Variance Reduction Techniques for Pricing Barrier Option

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1 Introduction

Barrier option is one of the most liquidly traded path dependent exotic option contract in foreign exchange market (FX) and over the counter market (OTC). There are two types of Barrier option: knock-out option and knock-in option. The knock-out option is extinguished if the underlying asset price reaches a prespecified level called the barrier or knock-out boundary. The knock-in option is activated if the underlying asset price reaches a prespecified level called the barrier or knock-in boundary. The existence of such barriers reduce the likelihood for the option to be excised, therefore, barrier option is less expensive than the corresponding regular options.

Barrier options are in a sense intermediate between standard European and American options. They resemble American options since their valuation depends on the entire path, but the valuation is simpler than that of American options, since the critical boundary of the underlying asset price is determined in advance and specified in the contract. Thus, unlike American option, barrier option has closed-form analytical pricing formula. The analytical formula for barrier option is derived using no-arbitrage argument and risk neutral expectation under the Black-Scholes environment. The risk neutral expectation could also be approximated by Monte Carlo simulation. The aim of this paper is to improve the efficiency of the Monte Carlo Method by using two variance reduction techniques (antithetic variates and control variates) and their implementation in C++.

Antithetic variates methods attempts to reduce the variance by introducing negative dependence between pairs of replications. It works as follows, given a sequence of paths $\{Y_i\}_{i=0}^n$, a parallel paths $\{\widehat{Y}_i\}_{i=0}^n$ could be generated by inverse CDF transformation $F^{-1}\big(1-F(Y_i)\big)$. Thus both $\{Y_i\}_{i=0}^n$ and $\{\widehat{Y}_i\}_{i=0}^n$ have the same distribution F but are antithetic to each other. For distribution symmetric about origin, the antithetic pairs have same magnitude but opposite signs. Within an antithetic pair, one large value is always accompanied by a small value. This suggest that any path with extreme values will always be balanced by the antithetic paths, resulting in a reduction in variance.

Control variates methods exploits information about the errors occurred during MC estimation of the expectation of a control variable, of which the expectation is already known. Use such information to reduce the MC estimation variance of a correlated variable of interest. In our case, value of vanilla option with known risk neutral expectation will be used as control variates to reduce the variance of estimation of barrier option. The effectiveness of a control variates depends on the correlation between variates of interest and the control variates: ρ_{XY}. The variance reduction factor $\frac{1}{1-\rho_{XY}^2}$ measures the computational speed up resulting from the use of control. Suppose the computational effort per replication is roughly same with and without control, then a 0.95 correlation produces a tenfold speed up, a 0.9 yields a fivefold speed up.

The rest of paper is structured as follows: section 2 set up the Black-Scholes environment, where underlying asset process follows geometric Brownian motion; and lay out the foundation for the general derivative pricing formula under risk neutral measure using noarbitrage argument; the joint density function between Maximum of Brownian motion and Brownian motion is derived using reflection principle. Section 3 discuss in detail of four different methods for pricing up and out barrier option and their implementation algorithm in C++. Section 3.1 derived the analytical solution using risk neutral measure. Section 3.2-3.4 using Monte Carlo simulation, with antithetic variates in 3.3 and control variates in 3.4. Section 4 evaluate different methods under different circumstances based on testing result.

2 Model Setting

2.1 Underlying Stock Price Process under Risk Neutral Measure

Suppose the underlying stock price follows Geometric Brownian with continuous dividend payment d.

$$\mathrm{d}S_t = (\alpha - d)S_t\mathrm{d}t \ + \ \sigma S_t \mathrm{d}W_t$$

The market money account risk free return is r. Let the market price of risk $\theta = \frac{\alpha - r}{\sigma}$, and apply Girsanov's Theorem to change the physical measure to risk neutral measure under which $\mathrm{d}\widetilde{W}_t = \theta\mathrm{d}t + \mathrm{d}W_t$ is a Brownian Motion. The Stock price under risk neutral measure is

$$\mathrm{d}S_t = (r - d)S_t\mathrm{d}t \ + \ \sigma S_t \mathrm{d}\widetilde{W}_t$$

The discounted portfolio of holding stock and money account is a martingale under risk neutral measure. Given any derivatives which payoff is $V(T)$ where $V(T)$ is $\mathcal{F}_T$-measurable. Applying martingale representation Theorem, we can find an adapted process $\Gamma(t)$ such that the short position of this derivative can be hedged. Under the no arbitrage condition, the price of the derivative should be the same as the value of the portfolio hedge the short position. Therefore,

$$D_t V_t = D_t X_t = \widetilde{\mathbb{E}}[\, D_T X_T \,|\, \mathcal{F}_t] = \widetilde{\mathbb{E}}[\, D_T V_T \,|\, \mathcal{F}_t] \tag{1}$$

Since r is constant, $D_t = \mathrm{e}^{-rt}$ is a deterministic discount factor independent of $\mathcal{F}_t$.

