databases, network systems, compilers, operating systems) are largely not relevant to AFTT.

Here is an example: To an academic computer scientist, the “science behind Haskell” is the theory of lambda-calculus, the type-theoretic “System $F\omega$ ”, and formal semantics. These theories guided the design of the Haskell language and define rigorously what a Haskell program “means” in a mathematical sense. Academic computer science courses teach these theories, although typically only at the graduate level.

However, a practicing Haskell or Scala programmer is not concerned with designing Haskell or Scala, or with proving any theoretical properties of those languages. A practicing programmer is mainly concerned with *using* a chosen programming language to *write code*.

Neither the theory of lambda-calculus, nor proofs of type-theoretical properties of “System $F\omega$ ”, nor theories of formal semantics will actually help a programmer to write code. So all these theories are not within the scope of AFTT. It appears that functional programming does not require graduate-level theoretical studies.

As an example of theoretical material that *is* within the scope of AFTT, consider the equational laws imposed on applicative functors (see Chapter 10). It is essential for a practicing functional programmer to be able to recognize and use applicative functors. An applicative functor is a data structure specifying declaratively a set of operations that run independently of each other. Programs can then easily combine these operations, for example, in order to execute them in parallel, or to refactor the program for better maintainability.

¹https://wiki.haskell.org/Research_papers/Functional_pearls

To use this functionality, the programmer must begin by checking whether a given data structure satisfies the laws of applicative functors. In a given application, a data structure may be dictated in part by the business logic rather than by a programmer's choice. The programmer first writes down the type of that data structure and the code implementing the required methods, and then checks that the laws hold. The data structure may need to be adjusted in order to fit the definition of an applicative functor or its laws.

This work is done using pen and paper, in a mathematical notation. Once the applicative laws are verified, the programmer proceeds to write code using that data structure.

Because of the mathematical proofs, it is assured that the data structure satisfies the known properties of applicative functors, no matter how the rest of the program is written. So, for example, it is assured that the relevant effects can be automatically parallelized and will still work correctly. In this way, AFTT directly guides the programmer and helps to write correct code.

Applicative functors were discovered by practitioners who were using Haskell for writing code, in applications such as parser combinators, compilers, and domain-specific languages for parallel computations. However, applicative functors are not a feature of Haskell: they are the same in Scala, OCaml, or any other functional programming language. And yet, no standard computer science textbook defines applicative functors, motivates their laws, explores their structure on basic examples, or shows data structures that are *not* applicative functors and explains why. (Books on category theory and type theory also do not mention applicative functors.)

18.2 AFTT is not category theory, type theory, or formal logic

So far it appears that AFTT includes a selection of certain areas of category theory, formal logic, and type theory. However, software engineers would not derive much benefit from following traditional academic courses in these subjects, because their presentation is too abstract and at the same time lacks specific results necessary for practical programming. In other words, the traditional academic courses answer questions that academic computer scientists have, not questions that software engineers have.

There exist several books intended as presentations of category theory “for computer scientists” or “for programmers”. However, these books do not explain certain concepts relevant to programming, such as applicative or traversable functors. Instead, these books contain purely theoretical topics such as limits, adjunctions, or toposes, – concepts that have no applications in practical functional programming today.

Typical questions in these books are “Is X an introduction rule or an elimination rule” and “Does the property Y hold in non-small categories, or only in the category of sets”. Questions a Scala programmer might have are “Can we compute a value of type `Either[Z, R => A]` from a value of type `R => Either[Z, A]`” and “Is the type constructor `F[A] = Option[(A,A,A)]` a monad or only an applicative functor”. The scope of AFTT includes answering the last two questions but *not* the first two.

A software engineer hoping to understand the foundations of functional programming will not find the concepts of filterable, applicative, or traversable functors in any books on category theory, including books intended for programmers. And yet, these concepts are necessary to obtain a mathematically correct implementation of such foundationally important operations as `filter`, `zip`, and `traverse` – operations that functional programmers often use in their code.

To compensate for the lack of AFTT textbooks, programmers have written many online tutorials for each other, trying to explain the theoretical concepts necessary for practical work. There are the infamous “monad tutorials”, but also tutorials about applicative functors, traversable functors, free monads, and so on. These tutorials tend to be hands-on (“run this code now and see what happens”) and narrow in scope, limited to one or two specific questions and specific applications. Such tutorials usually do not present sufficient mathematical insights to help programmers develop the necessary mathematical intuition.

For example, “free monads” became popular in the Scala community around 2015. Many talks about free monads were presented at Scala engineering conferences, each giving their own slightly

different implementation but never formulating rigorously the required properties for a piece of code to be a valid implementation of the free monad.

Without knowledge of mathematical principles behind free monads, a programmer cannot make sure that a given implementation is correct. However, books on category theory present free monads in a way that is unsuitable for programming applications: a free monad is just an adjoint functor to a forgetful functor into the category of sets. (“What’s your problem?” as the joke would go.) This definition is too abstract and, for instance, cannot be used to check whether a given implementation of the free monad in Scala is correct.

Perhaps the best selection of AFTT tutorial material can be found in the Haskell Wikibooks.² However, those tutorials are incomplete and limited to explaining the use of Haskell. Many of them are suitable neither as a first introduction nor as a reference on AFTT. Also, the Haskell Wikibooks tutorials rarely show any proofs or derivations of equational laws.

Apart from referring to some notions from category theory, AFTT also uses some concepts from type theory and formal logic. However, existing textbooks on type theory and formal logic focus on domain theory and proof theory – which is a lot of information that practicing programmers will have difficulty assimilating and yet will have no chance of ever applying in their daily work. At the same time, these books never mention practical techniques used in many functional programming libraries today, such as quantified types, types parameterized by type constructors, or partial type-level functions (known as “typeclasses”).

Theory can help the programmer with certain practical tasks, such as:

- Deciding whether two data types are equivalent and implementing the isomorphism transformations. For example, the Scala type `(A, Either[B,C])` is equivalent to `Either[(A,B), (A,C)]`.
- Checking whether a definition of a recursive type is “reasonable”, i.e. does not lead to a useless infinite recursion. An example of a useless recursive type definition is `case class Bad(x: Bad)`.
- Deciding whether a function with a given type signature can be implemented. For example, `def f[A, B]: (A => B) => A` cannot be implemented but `def g[A, B]: A => (B => A)` can be.
- Deriving an implementation of a function from its type signature and checking the required laws; for example, deriving the `flatMap` method for the `Reader` monad from the type signature

```
def flatMap[Z, A, B](r: Z => A)(f: A => Z => B): Z => B
```

and verifying that the monad laws hold.

These practical tasks are actual real-world-coding applications of domain theory and the Curry-Howard correspondence. However, existing books on type theory and logic do not give practical recipes for performing these tasks.

On the other hand, books such as *Scala with Cats*³ and *Functional programming, simplified*⁴ are focused on explaining the practical aspects of programming and do not adequately treat the equational laws that the mathematical structures require (such as the laws for applicative or monadic functors).

The only existing Scala-based AFTT textbook aiming at the proper scope is *Functional Programming in Scala*⁵, which balances practical considerations with theoretical developments such as equational laws. The present book, *Science of Functional Programming*⁶, is written at about the same level but goes deeper into the mathematical foundations and at the same time gives a wider range of examples. It is an attempt to delineate the proper scope of AFTT and to develop a rigorous yet clear and approachable presentation of the chosen material.

²<https://en.wikibooks.org/wiki/Haskell>

³<https://underscore.io/books/scala-with-cats/>

⁴<https://alvinalexander.com/scala/functional-programming-simplified-book>

⁵<https://www.manning.com/books/functional-programming-in-scala>

⁶<http://www.lulu.com/content/paperback-book/the-science-of-functional-programming/24915714>

19 Essay: Software engineers and software artisans

Let us look at the differences between the kind of activities we ordinarily call engineering, as opposed to artisanship or craftsmanship. It will then become apparent that today's computer programmers are better understood as "software artisans" rather than software engineers.

19.1 Engineering disciplines

Consider what kinds of process a mechanical engineer, a chemical engineer, or an electrical engineer follows in their work, and what kind of studies they require for proficiency in their work.

