DATA-DRIVEN DYNAMICAL SYSTEMS

DYNAMICAL SYSTEMS CONCERNS THE ANALYSIS, PREDICTION, AND UNDERSTANDING OF THE BEHAVIOR OF SYSTEMS OF DIFFERENTIAL EQUATIONS OR ITERATIVE MAPPING THAT DESCRIBE THE EVOLUTION OF THE STATE OF A SYSTEM.

HERE WE WILL FOCUS ON DISCOVERING DYNAMICS FROM DATA AND FINDING DATA-DRIVEN REPRESENTATIONS THAT MAKE NONLINEAR SYSTEMS AMENABLE TO LINEAR SYSTEM

LET'S CONSIDER AN AUTONOMOUS SYSTEM WITHOUT TIME DEPENDENCE OR PARAMETERS.

$$\frac{d}{dt}\underline{x}(t) = \underline{f}(\underline{x}(t)) \qquad (\cdot)$$

SINCE WE WANT WORKING WITH MEASUREMENTS WE HAVE TO MOVE FROM THE CONTINUOUS CASE TO THE DISCRETE ONE:

$$\underline{x}_{k+1} = \underline{F}(\underline{x}(t))$$

WE CAN OBTAIN DISCRETE-TIME DYNAMICS FROM CONTINUOUS-TIME DYNAMICS BY SAMPLING FROM THE TRAJECTORY OF $(\cdot)$ SO THAT $\underline{x}_k = \underline{x}(k\Delta t)$ AND:

$$\underline{F}_t(\underline{x}(t_0)) = \underline{x}(t_0) + \int_{t_0}^{t_0+t} \underline{f}(\underline{x}(\tau))\, d\tau$$

FOR PRACTICAL APPLICATION IS DESIRABLE TO WORK WITH LINEAR DINAMICS AS:

$$\frac{d}{dt}\underline{x} = \underline{\underline{A}}\,\underline{x}$$

LINEAR DYNAMICAL SYSTEMS ADMIT CLOSED-FORM SOLUTIONS AND A LOT OF TECHNIQUES EXIST FOR THEIR ANALYSIS, PREDICTION, ESTIMATION OR CONTROL.

$$\underline{x}(t_0+t) = e^{\underline{A}t}\,\underline{x}(t_0)$$

THE DYNAMICS IS COMPLETELY CHARACTERIZED BY THE EGVS AND THE EGVTS OF THE MATRIX $\underline{\underline{A}}$, OBTAINED BY SPECTRAL DECOMPOSITION:

$$\underline{\underline{A}}\,\underline{\underline{T}} = \underline{\underline{T}}\,\underline{\underline{\Lambda}}$$

EIGENVECTORS EIGENVALUES

SO, WE CAN WRITE:

$$\underline{\underline{A}} = \underline{\underline{T}}\,\underline{\underline{\Lambda}}\,\underline{\underline{T}}^{-1} \rightarrow \underline{x}(t_0+1) = \underline{\underline{T}}\, e^{\underline{\underline{\Lambda}} t}\, \underline{\underline{T}}^{-1} \underline{x}(t_0)$$

THE MATRIX $\underline{\underline{T}}$ ALLOWS TO TRANSFORM $\underline{x}$ INTO THE EIGENVECTORS COORDINATES, WHERE THE DYNAMICS IS DECOUPLED

$$\frac{d}{dt}\underline{z} = \underline{\underline{\Lambda}}\,\underline{z}$$

WHERE $\underline{z} = \underline{\underline{T}}^{-1}\underline{x}$

▷ DYNAMIC MODE DECOMPOSITION (DMD)

DMD WAS INVENTED IN THE FLUID DYNAMICS COMMUNITY TO IDENTIFY SPATIO-TEMPORAL COHERENT STRUCTURES FROM HIGH DIMENSIONAL DATA. IT PROVIDES A MODAL DECOMPOSITION WHERE EACH MODE CONSISTS OF SPATIALLY CORRELATED STRUCTURES THAT HAVE THE SAME LINEAR BEHAVIOR IN TIME. THUS, DMD PROVIDES, TOGETHER WITH A DIMENSIONALITY REDUCTION ALSO A MODEL OF HOW THESE MODES EVOLVE IN TIME.

DMD IS PURELY BASED ON MEASUREMENT DATA.