2.2 Payoff Function for Up and Out European Barrier Call Option

The payoff function for Up and Out European Barrier Call Option is:

$$
\begin{aligned}
V_T &= (S_T - K)^+ \, \mathbb{I}_{\left\{\max\limits_{0\le t\le T} S_t \le B\right\}} \\
&= \left(S_0 \mathrm{e}^{(r-d-\frac{1}{2}\sigma^2)T + \sigma \widetilde{W}_T} - K\right)^+ \mathbb{I}_{\left\{\max\limits_{0\le t\le T} S_0 \mathrm{e}^{(r-d-\frac{1}{2}\sigma^2)t+\sigma\widetilde{W}_t} \le B\right\}} \\
&= \left(S_0 e^{\sigma \widehat{W}_T} - K\right)^+ \mathbb{I}_{\left\{S_0 e^{\sigma \widehat{M}_T} \le B\right\}} \\
&= \left(S_0 e^{\sigma \widehat{W}_T} - K\right) \mathbb{I}_{\left\{S_0 e^{\sigma \widehat{W}_T} \ge K,\, S_0 e^{\sigma \widehat{M}_T} \le B\right\}}
\end{aligned} \tag{2}
$$

Where

$$
\widehat{W}_t = \frac{1}{\sigma}(r - d - \frac{1}{2}\sigma^2)t + \widetilde{W}_t, \quad \widehat{M}_T = \max_{0\le t\le T} \widehat{W}_t, \quad \max_{0\le t\le T} S_t = S_0 e^{\max\{\sigma \widehat{W}_t\}} = S_0 e^{\sigma \widehat{M}_T}
$$

2.3 Joint Probability Distribution of $\widehat{W}_t$ and $\widehat{M}_t$ under risk neutral measure $\widetilde{\mathbb{P}}$

2.3.1 Risk Neutral Formula

Given the payoff function, in order to set up a portfolio to hedge the short side of Barrier option, the initial capital for such a discounted portfolio is equivalent to the expected discounted value of the derivative at maturity under risk neutral measure. Substitute equation (2) into (1), we have

$$
\begin{aligned}
e^{-rt} X_t &= \widetilde{\mathbb{E}}\left[e^{-rT} X_T \,|\, \mathcal{F}_t\right] \\
&= \widetilde{\mathbb{E}}\left[e^{-rT} V_T \,|\, \mathcal{F}_t\right] \\
&= \widetilde{\mathbb{E}}\left[e^{-rT}\left(S_0 e^{\sigma \widehat{W}_T} - K\right) \mathbb{I}_{\left\{S_0 e^{\sigma \widehat{W}_T} \ge K,\, S_0 e^{\sigma \widehat{M}_T} \le B\right\}} \,|\, \mathcal{F}_t\right] \\
&= \widetilde{\mathbb{E}}\left[e^{-rT}\left(S_t e^{\sigma(\widehat{W}_T - \widehat{W}_t)} - K\right) \mathbb{I}_{\left\{S_t e^{\sigma(\widehat{W}_T - \widehat{W}_t)} \ge K,\, S_t e^{\sigma \max\left\{\widehat{W}_T - \widehat{W}_t\right\}} \le B,\, M_t \le B\right\}}\right] \\
&= \widetilde{\mathbb{E}}\left[e^{-rT}\left(S_t e^{\sigma \widehat{W}_\tau} - K\right) \mathbb{I}_{\left\{S_t e^{\sigma \widehat{W}_\tau} \ge K,\, S_t e^{\sigma \widehat{M}_\tau} \le B,\, M_t \le B\right\}}\right] \\
&= \mathbb{I}_{\{M_t \le B\}} \int_k^b \int_{w^+}^b e^{-rT} \left(S_t e^{\sigma w} - K\right) \tilde{f}_{\widehat{M}_\tau, \widehat{W}_\tau}(m, w) \, \mathrm{d}m \, \mathrm{d}w
\end{aligned} \tag{3}
$$

where

$$
k = \frac{1}{\sigma} \log \frac{K}{S_t}, \quad b = \frac{1}{\sigma} \log \frac{B}{S_t}, \quad w^+ = \max\{0, w\}, \quad M_t = \max_{0\le u\le t} S_u.
$$

2.3.2 Joint Densitiy under $\widetilde{\mathbb{P}}-$ Measure

The joint probablility density function for $\widehat{M_T}, \widehat{W_T}$ under risk neutral measure is

$$
\begin{aligned}
\widetilde{f}_{\widehat{M_T},\widehat{W_T}}(m,w) &= \frac{\partial^2}{\partial m \partial w}\widetilde{\mathbb{P}}\{\widehat{M_T} \leq m, \widehat{W_T} \leq w\} \\
&= \frac{\partial^2}{\partial m \partial w}\widetilde{\mathbb{E}}\left[\mathbb{I}_{\{\widehat{M_T} \leq m, \widehat{W_T} \leq w\}}\right] \\
&= \frac{\partial^2}{\partial m \partial w}\widehat{\mathbb{E}}\left[\frac{1}{\widehat{Z}_T}\mathbb{I}_{\{\widehat{M_T} \leq m, \widehat{W_T} \leq w\}}\right] \\
&= \frac{\partial^2}{\partial m \partial w}\int_{-\infty}^{w}\int_{-\infty}^{m} e^{\alpha y - \frac{1}{2}\alpha^2 T}\,\widehat{f}_{\widehat{M_T},\widehat{W_T}}(x,y)\,\mathrm{d}x\mathrm{d}y \\
&= e^{\alpha w - \frac{1}{2}\alpha^2 T}\,\widehat{f}_{\widehat{M_T},\widehat{W_T}}(m,w) \qquad (4)
\end{aligned}
$$

