Breaking the Nonsmooth Barrier: A Scalable Parallel Method for Composite Optimization

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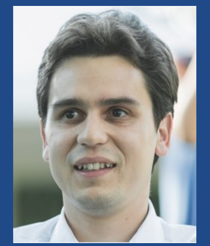


Summary

Optimization methods need to be adapted to the **parallel** setting to leverage modern computer architectures.

Highly efficient variants of stochastic gradient descent have been recently proposed, such as Hogwild [1], Kromagnon [2], ASAGA [3].

They assume that the objective function is smooth, so are inapplicable to problems such as Lasso, optimization with constraints, etc.

Contributions:

1. **Sparse Proximal SAGA**, a sparse variant of the linearly-convergent proximal SAGA algorithm.
2. **ProxASAGA**, the first parallel asynchronous variance-reduced method that supports composite objective functions.

Problem Setting

Objective: develop parallel asynchronous method for problems of the form

$$\underset{\boldsymbol{x}\in\mathbb{R}^p}{\text{minimize}}\ \frac{1}{n}\sum_{i=1}^n f_i(\boldsymbol{x}) + h(\boldsymbol{x}) ,$$

- f_i is differentiable with L-Lipschitz gradient.
- h is block-separable ($h(x) = \sum_B h_B([\boldsymbol{x}]_B)$) and "simple" in the sense that we have access to $\mathbf{prox}_{\gamma h} \stackrel{\text{def}}{=} \mathbf{arg\,min}_x\, \gamma h(x) + \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{z}\|^2$.

$\Longrightarrow$ includes Lasso, group Lasso or ERM with box constraints.

Variance-reduced stochastic gradient methods are natural candidates due to their state of the art performance and recent asynchronous variants.

The **SAGA algorithm** [4] maintains current iterate $\boldsymbol{x} \in \mathbb{R}^p$ and table of historical gradients $\boldsymbol{\alpha} \in \mathbb{R}^{n\times p}$. At each iteration, it samples an index $i \in \{1, \ldots, n\}$ and computes next iterate $(\boldsymbol{x}^+, \boldsymbol{\alpha}^+)$ as

$$\boldsymbol{x}^+ = \mathbf{prox}_{\gamma g}(\boldsymbol{x} - \gamma(\nabla f_i(\boldsymbol{x}) - \boldsymbol{\alpha}_i + \overline{\boldsymbol{\alpha}})) ,\ \ \boldsymbol{\alpha}_i^+ = \nabla f_i(\boldsymbol{x}) . \qquad \text{(SAGA)}$$

Difficulty of a Composite Extension

- Existing methods exhibit best performance when updates are sparse.
- Even in the presence of sparse gradients, the (SAGA) update is not sparse due to the presence of $\overline{\boldsymbol{\alpha}}$ and $\mathbf{prox}$.
- Existing convergence proofs bound noise from asynchrony using by the Lipschitz of the gradient. Property does not extend to composite case.

A new sequential algorithm: Sparse Proximal SAGA

The algorithm relies on the following quantities

- Extended support T_i: set of blocks that intersect with ∇f_i.

$$T_i \stackrel{\text{def}}{=} \{B : \text{supp}(\nabla f_i) \cap B \neq \varnothing,\ B \in \mathcal{B}\}$$

- For each block $B \in \mathcal{B}$, $d_B \stackrel{\text{def}}{=} n/n_B$, where $n_B := \sum_i \mathbb{1}\{B \in T_i\}$ is the number of times that $B \in T_i$.
- $\boldsymbol{D}_i$ is a diagonal matrix defined block-wise $[\boldsymbol{D}_i]_{B,B} \stackrel{\text{def}}{=} d_B \mathbb{1}\{B \in T_i\}\boldsymbol{I}_{|B|}$.
- φ_i is a block-wise reweighting of h: $\varphi_i \stackrel{\text{def}}{=} \sum_{B\in T_i} d_B h_B(\boldsymbol{x})$

Justification. The following properties are verified

$\varphi_i(\boldsymbol{x})$ is zero outside T_i $\quad$ $\boldsymbol{D}_i\boldsymbol{x}$ is zero outside T_i $\quad$ (sparsity)

