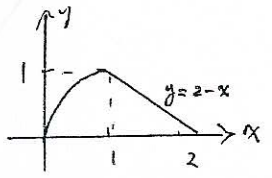
05年秋季《高数》A答案

一. (1) 原式 $= \lim\limits_{x\to 0} \dfrac{\sin x + 1 - e^x}{x \cdot \frac{3x}{2}} \overset{L'}{=} \lim\limits_{x\to 0} \dfrac{\cos x - e^x}{3x} \overset{L'}{=} \lim\limits_{x\to 0} \dfrac{-\sin x - e^x}{3} = -\dfrac{1}{3}$

(2) 两边微分. $dx + dy - dz = 2e^{x+y-z}dx + 2xe^{x+y-z}(dx+dy-dz)$ 将 $(0,1,1)$ 代入

$dx + dy - dz = 2dx \quad \therefore dz = -dx + dy$

(3)



原式 $= \int_0^1 dx \int_0^{\sqrt{1-(x-1)^2}} f\,dy + \int_1^2 dx \int_0^{2-x} f\,dy$

(4) $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \dfrac{\ln(n+1)}{(n+1)3^{n+1}} \cdot \dfrac{n3^n}{\ln n} = \dfrac{1}{3} \quad \therefore R = 3$. 收敛区间 $(-3,3)$ $\quad x=3$ 处发散 (积分判别法)

$x=-3$ 处 $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n \ln n}{n}$ 为Leibniz型. 收敛. ∴收敛域 $[-3,3)$

(5) $y = C_1 e^{2x} + C_2 e^x + 4e^{2x} - 2xe^x$ 或 $y = C_1 e^{2x} + C_2 e^x - 2xe^x$

(6) $(\sqrt{3}-i)^i = e^{i\,\mathrm{Ln}(\sqrt{3}-i)} = e^{i[\ln|\sqrt{3}-i| + i(\arg(\sqrt{3}-i) + 2k\pi)]} = e^{i[\ln 2 + i(-\frac{\pi}{6} + 2k\pi)]}$

其主值为 $e^{\frac{\pi}{6} + i\ln 2}$

二. (1) ∵ $\lim\limits_{\Delta x\to 0^+} \dfrac{|f(a+\Delta x)|}{\Delta x} \geq 0 \quad \lim\limits_{\Delta x\to 0^-} \dfrac{|f(a+\Delta x)|}{\Delta x} \leq 0 \quad \therefore \lim\limits_{\Delta x\to 0} \dfrac{|f(a+\Delta x)|}{\Delta x}\ \exists \Leftrightarrow f'(a) = 0$ 选(C).

(2) $I = \int_0^{2\pi} (e^{\sin x} - e^{-\sin x})dx \overset{令 x=t+\pi}{=} \int_{-\pi}^{\pi} (e^{\sin(t+\pi)} - e^{-\sin(t+\pi)})dt = \int_{-\pi}^{\pi} (e^{-\sin t} - e^{\sin t})dt$

∵ $f(t) = e^{-\sin t} - e^{\sin t}$ 为奇函数 $\quad \therefore I = 0$ 选(B)

(3) ∵ $0 \leq u_n < \dfrac{1}{n} \quad \therefore 0 \leq u_n^2 \leq \dfrac{1}{n^2}$ 故 $\sum\limits_{n=1}^{\infty} (-1)^n u_n^2$ 绝对收敛. 选(C)

(4) (A) 排除. (未必可微)

(B) 排除 (未必有切平面 ∴未必有法向量)

正确为(D) ∵ 曲线参数方程为 $\begin{cases} x = x \\ y = 0 \\ z = f(x,0) \end{cases} \quad \therefore \vec{a} = \{1, 0, f_x(0,0)\} = \{1, 0, 4\}$

三. (1) 令 $x=0$

得 $f(1) - 2f(1) = 0 \Rightarrow f(1) = 0$

$\lim\limits_{x\to 0} \dfrac{f(1+x) - 2f(1-x)}{x} = \lim\limits_{x\to 0} \dfrac{f(1+x) - f(1)}{x} + 2\lim\limits_{x\to 0} \dfrac{f(1-x) - f(1)}{-x} = 3f'(1) = \lim\limits_{x\to 0} \dfrac{3x + o(x)}{x} = 3$

∴ $f'(1) = 1$. ∴切线方程为 $y = x - 1$.

(2) $\int_0^2 (2+x)\sqrt{2x-x^2}\,dx = \int_0^2 (2+x)\sqrt{1-(x-1)^2}\,dx \overset{令 x-1=t}{=} \int_{-1}^{1} (3+t)\sqrt{1-t^2}\,dt = 3\int_{-1}^{1} \sqrt{1-t^2}\,dt = \dfrac{3\pi}{2}$.

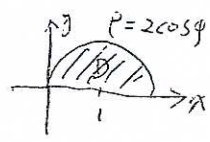
(3) 令 $u = xy$. $v = \dfrac{x^2-y^2}{2}$ $\quad g \begin{smallmatrix} u \\ v \end{smallmatrix} \begin{smallmatrix} x \\ y \end{smallmatrix}$

$\dfrac{\partial g}{\partial x} = \dfrac{\partial f}{\partial u} \cdot y + \dfrac{\partial f}{\partial v} \cdot x \qquad \dfrac{\partial^2 g}{\partial x^2} = \dfrac{\partial^2 f}{\partial u^2} \cdot y^2 + \dfrac{\partial^2 f}{\partial u \partial v} xy + \dfrac{\partial f}{\partial v} + \dfrac{\partial^2 f}{\partial v \partial u} xy + \dfrac{\partial^2 f}{\partial v^2} x^2$.

$\dfrac{\partial g}{\partial y} = \dfrac{\partial f}{\partial u} \cdot x + \dfrac{\partial f}{\partial v} \cdot (-y) \qquad \dfrac{\partial^2 g}{\partial y^2} = \dfrac{\partial^2 f}{\partial u^2} x^2 + \dfrac{\partial^2 f}{\partial u \partial v}(-xy) - \dfrac{\partial f}{\partial v} + \dfrac{\partial^2 f}{\partial v \partial u}(-xy) + \dfrac{\partial^2 f}{\partial v^2}(+y^2)$

$\therefore \dfrac{\partial^2 g}{\partial x^2} + \dfrac{\partial^2 g}{\partial y^2} = \dfrac{\partial^2 f}{\partial u^2}(x^2+y^2) + \dfrac{\partial^2 f}{\partial v^2}(x^2+y^2) = x^2+y^2$.

