NC STATE UNIVERSITY

Reciprocating Engine Design

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We have neither given nor received unauthorized aid on this assignment.

JL
SM
DO

- **Abstract:**

The objective of this project was to complete a vibrational analysis on an engine with three different systems: engine supported by a beam, engine supported by a beam with an undamped dynamic vibration absorber, and engine supported by a beam with a damped dynamic vibration absorber. This vibrational analysis involved setting up all necessary diagrams and equations of motion to find the steady state response of each system (amplitude/phase). With the system set up, our team was tasked with finding the best constants for the mass, damping, and spring of the dynamic vibration absorber. In the scope of this project, our objective was significant as it will allow for reduced vibration in a system to expand its lifespan, improve economic value and mechanical performance. In the scope of this course, the objective allowed us to build upon our knowledge of single and two degrees of freedom systems by working with a real world example that is likely to be a project of ours in the workfield one day. As we progress toward graduation, projects like these prepare us for our senior design work and future careers. Techniques used throughout this project include the following: problem formulation, consisting of modeling, free body diagrams, equations of motion with periodic force, and derivation using phase angles, amplitudes, and complex numbers, problem simulation consisting of matlab coding, amplitude interpretation, and economic analysis, and lastly, report writing. Assumptions made in the project include no air resistance, no friction, no fluctuation in mass, and the surrounding air damping is equivalent to 1 lbf-s/ft. Significant results found from our project completion include the spring constant of the beam, damping constants, and masses of the three different systems.

- Original Design: Spring constant of beam (k_1) is 14400 lbf/ft.
- Undamped Dynamic Vibration Absorber: Spring constant (k_2) is 1920.1 lbf/ft, mass (m_2) is 0.6211 slugs.
- Damped Dynamic Vibration Absorber: Spring constant (k_2) is 1920.1 lbf/ft, mass (m_2) is 0.6211 slugs, damping constant (c_2) is 5 lbf-s/ft.

- **Problem Formulation:**

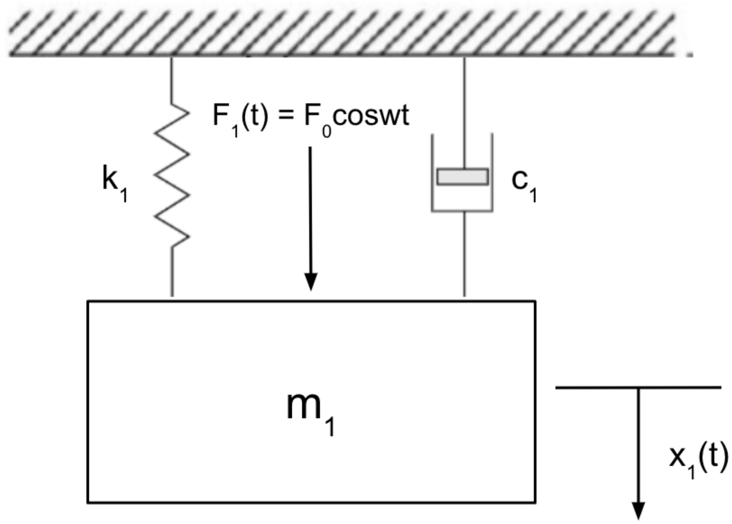

Figure 1: Model of System

Above is a model of our system without the dynamic vibration absorber. Originally, mass one, the engine, was mounted on top of a fixed-fixed beam in the system. We converted this beam into spring and damping elements. The spring constant could then be calculated using the dimensions of the beam, along with Young's Modulus. The damping constant is equivalent to the surrounding air damping constant. Our inertia reference was set in the direction of the harmonic force, as this allows our spring and damping force to act opposite the harmonic force. Lastly, we inverted the positioning of the objects in the model for ease of analysis.

