MEI

Mathematics in Education and Industry

Differential Equations Coursework

July, 2007

Differential Equations Coursework

Introduction

Differential Equations (DE) is an A2 module which may be taken as part of a Further Mathematics course. Marking standards should reflect this.

Teachers and/or candidates who have not attempted coursework assignments in Mechanics will nonetheless require a clear grasp of the process implicit in the Modelling Flowchart that appears in the Coursework Bank. Candidates should show an understanding of the application of that process to realistic problems by developing Differential Equations. It will not do to quote a standard law (e.g. Newton's cooling law) and to match it to the standard Physics experiment designed to illustrate that property.

The work is to be marked to one of two marking schemes, labelled Scheme A and Scheme B.

Scheme A is to be used when the emphasis is on the modelling process. Some data will be required to verify the quality of the model's predictions; this could come from a published source or from a demonstration experiment possibly performed by the teacher. The scheme requires that the candidate suggests possible amendments to the first model and, in addition, works these through in search of improved predictions. It is important that the first model allows scope for this; applying the best known theory to a standard situation will make it difficult to proceed to a second cycle of the process.

Scheme B is to be used when there is a rather greater emphasis on the experiment. The candidate should carry out an experiment and propose a model to match the situation and which requires differential equations to produce a solution. The predictions from this model are to be compared to the actual values obtained and the candidate must decide whether the discrepancies arise from inadequacies of experimental technique or from lack of refinement in the model, and attempt appropriate reworking.

Failure in either case to achieve a second cycle satisfactorily could mean that an able candidate does not qualify for full marks. The higher level of application expected for this module can be reached in either the initial modelling cycle or the improved second version.

In this new specification the marks for the coursework task for this module have been increased from 15 to 18. One mark has been added to the comparison domain thus raising the importance of possible variation in the parameters of the solution of the equation.

In addition, there are two marks for oral communication. Assessors are required to authenticate the work as being that of the student. The oral communication gives the assessor the opportunity to discuss the work with the student either in a formal interview or by ongoing discussion as the report is being written, allowing authenticity to be established. Alternatively, the student may give a presentation to the group.

Differential Equations Coursework Tasks

The following tasks have been compiled as suggestions for Differential Equations. Centres may develop their own tasks if they wish and may submit proposed tasks for comment and advice.

Centres who develop their own tasks are recommended to prepare sheets as has been done for the recommended tasks in this bank.

MEI would be pleased to consider the inclusion of such tasks in this bank.

Further tasks may be offered by MEI – Centres should add them to the list below.

DE 1 Cascades

DE 2 Damped Oscillations

DE 3 Parachutes

DE 4 Bungy Jumping

DE 5 Paper Cake Cups

DE 6 Aeroplane Landing

DE 7 Growth Functions

DE 8 Balloon Ascents

DE 9 Interacting Species

DE 10 Oscillating Spring

The format of assignments

1. Student’s Brief

This is intended to be given to students before any work is attempted. Work in the classroom should begin with the first ideas that students themselves can contribute. Discussion amongst themselves (within a teaching group), with teacher guidance, will usually bring out the significant features of the problem and a line of enquiry will be opened up.

Student’s briefs for the 10 tasks listed on the previous page are given at the back of this document.

2. Teacher’s Notes

These are designed to indicate to the teacher some possible features of this discussion. They may indicate certain pitfalls to be avoided, or starting points for work on the problem. These notes should not be shown to students; their content should be fed into the classroom by the teacher at appropriate moments.

These notes should not be shown to students.

3. Marking Notes

These indicate which marking scheme is appropriate in cases where there is a choice and how the criteria might be applied under the various headings of the marking scheme.

These notes should not be shown to students.

Teacher’s notes and marking notes are available to heads of mathematics departments by email from the MEI office (office@mei.org.uk).

A Code of Practice

It is recommended that all teachers follow this code for the implementation and assessment of coursework assignments.

Phase 1: Teacher preparation

This phase should be considered well in advance of the start of any work by the students.

