Acknowledgments and Dedication

To my parents, who have spent their entire life working hard just to provide me the best for everything, even at their own expenses. They have been teachers of life values that I have rarely found in someone else.

To my advisor, Professor Paola Mosconi, a guiding light in a stormy sea, without whom this final project would have been barely possible. When on the point of giving up, her call for perseverance was fundamental. I thank her for this new mindset she has helped me to develop, that will shape all my future professional life. I am very glad for having her as my advisor, I have learnt a lot during this journey.

To my professional and life mentor Ivan Vaghi, who, although his incommensurable knowledge in everything, being it business or coding, holds a very rare humility. I have had with him the best lunches of my life, with placemats becoming blackboards for his lessons on circuit logics and cryptography. I am very glad to have met him in a very delicate moment of my life.

A special thanks to my grandmother, who has taking care of me because of my parents' abnormal working hours. To her, life has never been easy, although she has never lost her big and warming smile. Even if financially demanding, while raising me, she has always avoided that I could experience her same lifelong regrets of not having undergone formal schooling as she did because of post-war poverty. Her curiosity is literally insane. She has always tried to instill in me her same passion in knowledge.

Special thanks to professor Marc Henrard, whose suggestions on the topic of rational models has been vital for the realization of this project. To him goes my deep admiration for his contribution to the opensource community by the realization of frameworks for issues in quantitative finance.

From here, my thanks to the millions of people improving the human heritage by freely sharing their knowledge worldwide through the Open Science and the Open Source communities. Expecially to my lifelong idle Aaron Swartz, with him the world has lost a great man.

Contents

1 Introduction

Interest rates in the last ten years have witnessed some major changes. As an effect of the 2007 crisis, which has seen the basis spreads between interbank offer rates with different tenors to widen, market players have started to use separate curves for forwarding and discounting purposes, thus paving the way to a transition from the single to the multi-curve framework for the term structure of interest rates. Later on, as the credit crisis spread to sovereign debts, with the decisions of FED Fund and the ECB to allow for almost vanishing interest rates, interbank rates (OIS, X-ibors,...) drastically dropped. This trend became even more severe after the end of year 2012, when the OIS-based curve for the EUR currency and then other X-ibors fell in the realm of negative forward rates.

Multi-curve setting and negative rates have had an enormous impact on the modelling of the term structure of interest rates and the evaluation methodologies of financial instruments. This project aims to address the two issues just mentioned, by studying in depth a class of models, the so called Rational Models, which, for their flexibility, could represent a good candidate to describe the actual interest rate environment. In the subsequent years to their introduction, in 1996 by Flesaker & Hughston, Rational Models had been investigated only marginally. Among them Rutkowski(1997)[12], Döberlein and Schweizer(2001)[13], Hunt and Kennedy(2004)[14], Brody and Hughston(2004)[15], Hughston and Rafailidis (2005)[17]. Brody, Hughston and Mackie(2012)[16],Akahori, Hishida, Teichmann and Tsuchiya(2014)[18], Filipović, Larsson and Trolle(2014)[19], Macrina and Parbhoo(2014)[20], Nguyen and Seifried(2014). However, with the increasing focus towards the multicurve setting, Rational Models, showing their potentials in dealing with multiple curves, have become the object of a renewed interest by Crépey, Macrina, Nguyen and Skovmand(2015)[1], Trillos, Henrard and Macrina(2016)[10].

Besides their flexibility, rational models allow to price non-plain vanilla products (e.g. basis swaps) and to naturally maintain price consistency under different probability measures (this feature being highly desirable for the management of risk exposures [Crépey(2015)[1]]. In this project we focus on a specific task, i.e. the calibration of rational models to the term structure of interest rates, exploiting the information that the market encodes into the quotes of swaption implied volatilities. We choose two instances of Rational Models (namely a one-factor and a two-factor lognormal models) and calibrate both to the market smile and to the term structure of co-terminal swaptions. We first perform this calculation at the same trading date used by Crépey[1] in order to check our results and frame the multi-curve environment and then repeat the calculation at a recent date in order to capture the effect of negative interest rates.

In section 2 we focus on the changes made by the multi-curve setting in the computation of forward swap rates. Rational Models are introduced in section 3, where, along with their specification, a general framework is presented. The pricing formulae exhibited here are proved in Appendix C using the techniques presented in Appendix B. The results of the calibration procedures are reported in section 4. Here, as regards the overall calibration, in subsection 4.1 along with a general algorithmic procedure we provide a summary table about the evolution of the relevant parameters. The main Matlab functions coded for the implementation of the procedure are reported in Appendix D.

2 Single vs Multi-curve

In this section we present the main implications that the introduction of the multi-curve setting has brought to the evaluation of forward swap rates. These quantities, being the strikes of the ATM swaptions we will deal with, are of great interest for our project. Although just the starting point and not the main focus of our research, for the sake of completeness we have decided to add in Appendix A an introduction to the basics of fixed income market and the methodology implemented under a single-curve setup for the computation of forward rates.

In a nutshell, the leading difference between the single and multi-curve approach refers to the number of curves used for the computation of discount factors and forward rates. Indeed, in the case of the former, as the name suggests, one single risk-free yield curve where everything is computed from should be expected. Oppositely, as regards the latter, different curves are expected to be used, one specific for the computation of discount factors and others for forward rates. In other words, they differ on the number of the ingredients needed. Counterparty risk and liquidity will mainly explain the roots of these differences

2.1 Forward Swap Rates

Consider the case of entering into an interest rate swap with a future starting date, known as forward starting swap. In the basic example two investors exchange floating against fixed amounts of money. Our aim is to find the equilibrium fixed rate that makes the value of the swap fair, meaning null price at inception. Such a quantity will be of high interest for our analysis since, as said above, represents the strike rate for plain vanilla ATM swaptions. The computation strictly depends on the curve setting we are dealing with, being it single or multi-curve.

