Q1 Given premises: $s \supset p,\ q \supset (p \supset r)$

Construct Hibert Proof for: $q \supset (s \supset r)$

The conditional Proof fact says that there is a proof from X to $A \supset B$ if and only if there is a proof from X, A to B.

Thus, to show that:

$$s \supset p,\ q \supset (p \supset r) \vdash q \supset (s \supset r) \quad \ldots\ (i)$$

We only need to show that:

$$s \supset p,\ q \supset (p \supset r),\ q \vdash (s \supset r) \quad \ldots\ (ii)$$

And to prove (ii) we only need to prove that:

$$s \supset p,\ q \supset (p \supset r),\ q,\ s \vdash r \quad \ldots\ (iii)$$

We draw from the above conclusions to get from (i) to (iii) using the conditional proof fact.

Thus, to show (i) we only need to show (iii)

Here we provide a Hilbert Proof for (iii) :

1.	$s \supset p$	{Premise}
2.	$q \supset (p \supset r)$	{Premise}
3.	q	{Premise}
4.	s	{Premise}
5.	$(p \supset r)$	{3,2 - (MP)}
6.	p	{4,1 - (MP)}
7.	r	{6,5 - (MP)}

Thus, by the conditional proof fact. Since, we've successfully proved (iii), we've also successfully proved (i).

In the last part of this question we will try to justify the conditional proof fact.

The conditional proof fact is a statement that contains an "if and only if". So we will need to prove 2 things:

(a) $X \vdash A \supset B$ then $X, A \vdash B$

(b) $X, A \vdash B$ then $X \vdash A \supset B$.

(a) If we already have a proof that starts with X and ends up with $A \supset B$. Such that:

$$\left.\begin{array}{c} X \\ \cdots \\ \cdots \\ \cdots \end{array}\right\} \text{Intermediate steps}$$

$$A \supset B$$

We can add the premise A to get:

$$\begin{array}{l} X \\ \boxed{A} \rightarrow \text{added in} \\ \cdots \\ \cdots \\ A \supset B \\ \underline{B} \quad \{MP\} \\ \hookrightarrow \text{result.} \end{array}$$

So we've got B from X, A.

(b) Now lets look at the proof that if $\mathcal{X}, A \vdash B$ then $\mathcal{X} \vdash A \supset B$.

Let the proof from $\mathcal{X}, A \vdash B$ contain n formulas $F_1, F_2, \ldots, F_n$.

$$F_1$$
$$F_2$$
$$\vdots$$
$$F_n$$

Now, each F_i where $1 \leq i \leq n-1$ is one of the following

(a) the formula A

(b) something in $\mathcal{X}$

(c) an axiom.

(d) follows from earlier formulas by MP.

F_n on the other hand is B.

Put $A \supset$ in front of each F_i such that we get

$$A \supset F_1$$
$$A \supset F_2$$
$$\vdots$$

$\vdots$

$A \supset F_n$

Now this new set of formulas is not a proof but it ends with what we want $A \supset B$. Now, all we need to do is fill in the gaps such that we can create a proof from this set of formulas

- If F_i is an axiom or in X then, then we can get $A \supset F_i$ from the axiom $F_i \supset (A \supset F_i)$ {an instance of weakening}. Now, since F_i {since axioms & premises are by definition true} we get $A \supset F_i$. Thus by adding a few extra lines we can get to $A \supset F_i$ if F_i is a part of X or an axiom.
- If F_i is A then we can show via Hilbert proofs that $A \supset A$. Thus we can fill in the proof for $A \supset A$.
- If F_i follows from F_j then $F_j \supset F_i$ but we already know that $A \supset F_j$.

We can get $A \supset F_i$ by using the distribution axiom $(A \supset F_j) \supset ((A \supset (F_j \supset F_i)) \supset (A \supset F_i))$. Now we can use MP to get $((A \supset (F_j \supset F_i)) \supset (A \supset F_i))$.
Since $(F_j \supset F_i)$ is true $A \supset (F_j \supset F_i)$ is an instance of weakening leading us to $(A \supset F_i)$.

Thus, here by filling in the holes after prefixing the F_is by $A \supset$ we can get a proof from X to $A \supset B$ as shown.

Thus since we've proved both directions here, we've successfully justified the conditional proof fact.

Q2

1. The set of all tautologies in a language made out of the atomic formulas p & q formed using the connectives $\neg$ and $\vee$.

Ans. Countably Infinite.

Justification: The number of tautologies will be infinite since we can make infinitely many formulas by suffixing any formula in the language to a tautology to produce even more tautologies. We can keep doing this process forever to produce even more tautologies.

We can enumerate all the sentences in the language to show that it is countably infinite. To show this we let p represent 1, q represent 2, $\vee$ represent 3, $\neg$ represent 4, (represent 5 &) represent 6.

In this way we can map each tautology to a natural number & since the set of natural numbers is countable we can show that all tautologies albeit infinite are countable.