Attempt one, or more, assignments from this bank yourself. In the case of a model, this allows you the opportunity to obtain a feel for the problem; in the case of an experiment, to assess the need for and availability of equipment. In neither case does this exclude the possibility of alternative approaches suggested subsequently by students, but it will allow you more easily to head them away from blind alleys. Read the Teacher's Notes also at this stage.

Phase 2: Student preparation

This should take place a week or two before work starts to enable students to consider the problem to be tackled.

Issue copies of the brief(s) to the students for the chosen problem(s), and copies of the marking scheme(s) to be used, as printed in the syllabus. (If subsequent work takes an unexpected direction such that one of the other marking schemes is more appropriate, this change should be explained to the students.)
The marking notes in the bank must not be shown to students.

Phase 3: Doing the Task

Discuss with students their suggested approach to the problem. Do not be afraid of a "void" at this point as facing this is a vital part of the student's learning process. Give time for students' understandable reticence to be overcome; by experience you will be able to draw out students' own ideas. You have the (confidential) Teacher's Notes to consult, but at this stage the students should be thinking for themselves.

Work begins, and continues with teacher guidance and discussion. Respond to requests for assistance, and intervene to avoid blind alleys, but try to leave the initiative with the students. It is not only acceptable but also in many cases beneficial for students to work in small groups, sharing the workload and pooling the results.

Make sure that the students understand the mark scheme. In doing this, it can be helpful for you to keep a list of important points that arise as work progresses. These can be allocated to the various domains and used later as a guide to allocating marks. This can considerably reduce the marking time.

You are particularly likely to have to explain the section headed "Variability". The ideas involved may well be new to students; they are rarely covered satisfactorily in a purely theoretical course.

Phase 4: Final write-up

The write-up of the work done should be completed individually by each student and it is **crucial** that you ensure that this is done, especially when students have worked on an experiment together and have produced the same data. The finished report should enable a fair assessment of what the student has achieved, either individually or as part of a group.

To enable you to authenticate the students' work as their own you may find it helpful to consult the JCQ document "Instructions for conducting coursework/portfolios" which contains advice on interim review of work, acknowledgment of sources and malpractice. It is available on www.jcq.org.uk

Collect work, suitably labelled; it is especially helpful to the assessment and external moderation process if the pages are numbered. The finished report should be held together in its entirety, at least by a treasury tag, a staple, or a folder. A copy of the appropriate mark scheme should be included at the front – students may fill in the details at the top of these, including their examination candidate number. This sheet has been used as a guide by the student, and will be required for assessment and external moderation.

Phase 5: Assessment of written work

When students hand in their work, the teacher's role changes from being an aid to the students' progress to being an assessor. Consequently it would be unethical to allow further amendment once the assessment has begun.

Read a representative sample of the work submitted. This process, together with any notes you have made during the course of work under the various marking scheme domains, should enable you to clarify your expectations for the marks in each domain.

Now you are ready to assess each piece of work. As you do so, use the assessment sheet to note significant features relevant to the various domains (by page number if possible) in the comments boxes provided. These comments and annotations on the script are extremely helpful to an external moderator; the more you show how you have come to make an assessment, the more likely is a moderator's agreement. If there is a disagreement, it is much easier to indicate the reason in a constructive way.

The marking schemes are designed to establish the extent to which a student's work shows mastery of the essential processes involved in the relationship between models and experience. They are not intended to distinguish between levels of excellence at the top end of the ability range. Beware of depressing marks of students who have essentially mastered the objectives of the task but who have not reached the standards of excellence established by one or more outstanding pieces of work. Note also that an experiment which eventually does not "work" does not thereby invalidate its write up for assessment, since evaluation of the reasons for failure is a valid part of the process.

Assessors need to note the following points:

- In a case where you are aware that the student has received a great deal of teacher input consider the extent to which the student now shows mastery of the area concerned.
- Cases of apparent similarity of two or more students' reports (that is, beyond stated pooling of results) should be challenged during the oral communication assessment.
- If the award of half marks in various domains results in a total which is not an integer then the mark must be rounded up or down, depending on your feeling about the value of the work.