To understand these differences, we will introduce firstly the case of a single yield curve and then, by comparison, move towards a scenario with more curves. Discount factors and forward rates are derived from the same set of rates in the single-curve approach. This will play the role of explaining why equilibrium swap rates will not depend on the frequency of the contract's payments. To highlight this point we now formalize the problem. We deal with a general case where T_0 is the first reset date of the swap that can be also the valuation date t, being it today or a future period. Define with T_0 and T_n respectively the occurrence of the first reset and the last payment. We also assume that the last payment of the floating leg coincides with the maturity of the underlying. We refer with Δ_{Fix} and Δ_{Fl} to the accrual factor of the fix and floating leg respectively. According to practice, we use the term payer (receiver) to indicate a swap where the fix rate has to be paid (received).The cash-flows of a payer swap are:

$$\sum_{i=1}^{n-1} L\left(T_i, T_{i+1}\right)\Delta_{Fl} - \sum_{j=1}^{n'-1} S\left(t, T_0, T_n\right)\Delta_{Fix}$$

Where $S\left(t, T_0, T_n\right)$ is the swap rate of a contract with inception t, first reset in T_0 and T_n as maturity of the underlying. To compute such a quantity a non-arbitrage reasoning is needed. We say that the equilibrium rate S is the one that makes null the current price. Starting from the cash-flows we can separately derive the value of the fixed and floating leg as follows:

$$\begin{aligned} PV\left(Fixed\right) &= \sum_{j=0}^{n'-1} S\left(t, T_0, T_n\right)\Delta_{Fix} P\left(t, T'_{j+1}\right) \\ PV\left(Float\right) &= \sum_{i=0}^{n-1} P\left(t, T_{i+1}\right)\Delta_{Fl} PV\left(L\left(T_i, T_{i+1}\right)\right) \end{aligned} \tag{2.1}$$

Where t is the valuation date, $T_0, T_1, \ldots.., T_n$ and $T'_1 \ldots.., T'_{n'}$ are the sets of relevant periods for the floating and the fixed leg respectively. Since they can differ from each other we have opted for a general notation. In fact, while for the floating leg the first relevant date is the reset, for the fixed one is the first payment. We would like to recall that $P\left(t, T\right)$ is the t-price of a zero coupon that pays a unitary amount in T. We use this quantity to discount future cash-flows. But, as reported in Appendix A.1 those should not be confused with discount factors, which are stochastic(at most deterministic).

The difference between prices and discounts that can rather seem redundant to the reader, it is actually vital to understand the multi-curve thesis. As stressed in the Appendix, the relation that links them together is:

$$P\left(t, T\right) = \mathbb{E}_t^{\mathbb{Q}}\left[D\left(t, T\right)\right] \tag{2.2}$$

The presence of the expected value in (2.2) shows that risk matters and that different type of rates, e.g. 3-month and 6-month can lead to different term structures: $T \to P^{3m}\left(t, T\right),\ T \to P^{6m}\left(t, T\right)$.

Back to the swap, one problem is to find the present value of the floating coupons, indicated as $PV\left(L\left(T_i, T_{i+1}\right)\right)$ in (2.1). To solve it, in the single-curve world, we can refer to the replicating strategy that is reported in Appendix A.2, where it is shown that the non-arbitrage present value of a future floating rate is equivalent to the current forward rates. We recall that:

$$PV\left(L\left(T_i, T_{i+1}\right)\right) = F\left(t, T_i, T_{i+1}\right) = \left(\frac{P\left(t, T_i\right)}{P\left(t, T_{i+1}\right)} - 1\right)\frac{1}{\Delta}$$

From it we can derive the value of the swap's floating leg as:

$$PV\left(Float\right) = \sum_{i=1}^{n-1} P\left(t, T_{i+1}\right)\Delta_{Fl}\left(\frac{P\left(t, T_i\right)}{P\left(t, T_{i+1}\right)} - 1\right)\frac{1}{\Delta_{Fl}} \tag{2.3}$$

Before moving forward, name the quantity $\sum_{j=1}^{n'-1} \Delta_{Fix} P\left(t, T_j'\right)$ as $A\left(t\right)$, standing for annuity. Once we are able to compute the present value of future floating rates, then we can solve for $S\left(t, T_0, T_n\right)$ the following equation: $PV\left(Float\right) - PV\left(Fixed\right) = 0$. From this the equilibrium swap rate is equal to:

$$S\left(t, T_0, T_n\right) = \frac{P\left(t, T_0\right) - P\left(t, T_n\right)}{A\left(t\right)} \tag{2.4}$$

Where the numerator, that is the value of the floating leg, happens to be independent from the payment frequencies. This is due to the fact that both the discounts and the forward rates are computed on the same curve as displayed in (2.3).

This is the way swap rates were computed before the introduction of multi-curve. But the divergences between market rates and equilibrium quantities arose some questions over whether the practice of using one single curve was sounded. In the multi-curve setting the forward swap rate is computed according to:

$$S\left(t, T_0, T_n\right) = \frac{\sum_{i=1}^{n-1} P\left(t, T_{i+1}\right)\Delta_{Fl}\left(\frac{P^f\left(t, T_i\right)}{P^f\left(t, T_{i+1}\right)} - 1\right)\frac{1}{\Delta_{Fl}}}{A\left(t\right)} \tag{2.5}$$

Where P and P^f are derived from different sets of rates and their difference makes the frequency counts in the valuation of the floating leg so that the numerator cannot be simplified as in (2.4).

2.2 Does Bloomberg quote in multi-curve?

The first step of our project was to find the setting used by Bloomberg to compute the forward rates and, consequently, the forward starting swap rates. Using the raw data of

the provider we have tried to replicate its swap quotes in order to speculate about its implemented algorithms. As valuation date we have chosen the 4^{th} of January 2011.
One way to collect information on those rates is to use the "Quick Pricer For Swaption", shown with the Bloomberg volatility cube(VCUB). In fact, we know that the ATM strike rate for a swaption is the forward starting swap rate $S(t, T_0, T_n)$.