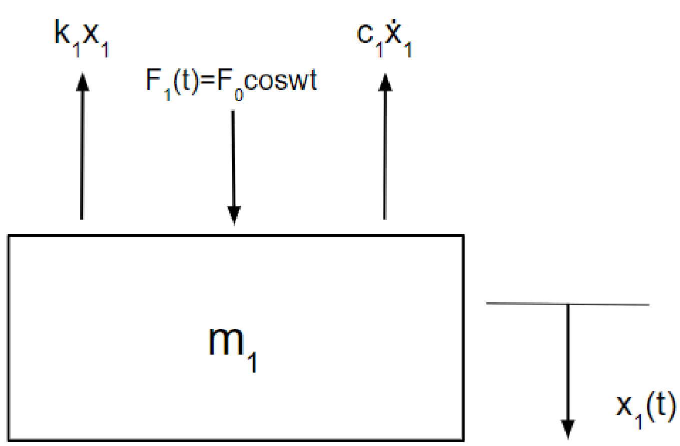

Figure 2: Free Body Diagram of System

The variables in the free body diagram above are as follows:

- m_1: Mass of the engine. Equivalent to 4.6584 slugs.
- $F_1(t)$: Harmonic force acting on the system. Equivalent to 50cosωt.
- k_1: Spring constant from the beam. Equivalent to 14400 lbf/ft.
- c_1: Damping constant from the beam, caused by surrounding air. Equivalent to 1 lbf-s/ft.
- x_1: Displacement of the system.
- $\dot{x}_1$: Velocity of the system.

The following is the equation of motion for this system:

$$F_1(t) - c_1\dot{x}_1 - k_1 x_1 = m_1\ddot{x}_1$$

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1 x_1 = F_1(t)$$

The following is the derivation of steady state response of the single degree of freedom system:

$$x_{p_1}(t) = \left|\tilde{X}_1\right| cos(\omega t - \phi)$$

$$\left|\tilde{X}_1\right| [(k_1 - m_1\omega^2)cos(\omega t - \phi) - (\omega sin(\omega t - \phi))] = F_0 cos(\omega t)$$

$$cos(\omega t - \phi) = cos(\omega t)cos(\phi) + sin(\omega t)sin(\phi)$$

$$sin(\omega t - \phi) = sin(\omega t)cos(\phi) - cos(\omega t)sin(\phi)$$

$$\left|\tilde{X}_1\right| [(k_1 - m_1\omega^2)cos(\phi) + c\omega sin(\phi)] = F_0$$

$$\left|\tilde{X}_1\right| [(k_1 - m_1\omega^2)sin(\phi) - c\omega cos(\phi)] = 0$$

$$\left|\tilde{X}_1\right| = \frac{F_0}{\sqrt{(k_1 - m_1\omega^2)^2 + {c_1}^2\omega^2}}$$

$$\left|\tilde{X}_1\right| = \frac{50}{\sqrt{[14400 - 150(55.6)^2]^2 + 1^2(55.6^2)}}$$