Phase 6: Oral communication assessment

There are various ways in which oral communication can be assessed, depending on the circumstances of the teaching group as much as on the problem itself.

(a) Presentation

A student (or students if they have worked together on the same task and have collected the same data) should be invited to give a presentation to the teaching group together with the assessor. While each student should contribute enough of the presentation to fulfil its purpose, the whole presentation should take no more than 10 minutes and include brief details of the preparation and execution of the task together with results and conclusions. The use of visual aids should be encouraged. An economic use of time for a large teaching group is that they should be split into a number of smaller groups (of, say, six students) so that the audience is not too large and so that each student should not have to listen to too many presentations! The Assessor may wish to ask pertinent questions at the end to check on the authenticity of the work.

(b) Interview

The Assessor conducts a short, one to one, interview with each student with the work available to enable each student to have the opportunity to talk about their work and for the Assessor to establish authenticity. Such an interview need take no more than 5 minutes.

(c) On-going discussion

If some of the report is written up during class time then the assessor can engage each student in a discussion of the report being written. This will have the same effect as the interview, but is carried out in a much less formal way.

A comment on the mark sheet should explain how the assessment was made and any points that will indicate why the mark given was awarded. Any special circumstances that led to the mark being awarded should also be noted.

Phase 7: Moderation

It is essential that as a result of the assessment the work of all the candidates in each group is correctly ranked and the Assessor must be satisfied that this is the case before the process of internal moderation begins.

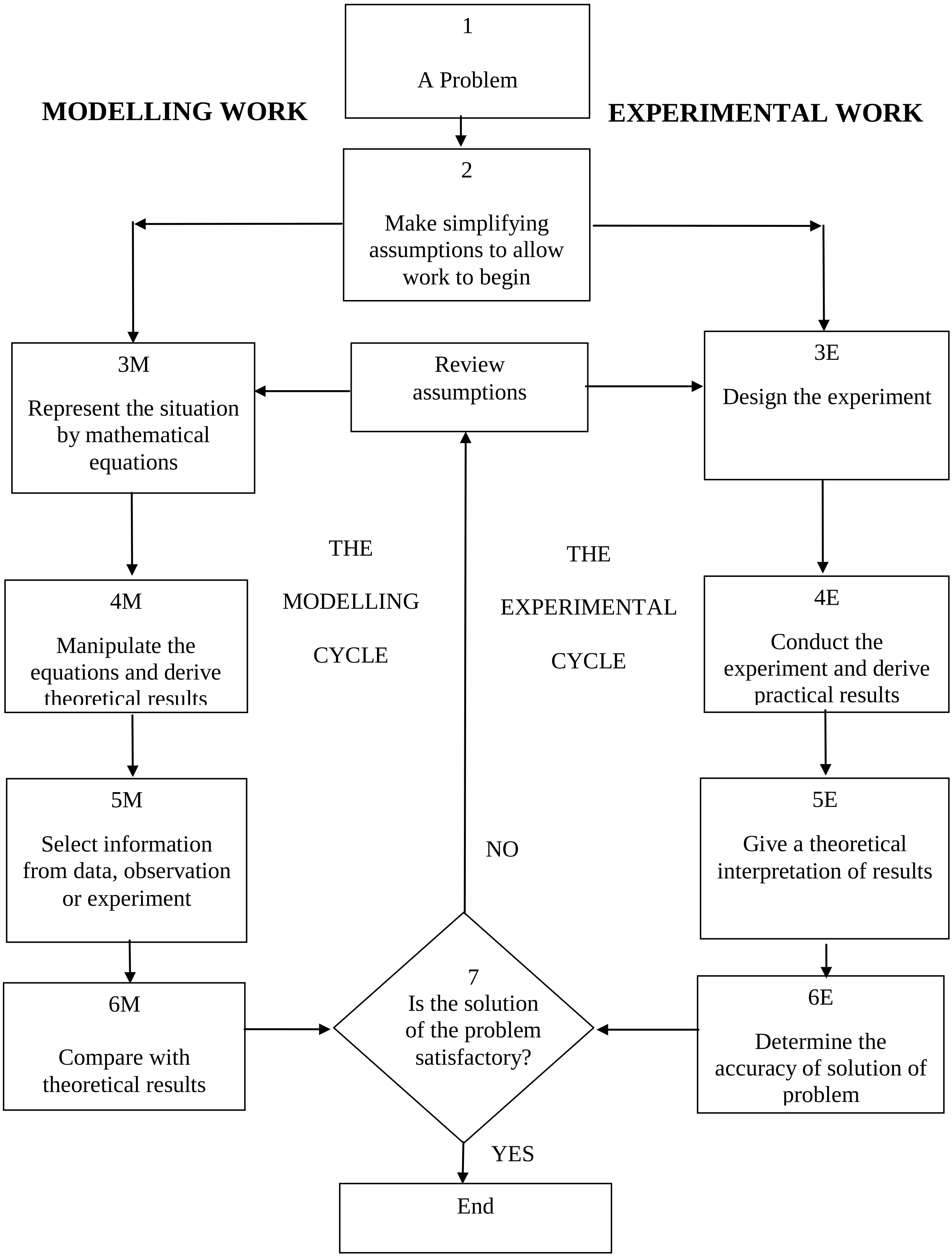
It is the role of the co-ordinator of internal moderation to see that standards are equalised between teaching groups in a Centre. Generally it will suffice to verify three scripts representing the range of marks from each marker, unless this produces evidence of disagreement.