(4) 记 $A = \iint\limits_D f(x,y)\,dx\,dy$. 两边对D上的二重积分得. $A = \iint\limits_D \sqrt{4-x^2-y^2}\,dx\,dy + \dfrac{4A}{\pi} \iint\limits_D dx\,dy$



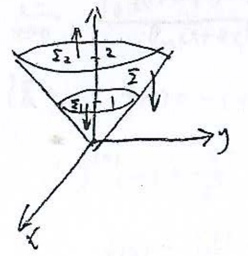
$\therefore A = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} \sqrt{4-\rho^2}\,\rho\,d\rho + 2A = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} \frac{1}{2}\sqrt{4-\rho^2}\,d\rho^2 + 2A = -\dfrac{16}{9} + \dfrac{4\pi}{3} + 2A \quad \therefore A = \dfrac{16}{9} - \dfrac{4\pi}{3}$

$\Rightarrow f(x,y) = \sqrt{4-x^2-y^2} + \dfrac{64}{9\pi} - \dfrac{16}{3}$

(5) $\oint_{|z|=3} \frac{z^3}{(z^2+1)(z+1)^2} dz = -2\pi i \,\mathrm{Res}[R(z), \infty] = 2\pi i \,\mathrm{Res}[R(\frac{1}{z})\frac{1}{z^2}, 0] = \cancel{2\pi i \left[\frac{1}{z(1+z^2)(1+z)^2}, 0\right]}$ (2°)

$= 2\pi i \,\mathrm{Res}\left[\frac{1}{z(1+z^2)(1+z)^2}, 0\right] = 2\pi i \lim_{z\to 0} \frac{z}{z(1+z^2)(1+z)^2} = 2\pi i$

四.



$原式 = \iint_{\Sigma} (x+e^{z^2})dy\wedge dz + y^2 dz\wedge dx + z^3 dx\wedge dy = \oiint_{\Sigma+\Sigma_1+\Sigma_2} - \iint_{\Sigma_1} - \iint_{\Sigma_2}$

$= \iiint_{\Omega} (1+2y+3z^2)dxdydz + \iint_{x^2+y^2\le 1} dxdy - \iint_{x^2+y^2\le 4} 8\, dx\wedge dy$

$= \iiint_{\Omega} (1+3z^2)dxdydz + \pi - 32\pi. = \int_1^2 (1+3z^2) z^2\pi\, dz - 31\pi = -\frac{151}{15}\pi.$

五. $\lim_{n\to\infty}\left|\frac{u_{n+1}(x)}{u_n(x)}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+2}}{(2n+3)4^{n+1}}\cdot\frac{(2n+1)4^n}{(-1)^n x^{2n}}\right| = \frac{x^2}{4} \gtrless 1. \Rightarrow |x|<2. \quad \therefore R=2$

又 $x=2$ 时 级数收敛. $x=-2$ 时 级数收敛. ∴收敛域为 $[-2, 2]$

$\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}\cdot\left(\frac{x}{2}\right)^{2n} = \frac{2}{x}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{x}{2}\right)^{2n+1}$

记 $s(t) = \sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}t^{2n+1}$ $\quad s'(t) = \sum_{n=1}^{\infty}(-1)^n\cdot t^{2n} = \frac{-t^2}{1+t^2}$ $\quad \therefore s(t) = \int_0^t \frac{-t^2}{1+t^2}dt = -t + \arctan t$

$\therefore \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)4^n} = \frac{2}{x}\left[-\frac{x}{2} + \arctan\frac{x}{2}\right] = -1 + \frac{2}{x}\arctan\frac{x}{2}$

六. $\begin{cases} V_y = \pi\int_0^y \varphi^2(y)dy & (1) \\ \frac{dV_y}{dt} = 2 & (2) \\ \frac{d(\varphi^2(y)\pi)}{dt} = 2\pi\varphi(y)\varphi'(y)\frac{dy}{dt} = \pi & (3) \end{cases}$

由(1) $\frac{dV_y}{dy} = \pi\varphi^2(y)$ $\quad \frac{(2)}{(3)}$ $\frac{dV_y}{dy} = 4\varphi(y)\varphi'(y)$

$\therefore \pi\varphi^2(y) = 4\varphi(y)\varphi'(y) \Rightarrow \int\frac{d\varphi(y)}{\varphi(y)} = \int\frac{\pi}{4}dy$

$\Rightarrow \ln\varphi(y) = \frac{\pi}{4}y + \tilde{c} \quad \Rightarrow \varphi(y) = ce^{\frac{\pi}{4}y} \quad \because \varphi(0)=2 \quad \therefore c=2$

故 $\varphi(y) = 2e^{\frac{\pi}{4}y}$ 即 $x = 2e^{\frac{\pi}{4}y}$ 或 $y = \frac{4}{\pi}\ln\frac{x}{2}$.

七. $F_1(x) = f(\xi_1)(x-a) \quad a\le\xi_1\le x$ $\qquad F_2(x) = \int_a^x f(\xi_1)(x-a)dx = f(\xi_2)\int_a^x(\cancel{x}-a)dx = f(\xi_2)\frac{(x-a)^2}{2} \quad a\le\xi_2\le x$

(注明. ∵ 若 $f_{min}=m$. $f_{max}=M$ 则 $m(x-a)\le f(\xi_1)(x-a)\le M(x-a) \Rightarrow m\int_a^x(x-a)dx \le \int_a^x f(\xi_1)(x-a)dx \le M\int_a^x(x-a)dx$

$\Rightarrow m \le \frac{\int_a^x f(\xi_1)(x-a)dx}{\int_a^x (x-a)dx} \le M$ 依介值Th $\exists\, \xi_2\in[a,x] \rightarrow f(\xi_2) = \frac{\int_a^x f(\xi_1)(x-a)dx}{\int_a^x(x-a)dx}$)

$\Rightarrow F_3(x) = f(\xi_3)\frac{(x-a)^3}{3!} \quad \xi_3\in[a,x]. \Rightarrow \cdots \quad F_n(x) = f(\xi_n)\frac{(x-a)^n}{n!}$ $\quad \because \sum_{n=0}^{\infty}\frac{(x-a)^n}{n!} = e^{x-a}$ 收敛