$$\left|\tilde{X}_1\right| = .00011128$$

$$\phi = tan^{-1}(\frac{c_1\omega}{k_1 - m_1\omega^2})$$

$$\phi = tan^{-1}(\frac{1(55.6)}{14400 - 150(55.6)^2}) = -.0071°$$

$$x_p(t) = .00011128\cos(55.6t + .0071)$$

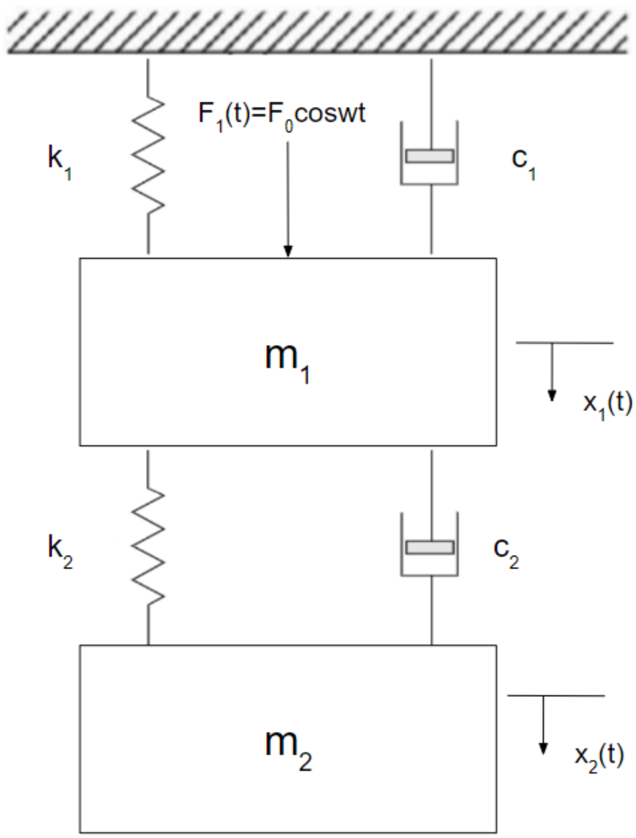

Figure 3: Model of System with Damped Absorber

Above is a model of our system with a damped dynamic vibration absorber. Similar to the first part, we converted the beam into spring and damping constants acting on the first mass, the engine, which countered the harmonic force. Below our first mass, a second mass was added, the dynamic vibration absorber, with both a spring and damping constant. This model is a two degree of freedom system.

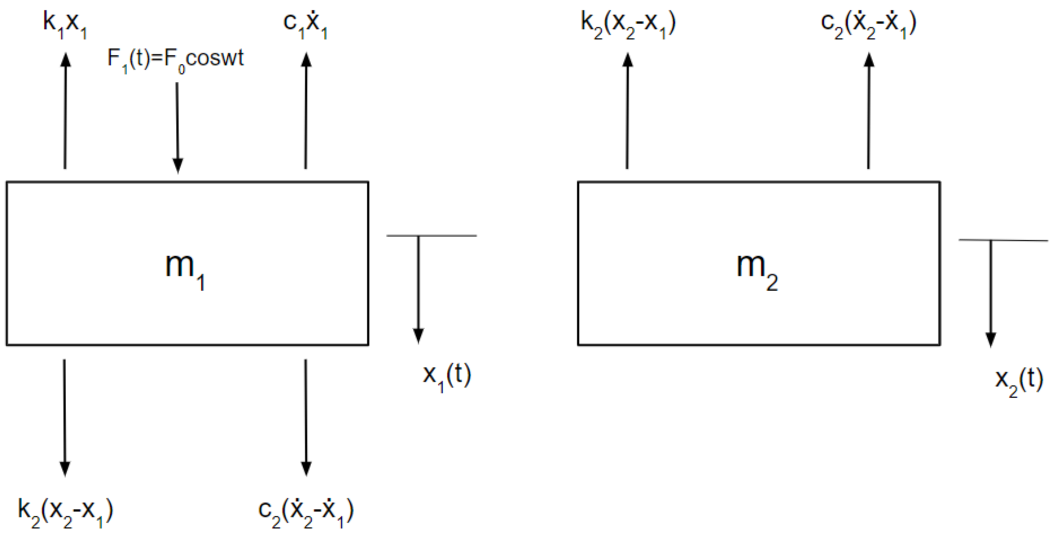

Figure 4: Free Body Diagram of System with Absorber

The variables in the free body diagram above are as follows:

- m_1: Mass of the engine. Equivalent to 4.6584 slugs.
- m_2: Mass of the dynamic vibration absorber.
- $F_1(t)$: Harmonic force acting on the system. Equivalent to 50cosωt.
- k_1: Spring constant from the beam. Equivalent to 14400 lbf/ft.
- c_1: Damping constant from the beam, caused by surrounding air. Equivalent to 1 lbf-s/ft.
- k_2: Spring constant from the dynamic vibration absorber.
- c_2: Damping constant from the dynamic vibration absorber.
- x_1: Displacement of the first mass.
- $\dot{x}_1$: Velocity of the first mass.
- x_2: Displacement of the second mass.
- $\dot{x}_2$: Velocity of the second mass.