WE CAN SAY THAT DMD CONNECTS THE ADVANTAGE OFFERED BY SVD OF THE DIMENSIONALITY REDUCTION AND THE ADVANTAGE OFFERED BY THE FFT OF THE TEMPORAL FREQUENCY IDENTIFICATION. EACH DMD MODE IS ASSOCIATED WITH A PARTICULAR EGV $\lambda = a + ib$

THE FIRST STEP TO APPLY DMD IS TO COLLECT A BUNCH OF SNAPSHOTS OF THE SYSTEM:

$$\left\{ \underline{x}(t_k),\ \underline{x}(t'_k) \right\}_{k=1}^{m}$$

WHERE $t'_k = t_k + \Delta t$ WITH A Δt ABLE TO RESOLVE THE HIGHEST FREQUENCIES IN THE DYNAMICS. THEN WE ARRANGE THESE SNAPSHOTS IN MATRICES

$$\underline{\underline{X}} = \begin{bmatrix} | & | & & | \\ \underline{x}_1 & \underline{x}_2 & \cdots & \underline{x}_{m-1} \\ | & | & & | \end{bmatrix}$$

EACH ROWS IS ONE TIME STEP AHEAD

$$\underline{\underline{X}} \quad \begin{bmatrix} | & | & & | \\ \underline{x}_2 & \underline{x}_3 & \cdots & \underline{x}_{m} \\ | & | & & | \end{bmatrix}$$

THE DMD SEEKS THE LEADING SPECTRAL DECOMPOSITION OF THE BEST-FIT LINEAR OPERATOR $\underline{\underline{A}}$ THAT RELATES THE TWO SNAPSHOT MATRICES:

$$\underline{\underline{X}}' \approx \underline{\underline{A}}\,\underline{\underline{X}}$$

BY ASSUMING UNIFORM SAMPLING IN TIME:

$$\underline{x}_{k+1} \approx \underline{\underline{A}}\,\underline{x}_k$$

THE BEST-FIT OPERATOR IS DEFINED AS:

$$\underline{\underline{A}} = \underset{\underline{\underline{A}}}{\operatorname{argmin}} \| \underline{\underline{X}}' - \underline{\underline{A}}\,\underline{\underline{X}} \|_F$$

$$= \underline{\underline{X}}\,\underline{\underline{X}}^*$$

FOR A HIGH-DIMENSIONAL VECTOR STATE $\underline{x}$ THIS OPERATION MAY BE INTRACTABLE. THE DMD ALGORITHM EXPLOITS DIMENSIONALITY REDUCTION TO COMPUTE THE DOMINANT EGVLS AND EGVTS OF $\underline{\underline{A}}$.

THE PSEUDO-INVERSE IS COMPUTED VIA SVD SINCE, USUALLY, $\underline{\underline{X}}$ HAS FAR FEWER COLUMNS THAN ROWS, i.e. $m \ll n$, AND HENCE THE RANK

OF MATRIX $\underline{\underline{A}}$ WILL BE m:

STEP 1. $\underline{\underline{X}} \approx \tilde{\underline{\underline{U}}} \tilde{\underline{\underline{\Sigma}}} \tilde{\underline{\underline{V}}}^*$ WHERE:

$$\left.\begin{array}{l} \tilde{\underline{\underline{U}}} \in \mathbb{C}^{n \times r} \\ \tilde{\underline{\underline{\Sigma}}} \in \mathbb{C}^{r \times r} \\ \tilde{\underline{\underline{V}}} \in \mathbb{C}^{m \times r} \end{array}\right\}$$

WHERE r DENOTES THE EXACT (m) OR APPROXIMATE RANK OF $\underline{\underline{X}}$

STEP 2. $\underline{\underline{A}} = \underline{\underline{X}}' \tilde{\underline{\underline{V}}} \tilde{\underline{\underline{\Sigma}}}^{-1} \tilde{\underline{\underline{U}}}^*$

NOW WE CAN PROJECT $\underline{\underline{A}}$ ONTO THE POD MODES

$$\tilde{\underline{\underline{A}}} = \tilde{\underline{\underline{U}}}^* \underline{\underline{A}} \tilde{\underline{\underline{U}}} = \tilde{\underline{\underline{U}}}^* \underline{\underline{X}}' \tilde{\underline{\underline{V}}} \tilde{\underline{\underline{\Sigma}}}^{-1}$$

THE REDUCED-ORDER MATRIX $\tilde{\underline{\underline{A}}}$ DEFINES A LINEAR MODEL FOR THE DYNAMICS OF THE VECTOR OF POD COEFFICIENTS $\tilde{\underline{x}}$.