We consider the case where T_0 is the maturity of the swaption and $(T_n - T_0)$ is the swap tenor, that is the length in time of the underlying. Using OIS as risk-free curve to discount future quantities, we have implemented formulae (2.4) and (2.5) to compute the forward starting swap rates both under the single and the multi-curve setting and both for 3-month and 6-month frequencies. In the case of the former we have used 3-month Euribor, while for the latter the 6-month version of the same rate. Our doubt was about whether Bloomberg used (2.4) or (2.5). From our findings reported in Appendix A.3 we can conclude that Bloomberg on the 4^{th} of January 2011 quoted under a multi-curve setting.

3 Rational Models

In this section we follow Crépey(2015) in the presentation of the models of interest, focusing on the one and two-factor lognormal frameworks. We report here the closed form-solutions that we have derived in Appendix C.

3.1 General Framework

Before going deeply into our class of models, a general framework for interest rate modelling is presented. Our starting point is a probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ where $\mathbb{P}$ stands for the real probability measure and $\{\mathcal{F}_t\}_{t\geq 0}$ denotes the market filtration, that is the set of information possessed up to the time of interest. Such a probability space is assumed to respect all the usual conditions. Under this setting the no-arbitrage pricing formula for an asset with a price process $\{S_{tT}\}_{T\geq t\geq 0}$ and a payoff at time T equal to S_{TT} is given by:

$$S_{tT} = \frac{1}{\pi_t}\mathbb{E}^{\mathbb{P}}[\pi_T S_{TT}|\mathcal{F}_t] \tag{3.1}$$

where the process $\{\pi_t\}_{T\geq t\geq 0}$ is known as the state price deflator or pricing kernel.
We have referred to the probability measure $\mathbb{P}$ using the word "real".

We would like to spend some words on this topic in order to give a clear view on the different probability measures. This is necessary to understand how we have derived the analytical solutions to the models presented next, where we have deeply made use of the the so called "change of measure" technique. The reader can refer to Appendix B for a more detailed discussion on the topic. Here we want to highlight that among all the words used for naming $\mathbb{P}$, the one that really helps in grasping the concept is: De Finetti's probability. The Italian mathematician is well known for having emphasized the role that subjectivity plays into the evaluation of probabilities.

Therefore, under $\mathbb{P}$, market participants have different expectations. This is why it is used for risk management purposes rather than pricing problems. Indeed, for ensuring the uniqueness of the prices, different probability measures need to be considered. To this aim, we introduce $\mathbb{Q}$ as being an equivalent martingale measure(EMM) to $\mathbb{P}$. We refer to $\mathbb{Q}$ as the "risk-neutral " measure since under it all the agents are assumed to be neutral to risk in economic terms. Under $\mathbb{Q}$ the numeraire is chosen to be the money market account $B\left(t,T\right)$, that is the value in T of one unit of currency invested in t.

The numeraire is such that the quantities relative to it are martingales under its specific measure. Where the term martingale simply means that, on average, the future values equal the current ones. For instance, a random variable X is said to be a martingale under a measure M if $X_t = \mathbb{E}^{\mathbb{M}}[X_T|\mathcal{F}_t]$, $\forall t < T$.

Being $\frac{1}{B}$ the discount factor we can easily see how discounted quantities are martingale under the measure $\mathbb{Q}$. In the case of a generic contingent claim, its price at time t is:

$$S_{tT} = B_t \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{S_{TT}}{B_T}\right| \mathcal{F}_t\right] \tag{3.2}$$

From now on, we use $\mathbb{E}_t^{\mathbb{M}}$ instead of $\mathbb{E}^{\mathbb{M}}[\ldots|\mathcal{F}_t]$. We show what the relation between (3.1) and (3.2) is and how we can move from $\mathbb{P}$ to $\mathbb{Q}$ using the Radon-Nikodym density process denoted by Z and defined as:

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = Z_t \tag{3.3}$$

By using Bayes' rule we can go from the risk-neutral to the subjective measure $\mathbb{P}$:

$$S_{tT} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{B_t}{B_T} S_{TT}\right] = \frac{\mathbb{E}_t^{\mathbb{P}}\left[Z_T \frac{B_t}{B_T} S_{TT}\right]}{Z_t} = \frac{\mathbb{E}_t^{\mathbb{P}}\left[\frac{Z_T}{B_T} S_{TT}\right]}{\frac{Z_t}{B_t}}$$

Where we can see that $\pi_t = \frac{Z_t}{B_t}$, the state-state price deflator previously encountered, has now a valid meaning. Furthermore, if we consider the case of an instrument that pays a single unitary amount in T, its value in t is:

$$P(t, T) = \frac{1}{\pi_t} \mathbb{E}_t^{\mathbb{P}}[\pi_T] \tag{3.4}$$

that is the no-arbitrage t-price of a T-zero coupon bond.

From this result we can derive a proposition whose usefulness was strongly defended before the appearance of negative interest rates. But now it easy to understand why in the current scenario of below-zero rates could be discarded. We report and prove it for consistency.

