PROBABILITY BY ALAN F. KARR

Solutions

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Contents

1 Probability

Exercise 1.1. Prove *de Morgan's law*:

$$(A \cup B)^c = A^c \cap B^c$$

and

$$(A \cap B)^c = A^c \cup B^c$$

both directly and using indicator functions.

Solution. Let's prove $(A \cup B)^c = A^c \cap B^c$, then $(A \cap B)^c = A^c \cup B^c$ can be proved in a similar way.

Directly
($\subseteq$) For an element $x \in (A \cup B)^c$, then $x \notin A \cup B = \{y | y \in A \vee y \in B\}$. In this case, $x \notin A$ and $x \notin B$, that is $x \in A^c$ and $x \in B^c$. Hence, $x \in A^c \cap B^c$. Given that this holds for all $x \in (A \cup B)^c$, we have $(A \cup B)^c \subseteq A^c \cap B^c$.
($\supseteq$) For an element $x \in A^c \cap B^c$, it means $x \in A^c$ and $x \in B^c$. In this case, $x \notin A$ and $x \notin B$, hence $x \notin A \cup B$ by definition. Therefore, $x \in (A \cup B)^c$. Given that this holds for all $x \in A^c \cap B^c$, we have $(A \cup B)^c \supseteq A^c \cap B^c$.

Indicator functions

$$\begin{aligned}
\mathbb{1}_{(A \cup B)^c} &= 1 - \mathbb{1}_{A \cup B} \\
&= 1 - \max\{\mathbb{1}_A, \mathbb{1}_B\} \\
&= \min\{1 - \mathbb{1}_A, 1 - \mathbb{1}_B\} \\
&= \min\{\mathbb{1}_{A^c}, \mathbb{1}_{B^c}\} \\
&= \mathbb{1}_{A^c \cap B^c}
\end{aligned}$$

□

Exercise 1.2. Prove that for events A and B, $A \Delta B = A^c \Delta B^c$.

Solution. By definition

$$\begin{aligned}
A \Delta B &= A \setminus B + B \setminus A \\
&= A \cap B^c + B \cap A^c \\
&= A^c \cap (B^c)^c + B^c \cap (A^c)^c \\
&= A^c \setminus B^c + B^c \setminus A^c \\
&= A^c \Delta B^c
\end{aligned}$$

□

Exercise 1.3. Let B and C be events (A_n) and let $A_n = B$ if n is odd and $A_n = C$ if n is even. Calculate $\limsup_n A_n$ and $\liminf_n A_n$.

Solution. Intuitively while n goes to ∞, $\limsup_n A_n$ is the set where A_n will *"never leave forever"*, and $\liminf_n A_n$ is the set where A_n will *"eventually stay forever"*. Since A_n is alternating between B and C, $\limsup_n A_n = B \cup C$ and $\liminf_n A_n = B \cap C$.

Now, let's prove it. By definition

$$\begin{aligned}\limsup_n A_n &= \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n \\ &= \bigcap_{k=1}^{\infty} (\bigcup_{n=2k-1}^{\infty} A_n \cup \bigcup_{n=2k}^{\infty} A_n) \\ &= \bigcap_{k=1}^{\infty} (B \cup C) \\ &= B \cup C\end{aligned}$$

And

$$\begin{aligned}\liminf_n A_n &= \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n \\ &= \bigcup_{k=1}^{\infty} (\bigcap_{n=2k-1}^{\infty} A_n \cap \bigcap_{n=2k}^{\infty} A_n) \\ &= \bigcup_{k=1}^{\infty} (B \cap C) \\ &= B \cap C\end{aligned}$$

□

Exercise 1.4. Prove part b) of proposition 1.9: Let $A_1, A_2, \ldots$ be subsets of Ω. If $A_1 \supseteq A_2 \supseteq \cdots$, then $A_n \to A = \cap_{n=1}^{\infty} A_n$. (This is written as $A_n \downarrow A$)

Solution. Let $A = \cap_{n=1}^{\infty} A_n$. To prove $A_n \to A$, we can show 1) $\liminf_n A_n = A$, and 2) $\limsup_n A_n = A$.