A mechanical engineer studies¹ calculus, linear algebra, differential geometry, and several areas of physics such as theoretical mechanics, thermodynamics, and elasticity theory, and then uses calculations to guide the design of a bridge, say. A chemical engineer studies² chemistry, thermodynamics, calculus, linear algebra, differential equations, some areas of physics such as thermodynamics and kinetic theory, and uses calculations to guide the design of a chemical process, say. An electrical engineer studies³ advanced calculus, linear algebra, and several areas of physics such as electrodynamics and quantum theory, and uses calculations to design an antenna or a microchip.

The pattern here is that an engineer uses mathematics and natural sciences in order to design new devices. Mathematical calculations and scientific reasoning are required *before* drawing a design, let alone building a real device or machine.

Some of the studies required for engineers include arcane abstract concepts such as a "rank-4 elasticity tensor"⁴ (used in calculations of elasticity of materials), "Lagrangian with non-holonomic constraints"⁵ (used in robotics), the "Gibbs free energy" (for chemical reactor design⁶), or the "Fourier transform of the delta function"⁷ and the "inverse Z-transform"⁸ (for digital signal processing).

To be sure, a significant part of what engineers do is not covered by any theory: the *know-how*, the informal reasoning, the traditional knowledge passed on from expert to novice, – all those skills that are hard to formalize are important. Nevertheless, engineering is crucially based on natural science and mathematics for some of its decision-making about new designs.

19.2 Artisanship: Trades and crafts

Now consider what kinds of things shoemakers, plumbers, or home painters do, and what they have to learn in order to become proficient in their profession.

A novice shoemaker, for example, begins by copying some drawings⁹ and goes on to cutting leather in a home workshop. Apprenticeships proceed via learning by doing, with comments and

¹<https://www.colorado.edu/mechanical/undergraduate-students/curriculum>

²<https://www.colorado.edu/engineering/sample-undergraduate-curriculum-chemical>

³<http://archive.is/XYLyE>

⁴https://serc.carleton.edu/NAGTWorkshops/mineralogy/mineral_physics/tensors.html

⁵<https://arxiv.org/abs/math/0008147>

⁶<https://www.amazon.com/Introduction-Chemical-Engineering-Kinetics-Reactor/dp/1118368258>

⁷<https://www.youtube.com/watch?v=KAbqISZ6SHQ>

⁸<http://archive.is/SsJqP>

⁹<https://youtu.be/cY5MY0czMAk?t=141>

instructions from an expert. After a few years of apprenticeship (for example, a painter apprenticeship in California¹⁰ can be as short as 2 years), a new specialist is ready to start productive work.

All these trades operate entirely from tradition and practical experience. The trades do not require academic study because there is no formal theory from which to proceed. Of course, there is *a lot* to learn in the crafts, and it takes prolonged effort to become a good artisan in any profession. But there are no rank-4 tensors to calculate, nor any differential equations to solve; no Fourier transforms to apply to delta functions, and no Lagrangians to check for non-holonomic constraints.

Artisans do not study any formal science or mathematics because their professions do not make use of any *formal computation* for guiding their designs or processes.

19.3 Programmers today are artisans, not engineers

Programmers are *not engineers* in the sense we normally see the engineering professions.

19.3.1 No requirement of formal study

According to this recent Stack Overflow survey¹¹, about half of the programmers do not have a degree in Computer Science. The author of this book is a self-taught programmer who has degrees in physics but never formally studied computer science or taken any academic courses in algorithms, data structures, computer networks, compilers, programming languages, or other CS topics.

A large fraction of successful programmers have no college degrees and perhaps *never* studied formally. They acquired all their knowledge and skills through self-study and practical work. Robert C. Martin¹² is a prominent example; an outspoken guru in the arts of programming who has seen it all, he routinely refers to programmers as artisans¹³ and uses the appropriate imagery: novices, trade and craft, the “honor of the guild”, etc. He compares programmers to plumbers, electricians, lawyers, and surgeons, but never to mathematicians, physicists, or engineers of any kind. According to one of his blog posts¹⁴, he started working at age 17 as a self-taught programmer, and then went on to more jobs in the software industry; he never mentions going to college. It is clear that R. C. Martin is an expert craftsman, and that he did *not* need academic study to master his craft.

Here is another opinion¹⁵ (emphasis is theirs):

Software Engineering is unique among the STEM careers in that it absolutely does *not* require a college degree to be successful. It most certainly does not require licensing or certification. *It requires experience.*

This description fits a career in crafts – but certainly not a career, say, in electrical engineering.

The high demand for software developers gave rise to “developer boot camps”¹⁶ – vocational schools that educate new programmers in a few months through purely practical training, with no formal theory or mathematics involved. These vocational schools are successful¹⁷ in job placement. But it is unimaginable that a 6-month crash course or even a 2-year vocational school could prepare an engineer to work successfully on say, quantum computers¹⁸ without ever having studied quantum physics or calculus.

¹⁰http://www.calapprenticeship.org/programs/painter_apprenticeship.php

¹¹<https://thenextweb.com/insider/2016/04/23/dont-need-go-college-anymore-programmer/>

¹²https://en.wikipedia.org/wiki/Robert_C._Martin

¹³<https://blog.cleancoder.com/uncle-bob/2013/02/01/The-Humble-Craftsman.html>

¹⁴<https://blog.cleancoder.com/uncle-bob/2013/11/25/Novices-Coda.html>

¹⁵<http://archive.is/tAKQ3>

¹⁶<http://archive.is/GkOL9>

¹⁷<http://archive.is/E9FXP>

¹⁸<https://www.dwavesys.com/quantum-computing>

19.3.2 No mathematical formalism guides software development

Most books on software engineering contain no formulas or equations, no mathematical derivations of any results, and no precise definitions of the various technical terms they are using (such as “object-oriented” or “software architecture”). Some of those books¹⁹ have almost no program code in them; instead they are filled with words and illustrative diagrams. These books talk about how programmers should approach their job, how to organize the work flow and the code architecture, etc., in vague and general terms: “code is about detail”, “you must never abandon the big picture”, “you should avoid tight coupling in your modules”, “a class must serve a single responsibility”, and so on. Practitioners such as R. C. Martin never studied any formalisms and do not think in terms of formalisms; instead they think in vaguely formulated, heuristic “principles”.²⁰

In contrast, textbooks on mechanical or electrical engineering include a significant amount of mathematics. The design of a microwave antenna is guided not by an “open and closed module principle” but by solving the relevant differential equations²¹ coming from electrodynamics.

Donald Knuth’s classic textbook is called “*The Art of Programming*”. It is full of tips and tricks about how to program; but it does not provide any formal theory that could guide programmers in actually *writing* programs. There is nothing in that book that would be similar to the way mathematical formalism guides designs in electrical or mechanical engineering. If Knuth’s books were based on such formalism, they would have looked quite differently: some theory would be first explained and then applied to help us write code.

Knuth’s books provide many rigorously derived algorithms. But algorithms are similar to patented inventions: they can be used immediately without further study. Understanding an algorithm is not similar to understanding a mathematical theory. Knowing one algorithm does not make it easier to develop another algorithm in an unrelated domain. In comparison, knowing how to solve differential equations will be applicable to thousands of different areas of science and engineering.

A book exists²² with the title “Science of Programming”, but the title is misleading. The author does not propose a science, similar to physics, at the foundation of the process of designing programs, similarly to how calculations in quantum physics predict the properties of a quantum device. The book claims to give precise methods that guide programmers in writing code, but the scope of proposed methods is narrow: the design of simple algorithms for iterative manipulation of data. The procedure suggested in that book is far from a formal mathematical *derivation* of programs from specifications. (A book with that title²³ also exists, and similarly disappoints.) In any case, programmers today are oblivious to these books and do not use the methods explained there.

Standard computer science courses today do not teach a true *engineering* approach to software construction. They do teach analysis of programs using formal mathematical methods; the main such methods are complexity analysis²⁴ (the “big-*O* notation”) and formal verification²⁵. But programs are analyzed or verified only *after* they are complete. Theory does not guide the actual *process* of writing code, does not suggest good ways of organizing the code (e.g. choosing which classes or functions or modules should be defined), and does not tell programmers which data structures or APIs would be best to implement. Programmers make these design decisions purely on the basis of experience and intuition, trial-and-error, copy-paste, and guesswork.