Proposition 1. *The term-structure $T \to P(t,T)$ implied by equation (3.4) is non-increasing for all $t \geq 0$ **if and only if** the state-price deflator π is a positive super-martingale with respect to $\mathbb{P}$*

Proof. Firstly, we show that if $P(t,T)$ is decreasing then π is a positive supermartingale:

$$1 = P(t,t) \geq P(t,T) = \frac{1}{\pi_t} \mathbb{E}_t^{\mathbb{P}}[\pi_T]$$

Hence,

$$\pi_t \geq \mathbb{E}_t^{\mathbb{P}}[\pi_T] \ , \ Vt \leq T$$

On the other hand, starting from the positive supermartingality assumption should lead us to the decreasing term structure. Indeed, if π is a positive supermartingale under $\mathbb{P}$, for any $t \leq T \leq S$:

$$P(t,S) = \frac{1}{\pi_t}\mathbb{E}_t^{\mathbb{P}}[\pi_S] = \frac{1}{\pi_t}\mathbb{E}_t^{\mathbb{P}}\left[\mathbb{E}_T^{\mathbb{P}}[\pi_S]\right] \leq \frac{1}{\pi_t}\mathbb{E}_t^{\mathbb{P}}[\pi_T] = P(t,T)$$

by using the law of iterated expectation, also known as tower property, we end up with a decreasing term structure. $\square$

3.1.1 Short rate

Typically the main variable for modelling purposes has been the so called short interest rate, defined as:

$$r_t = -(\delta_T \ln P(t,T))|_{T=t}$$

assuming the existence of the derivative with respect to the maturity term. According to proposition 1, the process $\{r_t\}$ is non-negative if $\{\pi_t\}$ is a $\mathbb{P}$-positive supermartingale and viceversa. The aim of researchers and practitioners is finding a suitable model for this quantity, such that their assumptions fit as best as possible the market reality. Moreover, the evolution of interest rates towards negative figures have made useless lots of the previous findings.

From here, our need for a setting capable of dealing with scenarios where the usefulness of proposition 1 is lost as a consequence of negative rates. In our work, we focus on a class of models, introduced by Flesaker&Hughston(1996)[9] and defined as “Rational Models”, that since its introduction have been analyzed only marginally by few researchers. In fact, only in the recent past years, due to their theoretical flexibility with respect to the sign of interest rates and the suitability to the multi-curve setting, they have been represented in the major academic journals.

3.1.2 Forward Libor Process

Consider a financial derivative having the Libor rate as its underlying, we are interested in deriving a dynamic process of the forward Libor, in order to price such an instrument. In other words, we would like to infer about the laws governing the forward rate's evolution in time. We focus our analysis on pricing problems under a multi-curve setting and introduce, for now, the case of a Forward Rate Agreement, FRA, to show what will be our main variables of interest.

For the sake of our argument, we do not, actually, consider FRA market standards. We just define them as instruments that exchange at a specified future date a fixed leg against a floating leg. We consider here the payer version, where the investor receives the floating amount and pays the fixed one. The relevant dates are $\{t, T_{i-1}, T_i\}$ with a payoff at time T_i equal to:

$$\Pi_T = N\Delta_i \left(L\left(T_{i-1}, T_i\right) - K\right)$$

where N is the notional and $\Delta_i = T_i - T_{i-1}$.

In Appendix A.2 we have shown how to derive the value of the stochastic future Libor under a single-curve setting, that is by implementing the no-arbitrage relationship with the replicating strategy. But after the shift of market standards towards the multi-curve, the assumption of the existence of a unique curve is no more consistent. That is why we now provide a general approach that, going beyond any types of setting, it is, actually, valid both for the single and the multi-curve case.

It is useful to see that: $L\left(T_{i-1}, T_i\right) = L\left(T_{i-1}, T_{i-1}, T_i\right)$ showing the equivalence between a spot rate starting at a future date and a forward rate where the inception and the reset fall in the same period, in this case T_{i-1}. By using the general pricing rules of the previous subsection, we have the following:

$$L\left(t, T_{i-1}, T_i\right) = \frac{1}{\pi_t}\mathbb{E}_t^{\mathbb{P}}\left[\pi_{T_i} L\left(T_{i-1}, T_{i-1}, T_i\right)\right] \tag{3.5}$$

where $L(T_{i-1}, T_{i-1}, T_i)$ is $\mathcal{F}_{T_{i-1}}$ measurable.

Under the single-curve we had a closed-form solution for $L\left(t, T_{i-1}, T_i\right)$. While now, mainly because of liquidity issues and credit risk, Libor process, that is how $L\left(t, T_{i-1}, T_i\right)$ evolves in time, needs to be modelled. The aim of rational models is to specify such process. Focusing on this quantity, it is clear, why this setup is a perfect choice for a multi-curve setting. With the term "rational" we refer to the fact that the variables of interest will be expressed as ratios of polynomials.

3.2 Model Specification

For the models here analyzed, the t-price of a financial asset with a payoff paid in T, is defined, as:

$$S_{tT} = \frac{S_{0T} + b_2(T) A_t^{(2)} + b_3(T) A_t^{(3)}}{P_{0t} + b_1(t) A_t^{(1)}} \tag{3.6}$$

where $P_{0t} = P(0,t)$ and S_{0T} are respectively the values in 0 of the t-zcb and the financial asset. For i=1,2,3, the b_i are deterministic functions and $A_t^{(i)} = f(t, X_t^i)$ are martingales under an equivalent martingale measure $\mathbb{M}$ and are driven by $\mathbb{M}$-Markov processes. In other words, those As processes are the only sources of randomness and are defined as function of a particular class of stochastic processes that possesses nice modelling properties.

There could have been more driving risky factors in the numerator, but adding more of them is not cost-free and could lead to over-fitting problems. We would like to have both a flexible and an efficient model. It can be shown, by using the change of measure, that the pricing kernel π can be rewritten as:

$$\pi_t = \frac{\pi_0}{M_0}\left[P_{0t} + b_1(t) A_t^{(1)}\right] M_t \tag{3.7}$$

where the process$\{M_t\}$ is a martingale under $\mathbb{P}$ and allows us to move to a different measure $\mathbb{M}$, under which the As are martingales.

Being P_{0t} and $b_1(t)$ deterministic functions, depending only on time, their definition is important for the properties possessed by $P_{0t} + b_1(t) A_t^{(1)}$. For instance if we want to work under a setting with a term structure consistent with proposition 1, then, they need to be specified such that this linear combination is a non-negative $\mathbb{M}$-martingale to ensure the $\mathbb{P}$-supermartingality of π_t.