The following are the equations of motion for this system:

$$Mass1 : m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 - c_2(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1) = F_0 cos\omega t$$

$$Mass2 : m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = 0$$

The following is the derivation of the steady state response of the two degree of freedom system:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} F_1(t) \\ 0 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix} e^{i\omega t}$$

$$\{x_p(t)\} = \{\widetilde{x}\} e^{i\omega t}$$

$$Impedance - Matrix : \left[-\omega^2 [m] + [k] + i\omega [c]\right] = [Z]$$

$$[Z] \{\widetilde{x}\} = \{F\}$$

$$\begin{bmatrix} -m_1\omega^2 + k_2 + i\omega c_2 & -k_2 - i\omega c_2 \\ -k_2 - i\omega c_2 & -m_2\omega^2 + k_2 + i\omega c_2 \end{bmatrix} \begin{Bmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

$$\{\widetilde{x}\} = [Z]^{-1} \{F\} = \frac{1}{det|Z|} \begin{bmatrix} Z_{22} & -Z_{12} \\ -Z_{21} & Z_{11} \end{bmatrix} \begin{Bmatrix} F_o \\ 0 \end{Bmatrix}$$

$$|\widetilde{x}_1| = \frac{Z_{22}F_0}{Z_{11}Z_{22} - Z_{12}^2} = \frac{-m_2\omega^2 + k_2 + i\omega c_2(F_0)}{(-m_1\omega^2 + k_1 + k_2 + i\omega(c_1 + c_2))(-m_2\omega^2 + k_2 + i\omega c_2) - (-k_2 - i\omega c_2)^2}$$

$$|\widetilde{x}_2| = \frac{-Z_{21}F_0}{Z_{11}Z_{22} - Z_{12}^2} = \frac{-k_2 - i\omega c_2(F_0)}{(-m_1\omega^2 + k_1 + k_2 + i\omega(c_1 + c_2))(-m_2\omega^2 + k_2 + i\omega c_2) - (-k_2 - i\omega c_2)^2}$$

$$x_{p1}(t) = |\widetilde{x}_1| e^{i\omega t}, x_{p2}(t) = |\widetilde{x}_2| e^{i\omega t}$$

- **Simulations and Results:**

Table 1: Properties of Original System

Mass, m (slugs)	Damping Constant, c (lbf-s/ft)	Spring Constant, k (lbf/ft)	Natural Frequency, w_n (rad/sec)
4.6584	1	14400	55.6