The theory of program analysis and verification is somewhat analogous to writing a mathematical equation for the surface of a shoe made by a fashion designer. True, the “shoe surface equations” are mathematically rigorous and can be “analyzed” or “verified”; but the equations are written after the fact and do not guide the fashion designers in actually making shoes. It is understandable that fashion designers do not study the mathematical theory of surfaces.

¹⁹E.g. <https://www.amazon.com/Object-Oriented-Software-Engineering-Unified-Methodology/dp/0073376256>

²⁰<https://blog.cleancoder.com/uncle-bob/2016/03/19/GivingUpOnTDD.html>

²¹<https://youtu.be/8KpfVsJ5Jw4?t=447>

²²<https://www.amazon.com/Science-Programming-Monographs-Computer/dp/0387964800>

²³<https://www.amazon.com/Program-Derivation-Development-Specifications-International/dp/0201416247>

²⁴<https://www.cs.cmu.edu/~adamchik/15-121/lectures/Algorithmic%20Complexity/complexity.html>

²⁵https://en.wikipedia.org/wiki/Formal_verification

19.3.3 Programmers avoid academic terminology

Programmers appear to dislike terminology such as “functor”, “monad”, or “lambda-functions”:

Those fancy words used by functional programmers purists really annoy me. Monads, functors... Nonsense!!! ²⁶

Perhaps only a small minority of software engineers actually complain about this; the vast majority remain unaware of “functors” or “monads”.

However, chemical engineers accept the need for studying differential equations and do not wince at the terms “phase diagram” or “Gibbs free energy”. Electrical engineers do not complain that the word “Fourier” is difficult to spell, or that “delta-function” is a weird thing to say. Mechanical engineers take it for granted that they need to calculate with “tensors” and “Lagrangians” and “non-holonomic constraints”. The arcane terminology seems to be the least of their difficulties, as their textbooks are full of complicated equations and long derivations.

Similarly, software engineers would not complain about the word “functor”, or about having to study the derivation of the algebraic laws for “monads,” – if they were actually *engineers*. True software engineers’ textbooks would be full of equations and derivations, which would be used to perform calculations required *before* starting to write code.

19.4 Towards software engineering

It is now clear that we do not presently have true software engineering. The people employed under that job title are actually artisans. They work using artisanal methods, and their culture and processes are that of a crafts guild.

One could point out that numerical simulations required for physics or the matrix calculations required for machine learning are “mathematical”. True, these programming *tasks* are mathematical in nature and require formal theory to be *formulated*. However, mathematical *subject matter* (aerospace control, physics or astronomy experiments, mathematical statistics, etc.) does not mean that the process of programming is a form of engineering. Data scientists, aerospace engineers, and natural scientists all write code nowadays – and they are all working as artisans when they write code.

True software engineering means having a theory that guides and informs the process of creating programs, – not theory that describes or analyzes programs after they are *somehow* written.

We expect that software engineers’ textbooks should be full of equations. What theory should those equations represent?

I believe this theory already exists, and I call it **applied functional type theory**. It is the algebraic foundation of the modern practice of functional programming, as implemented in languages such as OCaml, Haskell, and Scala. This theory is a blend of type theory, category theory, and logical proof theory. It has been in development since late 1990s and is still being actively worked on by a community of academic computer scientists and advanced software practitioners.

To appreciate that functional programming, unlike any other programming paradigm, *has a theory that guides coding*, we can look at some recent software engineering conferences such as “Scala By the Bay”²⁷ or BayHac²⁸, or at the numerous FP-related online tutorials and blogs. We cannot fail to notice that much time is devoted not to showing code but to a peculiar kind of mathematical reasoning. Rather than focusing on one or another API or algorithm, as it is often the case with other software engineering blogs or presentations, an FP speaker describes a *mathematical structure* – such as the “applicative functor”²⁹ or the “free monad”³⁰ – and illustrates its use for practical coding.

²⁶<http://archive.is/65K3D>

²⁷<http://2015.scala.bythebay.io/>

²⁸<http://bayhac.org/>

²⁹<http://www.youtube.com/watch?v=bmIxIslimVY>

³⁰<http://www.youtube.com/watch?v=U0lK0hnb4U>

These people are not graduate students showing off their theoretical research; they are practitioners, software engineers who use FP on their jobs. It is just the nature of FP that certain mathematical tools – coming from formal logic and category theory – are now directly applicable to practical programming tasks.

These mathematical tools are not mere tricks for a specific programming language; they apply equally to all FP languages. Before starting to write code, the programmer can jot down certain calculations in a mathematical notation (see Fig. 19.1). The results of those calculations will help design the code fragment the programmer is about to write. This activity is similar to that of an engineer who performs some mathematical calculations before embarking on a design project.

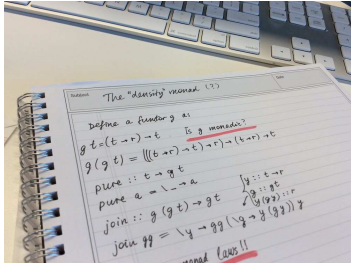


Figure 19.1: A programmer performs a derivation in Haskell before writing code.

A recent example of the hand-in-hand development of the functional type theory and its applications is seen in the “free applicative functor” construction. It was first described in a 2014 paper³¹; a couple of years later, a combined free applicative / free monad data type was designed and its implementation proposed³² as well as in Haskell³³. This technique allows programmers to implement declarative side-effect computations where some parts are sequential but other parts are computed in parallel, and to achieve the parallelism *automatically* while maintaining the composability of the resulting

programs. The new technique has distinct advantages over using monad transformers, which was the previous method of composing declarative side-effects.

The “free applicative / free monad” combination was designed and implemented by true software engineers. They first wrote down the types and derived the necessary algebraic properties; the obtained results directly guided them about how to proceed writing the library API.

Another example of a development in functional type theory is the “tagless final” encoding of data types, **first described in 2009**. This technique, developed from category theory and type theory motivations, has several advantages over the free monad technique and can improve upon it in a number of cases – just as the free monad itself was designed to cure certain problems with monad transformers³⁴. The new technique is also not a trick in a specific programming language; rather, it is a theoretical development that is available to programmers in any language (even in Java³⁵).

This example shows that we may need several more years of work before the practical aspects of using “functional type theory” are sufficiently well understood by the FP community. The theory is in active development, and its design patterns – as well as the exact scope of the requisite theoretical material – are still being figured out. If the 40-year gap hypothesis³⁶ holds, we should expect functional type theory (perhaps under a different name) to become mainstream by 2030. This book is a step towards a clear designation of the scope of that theory.

³¹<https://arxiv.org/pdf/1403.0749.pdf>

³²<https://github.com/typelevel/cats/issues/983>

³³<https://elvishjerricco.github.io/2016/04/08/applicative-effects-in-free-monads.html>

³⁴<http://blog.ezyang.com/2013/09/if-youre-using-lift-youre-doing-it-wrong-probably/>

³⁵<http://archive.is/rLAh9>

³⁶<http://archive.is/rJc4A>

19.5 Does software need engineers, or are artisans good enough?

The demand for programmers is growing. “Software developer” was #1 best job³⁷ in the US in 2018. But is there a demand for engineers or just for artisans?

We do not seem to be able³⁸ to train enough software artisans. So, it is probably impossible to train as many software engineers in the true sense of the word. Modern computer science courses do not actually train engineers in that sense; at best, they train academic researchers who write code as software artisans. Recalling the situation in construction business, with a few architects and hundreds of construction workers, we might also conclude that, perhaps, only a few software engineers are required per hundred software artisans.

What is the price of *not* having engineers, of replacing them with artisans?

Software practitioners have long bemoaned the mysterious difficulty of software development. Code “rots with time”, its complexity grows “out of control”, and operating systems have been notorious for ever-appearing security flaws³⁹ despite many thousands of programmers and testers employed. Clearly, we overestimated the capacity of the human brain for artisanal programming.

It is precisely in designing large and robust software systems that we would benefit from true engineering. Humanity has been building bridges and using chemical reactions long before mechanical or chemical engineering disciplines were developed and founded upon rigorous theory. Once the theory became available, humanity proceeded to create unimaginably more complicated and powerful structures and devices than ever before. It is clear that trial, error, and adherence to tradition is inadequate for the software development tasks in front of us.

To build large and reliable software, such as new mobile or embedded operating systems or distributed peer-to-peer trust architectures, we will most likely need the qualitative increase in productivity and reliability that can only come from replacing artisanal programming by a true engineering discipline. Functional type theory and functional programming are first steps in that direction.

