MA 01 LINEAR ALGEBRA I

REVIEW OF LECTURES – VIII

April 18 (Tue), 2023

Section: C5–6.
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§8. Matrix multiplication for the 3×3 case.

Today's agenda: Multiplications involving 3×3 matrices. As a starter:

$$\underline{\underline{\textit{The correct conversion of}}} \quad \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \quad \underline{\underline{\textit{is}}} \quad \begin{bmatrix} a_1 p + a_2 q + a_3 r \\ b_1 p + b_2 q + b_3 r \\ c_1 p + c_2 q + c_3 r \end{bmatrix}.$$

Like last time, we must *officially* declare this to be the rule that is going to be enforced throughout:

• **Rule.**
$$\boxed{\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} a_1 p + a_2 q + a_3 r \\ b_1 p + b_2 q + b_3 r \\ c_1 p + c_2 q + c_3 r \end{bmatrix}.}$$

Paraphrase:

$$\boxed{\begin{array}{l} A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \\ \\ \Longrightarrow \qquad A\boldsymbol{x} = \begin{bmatrix} a_1 p + a_2 q + a_3 r \\ b_1 p + b_2 q + b_3 r \\ c_1 p + c_2 q + c_3 r \end{bmatrix}. \end{array}}$$

• This one you could have easily guessed by extrapolating from the 2×2 case (the case A is 2×2 and $\boldsymbol{x}$ is 2×1, to be precise). It's just that three separate multiplications instead of two, every step of the way, and also there are three separate steps instead of two. Just in case, I want to offer the following breakdown:

Break-down. We are going to do

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \boxed{\diamondsuit} \\ \boxed{\spadesuit} \\ \boxed{\triangle} \end{bmatrix}.$$

(i) To find $\diamondsuit$, observe

$$\begin{bmatrix} \boxed{a_1 \quad a_2 \quad a_3} \\ b_1 \quad b_2 \quad b_3 \\ c_1 \quad c_2 \quad c_3 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p \\ q \\ r \end{matrix}} \end{bmatrix} = \begin{bmatrix} \boxed{a_1 p + a_2 q + a_3 r} \\ \boxed{\spadesuit} \\ \boxed{} \end{bmatrix}.$$

(ii) To find $\spadesuit$, observe

$$\begin{bmatrix} a_1 \quad a_2 \quad a_3 \\ \boxed{b_1 \quad b_2 \quad b_3} \\ c_1 \quad c_2 \quad c_3 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p \\ q \\ r \end{matrix}} \end{bmatrix} = \begin{bmatrix} \boxed{a_1 p + a_2 q + a_3 r} \\ \boxed{b_1 p + b_2 q + b_3 r} \\ \boxed{\triangle} \end{bmatrix}.$$

(iii) To find $\triangle$, observe

$$\begin{bmatrix} a_1 \quad a_2 \quad a_3 \\ b_1 \quad b_2 \quad b_3 \\ \boxed{c_1 \quad c_2 \quad c_3} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} p \\ q \\ r \end{matrix}} \end{bmatrix} = \begin{bmatrix} \boxed{a_1 p + a_2 q + a_3 r} \\ \boxed{b_1 p + b_2 q + b_3 r} \\ \boxed{c_1 p + c_2 q + c_3 r} \end{bmatrix}.$$

Example 1. For $A = \begin{bmatrix} 3 & -6 & 5 \\ -2 & 4 & 7 \\ -1 & 3 & 9 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, we have

$$\begin{aligned} A\boldsymbol{x} &= \begin{bmatrix} 3 & -6 & 5 \\ -2 & 4 & 7 \\ -1 & 3 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \cdot 2 + (-6) \cdot 3 + 5 \cdot 1 \\ (-2) \cdot 2 + 4 \cdot 3 + 7 \cdot 1 \\ (-1) \cdot 2 + 3 \cdot 3 + 9 \cdot 1 \end{bmatrix} = \begin{bmatrix} -7 \\ 15 \\ 16 \end{bmatrix}. \end{aligned}$$

Exercise 1. Perform each of the following multiplications:

(1) $\begin{bmatrix} 4 & 0 & 3 \\ 0 & 6 & 5 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}$. (2) $A\boldsymbol{x}$, where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$.

