# AEROPLANE LANDING

A Differential Equation Model


ABSTRACT

This document attempts to model the landing of an aeroplane and compares the model with given data.



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## Contents

# 1. The Scenario

## 1.1. Background

I have been given 27 data points of time which show 27 velocities of an aircraft after touchdown until it stops. During this time, we are told that after touchdown air-resistance slows it initially, and then when it is slow enough the air resistance is augmented by a constant force from the brakes. The velocities of the 27 seconds are given below:

| **t** | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **v** | 96 | 89 | 82 | 77 | 72 | 68 | 64 | 61 | 58 | 55 | 50 | 46 | 41 | 38 | 34 | 31 | 27 | 24 | 21 | 18 | 16 | 13 | 10 | 8 | 5 | 3 | 0 |

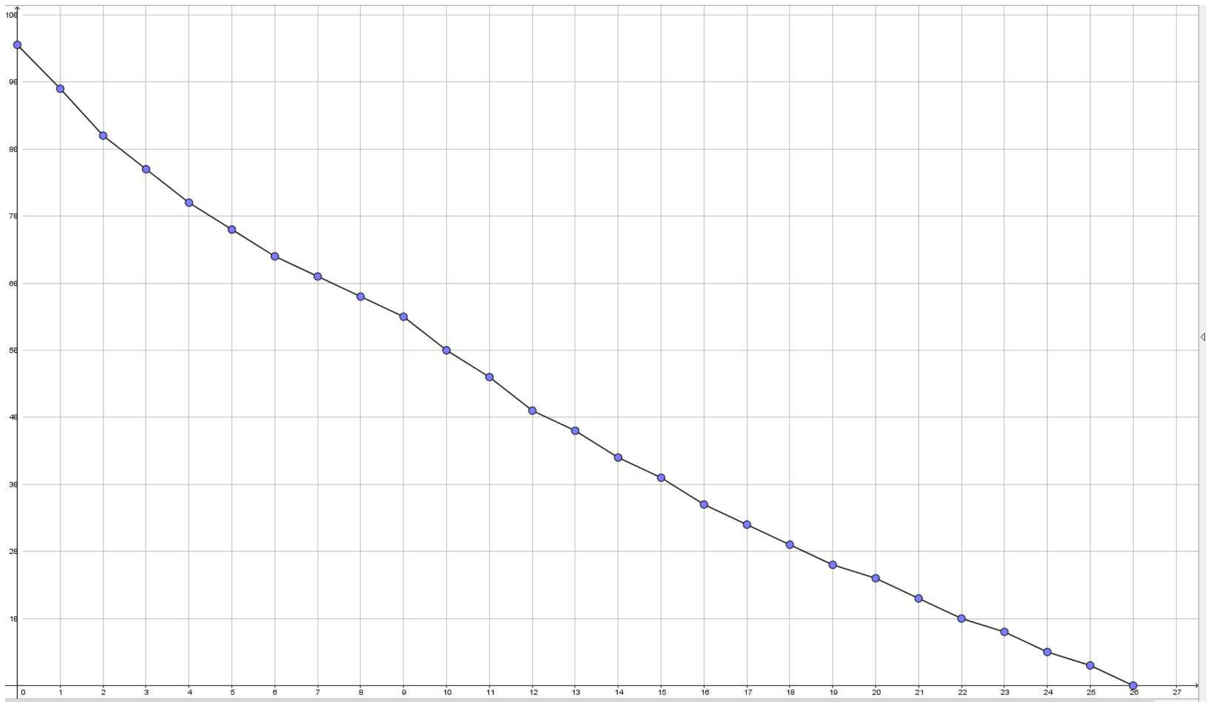
As all the velocities are given as integers it is reasonable to assume that these figures have been rounded to the nearest integer. I will assume that the times are given as exact numbers. Since an assumption has been made it seems appropriate to discuss some of the other assumptions that will be made in this work before going any further.

## 1.2. Assumptions used in modelling

Throughout this coursework, I will be making the following modelling assumptions:

- **The aeroplane can be modelled as a point particle** - this allows us to negate any complex resistances and turbulences that would occur. It is also appropriate to model it like this because we are not interested in any rotations of the aeroplane.
- **The runway is horizontal** - if we don't model the runway as horizontal we would need to consider a component of the weight and reaction force which would affect the calculations.
- **Friction is negligible** – we are told in the scenario that there is an air resistance that slows the plane initially and no mention of friction. To include friction would require a second model of resistance and for simplicity we ignore this. In the model(s) that we use, however, friction could be seen as being wrapped up with the air resistance model.
- **No vertical resultant force** - There are no vertical forces other than the weight of the aircraft and the reaction force from the ground, which are negated from this coursework as these will cancel with each other. If this weren't the case the aeroplane would either be sinking into the ground or lifting off the runway since the runway is assumed to be horizontal.
- **There is no driving force from the engines** – if there was a driving force we would need to incorporate this into the model.
- The only resistive force acting before the brakes are applied is air resistance.
- The only resistive forces acting after the brakes are applied are the air resistance and the constant braking force.
- As already mentioned, **the velocities have been rounded to the nearest integer** – I will assume the 27 values of t are exact, however, the probability that the values of v at these 27 points are also integers is essentially zero. By assuming the numbers are rounded to the nearest integer I will include a tolerance of ±0.5 $ms^{-1}$ from my models to the data.

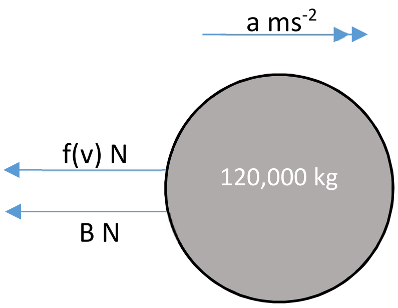
The other assumption I need to state is when I believe the brakes are applied as we haven't been told this. To do this we look at data given to us as a graph of velocity plotted against time:



The points suddenly drop between t = 9 and t = 10. We can confirm this by looking at the table - the gradient of the segments are as follows: -7, -7 ,-5 ,-5 ,-4 ,-4 ,-3 ,-3 ,-3 ,-5 ,-4 ,-5 ,-3 ,… The gradients are steadily increasing until t=9 therefore it is reasonable to assume that the brakes were applied at sometime between t = 9 and t = 10. For simplicity, we will model it as though the brakes were applied at t = 9.

### 1.3. Derivation of the model

I will now derive the model that will be used to model the aircraft landing. See the force diagram:




The diagram ignores the vertical forces that are acting on the aeroplane and consists of the horizontal acceleration, the air resistance (as a function of velocity) and the constant braking force, B Newtons.

By using the result from Newton's second law of motion, F = ma, we can state that, prior to the brakes coming on, we can model the situation with the differential equation:

$$120{,}000\frac{dv}{dt} = -f(v)$$

Where v is the velocity at time t and t=0 is when the aeroplane touches down. After the brakes are applied we have the differential equation:

$$120{,}000\frac{dv}{dt} = -f(v) - B$$

## 2. Model 1

### 2.1. Defining f(v)

For this first model, I will take the air resistance force to be proportional to the velocity. This seems like a suitable model as one would expect the resistance to increase as the velocity increases. Therefore, I assume:

$$f(v) \ \propto \ v$$

Hence,

$$f(v) \ = \ \alpha v$$

Where α is the constant of proportionality.

Therefore,

$$120{,}000\frac{dv}{dt} \ = \ \begin{cases} -\alpha v & 0 \le t < 9 \\ -\alpha v - B & 9 < t \le 26 \end{cases}$$

This is the first model which will be solved momentarily. First, I'd like to point out that both lines are not valid for t = 9, that is there is a discontinuity in the gradient of v at this point. I've done this on purpose because the model may be better modelled if the acceleration isn't continuous at t = 9, after all, the braking force wouldn't go from 0 to B Newtons instantaneously in reality either. I will, however, model the velocity as being continuous at t = 9 when the time comes.

### 2.2. General Solution

$0 \le t \le 9$

$$120{,}000\frac{dv}{dt} \ = -\alpha v$$

$$120{,}000\int v^{-1}\,dv \ = -\int \alpha\, d$$

$$120{,}000.\ln(v) \ = -\alpha t + c_1$$

$$v \ = \ A_1 e^{-\frac{\alpha}{120{,}000}t} \qquad\qquad Where\ A_1 = e^{\frac{c_1}{120{,}000}}$$

$9 \le t \le 26$

$$120{,}000\frac{dv}{dt} \ = -\alpha v - B$$

$$120{,}000\int (\alpha v + B)^{-1}\,dv \ = -\int 1\,dt$$

$$\frac{120{,}000}{\alpha}\int \alpha(\alpha v + B)^{-1}\,dv \ = -t$$

$$\frac{120{,}000}{\alpha}.\ln(\alpha v + B) \ = -t + c_2$$

$$\alpha v + B \ = \ e^{-\frac{\alpha}{120{,}000}t + \frac{c_2\alpha}{120{,}000}}$$

$$v \ = \ A_2 e^{-\frac{\alpha}{120{,}000}t} \ - \ \frac{B}{\alpha} \qquad\qquad Where\ A_2 = \frac{e^{\frac{c_2\alpha}{120{,}000}}}{\alpha}$$

Hence,

$$v \ = \ \begin{cases} A_1 e^{-\frac{\alpha}{120{,}000}t} & 0 \le t \le 9 \\ A_2 e^{-\frac{\alpha}{120{,}000}t} - \dfrac{B}{\alpha} & 9 \le t \le 26 \end{cases}$$

### 2.3. Particular Solution

This general solution has 4 unknown constants, the constant braking force, B, the constant of proportionality from the air resistance model, α, and 2 constants of integration from solving the 2 differential equations, $A_1$ and $A_2$. To find these constants we will use 3 data points from the data given to me and use the knowledge that v is continuous for all 0 ≤ v ≤ 26. But there are many combinations of choosing 3 valid data points; I do not intend to attempt all of them. To keep things simple, I will use the data point at t = 9 for both equations for v. This will also ensure the function is continuous.

#### Pre-Brakes

I will look at the first equation:

$$v = A_1 e^{-\frac{\alpha}{120{,}000}t} \qquad 0 \leq t \leq 9$$

And use the point (t = 9, v = 55):

$$55 = A_1 e^{-\frac{\alpha}{120{,}000}9}$$

I will then use the other 9 points that this equation is relevant for to calculate nine pairs of values for α and $A_1$, each pair passing through a different point for 0 ≤ t ≤ 8. I will do this as follows:

Take point ($t_0$, $v_0$):

$$v_0 = A_1 e^{-\frac{\alpha}{120{,}000}t_0}$$

Divide the two to eliminate $A_1$:

$$\frac{v_0}{55} = \frac{e^{-\frac{\alpha}{120{,}000}t_0}}{e^{-\frac{\alpha}{120{,}000}9}}$$

$$v_0 e^{-\frac{\alpha}{120{,}000}9} = 55 e^{-\frac{\alpha}{120{,}000}t_0}$$

$$\ln(v_0) - \frac{\alpha}{120{,}000}9 = \ln(55) - \frac{\alpha}{120{,}000}t_0$$

$$\frac{\alpha}{120{,}000}t_0 - \frac{\alpha}{120{,}000}9 = \ln\left(\frac{55}{v_0}\right)$$

$$\alpha = \frac{120{,}000}{t_0 - 9}\ln\left(\frac{55}{v_0}\right)$$

Hence,

$$v_0 = A_1 e^{-\frac{\ln\left(\frac{55}{v_0}\right)}{t_0-9}t_0}$$

$$A_1 = v_0 e^{\frac{\ln\left(\frac{55}{v_0}\right)}{t_0-9}t_0}$$

Using a spreadsheet package and the formulae derived on the previous page, I found values for pairs of α and $A_1$ and they are given as follows:

| $t_0$ | $v_0$ | α | $A_1$ |
|---|---|---|---|
| 0 | 96 | 7426.867 | 96.00000 |
| 1 | 89 | 7219.548 | 94.51885 |
| 2 | 82 | 6846.618 | 91.91181 |
| 3 | 77 | 6729.445 | 91.10763 |
| 4 | 72 | 6463.990 | 89.31170 |
| 5 | 68 | 6365.236 | 88.65264 |
| 6 | 64 | 6061.996 | 86.65917 |
| 7 | 61 | 6212.441 | 87.64252 |
| 8 | 58 | 6373.179 | 88.70548 |

We'll come back to these values in a short while, I'll first find solutions for $9 \leq t \leq 26$:

## Post-Brakes

Let's look at the second equation:

$$v = A_2 e^{-\frac{\alpha}{120{,}000}t} - \frac{B}{\alpha} \qquad 9 \leq t \leq 26$$

Using t = 9:

$$55 + \frac{B}{\alpha} = A_2 e^{-\frac{\alpha}{120{,}000}9}$$

Bearing in mind α has been found so we only have B and $A_2$ to find, we require 1 other data point. Again, I will use each one possible, this time in the interval $10 \leq t \leq 26$ yielding 17 results.