$\therefore \lim_{n\to\infty}\frac{(x-a)^n}{n!} = 0$ 即 $\lim_{n\to\infty}F_n(x) = \lim_{n\to\infty} f(\xi_n)\frac{(x-a)^n}{n!} = 0$

06年转系《高等数学》答案

一、

1. 原式 $=e^{\lim\limits_{x\to0}\frac{\ln(1+\frac{x^2}{3})}{\cos x-1}}=e^{\lim\limits_{x\to0}\frac{\frac{x^2}{3}}{-\frac{x^2}{2}}}=e^{-\frac{2}{3}}$

2. $\lim\limits_{x\to0}\dfrac{\int_0^x[(1+\alpha t^2)^{1/3}-1]dt}{\ln(1+2x^3)}=\lim\limits_{x\to0}\dfrac{\int_0^x[(1+\alpha t^2)^{1/3}-1]dt}{2x^3}=\lim\limits_{x\to0}\dfrac{(1+\alpha x^2)^{\frac{1}{3}}-1}{6x^2}=\lim\limits_{x\to0}\dfrac{\frac{\alpha}{3}x^2}{6x^2}=\dfrac{\alpha}{18}\stackrel{令}{=}1$ $\therefore\alpha=18$

3. 法一. $f(x)=-1+2(1-x)^{-1}$ $f'(x)=2(1-x)^{-2}$. $f''(x)=2\cdot2(1-x)^{-3}$. $f'''(x)=2\cdot3!(1-x)^{-4}$. $\cdots$ $f^{(n)}(x)=2\cdot n!(1-x)^{-(n+1)}$

$\therefore f^{(n)}(-1)=\dfrac{n!}{2^n}$

法二. $f(x)=\dfrac{x+1}{2}\dfrac{1}{1-\frac{x+1}{2}}=\dfrac{x+1}{2}+\dfrac{(x+1)^2}{2^2}+\dfrac{(x+1)^3}{2^3}+\cdots+\dfrac{(x+1)^n}{2^n}+\cdots$ $a_n=\dfrac{1}{2^n}=\dfrac{f^{(n)}(-1)}{n!}$ $\therefore f^{(n)}(-1)=\dfrac{n!}{2^n}$

4. 令 $F(x,y)=\int_0^x\sin t^2dt+\int_0^y e^{-t^2}dt$ $y'(x)=-\dfrac{\sin x^2}{e^{-y^2}}\stackrel{令}{=}0$ 在 $(0,\sqrt{2\pi})$ 内有唯一驻点 $x=\sqrt{\pi}$.

当 $x\in(0,\sqrt{\pi})$. $y'<0$. $\therefore y(\sqrt{\pi})=\min$.

$x\in(\sqrt{\pi},\sqrt{2\pi})$. $y'>0$

5. 方程的特征根 $r_{1,2}=-1\pm2i$. 特征方程 $(r+1-2i)(r+1+2i)=(r+1)^2-(2i)^2=r^2+2r+1+4=r^2+2r+5=0$

$\therefore$ 方程为: $y''+2y'+5y=0$

6. 令 $F(x,y,z)=\varphi(x-2z,z-y^2)-z$. $\dfrac{\partial z}{\partial x}=-\dfrac{\varphi_1}{-2\varphi_1+\varphi_2-1}$ $\dfrac{\partial z}{\partial y}=-\dfrac{-2y\varphi_2}{-2\varphi_1+\varphi_2-1}$ $\therefore\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}=\dfrac{-\varphi_1+2y\varphi_2}{-2\varphi_1+\varphi_2-1}$

二、

1. $\because\lim\limits_{x\to0^+}\dfrac{f(x)-f(0)}{x}=f'(0)>0$ $\therefore\exists\delta>0$ 当 $x\in(0,\delta)$. 有 $\dfrac{f(x)-f(0)}{x}>0\Rightarrow f(x)>f(0)$ (保号性). 选(C)

(但不选(A)的理由: 反例 $f(x)=\begin{cases}\sin x & x\text{为无理数}\\ x & x\text{为有理数}\end{cases}$ $f'(0)=\lim\limits_{x\to0}\dfrac{f(x)-0}{x}=\begin{cases}\lim\limits_{x\to0}\frac{\sin x}{x}=1 & x\text{为无理数}\\ \lim\limits_{x\to0}\frac{x}{x}=1 & x\text{为有理数}\end{cases}$

不满足 $\exists\delta>0$, $\Rightarrow x_1<x_2\in(0,\delta)$ $f(x_2)>f(x_1)$)

2. 二阶偏导数 $\exists$. 混合偏导数未必相等. (A)错. 显然(C)错. 否则如(C)对, 则(B)也对. ($\because$单选 $\therefore$不可能).

大致猜想选(B) 证明 $\lim\limits_{(\Delta x,\Delta y)\to(0,0)}[f(x+\Delta x,y+\Delta y)-f(x,y)]=\lim\limits_{(\Delta x,\Delta y)\to(0,0)}[f(x+\Delta x,y+\Delta y)-f(x,y+\Delta y)]$

$+\lim\limits_{(\Delta x,\Delta y)\to(0,0)}[f(x,y+\Delta y)-f(x,y)]\stackrel{中值Th}{=}\lim\limits_{(\Delta x,\Delta y)\to(0,0)}f_x(x+\theta_1\Delta x,y+\Delta y)\Delta x+\lim\limits_{(\Delta x,\Delta y)\to(0,0)}f_y(x,y+\theta_2\Delta y)\Delta y$

$=0+0=0$ $\therefore f$ 在D内连续

3. $(\vec{A})_{\vec{B}}=|\vec{A}|\cos(\vec{A},\vec{B})$

$=\dfrac{|\vec{A}||\vec{B}|\cos(\vec{A},\vec{B})}{|\vec{B}|}=\dfrac{\vec{A}\cdot\vec{B}}{\sqrt{\vec{B}\cdot\vec{B}}}=\dfrac{(2\vec{a}+3\vec{b})\cdot(\vec{a}-\vec{b})}{\sqrt{(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})}}=\dfrac{2|\vec{a}|^2-3|\vec{b}|^2+|\vec{a}||\vec{b}|\cos\frac{\pi}{3}}{\sqrt{|\vec{a}|^2+|\vec{b}|^2-2|\vec{a}||\vec{b}|\cos\frac{\pi}{3}}}=-\dfrac{9}{\sqrt{3}}=-3\sqrt{3}$ 选(A)