1) Given $A_1 \supseteq A_2 \supseteq \cdots$, for each k, $\cap_{n=k}^{\infty} A_n = A$. Hence by definition

$$\liminf_n A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n = \bigcup_{k=1}^{\infty} A = A.$$

2) On the other hand, for each k, $\cup_{n=k}^{\infty} A_n = A_k$, by definition

$$\limsup_n A_n = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n = \bigcap_{k=1}^{\infty} A_k = A.$$

□

Exercise 1.5. Prove that
a) if $t_n \downarrow t$, then $(-\infty, t_n] \downarrow (-\infty, t]$, but that
b) if $t_n \uparrow t$ and $t_n < t$ for each n, then $(-\infty, t_n] \uparrow (-\infty, t)$.

Solution. Let's prove a), then b) can be proved in a similar way. By Proposition 1.9,

$$(-\infty, t_n] \downarrow \bigcap_{n=1}^{\infty} (-\infty, t_n].$$

It is left to show that $\cap_{n=1}^{\infty}(-\infty, t_n] = (-\infty, t]$.

Let $x \in \cap_{n=1}^{\infty}(-\infty, t_n]$, then $x \leq t_n$ for all n. As $n \to \infty$, $t_n \to t$ and $x \leq t$, so $x \in (-\infty, t]$. On the other hand, if $x \in (-\infty, t]$, then $x \leq t \leq t_n$ for all n. Therefore, we have $\cap_{n=1}^{\infty}(-\infty, t_n] = (-\infty, t]$. $\square$

Exercise 1.6. Show that the disjointification procedure used to prove Proposition 1.24 actually yields $\cup_{n=1}^{k} A_n = \Sigma_{n=1}^{k} B_n$ for every k.

Solution. Let $k = 1$, then $A_1 = B_1$. Assuming $\cup_{n=1}^{k} A_n = \Sigma_{n=1}^{k} B_n$ holds for some $k > 1$, then

$$\begin{aligned}
\cup_{n=1}^{k+1} A_n &= A_{k+1} \cup (\cup_{n=1}^{k} A_n) \\
&= (A_{k+1} \backslash \cup_{n=1}^{k} A_n) + \cup_{n=1}^{k} A_n \\
&= B_{k+1} + \Sigma_{n=1}^{k} B_n \\
&= \Sigma_{n=1}^{k+1} B_n
\end{aligned}$$

Therefore, the statement is proved by induction. $\square$

Exercise 1.7. Prove (1.1) through (1.4):
(1.1) $\mathbb{1}_{A\cup B} = \max\{\mathbb{1}_A, \mathbb{1}_B\}$
(1.2) $\mathbb{1}_{A\cap B} = \min\{\mathbb{1}_A, \mathbb{1}_B\} = \mathbb{1}_A \mathbb{1}_B$
(1.3) $\mathbb{1}_{A^c} = 1 - \mathbb{1}_A$
(1.4) $\mathbb{1}_{A\Delta B} = |\mathbb{1}_A - \mathbb{1}_B|$

Solution. Note that both sides of the above equations are either or 0 or 1. If one side equals to 1, then it must be greater and equal to another side. Therefore, we just need to focus on the case when one side equals to 0. Let $\omega \in \Omega$.

(1.1) ($\geq$) If $\mathbb{1}_{A\cup B}(\omega) = 0$, by definition $\omega \notin (A \cup B)$. This means that $\omega \notin A$ and $\omega \notin B$, that is $\mathbb{1}_A(\omega) = \mathbb{1}_B(\omega) = 0$.
($\leq$) If $\max\{\mathbb{1}_A, \mathbb{1}_B\}(\omega) = 0$, it must be true that $\mathbb{1}_A(\omega) = \mathbb{1}_B(\omega) = 0$. By definition, $\omega \notin A$ and $\omega \notin B$. Hence $\omega \notin A \cup B$ and $\mathbb{1}_{A\cup B}(\omega) = 0$.

(1.2) ($\geq$) If $\mathbb{1}_{A\cap B}(\omega) = 0$, then $\omega \in A$ and $\omega \in B$ cannot happen simultaneously and $\min\{\mathbb{1}_A, \mathbb{1}_B\}(\omega) = 0$.
($\leq$) If $\min\{\mathbb{1}_A, \mathbb{1}_B\}(\omega) = 0$, then it is either $\mathbb{1}_A(\omega) = 0$ or $\mathbb{1}_B(\omega) = 0$. This means that either $\omega \notin A$ or $\omega \notin B$. In either of these cases, $\mathbb{1}_{A\cap B}(\omega) = 0$.