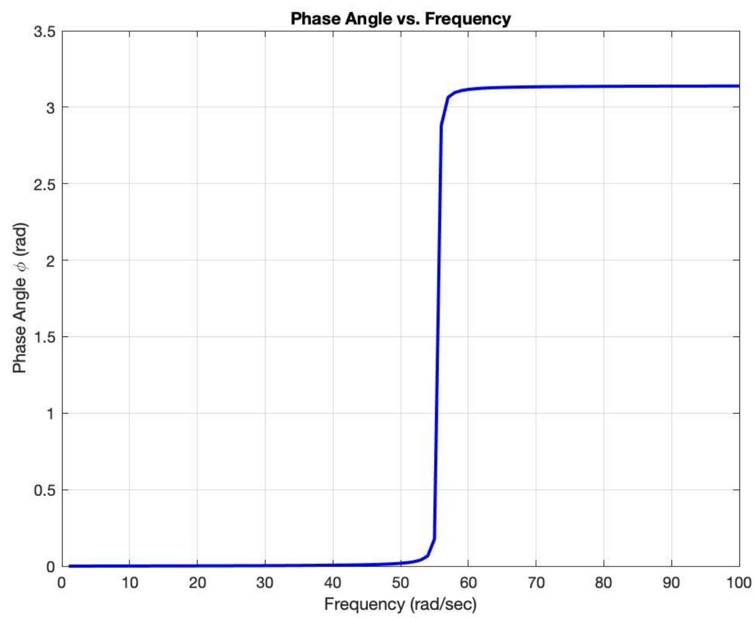

Figure 5: Frequency Spectrum (Phase) of Original System

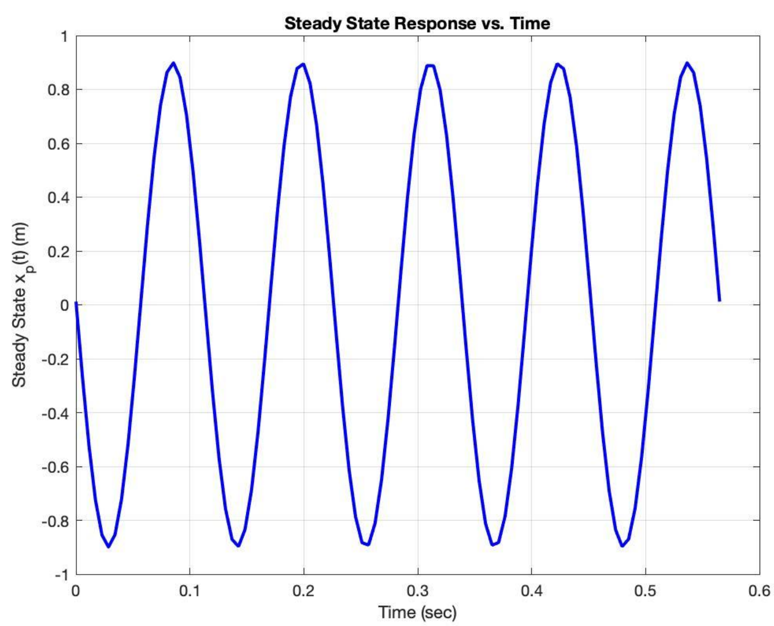

Figure 6: Steady State Motion

	Mass, m (slugs)	Damping Constant, c (lbf-s/ft)	Spring Constant, k (lbf/ft)
System 2	m_1=4.6584 m_2=0.6211	c_1=1 c_2=0	K_1=14,400, K_2=1920.1
System 3	m_1=4.6584 m_2=0.6211	c_1=1 c_2=5	K_1=14,400, K_2=1920.1

Table 2: Properties of Newly Designed System

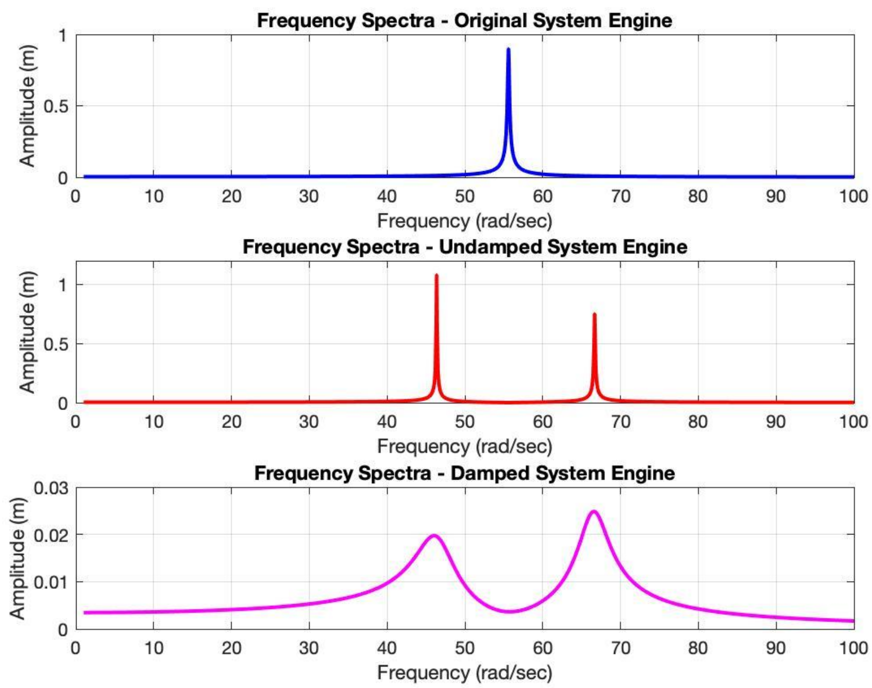

Figure 7: Frequency Spectra (Amplitude) for Engine - Original System, Undamped, Damped

The value for m_2 was chosen by picking an assortment of reasonable values that were tested out to see which would produce the most favorable results, the largest being a reduction in overall vibration for the system. k_2 was calculated using the formula, $k_2 = m_2(\omega)^2$, with ω being the operational ω for the engine. The chosen values lead to the smallest amplitude of vibration within the system, bringing extra value to the customer as the system will last longer with less vibration. All values are to a reasonable, obtainable degree, with no unrealistic outliers, providing us with a technically feasible system. With a longer lasting system, the economic viability increases as spending will decrease when parts last longer.