3* C内两个奇点 $z_1=1$ 和 $z_2=i$

$\therefore\oint_C\dfrac{dz}{(z-1)^2(z^2+1)}=2\pi i[\mathrm{Res}(R(z),1)+\mathrm{Res}(R(z),i)]=2\pi i\left[\lim\limits_{z\to1}\left(\dfrac{1}{z^2+1}\right)'+\lim\limits_{z\to i}\dfrac{1}{(z-1)^2(z+i)}\right]$

$=2\pi i\left[-\dfrac{2}{4}+\dfrac{1}{4}\right]=-\dfrac{\pi i}{2}$ 选(A)

4. (A)错. 反例 $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^2}$ (B)错. 反例 $\sum\limits_{n=1}^{\infty}\dfrac{1}{n\ln n}$ (积分判别法知发散) (D)错. 反例. $\sum\limits_{n=1}^{\infty}\dfrac{1}{\sqrt{n}}$

(C)对. $\because\lim\limits_{n\to\infty}nu_n=\lim\limits_{n\to\infty}\dfrac{u_n}{\frac{1}{n}}=l$. $\therefore\sum u_n$ 与 $\sum\frac{1}{n}$ 同为发散.

三、

1. $\int\dfrac{x}{\cos^2x\tan^3x}dx=\int\dfrac{x}{\tan^3x}d\tan x=-\dfrac{1}{2}\int x\,d\cot^2x=-\dfrac{1}{2}x\cot^2x+\dfrac{1}{2}\int\dfrac{\cos^2x}{\sin^2x}dx$

$=-\dfrac{1}{2}x\cot^2x+\dfrac{1}{2}\int\dfrac{1-\sin^2x}{\sin^2x}dx=-\dfrac{1}{2}x\cot^2x-\dfrac{1}{2}\cot x-\dfrac{1}{2}x+C$

2. $\dfrac{\partial z}{\partial x}=3x^2f+x^3[yf_1-\dfrac{y}{x^2}f_2]$

$\dfrac{\partial^2z}{\partial x\partial y}=3x^2[f_1\cdot x+f_2\cdot\dfrac{1}{x}]+x^3[f_1+y\cdot(f_{11}\cdot x+f_{12}\cdot\dfrac{1}{x})-\dfrac{1}{x^2}f_2-\dfrac{y}{x^2}(f_{21}\cdot x+f_{22}\cdot\dfrac{1}{x})]$

3. $f(x)\stackrel{令tx=u}{=}2\int_0^x f(u)du+e^{2x}\cos x$ 求导 $f'(x)=2f(x)+2e^{2x}\cos x-e^{2x}\sin x$

$f(x)=e^{\int2dx}[\int e^{2x}(2\cos x-\sin x)e^{-\int2dx}dx+C]=e^{2x}[2\sin x+\cos x+C]$ $\because f(0)=1$ $\therefore C=0$

故 $f(x)=e^{2x}[2\sin x+\cos x]$

(Ⅱ)

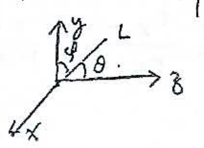
4. 设切点为(x_0, y_0, z_0) $\vec{n}=\{6x_0, 4y_0, -1\}$ 由条件 $\frac{6x_0}{3}=\frac{4y_0}{2}=\frac{-1}{1}$

$\therefore x_0=-\frac{1}{2}, y_0=-\frac{1}{2}$ 代入$z=3x^2+2y^2$ 得 $z_0=-\frac{5}{4}$ $\therefore$ 切平面方程为 $3(x+\frac{1}{2})+2(y+\frac{1}{2})+(z+\frac{5}{4})=0$

或 $12x+8y+4z+15=0$

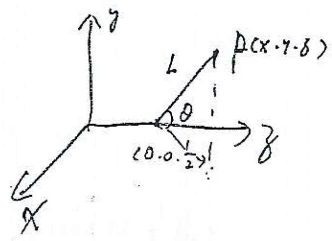
5. 直线方向向量 $\vec{a}=\begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 1 & -3 & -2 \end{vmatrix}=\{8, 4, -2\}$. 可取 $\vec{a}=\{4, 2, -1\}$. $\because$ 点 $(0, 0, \frac{1}{2})$ 满足直线 L 方程

$\therefore$ 直线方程为 $\cancel{\frac{x}{4}=\frac{y}{2}=\frac{z}{-1}}$ 直线 L: $\frac{x}{4}=\frac{y}{2}=\frac{z-\frac{1}{2}}{-1}$



设直线 L 与 xoz 面的夹角 $\sin\theta=\frac{|\vec{a}\cdot\{0,1,0\}|}{|a|\cdot|\{0,1,0\}|}=\cos\varphi$

$\therefore \sin\theta=\frac{|\{4,2,-1\}\cdot\{0,1,0\}|}{\sqrt{21}\cdot\sqrt{1}}=\frac{2}{\sqrt{21}}$



设 $p(x,y,z)$ 为旋转曲面上动点，则 $p(x,y,z)$ 满足：

$\sin\theta=\frac{|y|}{\sqrt{x^2+(z-\frac{1}{2})^2}}=\frac{2}{\sqrt{21}} \Rightarrow y^2=\frac{4}{21}[x^2+(z-\frac{1}{2})^2]$

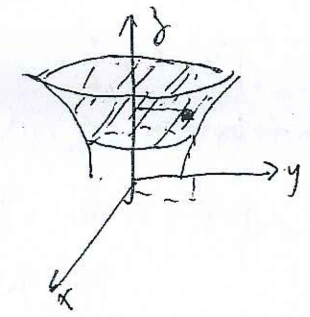
此即为所求的旋转曲面方程。

5*. $\frac{\partial v}{\partial x}=4x+1=-\frac{\partial u}{\partial y}$ (1) 由(2) $u=\int -4y\,dx=-4xy+\varphi(y)$ 求导 $\frac{\partial u}{\partial y}=-4x+\varphi'(y)$ 与(1) $\frac{\partial u}{\partial y}=-4x-1$

$\frac{\partial v}{\partial y}=-4y=\frac{\partial u}{\partial x}$ (2) 比较得 $\varphi'(y)=-1$. $\therefore \varphi(y)=-y+c$

$\Rightarrow u=-4xy-y+c$. $\therefore f(z)=-4xy-y+c+i(2x^2-2y^2+x)=c+i(2z^2+z)$

四.