Where $\widehat{Z}_T$ is the Radon–Nikodym derivative of $\widehat{\mathbb{P}}-$Measure with respect to $\widetilde{\mathbb{P}}-$Measure. According to the Girsanov's Theorem, $\widehat{W}$ is a Brownian under the $\widehat{\mathbb{P}}-$ Measure if $\widehat{Z}_T$ is defined as follow

$$
\widehat{Z}_T = \frac{\mathrm{d}\widehat{\mathbb{P}}}{\mathrm{d}\widetilde{\mathbb{P}}} = e^{-\alpha \widetilde{W}_T - \frac{1}{2}\alpha^2 T} = e^{-\alpha \widehat{W}_T + \frac{1}{2}\alpha^2 T}
$$

Where

$$
\alpha = \frac{1}{\sigma}(r - d - \frac{1}{2}\sigma^2), \quad \widehat{W_T} = \alpha T + \widetilde{W_T},
$$

2.3.3 Joint Densitiy under $\widehat{\mathbb{P}}-$ Measure

According to the Girsanov's theorem, under $\widehat{\mathbb{P}}-$ Measure, $\widehat{W_T}$ is the Brownian motion with zero drift. $\widehat{M_T}$ is in turn the Maximum value of the Brownian motion $\widehat{W_t}$ attained within 0 to T) The joint probablility density function for $\widehat{M_T}, \widehat{W_T}$ under $\widehat{\mathbb{P}}-$ Measure is:

$$
\begin{aligned}
\widehat{f}_{\widehat{M_T},\widehat{W_T}}(m,w) &= \frac{\partial^2}{\partial w \partial m}\int_{-\infty}^{m}\int_{-\infty}^{w} \widehat{f}_{\widehat{M_T},\widehat{W_T}}(x,y)\,\mathrm{d}y\mathrm{d}x \\
&= -\frac{\partial^2}{\partial w \partial m}\int_{m}^{\infty}\int_{-\infty}^{w} \widehat{f}_{\widehat{M_T},\widehat{W_T}}(x,y)\,\mathrm{d}y\mathrm{d}x \\
&= -\frac{\partial^2}{\partial m \partial w}\,\widehat{\mathbb{P}}\{\widehat{M_T} \geq m, \widehat{W_T} \leq w\} \\
&= -\frac{\partial^2}{\partial m \partial w}\,\widehat{\mathbb{P}}\{\widehat{W_T} \geq 2m - w\} \qquad \text{(by reflection principle)} \\
&= -\frac{\partial^2}{\partial m \partial w}\int_{2m-w}^{\infty} \frac{1}{\sqrt{2\pi T}}\,e^{-\frac{z^2}{2T}}\mathrm{d}z \\
&= \frac{\partial}{\partial w}\frac{\partial}{\partial m}\left(\int_{-\infty}^{2m-w} \frac{1}{\sqrt{2\pi T}}\,e^{-\frac{z^2}{2T}}\mathrm{d}z\right) \\
&= \frac{\partial}{\partial w}\left(\frac{2}{\sqrt{2\pi T}}\,e^{-\frac{(2m-w)^2}{2T}}\right) \\
&= \frac{2(2m-w)}{T\sqrt{2\pi T}}\,e^{-\frac{(2m-w)^2}{2T}} \qquad (5)
\end{aligned}
$$

2.3.4 Summary

Plug (5) into (4) we have (6). Which is the joint probablility density function for $\widehat{M_T}, \widehat{W_T}$ under risk neutral measure:

$$\begin{aligned} \widetilde{f}_{\widehat{M_T},\widehat{W_T}}(m,w) &= e^{\alpha w - \frac{1}{2}\alpha^2 T}\, \widehat{f}_{\widehat{M_T},\widehat{W_T}}(m,w) \\ &= \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{(2m-w)^2}{2T}} \end{aligned} \tag{6}$$

Under the no arbitrage assumption, the asset price for the Barrier Option at time t, V_t, should be same as the expected value of the short side hedging portfolio X_t. Plug (6) into (3) we derived the asset price for the up and out barrier option:

$$\begin{aligned} V_t &= X_t \\ &= \widetilde{\mathbb{E}}\left[e^{-r(T-t)} X_T \mid \mathcal{F}_t\right] \\ &= e^{-r(T-t)}\, \mathbb{I}_{\{M_t \le B\}} \int_k^b \int_{w^+}^b (S_t e^{\sigma w} - K)\, \tilde{f}_{\widehat{M_\tau},\widehat{W_\tau}}(m,w)\, \mathrm{d}m\, \mathrm{d}w \\ &= e^{-r\tau}\, \mathbb{I}_{\{M_t \le B\}} \int_k^b \int_{w^+}^b (S_t e^{\sigma w} - K)\, \frac{2(2m-w)}{\tau\sqrt{2\pi\tau}} e^{\alpha w - \frac{1}{2}\alpha^2 \tau - \frac{(2m-w)^2}{2\tau}}\, \mathrm{d}m\, \mathrm{d}w \end{aligned} \tag{7}$$