2. The set of all theories in a language made out of the atomic formulas p & q formed using the connectives $\neg$ & $\vee$.

Ans Finite

Justification: If we only had one literal p. We would have 4 possible theories: the theory that contains all tautologies and neither of p and $\neg p$, the theory that contains p & all the formulas it entails, the theory that contains $\neg p$ & all the formulas it entails & finally the theory that contains both p & $\neg p$ along with the formulas they entail.

In case we have p & q, we could have 4 possible theories for both. Combining this we get $4 \times 4 = 16$ possible theories using p & q.

3 The set of all models of a language with one two-place predicate R and the one constant (name) a on the domain $D = \{0, 1\}$ of two objects.

Ans: Finite

Justification: R is a two-place predicate & the domain D contains two elements. So, the possible combinations of inputs is $2^2 = 4$. For each of these possible inputs we can have the result as being True or False.

Thus, we can assign two values to each of the inputs. Thus, we are assigning either true or false for 4 variables. This can lead to $2^4 = 16$ different combinations for R. For each of these combinations of R, I(a) can be either of 0 or 1 leading to 2 models for each interpretation of R. Thus, giving $16 \times 2 = 32$ models, which is obviously finite.

4. The set of all models of a language with one two place predicate R on the domain $D = \{0, 1, 2, 3, \ldots\}$ of countably many objects.

Ans Uncountably infinite

Justification: We can convert a model in the given language to a stream of T & Fs by arranging the results of all $R(a, b)$ sorted in increasing order of $a+b$ {using a in case of ties to break the tie}.

This will give us an infinite stream that looks like T F T F T T T . . . . . . This is the exact same definition as the set of all infinite bitstreams.

Lets say we had n bitstreams we could create a $(n+1)$th stream by using diagonalisation to create a new bitstream that contradicts the ith bitstream at least at position i.

This is the same as Cantor's diagonalisation proof for infinite bitstreams. Thus, proving that the set of all models of the given language is uncountably infinite.

Question - 3

Let us start by stating the compactness theorem: The compactness theorem states that if we have a collections of sentences that is unsatisfiable (i.e they have no model) then there has to exist a finite subset of that collection of sentences which is also also unsatisfiable. This directly leads to the fact that any infinite collection of unsatifiable formulas has a finite subset which is also unsatisfiable. In logical notation the Compactness Theorem can be written out as: $X \models$ then $X' \models$ for some finite subset X' of X. We can prove compactness using the ideas of consistency and inconsistency since being satisfiable means the same thing as being consistent and being unsatisfiable means the same thing as being inconsistent. Thus, we try to show that if a tree for an infinite collection of sentences closes then it only uses a finite number of sentences to get to this closure. Logically we prove $X \vdash$ then $X' \vdash$ where X' is a finite subset of X (X may or may not be finite). If we start with X and construct a tree and a tree for X closes, then every branch in the tree would need to close. This would mean that every branch in the tree is finitely long. Moreover, we will only ever have a finite number of branches (no matter how large) since every rule for decomposition of formulas in a tree leads to either one or two branches (one in the case of conjunctions and two for disjunctions for example). Thus, since we are guaranteed to have a finite number of finite length branches we are guaranteed to have taken only a finite amount of steps during the construction of our proof in order to get to the point where our tree closes. This means that we've only used a finite amount of formulas from the potentially infinite amount of formulas that we could have used. We can thus, construct X' by putting all these formulas into it. This proves, that if $X \vdash$ **then** $X' \vdash$. Since inconsistency and unsatisfiability are essentially the same ideas because of soundness and completeness we've also proved $X \models$ then $X' \models$ which is the compactness theorem.

It follows from the compactness theorem that an infinite set of sentences is satisfiable if every finite subset of it is satisfiable (because unsatisfiability would need to come from a finite set and the lack of one leads of satisfiability). Thus, as a consequence of the compactness theorem we can create non-standard models like modelling infinitely small numbers (numbers that are > 0 but smaller than every whole fraction) which have applications in physics and calculus. We can model infinitesimal numbers as follows: Lets say we have a model M for the theory of real numbers, with X as the set of all sentences in model and the domain being all the numbers on the number line. We can construct another model M' such that M' is made up of M but also contains the set of formulas Y such that Y contains $0 < c, c < 1, c < \frac{1}{2}, ...$ where c is a constant. To prove that this model is satisfiable we need to show that M' is satisfiable by showing that $X \bigcup Y$ is satisfiable. But $X \bigcup Y$ is an infinite set of sentences since Y is always infinite. But we can prove it's satisfiability by proving the satisifiability of every finite subset of this model. If we take any subset of X it is satisfiable by definition and we can satisfy a subset of Y by choosing $c = \frac{1}{n+1}$ where n was the largest denominator in the subset of Y chosen. Thus, we can satisfy every finite subset and so the infinite set allowing us to model c as an infinitesimal.