(3) $A\boldsymbol{x}$, where $A = \begin{bmatrix} 7 & 4 & -4 \\ -5 & -2 & 5 \\ 2 & 2 & 3 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}$.

• Now we talk about multiplying a 3×3 matrix with another 3×3 matrix. Here is the rule that we hereby *officially* declare to enforce throughout:

Rule. $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}$ <u>is calculated as</u>

$$\boxed{\begin{bmatrix} a_1p_1 + a_2q_1 + a_3r_1 & a_1p_2 + a_2q_2 + a_3r_2 & a_1p_3 + a_2q_3 + a_3r_3 \\ b_1p_1 + b_2q_1 + b_3r_1 & b_1p_2 + b_2q_2 + b_3r_2 & b_1p_3 + b_2q_3 + b_3r_3 \\ c_1p_1 + c_2q_1 + c_3r_1 & c_1p_2 + c_2q_2 + c_3r_2 & c_1p_3 + c_2q_3 + c_3r_3 \end{bmatrix}.}$$

• Paraphrase:

$$\boxed{\begin{aligned} &A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}, \quad B = \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix} \\ &\Longrightarrow AB = \begin{bmatrix} a_1p_1 + a_2q_1 + a_3r_1 & a_1p_2 + a_2q_2 + a_3r_2 & a_1p_3 + a_2q_3 + a_3r_3 \\ b_1p_1 + b_2q_1 + b_3r_1 & b_1p_2 + b_2q_2 + b_3r_2 & b_1p_3 + b_2q_3 + b_3r_3 \\ c_1p_1 + c_2q_1 + c_3r_1 & c_1p_2 + c_2q_2 + c_3r_2 & c_1p_3 + c_2q_3 + c_3r_3 \end{bmatrix}. \end{aligned}}$$

This is a little bit more complicated than the 2×2 case, though, again, this could have been easily extrapolated from the case A and B are 2×2. In case, let me offer the following break-down:

- **Break-down:** First and foremost, acknowledge the following:

  A and B are both 3×3 matrices $\implies$ AB is a 3×3 matrix.

  In other words:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix} = \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}.$$

  (i) Let us find $\diamondsuit$ in

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{\diamondsuit} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}.$$

Since $\diamondsuit$ is in the top-left, accordingly highlight the portion of A and B, like

$$\begin{bmatrix} \boxed{a_1 \quad a_2 \quad a_3} \\ b_1 \quad b_2 \quad b_3 \\ c_1 \quad c_2 \quad c_3 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} p_1 \\ q_1 \\ r_1 \end{bmatrix} & \begin{matrix} p_2 & p_3 \\ q_2 & q_3 \\ r_2 & r_3 \end{matrix} \end{bmatrix}.$$

$\diamondsuit$ is $a_1p_1 + a_2q_1 + a_3r_1$:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{a_1p_1 + a_2q_1 + a_3r_1} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}.$$

(ii) Next, let's find $\heartsuit$ in

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{a_1p_1 + a_2q_1 + a_3r_1} & \boxed{\heartsuit} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}$$

Since $\heartsuit$ is the top-middle $\big($top-row & middle-column$\big)$, accordingly highlight the portion of A and B, like

$$\begin{bmatrix} \boxed{a_1 \quad a_2 \quad a_3} \\ b_1 \quad b_2 \quad b_3 \\ c_1 \quad c_2 \quad c_3 \end{bmatrix} \begin{bmatrix} p_1 & \boxed{\begin{matrix} p_2 \\ q_2 \\ r_2 \end{matrix}} & p_3 \\ q_1 & & q_3 \\ r_1 & & r_3 \end{bmatrix}.$$