(1.3) ($\geq$) If $\mathbb{1}_{A^c}(\omega) = 0$, then $\omega \notin A^c$ but $\omega \in A$, and $1 - \mathbb{1}_A(\omega) = 0$.
($\leq$) If $1 - \mathbb{1}_A(\omega) = 0$, then $\mathbb{1}_A(\omega) = 1$ and $\omega \in A$. Thus $\omega \notin A^c$ and $\mathbb{1}_{A^c}(\omega) = 0$.

(1.4) ($\geq$) If $\mathbb{1}_{A\Delta B}(\omega) = 0$, then it is either $\omega \in (A \cup B)^c$ or $\omega \in (A \cap B)$. The former case gives $\mathbb{1}_A(\omega) = \mathbb{1}_B(\omega) = 0$ and the latter case gives $\mathbb{1}_A(\omega) = \mathbb{1}_B(\omega) = 0$. Both cases yield $|\mathbb{1}_A - \mathbb{1}_B|(\omega) = 0$.
($\leq$) If $|\mathbb{1}_A - \mathbb{1}_B|(\omega) = 0$, similar to the previous case, it is either $\omega \in A$ and $\omega \in B$ occur simultaneously, or $\omega \notin A$ and $\omega \notin B$ occur simultaneously. In either case, $\mathbb{1}_{A\Delta B}(\omega) = 0$. □

Exercise 1.8. Let $A_1, A_2, \dots$ be events. Prove that $A_n \to A$ if and only if $\mathbb{1}_{A_n} \to \mathbb{1}_A$ as functions on Ω (that is, $\mathbb{1}_{A_n}(\omega) \to \mathbb{1}_A(\omega)$ for every ω).

Solution. Using (1.1) and (1.2) from Exercise 1.7, we have

$$\begin{aligned}
\mathbb{1}_{\liminf_n A_n} &= \mathbb{1}_{\cup_{k=1}\cap_{n=k}^{\infty} A_n} \\
&= \sup_{k\geq 1} \mathbb{1}_{\cap_{n=k}^{\infty} A_n} \quad \text{by (1.1)} \\
&= \sup_{k\geq 1} \inf_{n\geq k} \mathbb{1}_{A_n} \quad \text{by (1.2)} \\
&= \liminf_n \mathbb{1}_{A_n}
\end{aligned}$$

Similarly, we can prove $\mathbb{1}_{\limsup_n A_n} = \limsup_n \mathbb{1}_{A_n}$. Using these two results, the statement can be easily proved. □

Exercise 1.9. Given subsets A and B of Ω, identify all sets in $\sigma(A, B)$.

Solution. Let $A_1 = (A \cup B)^c, A_2 = A\backslash B, A_3 = A \cap B, A_4 = B\ A$ be finite partitions of Ω. Then

$$\begin{aligned}
\sigma(A, B) = \{&\emptyset, A_1, A_2, A_3, A_4, \\
&A_1 \cup A_2, A_1 \cup A_3, A_1 \cup A_4, A_2 \cup A_3, A_2 \cup A_4, A_3 \cup A_4, \\
&A_1 \cup A_2 \cup A_3, A_1 \cup A_2 \cup A_4, A_2 \cup A_3 \cup A_4, \Omega\}
\end{aligned}$$

□

Exercise 1.10. Prove that $\{x\}$ is a Borel set for every $x \in \mathbb{R}$.

Solution. By definition, a Borel set is an element of $\mathcal{B}(\mathbb{R})$, so the statement can be proved by representing $\{x\}$ by countable intersections and complement of intervals $(a, b]$, where $a < b$ in $\mathbb{R} \cup \{-\infty, +\infty\}$:

$$\{x\} = (-\infty, x] \backslash \cap_{n=1}^{\infty} (-\infty, x - \frac{1}{n}].$$

□

Exercise 1.11. Let A, B and C be disjoint events with $P(A) = .6$, $P(B) = .3$ and $P(C) = .1$. Calculate the probabilities of all events in the σ-algebra generated by $\{A, B, C\}$.