³⁷<http://archive.is/cGJ2T>

³⁸<http://archive.is/137b8>

³⁹<http://archive.fo/HtQzw>

20 Essay: Towards functional data engineering with Scala

Data engineering is among the most in-demand¹ novel occupations in the IT world today. Data engineers create software pipelines that process large volumes of data efficiently. Why did the Scala programming language emerge as a premier tool² for crafting the foundational data engineering technologies such as Spark or Akka? Why is Scala in such demand³ within the world of big data?

There are reasons to believe that the choice of Scala was not accidental.

20.1 Data is math

Humanity has been working with data at least since Babylonian tax tables⁴ and the ancient Chinese number books⁵. Mathematics summarizes several millennia's worth of data processing experience in a few fundamental tenets:

- Data is *immutable*, because facts are immutable.
- Each *type* of values – population count, land area, distances, prices, dates, times, – needs to be handled separately; e.g. it is meaningless to add a distance to a population count.
- Data processing should be performed according to *mathematical formulas*.

Violating these tenets produces nonsense (see Fig. 20.1 for a real-life illustration).

The power of the principles of mathematics extends over all epochs and all cultures; math is the same in Rio de Janeiro, in Kuala-Lumpur, and in Pyongyang (Fig. 20.2).

20.2 Functional programming is math

The functional programming paradigm is based on mathematical principles: values are immutable, data processing is coded through formula-like expressions, and each type of data is required to match correctly during the computations. The type-checking system automatically prevents programmers from making many kinds of coding errors. In addition, programming languages such as Scala and Haskell have a set of features adapted to building powerful



Figure 20.1: Mixing incompatible data types gives nonsense results.

¹<http://archive.is/mK59h>

²<https://www.slideshare.net/nootsab/scala-the-unpredicted-lingua-franca-for-data-science>

³<https://techcrunch.com/2016/06/14/scala-is-the-new-golden-child/>

⁴<https://www.nytimes.com/2017/08/29/science/trigonometry-babylonian-tablet.html>

⁵<http://quatr.us/china/science/chinamath.htm>

abstractions and domain-specific languages. This power of abstraction is not accidental. Since mathematics is the ultimate art of building abstractions, math-based functional programming languages capitalize on the advantage of several millennia of mathematical experience.

A prominent example of how mathematics informs the design of programming languages is the connection between constructive logic⁶ and the programming language's type system, called the Curry-Howard (CH) correspondence⁷. The main idea of the CH correspondence is to think of programs as mathematical formulas that compute a value of a certain type A . The CH correspondence is between programs and logical propositions: To any program that computes a value of type A , there corresponds a proposition stating that "a value of type A can be computed".

This may sound rather theoretical so far. To see the real value of the CH correspondence, recall that formal logic has operations "*and*", "*or*", and "*implies*". For any two propositions A, B , we can construct the propositions " A *and* B ", " A *or* B ", " A *implies* B ". These three logical operations are foundational; without one of them, the logic is *incomplete* (you cannot derive some theorems).

A programming language **obeys the CH correspondence** to the logic if for any two types A, B , the language also contains composite types corresponding to the logical formulas " A *or* B ", " A *and* B ", " A *implies* B ". In Scala, these composite types are `Either[A,B]`, the tuple `(A,B)`, and the function type, `A⇒B`. All modern functional languages such as OCaml, Haskell, Scala, F#, Swift, Elm, and PureScript support these three type constructions and thus are faithful to the CH correspondence. Having a *complete* logic in a language's type system enables declarative domain-driven code design⁸.



Figure 20.2: The Pyongyang method of error-free programming.

It is interesting to note that most older programming languages (C/C++, Java, JavaScript, Python) do not support some of these composite types. In other words, these programming languages have type systems based on an incomplete logic. As a result, users of these languages have to implement burdensome workarounds that make for error-prone code. Failure to follow mathematical principles has real costs (Figure 20.2).

20.3 The power of abstraction

Early adopters of Scala, such as Netflix, LinkedIn, and Twitter, were implementing what is now called "big data engineering". The required software needs to be highly concurrent,

distributed, and resilient to failure. These software companies used Scala as their main implementation language and reaped the benefits of functional programming.

What makes Scala suitable for big data tasks? The only reliable way of managing massively concurrent code is to use sufficiently high-level abstractions that make application code declarative. The two most important such abstractions are the "resilient distributed dataset" (RDD) of Apache Spark and the "reactive stream" used in systems such as Kafka, Akka Streams, and Apache Flink. While these abstractions are certainly implementable in Java or Python, a fully declarative and type-safe usage is possible only in a programming language with a sophisticated functional type system. Among the currently available mature functional languages, only Scala and Haskell are technically adequate for that task, due to their support for typeclasses and higher-order types. The early adopters of Scala were able to reap the benefits of the powerful abstractions Scala supports. In this way, Scala enabled those businesses to engineer reliably and to scale up their massively concurrent computations.

⁶https://en.wikipedia.org/wiki/Intuitionistic_logic

⁷https://en.wikipedia.org/wiki/Curry%E2%80%93Howard_correspondence

⁸<https://fsharpforfunandprofit.com/ddd/>

It remains to see why Scala and not, say, Haskell became the *lingua franca* of big data.

20.4 Scala is Java on math

The recently invented general-purpose functional programming languages can be grouped into “industrial” (F#, Scala, Swift) and “academic” (OCaml, Haskell).

The “academic” languages are clean-room implementations of well-researched mathematical principles of programming language design (the CH correspondence being one such principle). These languages are unencumbered by requirements of compatibility with any existing platform or libraries. Because of this, the “academic” languages are perfect playgrounds for taking various mathematical ideas to their logical conclusion. At the same time, software practitioners struggle to adopt these languages due to a steep learning curve, a lack of enterprise-grade libraries and tool support, and immature package management.

The languages from the “industrial” group are based on existing and mature software ecosystems: F# on .NET, Scala on JVM, and Swift on the MacOS/iOS platform. One of the important design requirements for these languages is 100% binary compatibility with their “parent” platforms and languages (F# with C#, Scala with Java, and Swift with Objective-C). Because of this, developers can immediately take advantage of the existing tooling, package management, and industry-strength libraries, while slowly ramping up the idiomatic usage of new language features. However, the same compatibility requirements necessitated certain limitations in the languages, making their design less than fully satisfactory from the functional programming viewpoint.

It is now easy to see why the adoption rate of the “industrial” group of languages is much higher⁹ than that of the “academic” languages. The transition to the functional paradigm is also made smoother for software developers because F#, Scala, and Swift seamlessly support the familiar object-oriented programming paradigm. At the same time, these new languages still have logically complete type systems, which gives developers an important benefit of type-safe domain modeling.

Nevertheless, the type systems of these languages are not equally powerful. For instance, F# and Swift are similar to OCaml in many ways but omit OCaml’s parameterized modules and some other features. Of all mentioned languages, only Scala and Haskell directly support typeclasses and higher-order types, which are helpful for expressing abstractions such as automatically parallelized data sets or asynchronous data streams.

To see the impact of these advanced features, consider LINQ, a domain-specific language for database queries on .NET, implemented in C# and F# through a special built-in syntax supported by Microsoft’s compilers. Analogous functionality is provided in Scala as a *library*, without need to modify the Scala compiler, by several open-source projects such as Slick, Squeryl, or Quill. Similar libraries exist for Haskell – but not in languages with less powerful type systems.

20.5 Summary

The decisive advantages of Scala over other contenders (such as OCaml, Haskell, F#, or Swift) are:

1. Functional collections in the standard library.
2. A sophisticated type system with support for typeclasses and higher-order types.
3. Seamless compatibility with a mature software ecosystem (JVM).

Based on this assessment, we may be confident in Scala’s great future as a main implementation language for big data engineering.

⁹<https://www.tiobe.com/tiobe-index/>, archived at <http://archive.is/RsNH8>

Part V

Appendixes

A Notations

Certain notations and terms were chosen in this book differently from what the functional programming community currently uses. The proposed notation is well adapted to reasoning about types and code, and especially for designing data types and proving the equational laws of typeclasses.