By changing measure and applying the definition of short rate, equation(3.8) can be easily found. We need to highlight that it holds in general and that any constraint is at researcher's discretion.

$$P_{tT} = \frac{P_{0T} + b_1(T) A_t^{(1)}}{P_{0t} + b_1(t) A_t^{(1)}} \qquad and \qquad r_t = -\frac{\dot{P}_{0t} + \dot{b}_1(t) A_t^{(1)}}{P_{0t} + b_1(t) A_t^{(1)}} \tag{3.8}$$

where the dot notation stands for the differentiation with respect to time.

By using equation (3.5) and (3.6) we can derive the forward libor process as follows:

$$L(t, T_{i-1}, T_i) = \frac{L(0, T_{i-1}, T_i) + b_2(T_{i-1}, T_i) A_t^{(2)} + b_3(T_{i-1}, T_i) A_t^{(3)}}{P_{0t} + b_1(t) A_t^{(1)}} \tag{3.9}$$

3.3 From the short-rate to the forward process

Following Crépey(2015) we now consider the lowest possible kernel and from there we will derive our results. Such an object is the already encountered short-rate. Using the right side of equation(3.8), we directly model it as:

$$r_t = -\frac{\dot{c}_1(t) + \dot{b}_1(t) A_t^{(1)}}{c_1(t) + b_1(t) A_t^{(1)}} \tag{3.10}$$

Assuming that $b_1(t)$ and $c_1(t)$, with $c_1(0) = 1$, are non-increasing deterministic functions, and that the $\mathcal{F}_t$ measurable $A_t^{(1)}$ is a $\mathbb{M}$-martingale with zero-initial point, $A_0^{(1)} = 0$, we have that:

$$h_t = c_1(t) + b_1(t) A_t^{(1)} \tag{3.11}$$

is $\mathcal{F}_t$-measurable and an $\mathbb{M}$-supermartingale for $t > 0$.

Note that none of these assumptions is necessary for working with the current market conditions. They have been introduced here because our first application of the model will be in a non-negative interest rates scenario in which the term-structure is still consistent with proposition 1. But the existence of rates below zero is no more utopian and can lead to non-trivial modelling problems. This can be seen, for instance, in the basic version of the CIR process with its diffusion coefficient showing the square root of the short rate. Here we are limited by the core nature of the model. Conversely, rational models can still be implemented despite of the sign of interest rates. Actually, in case of negative rates even less constraints to the driving processes are required.

Therefore, to summarize, we start from an idea on how the main quantities behave in general and then we add possible constraints such that the market reality is respected. This aspect will play a very important role in the calibration phase of section 4, in order to add the right constraints, if any, to fairly model the real world.
Now we move from $\mathbb{Q}$ to $\mathbb{M}$. As previously done, we use the Radon-Nikodym derivative to reach our purpose. Consider a process $\{\mu_t\}_{0 \le t \le T}$ where:

$$\mu_t = \left.\frac{d\mathbb{Q}}{d\mathbb{M}}\right|_{\mathcal{F}_t}$$

Denoting $D_t = e^{-\int_0^t r_s ds}$ as the discount factor under the risk-neutral measure, we have the following lemma:

Lemma 1. $h_t = D_t \mu_t$

For the proof of lemma 1 the reader can refer to page 7 of Crépey(2015).

This link between the two probability measures have been relevant for the derivation of our solutions. As already seen, the numeraire of the risk-neutral measure is the money market account $B(0,t) = \frac{1}{D_t}$. From this, we can derive the t-price of an OIS T-bond using lemma 1:

$$P_{tT} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{D_T}{D_t}\right] = \frac{1}{D_t\mu_t}\mathbb{E}_t^{\mathbb{M}}\left[D_T\mu_T\right] = \mathbb{E}_t^{\mathbb{M}}\left[\frac{h_T}{h_t}\right] = \frac{c_1(T) + b_1(T)A_t^{(1)}}{c_1(t) + b_1(t)A_t^{(1)}}$$

Comparing this result with (3.8) we see that $c_1(t) = P_{0t}$.

The change of measure between $\mathbb{Q}$ and $\mathbb{M}$ is such that the relative quantities are martingale under the specific setting. For instance:

$$P_{tT} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{D_T}{D_t}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{B(0,t)}{B(0,T)}\right] = B(0,t)\,\mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{B(0,T)}\right] \Rightarrow \frac{P_{tT}}{B(0,t)} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{B(0,T)}\right]$$

we see how the values, written relatively to the numeraire, are martingales under $\mathbb{Q}$. For moving to $\mathbb{M}$, we have started from the knowledge that h_t works as a state-price deflator for this measure and that the numeraire is equal to its reciprocal. Using our findings, we can now take a stochastic future quantity and price it either under $\mathbb{Q}$ or $\mathbb{M}$.

In our specification, we define the future Libor rate as a ratio of polynomials driven by a finite number of $\mathbb{M}$-martingale processes

$$L(T_{i-1},T_i) = L(T_{i-1},T_{i-1},T_i) = \frac{L(0,T_{i-1},T_i) + b_2(T_{i-1},T_i)A_{T_{i-1}}^{(2)} + b_3(T_{i-1},T_i)A_{T_{i-1}}^{(3)}}{P_{0t} + b_1(t)A_{T_{i-1}}^{(1)}}$$

where the only sources of randomness are the $\mathcal{F}_{T_{i-1}}$- measurable $A_{T_{i-1}}^{(i)}$ $\mathbb{M}$-martingales. We just consider at most two different possibly correlated As.

$L(T_{i-1},T_{i-1},T_i)$ is a stochastic quantity and our next purpose is the computation of its evolution in time. For $t < T_{i-1}$, the no-arbitrage pricing rule under the risk-neutral measure is such that:

$$L(t,T_{i-1},T_i) = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{D_{T_i}}{D_t}L(T_{i-1},T_{i-1},T_i)\right] \tag{3.12}$$

Equation (3.12) contains an expectation that it is not trivial to compute.