Take point $(t_0, v_0)$:

$$v_0 + \frac{B}{\alpha} = A_2 e^{-\frac{\alpha}{120{,}000}t_0}$$

Divide the two to eliminate $A_2$:

$$\frac{v_0 + \frac{B}{\alpha}}{55 + \frac{B}{\alpha}} = \frac{e^{-\frac{\alpha}{120{,}000}t_0}}{e^{-\frac{\alpha}{120{,}000}9}}$$

$$(\alpha v_0 + B)e^{-\frac{\alpha}{120{,}000}9} = e^{-\frac{\alpha}{120{,}000}t_0}(55\alpha + B)$$

$$Be^{-\frac{\alpha}{120{,}000}9} - Be^{-\frac{\alpha}{120{,}000}t_0} = 55\alpha e^{-\frac{\alpha}{120{,}000}t_0} - \alpha v_0 e^{-\frac{\alpha}{120{,}000}9}$$

$$B = \frac{55\alpha e^{-\frac{\alpha}{120{,}000}t_0} - \alpha v_0 e^{-\frac{\alpha}{120{,}000}9}}{e^{-\frac{\alpha}{120{,}000}9} - e^{-\frac{\alpha}{120{,}000}t_0}}$$

Hence,

$$v_0 = A_2 e^{-\frac{\alpha}{120{,}000}t_0} - \frac{55e^{-\frac{\alpha}{120{,}000}t_0} - v_0 e^{-\frac{\alpha}{120{,}000}9}}{e^{-\frac{\alpha}{120{,}000}9} - e^{-\frac{\alpha}{120{,}000}t_0}}$$

$$A_2 = \left(v_0 + \frac{55e^{-\frac{\alpha}{120{,}000}t_0} - v_0 e^{-\frac{\alpha}{120{,}000}9}}{e^{-\frac{\alpha}{120{,}000}9} - e^{-\frac{\alpha}{120{,}000}t_0}}\right) e^{\frac{\alpha}{120{,}000}t_0}$$

Using spreadsheet software, I can calculate approximations for pairs of values for B and $A_2$:

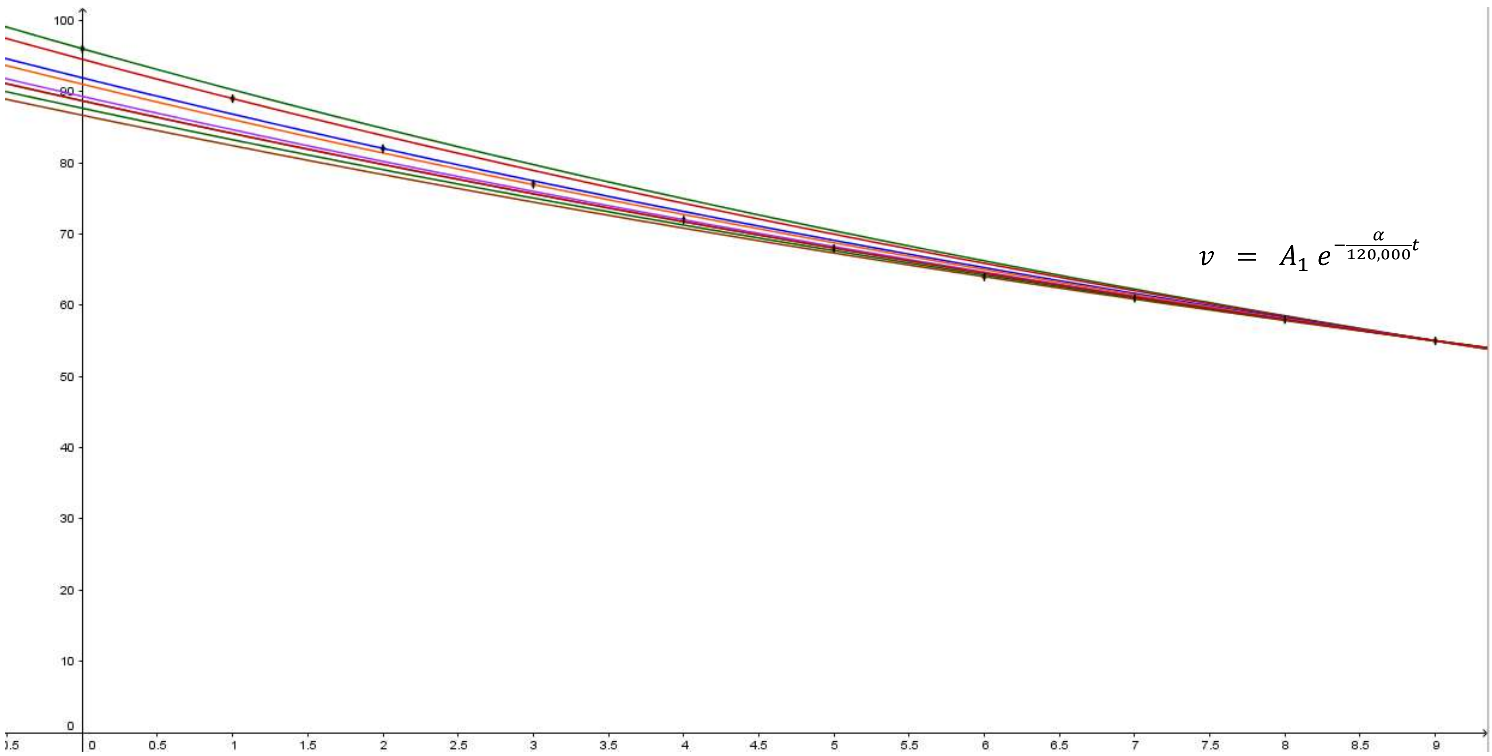
| | | $\alpha_0$ = 7,426.867 | | $\alpha_1$ = 7,219.548 | | $\alpha_2$ = 6,846.618 | | $\alpha_3$ = 6,729.445 | | $\alpha_4$ = 6,463.990 | | $\alpha_5$ = 6,365.236 | | $\alpha_6$ = 6,061.996 | | $\alpha_7$ = 6,212.441 | | $\alpha_8$ = 6,373.179 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $t_0$ | $v_0$ | B | $A_2$ | B | $A_2$ | B | $A_2$ | B | $A_2$ | B | $A_2$ | B | $A_2$ | B | $A_2$ | B | $A_2$ | B | $A_2$ |
| 10 | 50 | 210,281 | 145.4200 | 221,155 | 147.1620 | 240,715 | 150.6656 | 246,861 | 151.8744 | 260,786 | 154.8249 | 265,966 | 156.0031 | 281,873 | 159.9230 | 273,981 | 157.9190 | 265,549 | 155.9066 |
| 11 | 46 | 165,633 | 134.9268 | 176,064 | 136.4288 | 194,832 | 139.4663 | 200,729 | 140.5185 | 214,091 | 143.0944 | 219,062 | 144.1256 | 234,329 | 147.5653 | 226,754 | 145.8053 | 218,662 | 144.0411 |
| 12 | 41 | 205,118 | 144.2067 | 214,981 | 145.6925 | 232,729 | 148.7163 | 238,307 | 149.7686 | 250,947 | 152.3532 | 255,650 | 153.3908 | 270,096 | 156.8619 | 262,928 | 155.0840 | 255,272 | 153.3057 |
| 13 | 38 | 167,253 | 135.3076 | 176,750 | 136.5920 | 193,843 | 139.2253 | 199,216 | 140.1465 | 211,396 | 142.4175 | 215,928 | 143.3321 | 229,851 | 146.4016 | 222,943 | 144.8277 | 215,564 | 143.2571 |
| 14 | 34 | 177520 | 137.7206 | 186,525 | 138.9188 | 202,739 | 141.3964 | 207,838 | 142.2683 | 219,395 | 144.4270 | 223,698 | 145.2996 | 236,918 | 148.2384 | 230,357 | 146.7295 | 223,352 | 145.2279 |
| 15 | 31 | 166,148 | 135.0479 | 174,760 | 136.1184 | 190,274 | 138.3539 | 195,154 | 139.1462 | 206,220 | 141.1170 | 210,340 | 141.9169 | 223,003 | 144.6217 | 216,718 | 143.2311 | 210,008 | 141.8511 |
| 16 | 27 | 182,983 | 139.0044 | 191,072 | 140.0012 | 205,652 | 142.1074 | 210,241 | 142.8600 | 220,650 | 144.7422 | 224,527 | 145.5096 | 236,450 | 148.1166 | 230,532 | 146.7742 | 224,215 | 145.4465 |
| 17 | 24 | 181,100 | 138.5619 | 188,770 | 139.4532 | 202,604 | 141.3634 | 206,960 | 142.0524 | 216,846 | 143.7866 | 220,530 | 144.4974 | 231,863 | 146.9245 | 226,237 | 145.6725 | 220,234 | 144.4389 |
| 18 | 21 | 182,773 | 138.9552 | 190,013 | 139.7490 | 203,080 | 141.4795 | 207,197 | 142.1107 | 216,546 | 143.7112 | 220,032 | 144.3712 | 230,760 | 146.6377 | 225,432 | 145.4663 | 219,751 | 144.3167 |
| 19 | 18 | 187,002 | 139.9491 | 193,799 | 140.6503 | 206,078 | 142.2114 | 209,951 | 142.7885 | 218,749 | 144.2646 | 222,031 | 144.8774 | 232,139 | 146.9964 | 227,119 | 145.8988 | 221,767 | 144.8268 |
| 20 | 16 | 178,108 | 137.8587 | 184,576 | 138.4550 | 196,274 | 139.8184 | 199,966 | 140.3307 | 208,359 | 141.6544 | 211,491 | 142.2085 | 221,146 | 144.1389 | 216,349 | 143.1364 | 211,239 | 142.1626 |
| 21 | 13 | 186,617 | 139.8585 | 192,622 | 140.3700 | 203,495 | 141.5809 | 206,930 | 142.0451 | 214,748 | 143.2594 | 217,668 | 143.7726 | 226,676 | 145.5763 | 222,199 | 144.6369 | 217,433 | 143.7300 |
| 22 | 10 | 196,181 | 142.1064 | 201,711 | 142.5337 | 211,741 | 143.5935 | 214,914 | 144.0103 | 222,142 | 145.1171 | 224,845 | 145.5901 | 233,192 | 147.2700 | 229,042 | 146.3922 | 224,628 | 145.5508 |
| 23 | 8 | 193,805 | 141.5479 | 198,981 | 141.8839 | 208,384 | 142.7742 | 211,363 | 143.1362 | 218,156 | 144.1155 | 220,698 | 144.5399 | 228,558 | 146.0654 | 224,648 | 145.2651 | 220,493 | 144.5045 |
| 24 | 5 | 205,518 | 144.3006 | 210,199 | 144.5541 | 218,723 | 145.2978 | 221,429 | 145.6141 | 227,609 | 146.4905 | 229,926 | 146.8767 | 237,099 | 148.2854 | 233,529 | 147.5430 | 229,739 | 146.8443 |
| 25 | 3 | 205,981 | 144.4095 | 210,293 | 144.5765 | 218,163 | 145.1611 | 220,667 | 145.4263 | 226,392 | 146.1847 | 228,541 | 146.5261 | 235,207 | 147.7936 | 231,887 | 147.1220 | 228,368 | 146.4974 |
| 26 | 0 | 219,167 | 147.5085 | 222,965 | 147.5929 | 229,924 | 148.0316 | 232,144 | 148.2515 | 237,234 | 148.9083 | 239,149 | 149.2122 | 245,103 | 150.3658 | 242,135 | 149.7505 | 238,995 | 149.1866 |

Where I have used the 9 different α values obtained at the top of the previous page for each data point $t \geq 10$ totalling 153 pairs of values for B and $A_2$.

## 2.4. Graphs and Analysis

### $0 \leq t \leq 9$

I'll now look at the nine pairs of values I obtained for the constants α and $A_1$ for $0 \leq t \leq 9$, plot them and look at some of the errors these models produce. First, here are the 9 graphs:



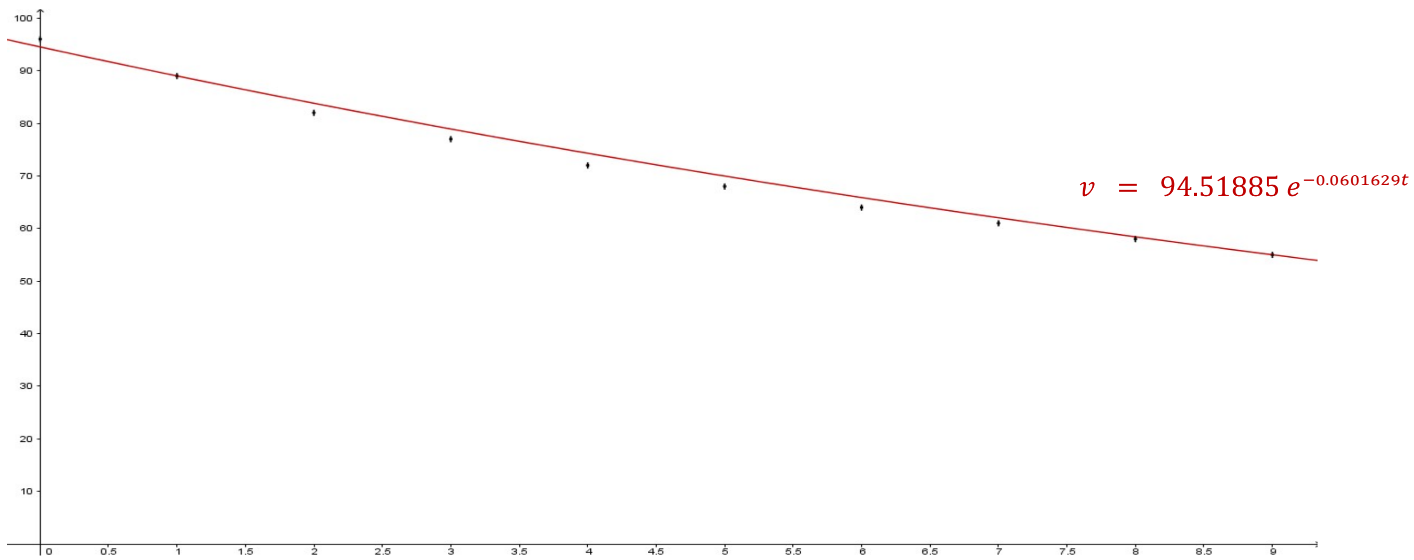

Where the black points are the given data and the vertical lines show ±0.5 error; my ideal tolerance.