The external moderator has two tasks:

- To check from a representative sample of a Centre’s work whether a valid order of merit has been achieved,

- To check that the Centre is marking to the same standard as other Centres.

Modelling Flowchart



Differential Equations (DE) Coursework: Assessment Sheet (A)
Work based on the Modelling Cycle

Task: Candidates will model a real-life situation of their own choice which requires the use of differential equations.

Coursework Title			
Candidate Name		**Candidate Number**	
Centre Number		**Date**	

Domain	Mark	Description	Comment	Mark
Simplifying the situation and setting up the model. (3)	1 1 1	There are clear statements of the assumptions made and an awareness is shown of the relative importance of these assumptions. The relevance of the assumptions to the initial equations of the model is explained clearly. Differential equation(s) to model the situation are established and justified.		
Manipulating the model. (5)	1 1 1 1 1	A correct methods of solution is applied to the differential equation(s). A solution to the differential equation(s) is obtained. Values for the parameters of the equation(s) are chosen and justified. There is a quantitative consideration of the effects of possible variation in the values of the parameters. A set of predictions is produced from the equation(s).		
The collection of data to verify the model. (2)	1 1	The source and means of collecting the data are clearly described and their relevance to the model is demonstrated. The data are presented in a form suitable for comparison with the predictions of the model.		
Comparison between the data collected and the predictions of the model. (2)	1 1	A clear comparison is made, using diagrams and graphs where appropriate. The effects of variation in the parameters are taken into account.		
Revision of the model. (2)	1 1	In the light of the comparison, proposals are made for amending the initial assumptions to improve the quality of the model. New equations are established as a result of the amended assumptions.		
Assessment of the improvement obtained. (2)	1 1	The new equations are manipulated to produce new predictions. A comparison is made between the data and the new predictions, and comments made on whether an improvement has been obtained.		
Oral communication. (2)	2	Presentation ☐ Interview ☐ Discussion ☐	Please tick at least one box and give a brief report.	
			Total	18

Half marks may be awarded but the overall total must be an integer.
Please report overleaf on any help that the candidate has received beyond the guidelines.

Authentication

Teachers should ensure that an OCR Declaration sheet is completed for all candidates and sent with the tasks to the moderator.

Coursework must be available for moderation by OCR.

DEcoursework/MEIDraft

Differential Equations (DE) Coursework: Assessment Sheet (B)
Work based on a combination of the Modelling and Experimental Cycles

Task: Candidates will model a real-life situation of their own choice which requires the use of differential equations.