The same k_2 and m_2 values were used for the damped DVA. For c_2, a value that was between 0 and 50% of c_{cr} was chosen. As the c value was increased, the amplitude of the system decreased which reduces the vibration in the system. Therefore, c_2 was chosen so that it reduced the amplitudes of the graph without overdamping the system. All constants were found to reasonable degrees, following all laws of vibrations, making our system technically feasible. As stated earlier, reducing the amplitude of vibration allows for higher utility of a system. As vibration is reduced, a system will last longer, therefore increasing customer value and economic viability, as a longer lasting system decreases long term costs.

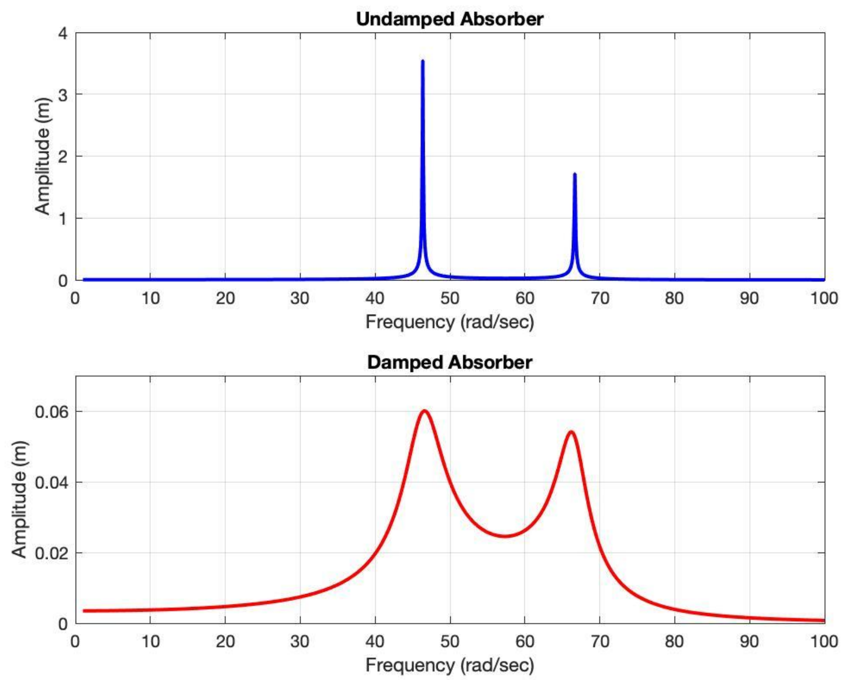

Figure 8: Frequency Spectra (Amplitude) for Absorber - Undamped and Damped

Analyzing the amplitude spectrum of the undamped absorber, we are able to observe the overall system stability and optimize based on the resonance peaks. With the goal of reducing vibration in mind, we can reduce these peaks depending on our chosen k_2 and m_2 to determine what would be most viable for our system. The dips and peaks throughout the system allow us to understand the performance of the dynamic vibration absorber and reduce the response.

Similarly, analyzing the amplitude spectrum of the damped absorber, we are able to observe the overall system stability and optimize based on the resonance peaks. Choosing the appropriate m_2, c_2, and k_2 allows us to reduce these peaks and dips and improve the response of the system with the use of the dynamic vibration absorber. The presence of damping, when compared to the undamped system, increases the period of the waves of the system.

- **Conclusion:**

The purpose of this project was to design a dynamic vibration absorber (DVA) that minimizes the effects of an engine's vibration on a support beam. A multitude of techniques were used throughout this project, beginning with problem formulation, including identifying our model, its free body diagram, and its equation of motion. Following this, we derived the steady-state response of the system with no DVA, an undamped DVA, and a damped DVA. Part one was accomplished using the method of amplitude and phase angles. Parts two and three took advantage of the matrix method, including the impedance matrix in order to derive the amplitude and steady-state response. Assumptions made in the project include no air resistance, no friction, no fluctuation in mass, and the value of the surrounding air damping. The final design choices reduced the vibration, in the most economical and practical sense, which can be seen in the graphs as the amplitude is smaller than the original design without the DVA.