First notice that all 9 solutions pass through the point (9,55), this is because this coordinate was used in all 9 calculations and ensures the velocity function is continuous at t=9. The difference between these models comes from the second chosen point; notice that each of these solution curves passes through a different data point. The green on the top, for example, passes through (0,96), the red one below it passes through (1,89), and so on… I can see that these estimates produce an appropriate approximation to the given data, however, I now look at the error each estimate gives to allow me to paint an overall picture of these models and aim to conclude which one is best:

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| α | 7,426.87 | α | 7,219.55 | α | 6,846.62 | α | 6,729.45 | α | 6,463.99 | α | 6,365.24 | α | 6,062.00 | α | 6,212.44 | α | 6,373.18 |
| $A_1$ | 96.0000 | $A_1$ | 94.5189 | $A_1$ | 91.9118 | $A_1$ | 91.1076 | $A_1$ | 89.3117 | $A_1$ | 88.6526 | $A_1$ | 86.6592 | $A_1$ | 87.6425 | $A_1$ | 88.7055 |
| **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** |
| 96.000 | 0.000 | 94.519 | 1.481 | 91.912 | 4.088 | 91.108 | 4.892 | 89.312 | 6.688 | 88.653 | 7.347 | 86.659 | 9.341 | 87.643 | 8.357 | 88.705 | 7.295 |
| 90.239 | 1.239 | 89.000 | 0.000 | 86.815 | 2.185 | 86.139 | 2.861 | 84.628 | 4.372 | 84.073 | 4.927 | 82.390 | 6.610 | 83.221 | 5.779 | 84.117 | 4.883 |
| 84.823 | 2.823 | 83.803 | 1.803 | 82.000 | 0.000 | 81.441 | 0.559 | 80.190 | 1.810 | 79.729 | 2.271 | 78.331 | 3.669 | 79.022 | 2.978 | 79.766 | 2.234 |
| 79.732 | 2.732 | 78.910 | 1.910 | 77.452 | 0.452 | 77.000 | 0.000 | 75.985 | 1.015 | 75.610 | 1.390 | 74.473 | 2.527 | 75.035 | 1.965 | 75.641 | 1.359 |
| 74.947 | 2.947 | 74.303 | 2.303 | 73.157 | 1.157 | 72.801 | 0.801 | 72.000 | 0.000 | 71.704 | 0.296 | 70.804 | 1.196 | 71.249 | 0.751 | 71.728 | 0.272 |
| 70.449 | 2.449 | 69.964 | 1.964 | 69.100 | 1.100 | 68.831 | 0.831 | 68.224 | 0.224 | 68.000 | 0.000 | 67.316 | 0.684 | 67.655 | 0.345 | 68.018 | 0.018 |
| 66.221 | 2.221 | 65.879 | 1.879 | 65.268 | 1.268 | 65.077 | 1.077 | 64.646 | 0.646 | 64.487 | 0.487 | 64.000 | 0.000 | 64.241 | 0.241 | 64.500 | 0.500 |
| 62.247 | 1.247 | 62.033 | 1.033 | 61.648 | 0.648 | 61.528 | 0.528 | 61.256 | 0.256 | 61.156 | 0.156 | 60.847 | 0.153 | 61.000 | 0.000 | 61.164 | 0.164 |
| 58.512 | 0.512 | 58.411 | 0.411 | 58.229 | 0.229 | 58.172 | 0.172 | 58.044 | 0.044 | 57.996 | 0.004 | 57.850 | 0.150 | 57.922 | 0.078 | 58.000 | 0.000 |
| 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 |
| | 1.617 | | 1.278 | | 1.113 | | 1.172 | | 1.506 | | 1.688 | | 2.433 | | 2.049 | | 1.672 |
| | 1.107 | | 0.817 | | 1.181 | | 1.466 | | 2.145 | | 2.392 | | 3.058 | | 2.734 | | 2.372 |

Where |E| is the absolute error between the model and the given value.

The top number at the bottom of each column shows the average of the absolute errors and the bottom number at the bottom of each column shows the standard deviation of the absolute errors. At first glance, the third model in from the left looks to be one of the best, it has the lowest average of the errors and the standard deviation isn't too bad compared to the others either, however, the reason the standard deviation is so high is because the model gives an absolute error of almost 4.1 when t=0, and this is reflected in the graph. The models then proceed to give worse and worse estimates for v at t=0. Due to the lowest standard deviation, the third lowest mean and the accuracy of the graph I conclude that out of these 9 models the second model from the left is the most accurate despite only having 1 value within the 0.5 tolerance level. Therefore, I state that the values of α and $A_1$ are 7,219.548 (7sf) and 94.51885 (7sf) respectively. The respective graph is shown below:



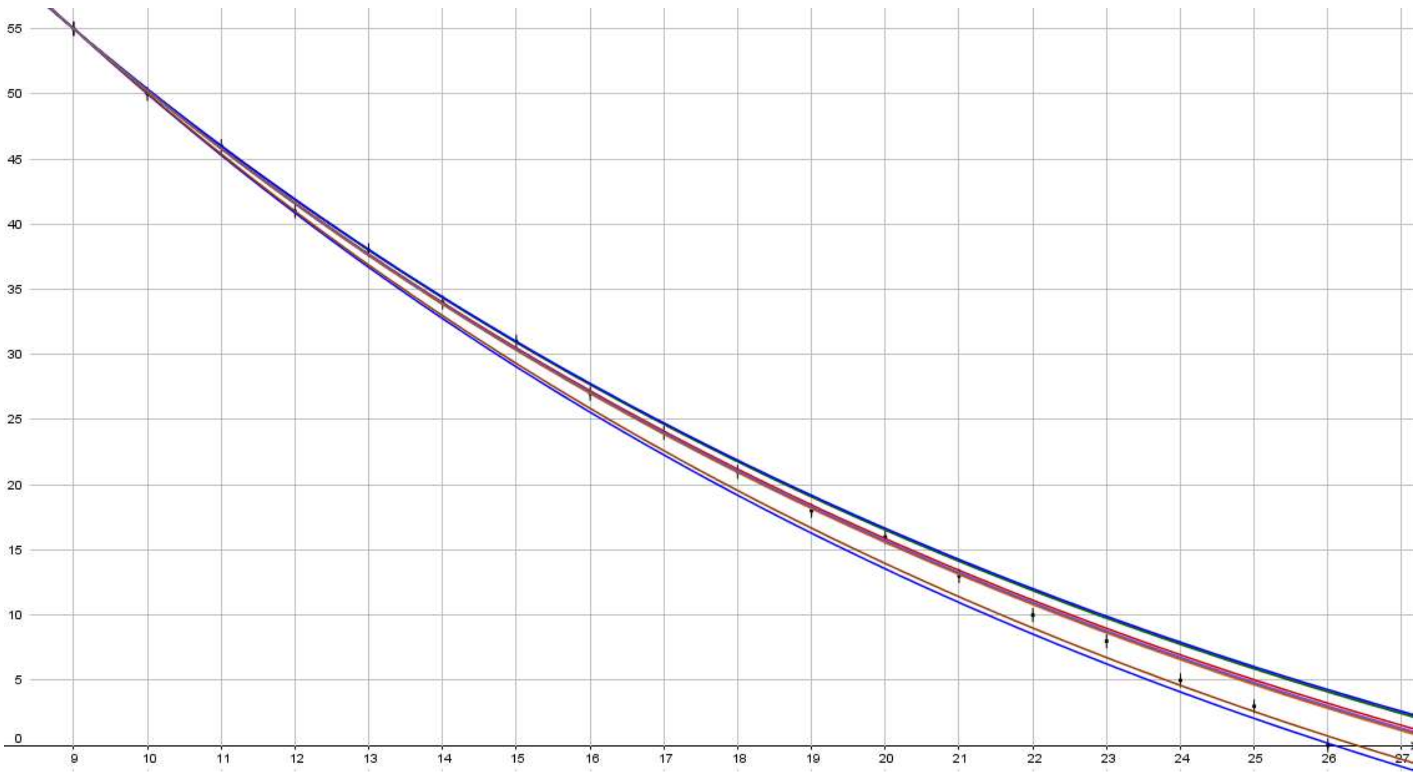
## $9 \le t \le 26$

Now I shall have a look for the latter part of the model, where the brakes have been applied. For this part of the solution I obtained 153 pairs of values for B and $A_2$ because of the 17 data points and the 9 values of α. Well, now that I have decided on a value for α, I will cut these 153 pairs back down to the 17 pairs of values which correspond to α = 7,219.548. The data in the table to the right takes just the $\alpha_1$ columns from the table at the top of page 7 along with the $t_0$ and $v_0$ columns.

| | | $\alpha_1$ = 7,219.548 | |
|---|---|---|---|
| $t_0$ | $v_0$ | B | $A_2$ |
| 10 | 50 | 221,154 | 147.1620 |
| 11 | 46 | 176,064 | 136.4288 |
| 12 | 41 | 214,981 | 145.6925 |
| 13 | 38 | 176,750 | 136.5920 |
| 14 | 34 | 186,525 | 138.9188 |
| 15 | 31 | 174,760 | 136.1184 |
| 16 | 27 | 191,072 | 140.0012 |
| 17 | 24 | 188,770 | 139.4532 |
| 18 | 21 | 190,013 | 139.7490 |
| 19 | 18 | 193,799 | 140.6503 |
| 20 | 16 | 184,576 | 138.4550 |
| 21 | 13 | 192,622 | 140.3700 |
| 22 | 10 | 201,711 | 142.5337 |
| 23 | 8 | 198,981 | 141.8839 |
| 24 | 5 | 210,199 | 144.5541 |
| 25 | 3 | 210,293 | 144.5765 |
| 26 | 0 | 222,965 | 147.5929 |

Just as was done for $0 \le t \le 9$ the 17 pairs of values for B and $A_2$ along with the respective α value shall be plotted for $9 \le t \le 26$. An error table will then be calculated and a best fit model will be concluded.

First then, the 17 graphs:




This graph shows the first 9 curves of the 17 along with the data points, again, all of these curves pass through the point (9,55) and each curve passes through another one of the other data points. Only 9 have been shown here because all 17 on the same graph was too congested. The other 8 curves are shown on the graph at the top of the next page.

As before the data points have been included in black and a ±0.5 vertical tolerance line has been included at each point.


The next page consists of a table which evaluates all the estimates for v of the 17 different models at the times $9 \le t \le 26$. It then calculates the absolute error between these estimates and the values of v we have been given for this coursework. As before, those estimates that are exact due to the use of that point are coloured green and those estimates that are within the ±0.5 tolerance level are highlighted in green.