Coursework Title			
Candidate Name		**Candidate Number**	
Centre Number		**Date**	

Domain	Mark	Description	Comment	Mark
Simplifying the situation and setting up the model. (3)	1 1 1	There are clear statements of the assumptions made and an awareness is shown of the relative importance of these assumptions. The relevance of the assumptions to the initial equations of the model is explained clearly. Differential equation(s) to model the situation are established and justified.		
Manipulating the model. (4)	1 1 1 1	A correct method of solution is applied to the differential equation(s). A solution to the differential equation(s) is obtained. Values for the parameters of the equation(s) are chosen and justified. A set of predictions is produced from the equation(s).		
Conducting the experiment. (4)	1 1 1 1	The relevance of the assumptions to the design of the experiment is explained clearly. The conduct of the experiment is described clearly, including diagrams of the apparatus used. Any steps taken to reduce experimental error are also described. Sufficient results are obtained, and these are presented clearly and concisely, in a form suitable for comparison with the predictions of the model. There is a discussion of the variability in the measurements taken.		
Comparison between the experimental results and the predictions of the model. (2)	1 1	A clear comparison is made, using diagrams and graphs where appropriate. The effects of variation in the parameters and in the measurements are taken into account.		
Revision of the process. (2)	1 1	In the light of the comparison, a decision is made, and justified, on whether further revision is needed to the modelling process, or to the conduct of the experiment, or both. If revision is needed (or if the original assumptions led to a trivial piece of modelling), detailed proposals are made for amending the assumptions and/or the conduct of the experiment. If not, there is a serious discussion of how the work could be extended to related situations.		
Assessment of the improvement obtained. (1)	1	There is an investigation into applying the effects of the proposed revision/extension. [Note: it is not necessary for the candidate to rework the model completely unless the initial differential equation(s) was (were) trivial.]		
Oral communication. (2)	2	Presentation ☐ Interview ☐ Discussion ☐	Please tick at least one box and give a brief report.	

Half marks may be awarded but the overall total must be an integer.
Please report overleaf on any help that the candidate has received beyond the guidelines.

Total	/18

Authentication
Teachers should ensure that an OCR Declaration sheet is completed for all candidates and sent with the tasks to the moderator.

Coursework must be available for moderation by OCR.

DEcoursework/MEIDraft

Student briefs for the tasks

DE 1	Cascades
DE 2	Damped Oscillations
DE 3	Parachutes
DE 4	Bungy Jumping
DE 5	Paper Cake Cups
DE 6	Aeroplane Landing
DE 7	Growth Functions
DE 8	Balloon Ascents
DE 9	Interacting Species
DE 10	Oscillating Spring

Cascades

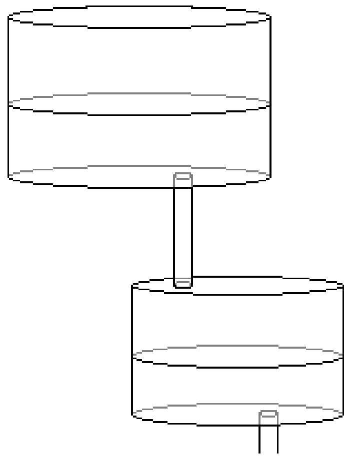
Student's Brief

The purpose of this task is to investigate the rate of flow of water through a small hole, using appropriate techniques to model this situation and to make predictions, which can then be tested experimentally.

To begin with, you should look at the flow from one container; how does the rate of flow depend on the depth of the liquid (assuming a constant cross-section)? Is the rate of flow constant? Are the rate of flow and depth directly proportional? Each suggestion could be used to predict the time to empty from various depths and these predictions tested using simple apparatus.
What simplifying assumptions are needed at this stage?

The next step is to consider the case where one container empties into another, which empties in turn. The minimum equipment needed is:

(i) Two cans (one larger than the other works well),
(ii) A stop clock,
(iii) Some means of measuring the volume or depth of liquid,
(iv) Some means of making holes in the cans.