A.1 Summary of notations for types and code

F^A type constructor F with type argument A . In Scala, `F[A]`

$x:A$ value x has type A ; in Scala, `x:A`

$\mathbb{1}$, 1 the unit type and its value; in Scala, `Unit` and `()`

$\mathbb{0}$ the void type. In Scala, `Nothing`

$A + B$ a disjunctive type. In Scala, this type is `Either[A, B]`

$x:A + \mathbb{0}:B$ a value of a disjunctive type $A + B$. In Scala, `Left(x)`

$A \times B$ a product (tuple) type. In Scala, this type is `(A,B)`

$a:A \times b:B$ value of a tuple type $A \times B$. In Scala, `(a, b)`

$A \Rightarrow B$ the function type, mapping from A to B

$x:A \Rightarrow f$ a nameless function (as a value). In Scala, `{ x:A => f }`

id the identity function; in Scala, `identity[A]`

\triangleq “equal by definition”

\cong for types, a natural isomorphism between types; for values, “equivalent” values according to an established isomorphism

$A:F^B$ type annotation, used for defining unfunctors (GADTs)

\wedge logical conjunction; $\alpha \wedge \beta$ means “both α and β are true”

\vee logical disjunction; $\alpha \vee \beta$ means “either α or β or both are true”

\Rightarrow logical implication; $\alpha \Rightarrow \beta$ means “if α is true then β is true”

fmap_F the standard method `fmap` pertaining to a functor F . In Scala, `Functor[F].fmap`

flm_F the standard method `flatMap` pertaining to a monad F . In Scala, `Monad[F].flatMap`

ftn_F the standard method `flatten` pertaining to a monad F . In Scala, `Monad[F].flatten`

pu_F the standard method `pure` of a monad F . In Scala, `Monad[F].pure`

F^\bullet the type constructor F understood as a type-level function. In Scala, `F[_]`

$F^\bullet \rightsquigarrow G^\bullet$ or $F^A \rightsquigarrow G^A$ a natural transformation between functors F and G . In Scala, `F ~> G`

$\forall A. P^A$ a universally quantified type expression. In Scala 3, `[A] => P[A]`

$\exists A. P^A$ an existentially quantified type expression. In Scala, `{ type A; val x: P[A] }`

\circ the forward composition of functions: $f \circ g$ is $x \Rightarrow g(f(x))$. In Scala, `f andThen g`

\circ the backward composition of functions: $f \circ g$ is $x \Rightarrow f(g(x))$. In Scala, `f compose g`

\circ the backward composition of type constructors: $F \circ G$ is F^G

\triangleright use a value as the argument of a function: $x \triangleright f$ is $f(x)$. In Scala, `x.pipe(f)`

$f^{\uparrow G}$ a function f raised to a functor G ; same as `fmapGf`

$f^{\uparrow G \uparrow H}$ a function raised first to G and then to H . In Scala, `h.map(_ .map(f))`

$f^{\downarrow H}$ a function f raised to a contrafunctor

\diamond_M the Kleisli product operation for the monad M

\oplus the binary operation of a monoid. In Scala, `x |+| y`

Δ the “diagonal” function of type $\forall A. A \Rightarrow A \times A$

$\nabla_1, \nabla_2, \dots$ the projections from a tuple to its first, second, ..., parts

\boxtimes pair product of functions, $(f \boxtimes g)(a \times b) = f(a) \times g(b)$

$[a, b, c]$ an ordered sequence of values. In Scala, `Seq(a, b, c)`

$\left\| \begin{array}{cc} x \Rightarrow x & 0 \\ 0 & a \Rightarrow a \times a \end{array} \right\|$ a function that works with disjunctive types

A.2 Detailed explanations

F^A means a type constructor F with a type parameter A . In Scala, this is `F[A]`. Type constructors with multiple type parameters are denoted by $F^{A,B,C}$.

$x^{:A}$ means a value x that has type A ; this is a **type annotation**. In Scala, a type annotation is `x:A`. The colon symbol, `:`, in the superscript shows that A is not a type argument (as it would be in a type constructor, F^A). The notation $x : A$ can be used as well, but $x^{:A}$ is easier to read when x is inside a larger code expression.

$\mathbb{1}$ means the unit type, and 1 means the value of the unit type. In Scala, the unit type is `Unit`, and its value is `()`. Example of using the unit type is $\mathbb{1} + A$, which corresponds to `Option[A]` in Scala.

$\mathbb{0}$ means the void type (the type with no values). In Scala, this is the type `Nothing`. Example of using the void type is to denote the empty part of a disjunction. For example, in the disjunction $\mathbb{1} + A$ the non-empty part is $\mathbb{0} + A$, which in Scala corresponds to `Some[A]`. The empty part $\mathbb{1} + \mathbb{0}$ corresponds to `None`. Similarly, $A + \mathbb{0}$ denotes the left part of the type $A + B$ (in Scala, `Left[A]`), while $\mathbb{0} + B$ denotes its right part (in Scala, `Right[A]`). Values of disjunctive types are denoted similarly. For instance, $x^{:A} + \mathbb{0}^{:B}$ denotes a value of the left part of the type $A + B$; in Scala, this value is written as `Left[A,B](x)`.

$A + B$ means the disjunctive type made from types A and B (or, a disjunction of A and B). In Scala, this is the type `Either[A, B]`.

$x^{:A} + \mathbb{0}^{:B}$ denotes a value of a disjunctive type $A + B$, where x is the value of type A , which is the chosen case, and $\mathbb{0}$ stands for other possible cases. For example, $x^{:A} + \mathbb{0}^B$ is `Left[A,B](x)` in Scala. Type annotations $^{:A}$ and $^{:B}$ may be omitted if the types are unambiguous from the context.

$A \times B$ means the product type made from types A and B . In Scala, this is the tuple type `(A,B)`.

$a{:}^A \times b{:}^B$ means a value of a tuple type $A \times B$; in Scala, this is the tuple value `(a, b)`. Type annotations $:^A$ and $:^B$ may be omitted if the types are unambiguous from the context.

$A \Rightarrow B$ means a function type from A to B . In Scala, this is the function type `A => B`.

$x{:}^A \Rightarrow y$ means a nameless function with argument x of type A and function body y . (Usually, the body y will be an expression that uses x . In Scala, this is `{ x: A => y }`). Type annotation $:^A$ may be omitted if the type is unambiguous from the context.

id means the identity function. The type of its argument should be either specified as id^A or $\text{id}^{A \Rightarrow A}$, or else should be unambiguous from the context. In Scala, `identity[A]` corresponds to id^A .

\triangleq means “equal by definition”. Examples:

- $f \triangleq (x{:}^{\text{Int}} \Rightarrow x + 10)$ is a definition of a function f . In Scala, this is `val f = { x: Int => x + 10 }`.
- $F^A \triangleq 1 + A$ is a definition of a type constructor F . In Scala, this is `type F[A] = Option[A]`.

\cong for types means an equivalence (an isomorphism) of types. For example, $A + A \times B \cong A \times (1 + B)$. The same symbol \cong for *values* means “equivalent” according to an equivalence relation that needs to be established in the text. For example, if we have established an equivalence that allows nested tuples to be reordered whenever needed, we can write $(a \times b) \times c \cong a \times (b \times c)$, meaning that these values are mapped to each other by the established isomorphism functions.