$\mathbb{E}_i\, \varphi_i = h$ $\quad$ $\mathbb{E}_i\, \boldsymbol{D}_i = \boldsymbol{I}$ $\quad$ (unbiasedness)

Algorithm. As SAGA, it maintains current iterate $\boldsymbol{x} \in \mathbb{R}^p$ and table of historical gradients $\boldsymbol{\alpha} \in \mathbb{R}^{n\times p}$. At each iteration, it samples an index $i \in \{1, \ldots, n\}$ and computes next iterate $(\boldsymbol{x}^+, \boldsymbol{\alpha}^+)$ as

$$\boldsymbol{v}_i = \nabla f_i(\boldsymbol{x}) - \boldsymbol{\alpha}_i + \boldsymbol{D}_i\overline{\boldsymbol{\alpha}}\,;\ \boldsymbol{x}^+ = \mathbf{prox}_{\gamma\varphi_i}\left(\boldsymbol{x} - \gamma\boldsymbol{v}_i\right)\,;\ \boldsymbol{\alpha}_i^+ = \nabla f_i(\boldsymbol{x})$$

Features

- Iteration cost $\mathcal{O}(|T_i|)$ (only coefficients in T_i are updated).
- Easy to implement (compared to the lagged update approach [5]).
- Parallelizable.

Convergence Analysis

For step size $\gamma = \frac{1}{5L}$ and f μ-strongly convex ($\mu > 0$), Sparse Proximal SAGA converges geometrically in expectation. At iteration t we have

$$\mathbb{E}\|\boldsymbol{x}_t - \boldsymbol{x}^*\|^2 \leq (1 - \tfrac{1}{5}\min\{\tfrac{1}{n}, \tfrac{1}{\kappa}\})^t C_0 ,$$

with $C_0 = \|\boldsymbol{x}_0 - \boldsymbol{x}^*\|^2 + \frac{1}{5L^2}\sum_{i=1}^n \|\boldsymbol{\alpha}_i^0 - \nabla f_i(\boldsymbol{x}^*)\|^2$ and $\kappa = \frac{L}{\mu}$ (condition number).

Implications

- Adaptivity to strong convexity, i.e., no need to know strong convexity parameter to obtain linear convergence.
- In the "big data regime" ($n \geq \kappa$): convergence rate is $\mathcal{O}(1/n)$.
- In the "ill-conditioned regime" ($n \leq \kappa$): convergence rate is $\mathcal{O}(1/\kappa)$.

A New Parallel Algorithm: Proximal Asynchronous SAGA

The Proximal Asynchronous SAGA (ProxASAGA) runs

Inconsistent read: no locks while reading, i.e., read the vector while another core might be writing to it.

Algorithm 1 ProxASAGA

Initialize shared variables $\boldsymbol{x}$, $(\boldsymbol{\alpha}_i)_{i=1}^n$, $\overline{\boldsymbol{\alpha}}$
keep doing in parallel
 Sample i uniformly in $\{1, ..., n\}$
 $[\hat{\boldsymbol{x}}]_{T_i}$ = inconsistent read of $\boldsymbol{x}$ on T_i
 $\hat{\boldsymbol{\alpha}}_i$ = inconsistent read of $\boldsymbol{\alpha}_i$
 $[\overline{\boldsymbol{\alpha}}]_{T_i}$ = inconsistent read of $\overline{\boldsymbol{\alpha}}$ on T_i
 $[\delta\boldsymbol{\alpha}]_{S_i} = [\nabla f_i(\hat{\boldsymbol{x}})]_{S_i} - [\hat{\boldsymbol{\alpha}}_i]_{S_i}$
 $[\hat{\boldsymbol{v}}]_{T_i} = [\delta\boldsymbol{\alpha}]_{T_i} + [\boldsymbol{D}_i\overline{\boldsymbol{\alpha}}]_{T_i}$
 $[\delta\boldsymbol{x}]_{T_i} = [\mathbf{prox}_{\gamma\varphi_i}(\hat{\boldsymbol{x}} - \gamma\hat{\boldsymbol{v}})]_{T_i} - [\hat{\boldsymbol{x}}]_{T_i}$
 for B **in** T_i **do**
  for b **in** B **do**
   $[\boldsymbol{x}]_b \leftarrow [\boldsymbol{x}]_b + [\delta\boldsymbol{x}]_b$ $\quad$ ▷ atomic
   if $b \in$ support(∇f_i) **then**
    $[\overline{\boldsymbol{\alpha}}]_b \leftarrow [\overline{\boldsymbol{\alpha}}]_b + {}^1\!/\!{}_n[\delta\boldsymbol{\alpha}]_b$ $\quad$ ▷ atomic
   end if
  end for
 end for
 $\boldsymbol{\alpha}_i \leftarrow \nabla f_i(\hat{\boldsymbol{x}})$ $\quad$ (scalar update) $\quad$ ▷ atomic
end parallel loop