$\heartsuit$ is $a_1p_2 + a_2q_2 + a_3r_2$:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{a_1p_1 + a_2q_1 + a_3r_1} & \boxed{a_1p_2 + a_2q_2 + a_3r_2} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}.$$

(iii) Similarly, we can find ♣ in

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{a_1p_1 + a_2q_1 + a_3r_1} & \boxed{a_1p_2 + a_2q_2 + a_3r_2} & \boxed{\clubsuit} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}$$

as

$$\begin{bmatrix} \boxed{a_1 \quad a_2 \quad a_3} \\ b_1 \quad b_2 \quad b_3 \\ c_1 \quad c_2 \quad c_3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & \boxed{p_3} \\ q_1 & q_2 & \boxed{q_3} \\ r_1 & r_2 & \boxed{r_3} \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{a_1p_1 + a_2q_1 + a_3r_1} & \boxed{a_1p_2 + a_2q_2 + a_3r_2} & \boxed{a_1p_3 + a_2q_3 + a_3r_3} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}.$$

(iv) Next, we can find ♠ in

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{a_1p_1 + a_2q_1 + a_3r_1} & \boxed{a_1p_2 + a_2q_2 + a_3r_2} & \boxed{a_1p_3 + a_2q_3 + a_3r_3} \\ \boxed{\spadesuit} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}$$

as

$$
\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}
\begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}
$$

$$
= \begin{bmatrix} \boxed{a_1p_1 + a_2q_1 + a_3r_1} & \boxed{a_1p_2 + a_2q_2 + a_3r_2} & \boxed{a_1p_3 + a_2q_3 + a_3r_3} \\ \boxed{b_1p_1 + b_2q_1 + b_3r_1} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{bmatrix}.
$$

Now, the rest goes the same way. The following is the end result:

$$
\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}
\begin{bmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{bmatrix}
$$

$$
= \begin{bmatrix} \boxed{a_1p_1 + a_2q_1 + a_3r_1} & \boxed{a_1p_2 + a_2q_2 + a_3r_2} & \boxed{a_1p_3 + a_2q_3 + a_3r_3} \\ \boxed{b_1p_1 + b_2q_1 + b_3r_1} & \boxed{b_1p_2 + b_2q_2 + b_3r_2} & \boxed{b_1p_3 + b_2q_3 + b_3r_3} \\ \boxed{c_1p_1 + c_2q_1 + c_3r_1} & \boxed{c_1p_2 + c_2q_2 + c_3r_2} & \boxed{c_1p_3 + c_2q_3 + c_3r_3} \end{bmatrix}.
$$

Example 2. For $A = \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix}$, we have

$$
\begin{aligned}
AB
&= \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 1\cdot 1 + (-1)\cdot 2 + 7\cdot 1 & 1\cdot 1 + (-1)\cdot 1 + 7\cdot(-3) & 1\cdot 2 + (-1)\cdot 1 + 7\cdot 2 \\ 2\cdot 1 + (-1)\cdot 2 + 8\cdot 1 & 2\cdot 1 + (-1)\cdot 1 + 8\cdot(-3) & 2\cdot 2 + (-1)\cdot 1 + 8\cdot 2 \\ 3\cdot 1 + 1\cdot 2 + (-1)\cdot 1 & 3\cdot 1 + 1\cdot 1 + (-1)\cdot(-3) & 3\cdot 2 + 1\cdot 1 + (-1)\cdot 2 \end{bmatrix} \\
&= \begin{bmatrix} 6 & -21 & 15 \\ 8 & -23 & 19 \\ 4 & 7 & 5 \end{bmatrix},
\end{aligned}
$$