What do expect to happen in the depth in the lower container during the emptying of the upper one? Can you predict from your model what the maximum depth attained ought to be?

The situation should then be remodelled in order to achieve a better match between the predictions and the experimental results, perhaps by changing the form of dependency of the rate of flow on the depth. The effects of different sizes of hole could also be considered.

The differential equations obtained may well turn out to have no analytical solution. In that case it would be appropriate to make use of computer software to plot the solution step by step and compare this to the experimental results.

Before attempting this work, make sure that you have a copy of the mark scheme used for its assessment. As the work progresses, bear in mind the criteria to be used in assessing it.

Damped Oscillations

Student's Brief

You are to investigate the motion of a system of damped oscillations, and attempt to model it using differential equations.

Here are some ideas:

A mass on the end of a spring oscillating in a thin liquid.

A mass on the end of a spring, with a large piece of card (or plasticard) attached to increase wind resistance.

A pendulum swinging with its bob in a thin liquid (one Centre has used a swimming pool for this!)

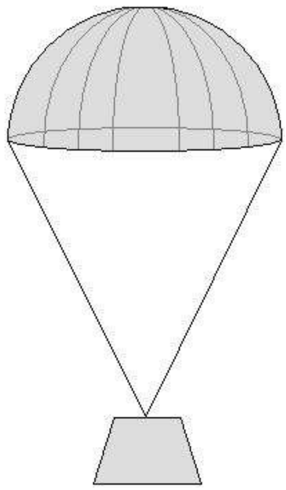
Begin by discussing with your teacher the system that you will investigate.

The first part of the work should be to make measurements of the movement of the system under consideration. Think how a model will apply to it so that the measurements taken can be compared with the later results from the model. You can then move on to propose appropriate differential equations. An important link between the two stages will be the determining of parameters to use in the model; this may require some subsidiary experiments.

Before attempting this work, make sure that you have a copy of the mark scheme used for its assessment. As the work progresses, bear in mind the criteria to be used in assessing it.

Parachutes

Student's Brief



Investigate how differential equations can be used to model the motion of a parachute. For the experiment work use a "toy" parachute; you will need a draught-free dropping place to test its motion.

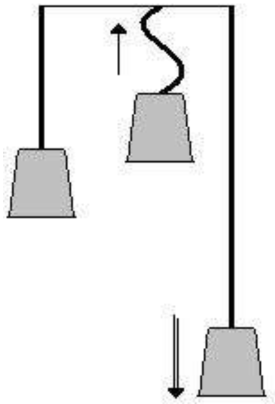
Start by making the parachute; your teacher will give you some ideas. Drop it and make measurements suitable for matching to the modelling work which follows. Note that time errors can easily occur in this work, and several drops are desirable to get a clear picture of what is going on. If you are using a stopwatch then some consideration of reaction times is also necessary.

Now model the observed motion using differential equations. You will need the values of some parameters. These may be obtainable from the experiment that you have already done, or it may need a subsidiary experiment for their determination. It is important that you relate the work you do to the specific motion being observed. When the predicted results have been obtained you need to consider how good the model is at explaining your observations, and what should be done to improve it.

Before attempting the work, make sure that you have a copy of the mark scheme to be used for its assessment. As the work proceeds, bear in mind the criteria that will be used for assessing it.

Bungy Jumping

Student's Brief



You need to know what bungy jumping is before starting the work.

You will be working with a small-scale simulation just in case of inaccuracies in your modelling! This can use something quite small like a lead toy soldier (plastic ones are not really heavy enough), something like a teddy bear, or just a weight.
You will also need to know what set-up you will be using. A toy soldier works well off a table or bench, but times involved are small and may make measuring difficult. A teddy bear may need the well of a staircase, but timing problems are eased. Or you can just use a mass of any kind and then match the "bridge" to the mass involved.