$$\tilde{\underline{x}}_{k+1} = \tilde{\underline{\underline{A}}} \tilde{\underline{x}}_k$$

$$\underline{x} = \tilde{\underline{\underline{U}}} \tilde{\underline{x}}$$

STEP 3. THE SPECTRAL DECOMPOSITION OF $\tilde{\underline{\underline{A}}}$ IS COMPUTED:

$$\tilde{\underline{\underline{A}}} \underline{\underline{W}} = \underline{\underline{W}} \underline{\underline{\Lambda}}$$

DMD EGVS

THE POD EGVS ARE THE SAME EGVS OF THE MATRIX $\underline{\underline{A}}$ AND THE EIGENVECTORS $\underline{\underline{W}}$ WILL PROVIDE A DIAGONALIZATION OF THE MATRIX. WE CAN THINK TO THE COLUMNS OF $\underline{\underline{W}}$ AS A LINEAR COMBINATIONS OF POD MODES

STEP 4. THE HIGH DIMENSIONAL DMD MODES $\underline{\underline{\Phi}}$ ARE RECONSTRUCTED USING THE EIGENVECTORS $\underline{\underline{W}}$

$$\underline{\underline{\Phi}} = \underline{\underline{X}}^{-1} \underline{\underline{\tilde{V}}} \underline{\underline{\tilde{\Sigma}}}^{-1} \underline{\underline{W}}$$

$$\underline{\underline{A}}\,\underline{\underline{\Phi}} = \underbrace{(\underline{\underline{X}}' \underline{\underline{\tilde{V}}} \underline{\underline{\tilde{\Sigma}}}^{-1} \underline{\underline{\tilde{U}}}^{*})(\underline{\underline{X}}^{-1} \underline{\underline{\tilde{V}}} \underline{\underline{\tilde{\Sigma}}}^{-1}}_{\underline{\underline{\tilde{A}}}} \underline{\underline{W}}) = \underline{\underline{X}}' \underline{\underline{\tilde{V}}} \underline{\underline{\tilde{\Sigma}}}^{-1} \underline{\underline{\tilde{A}}}\, \underline{\underline{W}} =$$

$$= \underline{\underline{X}}' \underline{\underline{\tilde{V}}} \underline{\underline{\tilde{\Sigma}}}^{-1} \underline{\underline{W}}\, \underline{\underline{\Lambda}} = \underline{\underline{\Phi}}\, \underline{\underline{\Lambda}}$$

▷ DMD EXPANSION

$$\underline{x}_k = \sum_{j=1}^{r} \underline{\phi}_j \lambda_j^{k-1} b_j = \underline{\underline{\Phi}}\, \underline{\underline{\Lambda}}^{k-1} \underline{b} \qquad (\cdot)$$

WHERE ϕ_j ARE DMD MODES, λ_j ARE DMD EIGENVALUES, AND b_j IS THE MODE AMPLITUDE:

$$\underline{b} = \underline{\underline{\Phi}}^{*} \underline{x}_1$$

THIS COMPUTATION IS USUALLY EXPENSIVE SO, IT IS BETTER TO EXPLOIT POD PROJECTED DATA:

$$\underline{\underline{\tilde{U}}}\, \underline{\tilde{x}}_1 = \underline{\underline{X}}' \underline{\underline{\tilde{V}}} \underline{\underline{\Sigma}}^{-1} \underline{\underline{W}}\, \underline{b}$$

$$\underline{\tilde{x}}_1 = \underline{\underline{\tilde{U}}}^{*} \underline{\underline{X}}' \underline{\underline{\tilde{V}}} \underline{\underline{\Sigma}}^{-1} \underline{\underline{W}}\, \underline{b}$$

$$\underline{\tilde{x}}_1 = \underline{\underline{\tilde{A}}}\, \underline{\underline{W}}\, \underline{b}$$

$$\underline{\tilde{x}}_1 = \underline{\underline{W}}\, \underline{\underline{\Lambda}}\, \underline{b}$$

AND SO:

$$\underline{b} = (\underline{\underline{W}}\, \underline{\underline{\Lambda}})^{-1} \underline{\tilde{x}}_1$$

WE CAN ALSO WRITE THE SPECTRAL EXPANSION $(\cdot)$ IN CONTINUOUS TIME BY INTRODUCING THE CONTINUOUS EGV. $\omega = \log(\lambda)/\Delta t$

$$\underline{x}(t) = \sum_{j=1}^{r} \underline{\phi}_j e^{\omega_j t} b_j = \underline{\underline{\Phi}} \exp(\underline{\underline{\Omega}} t) \underline{b}$$

▷ KOOPMAN OPERATOR THEORY

LET CONSIDER $\underline{x}(t) \in \mathcal{M}$, AN M-DIMENSIONAL STATE THAT LIVES IN A SMOOTH MANIFOLD $\mathcal{M}$. IN 1931 WAS DEMONSTRATED THAT IT IS POSSIBLE TO REPRESENT A NONLINEAR DYNAMICAL SYSTEM IN TERMS OF AN ∞-DIMENSIONAL LINEAR OPERATOR ACTING ON AN HILBER SPACE OF MEASUREMENT FUNCTIONS OF THE STATE OF THE SYSTEM. THIS OPERATOR IS CALLED KOOPMAN OPERATOR AND IT IS LINEAR AND ITS SPECTRAL DECOMPOSITION COMPLETELY CHARACTERIZES THE BEHAVIOR OF A NONLINEAR SYSTEM.