We do not know what the distribution of $L(T_{i-1},T_{i-1},T_i)$ under $\mathbb{Q}$ is. Even worse, it is far from easy evaluating how L and D jointly behave under this measure. Notwithstanding, we can directly changing the probability measure by mean of Radon-Nikodym and Bayes's theory:

$$L(t,T_{i-1},T_i) = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{D_{T_i}}{D_t}L(T_{i-1},T_{i-1},T_i)\right] = \mathbb{E}_t^{\mathbb{M}}\left[\frac{D_T\mu_T}{D_t\mu_t}L(T_{i-1},T_{i-1},T_i)\right] =$$
$$= \mathbb{E}_t^{\mathbb{M}}\left[\frac{\mathbb{E}_{T_{i-1}}^{\mathbb{M}}[h_T]}{h_t}L(T_{i-1},T_{i-1},T_i)\right]$$

where in the last passage the tower property has been used.

We derive the Libor process as:

$$L(t,T_{i-1},T_i) = \frac{L(0,T_{i-1},T_i) + b_2(T_{i-1},T_i)A_t^{(2)} + b_3(T_{i-1},T_i)A_t^{(3)}}{P_{0t} + b_1(t)A_t^{(1)}}$$

That is equal to what we had already found in (3.9). Again here, we stress how, under rational models, the positivity of the Libor forward rates strictly depends on the class of processes used for the specification of the As appearing in the numerator.

For instance, if for $i = 1,2,3$; $A_t^{(i)} = S_t^{(i)} - 1$, with $S_t^{(i)}$ being a positive $\mathbb{M}$-martingale and $S_0^{(i)} = 1$, as in the case of a unit-initialized Lévy martingale, then the positivity of the Libor is ensured if:

$$0 \leq b_2(T_{i-1},T_i) + b_3(T_{i-1},T_i) \leq L(0,T_{i-1},T_i) \tag{3.13}$$

3.3.1 Pricing Formulae

Our next question is how to find a price for a derivative with an underlying dependent on the Libor rate. For the scope of this project we deal with plain vanilla products of European type, with the aim to find closed form formulae for the pricing problems. In this way, we are able to estimate the parameters that minimize, in statistical terms, the difference between models and markets quotes. This will be presented in the section on calibration. Once those parameters are estimated then the model can be used for pricing more exotic products.

Here we consider the case of a European payer swaption, where the investor buys the right to enter in a spot-starting swap by paying a strike equal to K. We deal with options on spot starting swap, meaning that the inception of the swap and its reset date coincide with the maturity of the swaption. The duration of the swap contract will take the name of tenor. To respect the practical application of the proposed models, here we consider a general case, where the floating and the fixed leg can have different time structures.

At maturity the payoff of a swaption is equal to:

$$\Pi\mathrm{swpn}_{T_k} = N\left(\sum_{i=1}^{l} L\left(T_k, T_{i-1}, T_i\right)\delta_i^{Float} - \sum_{j=1}^{m} KP_{T_k T_j'}\delta_j^{Fix}\right)^+ \tag{3.14}$$

We now want to price such a payoff. For completeness, all the passages will be reported, in order to highlight and stress the usefulness of the change of measure. Such a technique has played a major role in solving lots of pricing issues in the fixed income market. As always, we start from what we know.

Therefore, our starting point will be the risk-neutral $\mathbb{Q}$, from there we will move towards other measures. According to our notation, the price in t of a swaption expiring in T_k is:

$$Swn_{tT_k} = \frac{N}{D_t}\mathbb{E}_t^{\mathbb{Q}}\left[D_{T_k}\left(\sum_{i=1}^{l} L\left(T_k, T_{i-1}, T_i\right)\delta_i^{Float} - \sum_{j=1}^{m} KP_{T_k T_j'}\delta_j^{Fix}\right)^+\right]$$

where T_k is the maturity of the swaption and $T_n - T_k$ is the tenor of the underlying. The indexes l and m could be both equal to n if the last payment dates for both the fixed and the floating leg coincide with the expiration of the underlying swap. With T_0 we define the first reset date and with T_1 the occurrence of the first payment for the floating leg. While with T_1' we refer to the first payment date of the fixed leg. Therefore we are in a general case where the fixed and the floating leg may pay at different times. This justifies the presence of two different accrual factors δ. According to our notation, the price in t of a swaption expiring in T_k is:

$$\begin{aligned}
Swn_{tT_k} &= \frac{N}{D_t}\mathbb{E}_t^{\mathbb{Q}}\left[D_{T_k}\left(\sum_{i=1}^{l} L\left(T_k, T_{i-1}, T_i\right)\delta_i^{Float} - \sum_{j=1}^{m} KP_{T_k T_j'}\delta_j^{Fix}\right)^+\right] = \\
&= \frac{N}{D_t\mu_t}\mathbb{E}_t^{\mathbb{M}}\left[D_{T_k}\mu_{T_k}\left(\sum_{i=1}^{l} L\left(T_k, T_{i-1}, T_i\right)\delta_i^{Float} - \sum_{j=1}^{m} KP_{T_k T_j'}\delta_j^{Fix}\right)^+\right] = \\
&= \frac{N}{h_t}\mathbb{E}_t^{\mathbb{M}}\left[h_{T_k}\left(\sum_{i=1}^{l} L\left(T_k, T_{i-1}, T_i\right)\delta_i^{Float} - \sum_{j=1}^{m} KP_{T_k T_j'}\delta_j^{Fix}\right)^+\right] = \\
&= \frac{N}{P_{0t} + b_1\left(t\right)A_t{}^{(1)}}\mathbb{E}_t^{\mathbb{M}}\left[\left(Float - Fix\right)^+\right)\right]
\end{aligned}$$