3 Pricing the Barrier Option

3.1 Closed Form Formula

The closed form solution for the stock price SDE under risk neutral measure

$$\mathrm{d}S_t = (r-d)S_t\mathrm{d}t \ + \ \sigma S_t \mathrm{d}\widetilde{W}_t$$

is

$$S_t = S_0 \mathrm{e}^{(r-d-\frac{1}{2}\sigma^2)t+\sigma\widetilde{W}_t} = S_0 \mathrm{e}^{(r-d-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}\mathbf{Z}}, \quad \text{where } \mathbf{Z} \sim \text{Normal}(0,1)$$

Substitute into the risk neutral pricing formula (7), we have

$$\begin{aligned} V_t &= e^{-r\tau}\,\mathbb{I}_{\{M_t\le B\}} \int_k^b \int_{w^+}^b (S_t e^{\sigma w}-K)\,\frac{2(2m-w)}{\tau\sqrt{2\pi\tau}} e^{\alpha w-\frac{1}{2}\alpha^2\tau-\frac{(2m-w)^2}{2\tau}}\,\mathrm{d}m\,\mathrm{d}w \\ V_0 &= e^{-rT}\,\mathbb{I}_{\{S_0\le B\}} \int_k^b \int_{w^+}^b (S_0 e^{\sigma w}-K)\,\frac{2(2m-w)}{T\sqrt{2\pi T}} e^{\alpha w-\frac{1}{2}\alpha^2 T-\frac{(2m-w)^2}{2T}}\,\mathrm{d}m\,\mathrm{d}w \end{aligned} \tag{8}$$

Where

$$\alpha = \frac{1}{\sigma}(r-d-\frac{1}{2}\sigma^2), \quad k = \frac{1}{\sigma}\log\frac{K}{S_0}, \quad b = \frac{1}{\sigma}\log\frac{B}{S_t}, \quad w^+ = \max\{0,w\}.$$

The calculation of the integral in (8) lead us to the formula for the barrier call option:

$$\begin{aligned} V_0 =& S_0 I_1 - K I_2 - S_0 I_3 + K I_4 \\ =& e^{-dT} S_0 \left[N\left(\delta_+\left(T,\frac{S_0}{K}\right)\right) - N\left(\delta_+\left(T,\frac{S_0}{B}\right)\right)\right] \\ & - e^{-rT} K \left[N\left(\delta_-\left(T,\frac{S_0}{K}\right)\right) - N\left(\delta_-\left(T,\frac{S_0}{B}\right)\right)\right] \\ & - e^{-dT} B \left(\frac{S_0}{B}\right)^{-\frac{2(r-d)}{\sigma^2}} \left[N\left(\delta_+\left(T,\frac{B^2}{KS_0}\right)\right) - N\left(\delta_+\left(T,\frac{B}{S_0}\right)\right)\right] \\ & + e^{-rT} K \left(\frac{S_0}{B}\right)^{-\frac{2(r-d)}{\sigma^2}+1} \left[N\left(\delta_-\left(T,\frac{B^2}{KS_0}\right)\right) - N\left(\delta_-\left(T,\frac{B}{S_0}\right)\right)\right] \end{aligned}$$

Where

$$\delta_\pm(\tau,s) = \frac{1}{\sigma\sqrt{\tau}}\left[\log s + \left(r-d\pm\frac{1}{2}\sigma^2\right)\tau\right]$$

plug into all the given parameters and use the "CumulativeNormal(double x)" function in the "Normals.h" to compute $N(\pm d_\pm)$ we have the option price calculated via explicit formula written in file "BarrierFormula.hpp" and "BarrierFormula.cpp".

3.2 Monte Carlo Simulation

According to the strong law of large number. The average of a sequence of random variable converge almost surely to the expected value. So we can estimate

$$e^{-rT}\widetilde{\mathbb{E}}\left[\left(S_0 \mathrm{e}^{(r-d-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}\mathbf{Z}} - K\right)^+ \mathbb{I}_{\left\{\max\limits_{0\le t\le T} S_0 \mathrm{e}^{(r-d-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}\mathbf{Z}} \le B\right\}}\right]$$

by simulate a large number of underlying equity pathes following $S_0 \mathrm{e}^{(r-d-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}\mathbf{Z}}$ using Euler scheme. The Euler Scheme for the approximation of SDE for log spot is

$$\log S_{t+1}^{(i)} = \log S_t^{(i)} + (r-d-\frac{1}{2}\sigma^2)h + \sigma\sqrt{h}\mathbf{Z}_t^{(i)},\ t=0,\cdots 251$$

where $\log S_0 = \log(100),\ h = \frac{1}{252},\ \mathbf{Z}_t^{(i)} \sim \text{Normal}(0,1)$. Then the payoff for each path could be computed by exponentiate each log spot and check if any of them exceed the Barrier. The risk neutral formula could then be approximated by the sample mean payoffs of each simulated path. $\mathbf{Z}_t^{(i)}$ for each subinterval in Euler Scheme is generated by Park Miller linear congruential random generator implemented by "random2" and "ParkMiller". This Monte Carlo option pricing algorithm is implemented in the files "ExoticBSEngine" and "PathDependentBarrier". I write a statistc getherer in "MCSamples" in order to record the simulation result and compute the standard error(i.e standard deviation for sample mean).