THE KOOPMAN OPERATOR IS AN ∞-DIMENSIONAL LINEAR OPERATOR THAT ACTS ON A MEASUREMENT FUNCTIONS $g: \mathcal{M} \rightarrow \mathbb{R}$

$$\mathcal{K} g = g \circ F_t$$

WHERE "$\circ$" IS THE COMPOSITION OPERATOR. FOR A DISCRETE TIME SYSTEM WITH TIMESTEP Δt:

$$\mathcal{K} g(\underline{x}_k) = g(F(\underline{x}_k)) = g(\underline{x}_{k+1})$$

FOR SUFFICIENTLY SMOOTH DYNAMICAL SYSTEMS, IT IS POSSIBLE TO DEFINE THE CONTINUOUS TIME ANALOGUE:

$$\frac{d}{dt} g = \mathcal{K} g$$

WE TAKE MEASUREMENTS FROM THE SYSTEM:

$$X = \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \dots & x_{m-1} \\ | & | & & | \end{bmatrix} \qquad X' = \begin{bmatrix} | & | & & | \\ x_2 & x_3 & \dots & x_m \\ | & | & & | \end{bmatrix}$$

FOLLOWING THE DMD PATH:

THIS IS THE CONNECTION OF DMD TO KOOPMAN

$$g(x) = x \rightarrow A_x = X'X^* \rightarrow x_{t+1} = A_x x_t$$

FOLLOWING THE KOOPMAN PATH WE ENRICH OUR MEASUREMENTS. INSTEAD OF DOING REGRESSION ON DIRECT MEASUREMENTS WE PERFORM REGRESSION ON AN AUGMENTED VECTOR.

$$Y = \begin{bmatrix} | & | & & | \\ g(x_1) & g(x_2) & \dots & g(x_{m-1}) \\ | & | & & | \end{bmatrix} \qquad Y' = \begin{bmatrix} | & & | \\ g(x_2) & \dots & g(x_m) \\ | & & | \end{bmatrix}$$

AND THEN WE COMPUTE THE BEST FIT OPERATOR TO MAP Y INTO Y':

$$A_Y = \underset{A_Y}{\operatorname{argmin}} \| Y' - A_Y Y \| = Y' Y^*$$

IT IS IMPORTANT TO REMARK THAT WE NEED ALSO TO COMPUTE g^{-1} IN ORDER TO GET BACK TO x_k

$$x_k = g^{-1}(y_k)$$

JUST BECAUSE MEASURE SOMETHING, THIS DOESN'T IMPLY THAT THEY ARE THE CORRECT VARIABLES. FOR KOOPMAN THE GOOD COORDINATES SYSTEM IS THE ONE THAT MAKES THE SYSTEM MORE LINEAR

LET'S CONSIDER AN EXAMPLE:

$$\dot{x}_1 = \mu x_1$$
$$\dot{x}_2 = \lambda(x_2 - x_1^2)$$

IT IS POSSIBLE TO AUGMENT THE STATE WITH THE NONLINEAR MEASUREMENT $g = x_1^2$

$$y_1 = x_1 \qquad y_2 = x_2 \qquad y_3 = x_1^2$$

$$\dot{y} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \lambda & -\lambda \\ 0 & 0 & 2\mu \end{bmatrix} y$$

NOW THE SYSTEM IS LINEAR AND IT IS EXACTLY WHAT WE WANT TO DO WITH KOOPMAN INVARIANT. THIS PROBLEM WAS TRIVIAL AND IN THEORY, MORE DIFFICULT PROBLEMS CAN REQUIRE INFINITE TERMS.