Where

$$Float = \sum_{i=1}^{l}\left(L\left(0, T_{i-1}, T_i\right) + b_2\left(T_{i-1}, T_i\right)A_{T_k}{}^{(2)} + b_3\left(T_{i-1}, T_i\right)A_{T_k}{}^{(3)}\right)\delta_i^{Float}$$

$$Fix = \sum_{j=1}^{m} \left[K \left(P_{0T_j'} + b_1 \left(T_j' \right) A_{T_k}{}^{(1)} \right) \right] \delta_j^{Fix}$$

In case the valuation date is $t = 0$, and recalling $A_t^{(i)} = S_t^{(i)} - 1$, we have:

$$Swn_{0T_k} = N\mathbb{E}^{\mathbb{M}} \left[\left(c_2 A_{T_k}^{(2)} + c_3 A_{T_k}^{(3)} - c_1 A_{T_k}^{(1)} + c_0 \right)^+ \right] = N\mathbb{E}^{\mathbb{M}} \left[\left(c_2 S_{T_k}^{(2)} + c_3 S_{T_k}^{(3)} - c_1 S_{T_k}^{(1)} + \widetilde{c_0} \right)^+ \right] \tag{3.15}$$

where

$$c_2 = \sum_{i=1}^{l} b_2 \left(T_{i-1}, T_i \right) \delta_i^{Float}, \quad c_3 = \sum_{i=1}^{l} b_3 \left(T_{i-1}, T_i \right) \delta_i^{Float}, \quad c_1 = K \sum_{j=1}^{m} b_1 \left(T_j' \right) \delta_j^{Fix}$$

$$c_0 = \sum_{i=1}^{l} L \left(0, T_{i-1}, T_i \right) \delta_i^{Float} - \sum_{j=1}^{m} K P_{0T_j'} \delta_j^{Fix}, \qquad \widetilde{c_0} = c_0 + c_1 - c_2 - c_3$$

3.4 One-factor Lognormal Model

Under the one-factor lognormal framework the stochasticity is injected by just one driving risky process $\left\{ A_t^{(2)} \right\}_{0 \le t \le T}$ while for the remaining quantities we have $\left\{ A_t^{(1)} \right\}_{0 \le t \le T} = \left\{ A_t^{(3)} \right\}_{0 \le t \le T} = 0$.
The risky term is defined as:

$$A_t^{(2)} = e^{\left(a_2 X_t^{(2)} - \frac{1}{2} a_2^2 t \right)} - 1 \tag{3.16}$$

where $\left\{ X_t^{(2)} \right\}_{0 \le t \le T}$ is a $\mathbb{M}$-Standard Brownian Motion and a_2 is a real constant. In this case we have that $\widetilde{c_0} = c_0 - c_2$.
Using (3.15) it can be show that the 0-price of a T_k-swaption under this setting is equivalent to:

$$Swn_{0T_k} = N \left(c_2 \Phi \left(\frac{\frac{1}{2} a_2^2 T_k - \ln \left(-\widetilde{c_0} / c_2 \right)}{a_2 \sqrt{T_k}} \right) + \widetilde{c_0} \Phi \left(\frac{-\frac{1}{2} a_2^2 T_k - \ln \left(-\widetilde{c_0} / c_2 \right)}{a_2 \sqrt{T_k}} \right) \right) \tag{3.17}$$

where Φ stands for the standard normal cumulative function.

In case $\widetilde{c_0} \ge 0$ then: $Swn_{0T_k} = N c_0$. All the relevant passages for deriving (3.17) have been reported in Appendix C.1.

3.5 Two-factor Lognormal Model

We now increase the number of risky processes. We assume that the stochasticity comes from two different $\mathbb{M}$-martingales, which can show some degree of linear dependence measured by a parameter ρ. For $i = 1, 2, 3$ we have:

$$A_t^{(i)} = e^{\left(a_i X_t^{(i)} - \frac{1}{2} a_i^2 t\right)} - 1 \tag{3.18}$$

for a real constant a_i and standard Brownian motions $\left\{X_t^{(1)}\right\}_{0 \le t \le T} = \left\{X_t^{(3)}\right\}_{0 \le t \le T}$ and $\left\{X_t^{(2)}\right\}_{0 \le t \le T}$ with a correlation equal to ρ. In this case the price in $t = 0$ of a swaption expiring in T_k is:

$$Swn_{0T_k} = N\mathbb{E}^{\mathbb{M}}\left[\left(c_2 e^{\left(a_2 X\sqrt{T_k} - \frac{1}{2}a_2^2 T_k\right)} + c_3 e^{\left(a_3 Y\sqrt{T_k} - \frac{1}{2}a_3^2 T_k\right)} - c_1 e^{\left(a_1 Y\sqrt{T_k} - \frac{1}{2}a_1^2 T_k\right)} + \widetilde{c_0} \right)^+\right] \tag{3.19}$$

where $X \sim \mathcal{N}(0,1)$, $Y \sim \mathcal{N}(0,1)$ and $(X|Y) = y \sim \mathcal{N}(\rho y, (1-\rho^2))$.