A^{F^B} in type expressions means that the type constructor F^\bullet assigns the type F^B to the type expression A . This notation is used for defining unfunctors (GADTs). For example, the Scala code

```
sealed trait F[A]
case class F1() extends F[Int]
case class F2[A](a: A) extends F[(A, String)]
```

defines an unfunctor, which is denoted by $F^A \triangleq 1{:}^{F^{\text{Int}}} + A{:}^{F^{A \times \text{String}}}$.

fmap_F means the standard method `fmap` of the `Functor` typeclass, implemented for the functor F . In Scala, this may be written as `Functor[F].fmap`. Since each functor F has its own specific implementation of fmap_F , the subscript “ F ” is not a type parameter of fmap_F . The method fmap_F actually has *two* type parameters, which can be written out as $\text{fmap}_F^{A,B}$. Then the type signature of fmap is written in full as $\text{fmap}_F^{A,B} : (A \Rightarrow B) \Rightarrow F^A \Rightarrow F^B$. For clarity, we may sometimes write out the type parameters A, B in the expression $\text{fmap}_F^{A,B}$, but in most cases these type parameters A, B can be omitted without loss of clarity.

pu_F denotes a monad F ’s method `pure`. This function has type signature $A \Rightarrow F^A$ that contains a type parameter A . In the code notation, the type parameter may be either omitted or denoted as pu_F^A . If we are using the `pure` method with a complicated type, e.g. $1 + P^A$, instead of the type parameter A , we might want to write this type parameter for clarity and write $\text{pu}_F^{1+P^A}$. The type signature of that function is then

$$\text{pu}_F^{1+P^A} : 1 + P^A \Rightarrow F^{1+P^A} .$$

But in most cases we will not need to write out the type parameters.

flm_F denotes a monad F ’s method `flatMap` with the type signature

$$\text{flm}_F : (A \Rightarrow F^B) \Rightarrow F^A \Rightarrow F^B .$$

Note that Scala’s standard `flatMap` type signature is not curried. The curried method flm_F is easier to use in calculations involving the monad’s laws.

ftn_F denotes a monad F ’s method `flatten` with the type signature

$$\text{ftn}_F : F^{F^A} \Rightarrow F^A .$$

F^\bullet means the type constructor F understood as a type-level function, – that is, with a type argument unspecified. In Scala, this is `F[_]`. The bullet symbol, \bullet , is used as a placeholder for the missing type parameter. I also simply write F when no type argument is needed, and it means the same as

F^\bullet . (For example, “a functor F ” and “a functor F^\bullet ” mean the same thing.) However, it is useful for clarity to be able to indicate the place where the type argument would appear. For instance, functor composition is denoted as F^G ; in Scala, this is `F[G[?]]` when using the “kind projector” plugin.¹ As another example, $T_L^{M,\bullet}$ denotes a monad transformer for the base monad L and the foreign monad M . The foreign monad M is a type parameter in $T_L^{M,\bullet}$, and so is the missing type parameter denoted by the placeholder symbol \bullet . (However, the base monad L is not a type parameter in $T_L^{M,\bullet}$ because the construction of the monad transformer depends sensitively on the internal details of L .)

$F^\bullet \rightsquigarrow G^\bullet$ or $F^A \rightsquigarrow G^A$ means a natural transformation between two functors F and G . In some Scala libraries, this is denoted by `F ~> G`.

$\forall A. P^A$ is a universally quantified type expression, in which A is a bound type parameter.

$\exists A. P^A$ is an existentially quantified type expression, in which A is a bound type parameter.

\circ means the forward composition of functions: $f \circ g$ (reads “ f before g ”) is the function defined as $x \Rightarrow g(f(x))$.

\circ means the backward composition of functions: $f \circ g$ (reads “ f after g ”) is the function defined as $x \Rightarrow f(g(x))$.

\circ with type constructors means their (backward) composition, for example $F \circ G$ denotes the type constructor F^G . In Scala, this is `F[G[A]]`.

$x \triangleright f$ means that x is inserted as the argument into the function f . This **pipe notation**, $x \triangleright f$, means the same expression as $f(x)$ or $f\ x$. In Scala, the expression $x \triangleright f$ is written as `x.pipe(f)` or, if `f` is a method, `x.f`. This syntax is used with many standard methods such as `.size` or `.toSeq`. Because the function f is to the *right* of x in this notation, forward compositions of functions such as $x \triangleright f \triangleright g$ are naturally grouped to the left, similarly to how this is done in Scala code, for example `x.toSeq.sorted`. The operation \triangleright (pronounced “pipe”) groups weaker than the forward composition (\circ), and so we have $x \triangleright f \circ g = x \triangleright f \triangleright g$ in this notation. Reasoning about code in the forwarding notation uses the identities

$$\begin{aligned} x \triangleright f &= f(x), & (x \triangleright f) \triangleright g &= x \triangleright f \triangleright g, \\ x \triangleright f \circ g &= x \triangleright (f \circ g), & x \triangleright f \triangleright g &= x \triangleright f \circ g. \end{aligned}$$

Some examples of reasoning in the pipe notation:

$$\begin{aligned} (a \Rightarrow a \triangleright f) &= (a \Rightarrow f(a)) = f, \\ f \triangleright (y \Rightarrow a \triangleright y) &= a \triangleright f = f(a), \\ f(y(x)) &= x \triangleright y \triangleright f \neq x \triangleright (y \triangleright f) = f(y)(x). \end{aligned}$$

The correspondence between the forward composition and the backward composition:

$$\begin{aligned} f \circ g &= g \circ f, \\ x \triangleright (f \circ g) &= x \triangleright f \circ g = x \triangleright f \triangleright g = g(f(x)) = (g \circ f)(x). \end{aligned}$$

$f^{\uparrow G}$ means a function f lifted to a functor G . For a function $f^{A \Rightarrow B}$, the application of $f^{\uparrow G}$ to a value g^{G^A} is written as $f^{\uparrow G}(g)$ or as $g \triangleright f^{\uparrow G}$. In Scala, this is `g.map(f)`. Nested lifting (i.e. lifting to the functor composition $H \circ G$) can be written as $f^{\uparrow G \uparrow H}$, which means $(f^{\uparrow G})^{\uparrow H}$ and produces a function of type $H^{G^A} \Rightarrow H^{G^B}$. Applying a nested lifting to a value h of type H^{G^A} is written as $f^{\uparrow G \uparrow H} h$ or $h \triangleright f^{\uparrow G \uparrow H}$. In Scala, this is `h.map(_ . map(f))`. The functor composition law is written as

$$p^{\uparrow G} \circ q^{\uparrow G} = (p \circ q)^{\uparrow G}.$$

Note the similarity between Scala code `x.map(p).map(q)` and the notation $x \triangleright p^{\uparrow G} \triangleright q^{\uparrow G}$.

¹<https://github.com/typelevel/kind-projector>

$f^{\downarrow H}$ means a function f lifted to a contrafunctor H . For a function $f:A \Rightarrow B$, the application of $f^{\downarrow H}$ to a value $h : H^B$ is written as $f^{\downarrow H} h$ or $h \triangleright f^{\downarrow H}$, and yields a value of type H^A . In Scala, this is `h.contramap(f)`. Nested lifting is denoted as $f^{\downarrow H \uparrow G} \triangleq (f^{\downarrow H})^{\uparrow G}$.

\diamond_M means the Kleisli product operation for a given monad M . This is a binary operation working on two Kleisli functions of types $A \Rightarrow M^B$ and $B \Rightarrow M^C$ and yields a new function of type $A \Rightarrow M^C$.

\oplus means the binary operation of a monoid, for example $x \oplus y$. The specific monoid type should be defined for this expression to make sense. For example, in Scala the monoidal operation is usually denoted by `x |+| y`.

Δ means the “diagonal” function of type $\forall A. A \Rightarrow A \times A$. There is only one implementation of this type signature,

```
def delta[A](a: A): (A, A) = (a, a)
```

$\nabla_1, \nabla_2, \dots$ denote the standard projection functions from a tuple to its first, second, ..., parts. In Scala, ∇_1 is `_.1`.

\boxtimes means the “pair product” of functions, where the result is a pair of the values of the two functions: $(f \boxtimes g)(a \times b) = f(a) \times g(b)$. In Scala, this operation can be defined by

```
def boxtimes[A,B,P,Q](f: A => P, g: B => Q): ((A, B)) => (P, Q) = {
  case (a, b) => (f(a), g(b))
}
```

The operations Δ , ∇_i (where $i = 1, 2, \dots$), and \boxtimes allow us to express any function operating on tuples. Useful properties for reasoning about code of such functions:

$$\begin{aligned}
&\text{identity law : } \Delta \circ \nabla_i = \text{id} \quad , \\
&\text{duplication law : } f \circ \Delta = \Delta \circ (f \boxtimes f) \quad , \\
&\text{projection law : } (f \boxtimes g) \circ \nabla_1 = \nabla_1 \circ f \quad , \\
&\text{projection law : } (f \boxtimes g) \circ \nabla_2 = \nabla_2 \circ g \quad , \\
&\text{composition law : } (f \boxtimes g) \circ (p \boxtimes q) = (f \circ p) \boxtimes (g \circ q) \quad ,
\end{aligned}$$

as well as the naturality laws for Δ and ∇_i :

$$\begin{aligned}
f^{\uparrow F} \circ \Delta &= \Delta \circ f^{\uparrow(F \times F)} = \Delta \circ (f^{\uparrow F} \boxtimes f^{\uparrow F}) \quad , \\
(f^{\uparrow F} \boxtimes f^{\uparrow G}) \circ \nabla_1 &= f^{\uparrow(F \times G)} \circ \nabla_1 = \nabla_1 \circ f^{\uparrow F} \quad .
\end{aligned}$$

$[a, b, c]$ means an ordered sequence of values, such as a list or an array. In Scala, this can be `List(a, b, c)`, `Vector(a, b, c)`, `Array(a, b, c)`, or another collection type.