LET NOW CONSIDER ANOTHER EXAMPLE, THE BURGER' EQUATION:

$$u_t + u u_x = \nu u_{xx}$$

WHICH IS A 1D EXAMPLE OF THE NONLINEAR CONVECTION AND DIFFUSION THAT GIVES RISE TO SHOCK WAVES IN FLUIDS. WE CAN LINEARIZE THE PROBLEM THANKS TO THE COLE-HOPF TRANSFORM

CONSIDERING THE SHRODINGER EQUATION

$$i u_t + \frac{1}{2} u_{xx} + |u|^2 u = 0$$

WE CAN AUGMENT THE MEASUREMENTS OF u WITH A CUBIC TERM.

$$\text{IN DMD: } X = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \dots & u_m \\ | & | & & | \end{bmatrix} \rightarrow \text{DMD}$$

$$\text{IN KOOPMAN: } Y = \begin{bmatrix} | & & | \\ u_1 & & u_m \\ | & \dots & | \\ |u_1|^2 u_1 & & |u_m|^2 u_m \\ | & & | \end{bmatrix} \rightarrow \text{DMD}$$

THIS IS THE KOOPMAN OPERATOR

WITH THIS MEASUREMENTS AUGMENTATION WE OBTAIN A PERFECT LINEAR EMBEDDING OF THE SYSTEM. IT IS IMPORTANT TO HIGHLIGHT THAT A WRONG MEASUREMENTS AUGMENTATION WE CAN OBTAIN WORSE RESULTS THAN SIMPLE DMD EXPANSION.

WHEN WE USE THE CORRECT KOOPMAN OPERATOR WE HAVE:

$$\frac{dx}{dt} = f(x) \rightarrow y = g(x) \rightarrow \frac{dy}{dt} = Ly$$

MORE REALISTICALLY WE HAVE:

$$\frac{dy}{dt} = Ly + \varepsilon g$$

WITH $\varepsilon \ll 1$.

▷ TIME DELAY EMBEDDINGS

LET'S CONSIDER A TIME-SERIES MEASUREMENT $x(t)$. WE BUILD THE HANKEL MATRIX $\underline{\underline{H}}$

$$\underline{\underline{H}} = \begin{bmatrix} x_1 & x_2 & \cdots & x_p \\ x_2 & & & \\ x_3 & & & \\ \vdots & & & \\ x_q & & & x_{p+q} \end{bmatrix} = \underline{\underline{U}}\,\underline{\underline{\Sigma}}\,\underline{\underline{V}}^*$$

AND HERE WE APPLY DMD. THESE IS A METHOD OF FINDING A GOOD SET OF COORDINATES SYSTEM

THE SOLUTION OF A LINEAR SYSTEM SHOULD BE AN EXPONENT OF THE FORM $e^{i\omega t}$. AN EXPONENTIAL SOLUTION IS NOTHING MORE THAN A COSINE AND SINE. IN THE LIMIT OF AN ∞-TIME EMBEDDING WE OBTAIN THE FOURIER MODES SO COSINE AND COSINE. INSTEAD OF USING ALL THE FOURIER MODES, WITH THIS METHOD WE DO A STEP FORWARD SINCE WE DISCOVER JUST THE REQUIRED MODES!

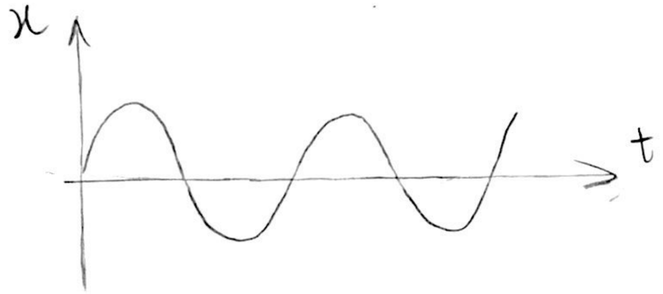
ONE OF THE MOST IMPORTANT APPLICATIONS OF DMD MODEL IS THE CONTROL. INSTEAD OF WORKING ON A NONLINEAR SYSTEM:

$$\dot{x} = f(x) + Bu$$

WE CAN TAKE A LOT OF MEASUREMENTS AND WORK ON:

$$\dot{x} = Ax + Bu$$

LET'S CONSIDER THE FOLLOWING MEASUREMENT



THIS FUNCTION CAN BE REPRESENTED WITH $e^{\pm i\omega t} = \cos\omega t \pm i \sin\omega t$
BUT

$$X = [x_1 \; x_2 \; \dots \; x_m]$$

CAN PROVIDES JUST ONE MODES, CAUSING RANK DEFICIENCY. WE NEED TO AUGMENT THE DATA WITH THE HANKEL MATRIX. EVERY TIME DELAYED EMBEDDING PROVIDES A SINGLE MODE. IF WE NEED TO REPRESENT A SIGNAL WITH 2 FREQUENCIES WE NEED 4 TIME DELAYED EMBEDDING.