Perturbed iterates framework

Problem: Analysis of parallel algorithms is *hard*.
Solution: Cast them as sequential algorithms working on *perturbed* inputs.
Distinguish:

- $\hat{\boldsymbol{x}}_t$: inconsistent quantity read by the cores
- $\boldsymbol{x}_t$: the *virtual* iterate defined by

$$\boldsymbol{x}_{t+1} \stackrel{\text{def}}{=} \boldsymbol{x}_t - \gamma\boldsymbol{g}_t \ \text{ with } \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{v}, i) = \tfrac{1}{\gamma}\big(\hat{\boldsymbol{x}}_t - \mathbf{prox}_{\gamma\varphi_i}(\hat{\boldsymbol{x}}_t - \gamma\hat{\boldsymbol{v}}_{i_t})\big) ,$$

Interpret $\hat{x}_t$ as a noisy version of $\boldsymbol{x}_t$ due to asynchrony.

Analysis preliminaries

Definition (measure of sparsity). Let $\Delta := \max_{B\in\mathcal{B}} |\{i : T_i \ni B\}|/n$. This is the normalized maximum number of times that a block appears in the extended support. We always have ${}^1\!/\!{}_n \leq \Delta \leq 1$.
Definition (delay bound). τ is a uniform bound on the *maximum delay* between two iterations processed concurrently.

Convergence guarantee of ProxASAGA

Suppose $\tau \leq \frac{1}{10\sqrt{\Delta}}$. For any step size $\gamma = \frac{a}{L}$ with $a \leq a^*(\tau) := \frac{1}{36}\min\{1, \frac{6\kappa}{\tau}\}$, the inconsistent read iterates of ProxASAGA converge in expectation at a geometric rate factor of at least: $\rho(a) = \frac{1}{5}\min\left\{\frac{1}{n}, a\frac{1}{\kappa}\right\}$, i.e. $\mathbb{E}\|\hat{\boldsymbol{x}}_t - \boldsymbol{x}^*\|^2 \leq (1-\rho)^t\,\tilde{C}_0$, where $\tilde{C}_0$ is a constant independent of t ($\approx \frac{n\kappa}{a}C_0$ with C_0 as defined in Theorem **??**).