BA

$$
\begin{aligned}
&= \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 7 \\ 2 & -1 & 8 \\ 3 & 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1\cdot 1 + 1\cdot 2 + 2\cdot 3 & 1\cdot(-1) + 1\cdot(-1) + 2\cdot 1 & 1\cdot 7 + 1\cdot 8 + 2\cdot(-1) \\ 2\cdot 1 + 1\cdot 2 + 1\cdot 3 & 2\cdot(-1) + 1\cdot(-1) + 1\cdot 1 & 2\cdot 7 + 1\cdot 8 + 1\cdot(-1) \\ 1\cdot 1 + (-3)\cdot 2 + 2\cdot 3 & 1\cdot(-1) + (-3)\cdot(-1) + 2\cdot 1 & 1\cdot 7 + (-3)\cdot 8 + 2\cdot(-1) \end{bmatrix} \\
&= \begin{bmatrix} 9 & 0 & 13 \\ 7 & -2 & 21 \\ 1 & 4 & -19 \end{bmatrix}.
\end{aligned}
$$

So

$$
AB = \begin{bmatrix} 6 & -21 & 15 \\ 8 & -23 & 19 \\ 4 & 7 & 5 \end{bmatrix}, \qquad BA = \begin{bmatrix} 9 & 0 & 13 \\ 7 & -2 & 21 \\ 1 & 4 & -19 \end{bmatrix}.
$$

So, once again, (just like the 2×2 case) in general, AB and BA are not equal.

Exercise 2. Calculate AB and BA:

(1) $A = \begin{bmatrix} 2 & 1 & 3 \\ -2 & 2 & 3 \\ 0 & -1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 3 & 1 \\ -1 & 2 & -1 \end{bmatrix}.$

(2) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$

(3) $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -4 & 0 \\ -1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}.$

(4) $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$

- **The inverse matrix A^{-1}.**

Next, let's revisit the inverse of 3×3 matrices. Remember that the following was thrown at the end of "Review of Lectures — III":

Inverse of a 3×3 matrix.

Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$. The <u>inverse</u> A^{-1} of A is the following matrix:

$$\boxed{A^{-1} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}^{-1} = \frac{1}{\det A} \text{ adj } A,}$$

where

$$\det A = a_1 b_2 c_3 - a_1 b_3 c_2 - a_2 b_1 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1,$$

and

$$\text{adj } A = \begin{bmatrix} + \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} & - \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} & + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \\ - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} & + \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} & - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \\ + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} & - \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix} & + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \end{bmatrix}.$$

A^{-1} exists, provided $\det A \neq 0$.

The above inversion method was touched — if ever so briefly — at the end of "Review of Lectures – III". It would have been too much for one lecture to include this remark so I left it out, but there is something we have to be super-meticulous about. Actually I have already made the same remark for the 2×2 case (in page 3–4 of "Review of Lectures – III"), so the following is a mere extrapolation. In the previous page, inside the smaller highlighted box,

- ○ the part $\dfrac{1}{\det A}$ is a scalar,

whereas

- ○ the part $\operatorname{adj} A$ is a matrix.

Those two ingredients are being juxtaposed. It signifies

" $\underline{\underline{\text{a scalar being multiplied to}}}$ a 3×3 matrix ".

We haven't *officially* defined it yet, which we must now. Here we go:

- **Definition (Scalar multiplied to a matrix).** Let s be a scalar. Then

$$\boxed{s \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} sa_1 & sa_2 & sa_3 \\ sb_1 & sb_2 & sb_3 \\ sc_1 & sc_2 & sc_3 \end{bmatrix}.}$$

Paraphrase:

$$\boxed{\begin{array}{l} \text{If} \quad A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \qquad \text{and} \qquad s : \text{ a scalar} \\ \qquad \Longrightarrow \qquad sA = \begin{bmatrix} sa_1 & sa_2 & sa_3 \\ sb_1 & sb_2 & sb_3 \\ sc_1 & sc_2 & sc_3 \end{bmatrix}. \end{array}}$$

I trust you have been circumspect about this point — however minute — when you tried Exercise 5 in page 14 of "Review of Lextures — III'. Speaking of, I think this is a good place to revisit that exercise, so let me pull one of the questions therein:

Example 3. For

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 5 & -4 & -1 \\ 1 & -3 & 4 \end{bmatrix}$$

$\big($= part (1) of Ecercise 5, in "Review of Lextures — III"$\big)$, let's find its inverse A^{-1}.

