Q1 Given premises : $s \supset p,\ q \supset (p \supset r)$

Construct Hibert Proof for : $q \supset (s \supset r)$

The conditional Proof fact says that there is a proof from X to $A \supset B$ if and only if there is a proof from X, A to B.

Thus, to show that :

$$s \supset p,\ q \supset (p \supset r) \vdash q \supset (s \supset r) \ \ldots\ (i)$$

We only need to show that :

$$s \supset p,\ q \supset (p \supset r),\ q \vdash (s \supset r) \ \ldots\ (ii)$$

And to prove (ii) we only need to prove that :

$$s \supset p,\ q \supset (p \supset r),\ q,\ s \vdash r \ \ldots\ (iii)$$

We draw from the above conclusions to get from (i) to (iii) using the conditional proof fact.

Thus, to show (i) we only need to show (iii)

Here we provide a Hilbert Proof for (iii) :

1.	$s \supset p$	{Premise}
2.	$q \supset (p \supset r)$	{Premise}
3.	q	{Premise}
4.	s	{Premise}
5.	$(p \supset r)$	{3,2 - (MP)}
6.	p	{4,1 - (MP)}
7.	r	{6,5 - (MP)}

Thus, by the conditional proof fact. Since, we've successfully proved (iii), we've also successfully proved (i).

In the last part of this question we will try to justify the conditional proof fact.

The conditional proof fact is a statement that contains an "if and only if". So we will need to prove 2 things:

(a) $X \vdash A \supset B$ then $X, A \vdash B$

(b) $X, A \vdash B$ then $X \vdash A \supset B$.

(a) If we already have a proof that starts with X and ends up with $A \supset B$. Such that:

$$\left.\begin{array}{l} X \\ \ldots \\ \ldots \\ \ldots \end{array}\right\} \text{Intermediate steps}$$

$$A \supset B$$

We can add the premise A to get:

$$\begin{array}{l} X \\ \boxed{A} \to \text{added in} \\ \ldots \\ \ldots \\ A \supset B \\ \underline{B} \quad \{MP\} \\ \hookrightarrow \text{result.} \end{array}$$

So we've got B from X, A.

(b) Now lets look at the proof that if $\mathcal{X}, A \vdash B$ then $\mathcal{X} \vdash A \supset B$.

Let the proof from $\mathcal{X}, A \vdash B$ contain n formulas $F_1, F_n, \dots, F_n$.

$$F_1$$
$$F_2$$
$$\vdots$$
$$F_n$$

Now, each F_i where $1 \leq i \leq n-1$ is one of the following

(a) the formula A

(b) something in $\mathcal{X}$

(c) an axiom.

(d) follows from earlier formulas by MP.

F_n on the other hand is B.

Put $A \supset$ in front of each F_i such that we get

$$A \supset F_1$$
$$A \supset F_2$$
$$\vdots$$

,

.

$A \supset F_n$

Now this new set of formulas is not a proof but it ends with what we want $A \supset B$. Now, all we need to do is fill in the gaps such that we can create a proof from this set of formulas

- If F_i is an axiom or in X then, then we can get $A \supset F_i$ from the axiom $F_i \supset (A \supset F_i)$ {an instance of weakening}. Now, since F_i {since axioms & premises are by definition true} we get $A \supset F_i$. Thus by adding a few extra lines we can get to $A \supset F_i$ if F_i is a part of X or an axiom.
- If F_i is A then we can show via Hilbert proofs that $A \supset A$. Thus we can fill in the proof for $A \supset A$.
- If F_i follows from F_j then $F_j \supset F_i$ but we already know that $A \supset F_j$.

We can get $A \supset F_i$ by using the distribution axiom $(A \supset F_j) \supset ((A \supset (F_j \supset F_i)) \supset (A \supset F_i))$. Now we can use MP to get $((A \supset (F_j \supset F_i)) \supset (A \supset F_i))$.
Since $(F_j \supset F_i)$ is true $A \supset (F_j \supset F_i)$ is an instance of weakening leading us to $(A \supset F_i)$.

Thus, here by filling in the holes after prefixing the F_is by $A \supset$ we can get a proof from X to $A \supset B$ as show.

Thus since we've proved both directions here, we've successfully justified the conditional proof fact.