$f:Z+A \Rightarrow Z+A \times A \triangleq \left\| \begin{array}{cc} z \Rightarrow z & \emptyset \\ \emptyset & a \Rightarrow a \times a \end{array} \right\|$ is the “matrix notation” for a function whose input and/or

output type is a disjunctive type. In Scala, the function f is implemented as

```
def f[Z, A]: Either[Z, A] => Either[Z, (A, A)] = {
  case Left(z)   => Left(z) // Identity function on Z.
  case Right(a)  => Right((a, a)) // Delta on A.
}
```

At the initial stages of reasoning The rows of the matrix indicate the different cases in the function’s code, corresponding to the different parts of the input disjunctive type. If the input type is not disjunctive, there will be only one row. The columns of the matrix indicate the parts of the output disjunctive type. If the the output type is not disjunctive, there will be only one column.

An “annotated matrix” writes out all parts of the disjunctive types in a separate “type row” and “type column”:

$$f \triangleq \begin{array}{c|c|cc} & & Z & A \times A \\ \hline Z & & \text{id} & \emptyset \\ A & & \emptyset & a \Rightarrow a \times a \end{array} . \quad (\text{A.1})$$

This notation clearly indicates the input and the output types of the function and is useful at the initial stages of reasoning about the code. The vertical double line separates input types from the function code. In the code above, the “type column” shows the parts of the input disjunction type $Z + A$. The “type row” shows the parts of the output disjunction type $Z + A \times A$.

The matrix notation is adapted to *forward* function compositions ($f \circ g$). Assume that A is a monoid type, and consider the composition of the function f shown above and the function g defined as

```
def g[Z, A: Monoid]: Either[Z, (A, A)] => A = {
  case Left(_)    => Monoid[A].empty
  case Right((a1, a2)) => a1 |+| a2
}
```

In the matrix notation, the function g is written (with and without types) as

$$g \triangleq \left| \begin{array}{c|c} & A \\ \hline Z & _ \Rightarrow e^A \\ A \times A & a_1 \times a_2 \Rightarrow a_1 \oplus a_2 \end{array} \right|, \quad g \triangleq \left| \begin{array}{c} _ \Rightarrow e^A \\ a_1 \times a_2 \Rightarrow a_1 \oplus a_2 \end{array} \right|.$$

The forward composition $f \circ g$ is computed by forward-composing the matrix elements using the rules of the ordinary matrix multiplication, omitting any terms containing $\mathbb{0}$:

$$\begin{aligned} f \circ g &= \left| \begin{array}{c|c} \text{id} & \mathbb{0} \\ \hline \mathbb{0} & a \Rightarrow a \times a \end{array} \right| \circ \left| \begin{array}{c} _ \Rightarrow e^A \\ a_1 \times a_2 \Rightarrow a_1 \oplus a_2 \end{array} \right| \\ &= \left| \begin{array}{c} \text{id} \circ (_ \Rightarrow e^A) \\ (a \Rightarrow a \times a) \circ (a_1 \times a_2 \Rightarrow a_1 \oplus a_2) \end{array} \right| = \left| \begin{array}{c} _ \Rightarrow e^A \\ a \Rightarrow a \oplus a \end{array} \right|. \end{aligned}$$

Applying a function to a value of a disjunctive type such as $x : Z + A$ is computed by writing x as a single-row matrix, for example

$$x = z^Z + \mathbb{0}^A = \left| \begin{array}{c} z^Z \quad \mathbb{0} \end{array} \right|,$$

and the computation $x \triangleright f \circ g$ again follows the rules of matrix multiplication:

$$x \triangleright f \circ g = \left| \begin{array}{c} z^Z \quad \mathbb{0} \end{array} \right| \triangleright \left| \begin{array}{c} _ \Rightarrow e^A \\ a \Rightarrow a \oplus a \end{array} \right| = z \triangleright (_ \Rightarrow e) = e.$$

Since the standard rules of matrix multiplication are associative, the properties of the \triangleright -notation such as $x \triangleright (f \circ g) = (x \triangleright f) \triangleright g$ are guaranteed to hold.

To use the matrix notation with backward compositions ($f \circ g$), all function matrices need to be transposed. (A standard identity of matrix calculus is that the transposition reverses the order of composition, $(AB)^T = B^T A^T$.) The argument types will then appear in the top row and the result types in the left column; the double line is above the matrix since that is where the function inputs come from. The above calculations are then rewritten as

$$\begin{aligned} g \circ f &= \left| \begin{array}{c|c} & Z \quad A \times A \\ \hline A & _ \Rightarrow e^A \quad a_1 \times a_2 \Rightarrow a_1 \oplus a_2 \end{array} \right| \circ \left| \begin{array}{c|c} & Z \quad A \\ \hline Z & \text{id} \quad \mathbb{0} \\ A \times A & \mathbb{0} \quad a \Rightarrow a \times a \end{array} \right| \\ &= \left| \begin{array}{c} \text{id} \circ (_ \Rightarrow e^A) \quad (a \Rightarrow a \times a) \circ (a_1 \times a_2 \Rightarrow a_1 \oplus a_2) \end{array} \right| = \left| \begin{array}{c} _ \Rightarrow e^A \quad a \Rightarrow a \oplus a \end{array} \right|. \\ (g \circ f)(x) &= \left| \begin{array}{c} _ \Rightarrow e^A \quad a \Rightarrow a \oplus a \end{array} \right| \left| \begin{array}{c} z^Z \\ \mathbb{0} \end{array} \right| = (_ \Rightarrow e^A)(z) = e. \end{aligned}$$

The forward composition seems to be easier to read and to reason about in the matrix notation.

B Glossary of terms

Code notation A mathematical notation developed in this book for deriving properties of code in functional programs. Variables have optional type annotations, such as x^A or $f^{A \Rightarrow B}$. Nameless functions are denoted by $x^A \Rightarrow f$, products by $a \times b$, and values of a disjunctive type $A + B$ are written as $x^A + 0^B$ or $0^A + y^B$. Functions working with disjunctive types are denoted by matrices. Lifting of functions to functors, such as $\text{fmap}_L(f)$, is denoted by $f^{\uparrow L}$; function compositions are denoted by $f \circ g$ (forward composition) and $f \circ g$ (backward composition); and function applications by $f(x)$ or equivalently $x \triangleright f$. See Appendix A for details.

Contrafunctor A type constructor having the properties of a contravariant functor with respect to a type parameter. Instead of saying “contravariant functor”, this book uses the shorter name “contrafunctor”.

Disjunctive type A type representing one of several distinct possibilities. In Scala, this is usually implemented as a sealed trait extended by several case classes. The standard Scala disjunction types are `Option[A]` and `Either[A, B]`. Also known as **sum type**, **tagged union type**, **co-product type**, and **variant type** (in Object Pascal and in OCaml). The shortest name is “sum type,” but the English word “sum” is more ambiguous to the ear than “disjunctive”.

Exponential-polynomial type A type constructor built using products, disjunctions (sums or co-products), and function types (“exponentials”), as well as type parameters and fixed types. For example, `type F[A] = Either[(A,A), Int=>A]` is an exponential-polynomial type constructor. Such type constructors are always profunctors and can also be functors or contrafunctors.

Functor block A short syntax for composing several `.map`, `.flatMap`, and `.filter` operations applied to a functor-typed value. The type constructor corresponding to that value must therefore be fixed throughout the entire functor block. (The type constructor *must* be a functor and may additionally be filterable and/or monadic.) For example, in Scala the code

```
for { x <- List(1,2,3); y <- List(10, x); if y > 2 }  
  yield 2 * y
```

is equivalent to the code

```
List(1, 2, 3).flatMap(x => List(10, x))  
  .filter(y => y > 1).map(y => 2 * y)
```

and computes the value `List(20, 20, 6)`. This is a functor block that “raises” computations to the `List` functor. Similar syntax exists in a number of languages and is called a “**for-comprehension**” or a “list comprehension” in Python, “**do-notation**” in Haskell, and “**computation expressions**” in F#. I use the name “functor block” in this book because it is shorter and more descriptive. (The type constructor used in a functor block needs to be at least a functor but does not have to be a monad.)