3.3 Variance Reduction with Antithetic Variates

The pair $(Y, \widetilde{Y})$ is antithetic if $Y = g(Z)$ and $\widetilde{Y} = g(Z)$, where Z is a standard normal random variable and $g(z)$ a deterministic function. Using the antithetic pair could enhance the simulation efficiency and possibly reduce the standard deviation of the Monte Carlo sample mean estimator. The underlying log equity price process is simulated by Euler Scheme:

$$\log S_{t+1}^{(i)} = \log S_t^{(i)} + (r-d-\frac{1}{2}\sigma^2)h + \sigma\sqrt{h}\mathbf{Z}_t^{(i)},\ t=0,\cdots 251$$

$$\log \widetilde{S}_{t+1}^{(i')} = \log \widetilde{S}_t^{(i')} + (r-d-\frac{1}{2}\sigma^2)h - \sigma\sqrt{h}\mathbf{Z}_t^{(i)},\ t=0,\cdots 251$$

where $\log S_0^{(i)} = \log \widetilde{S}_0^{(i')} = \log(100),\ h = \frac{1}{252},\ \mathbf{Z}_t^{(i)} \sim \text{Normal}(0,1)$. Let

$$\begin{aligned} Y_i &= \mathrm{e}^{-rT}\left(S_T^{(i)} - K\right)^+ \mathbb{I}_{\left\{\max\limits_{0\le t\le T} S_t \le B\right\}} \\ \widetilde{Y}_i &= \mathrm{e}^{-rT}\left(\widetilde{S}_T^{(i')} - K\right)^+ \mathbb{I}_{\left\{\max\limits_{0\le t\le T} \widetilde{S}_t^{(i')} \le B\right\}} \end{aligned}$$

As a result the antithetic Monte Carlo estimator is:

$$\widehat{Y}_{\text{AV}} := \frac{2}{N}\sum_{i=1}^{\frac{N}{2}}\left(\frac{Y_i + \widetilde{Y}_i}{2}\right), \quad \text{where } N \text{ is the number of path.}$$

Note that number of $\mathbf{Z}_t^{(i)}$ generated is just the half of (b) does since the diffusion part for path i' is just the negative of that in path i. The antithetic variates is implemented in "AntiThetic".

3.4 Variance Reduction with Control Variates

The control variates is another variance reduction techniques used in Monte Carlo Simulation. It exploits information about the errors in estimates of known quantities (Vanila Option) to reduce the error of an estimate of an unknown quantity (Barrier Option). The risk neutral formula for Vanila and Barrier call option is as follows:

$$
\begin{aligned}
C_0 &= \mathrm{e}^{-rT}\widetilde{\mathbb{E}}\left[(S_T-K)^+\right] \\
V_0 &= \mathrm{e}^{-rT}\widetilde{\mathbb{E}}\left[(S_T-K)^+\,\mathbb{I}_{\left\{\max\limits_{0\le t\le T} S_t\le B\right\}}\right]
\end{aligned}
$$

The expectation under the risk neutral measure could be estimated by Monte Carlo technique. Where for each path, the underlying equity stochastic process is approximated by the Euler Scheme

$$
\log S_{t+1}^{(i)} = \log S_t^{(i)} + (r-d-\frac{1}{2}\sigma^2)h + \sigma\sqrt{h}\mathbf{Z}_t^{(i)},\ t=0,\cdots 251
$$

where $\log S_0 = \log(100),\ h=\frac{1}{252},\ \mathbf{Z}_t^{(i)} \sim \text{Normal}(0,1)$. Let

$$
\begin{aligned}
Y_i &= \mathrm{e}^{-rT}\left(S_T^{(i)}-K\right)^+\mathbb{I}_{\left\{\max\limits_{0\le t\le T} S_t\le B\right\}} \\
X_i &= \mathrm{e}^{-rT}\left(S_T^{(i)}-K\right)^+
\end{aligned}
$$

The control variate estimator is

$$
\begin{aligned}
\bar{Y}(b) &= \bar{Y} - b(\bar{X}-\mathbb{E}[X]) \\
&= \frac{1}{N}\sum_{i=1}^{N}\left(Y_i - b\left(X_i-\mathbb{E}[X]\right)\right)
\end{aligned}
$$

Where $\mathbb{E}[X]$ follows the Black-Scholes formula

$$
\mathbb{E}[X] = \mathrm{e}^{-rT}\left[S_0\mathrm{e}^{(r-d)T}\Phi(d_+) - K\Phi(d_-)\right],\quad d_\pm = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{S_0}{K} + (r-d\pm\frac{1}{2}\sigma^2)T\right]
$$

b is chosen to minimize the variance of $\bar{Y}(b)$

$$
\begin{aligned}
\text{Var}[\bar{Y}(b)] &= \frac{1}{n}\text{Var}\left[\bar{Y}_i(b)\right] = \frac{1}{n}\left(\sigma_Y^2 - 2b\sigma_X\sigma_Y\rho_{XY} + b^2\sigma_X^2\right) \\
b^* &= \underset{b}{\text{argmin}}\,\text{Var}[\bar{Y}(b)] = \frac{\sigma_Y}{\sigma_X}\rho_{XY} = \frac{\text{Cov}[X,Y]}{\text{Var}[X]} \\
\text{Var}[\bar{Y}(b^*)] &= \frac{\sigma_Y^2}{N}(1-\rho_{XY}^2) = \text{Var}[\bar{Y}](1-\rho_{XY}^2)
\end{aligned}
$$

In practice, we use the sample covariance and the sample variance to replace the associated covariance and variance. So we have $\widehat{b}_N$ and $\widehat{\rho}_{XY}$ as follow:

$$
\widehat{b}_N = \frac{\sum_{i=1}^{N}\left(X_i-\bar{X}\right)\left(Y_i-\bar{Y}\right)}{\sum_{i=1}^{N}\left(X_i-\bar{X}\right)^2},\quad \widehat{\rho}_{XY} = \frac{\hat{b}_N}{\hat{\sigma}_Y\hat{\sigma}_X} = \frac{\sum_{i=1}^{n}\left(X_i-\bar{X}\right)\left(Y_i-\bar{Y}\right)}{\sqrt{\sum_{i=1}^{n}\left(Y_i-\bar{Y}\right)^2\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^2}}
$$

The algorithm is implemented in "ExoticBSEngineCtrl", "PathDependentBarrier","Statistics" etc.