Start with a small piece of elastic and investigate its elastic properties. With these data and knowledge of the "bridge", work out the length of elastic needed to give your toy a safe jump. If the toy is heavy you may need more than one strand of elastic. Think of how you are going to launch "him".

Your modelling does need to include work with differential equations, so consider time of fall as well as length of elastic needed for a safe jump.

Once your experiments are over, investigate what improvements you can make to the model or to the experiment to achieve a much better match between experimental and predicted values.

Some other ideas

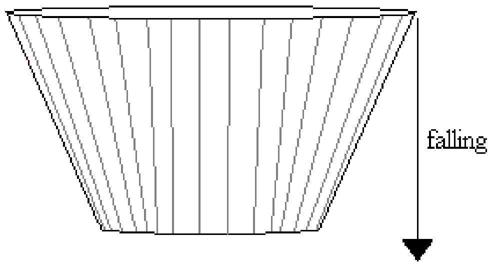
The toy will go up and down several times before coming to rest. Can you model this motion?
Investigate the maximum height of each "bounce", or estimate the time until "he" comes to rest.
You have worked with a simulation; what happens if the mass dropped is doubled? Have you any confidence you could scale up your results to work with an 80 kilogram person?
Before attempting the work, make sure that you have a copy of the mark scheme to be used for its assessment. As the work proceeds, bear in mind the criteria that will be used for assessing it.

Paper Cake Cups

Student's Brief

The purpose of this task is to investigate the effects of air resistance on falling objects, which have the same dimensions but different masses, using appropriate techniques to model this situation and to make predictions, which can then be tested experimentally.

The use of stacks of cake cups, perhaps with added weights, will allow you to vary the mass of the object, without significantly changing its profile as it falls. (See diagram)



A simple model is that air resistance is constant. Others are that it is directly proportional to the speed or to the square of the speed. You could predict the time of fall for a fixed height or you could predict the equation for s against t once terminal velocity had been reached. You would then need to test your prediction by doing an experiment.

You are advised to start with a simple model first and then adjust it to improve the match between prediction and experiment. You should also consider how the design of your experiment might be improved.

Before attempting the work, make sure that you have a copy of the mark scheme to be used for its assessment. As the work proceeds, bear in mind the criteria that will be used for assessing it.

Aeroplane Landing

Student's Brief

An aeroplane of mass 120 000 kg comes in to land at a runway. After touchdown air resistance slows it initially, and then when it is slow enough this is augmented by a constant force from the wheel brakes.

The following able gives its speed, v ms^{-1}, t seconds after touchdown.

v	96	89	82	77	72	68	64	61	58	55	50	46	41	38
t	0	1	2	3	4	5	6	7	8	9	10	11	12	13

v	34	31	27	24	21	18	16	13	10	8	5	3	0
t	14	15	16	17	18	19	20	21	22	23	24	25	26

Investigate a suitable mathematical model to explain the nature of the forces acting on the plane during these 26 seconds.

Can you recommend a minimum length of runway that this plane requires with any confidence?

Before attempting the work, make sure that you have a copy of the mark scheme used for its assessment. As the work proceeds, bear in mind the criteria to be used in assessing it.

Growth Functions

Student's Brief

A large number of events show considerable variation with time. Often it is reasonable to suppose that there is a fairly simple rule which explains the growth rate with respect to time.
An example of this is the cooling of liquid where Newton postulated that the rate of cooling was proportional to the difference between the temperature of the liquid and the background air temperature.

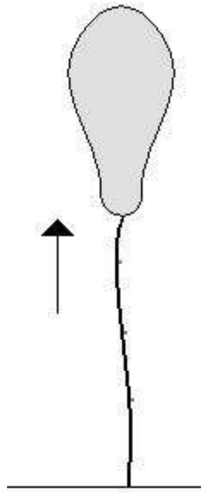
This tasks asks you to investigate a situation where there is no generally known "law of growth" to see if you can find a differential equation that could explain the process adequately. There are data available for the increase of population in laboratory conditions of a population of fruit flies, but you may well wish to collect your own data from any area that interests you.