$dJ=(x^2+y^2)\rho\,dv$ $\therefore J=\iiint_\Omega \rho(x^2+y^2)\,dv \overset{切片}{=} \rho\int_{\ln 4}^{\ln 9}dz\iint_{x^2+y^2\le e^z}(x^2+y^2)\,dxdy$

$=\rho\int_{\ln 4}^{\ln 9}dz\int_0^{2\pi}d\varphi\int_0^{\sqrt{e^z}}\rho^2\cdot\rho\,d\rho=2\pi\rho\int_{\ln 4}^{\ln 9}\frac{\rho^4}{4}\Big|_0^{\sqrt{e^z}}dz=\frac{2\pi\rho}{4}\int_{\ln 4}^{\ln 9}e^{2z}dz$

$=\frac{\pi\rho}{4}e^{2z}\Big|_{\ln 4}^{\ln 9}=\frac{65}{4}\pi\rho$

五. 由$f''(x)>0 \Rightarrow f'(x)\uparrow$.

$(\frac{f(x)}{x})'=\frac{xf'(x)-f(x)}{x^2}=\frac{xf'(x)-[f(x)-f(0)]}{x^2}$ $(\because f(0)=0)$

$\overset{中值Th}{=}\frac{xf'(x)-xf'(\xi)}{x^2}$ $0<\xi<x$

$=\frac{f'(x)-f'(\xi)}{x}>0$ $\therefore \frac{f(x)}{x}\uparrow$.

六. 设C_1, C_2为围绕原点的任意两条光滑闭曲线，且C_2整个位于C_1内，则在C_1, C_2所围的复连通域内，

$\oint_{C_1^+ + C_2^-}\frac{-y\,dx+x\,dy}{\varphi(x)+y^2}\overset{Green}{=}\iint_D(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dxdy=A-A=0$ $\therefore \frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$ 即 $\frac{\varphi(x)+y^2-x\varphi'(x)}{(\varphi(x)+y^2)^2}=\frac{-(\varphi(x)+y^2)+y\cdot 2y}{(\varphi(x)+y^2)^2}$

$\Rightarrow x\varphi'(x)-2\varphi(x)=-2y^2+2y^2$ 即 $\varphi'(x)=\frac{2}{x}\varphi(x)$ $\Rightarrow \int\frac{d\varphi}{\varphi}=\int\frac{2}{x}dx$ $\therefore \varphi(x)=cx^2$ 由$\varphi(1)=1 \Rightarrow c=1$ $\therefore \varphi(x)=x^2$

取C为$x^2+y^2=1$. $\oint_C\frac{-y\,dx+x\,dy}{x^2+y^2}=\oint_C -y\,dx+x\,dy \overset{Green}{=}\iint_{x^2+y^2\le 1}2\,dxdy=2\pi=A$ $\therefore A=2\pi$.

七. $a_1=\int_0^\pi x\sin x\,dx=-\int_0^\pi x\,d\cos x=-x\cos x\Big|_0^\pi+\int_0^\pi\cos x\,dx=\pi$

$a_n=\sum_{k=1}^{n}\int_{(k-1)\pi}^{k\pi}x|\sin x|\,dx$ 其中 $\int_{(k-1)\pi}^{k\pi}x|\sin x|\,dx \overset{令x=t+(k-1)\pi}{=} \int_0^\pi(t+(k-1)\pi)|\sin t|\,dt$

$=a_1+(k-1)\pi\int_0^\pi\sin x\,dx=(2k-1)\pi$ $\therefore a_n=\pi+3\pi+5\pi+\cdots+(2n-1)\pi=n^2\pi$.

$\therefore \frac{a_1}{1^2}+\frac{a_2}{2^2}+\cdots+\frac{a_n}{n^2}=\pi+\pi+\cdots+\pi=n\pi$. 故 $\lim_{n\to\infty}(\frac{a_1}{1^2}+\frac{a_2}{2^2}+\cdots+\frac{a_n}{n^2})/n\pi=\lim_{n\to\infty}\frac{n\pi}{n\pi}=1$

08期末《高数》A答案

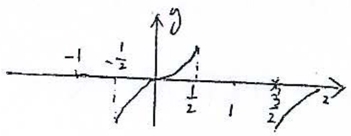
一 (1) $f(x)=\begin{cases} x^2 & x>1 \\ \frac{1+a}{2} & x=1 \\ ax & x<1 \end{cases}$ $\quad \therefore a=1$

(2) $f'(x)=x\ln(1+x^2)$ $\quad \therefore f(x)=\int x\ln(1+x^2)dx=\frac{1}{2}\int\ln(1+x^2)dx^2=\frac{x^2\ln(1+x^2)}{2}-\frac{1}{2}\int\frac{2x^3}{1+x^2}dx$

$=\frac{x^2\ln(1+x^2)}{2}-\int(x-\frac{x}{1+x^2})dx=\frac{x^2\ln(1+x^2)}{2}-\frac{x^2}{2}+\frac{1}{2}\ln(1+x^2)+C$ $\quad \because f(0)=-\frac{1}{2} \Rightarrow C=-\frac{1}{2}$

$\therefore f(x)=\frac{1}{2}(x^2+1)\ln(1+x^2)-\frac{1}{2}(x^2+1)$

(3) $\lim\limits_{x\to0}\frac{\int_0^{x^2}2\ln(1+t)dt}{x^\alpha(e^x-1)}=\lim\limits_{x\to0}\frac{\int_0^{x^2}2\ln(1+t)dt}{x^{\alpha+1}}\overset{L'}{=}\lim\limits_{x\to0}\frac{2\ln(1+x^2)\cdot2x}{(\alpha+1)x^\alpha}=\frac{4}{\alpha+1}\lim\limits_{x\to0}\frac{x^3}{x^\alpha}\overset{令}{=}1$ $\quad \therefore \alpha=3$

(4)  $\quad S(\frac{3}{2})=\frac{0-\frac{1}{4}}{2}=-\frac{1}{8}$

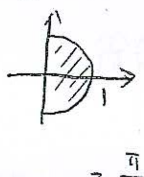
(5) $\frac{\partial z}{\partial x}=-\frac{3\varphi_2-2\varphi_3}{2\varphi_1-4\varphi_2}$ $\quad \frac{\partial z}{\partial y}=-\frac{-3\varphi_1+4\varphi_3}{2\varphi_1-4\varphi_2}$ $\quad \therefore 4\frac{\partial z}{\partial x}+2\frac{\partial z}{\partial y}=\frac{6(\varphi_1-2\varphi_2)}{2(\varphi_1-2\varphi_2)}=3$