4 Results and Discussions

4.1 Barrier Option Pricing Result

By using the above initial parameters, the testing program price the barrier option using analytical formula, Monte Carlo method, antithetic variance reduction method, and control variance reduction method. The result is as follows:

Table 1: Barrier Option with Different Npaths. Spot price $S_0 = 100$, risk free interest rate $r = 0.05$, dividend rate $d = 0.02$, annualized volatility $\sigma = 0.3$, strike $K = 110$, expire in one year $T = 1$, number of observation dates steps = 252, barrier $B = 120$.

	Analytical	MC		AV		CV	
nPaths	μ	μ	σ	μ	σ	μ	σ
10,000	0.0507592	0.0824091	0.00660106	0.0751414	0.00603297	0.0822486	0.00698471
100,000	0.0507592	0.0714281	0.00189828	0.0695315	0.00187706	0.0701599	0.00197691

Table 2: Barrier Option at Different Barrier. Spot price $S_0 = 100$, risk free interest rate $r = 0.05$, dividend rate $d = 0.02$, annualized volatility $\sigma = 0.3$, strike $K = 110$, expire in one year $T = 1$, number of observation dates steps = 252, number of simulated path nPaths = $10,000$

	Analytical	MC		AV		CV	
B	μ	μ	σ	μ	σ	μ	σ
120	0.0507592	0.0824091	0.00660106	0.0751414	0.00603297	0.0822486	0.00698471
200	7.44474	7.54425	0.146756	7.37878	0.146577	7.55196	0.104141
1000	9.05705	9.13188	0.186443	8.9043	0.184485	9.05705	5.6757e-15

From Table 1 we observed that the Monte Carlo and variance reduction methods are close to the analytical formula but there exist a discrepancy which is caused by narrow bandwidth between barrier and strike and the lack of simulated paths (only 10,000). As the path number increase to 100,000, the discrepancy slightly alleviated, average price becomes to 0.0714281. As the barrier raised in Table 2 from 120 to 200 and 1000, the discrepancy almost eliminated. This is because previous narrow bandwidth between barrier and strike result in lack of non-zero samples. Increase the widthband allows more meaningful samples thus improve the MC accuracy.

From Table 1, the Antithetic Variate always achieves a higher accuracy and lower standard error as expected. The performance for Control Variate method is slightly better than MC for small number of path. It becomes significantly better than pure MC method when the sample size increases to 100,000. However, the CV method always had the highest Variance, which is, at first glance, quit contrary to our intuition. The reason is still caused by the narrow band between the S_0 and B, which leads to a frequent strike out of Barrier option, which in turn leads to the weak negative correlation between X, Y and even fewer meaningful samples (most of the Y are zero) to characterize the true relationship. As a result the estimation for $\hat{b}$ is severely biased and not statistical significant. Because of this, the variance of $\bar{Y}(b)$ is not only unoptimized but also introducing additional disturbance from the control X.

From Table 2, we confirmed that when the Barrier becomes larger (i.e, the bandwidth between initial price and the knock out price becomes large), the control variate would achieve better performance. The control variate had the lowest variance in the cases of 200 and 1000 when the Barrier is large. This is because the correlation between control X and Y is high when barrier is large (less likily to knock out). The estimation for the correlation and the optimal control coefficients for Barrier equals to 120, 200, 100 shows in Table 3. To the extreme case, when correlation equals to 1, then Barrier is too high to be strike out, the Barrier option is in fact the vanila option.

Table 3: Optimal Control Coefficients at Different Barrier. Spot price $S_0 = 100$, risk free interest rate $r = 0.05$, dividend rate $d = 0.02$, annualized volatility $\sigma = 0.3$, strike $K = 110$, expire in one year $T = 1$, number of observation dates steps = 252, number of simulated path nPaths = $10,000$.

Barrier	ρ	$\widehat{b}$
120	-0.0247835	-0.000956555
200	0.506527	0.749849
1000	1	1

4.2 Barrier Option at Different Strikes

The barrier option is priced at different strikes and the result is summarized in Table 4. It shows that the MC valuation of the deep in the money call option (with strike significantly less than current price) gives the better performance especially control variates. This because larger bandwidth between strike and barrier and strong negative correlation between vanilla and barrier option. The larger bandwidth allows more non-zero samples which improves the MC valuation in general. Given a lower strike, a sample path with higher price will have higher vanilla valuation while having higher chance to be knocked out and end up with 0 barrier valuation. Table 6 shows that such negative correlation decreases as strike increases. The effectiveness of a control variates depends on the correlation between variates of interest and the control variates: ρ_{XY}. Therefore, it should not be supprised that control variates gives the best performance when strike is smallest. Antithetic variates always perform better than pure MC method in Table 4.

