Single Variable Calculus – Micheal Spivak
3rd Edition
Part I: Prologue
Ch 01: Basic Properties of Numbers

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***Abstract*— This document is for answering Part I: Prologue, Ch 01: "Basic Properties of Numbers" problems.**

I. Problems

A. Problem (1) Prove the following:

*1) **If** $ax = a$ **for some number** $a \neq 0$**, then** $x = 1$.*

Proof:

$$
\begin{aligned}
a.x &= a && (1)\\
a^{-1}.a.x &= a.a^{-1} && \text{From P(7) Multiplication Inverse} \quad (2)\\
1.x &= 1 && (3)\\
x &= 1 && (4)
\end{aligned}
$$

■

2) $\boldsymbol{x^2 - y^2 = (x - y)(x + y)}$

Proof:

$$
\begin{aligned}
x^2 - y^2 &= (x - y)(x + y) && (5)\\
&= [x + (-y)](x + y) && \text{From P(9) Distributive Law} \quad (6)\\
&= x(x + y) + (-y)(x + y) && (7)\\
&= x(x + y) - [y(x + y)] && (8)\\
&= x^2 + xy - [yx + y^2] && \text{From P(8) Commutative law for Multiplication} \quad (9)\\
&= x^2 + xy - xy - y^2 && \text{From P(2) Addition Inverses} \quad (10)\\
&= x^2 + 0 - y^2 && (11)\\
&= x^2 - y^2 && (12)
\end{aligned}
$$

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*3) **If** $x^2 = y^2$**, then** $x = y$ **or** $x = -y$.*

Proof:

Has been proven in Problem (6), (13)

in a more generic way, with different forms. (14)

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Problem (5) answers are from Ref [1]

B. Problem (5) Prove the following:

*1) **If** $a < b$ **and** $c < d$**, then** $a + c < b + d$.*

Proof:

$$
\begin{aligned}
& a < b \rightarrow b - a > 0 && (15)\\
& c < d \rightarrow d - c > 0 && (16)\\
& \text{Since } b - a, \text{ and } d - c \text{ are both positive,} && (17)\\
& b + d - (c + a) > 0 && \text{By P(11) Closure under addition} \quad (18)\\
\text{Therefore: } & a + c < b + d && (19)
\end{aligned}
$$

■

*2) **If** $a < b$ **and** $c > d$**, then** $a - c < b - d$.*

Proof:

$$
\begin{aligned}
& a < b \rightarrow b - a > 0 && (20)\\
& c > d \rightarrow c - d > 0 && (21)\\
\text{Since: } & b - a, \text{ and } c - d \text{ are both positive,} && (22)\\
& b - a + (c - d) > 0 && (23)\\
& -a + c + b - d > 0 && (24)\\
& (-a + c) + (b - d) > 0 && (25)\\
\text{Therefore: } & b - d > -[-a + c] && (26)\\
& a - c < b - d && (27)
\end{aligned}
$$

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*3) **If** $a < b$**, then** $-b < -a$.*

Proof:

$$
\begin{aligned}
& a < b \rightarrow b - a > 0 && (28)\\
& \text{Then } b - a \quad \text{is positive.} && (29)\\
& \text{Then } -(b - a) && (30)\\
& \text{Then } a - b \quad \text{is negative.} && (31)\\
& \text{Since } a - b < 0 && (32)\\
& \text{Therefore } -b < -a && (33)
\end{aligned}
$$

■

*4) **If** $a < b$ **and** $c > 0$**, then** $a.c < b.c$.*

Proof:

$$a < b \rightarrow b - a > 0 \quad (34)$$

$$\text{Since: } c > 0 \quad (35)$$

$$c.(b - a) > 0.c \quad (36)$$

$$c.(b - a) > 0 \quad (37)$$

$$c.b - c.a > 0 \quad \text{By P(9) Distributive law} \quad (38)$$

$$c.b - c.a + c.a > 0 + c.a \quad \text{By P(3) Additive inverse.} \quad (39)$$

$$c.b > c.a \quad \text{By P(2) Additive identity.} \quad (40)$$

$$b.c > a.c \quad \text{By P(8) Commutative law for multiplication.} \quad (41)$$

$$\text{Therefore: } a.c < b.c \quad (42)$$

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*5) **If** $a < b$ **and** $c < 0$**, then** $a.c > b.c$*

Proof:

$$a < b \rightarrow b - a > 0 \quad (43)$$

$$c < 0 \quad (44)$$

$$c.(b - a) > 0.c \quad \text{Multiply both sides with } c \quad (45)$$

$$c.b - a.c > 0 \quad \text{By P(9) Distributive law.} \quad (46)$$

$$c.b - a.c + a.c > 0 + a.c \quad \text{By P(3) Additive inverses.} \quad (47)$$

$$\text{Therefore: } c.b > a.c \equiv a.c < b.c \quad (48)$$

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*6) **If** $a > 1$**, then** $a^2 > a$.*

Proof:

$$\because a > 1 \quad (49)$$

$$a.a = a^2 > a \quad (50)$$

$$\because a > 1 \quad (51)$$

$$\therefore a > 0 \quad (52)$$

$$\because c > b \wedge d > 0 \therefore c.d > b.d \quad \text{Proven at P(5.1)} \quad (53)$$

$$\text{let: } d = c = a \wedge b = 1 \quad (54)$$

$$\therefore a.a > a.1 \rightarrow a^2 > a \quad (55)$$

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Problem (6) answers are from Ref [1]

C. Problem (6) Prove the following:

*1) **Prove that if** $0 \leq x < y$**, then** $x^n < y^n, n = 1, 2, 3, \ldots$*

Proof:

Has been proven in Problem (6), (56)

in a more generic way, with different forms. (57)

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ACKNOWLEDGMENT

I would like to thank so much ...

REFERENCES

[1] ETOIX, CALCULUS BY SPIVAK, CHAPTER 1, PROBLEM 6, SEPTEMBER 30, 2014 link