(6) $e^z=(1+i)^i \Rightarrow z=i\operatorname{Ln}(1+i)=i[\ln|1+i|+i(\arg(1+i)+2k\pi)]$ $\quad \therefore \operatorname{Im}(z)=\ln|1+i|=\frac{1}{2}\ln2$

二 (1) 原式 $=e^{\lim\limits_{x\to0}\frac{\ln x-\ln\sin x}{x^2}}=e^{\lim\limits_{x\to0}\frac{\frac{1}{x}-\cot x}{2x}}=e^{\lim\limits_{x\to0}\frac{\sin x-x\cos x}{2x^3}}=e^{\lim\limits_{x\to0}\frac{\cos x-\cos x+x\sin x}{6x^2}}=e^{\frac{1}{6}}$

(2) $\int_0^{\frac{\pi}{4}}\frac{x}{1+\cos2x}dx=\frac{1}{2}\int_0^{\frac{\pi}{4}}x\,d\tan x=\frac{1}{2}x\tan x\Big|_0^{\frac{\pi}{4}}-\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos x}dx=\frac{\pi}{8}+\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{d\cos x}{\cos x}=\frac{\pi}{8}+\frac{1}{2}\ln|\cos x|\Big|_0^{\frac{\pi}{4}}$

$=\frac{\pi}{8}-\frac{1}{4}\ln2$

(3)  由对称性. 原式 $=\iint\limits_D\frac{dxdy}{1+(x^2+y^2)^2}\overset{“极”}{=}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\varphi\int_0^1\frac{\rho}{1+\rho^4}d\rho=\pi\cdot\frac{1}{2}\int_0^1\frac{d\rho^2}{1+(\rho^2)^2}$

$=\frac{\pi}{2}\arctan\rho^2\Big|_0^1=\frac{\pi^2}{8}$

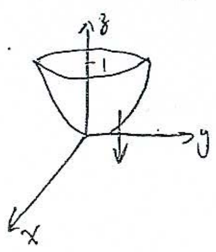
(4) 设$f(z)$定义在区域D上. $z_0\in D$. 若$f(z)$在z_0的某个含于D内的邻域内可导. 称$f(z)$在点z_0处解析. 由C-R条件: $u_x=6x^2=v_y=9y^2$, $u_y=0=-v_x=0$ $\quad \therefore f(z)$仅在$x=\pm\sqrt{\frac{3}{2}}y$上可导. 处处不解析

三. 原级数 $=\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{2n-1}(\sqrt{\frac{3}{4}})^{2n}$ $\quad$ 令 $S(x)=\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{2n-1}x^{2n}=x\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{2n-1}x^{2n-1}=x\tilde{S}(x)$ $\quad x\in[-1,1]$

则原级数 $=S(\sqrt{\frac{3}{4}})$

$\tilde{S}'(x)=\sum\limits_{n=1}^{\infty}(-1)^n x^{2n-2}=-\sum\limits_{n=1}^{\infty}(-x^2)^{n-1}=-\frac{1}{1+x^2}$

$\therefore \tilde{S}(x)=\int_0^x\frac{-1}{1+x^2}dx=-\arctan x \Rightarrow S(x)=x\tilde{S}(x)=-x\arctan x$ $\quad \therefore \sum\limits_{n=1}^{\infty}\frac{(-1)^n}{2n-1}(\frac{3}{4})^n=-\sqrt{\frac{3}{4}}\arctan\sqrt{\frac{3}{4}}$.

四.  补面 $\Sigma_补 \begin{cases} x^2+y^2\le1 \\ z=1 \end{cases}$ (上侧) $\quad I=\oiint\limits_{\Sigma+\Sigma_补}-\iint\limits_{\Sigma_补}\overset{Gauss}{=}\iiint\limits_{\Omega}(y+2x^2z)dV-\iint\limits_{x^2+y^2\le1}x^2dxdy$

$\overset{对称}{=}\iiint\limits_{\Omega}2x^2z\,dV-\frac{1}{2}\iint\limits_{x^2+y^2\le1}(x^2+y^2)dxdy=\int_0^1z\,dz\iint\limits_{x^2+y^2\le z}(x^2+y^2)dxdy-\frac{1}{2}\int_0^{2\pi}d\varphi\int_0^1\rho^2\cdot\rho\,d\rho$

$=\int_0^1z\,dz\int_0^{2\pi}d\varphi\int_0^{\sqrt{z}}\rho^2\cdot\rho\,d\rho-\frac{\pi}{4}=\int_0^1z\cdot2\pi\cdot\frac{z^2}{4}dz-\frac{\pi}{4}=\frac{\pi}{8}-\frac{\pi}{4}=-\frac{\pi}{8}$

五. 法一. $I\overset{Stokes公式}{=}2\iint\limits_{\Sigma}dy\wedge dz+dz\wedge dx+dx\wedge dy$ $\quad \Sigma$: 以L为边界的平面 $x-y+z=2$ 区域 (上侧)

转换第一型 $-\frac{2}{\sqrt3}\iint\limits_{\Sigma}dA=-\frac{2}{\sqrt3}\iint\limits_{x^2+y^2\le1}\sqrt{1+1+1}\,dxdy=-2\pi$.

法二. $L:\begin{cases} x=\cos\varphi \\ y=\sin\varphi \\ z=2-\sin\varphi+\cos\varphi \end{cases}$ $\quad 0\le\varphi\le2\pi$

原式 $= \int_0^{2\pi} \{(\sin\varphi - 2 + \sin\varphi - \cos\varphi)(-\sin\varphi) + (2 - \sin\varphi + \cos\varphi - \cos\varphi)\cos\varphi + (\cos\varphi - \sin\varphi)(-\cos\varphi - \sin\varphi)\} d\varphi$

$= \int_0^{2\pi} (-3\sin\varphi + \sin\varphi\cos\varphi + 2\cos\varphi - 1)\, d\varphi = -2\pi$.