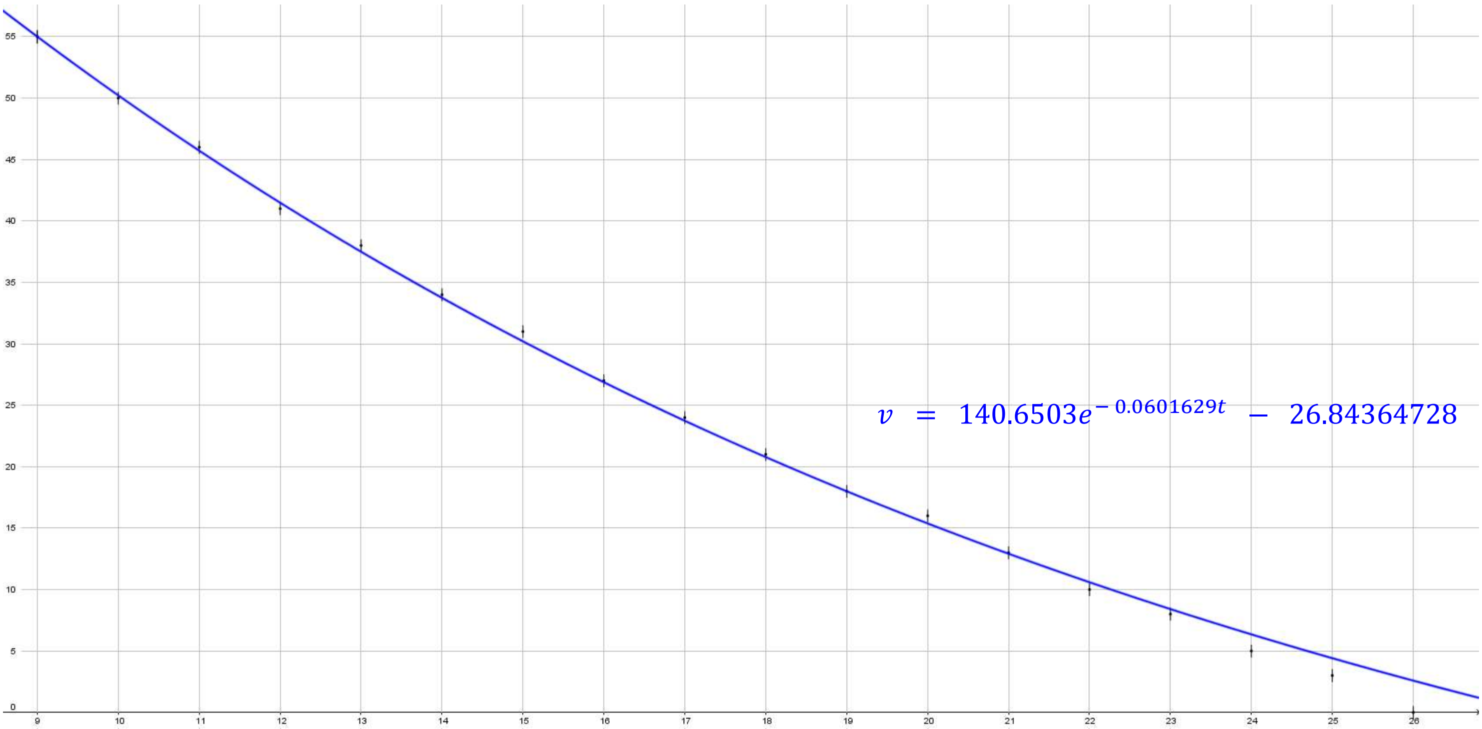
| B | 221,155 | B | 176,064 | B | 214,981 | B | 176,750 | B | 186,525 | B | 174,760 | B | 191,072 | B | 188,770 | B | 190,013 | B | 193,799 | B | 184,576 | B | 192,621 | B | 201,711 | B | 198,981 | B | 210,199 | B | 210,293 | B | 222,965 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_2$ | 147.16 | $A_2$ | 136.43 | $A_2$ | 145.69 | $A_2$ | 136.59 | $A_2$ | 138.92 | $A_2$ | 136.12 | $A_2$ | 140.00 | $A_2$ | 139.45 | $A_2$ | 139.75 | $A_2$ | 140.65 | $A_2$ | 138.45 | $A_2$ | 140.37 | $A_2$ | 142.53 | $A_2$ | 141.88 | $A_2$ | 144.55 | $A_2$ | 144.58 | $A_2$ | 147.59 |
| **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** |
| 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 |
| 50.000 | 0.000 | 50.365 | 0.365 | 50.050 | 0.050 | 50.359 | 0.359 | 50.280 | 0.280 | 50.375 | 0.375 | 50.243 | 0.243 | 50.262 | 0.262 | 50.252 | 0.252 | 50.221 | 0.221 | 50.296 | 0.296 | 50.231 | 0.231 | 50.157 | 0.157 | 50.179 | 0.179 | 50.089 | 0.089 | 50.088 | 0.088 | 49.985 | 0.015 |
| 45.292 | 0.708 | 46.000 | 0.000 | 45.389 | 0.611 | 45.989 | 0.011 | 45.836 | 0.164 | 46.020 | 0.020 | 45.764 | 0.236 | 45.800 | 0.200 | 45.781 | 0.219 | 45.722 | 0.278 | 45.866 | 0.134 | 45.740 | 0.260 | 45.597 | 0.403 | 45.640 | 0.360 | 45.464 | 0.536 | 45.463 | 0.537 | 45.264 | 0.736 |
| 40.859 | 0.141 | 41.890 | 0.890 | 41.000 | 0.000 | 41.874 | 0.874 | 41.651 | 0.651 | 41.920 | 0.920 | 41.547 | 0.547 | 41.600 | 0.600 | 41.571 | 0.571 | 41.485 | 0.485 | 41.695 | 0.695 | 41.511 | 0.511 | 41.304 | 0.304 | 41.366 | 0.366 | 41.109 | 0.109 | 41.107 | 0.107 | 40.817 | 0.183 |
| 36.684 | 1.316 | 38.020 | 0.020 | 36.867 | 1.133 | 38.000 | 0.000 | 37.710 | 0.290 | 38.059 | 0.059 | 37.576 | 0.424 | 37.644 | 0.356 | 37.607 | 0.393 | 37.495 | 0.505 | 37.768 | 0.232 | 37.530 | 0.470 | 37.261 | 0.739 | 37.341 | 0.659 | 37.009 | 0.991 | 37.006 | 0.994 | 36.631 | 1.369 |
| 32.754 | 1.246 | 34.376 | 0.376 | 32.976 | 1.024 | 34.352 | 0.352 | 34.000 | 0.000 | 34.423 | 0.423 | 33.836 | 0.164 | 33.919 | 0.081 | 33.874 | 0.126 | 33.738 | 0.262 | 34.070 | 0.070 | 33.781 | 0.219 | 33.454 | 0.546 | 33.552 | 0.448 | 33.148 | 0.852 | 33.145 | 0.855 | 32.689 | 1.311 |
| 29.053 | 1.947 | 30.945 | 0.055 | 29.312 | 1.688 | 30.916 | 0.084 | 30.506 | 0.494 | 31.000 | 0.000 | 30.315 | 0.685 | 30.412 | 0.588 | 30.360 | 0.640 | 30.201 | 0.799 | 30.588 | 0.412 | 30.250 | 0.750 | 29.869 | 1.131 | 29.983 | 1.017 | 29.513 | 1.487 | 29.509 | 1.491 | 28.977 | 2.023 |
| 25.568 | 1.432 | 27.714 | 0.714 | 25.862 | 1.138 | 27.682 | 0.682 | 27.216 | 0.216 | 27.777 | 0.777 | 27.000 | 0.000 | 27.110 | 0.110 | 27.050 | 0.050 | 26.870 | 0.130 | 27.309 | 0.309 | 26.926 | 0.074 | 26.494 | 0.506 | 26.623 | 0.377 | 26.089 | 0.911 | 26.085 | 0.915 | 25.482 | 1.518 |
| 22.286 | 1.714 | 24.672 | 0.672 | 22.613 | 1.387 | 24.636 | 0.636 | 24.119 | 0.119 | 24.741 | 0.741 | 23.878 | 0.122 | 24.000 | 0.000 | 23.934 | 0.066 | 23.734 | 0.266 | 24.222 | 0.222 | 23.796 | 0.204 | 23.315 | 0.685 | 23.460 | 0.540 | 22.866 | 1.134 | 22.861 | 1.139 | 22.191 | 1.809 |
| 19.196 | 1.804 | 21.808 | 0.808 | 19.554 | 1.446 | 21.768 | 0.768 | 21.202 | 0.202 | 21.883 | 0.883 | 20.939 | 0.061 | 21.072 | 0.072 | 21.000 | 0.000 | 20.781 | 0.219 | 21.315 | 0.315 | 20.849 | 0.151 | 20.323 | 0.677 | 20.481 | 0.519 | 19.831 | 1.169 | 19.826 | 1.174 | 19.092 | 1.908 |
| 16.287 | 1.713 | 19.111 | 1.111 | 16.674 | 1.326 | 19.068 | 1.068 | 18.455 | 0.455 | 19.192 | 1.192 | 18.171 | 0.171 | 18.315 | 0.315 | 18.237 | 0.237 | 18.000 | 0.000 | 18.578 | 0.578 | 18.074 | 0.074 | 17.505 | 0.495 | 17.675 | 0.325 | 16.973 | 1.027 | 16.967 | 1.033 | 16.174 | 1.826 |
| 13.547 | 2.453 | 16.571 | 0.571 | 13.961 | 2.039 | 16.525 | 0.525 | 15.869 | 0.131 | 16.658 | 0.658 | 15.564 | 0.436 | 15.719 | 0.281 | 15.635 | 0.365 | 15.382 | 0.618 | 16.000 | 0.000 | 15.461 | 0.539 | 14.851 | 1.149 | 15.034 | 0.966 | 14.282 | 1.718 | 14.276 | 1.724 | 13.426 | 2.574 |
| 10.968 | 2.032 | 14.179 | 1.179 | 11.407 | 1.593 | 14.130 | 1.130 | 13.434 | 0.434 | 14.272 | 1.272 | 13.110 | 0.110 | 13.274 | 0.274 | 13.186 | 0.186 | 12.916 | 0.084 | 13.573 | 0.573 | 13.000 | 0.000 | 12.353 | 0.647 | 12.547 | 0.453 | 11.748 | 1.252 | 11.741 | 1.259 | 10.839 | 2.161 |
| 8.539 | 1.461 | 11.927 | 1.927 | 9.003 | 0.997 | 11.876 | 1.876 | 11.141 | 1.141 | 12.025 | 2.025 | 10.800 | 0.800 | 10.973 | 0.973 | 10.879 | 0.879 | 10.595 | 0.595 | 11.288 | 1.288 | 10.683 | 0.683 | 10.000 | 0.000 | 10.205 | 0.205 | 9.362 | 0.638 | 9.355 | 0.645 | 8.403 | 1.597 |
| 6.252 | 1.748 | 9.807 | 1.807 | 6.738 | 1.262 | 9.753 | 1.753 | 8.982 | 0.982 | 9.910 | 1.910 | 8.624 | 0.624 | 8.805 | 0.805 | 8.707 | 0.707 | 8.409 | 0.409 | 9.136 | 1.136 | 8.501 | 0.501 | 7.785 | 0.215 | 8.000 | 0.000 | 7.115 | 0.885 | 7.108 | 0.892 | 6.109 | 1.891 |
| 4.098 | 0.902 | 7.810 | 2.810 | 4.606 | 0.394 | 7.754 | 2.754 | 6.949 | 1.949 | 7.918 | 2.918 | 6.575 | 1.575 | 6.764 | 1.764 | 6.662 | 1.662 | 6.350 | 1.350 | 7.110 | 2.110 | 6.447 | 1.447 | 5.699 | 0.699 | 5.924 | 0.924 | 5.000 | 0.000 | 4.992 | 0.008 | 3.949 | 1.051 |
| 2.070 | 0.930 | 5.931 | 2.931 | 2.599 | 0.401 | 5.872 | 2.872 | 5.035 | 2.035 | 6.042 | 3.042 | 4.646 | 1.646 | 4.843 | 1.843 | 4.736 | 1.736 | 4.412 | 1.412 | 5.202 | 2.202 | 4.513 | 1.513 | 3.735 | 0.735 | 3.968 | 0.968 | 3.008 | 0.008 | 3.000 | 0.000 | 1.915 | 1.085 |
| 0.161 | 0.161 | 4.160 | 4.160 | 0.708 | 0.708 | 4.099 | 4.099 | 3.232 | 3.232 | 4.276 | 4.276 | 2.829 | 2.829 | 3.033 | 3.033 | 2.923 | 2.923 | 2.587 | 2.587 | 3.405 | 3.405 | 2.692 | 2.692 | 1.885 | 1.885 | 2.127 | 2.127 | 1.132 | 1.132 | 1.124 | 1.124 | 0.000 | 0.000 |
| | 1.206 | | 1.133 | | 0.955 | | 1.102 | | 0.710 | | 1.194 | | 0.593 | | 0.642 | | 0.612 | | 0.568 | | 0.776 | | 0.573 | | 0.610 | | 0.580 | | 0.774 | | 0.777 | | 1.281 |
| | 0.732 | | 1.138 | | 0.591 | | 1.119 | | 0.848 | | 1.172 | | 0.716 | | 0.785 | | 0.749 | | 0.627 | | 0.904 | | 0.667 | | 0.442 | | 0.484 | | 0.525 | | 0.527 | | 0.781 |

Where |E| is the absolute error between the model and the given value.

Based on this table there is a clear best model in my opinion, and that is the one that is in the 10th column from the left. This column is the column that contains the lowest average of the absolute errors, a column that contains 9 points within the tolerance level (11 if the exact points were included) and a column that has a very low standard deviation implying that there are very few points (if any) of a significant distance away from the true value, in fact, looking at the entire column, the only point that is significantly far away (1.5 or more) is when t=26. Therefore, based on the analysis that has gone before I state that the values of these latter two constants are:

- B = 193,799 (6sf)
- $A_2$ = 140.6503 (7sf)

Which results in the following graph:



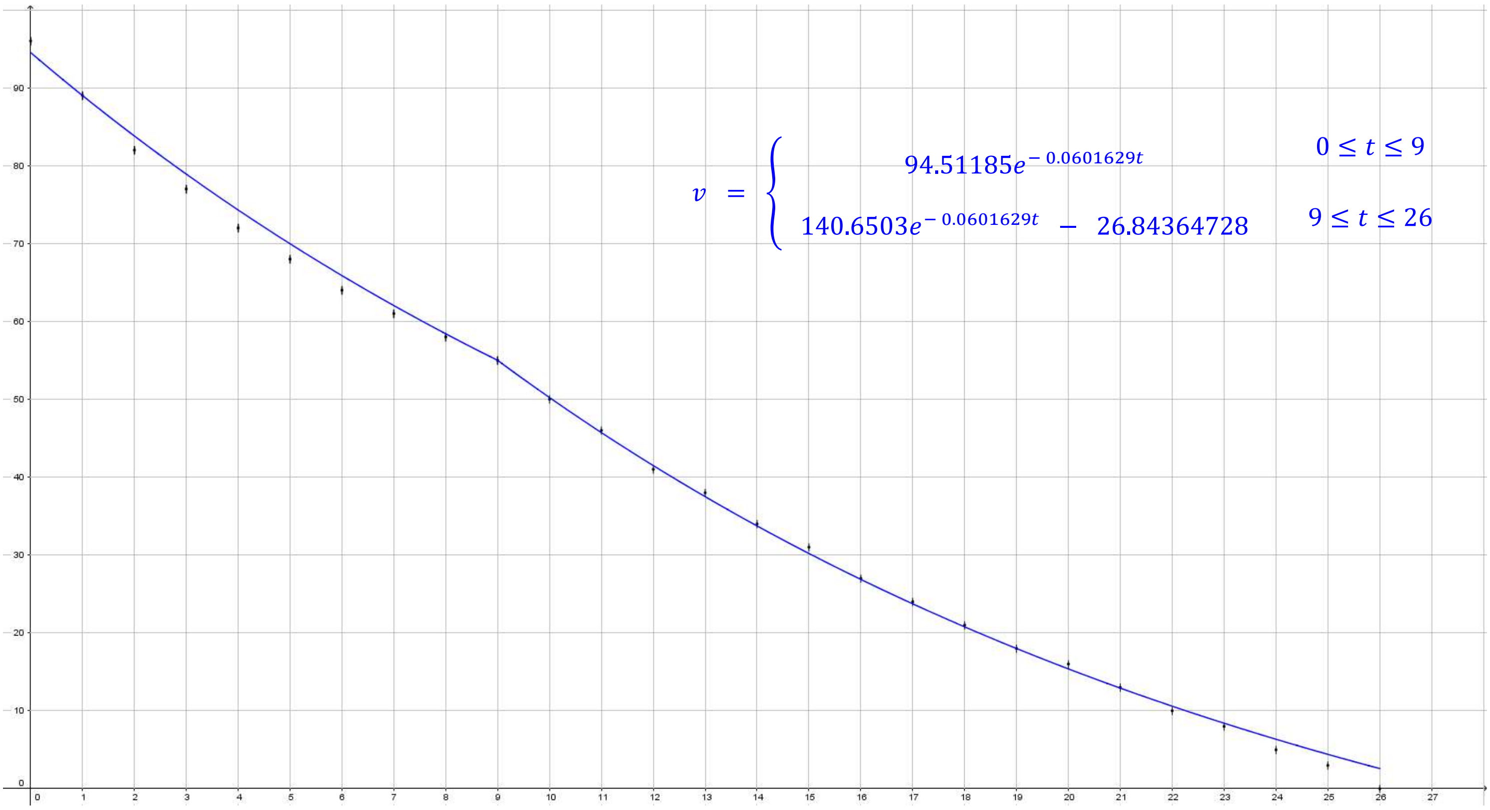

## 2.5. Conclusion of Model 1

Now that the analysis of the solutions has been done, all four of the constants, B, α, $A_1$ and $A_2$ have been found to a good degree of accuracy. They are shown below with the resultant function for v in terms of t:

- $A_1$ = 94.51885
- α = 7,219.548
- B = 193,799
- $A_2$ = 140.6503

$$v = \begin{cases} 94.51185e^{-0.0601629t} & 0 \le t \le 9 \\ 140.6503e^{-0.0601629t} - 26.84364728 & 9 \le t \le 26 \end{cases}$$

Which gives the following graph:




We can use the function for v to estimate the length of runway required for the aeroplane to land; since $\dot{x} = v$, $x$ is given by the integral:

$$x = \int_0^{26} v\,dt \approx \int_0^{9} 94.51185e^{-0.0601629t}\,dt + \int_9^{26} (140.6503e^{-0.0601629t} - 26.84364728)\,dt$$

$$x \approx \left[-1{,}570.932e^{-0.0601629t}\right]_0^9 + \left[-2{,}337.824e^{-0.0601629t} - 26.84364728t\right]_9^{26}$$

$$x \approx 656.8153 + 414.8401 = 1{,}071.6554$$

This means, according to model 1, the aeroplane stops in approximately 1,070 metres. Therefore, I would advise, based on this result, the runway needs to be 1,600m long minimum. This figure is approximately 1.5 times the stopping distance and is a value which would allow for conditions which could require a longer runway, i.e. wet runway, or oil on the runway, or a brake failure, etc…

Ideally, however, I would be looking for absolute errors to be 0.5 or less, that is, I would be looking to see the solution curve v(t) to lie within the small black vertical lines at each point. This is because I am assuming the velocities that were given to me have been rounded to the nearest integer and could therefore contain an error of up to ±0.5.

Although most of the points for $9 \le t \le 23$ are very close to the curve in the graph at the top of the page, I can see that the data points for $2 \le t \le 7$ and t = 24, 25, 26 have a significant amount of room for improvement, therefore, I will now revise the model to see if a better function for v can be found.

# 3. Model 2

## 3.1. Refining the Model

In the previous section I made the modelling assumption that the force due to air resistance is directly proportional to the velocity. I now intend on changing this assumption to see if the model can be improved. In the graphs of the previous model we can see that the curve doesn't "bend" enough to get close to all of the points, this could imply that the resistance model we used is not a powerful enough relationship; a more powerful resistance model should allow the curve to better fit the data, therefore, I will now model the resistance as being proportional to the square of the velocity:

$$f(v) \;=\; \beta v^2$$

Hence,

$$120{,}000\frac{dv}{dt} \;=\; \begin{cases} -\beta v^2 & 0 \le t < 9 \\ -\beta v^2 - B' & 9 < t \le 26 \end{cases}$$

Where B' is used only to emphasise the difference between B, from model 1, and B' from model 2.

## 3.2. General Solution

### $0 \le t \le 9$

$$120{,}000\frac{dv}{dt} \;=\; -\beta v^2$$

$$120{,}000\int v^{-2}\,dv \;=\; -\int \beta\,dt$$

$$-120{,}000.\,v^{-1} \;=\; -\beta t + c_3$$

$$v \;=\; \frac{120{,}000}{A_3 + \beta t} \qquad\qquad \mathit{Where}\ A_3 \;=\; -c_3$$

### $9 \le t \le 26$

$$120{,}000\frac{dv}{dt} \;=\; -\beta v^2 - B'$$

$$120{,}000\int \frac{1}{\beta v^2 + B'}\,dv \;=\; -\int 1\,dt$$

$$\frac{120{,}000}{\beta}\int \frac{1}{v^2 + \frac{B'}{\beta}}\,dv \;=\; -t$$

$$\frac{120{,}000}{\beta}.\sqrt{\frac{\beta}{B'}}.\arctan\left(\sqrt{\frac{\beta}{B'}}v\right) \;=\; -t + c_4$$

$$\arctan\left(\sqrt{\frac{\beta}{B'}}v\right) \;=\; \frac{\sqrt{\beta B'}(c_4 - t)}{120{,}000}$$

$$v \;=\; \sqrt{\frac{B'}{\beta}}\,tan\left(A_4 \;-\; \frac{\sqrt{\beta B'}}{120{,}000}t\right) \qquad\qquad \mathit{Where}\ A_4 = \frac{\sqrt{\beta B'}c_4}{120{,}000}$$

Hence,

$$v \;=\; \begin{cases} \dfrac{120{,}000}{A_3 + \beta t} & 0 \le t \le 9 \\ \sqrt{\dfrac{B'}{\beta}}\,tan\left(A_4 \;-\; \dfrac{\sqrt{\beta B'}}{120{,}000}t\right) & 9 \le t \le 26 \end{cases}$$

### 3.3. Particular Solution

Again, as before, I will be using (t = 9, v = 55) consistently across the 2 equations, to ensure the velocity is continuous, and I will use 1 other point each side of t = 9. This will enable me to calculate values for the 4 unknown constants, B', β, $A_3$ and $A_4$. Note that I will be required to use a numerical method to find values for B' in this model as the unknown B' appears inside and outside of the tan function.

#### Pre-Brakes

First, I look at the equation

$$v = \frac{120{,}000}{A_3 + \beta t} \qquad 0 \le t \le 9$$

And use the point (t = 9, v = 55):

$$55 = \frac{120{,}000}{A_3 + 9\beta}$$

I will then use the other 9 points that this equation is relevant for to calculate nine pairs of values for β and $A_3$, each pair passing through a different point for 0 ≤ t ≤ 8. I will do this as follows:

Take point ($t_0$, $v_0$):

$$v_0 = \frac{120{,}000}{A_3 + \beta t_0}$$

Eliminate $A_3$:

$$A_3 = \frac{120{,}000}{55} - 9\beta = \frac{120{,}000}{v_0} - \beta t_0$$

$$\frac{24{,}000}{11} - \frac{120{,}000}{v_0} = 9\beta - \beta t_0$$

$$\frac{24{,}000(v_0 - 55)}{11 v_0} = (9 - t_0)\beta$$

$$\beta = \frac{24{,}000(v_0 - 55)}{11 v_0 (9 - t_0)}$$

Hence,

$$A_3 = \frac{120{,}000}{55} - 9\frac{120{,}000(v_0 - 55)}{55 v_0 (9 - t_0)}$$

$$A_3 = \frac{24{,}000}{11}\left(1 - \frac{9(v_0 - 55)}{v_0 (9 - t_0)}\right)$$

Using a spreadsheet package I can quickly obtain 9 pairs of values for β and $A_3$, the results are below:

| $t_0$ | $v_0$ | β | $A_3$ |
|---|---|---|---|
| 0 | 96 | 103.5354 | 1250.000 |
| 1 | 89 | 104.1879 | 1244.127 |
| 2 | 82 | 102.6291 | 1258.156 |
| 3 | 77 | 103.8961 | 1246.753 |
| 4 | 72 | 103.0303 | 1254.545 |
| 5 | 68 | 104.2781 | 1243.316 |
| 6 | 64 | 102.2727 | 1261.364 |
| 7 | 61 | 107.3025 | 1216.095 |
| 8 | 58 | 112.8527 | 1166.114 |

## Post-Brakes

Now I look at the second equation:

$$v = \sqrt{\frac{B'}{\beta}}\, tan\left(A_4 - \frac{\sqrt{\beta B'}}{120{,}000}t\right) \qquad 9 \le t \le 26$$

Using t = 9:

$$55 = \sqrt{\frac{B'}{\beta}}\, tan\left(A_4 - \frac{9\sqrt{\beta B'}}{120{,}000}\right)$$

Bearing in mind β has been found so we only have B' and $A_4$ to find, we require 1 other data point.

Take point ($t_0$, $v_0$):

$$v_0 = \sqrt{\frac{B'}{\beta}}\, tan\left(A_4 - \frac{\sqrt{\beta B'}}{120{,}000}t_0\right)$$

Eliminate $A_4$:

$$A_4 = \arctan\left(\sqrt{\frac{\beta}{B'}}55\right) + \frac{9\sqrt{\beta B'}}{120{,}000} = \arctan\left(\sqrt{\frac{\beta}{B'}}v_0\right) + \frac{\sqrt{\beta B'}}{120{,}000}t_0$$

$$Let \quad F(B') = \arctan\left(\sqrt{\frac{\beta}{B'}}55\right) + \frac{9\sqrt{\beta B'}}{120{,}000} - \arctan\left(\sqrt{\frac{\beta}{B'}}v_0\right) - \frac{\sqrt{\beta B'}}{120{,}000}t_0$$

$$F(B') = \arctan\left(\sqrt{\frac{\beta}{B'}}55\right) - \arctan\left(\sqrt{\frac{\beta}{B'}}v_0\right) + \frac{\sqrt{\beta B'}}{120{,}000}(9 - t_0)$$

$$F'(B') = -\frac{55\sqrt{\beta}}{2\sqrt{B'}\,(3025\beta + B')} + \frac{v_0\sqrt{\beta}}{2\sqrt{B'}\,({v_0}^2\beta + B')} + \frac{\sqrt{\beta}}{240{,}000\sqrt{B'}}(9 - t_0)$$

Hence the Newton Raphson formula to find B' is as follows:

$$B'_{n+1} = B'_n - \frac{\arctan\left(\sqrt{\frac{\beta}{B'_n}}55\right) - \arctan\left(\sqrt{\frac{\beta}{B'_n}}v_0\right) + \frac{\sqrt{\beta B'_n}}{120{,}000}(9 - t_0)}{\frac{v_0\sqrt{\beta}}{2\sqrt{B'_n}({v_0}^2\beta + B'_n)} - \frac{55\sqrt{\beta}}{2\sqrt{B'_n}(3025\beta + B'_n)} + \frac{\sqrt{\beta}}{240{,}000\sqrt{B'_n}}(9 - t_0)}$$

This could be tidied up to give something much neater, but the spreadsheet package I'm using can more than cope with this formula so I'll leave it as it is.

I wouldn't expect the values of B' to be too much different in model 2 as the B values were in model 1, therefore, I will take $B'_0 = 200{,}000$ and run 5 iterations of the Newton Raphson iteration method to obtain values for B'. From there we can use the following to obtain corresponding values for $A_4$:

$$A_4 = \arctan\left(\sqrt{\frac{\beta}{B'_5}}\,55\right) + \frac{9\sqrt{\beta B'_5}}{120{,}000}$$

From model 1 we obtained 153 pairs of numbers for B and if I were to do the same here then this would require computing 765 iterations of the Newton Raphson Method. To cut back on this figure I will average the first 8 β values obtained at the top of the previous page. The reason for only taking the first 8 is because these are all fairly similar numbers whereas the $9^{th}$ value (which is from the point (t = 8, v = 58)) gives values for β and $A_3$, which are radically different from the other 8, therefore, I will take β = 103.89152 and calculate 17 pairs of values for B' and $A_4$. Results are as follows:

| $t_0$ | $v_0$ | $B'_0$ | $B'_1$ | $B'_2$ | $B'_3$ | $B'_4$ | $B'_5$ | $A_4$ |
|---|---|---|---|---|---|---|---|---|
| 10 | 50 | 200,000 | 319,831 | 313,842 | 313,845 | 313,845 | 313,845 | 1.213999 |
| 11 | 46 | 200,000 | 277,670 | 275,721 | 275,721 | 275,721 | 275,721 | 1.219500 |
| 12 | 41 | 200,000 | 324,544 | 321,817 | 321,819 | 321,819 | 321,819 | 1.213133 |
| 13 | 38 | 200,000 | 287,802 | 287,218 | 287,218 | 287,218 | 287,218 | 1.217586 |
| 14 | 34 | 200,000 | 299,860 | 300,581 | 300,582 | 300,582 | 300,582 | 1.215646 |
| 15 | 31 | 200,000 | 289,156 | 290,721 | 290,722 | 290,722 | 290,722 | 1.217049 |
| 16 | 27 | 200,000 | 304,066 | 308,153 | 308,163 | 308,163 | 308,163 | 1.214672 |
| 17 | 24 | 200,000 | 300,452 | 305,753 | 305,771 | 305,771 | 305,771 | 1.214970 |
| 18 | 21 | 200,000 | 299,245 | 305,958 | 305,990 | 305,990 | 305,990 | 1.214943 |
| 19 | 18 | 200,000 | 299,388 | 307,732 | 307,786 | 307,786 | 307,786 | 1.214719 |
| 20 | 16 | 200,000 | 289,057 | 296,606 | 296,653 | 296,653 | 296,653 | 1.216186 |
| 21 | 13 | 200,000 | 291,542 | 300,917 | 300,997 | 300,997 | 300,997 | 1.215591 |
| 22 | 10 | 200,000 | 293,913 | 305,239 | 305,365 | 305,365 | 305,365 | 1.215022 |
| 23 | 8 | 200,000 | 287,930 | 298,631 | 298,750 | 298,750 | 298,750 | 1.215895 |
| 24 | 5 | 200,000 | 290,676 | 303,305 | 303,481 | 303,481 | 303,481 | 1.215263 |
| 25 | 3 | 200,000 | 286,209 | 298,258 | 298,425 | 298,425 | 298,425 | 1.215940 |
| 26 | 0 | 200,000 | 288,831 | 302,627 | 302,855 | 302,855 | 302,855 | 1.215345 |

## 3.4. Graphs and Analysis

### 0 ≤ t ≤ 9

The idea was to now graph the 8 graphs that have been obtained for the 8 pairs of β and $A_3$ values shown at the top of the previous page, however, to calculate my values of B' I have fixed the value of β to be 103.89152 (8sf). This means I will need to use this value to recalculate a value for $A_3$ to ensure v = 55 when t = 9. At this stage I hope this will give a model which is within the tolerance level otherwise I'll need to adapt my current strategy.

