Q2

1. The set of all tautologies in a language made out of the atomic formulas p & q formed using the connectives $\neg$ and $\vee$.

Ans. Countably Infinite.

Justification: The number of tautologies will be infinite since we can make infinitely many formulas by suffixing any formula in the language to a tautology to produce even more tautologies. We can keep doing this process forever to produce even more tautologies.

We can enumerate all the sentences in the language to show that it is countably infinite. To show this we let p represent 1, q represent 2, $\vee$ represent 3, $\neg$ represent 4, (represent 5 &) represent 6.

In this way we can map each tautology to a natural number & since the set of natural numbers is countable we can show that all tautologies albeit infinite are countable.

2. The set of all theories in a language made out of the atomic formulas p & q formed using the connectives $\neg$ & $\vee$.

Ans Finite

Justification: If we only had one literal p. We would have 4 possible theories: the theory that contains all tautologies and neither of p and $\neg p$, the theory that contains p & all the formulas it entails, the theory that contains $\neg p$ & all the formulas it entails & finally the theory that contains both p & $\neg p$ along with the formulas they entail.
In case we have p & q, we could have 4 possible theories for both. Combining this we get $4 \times 4 = 16$ possible theories using p & q.

3 The set of all models of a language with one two-place predicate R and the one constant (name) a on the domain $D = \{0, 1\}$ of two objects.

Ans: Finite

Justification: R is a two-place predicate & the domain D contains two elements. So, the possible combinations of inputs is $2^2 = 4$. For each of these possible inputs we can have the result as being True or False.

Thus, we can assign two values to each of the inputs. Thus, we are assigning either true or false for 4 variables. This can lead to $2^4 = 16$ different combinations for R. For each of these combinations of R, $I(a)$ can be either of 0 or 1 leading to 2 models for each interpretation of R. Thus, giving $16 \times 2 = 32$ models, which is obviously finite.

4. The set of all models of a language with one two place predicate R on the domain $D = \{0, 1, 2, 3, \ldots\}$ of countably many objects.

Ans Uncountably infinite

Justification: We can convert a model in the given language to a stream of T & Fs by arranging the results of all $R(a, b)$ sorted in increasing order of $a+b$ {using a in case of ties to break the tie}.

This will give us an infinite stream that looks like TFTFTTT...... This is the exact same definition as the set of all infinite bitstreams.

Lets say we had n bitstreams we could create a $(n+1)$th stream by using diagonalisation to create a new bitstream that contradicts the ith bitstream at least at position i.

This is the same as Cantor's diagonalisation proof for infinite bitstreams. Thus, proving that the set of all models of the given language is uncountably infinite.