Extending our work into the future, our next steps would be to continue following the engineering method. We have successfully defined a problem, conducted research, and derived/conceptualized a solution to our issue. Our next step would be to create a working prototype. With this prototype, extensive testing would need to be conducted to ensure repeatability of our results, and to ensure safety for the public market. After the prototype has been mastered, our final product would be manufactured and ready for widespread market use. Over time, any issues that may arise could be addressed, such as improving the lifespan of our materials or continuing to find new, effective ways to solve the constant influx of changes and discoveries within the engineering community.

- **Appendix:**

```
%% MAE 315 Project 2
% SM, JL, DO

%% Problem Setup

%% Given Values

m1 = 150; %lbm
l = 100; %in
a = 20; %in
t = 0.5; %in
E = 30e6; %psi
c1 = 1; %lbf-s/ft
F0 = 50; %lbf
%wop = 50:60; %rad/s (range)

%% Conversions

m1 = m1/32.2; %lbm to slugs
l = l/12; %in to ft
a = a/12; %in to ft
t = t/12; %in to ft
E = E*144; %psi to lbf/ft^2

%% Part I #1


% Find k
k1 = 16*E*a*(t^3)/(l^3); %lbf/ft

% Frequency Response

wop = 1:.001:100;
xtilde1 = zeros(1,100);
x1 = 1:.001:100;
counter = 1;

for i = 1:.001:100

    xtilde1(counter) = F0./sqrt( (k1-m1*(i.^2)).^2+(c1*i).^2 );

    counter = counter + 1;

end

% Phase Angle

phi1 = zeros(1, 100);
counter = 1;
```

```
for i = 1:100

  phi1(counter) = atan( (c1*i)/(k1-m1*(i.^2)) );

  if (k1-m1*(i.^2)) < 0
    phi1(i) = phi1(i) + pi;
  end
  counter = counter + 1;

end

%% Plots

% Phase Angle

x1_phi = 1:100;

figure(1);
plot(x1_phi, phi1, 'b','LineWidth',2,'MarkerSize', 15);
xlabel('Frequency (rad/sec)');
ylabel('Phase Angle {\phi} (rad)');
title('Phase Angle vs. Frequency');
grid on;

%% Part I #2

% Given Values
w = 55.6; %rad/s
time = linspace(0,10*pi/w,100); %sec

% Steady-State Response
xtilde1_state = F0/sqrt( (k1-m1*(w^2))^2+(c1*w)^2 );
phi1_state = atan( (c1*w)/(k1-m1*(w^2)) );
xp1 = xtilde1_state * exp(1i*(w*time-phi1_state));


%% Plot

% Steady State

figure(2);
plot(time, xp1, 'b','LineWidth',2,'MarkerSize', 15);
xlabel('Time (sec)');
ylabel('Steady State x_{p}(t) (m)');
title('Steady State Response vs. Time');
grid on;

%% Part II

% Givens

w = 55.6; %rad/sec
m2 = 20/32.2; %lbm to slugs
k2 = m2*(w^2); %lbf/ft
```

```
% Impedance Matrix

%Z11 =  -m1*(w^2) + k1 + k2 + 1i*w*c1;
%Z12 = -k2;
%Z21 = Z12;
%Z22 = -m2*(w^2) + k2;

% Xtilde

%xtilde21 = ( Z22*F0 ) / ( Z11*Z22 - (Z12^2) );
%xtilde22 = ( -Z21*F0 ) / ( Z11*Z22 - (Z12^2) );


wop = 1:.001:100;
xtilde21 = zeros(1,100);
xtilde22 = zeros(1,100);
x1 = 1:.001:100;
counter = 1;

for i = 1:.001:100

   Z11 =  -m1*(i.^2) + k1 + k2 + 1i*i*c1;
   Z12 = -k2;
   Z21 = Z12;
   Z22 = -m2*(i.^2) + k2;

   xtilde21(counter) = ( Z22*F0 ) / ( Z11*Z22 - (Z12^2) );
   xtilde22(counter) = ( -Z21*F0 ) / ( Z11*Z22 - (Z12^2) );