Recall, from page 15, that

$$55 = \frac{120{,}000}{A_3 + 9\beta}$$

On page 15 I then used another point to solve simultaneous equations in $A_3$ and β. I then used these values of β to find an average to define a new value for β which was used to find values for B' and $A_4$. Now, I need to use this new value of β in the equation above to define a corresponding value for $A_3$.

$$A_3 = \left(\frac{120{,}000}{55} - 9\beta\right)\Bigg|_{\beta=103.89152} = 1{,}246.795$$

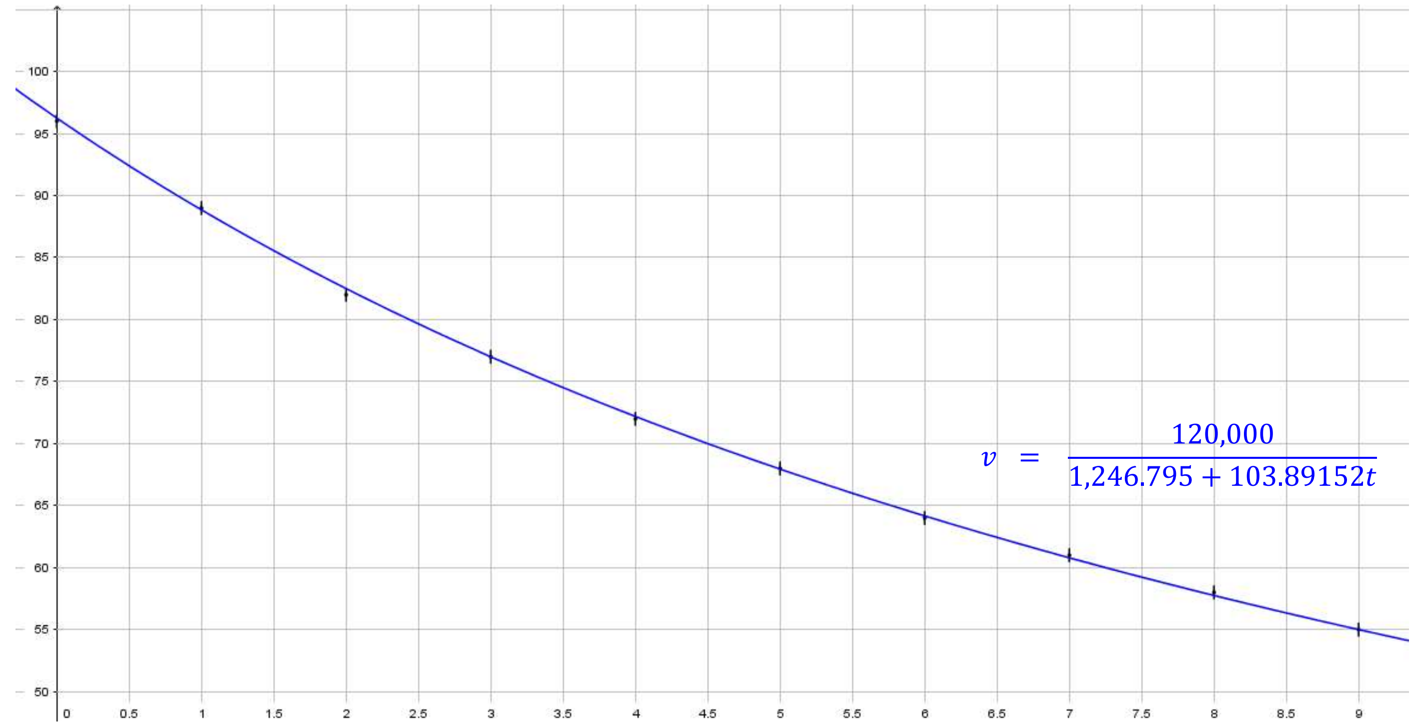
Below I have included the values of the models for the first 7 pairs of $A_3$ and β given at the top of page 16, along with this new pair described above.

| β | 103.5354 | β | 104.1879 | β | 102.6291 | β | 103.8961 | β | 103.0303 | β | 104.2781 | β | 102.2727 | β | 103.8915 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_3$ | 1,250.000 | $A_3$ | 1,244.127 | $A_3$ | 1,258.156 | $A_3$ | 1,246.753 | $A_3$ | 1,254.545 | $A_3$ | 1,243.316 | $A_3$ | 1,261.364 | $A_3$ | 1,246.795 |
| **V** | **\|Err\|** | **V** | **\|Err\|** | **V** | **\|Err\|** | **V** | **\|Err\|** | **V** | **\|Err\|** | **V** | **\|Err\|** | **V** | **\|Err\|** | **V** | **\|Err\|** |
| 96.000 | 0.000 | 96.453 | 0.453 | 95.378 | 0.622 | 96.250 | 0.250 | 95.652 | 0.348 | 96.516 | 0.516 | 95.135 | 0.865 | 96.247 | 0.247 |
| 88.657 | 0.343 | 89.000 | 0.000 | 88.184 | 0.816 | 88.846 | 0.154 | 88.393 | 0.607 | 89.048 | 0.048 | 88.000 | 1.000 | 88.844 | 0.156 |
| 82.357 | 0.357 | 86.616 | 0.616 | 82.000 | 0.000 | 82.500 | 0.500 | 82.158 | 0.158 | 82.652 | 0.652 | 81.860 | 0.140 | 82.498 | 0.498 |
| 76.893 | 0.107 | 77.087 | 0.087 | 76.626 | 0.374 | 77.000 | 0.000 | 76.744 | 0.256 | 77.113 | 0.113 | 76.522 | 0.478 | 76.999 | 0.001 |
| 72.109 | 0.109 | 72.251 | 0.251 | 71.913 | 0.087 | 72.188 | 0.188 | 72.000 | 0.000 | 72.271 | 0.271 | 71.837 | 0.163 | 72.187 | 0.187 |
| 67.886 | 0.114 | 67.986 | 0.014 | 67.747 | 0.253 | 67.941 | 0.059 | 67.808 | 0.192 | 68.000 | 0.000 | 67.692 | 0.308 | 67.940 | 0.060 |
| 64.130 | 0.130 | 64.197 | 0.197 | 64.037 | 0.037 | 64.167 | 0.167 | 64.078 | 0.078 | 64.206 | 0.206 | 64.000 | 0.000 | 64.166 | 0.166 |
| 60.767 | 0.233 | 60.807 | 0.193 | 60.712 | 0.288 | 60.789 | 0.211 | 60.736 | 0.264 | 60.813 | 0.187 | 60.690 | 0.310 | 60.789 | 0.211 |
| 57.740 | 0.260 | 57.758 | 0.242 | 57.715 | 0.285 | 57.750 | 0.250 | 57.726 | 0.274 | 57.761 | 0.239 | 57.705 | 0.295 | 57.750 | 0.250 |
| 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 |
| | 0.165 | | 0.205 | | 0.276 | | 0.178 | | 0.218 | | 0.223 | | 0.356 | | 0.178 |
| | 0.121 | | 0.192 | | 0.258 | | 0.139 | | 0.171 | | 0.204 | | 0.322 | | 0.138 |

All the errors from the latter column are within the tolerance level.

I do believe that the left most column is the best model for this data; I have judged this on the fact that all data points are within the tolerance **and** that the mean and standard deviation of the absolute errors are both less than that of the right-hand column, however, I feel I would be doing an injustice to the model if I were to use a different β value for before the brakes are applied to after the brakes are applied. This is because the coefficient of proportionality, β, has no reason to change within the model regardless of the application of the brakes.

Therefore, since all data points lie within the tolerance for the right-most column, I feel confident enough to conclude that β is 103.89152 (8sf) and $A_3$ is 1,246.795 (7sf). This results in the following graph:

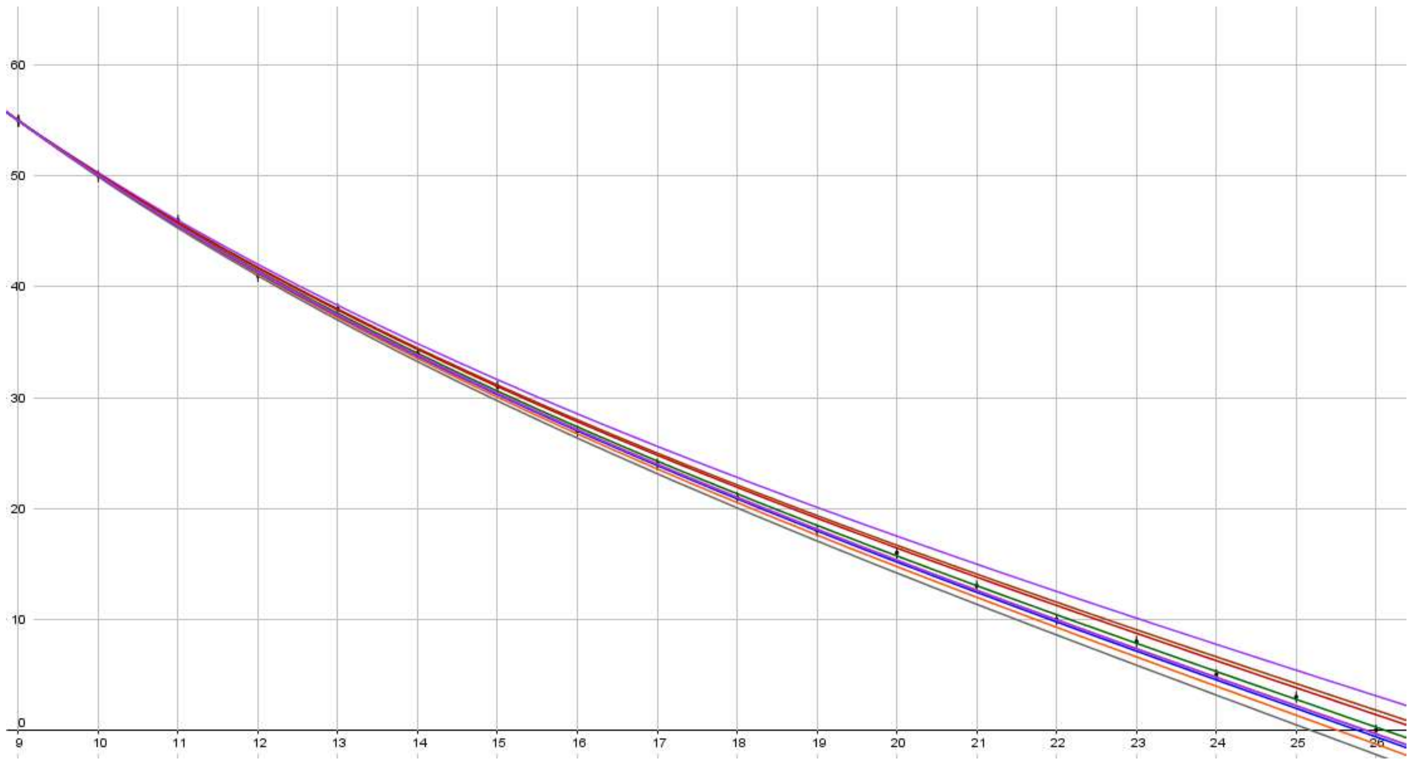


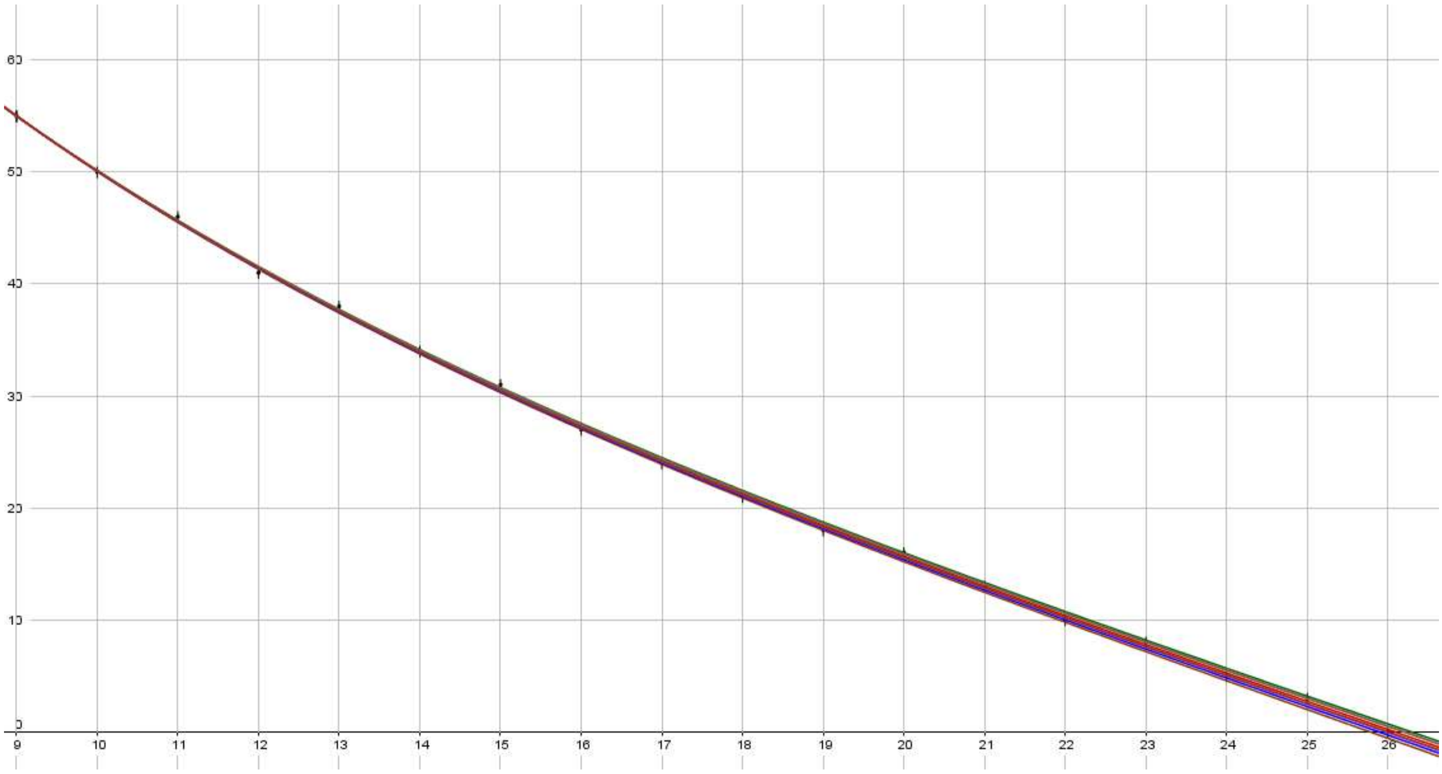

## $9 \leq t \leq 26$

Recall, for this portion of the model, we have

$$v = \sqrt{\frac{B'_5}{\beta}}\, tan\left(A_4 - \frac{\sqrt{\beta B'_5}t}{120{,}000}\right)$$

Where β is defined as 103.89152 and the 17 pairs of $B'_5$ and $A_4$ values have been given on page 17. Below are the graphs of the first 9 pairs of values. The other 8 are in the graph at the top of the next page; split only because of graph congestion.






The variation in the values of the parameters appear to affect the curve less here than they did in model 1. On the next page is a table of the 17 models across the 18 values of t such that 9 ≤ t ≤ 26 along with the absolute errors between the model predictions and the values given to me in the initial data. The parts of the table written in green are exactly the same as the given data (because these were the points I used in the calculations of the parameters), the parts of the table highlighted in green are within the ±0.5 tolerance level I described in the assumptions section of this work.

| $B'_5$ | 313,845 | $B'_5$ | 275,721 | $B'_5$ | 321,819 | $B'_5$ | 287,218 | $B'_5$ | 300,582 | $B'_5$ | 290,722 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_4$ | 1.21400 | $A_4$ | 1.21950 | $A_4$ | 1.21313 | $A_4$ | 1.21759 | $A_4$ | 1.21565 | $A_4$ | 1.21705 |
| **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** |
| 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 |
| 50.000 | 0.000 | 50.304 | 0.304 | 49.936 | 0.064 | 50.212 | 0.212 | 50.106 | 0.106 | 50.184 | 0.184 |
| 45.415 | 0.585 | 46.000 | 0.000 | 45.293 | 0.707 | 45.824 | 0.176 | 45.619 | 0.381 | 45.770 | 0.230 |
| 41.178 | 0.178 | 42.026 | 1.026 | 41.000 | 0.000 | 41.770 | 0.770 | 41.473 | 0.473 | 41.692 | 0.692 |
| 37.232 | 0.768 | 38.331 | 0.331 | 37.002 | 0.998 | 38.000 | 0.000 | 37.615 | 0.385 | 37.899 | 0.101 |
| 33.533 | 0.467 | 34.874 | 0.874 | 33.252 | 0.748 | 34.470 | 0.470 | 34.000 | 0.000 | 34.347 | 0.347 |
| 30.043 | 0.957 | 31.619 | 0.619 | 29.712 | 1.288 | 31.145 | 0.145 | 30.592 | 0.408 | 31.000 | 0.000 |
| 26.730 | 0.270 | 28.538 | 1.538 | 26.350 | 0.650 | 27.994 | 0.994 | 27.360 | 0.360 | 27.828 | 0.828 |
| 23.566 | 0.434 | 25.606 | 1.606 | 23.137 | 0.863 | 24.993 | 0.993 | 24.278 | 0.278 | 24.806 | 0.806 |
| 20.530 | 0.470 | 22.801 | 1.801 | 20.052 | 0.948 | 22.119 | 1.119 | 21.323 | 0.323 | 21.911 | 0.911 |
| 17.599 | 0.401 | 20.105 | 2.105 | 17.071 | 0.929 | 19.353 | 1.353 | 18.475 | 0.475 | 19.123 | 1.123 |
| 14.757 | 1.243 | 17.501 | 1.501 | 14.178 | 1.822 | 16.678 | 0.678 | 15.717 | 0.283 | 16.427 | 0.427 |
| 11.986 | 1.014 | 14.975 | 1.975 | 11.354 | 1.646 | 14.080 | 1.080 | 13.033 | 0.033 | 13.806 | 0.806 |
| 9.273 | 0.727 | 12.513 | 2.513 | 8.586 | 1.414 | 11.544 | 1.544 | 10.408 | 0.408 | 11.247 | 1.247 |
| 6.602 | 1.398 | 10.104 | 2.104 | 5.858 | 2.142 | 9.058 | 1.058 | 7.831 | 0.169 | 8.737 | 0.737 |
| 3.962 | 1.038 | 7.737 | 2.737 | 3.158 | 1.842 | 6.611 | 1.611 | 5.289 | 0.289 | 6.266 | 1.266 |
| 1.340 | 1.660 | 5.402 | 2.402 | 0.473 | 2.527 | 4.192 | 1.192 | 2.769 | 0.231 | 3.820 | 0.820 |
| -1.275 | 1.275 | 3.088 | 3.088 | -2.210 | 2.210 | 1.790 | 1.790 | 0.262 | 0.262 | 1.391 | 1.391 |
| | 0.716 | | 1.474 | | 1.155 | | 0.844 | | 0.270 | | 0.662 |
| | 0.473 | | 0.932 | | 0.741 | | 0.553 | | 0.149 | | 0.433 |

| $B'_5$ | 308,163 | $B'_5$ | 305,771 | $B'_5$ | 305,990 | $B'_5$ | 307,786 | $B'_5$ | 296,653 | $B'_5$ | 300,997 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_4$ | 1.21467 | $A_4$ | 1.21497 | $A_4$ | 1.21494 | $A_4$ | 1.21472 | $A_4$ | 1.21619 | $A_4$ | 1.21559 |
| **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** |
| 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 |
| 50.045 | 0.045 | 50.064 | 0.064 | 50.063 | 0.063 | 50.048 | 0.048 | 50.137 | 0.137 | 50.102 | 0.102 |
| 45.502 | 0.498 | 45.539 | 0.461 | 45.536 | 0.464 | 45.508 | 0.492 | 45.679 | 0.321 | 45.612 | 0.388 |
| 41.304 | 0.304 | 41.357 | 0.357 | 41.353 | 0.353 | 41.313 | 0.313 | 41.560 | 0.560 | 41.464 | 0.464 |
| 37.396 | 0.604 | 37.465 | 0.535 | 37.459 | 0.541 | 37.407 | 0.593 | 37.728 | 0.272 | 37.603 | 0.397 |
| 33.733 | 0.267 | 33.817 | 0.183 | 33.810 | 0.190 | 33.746 | 0.254 | 34.138 | 0.138 | 33.985 | 0.015 |
| 30.278 | 0.722 | 30.377 | 0.623 | 30.368 | 0.632 | 30.294 | 0.706 | 30.755 | 0.245 | 30.575 | 0.425 |
| 27.000 | 0.000 | 27.114 | 0.114 | 27.103 | 0.103 | 27.018 | 0.018 | 27.547 | 0.547 | 27.341 | 0.341 |
| 23.872 | 0.128 | 24.000 | 0.000 | 23.988 | 0.012 | 23.892 | 0.108 | 24.489 | 0.489 | 24.256 | 0.256 |
| 20.870 | 0.130 | 21.013 | 0.013 | 21.000 | 0.000 | 20.893 | 0.107 | 21.557 | 0.557 | 21.298 | 0.298 |
| 17.975 | 0.025 | 18.133 | 0.133 | 18.119 | 0.119 | 18.000 | 0.000 | 18.734 | 0.734 | 18.448 | 0.448 |
| 15.169 | 0.831 | 15.342 | 0.658 | 15.326 | 0.674 | 15.196 | 0.804 | 16.000 | 0.000 | 15.687 | 0.313 |
| 12.436 | 0.564 | 12.624 | 0.376 | 12.607 | 0.393 | 12.465 | 0.535 | 13.341 | 0.341 | 13.000 | 0.000 |
| 9.761 | 0.239 | 9.965 | 0.035 | 9.947 | 0.053 | 9.793 | 0.207 | 10.743 | 0.743 | 10.373 | 0.373 |
| 7.130 | 0.870 | 7.352 | 0.648 | 7.332 | 0.668 | 7.165 | 0.835 | 8.193 | 0.193 | 7.793 | 0.207 |
| 4.532 | 0.468 | 4.771 | 0.229 | 4.750 | 0.250 | 4.570 | 0.430 | 5.679 | 0.679 | 5.247 | 0.247 |
| 1.955 | 1.045 | 2.212 | 0.788 | 2.189 | 0.811 | 1.995 | 1.005 | 3.189 | 0.189 | 2.725 | 0.275 |
| -0.614 | 0.614 | -0.337 | 0.337 | -0.362 | 0.362 | -0.570 | 0.570 | 0.713 | 0.713 | 0.214 | 0.214 |
| | 0.409 | | 0.308 | | 0.316 | | 0.390 | | 0.381 | | 0.265 |
| | 0.320 | | 0.253 | | 0.259 | | 0.310 | | 0.244 | | 0.147 |

