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Numerical Integration

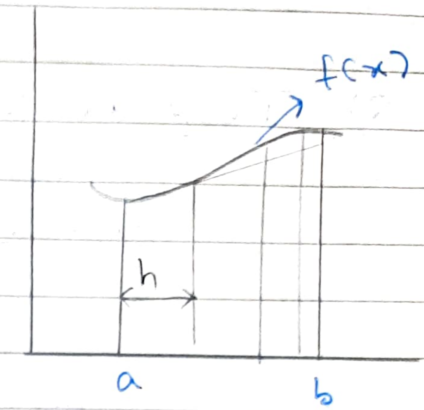
① Integration is area under a curve. $\int_a^b f(x)dx$

② Single application

→ Trapezoidal Rule

→ Simpson's $1/3^{rd}$ Rule

→ Simpson's $3/8^{th}$ Rule



Consider

$$I = \int_a^b f(x)\,dx \qquad \text{let } y = f(x)$$

Let $(b-a)$ be divided into 'n' equal parts each of the width 'h'.

$$h = \frac{b-a}{n}$$

Let $x_0 = a$ $\quad x_1 = a+h$ $\quad x_2 = a+2h$ $\ldots$ $x_n = a+nh$

lly y_0 $\quad y_1$ $\quad y_2$ $\ldots$ y_n

- Trapezoidal Rule [for any value of n]

$$I = h/2\left[(y_0+y_n) + 2(y_1+y_2+y_3+\ldots y_{n-1})\right]$$

- Simpson's $1/3^{rd}$ Rule [where n is multiple of 2]

$$I = \frac{h}{3}\left[(y_0+y_n) + 4\underbrace{(y_1+y_3+\ldots y_{n-1})}_{\text{odd term}} + 2\underbrace{(y_2+y_4+\ldots+y_{n-2})}_{\text{Even}}\right]$$

Simpson's 3/8th Rule [where n is multiple of 3]

$$I = \frac{3h}{8}\Big[(y_0+y_n) + 3(y_1+y_2+y_4+\dots+y_{n-1}) + 2(y_3+y_6+\dots+y_{n-3})\Big]$$

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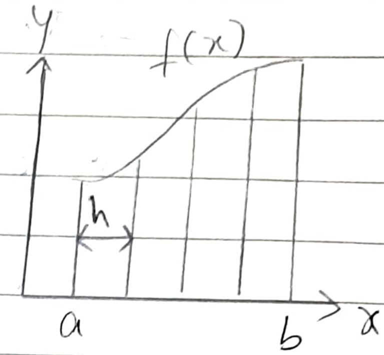
① **Trapezoidal Rule**

$$h = \frac{b-a}{n} \qquad I = \int_a^b f(x)\,dx$$



$$I = \frac{h}{2}\Big[(y_0+y_n) + 2(y_1+y_2+y_3+\dots+y_{n-1})\Big]$$

Given that ;

	x_0	x_1	x_2	x_3	x_4
x	0.25	0.26	0.27	0.28	0.29
$f(x)$	0.2474	0.2571	0.667	0.764	0.2860

$\therefore n = 4$

If Trapezoidal rule is applied then $\int_{0.25}^{0.27} f(x)\,dx = ?$

Solution:

$$h = 0.26 - 0.25 = 0.01$$

$$I = \frac{h}{2}\Big[(y_0+y_n) + 2(y_1+y_2+y_3+\dots+y_{n-1})\Big]$$

$$= \frac{h}{2}\Big[(y_0+y_4) + 2(y_1+y_2+y_3+\cancel{y_4})\Big]$$

$$= \frac{0.01}{2}\Big[(0.2474+0.2860) + 2(0.2571+0.2667+0.2764)\Big]$$

$$= \frac{0.01}{2}\left[(0.2474+0.2860)+1.6004\right]$$

$$= \frac{0.01}{2}\left[0.5334+1.6004\right]$$

$$= 0.010669$$

$$= 0.0107$$

Q The value of integral $\int_0^6 \frac{dx}{1+x^2}$ by trapezoidal rule [Take h=1]