Solution. Given A, B, C are disjoint events, $\sigma(\{A, B, C\}) = \{\emptyset, A, B, C, A \cup B, A \cup C, B \cup C, A \cup B \cup C\}$. Then

$$\begin{aligned} P(\emptyset) &= 0 \\ P(A) &= .6 \\ P(B) &= .3 \\ P(C) &= .1 \\ P(A \cup B) &= P(A) + P(B) = .6 + .3 = .9 \\ P(A \cup C) &= P(A) + P(C) = .6 + .1 = .7 \\ P(B \cup C) &= P(B) + P(C) = .3 + .1 = .4 \\ P(A \cup B \cup C) &= P(A) + P(B) + P(C) = .9 + .6 + .3 = 1 \end{aligned}$$

□

Exercise 1.12. Verify that (1.8) defines a probability.

Solution. To verify ε_ω (the point mass at $\omega \in \Omega$) is a probability, we need to verify the three conditions in the Definition 1.19.
a) As ε_ω has values in $\{0, 1\}$, $\varepsilon_\omega(A) \geq 0$ for all $A \in \mathcal{F}$.
b) As $\omega \in \Omega$, $\varepsilon_\omega(\Omega) = 1$.
c) Let $A_1, A_2, \ldots$ be disjoint sets in $\mathcal{F}$. It is either $\omega \in \Sigma_{n=1}^{\infty} A_n$ or $\omega \notin \cup_{n=1}^{\infty} A_n$.
In the former case, $\varepsilon_\omega(\Sigma_{n=1}^{\infty} A_n) = 1$, and ω must exist in only A_k for some $k \geq 1$, so $\Sigma_{n=1}^{\infty} P(A_n) = P(A_k) + \Sigma_{n \neq k} P(A_n) = 1 + 0 = 1$. □

Exercise 1.13. Confirm that (1.9) defines a probability.

Solution. a) Since p_i are defined to be positive, $\Sigma_{i \in I} p_i \geq 0$ for all $I \subseteq \{1, \ldots, n\}$.
b) By definition of p_i, $\Sigma_{i=1}^{n} p_i = 1$.
c) The countable distjoint sets can only be finite in this problem. By definition of p_i, c) can be easily verified. □

Exercise 1.14. Prove that for all events A and B,

$$\begin{aligned} P(A \Delta B) &= P(A \cup B) - P(A \cap B) \\ &= P(A) + P(B) - 2P(A \cap B). \end{aligned}$$

Solution. For the first equality, since $A \cup B = A \Delta B + A \cup B$, the sum of two disjoint sets, we have $P(A \cup B) = P(A \Delta B) + P(A \cup B)$. Simply rearranging gives the first equality.
For the second equality, given $A = A \backslash B + A \cap B$ and $B = B \backslash A + A \cap B$, we can get $P(A \backslash B) = P(A) - P(A \cap B)$ and $P(B \backslash A) = P(B) - P(A \cap B)$. Since $A \Delta B = A \backslash B + B \backslash A$, we have $P(A \Delta B) = P(A \backslash B) + P(B \backslash A) = P(A) + P(B) - 2P(A \cap B)$. □

Exercise 1.15. Let Ω be a finite set and let $\mathcal{F} = \mathcal{P}(\otimes)$ be the σ-algebra of all subsets of Ω. For $A \subseteq \Omega$, let $|A|$ be the number of elements in A. Prove that the formula $P(A) = |A|/|\Omega|$ defines a probability on $(\Omega, \mathcal{F})$ (the uniform distribution on Ω).

Solution. Assume Ω is not empty.
a) Since $|A| \geq 0$ and $|\Omega| \geq 1$, $P(A) \geq 0$ for all $A \in \mathcal{F}$.
b) $P(\Omega) = |\Omega|/|\Omega| = 1$.
c) Let $A_1, A_2, \ldots, A_n$ be a finitne number of disjoint sets in $\mathcal{F}$. The number of elements in $\Sigma_{k=1}^{n} A_k$ must be equal to the total number of elements in A_k for all $k \in \{1, \ldots, n\}$, i.e. $|\Sigma_{k=1}^{n} A_k| = \Sigma_{k=1}^{n} |A_k|$. Thus, c) can be easily verified. □

Exercise 1.16. Prove that (1.15) and (1.16) are equivalent.

