Question - 3

Let us start by stating the compactness theorem: The compactness theorem states that if we have a collections of sentences that is unsatisfiable (i.e they have no model) then there has to exist a finite subset of that collection of sentences which is also also unsatisfiable. This directly leads to the fact that any infinite collection of unsatifiable formulas has a finite subset which is also unsatisfiable. In logical notation the Compactness Theorem can be written out as: $X \models$ then $X' \models$ for some finite subset X' of X. We can prove compactness using the ideas of consistency and inconsistency since being satisfiable means the same thing as being consistent and being unsatisfiable means the same thing as being inconsistent. Thus, we try to show that if a tree for an infinite collection of sentences closes then it only uses a finite number of sentences to get to this closure. Logically we prove $X \vdash$ then $X' \vdash$ where X' is a finite subset of X (X may or may not be finite). If we start with X and construct a tree and a tree for X closes, then every branch in the tree would need to close. This would mean that every branch in the tree is finitely long. Moreover, we will only ever have a finite number of branches (no matter how large) since every rule for decomposition of formulas in a tree leads to either one or two branches (one in the case of conjunctions and two for disjunctions for example). Thus, since we are guaranteed to have a finite number of finite length branches we are guaranteed to have taken only a finite amount of steps during the construction of our proof in order to get to the point where our tree closes. This means that we've only used a finite amount of formulas from the potentially infinite amount of formulas that we could have used. We can thus, construct X' by putting all these formulas into it. This proves, that if $X \vdash$ **then** $X' \vdash$. Since inconsistency and unsatisfiability are essentially the same ideas because of soundness and completeness we've also proved $X \models$ then $X' \models$ which is the compactness theorem.

It follows from the compactness theorem that an infinite set of sentences is satisfiable if every finite subset of it is satisfiable (because unsatisfiability would need to come from a finite set and the lack of one leads of satisfiability). Thus, as a consequence of the compactness theorem we can create non-standard models like modelling infinitely small numbers (numbers that are > 0 but smaller than every whole fraction) which have applications in physics and calculus. We can model infinitesimal numbers as follows: Lets say we have a model M for the theory of real numbers, with X as the set of all sentences in model and the domain being all the numbers on the number line. We can construct another model M' such that M' is made up of M but also contains the set of formulas Y such that Y contains $0 < c, c < 1, c < \frac{1}{2}, ...$ where c is a constant. To prove that this model is satisfiable we need to show that M' is satisfiable by showing that $X \bigcup Y$ is satisfiable. But $X \bigcup Y$ is an infinite set of sentences since Y is always infinite. But we can prove it's satisfiability by proving the satisifiability of every finite subset of this model. If we take any subset of X it is satisfiable by definition and we can satisfy a subset of Y by choosing $c = \frac{1}{n+1}$ where n was the largest denominator in the subset of Y chosen. Thus, we can satisfy every finite subset and so the infinite set allowing us to model c as an infinitesimal.