   counter = counter +1;

end

xtilde21 = abs(xtilde21);
xtilde22 = abs(xtilde22);

%% Part III

% Givens
c2 = 5; %lbf-s/ft

% Impedance Matrix

%Z11 =  -m1*(w^2) + k1 + k2 + 1i*w*(c1+c2);
%Z12 = -k2 - 1i*c2;
%Z21 = Z12;
%Z22 = -m2*(w^2) + k2 + 1i*w*c2;

% Xtilde

%xtilde31 = ( Z22*F0 ) / ( Z11*Z22 - (Z12^2) );
%xtilde32 = ( -Z21*F0 ) / ( Z11*Z22 - (Z12^2) );
```

```
wop = 1:.001:100;
xtilde31 = zeros(1,100);
xtilde32 = zeros(1,100);
x1 = 1:.001:100;
counter = 1;

for i = 1:.001:100

   Z11 =  -m1*(i.^2) + k1 + k2 + 1i*i*(c1+c2);
   Z12 = -k2 - 1i*c2;
   Z21 = Z12;
   Z22 = -m2*(i.^2) + k2 + 1i*i*c2;

   xtilde31(counter) = ( Z22*F0 ) / ( Z11*Z22 - (Z12^2) );
   xtilde32(counter) = ( -Z21*F0 ) / ( Z11*Z22 - (Z12^2) );

   counter = counter + 1;
end

xtilde31 = abs(xtilde31);
xtilde32 = abs(xtilde32);

%% Frequency Spectra of Systems (Fig.7)

figure(3);

subplot(3,1,1)
plot(x1,xtilde1, 'b','LineWidth',2,'MarkerSize', 15);
xlabel('Frequency (rad/sec)');
ylabel('Amplitude (m)');
title('Frequency Spectra - Original System Engine');
ylim([0,1]);
grid on;

subplot(3,1,2)
plot(x1,xtilde21, 'r','LineWidth',2,'MarkerSize', 15);
xlabel('Frequency (rad/sec)');
ylabel('Amplitude (m)');
title('Frequency Spectra - Undamped System Engine');
ylim([0,1.2]);
grid on;

subplot(3,1,3)
plot(x1,xtilde31, 'm','LineWidth',2,'MarkerSize', 15);
xlabel('Frequency (rad/sec)');
ylabel('Amplitude (m)');
title('Frequency Spectra - Damped System Engine');
ylim([0,0.03]);
grid on;

%% System Amps Separate

figure(4);
plot(x1,xtilde1, 'b','LineWidth',2,'MarkerSize', 15);
```

```
hold on;
xlabel('Frequency (rad/sec)');
ylabel('Amplitude (m)');
title('Original System Frequency Spectra');
ylim([0,1.5]);
grid on;

figure(5);
plot(x1,xtilde21, 'b','LineWidth',2,'MarkerSize', 15);
xlabel('Frequency (rad/sec)');
ylabel('Amplitude (m)');
title('Undamped System Frequency Spectra');
ylim([0,1.5]);
grid on;

figure(6);
plot(x1,xtilde31, 'b','LineWidth',2,'MarkerSize', 15);
xlabel('Frequency (rad/sec)');
ylabel('Amplitude (m)');
title('Damped System Frequency Spectra');
ylim([0,1.5]);
grid on;

%% Damped & Un-damped Plots (Fig.8)

figure(7);

subplot(2,1,1)
plot(x1,xtilde22, 'b','LineWidth',2,'MarkerSize', 15);
xlabel('Frequency (rad/sec)');
ylabel('Amplitude (m)');
title('Undamped Absorber');
ylim([0,4]);
grid on;

subplot(2,1,2)
plot(x1,xtilde32, 'r','LineWidth',2,'MarkerSize', 15);
xlabel('Frequency (rad/sec)');
ylabel('Amplitude (m)');
title('Damped Absorber');
ylim([0,0.07]);
grid on;

%% Absorber Amps Separate

figure(8);
plot(x1,xtilde22, 'b','LineWidth',2,'MarkerSize', 15);
xlabel('Frequency (rad/sec)');
ylabel('Amplitude (m)');
title('Undamped Absorber Frequency Spectra');
ylim([0,1.5]);
grid on;

figure(9);
```

```
plot(x1,xtilde32, 'b','LineWidth',2,'MarkerSize', 15);
xlabel('Frequency (rad/sec)');
ylabel('Amplitude (m)');
title('Damped Absorber Frequency Spectra');
ylim([0,1.5]);
grid on;
```