Solution. (1.15) $\Rightarrow$ (1.16):

$$\limsup_{n\to\infty} P(A_n) = 1 - \liminf_{n\to\infty} P(A_n^c) \leq 1 - P(\liminf_n A_n^c) = P(\limsup_n A_n)$$

(1.16) $\Rightarrow$ (1.15):

$$P(\liminf_n A_n) = 1 - P(\limsup_n A_n^c) \leq 1 - \limsup_{n\to\infty} P(A_n^c) = \liminf_{n\to\infty} P(A_n)$$

□

Exercise 1.17. Prove that there does not exist a uniform distribution on the set $\mathbb{N} = \{0, 1, \ldots\}$.

Solution. Assume there exists a uniform distribution on $\mathbb{N}$, which has the probability $P(A) = k|A|$ for some constant $k \geq 0$.
If $k = 0$, $P(\mathbb{N}) = 0$ contradicts the definition of probability.
If $k > 0$, $P(\mathbb{N}) = k|\mathbb{N}| = +\infty > 1$ also contradicts the definition of probability.
Therefore, by contradiction, there does not exist a uniform distribution on $\mathbb{N}$. □

Exercise 1.18. Suppose that P_1 and P_2 are probabilities on $(\Omega, \mathcal{F})$ and that $0 \leq \alpha \leq 1$. Prove that the set function

$$P(A) = \alpha P_1(A) + (1-\alpha) P_2(A)$$

is also a probability.

Solution. a) As $\alpha \geq 0$ and $1 - \alpha \geq 0$, $P(A) \geq 0$ for all $A \in \mathcal{F}$.
b) $P(\Omega) = \alpha P_1(\Omega) + (1-\alpha) P_2(\Omega) = \alpha + (1-\alpha) = 1$.
c) Let $A_1, A_2, \ldots$ are disjoint sets in $\mathcal{F}$,

$$\begin{aligned}
P(\Sigma_{n=1}^{\infty} A_n) &= \alpha P_1(\Sigma_{n=1}^{\infty} A_n) + (1-\alpha) P_2(\Sigma_{n=1}^{\infty} A_n) \\
&= \alpha \Sigma_{n=1}^{\infty} P_1(A_n) + (1-\alpha) \Sigma_{n=1}^{\infty} P_2(A_n) \\
&= \Sigma_{n=1}^{\infty} (\alpha P_1(A_n) + (1-\alpha) P_2(A_n)) \\
&= \Sigma_{n=1}^{\infty} P(A_n)
\end{aligned}$$

□

Exercise 1.19. Prove that if $P(A_i) = 1$ for each i, then

$$P(\cap_{i=1}^{\infty} A_i) = 1.$$

Solution. By Boole's inequility, $P(\cup_{i=1}^{\infty} A_i^c) \leq \Sigma_{i=1}^{\infty} P(A_i^c) = 0$. Since $P \geq 0$ by difinition, $P(\cup_{i=1}^{\infty} A_i^c) = 0$. Thus

$$P(\cap_{i=1}^{\infty} A_i) = 1 - P(\cup_{i=1}^{\infty} A_i^c) = 1.$$

□

Exercise 1.20. Show that for events A and B,

$$P(A \cap B) \geq P(A) + P(B) - 1.$$

Solution. By Boole's inequality,

$$P(A \cap B) = 1 - P(A^c \cup B^c) \geq 1 - P(A^c) - P(B^c) = P(A) + P(B) - 1.$$

□

Exercise 1.21. Let P be a probability on $\mathbb{R}$. Prove that for every $\varepsilon > 0$ there is a compact set K such that $P(K) > 1 - \varepsilon$.

Solution. Consider compact intervals $K_n = [-n, n]$, $n \in \mathbb{N}$. As $n \to \infty$, $K_n \uparrow \mathbb{R}$ and $P(K_n) \uparrow 1$. For an arbitrary $\varepsilon > 0$, there must exist a sufficiently large n such that $P(K_n) > 1 - \varepsilon$. □

Exercise 1.22. Prove that a distribution function on $\mathbb{R}$ has at most countably many points of discontinuity.

Solution. TODO □

Exercise 1.23. Prove that if $P(A) > 0$, then the set function

$$P_A(B) = P(B|A)$$

is a probability on $(\Omega, \mathcal{F})$ satisfying $P_A(A^c) = 0$.

Solution. By Bayes' theorem,

$$P_A(A^c) = \frac{P(A^c \cap A)}{P(A)} = \frac{P(\emptyset)}{P(A)} = 0.$$

□

Exercise 1.24. Let P be the uniform distribution on a fininte set Ω and let A be a subset Ω. Prove that $P(\cdot|A)$ is the uniform distribution on A.