Table 4: Barrier Option at Different Strikes. Spot price $S_0 = 100$, risk free interest rate $r = 0.05$, dividend rate $d = 0.02$, annualized volatility $\sigma = 0.3$, expire in one year $T = 1$, number of observation dates steps = 252, number of simulated path nPaths = $10,000$, barrier $B = 120$.

	Analytical		MC		AV		CV	
K	Vanilla	Barrier	μ	σ	μ	σ	μ	σ
20	78.9953	27.7349	29.5966	0.100075	29.4557	0.0999527	29.5717	0.0922646
80	24.7833	3.3827	3.80596	0.0246429	3.76365	0.0245253	3.79384	0.0258043
100	13.0203	0.422803	0.516688	0.00694228	0.511714	0.00688454	0.512052	0.00727285
110	9.05705	0.0507592	0.0714281	0.00189828	0.0695315	0.00187706	0.0701599	0.00197691

4.3 Speed of Variance Reduction Methods

The antithetic variate achieve the fastest speed in each and every cases. The speed for Pure MC and control variate are approximately same. The reason behind is antithetic only need to generate

Table 5: Optimal Control Coefficients at Different Strikes. Spot price $S_0 = 100$, risk free interest rate $r = 0.05$, dividend rate $d = 0.02$, annualized volatility $\sigma = 0.3$, expire in one year $T = 1$, number of observation dates steps = 252, number of simulated path nPaths = $10,000$, barrier $B = 120$.

	CV	
K	ρ	b
20	-0.483671	-0.502639
80	-0.0910105	-0.0269601
100	-0.0344363	-0.00370545
110	-0.0252882	-0.000892074

half the number of random Gaussians. Theoretically the speed for pure MC should be slightly faster than control variate, since control variate have addition computations to implement such as estimation for the optimal control coefficient b^*, but these computation time is negligiblely small compared to the time spend on generating random variables and construct Euler Scheme pathes. Therefore they have the same order, while antithetic variate could significantly reduce the time. However, if the aim is to achieve the same level of accuracy, then the reduction variance method (control variate and antithetic variate) outperform the pure MC for sure. Since they could achieve the same level of variance with fewer path generated thus less time to take.

Table 6: Computational Speed. Spot price $S_0 = 100$, risk free interest rate $r = 0.05$, dividend rate $d = 0.02$, annualized volatility $\sigma = 0.3$, expire in one year $T = 1$, number of observation dates steps = 252, number of simulated path nPaths = $10,000$, barrier $B = 120$.

	MC	AV	CV
K	Time	Time	Time
20	2.237	1.792	1.858
80	2.116	1.765	1.85
100	2.369	1.886	1.778
110	2.399	1.742	1.753