Kleisli function Also called a Kleisli morphism or a Kleisli arrow. A function with type signature $A \Rightarrow M^B$ for some fixed monad M . More verbosely, “a morphism from the Kleisli category corresponding to the monad M ”. The standard monadic method `pureM` : $A \Rightarrow M^A$ has the type signature of a Kleisli function. The Kleisli product operation, \diamond_M , is a binary operation that combines two Kleisli functions (of types $A \Rightarrow M^B$ and $B \Rightarrow M^C$) into a new Kleisli function (of type $A \Rightarrow M^C$).

Method This word is used in two ways: 1) A method₁ is a Scala function defined as a member of a typeclass. For example, `flatMap` is a method defined in the `Monad` typeclass. 2) A method₂ is a Scala function defined as a member of a data type declared as a Java-compatible `class` or `trait`. Trait methods₂ are necessary in Scala when implementing functions whose arguments have type parameters (because Scala function values defined via `val` cannot have type parameters). So, many typeclasses such as `Functor` or `Monad`, whose methods₁ require type parameters, will use Scala traits with methods₂ for their implementation. The same applies to type constructions with quantified types, such as the Church encoding.

Nameless function An expression of function type, representing a function. For example, `(x: Int) => x * 2`. Also known as function expression, function literal, anonymous function, closure, lambda-function, lambda-expression, or simply a “lambda”.

Partial type-to-value function (PTVF) A function with a type parameter but defined only for a certain subset of types. In Scala, PTVFs are implemented via a typeclass constraint:

```
def double[T: Semigroup](t: T): T = implicitly[Semigroup[T]].combine(t, t)
```

This PTVF is defined only for types τ for which a `Semigroup` typeclass instance is available.

Polynomial functor A type constructor built using disjunctions (sums), products (tuples), type parameters and fixed types. For example, in Scala, `type F[A] = Either[(Int, A), A]` is a polynomial functor with respect to the type parameter `A`, while `Int` is a fixed type (not a type parameter). Polynomial functors are also known as **algebraic data types**.

Product type A type representing several values given at once. In Scala, product types are the tuple types, for example `(Int, String)`, and case classes. Also known as **tuple type**, **struct** (in C and C++), and **record**.

Profunctor A type constructor whose type parameter occurs in both covariant and contravariant positions and satisfying the appropriate laws; see Section 6.4.1.

Type notation A mathematical notation for type expressions developed in this book for easier reasoning about types in functional programs. Disjunctive types are denoted by $+$, product types by \times , and function types by \Rightarrow . The unit type is denoted by $\mathbb{1}$, and the void type by $\mathbb{0}$. The function arrow \Rightarrow groups weaker than $+$, which in turn groups weaker than \times . This means

$$Z + A \Rightarrow Z + A \times A \quad \text{is the same as} \quad (Z + A) \Rightarrow (Z + (A \times A)) \quad .$$

Type parameters are denoted by superscripts. As an example, the Scala definition

```
type F[A] = Either[(A, A => Option[Int]), String => List[A]]
```

is written in the type notation as

$$F^A \triangleq A \times (A \Rightarrow \mathbb{1} + \text{Int}) + (\text{String} \Rightarrow \text{List}^A) \quad .$$

Unfunctor A type constructor that cannot possibly be a functor, nor a contrafunctor, nor a profunctor. An example is a type constructor with explicitly indexed type parameters, such as $F^A \triangleq (A \times A)^{:F^{\text{Int}}} + (\text{Int} \times A)^{:F^{\mathbb{1}}}$. The Scala code for this type constructor is

```
sealed trait F[A]
final case class F1[A](x: A, y: A) extends F[Int]
final case class F2[A](s: Int, t: A) extends F[Unit]
```

This can be seen as a **GADT** (generalized algebraic data type) that uses specific values of type parameters instead of the type parameter `A` in at least some of its case classes.

B.1 On the current misuse of the term “algebra”

This book avoids using the terms “algebra” or “algebraic” because these terms are too ambiguous. The functional programming community uses the word “algebra” in at least *four* incompatible ways.

Definition 0. In mathematics, an “algebra” is a vector space with multiplication and certain standard properties. For example, we need $1 * x = x$, the addition must be commutative, the multiplication must be distributive over addition, etc. The set of all 10×10 matrices with real coefficients is an “algebra” in this sense. These matrices form a 100-dimensional vector space, and they can be multiplied and added. This definition of “algebra” is not used in functional programming.

Definition 1. An “algebra” is a function with type signature $F^A \Rightarrow A$, where F^A is some fixed functor. This definition comes from category theory, where such types are called *F-algebras*. There is no direct connection between this “algebra” and Definition 0, except when the functor F is defined by $F^A \triangleq A \times A$, and then a function of type $A \times A \Rightarrow A$ may be interpreted as a “multiplication” operation (however, A is a type and not a vector space, and there are no distributivity or commutativity laws). It is better to call such functions “*F*-algebras”, emphasizing that they characterize and depend on a chosen functor F . However, knowing how to reason about the theoretical properties of *F*-algebras does not help in practical programming, and so they are not used in this book.

Definition 2. Polynomial functors are often called “algebraic data types”. However, they are not “algebraic” in the sense of Definitions 0 or 1. For example, consider the “algebraic data type” `Either[Option[A], Int]`, which is $F^A \triangleq 1 + A + \text{Int}$ in the type notation. The set of all values of the type F^A does not have the addition and multiplication operations required by the mathematical definition of “algebra”. The type F^A may admit some binary or unary operations (e.g. that of a monoid), but these operations will not be commutative or distributive. Also, there is not necessarily a function with type $F^A \Rightarrow A$, as required for Definition 1. Rather, the word “algebra” refers to “school-level algebra” with polynomials, since these data types are built from sums and products of types. If the data contains a function type, e.g. `Option[Int => A]`, the type is no longer polynomial. So this book uses more precise terms: “polynomial type” and “exponential-polynomial type”.

Definition 3. People talk about the “algebra” of properties of functions such as `map` or `flatMap`, meaning that these functions satisfy certain laws (e.g. the composition, naturality, or associativity laws). But these laws do not make functions `map` or `flatMap` into an “algebra” in the sense of Definition 0 or in the sense of Definition 1. There is also no relation to “algebraic data types” of Definition 2. So, this is a different usage of the word “algebra”. However, there is no general “algebra of laws” that we can use; every derivation proceeds in a different way, specific to the laws being proved. In mathematics, “algebraic equations” are distinguished in another sense from “differential” or “integral” equations. But the laws in functional programming are always “algebraic”: they are equations with compositions and applications of functions. We find that it is not useful to refer to “algebraic” laws in any of these two senses. This book talks about “equational laws” or just “laws”.

Definition 4. The Church encoding of a free monad (also known as the “final tagless style of programming”) is the type expression $\forall E^\bullet. (S^{E^\bullet} \rightsquigarrow E^\bullet) \Rightarrow E^A$ that uses a higher-order type constructor S parameterized by a *type constructor* parameter E . The sub-expression $S^{E^\bullet} \rightsquigarrow E^\bullet$ can be viewed as an *S*-algebra in the category of type constructors (functors in a category of types without non-trivial morphisms). So, Definition 4 is related to Definition 1, with a specific choice of a category. However, the knowledge that $S^{E^\bullet} \rightsquigarrow E^\bullet$ is an *S*-algebra in the category of type constructors does not provide any help or additional insights for practical work with the Church encoding of a free monad.

The higher-order type constructor S is used to parameterize the effects described by a Church-encoded free monad, so this book calls it the “effect constructor”.

So, it seems that the current usage of the word “algebra” in functional programming is both inconsistent and unhelpful to practitioners. In this book, the word “algebra” always denotes the branch of mathematics, as in “school-level algebra”. Instead of “algebra” as in Definitions 1 to 4, this book talks about “*F*-algebras” with a specific functor F ; “polynomial types” or “polynomial functors” or “exponential-polynomial functors” etc.; “equational laws”; and an “effect constructor” S .

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