$$h = \frac{b-a}{n}$$

$f(x) = \frac{1}{1+x^2}$ $\quad n = \frac{b-a}{h} = \frac{6-0}{1} = 6$

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	0.0588	0.0385	0.0270

$$I = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\cdots+y_{n-1})\right]$$

$$= \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{2}\left[(1+0.027)+2(0.5+0.2+0.1+0.0588+0.0385)\right]$$

$$= \frac{1}{2}\left[1.027+2(0.8973)\right]$$

$$= \frac{1}{2}\left[2.8216\right]$$

$$I = 1.4108$$

Simpson's $1/3^{rd}$ Rule :

$$I = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+\cdots+y_{n-1}) + 2(y_2+y_4+\cdots+y_{n-2})\right]$$

Q. $f(x) = 5x^3 - 3x^2 + 2x + 1$ from $x = -1$ & $x = 1$ where $(h=1)$ using Simpson's $1/3^{rd}$ Rule

$x_0 = -1 \qquad x_n = 1$

$$h = \frac{1-(-1)}{n}$$

$$\boxed{n = 2}$$

	x_0	x_1	x_2
x	-1	0	1
$f(x)$	-9	1	5

$$I = \frac{h}{3}\left[(y_0+y_2) + 4(y_1)\right] \cancel{+2(y_2+}$$

$$= \frac{1}{3}\left[(-9+5) + 4(1)\right]$$

$$= \frac{1}{3}\left[-4+4\right]$$

$$I = 0$$

Numerical Integration

Q A curve is drawn passing through the points given by

Given:

	x_0	x_1	x_2	x_3	x_4	x_5	x_6
x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

$x_0 = 1 \qquad x_n = 4$

$$I = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+y_5+\dots+y_{n-1}) + 2(y_2+y_4+\dots+y_{n-2})\right]$$

$$= \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

$$= \frac{0.5}{3}\left[(2+2.1)+4(2.4+2.8+2.6)+2(2.7+3)\right]$$

$$= \frac{0.5}{3}\left[(4.1) + 31.2 + 11.4\right]$$

$$I = 7.7833$$

Error Bound

Trapezoidal Rule

for $I = \int_a^b f(x)\,dx \qquad n = \frac{b-a}{h}$

$$|E_T| \le \frac{M_2 (b-a)^3}{12n^2} = \frac{M_2 n h^3}{12}$$

where M_2 = maximum value of $|f''(x)|$ in the interval (a,b).

Important Note:

→ Trapezoidal Rule gives exact Value at integral if $f(x)$ is a polynomial function of degree 0 (or) 1.

→ If the interval at which $f(x)$ is tabulated is halved, the error would be reduced on eighth.

Simpson's Rule:

→ The error term due to Simpson's rule is

$$|E_S| \leq \frac{M_4 (b-a)^5}{180 n^4} = \frac{M_4 n h^5}{180}$$

Where M_4 = maximum value of $|f^{IV}(x)|$ in the interval (a,b).

Important Note:

→ If $f(x)$ is a polynomial function of degree ≥ 3 then Simpson's rule gives exact Value of integral.

→ If the tabulated interval is halved the error is reduced by a factor 32.

→ In general, Simpson's rule is more acurate than a trapezoidal rule with 'n' points gives about as much accuracy as Trapezoidal rule with 2n.

Q. $f(x) = 5x^3 - 3x^2 + 2x + 1$ from $x = -1$ to $x = 1$

$$I = \int_{-1}^{1} (5x^3 - 3x^2 + 2x + 1)\, dx$$

$$E_s \leq \frac{M_4 (b-a)^5}{180 n^4}$$

$M_4 = f^{IV}(x)$

$E_s = \dfrac{0 \times (2)^5}{180 \times n^4}$

$= 0$

$f(x) = 5x^3 - 3x^2 + 2x + 1$

$f'(x) = 15x^2 - 6x + 2$

$f''(x) = 30x - 6$

$f'''(x) = 30$

$f^{IV}(x) = 0$

$\therefore \quad E_s = 0$