4.4 Convergence Diagnostics

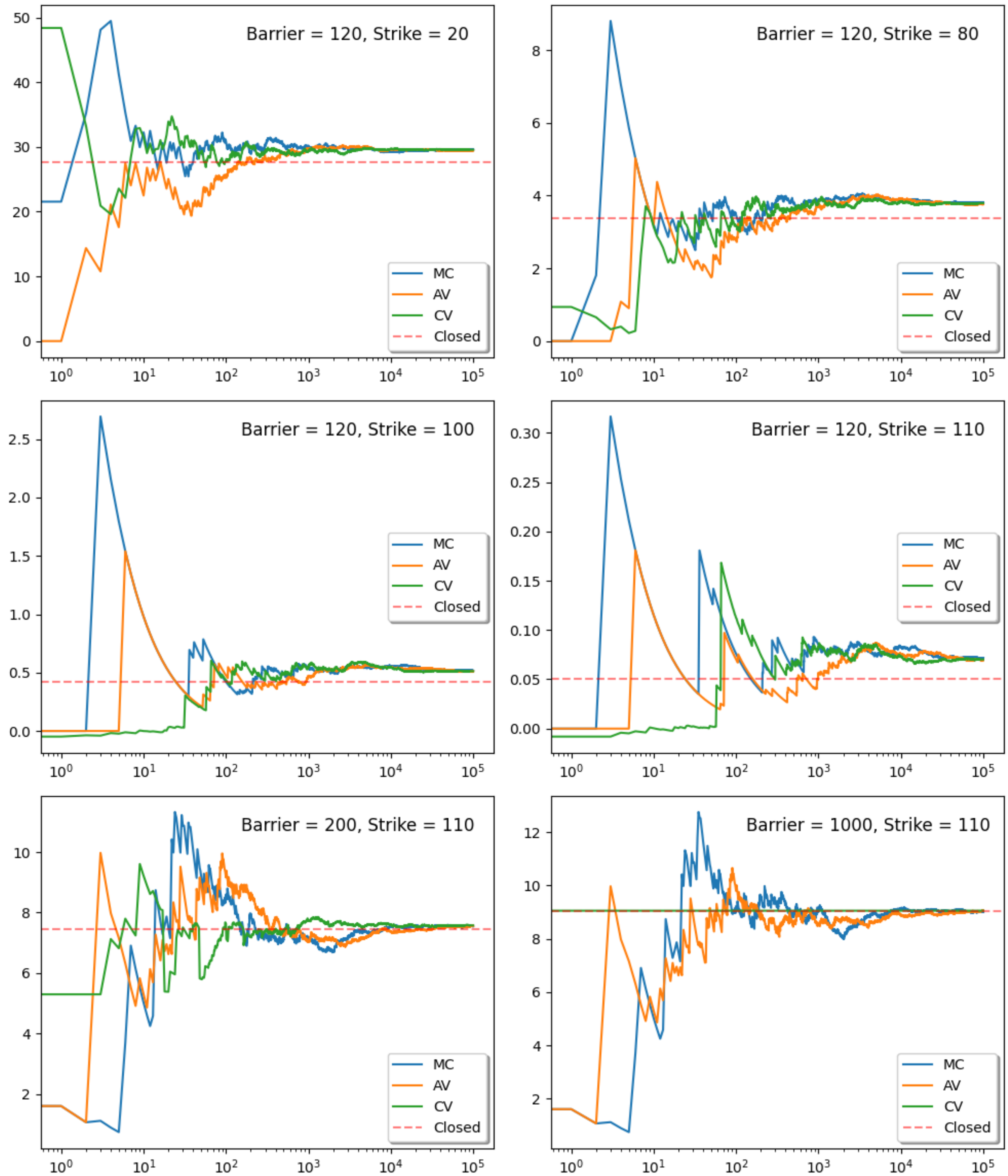

Figure 1: Convergence Diagnostics for Mean

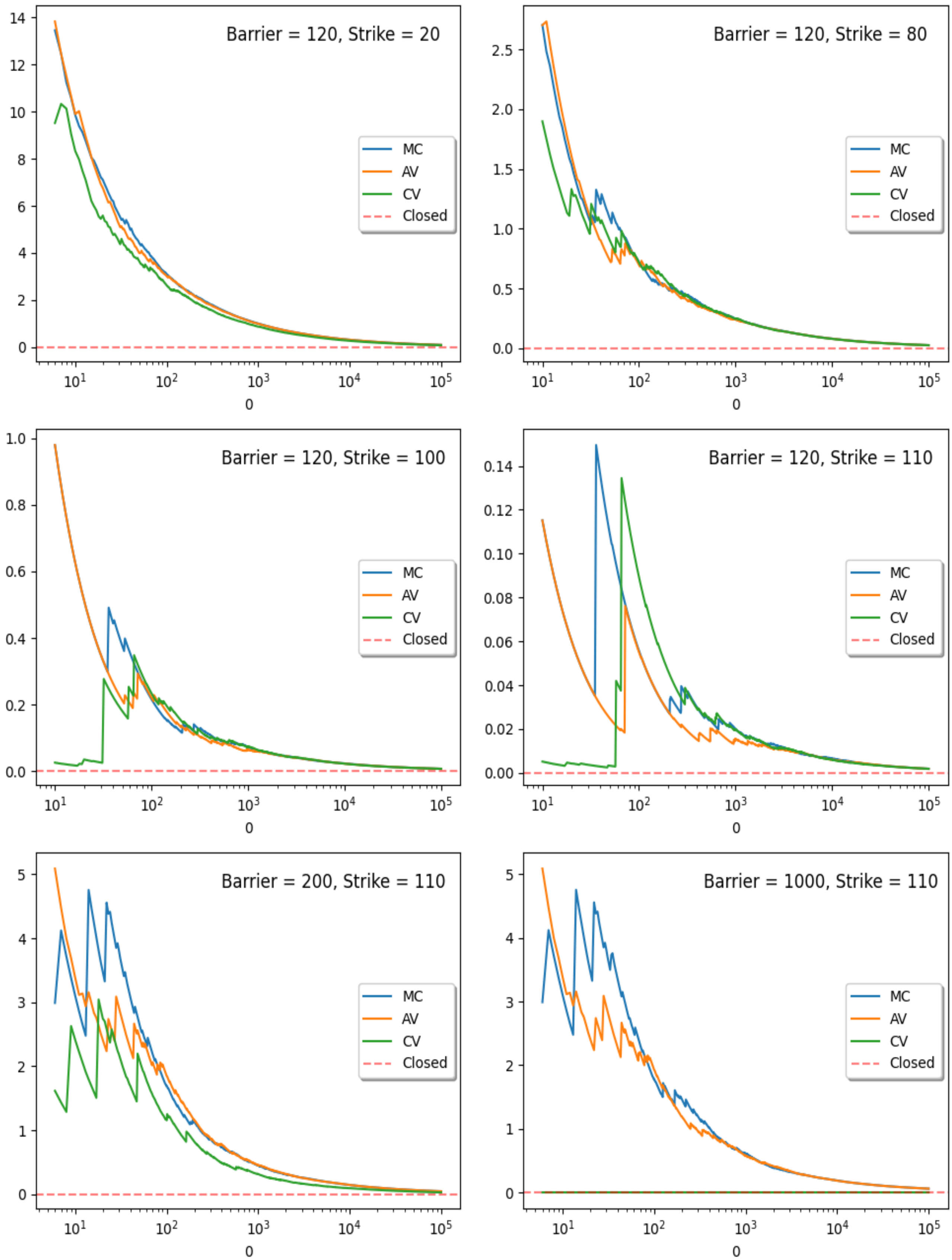

Figure 2: Convergence Diagnostics for Standard Deviation

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