六. (1) $\dfrac{d\left(\frac{1}{\frac{dy}{dx}}\right)}{dy} = -(y + \sin x)\left(\dfrac{1}{\frac{dy}{dx}}\right)^3 \Rightarrow \dfrac{-\frac{d\left(\frac{dy}{dx}\right)}{dy}}{\left(\frac{dy}{dx}\right)^2} = -(y+\sin x)\dfrac{1}{\left(\frac{dy}{dx}\right)^3} = \dfrac{\cancel{dy}}{dx} \cdot \dfrac{d\left(\frac{dy}{dx}\right)}{\cancel{dy}} = y + \sin x$

$\Rightarrow \dfrac{d^2y}{dx^2} = y + \sin x$ 即 $y'' - y = \sin x$

(2) $r^2 - 1 = 0$ 通解 $y = C_1 e^{x} + C_2 e^{-x} - \frac{1}{2}\sin x$ 由 $y(0) = 0$. $y'(0) = \frac{3}{2} \Rightarrow C_1 = 1$. $C_2 = -1$

$\therefore$ 满足初始条件的解为 $y = e^{x} - e^{-x} - \frac{1}{2}\sin x$

七. $\because f'(x_0) = 0 \quad \therefore x_0 f''(x_0) = 1 - e^{-x_0} \Rightarrow f''(x_0) = \dfrac{1 - e^{-x_0}}{x_0}$

当 $x_0 > 0 \quad \because e^{-x_0} < 1 \quad \therefore f''(x_0) > 0$

当 $x_0 < 0 \quad \because e^{-x_0} > 1 \quad \therefore f''(x_0) > 0$

$\therefore f(x)$ 在 $x = x_0$ 处取得极小值.

八. $\because \sum_{n=1}^{\infty} a_n$ 条件收敛 $\therefore \forall N$, $n > N$ 时. a_n 有正有负. 故若 $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \exists r$. 则 $r \le 0$

~~否则. 若 $r>0$. 由保号性 $n>N$ 时 $\frac{a_{n+1}}{a_n} > 0 \Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r > 0$ ... $r<1$ $\therefore \exists$~~

考虑 $\sum_{n=1}^{\infty} a_n x^n$ $\because$ 在 $x=1$ 处条件收敛. $\therefore$ 其收敛半径 $R = 1$ $\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |r| = 1 \Rightarrow r = -1$.

符合上述条件的数项级数为 $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

09年转系试卷一 高数A 答案

一．填空

1. $F'(x)=2-\frac{1}{\sqrt{x}}<0 \Rightarrow x<\frac{1}{4}$，又 $x>0$，∴答 $(0,\frac{1}{4})$

2. $\int_{\frac{\pi}{2}}^{\pi} x f'(x)dx=\int_{\frac{\pi}{2}}^{\pi} x\, df(x) = x f(x)\Big|_{\frac{\pi}{2}}^{\pi} - \int_{\frac{\pi}{2}}^{\pi} f(x)dx = x\left(\frac{\sin x}{x}\right)'\Big|_{\frac{\pi}{2}}^{\pi} - \frac{\sin x}{x}\Big|_{\frac{\pi}{2}}^{\pi}$

$= \frac{x\cos x-\sin x}{x}\Big|_{\frac{\pi}{2}}^{\pi}+\frac{2}{\pi}=\frac{4}{\pi}-1$

3. $\frac{\partial z}{\partial x}=-\frac{2xF_1}{-F_1-F_2}$

4. 原式 $=\frac{1}{2}\iint_{x^2+y^2\le 1}(x^2+y^2)d\sigma=\frac{1}{2}\int_0^{2\pi}d\varphi\int_0^1\rho^3 d\rho=\frac{\pi}{4}$

5. $f(z)=(1+z+z^2+z^3+\cdots)(1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\cdots)$

∴ $C_{-1}=1+\frac{1}{2!}+\frac{1}{3!}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n!}$　即 $Res[f(z),0]=\sum_{n=1}^{\infty}\frac{1}{n!}\overset{或}{=}e-1$.

二．选择

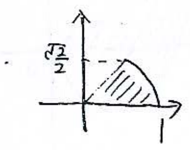
6. 由无穷小×有界量=无穷小，一眼看出 (A)×，(B)×。

∴考虑 $f'(0)=\lim_{x\to 0}\frac{|x|\sin\frac{1}{x^2}}{x}$ 极限不存在，∴选 (C)

7. 椭球曲面方程：$3x^2+2y^2+2z^2=12$．点 $(0,\sqrt{3},\sqrt{2})$ 处的指向外侧的法矢．

$\vec{n}=\{6x,4y,4z\}\big|_{(0,\sqrt{3},\sqrt{2})}=\{0,4\sqrt{3},4\sqrt{2}\}$．∴ $\vec{n}_0=\{0,\frac{4\sqrt{3}}{\sqrt{80}},\frac{4\sqrt{2}}{\sqrt{80}}\}=\{0,\frac{\sqrt{15}}{5},\frac{\sqrt{10}}{5}\}$

选 (D)

8. 

原式 $=\int_0^{\frac{\sqrt{2}}{2}}dy\int_0^{\sqrt{1-y^2}}f(x,y)dx$．选 (A)

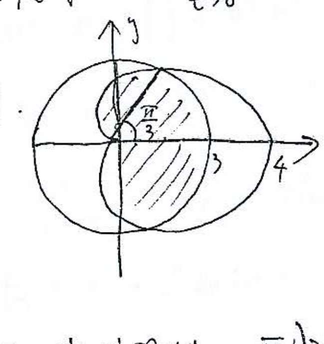
9. 主要部分 $\sum_{n=1}^{\infty}(-1)^n\frac{1}{(z-2)^n}$ 的收敛域　$\left|\frac{1}{z-2}\right|<1$，即 $1<|z-2|$

解析部分 $\sum_{n=0}^{\infty}(-1)^n(1-\frac{z}{2})^n$ ----　$|1-\frac{z}{2}|<1$，即 $|z-2|<2$

∴ Laurent级数收敛域为 $1<|z-2|<2$．(选 (D))

三

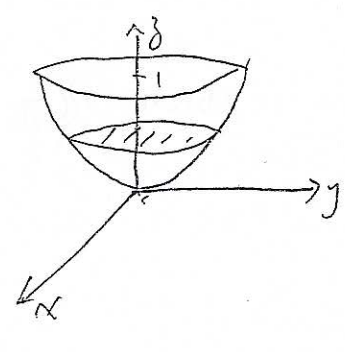
10. 原式 $\overset{令\frac{1}{x}=t}{=}\lim_{t\to 0}(\sin 2t+\cos t)^{\frac{1}{t}}=e^{\lim_{t\to 0}\frac{\ln(\sin 2t+\cos t)}{t}}\overset{L'}{=}e^{\lim_{t\to 0}\frac{2\cos 2t-\sin t}{\sin 2t+\cos t}}=e^2$

11. 