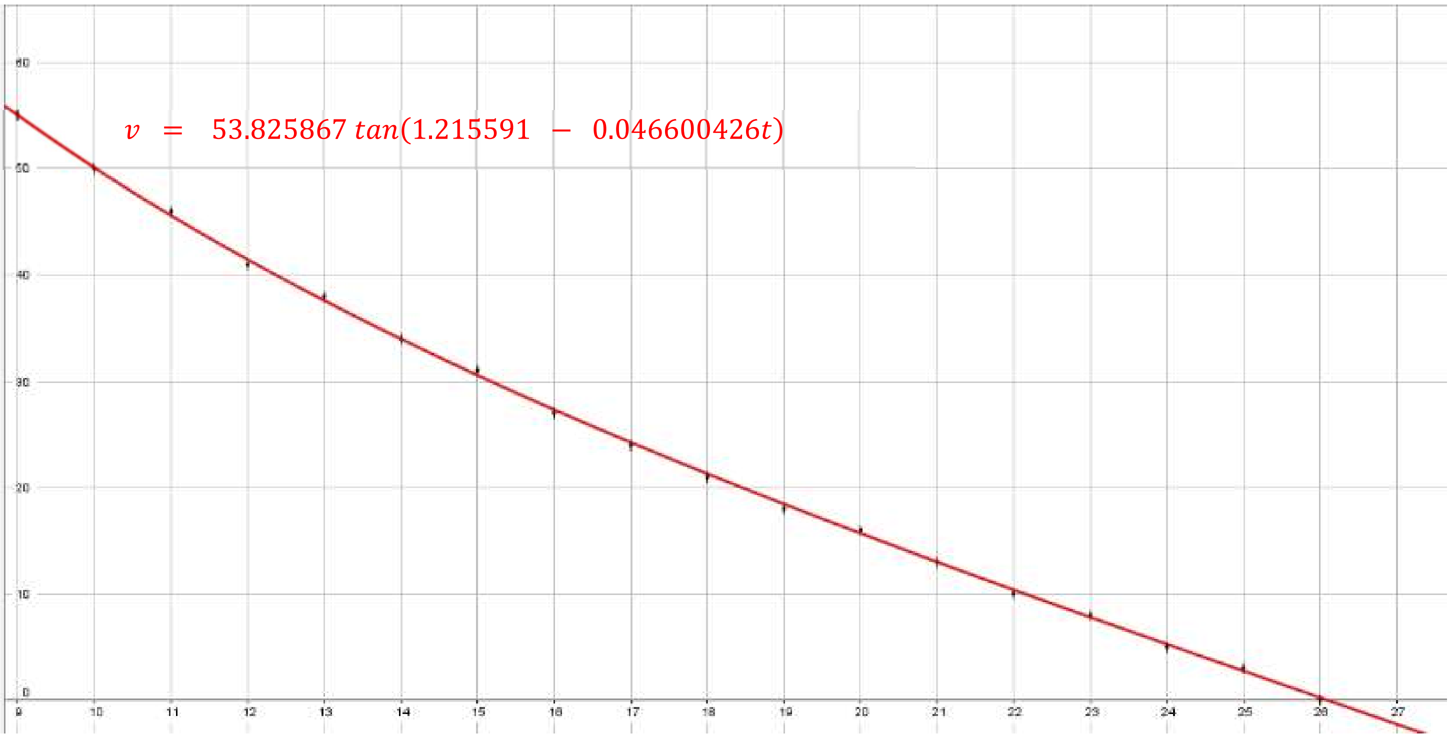
| $B'_5$ | 305,365 | $B'_5$ | 298,750 | $B'_5$ | 303,481 | $B'_5$ | 298,425 | $B'_5$ | 302,855 |
|---|---|---|---|---|---|---|---|---|---|
| $A_4$ | 1.21502 | $A_4$ | 1.21590 | $A_4$ | 1.21526 | $A_4$ | 1.21594 | $A_4$ | 1.21535 |
| **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** | **V** | **\|E\|** |
| 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 | 55.000 | 0.000 |
| 50.068 | 0.068 | 50.120 | 0.120 | 50.083 | 0.083 | 50.123 | 0.123 | 50.088 | 0.088 |
| 45.545 | 0.455 | 45.647 | 0.353 | 45.574 | 0.426 | 45.652 | 0.348 | 45.584 | 0.416 |
| 41.366 | 0.366 | 41.514 | 0.514 | 41.408 | 0.408 | 41.521 | 0.521 | 41.422 | 0.422 |
| 37.477 | 0.523 | 37.668 | 0.332 | 37.531 | 0.469 | 37.677 | 0.323 | 37.549 | 0.451 |
| 33.832 | 0.168 | 34.064 | 0.064 | 33.898 | 0.102 | 34.076 | 0.076 | 33.920 | 0.080 |
| 30.394 | 0.606 | 30.668 | 0.332 | 30.472 | 0.528 | 30.681 | 0.319 | 30.498 | 0.502 |
| 27.133 | 0.133 | 27.447 | 0.447 | 27.223 | 0.223 | 27.463 | 0.463 | 27.252 | 0.252 |
| 24.022 | 0.022 | 24.376 | 0.376 | 24.123 | 0.123 | 24.394 | 0.394 | 24.156 | 0.156 |
| 21.037 | 0.037 | 21.432 | 0.432 | 21.150 | 0.150 | 21.452 | 0.452 | 21.187 | 0.187 |
| 18.160 | 0.160 | 18.596 | 0.596 | 18.284 | 0.284 | 18.617 | 0.617 | 18.325 | 0.325 |
| 15.371 | 0.629 | 15.849 | 0.151 | 15.508 | 0.492 | 15.872 | 0.128 | 15.553 | 0.447 |
| 12.656 | 0.344 | 13.177 | 0.177 | 12.805 | 0.195 | 13.202 | 0.202 | 12.854 | 0.146 |
| 10.000 | 0.000 | 10.565 | 0.565 | 10.161 | 0.161 | 10.592 | 0.592 | 10.214 | 0.214 |
| 7.389 | 0.611 | 8.000 | 0.000 | 7.564 | 0.436 | 8.030 | 0.030 | 7.621 | 0.379 |
| 4.812 | 0.188 | 5.471 | 0.471 | 5.000 | 0.000 | 5.503 | 0.503 | 5.062 | 0.062 |
| 2.256 | 0.744 | 2.965 | 0.035 | 2.458 | 0.542 | 3.000 | 0.000 | 2.526 | 0.474 |
| -0.290 | 0.290 | 0.473 | 0.473 | -0.072 | 0.072 | 0.510 | 0.510 | 0.000 | 0.000 |
| | 0.297 | | 0.302 | | 0.261 | | 0.311 | | 0.256 |
| | 0.240 | | 0.195 | | 0.183 | | 0.205 | | 0.169 |

Where |E| is the absolute error between the model and the given value.

Here I have a model which is clearly better than the others because it is within 0.5 of **all** of the data points and it has the lowest mean absolute error and the lowest standard deviation of the absolute error. It is the column 6th from the right. Another really good model is given in the 5th column from the left; this model is also within the tolerance of all of the points, however, the average of the absolute errors was slightly higher, therefore, I claim that the values of the 2 constants B' and $A_4$ are as follows:

- $B'_5$ = 300,997 (6sf)
- $A_4$ = 1.215591 (7sf)
- And β = 103.89152 (8sf)

Which gives the following graph:



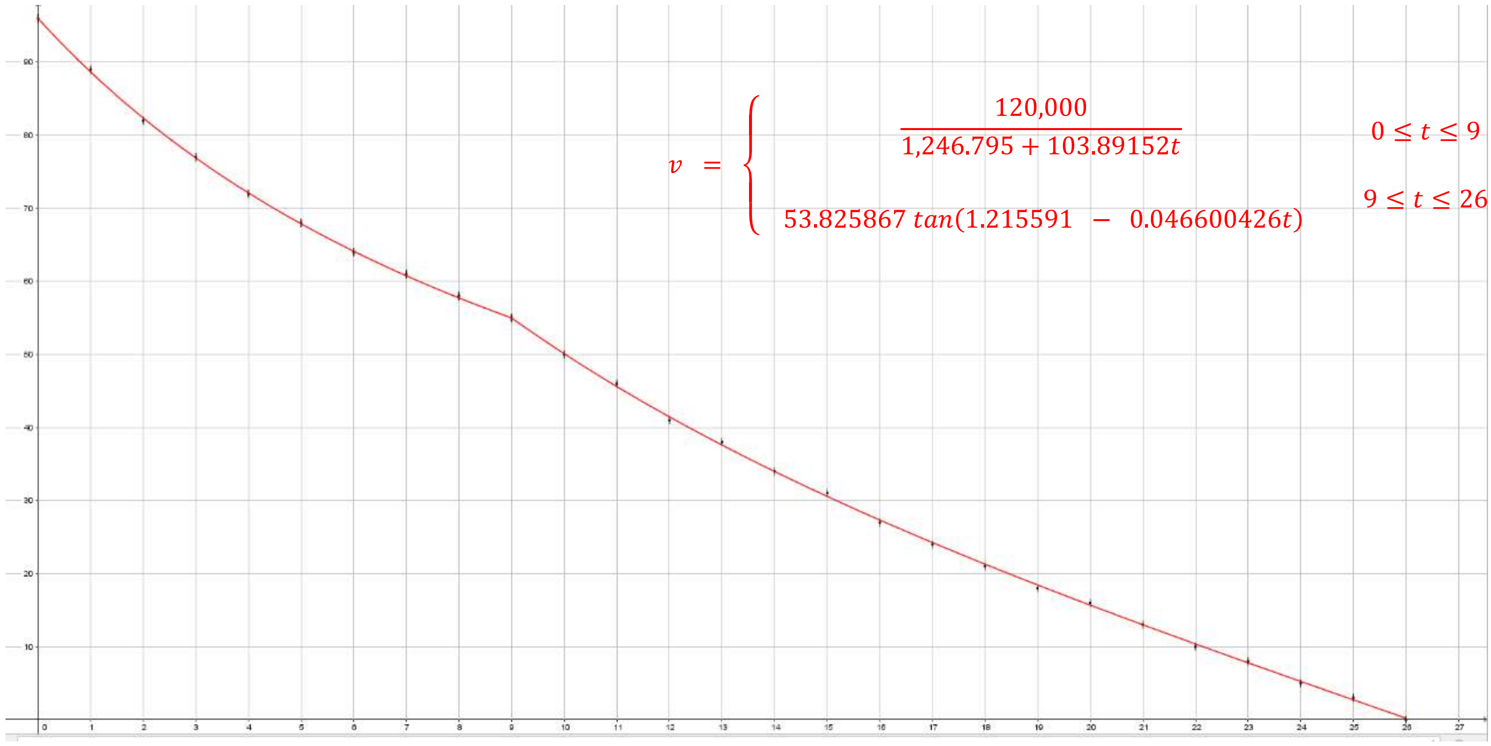

## 3.5. Conclusion of Model 2

Now that the analysis of the solutions has been done, all four of the constants, B', β, $A_3$ and $A_4$ have been found to a good degree of accuracy. They are shown below with the resultant function for v in terms of t:

- $A_3$ = 1,246.795
- β = 103.89152
- B' = 300,997
- $A_4$ = 1.215591

$$v = \begin{cases} \dfrac{120{,}000}{1{,}246.795 + 103.89152t} & 0 \le t \le 9 \\ 53.825867\,tan(1.215591 - 0.046600426t) & 9 \le t \le 26 \end{cases}$$

Which gives the following graph:




We can use the function for v to estimate the length of runway required for the aeroplane to land since $\dot{x} = v$ and therefore $x$ is given by the integral:

$$x = \int_0^{26} v\,dt \approx \int_0^9 \frac{120{,}000}{1{,}246.795 + 103.89152t}dt + \int_9^{26} 53.825867\,tan(1.215591 - 0.0466004261t)\,dt$$

$$x \approx \left[\ 1{,}155.051\ln(1{,}246.795 + 103.89152t)\right]_0^9 + \left[-1{,}155.051\ln\big(sec(1.215591 - 0.0466004261t)\big)\right]_9^{26}$$

$$x \approx 646.3463 + 412.8983 = 1{,}059.2446$$

This means, according to model 2, the aeroplane stops in approximately 1,060 metres. Therefore, I would still advise, based on this result, the runway needs to be 1,600m long, minimum. This figure is approximately 1.5 times the stopping distance and is a value which would allow for conditions which could require a longer runway, i.e. wet runway, or oil on the runway, or a brake failure, etc…

# 4. Final Conclusions

## 4.1. Prediction

Based on the models I predict the aeroplane stopped in 1,060 metres and therefore recommend a length of runway to be absolutely no less than 1,600 metres for the reasons mentioned in this work.

## 4.2. Comparing the Models

| Data | | Model 1 | | Model 2 | |
|---|---|---|---|---|---|
| t | v | V | \|Err\| | V | \|Err\| |
| 0 | 96 | 94.519 | 1.481 | 96.247 | 0.247 |
| 1 | 89 | 89.000 | 0.000 | 88.844 | 0.156 |
| 2 | 82 | 83.803 | 1.803 | 82.498 | 0.498 |
| 3 | 77 | 78.910 | 1.910 | 76.999 | 0.001 |
| 4 | 72 | 74.303 | 2.303 | 72.187 | 0.187 |
| 5 | 68 | 69.964 | 1.964 | 67.940 | 0.060 |
| 6 | 64 | 65.879 | 1.879 | 64.166 | 0.166 |
| 7 | 61 | 62.033 | 1.033 | 60.789 | 0.211 |
| 8 | 58 | 58.411 | 0.411 | 57.750 | 0.250 |
| 9 | 55 | 55.000 | 0.000 | 55.000 | 0.000 |
| 10 | 50 | 50.221 | 0.221 | 50.102 | 0.102 |
| 11 | 46 | 45.722 | 0.278 | 45.612 | 0.388 |
| 12 | 41 | 41.485 | 0.485 | 41.464 | 0.464 |
| 13 | 38 | 37.495 | 0.505 | 37.603 | 0.397 |
| 14 | 34 | 33.738 | 0.262 | 33.985 | 0.015 |
| 15 | 31 | 30.201 | 0.799 | 30.575 | 0.425 |
| 16 | 27 | 26.870 | 0.130 | 27.341 | 0.341 |
| 17 | 24 | 23.734 | 0.266 | 24.256 | 0.256 |
| 18 | 21 | 20.781 | 0.219 | 21.298 | 0.298 |
| 19 | 18 | 18.000 | 0.000 | 18.448 | 0.448 |
| 20 | 16 | 15.382 | 0.618 | 15.687 | 0.313 |
| 21 | 13 | 12.916 | 0.084 | 13.000 | 0.000 |
| 22 | 10 | 10.595 | 0.595 | 10.373 | 0.373 |
| 23 | 8 | 8.409 | 0.409 | 7.793 | 0.207 |
| 24 | 5 | 6.350 | 1.350 | 5.247 | 0.247 |
| 25 | 3 | 4.412 | 1.412 | 2.725 | 0.275 |
| 26 | 0 | 2.587 | 2.587 | 0.214 | 0.214 |
| | | | 0.852 | | 0.242 |

To the left is a table of all the values of v the 2 models give along with the given values of v and the absolute error between the models and the given values.

As mentioned on page 2, in the initial assumptions, I have assumed that the values for the velocity that were given to me have been rounded to the nearest integer. This gives me a ±0.5 error tolerance in my models.

13 out of 27 data points lie within the tolerance level in model 1.

All 27 data points lie within the tolerance level in model 2.

Moreover, the average absolute error for model 1 is over 3 times greater than that for model 2. These are shown at the bottom of the absolute error columns.

Therefore, I declare, based on this analysis, that model 2 is a significant improvement from model 1.