Step 1. First find the determinant of A, as follows:

$$\det A = 2 \cdot \begin{vmatrix} -4 & -1 \\ -3 & 4 \end{vmatrix} - 1 \cdot \begin{vmatrix} 5 & -1 \\ 1 & 4 \end{vmatrix} + \big(-2\big) \cdot \begin{vmatrix} 5 & -4 \\ 1 & -3 \end{vmatrix}$$

$$= 2 \cdot \big(-19\big) - 1 \cdot \big(-21\big) + \big(-2\big) \cdot \big(-11\big)$$

$$= -38 - 21 + 22 = -37.$$

Step 2. Second find the adjoint matrix $\operatorname{adj} A$ of A as follows:

$$\operatorname{adj} A = \begin{bmatrix} +\begin{vmatrix} -4 & -1 \\ -3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ -3 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & -2 \\ -4 & -1 \end{vmatrix} \\ -\begin{vmatrix} 5 & -1 \\ 1 & 4 \end{vmatrix} & +\begin{vmatrix} 2 & -2 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & -2 \\ 5 & -1 \end{vmatrix} \\ +\begin{vmatrix} 5 & -4 \\ 1 & -3 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 5 & -4 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -19 & 2 & -9 \\ -21 & 10 & -8 \\ -11 & 7 & -13 \end{bmatrix}.$$

To conclude,

$$A^{-1} = \frac{1}{-37} \begin{bmatrix} -19 & 2 & -9 \\ -21 & 10 & -8 \\ -11 & 7 & -13 \end{bmatrix}$$

$$= \frac{1}{37} \begin{bmatrix} 19 & -2 & 9 \\ 21 & -10 & 8 \\ 11 & -7 & 13 \end{bmatrix}$$

$$\left(= \begin{bmatrix} \frac{19}{37} & \frac{-2}{37} & \frac{9}{37} \\ \frac{21}{37} & \frac{-10}{37} & \frac{8}{37} \\ \frac{11}{37} & \frac{-7}{37} & \frac{13}{37} \end{bmatrix} \right).$$

- Let me do another example (not from the past exercises):

Example 4. For

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 3 & -5 & 2 \\ 6 & -6 & 2 \end{bmatrix},$$

let's find its inverse A^{-1}.

Step 1. First find the determinant of A, as follows:

$$\det A = 1 \cdot \begin{vmatrix} -5 & 2 \\ -6 & 2 \end{vmatrix} - \left(-3\right) \cdot \begin{vmatrix} 3 & 2 \\ 6 & 2 \end{vmatrix} + 2 \cdot \begin{vmatrix} 3 & -5 \\ 6 & -6 \end{vmatrix}$$

$$= 1 \cdot 2 - \left(-3\right) \cdot \left(-6\right) + 2 \cdot 12$$

$$= 2 - 18 + 24 = 8.$$

Step 2.

$$\operatorname{adj} A = \begin{bmatrix} +\begin{vmatrix} -5 & 2 \\ -6 & 2 \end{vmatrix} & -\begin{vmatrix} -3 & 2 \\ -6 & 2 \end{vmatrix} & +\begin{vmatrix} -3 & 2 \\ -5 & 2 \end{vmatrix} \\ \\ -\begin{vmatrix} 3 & 2 \\ 6 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 6 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} \\ \\ +\begin{vmatrix} 3 & -5 \\ 6 & -6 \end{vmatrix} & -\begin{vmatrix} 1 & -3 \\ 6 & -6 \end{vmatrix} & +\begin{vmatrix} 1 & -3 \\ 3 & -5 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -6 & 4 \\ 6 & -10 & 4 \\ 12 & -12 & 4 \end{bmatrix}.$$

To conclude,

$$\begin{aligned} A^{-1} &= \frac{1}{8} \begin{bmatrix} 2 & -6 & 4 \\ 6 & -10 & 4 \\ 12 & -12 & 4 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 1 & -3 & 2 \\ 3 & -5 & 2 \\ 6 & -6 & 2 \end{bmatrix} \\ &\left(= \begin{bmatrix} \frac{1}{4} & \frac{-3}{4} & \frac{1}{2} \\ \frac{3}{4} & \frac{-5}{4} & \frac{1}{2} \\ \frac{3}{2} & \frac{-3}{2} & \frac{1}{2} \end{bmatrix} \right). \end{aligned}$$

Note. Realize that, in this example, A^{-1} equals $\dfrac{1}{4}A$. This happens rarely.