$A=2\int_0^{\frac{\pi}{3}}d\varphi\int_0^3\rho d\rho+2\int_{\frac{\pi}{3}}^{\pi}d\varphi\int_0^{2(1+\cos\varphi)}\rho d\rho$

$=3\pi+4\int_{\frac{\pi}{3}}^{\pi}(1+2\cos\varphi+\cos^2\varphi)d\varphi=3\pi+\frac{8\pi}{3}-4\sqrt{3}+\frac{4\pi}{3}-\frac{\sqrt{3}}{2}$

$=7\pi-\frac{9\sqrt{3}}{2}$



12. 由对称性，原式 $=\iiint_{\Omega}z\,dv\overset{切片}{=}\int_0^1 z\cdot z\pi\,dz=\frac{\pi}{3}$

13. $\frac{\partial(u-v)}{\partial x}=e^x(\cos y-\sin y)-1=\frac{\partial(u+v)}{\partial y}$　(1)

$\frac{\partial(u-v)}{\partial y}=e^x(-\sin y-\cos y)-1=-\frac{\partial(u+v)}{\partial x}$

即 $\frac{\partial(u+v)}{\partial x}=e^x(\cos y+\sin y)+1$　(2)

由(1) $u+v=\int[e^{x}(\cos y-\sin y)-1]dy=e^{x}(\sin y+\cos y)-y+\varphi(x)$

对x求导： $\frac{\partial(u+v)}{\partial x}=e^{x}(\sin y+\cos y)+\varphi'(x)$ 与(2)式比较

$\varphi'(x)=1$ $\therefore \varphi(x)=x+c$

故 $u+v=e^{x}(\sin y+\cos y)-y+x+c$ (3)

由 $u-v=e^{x}(\cos y-\sin y)-x-y$ (5)

$\Rightarrow u=e^{x}\cos y-y+\frac{c}{2}$ $v=e^{x}\sin y+x+\frac{c}{2}$

$\therefore f(z)=e^{x}\cos y-y+\frac{c}{2}+i(e^{x}\sin y+x+\frac{c}{2})=e^{z}+\frac{c}{2}+i(z+\frac{c}{2})$

14. 由 $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$ 即 $f''(x)+f(x)=x^{2}$ $r^{2}+1=0$ $r=\pm i$

$\therefore f(x)=c_1\cos x+c_2\sin x+x^{2}-2$ 由 $f(0)=0$ $c_1=2$. $f'(0)=1$. $c_2=1$ $(f'(x)=-c_1\sin x+c_2\cos x+2x)$

$\therefore f(x)=2\cos x+\sin x+x^{2}-2$

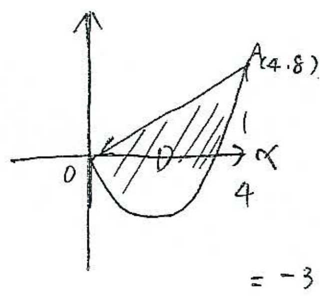
$\therefore$ 原全微分方程为 $xy^{2}dx-2\cos x\cdot ydx+\sin x\cdot ydx+2ydx-2\sin xdy+\cos xdy+2xdy+x^{2}ydy=0$

$\Rightarrow(\frac{1}{2}y^{2}dx^{2}+\frac{1}{2}x^{2}dy^{2})-(2\cdot yd\sin x+2\sin xdy)+(yd\cos x+\cos xdy)+(2ydx+2xdy)=0$

$\Rightarrow\frac{1}{2}d(y^{2}x^{2})-2d(y\sin x)+d(y\cos x)+2d(xy)=0$

$\therefore$ 通解为 $\frac{1}{2}y^{2}x^{2}-2y\sin x+y\cos x+2xy=c$

15.



补线 $\overline{AO}$，方程 $y=2x$

原式 $=\iint_D(2x-5x)d\sigma-\int_4^0(10x^{2}+e^{2x}+2x^{2}-e^{2x})dx$

$=-3\int_0^4dx\int_{x^2-2x}^{2x}xdy+\int_0^4 12x^{2}dx$

$=-3\int_0^4 x(2x-x^{2}+2x)dx+4x^{3}\Big|_0^4$

$=-3\int_0^4(4x^{2}-x^{3})dx+4x^{3}\Big|_0^4=-4x^{3}\Big|_0^4+\frac{3}{4}x^{4}\Big|_0^4+4x^{3}\Big|_0^4=3\cdot4^{3}=192$

16. $u_n(x)=e^{\int dx}[\int x^{n-1}e^{x}\cdot e^{-\int dx}dx+c]=e^{x}[\frac{x^{n}}{n}+c]$ $\because u_n(1)=\frac{e}{n}$ $\therefore c=0$

故 $u_n(x)=\frac{x^{n}}{n}e^{x}$ $S(x)=\sum_{n=1}^{\infty}u_n(x)=\sum_{n=1}^{\infty}\frac{x^{n}}{n}e^{x}=e^{x}\sum_{n=1}^{\infty}\frac{x^{n}}{n}$

记 $\sum_{n=1}^{\infty}\frac{x^{n}}{n}=T(x)$. $T'(x)=\sum_{n=1}^{\infty}x^{n-1}=\frac{1}{1-x}$ $\therefore T(x)=\int\frac{1}{1-x}dx=-\ln(1-x)$

故 $S(x)=-e^{x}\ln(1-x)$ $-1\le x<1$.

17. $\vec{n}=\frac{1}{\sqrt{3}}\{1,-1,1\}$. 化为第一型. 原式 $=\frac{1}{\sqrt{3}}\iint_\Sigma[f+x-2f-y+f+z]dA=\frac{1}{\sqrt{3}}\iint_\Sigma(x-y+z)dA$

$=\frac{1}{\sqrt{3}}\iint_\Sigma dA=\frac{1}{\sqrt{3}}\times\Sigma$面的面积 $=\frac{1}{\sqrt{3}}\times\frac{\sqrt{3}}{2}=\frac{1}{2}$.