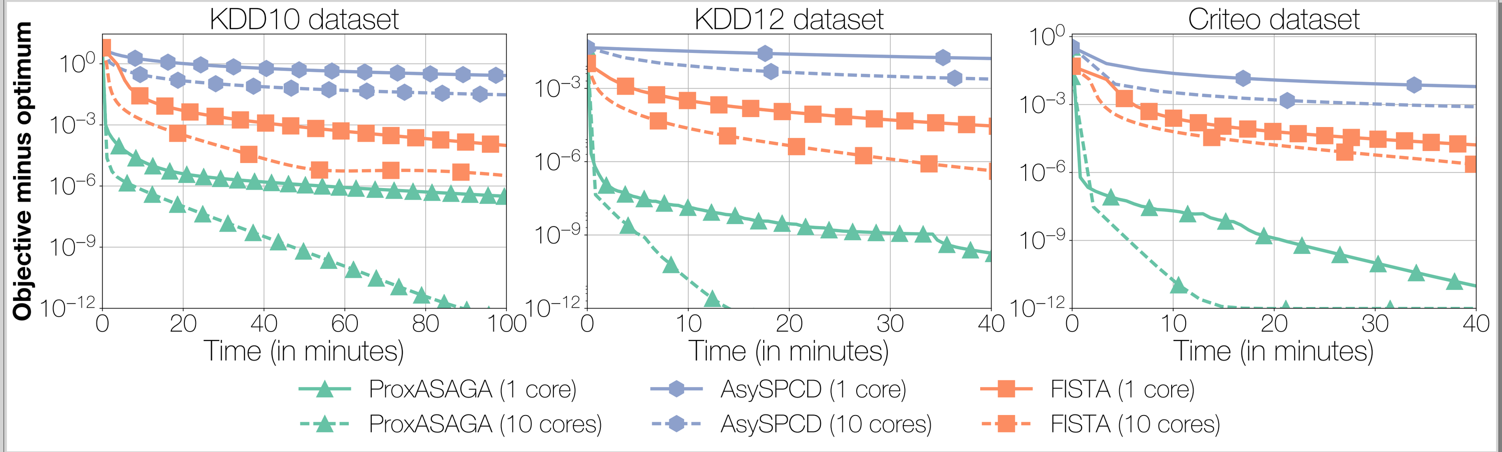
Experimental results

Comparison on 3 large-scale datasets

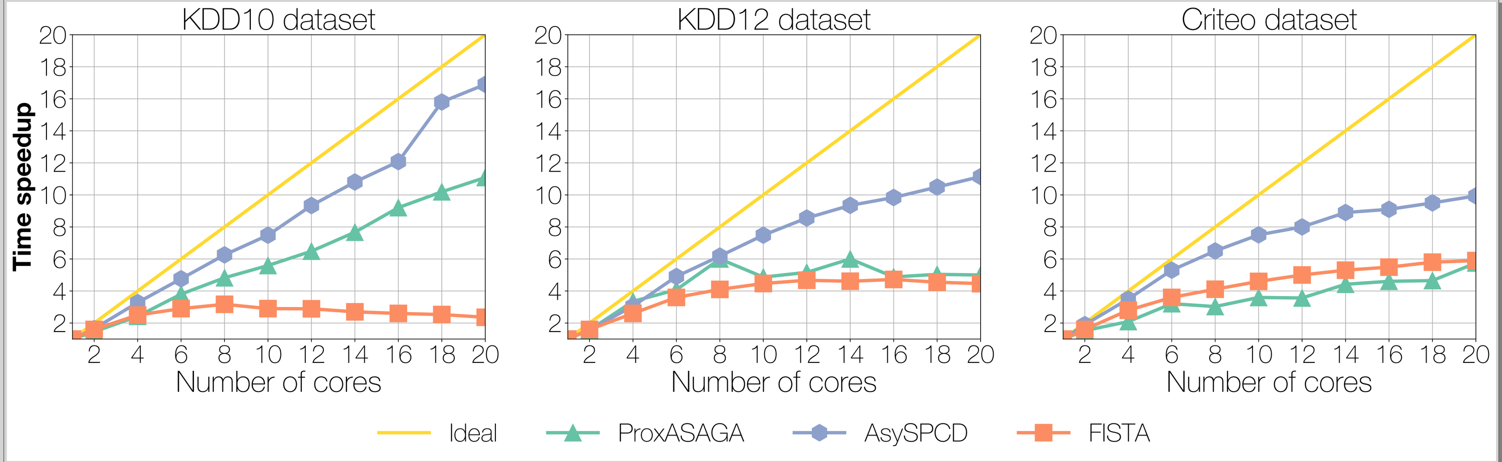
Large-scale logistic regression

Dataset	n	p	density	L	Δ
KDD 2010	19,264,097	1,163,024	10^{-6}	28.12	0.15
KDD 2012	149,639,105	54,686,452	2×10^{-7}	1.25	0.85
Criteo	45,840,617	1,000,000	4×10^{-5}	1.25	0.89

$$\frac{1}{n}\sum_{i=1}^n \log\left(1 + \exp(-b_i\boldsymbol{a}_i^\intercal\boldsymbol{x})\right) + \frac{\lambda_1}{2}\|\boldsymbol{x}\|_2^2 + \lambda_2\|\boldsymbol{x}\|_1 ,$$



Speedup



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