Before spending time doing this, consult your teacher to check that the data are likely to be suitable.

In both cases, you are not being asked to find an equation that will fit the data adequately. Rather, you are to postulate a law that might govern the situation, translate it into a differential equation, and then see whether it does so. The law you try can come from common sense, or from knowledge that you have of the situation – as a result of previous experience, or of your academic studies.

Before attempting the work, make sure that you have a copy of the mark scheme used for its assessment. As the work proceeds, bear in mind the criteria to be used in assessing it.

Balloon Ascents



Student's Brief

The aim of this task is to model the motion in still air of the ascent of a balloon filled with helium (or some other light gas), which carries with it a tail of string uncoiling from the ground.

You will need a balloon, string, scissors, marker pen, metre rule, stop-watch with multiple lap facility, an accurate weighing device and some bluetack".

Consider the underlying assumptions and information that you will need to model this motion. Design the experiment so that you collect all the numerical quantities that you consider you need.

You will need a room with a high ceiling (an assembly hall or gym). Take the balloon, tie on the string, and add extra weights until it will rise but only just gets up to the ceiling. Mark the string at regular intervals to aid the timing of the motion as it ascends. Collect several sets of data. Repeat with a different weight on the balloon.

Before attempting the work, make sure that you have a copy of the mark scheme to be used for its assessment. As the work proceeds, bear in mind the criteria that will be used for assessing it.

Interacting Species

Student's Brief

Two-species interaction can take a number of forms, some of which are predator-prey, competing for food and symbiosis.

A simple model of interacting species is given by

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy \qquad a,\ b,\ c,\ d \text{ constants.}$$

For example, the following are examples of simple predator-prey models with x representing the predator population and y representing the prey population.

$$\dot{x} = x + y \qquad\qquad \text{or} \qquad \dot{x} = -x + y$$
$$\dot{y} = -x + y \qquad\qquad\qquad \dot{y} = -x + 3y$$

On the other hand the following example could represent two species competing for food.

$$\dot{x} = x - y$$
$$\dot{y} = -x + y$$

A more complicated model, attributed by A.J.Lotka, is given by

$$\dot{x} = ax + bxy$$
$$\dot{y} = cy + dxy$$

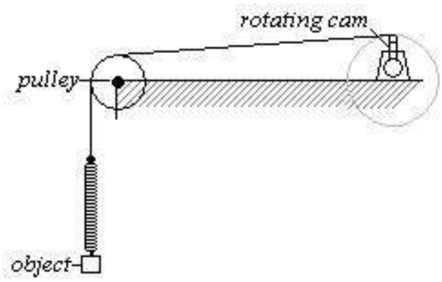
where, for example, $\begin{array}{l}\dot{x} = -2x + xy \\ \dot{y} = 3y - 2xy\end{array}$ represents predator-prey interaction.

Investigate both models. You should interpret what the differential equations are saying in terms of the population. Investigate how the populations change over time – do they always grow or die? Are there any stable populations? Is there any equilibrium point where no changes take place?

Before attempting the work, make sure that you have a copy of the mark scheme used for its assessment. As the work proceeds, bear in mind the criteria to be used in assessing it.

Oscillating Spring

Student's Brief



For this experiment you will be given access to the apparatus shown in the sketch. The electric motor is equipped with a hook which can be made to rotate in a circle. To the hook a thread is attached which passes over a pulley to a spring with a suspended mass. The motor can be made to rotate at different speeds by a continuous control mechanism.

As the motor rotates slowly, it forces the mass to oscillate. Can you explain this? What happens when the speed of the motor is slowly increased?

In a more advanced version of this experiment the spring is attached to a mass hanger and lower end of which bears a damper disc which is suspended in a container of water. Here the oscillations will be damped so the motion of the mass on the spring is more complex.

Consider the underlying assumptions and information that you will need in order to model this motion.

Before attempting the work, make sure that you have a copy of the mark scheme to be used for its assessment. As the work proceeds, bear in mind the criteria that will be used for assessing it.