- **The 3×3 identity matrix.**

Recall that 2×2 identity matrix was $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. What would be its 3×3 counterpart? Yes, it is

$$\boxed{I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.}$$

We call it the 3×3 identity matrix. If you want to be meticuous, you can denote it I_3 to indicate the size. The following two facts are in sync with the 2×2 case:

Fact 1. For $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, we have

$$\boxed{IA = A, \quad \text{and} \quad AI = A.}$$

Fact 2. For $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, suppose

$$\det A \neq 0.$$

Then

$$\boxed{AA^{-1} = I, \quad \text{and} \quad A^{-1}A = I.}$$

Exercise 3. Prove Fact 1 and Fact 2 above $\big($brute-force calculation$\big)$.

- **Gaussian elimination.**

One more stuff before wrapping up today's session. You probably remember the name "Gaussian elimination method", as I mentioned it several times over the course of the semester. I alluded that we are going to cover it at some point. That's going to happen in the next lecture. $\big($Hooray!$\big)$ I thought spending the last ten or so minutes to give some sneak preview of it wouldn't hurt. The above way of finding A^{-1} involves a lot of calculations. Good news: There is a way to reduce the amount of work when A is a concrete matrix filled by numbers, and that's "Gaussian elimination method". Before full disclosure, I suggest we look at some archetypal examples of "Gaussian elimination method". What's potentially confusing is, such $\big($what I would call$\big)$ archetypal examples — such as Example 5 below — make no direct reference to the inverse of a matrix. So don't get freaked out the following example may appear to have nothing to do with inverting a matrix. I will explain everything in the forthcoming lectures, how "Gaussian elimination method" has a bearing on the business of inverting matrices $\big($see "Review of Lectures – IX"; page 6–10$\big)$. Below is a kind that you are all familiar with from high school, yet it best captures the essence of "Gaussian elimination method".

Example 5. Consider the following system of linear equations

$$\begin{cases} x + y + z = 2, \\ -x + 3y + 2z = 8, \\ 4x + y = 4. \end{cases}$$

Let's solve this system *brute-force*, without relying on any formula whatsoever. <u>It goes step-by-step</u>.

Step 1. Multiply 2 to the first equation in the system sidewise. The result is

$$2x + 2y + 2z = 4.$$

Step 2. Subtract it from the second equation in the given system sidewise. The result is

$$-3x + y = 4.$$

Step 3. Subtract it from the third equation in the given system sidewise. The result is

$$7x \;=\; 0.$$

Step 4. Multiply $\dfrac{1}{7}$ to the two sides. The result is

$$x \;=\; 0.$$

Step 5. Go back to Step 2:

$$-3x \,+\, y \;=\; 4.$$

Substitute the outcome of Step 4: $x = 0$. The result is

$$y \;=\; 4.$$

Step 6. Go back to the first equation in the original system:

$$x + y + z \;=\; 2.$$

Substitute the outcomes of Step 4 and Step 5: $x = 0$, $y = 4$. The result is

$$4 \,+\, z \;=\; 2.$$

Solve it for z:

$$z \;=\; -2.$$

In sum, we have obtained the solution

$$\Big(x,\;\; y,\;\; z\Big) \;\;=\;\; \Big(0,\;\; 4,\;\; -2\Big).$$

• And that was some elementary stuff. But like I said, this example fairly depicts the flavor of the "Gaussian elimination method". Our next job is to do exactly the same but using matrices. — To be continued.