Solution. For an arbitrary set $B \subseteq A$, by Bayes' theorem,

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)}{P(A)} = \frac{|B|}{|A|},$$

which coincides the probability of a uniform distribution on A. □

Exercise 1.25. Let $A_1, \dots, A_n$ be events, and for $J \subseteq \{1, \dots, n\}$, let $B_J = \cap_{j \in J} A_j$. For $k \geq 1$, let $S_k = \Sigma_{|J|=k} P(B_J)$, where the sum is over all subsets J of $\{1, \dots, n\}$ with $|J| = k$.
a) Prove the *inclusion-exclusion principle*:

$$P(\bigcup_{i=1}^{n} A_i) = \Sigma_{k=1}^{n} (-1)^{k-1} S_k.$$

b) Suppose that $P(B_J)$ depends only on $|J|$, i.e., there are numbers $q_0, \dots, q_n$ such that $P(B_J) = q_k$ whenever $|J| = k$. (This is true, for example, in Exercise 1.15.) Prove that $S_k = \binom{n}{k} q_k$.

Solution. a) Prove the inclusion-exclusion principle by induction is such a pain. Here we show a simpler proof by borrowing Definition 4.1 of expectation.
Using indicator functions, we have

$$\mathbb{1}_{\cup_{i=1}^{n} A_i} = 1 - \mathbb{1}_{\cap_{i=1}^{n} A_i^c} = 1 - \prod_{i=1}^{n} (1 - \mathbb{1}_{A_i}).$$

Expanding this equation, we can get

$$\mathbb{1}_{\cup_{i=1}^{n} A_i} = \Sigma_{k=1}^{n} (-1)^{k-1} \Sigma_{|J|=k} \prod_{j \in J} \mathbb{1}_{A_j}.$$

Taking expectation from both sides tives the inclusion-exclusion principle.
b) As $\binom{n}{k}$ represents the number of combinations of choosing k from n:

$$S_k = \Sigma_{|J|=k} P(B_J) = \Sigma_{|J|=k} q_k = \binom{n}{k} q_k.$$

□

Exercise 1.26. This is known as the *coupon collector's problem.* There are t different types of coupons available and the collector is seeking to collect one of each (for example, in order to win some premium). Show that if n coupons have been collected, then the probability p_n of having at least one of each type is

$$p_n = \Sigma_{k=1}^{t} (-1)^{k-1} \binom{t}{k} \left(1 - \frac{k}{t}\right)^n.$$

Solution. Let A_i be the event that the i-th type coupon is missing when n coupons have been collected, and $\cup_{i=1}^{t} A_i$ is the event that at least one type is missing. Then the event of having at least one of each type is the complement of $\cup_{i=1}^{t} A_i$. By applying the inclusion-exclusion principle (from exercise 1.25),

$$P(\cup_{i=1}^{n} A_i) = \Sigma_{k=1}^{n} (-1)^{k-1} S_k$$

where $S_k = \Sigma_{|J|=k} P(B_J)$ and $B_J = \cap_{j \in J} A_j$. In this problem, B_J is the event that all the types in J are missing once n coupons have been collected, and

$$P(B_J) = \left(1 - \frac{k}{t}\right)^n.$$

Since $P(B_J)$ depends only on $|J| = k$, applying exercise 1.25 b), $S_k = \binom{n}{k} \left(1 - \frac{k}{t}\right)^n$ and

$$P(\cup_{i=1}^n A_i) = \Sigma_{k=1}^n (-1)^{k-1} \binom{n}{k} \left(1 - \frac{k}{t}\right)^n.$$

Therefore, the probability p_n of having at least one of each type is

$$p_n = 1 - P(\cup_{i=1}^n A_i) = 1 - \Sigma_{k=1}^n (-1)^{k-1} \binom{n}{k} \left(1 - \frac{k}{t}\right)^n.$$

(Note: It seems like there is an erratum in this exercise.) □

Exercise 1.27. Consider an urn that initially contains r red balls and b black balls. At each trial one ball is drawn. It is replaced and $c \geq 0$ balls of the same color added to the urn. Let A_j be the event that the j th ball drawn is black. Show that $P(A_j) = b/(b+r)$ for every j.

Solution. TODO □