Since the quantities inside the expectation in (3.19) are continuous processes, the expected value is computed by mean of a double integral. Recalling that the jointly probability distribution density $f(x,y)$ can be written as $f(x,y) = f(x|y) f(y)$. We have that:

$$\begin{aligned}
Swn_{0T_k} &= N \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(c_2 e^{\left(a_2 x\sqrt{T_k} - \frac{1}{2}a_2^2 T_k\right)} - K(y)\right)^+ f(x|y) f(y)\, dx dy = \\
&= N \int_{K(y)>0} \left(\int_{-\infty}^{\infty} \left(c_2 e^{\left(a_2 x\sqrt{T_k} - \frac{1}{2}a_2^2 T_k\right)} - K(y)\right)^+ f(x|y)\, dx \right) f(y)\, dy + \\
&+ N \int_{K(y)>0} \left(\int_{-\infty}^{\infty} \left(c_2 e^{\left(a_2 x\sqrt{T_k} - \frac{1}{2}a_2^2 T_k\right)} - K(y)\right) f(x|y)\, dx \right) f(y)\, dy
\end{aligned} \tag{3.20}$$

where

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}; \quad f(x|y) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{(x-\rho y)^2}{2(1-\rho y)}}$$

$$K(y) = c_1\left(e^{\left(a_1 y\sqrt{T_k} - \frac{1}{2}a_1^2 T_k\right)} - 1\right) - c_3\left(e^{\left(a_3 y\sqrt{T_k} - \frac{1}{2}a_3^2 T_k\right)} - 1\right) - c_0 + c_2$$

Note that $K(y)$ is random by construction. But it becomes a real constant when we consider $f(x|y)$, since in this case y is treated as known.

The swaption price can be simplified further to a single integral as follows:

$$\begin{aligned}
Swn_{0T_k} &= \int_{K>0} \left[c_2 e^{a_2\sqrt{T_k}\rho y - \frac{1}{2}a_2^2 T_k \rho^2} \Phi^{\mathbb{M}}(d_1) - K(y)\Phi^{\mathbb{M}}(d_2)\right] f(y) dy + \\
&+ \int_{K<0} \left(c_2 e^{a_2\sqrt{T_k}\rho y - \frac{1}{2}a_2^2 T_k \rho^2} - K(y)\right) f(y)\, dy
\end{aligned} \tag{3.21}$$

where $\Phi^{\mathbb{M}}$ means that the values of d_1 and d_2 have been computed under the measure $\mathbb{M}$ and are equal to:

$$d_1 = \left(\frac{\ln \frac{c_2}{K(y)} + a_2^2 T_k \left(\frac{1}{2} - \rho^2\right) + a_2\sqrt{T_k}\rho y}{a_2\sqrt{T_k}\sqrt{(1-\rho^2)}} \right)$$

$$d_2 = \left(\frac{\ln \frac{c_2}{K(y)} - \frac{1}{2} a_2^2 T_k + a_2\sqrt{T_k}\rho y}{a_2\sqrt{T_k}\sqrt{(1-\rho^2)}} \right)$$

The proof for (3.21) has been reported into Appendix C.2, where we have implemented a Black Like approach. For validating our results we have then also used the formulae reported in AppendixE of Brigo&Mercurio[3].

3.6 Advantages of Rational Models

In our setting we have seen that both the Libor process and the zero coupon price present at their denominator the quantity h, that we have shown to be the pricing kernel under the $\mathbb{M}$ measure. As far as the main advantages of this model are concerned, three can be highlighted:

1. The rational form of $\{L(t, T_{i-1}, T_i)\}$ and $\{P_{tT}\}$ is such that when discounted under $\mathbb{M}$ they become a linear combination of the $\mathbb{M}$-martingales placed in the numerator.

2. The Libor process and OIS discount factor originate from different sources. The former coming from the stochasticity of the numerator and the latter from that of the denominator.

3. Modelling the Libor process directly allows for higher flexibility.

4 Calibration

4.1 Procedure

The frameworks presented and derived in section 3 have been calibrated on two different valuation dates, respectively the 4^{th} of January 2011 and the 4^{th} of January 2016. The choice of these dates has been driven by the aim of comparatively evaluate models' behaviour between a positive and a negative interest rates scenario.

Our calibration procedure is focused on 3-month tenor European swaption products, where the frequencies for the floating and the fixed leg are respectively quarterly and annually. The relevant discounting and forwarding curves have been computed under a multi-curve setting, calibrating on different sets of raw data and with a cubic spline interpolation.
In order to be consistent with market standards an ACT/360 convention has been used for the computations of the relevant quantities.
We have run our calibration procedure on prices rather than volatilities[1].

The estimation of the full set of parameters that optimally identifies the models defined in section 3 has been performed in two different sub-procedures. At first, we have considered the case of a 9x1 swaption with nine different strikes centered around the forward starting swap rate. Then we have focused on the coterminal ATM swaptions from 9x1 to 1x9. The objective function to be minimized in the procedure can be generally defined as:

$$\sum_{i=1}^{n} \left(P_i^{Mkt} - P_i^{Model}\right)^2 \tag{4.1}$$

where P_i^{Mkt} is the i^{th}-quote computed by mean of Black76, or Bachelier, on the i^{th}-volatility quoted in the market. P_i^{Model} is the price derived by the application of our model to a swaption with the same characteristics. The specification of P_i^{Model} depends on the frameworks of interest, being it one-factor or two-factor.

Moving forward to the implementation of the procedure, in the calibration of the 9x1 smile the relevant strikes have been taken at the following distances, in basis points, from the ATM : [-200 -100 -50 -25 0 25 50 100 200]. We refer to this step as smile calibration. In this case $n = 9$ and we define the parameters set to be estimated with Φ_1. Therefore,

[1]The objective function can also be defined comparing market and model implied volatilities. We have preferred quotes since they are unambiguously defined.

the completion of this step has resulted in the computation of the estimate $\hat{\Phi}_1$ that has been stored since required next.

After the 9x1 smile, we have then moved towards the estimation of the term structure of the co-terminal ATM swaptions. This is made by the $(10 - T, T)$ products, with $T \in [1, 9]$. Since already performed, for the ATM 9x1 option the previous founded quote has been used. Therefore, in the co-terminal sub-procedure, to estimate the remaining parameters we have considered from the 8x2 to the 1x9 option. Having the ATM swaptions different timing among each other, the calibration of their term structure is done sequentially. Meaning that the ATM swaptions are considered one at a time. In this case we have $n = 1$ in (4.1) at each step of the co-terminal procedure.

In order to deal sequentially with the co-terminal ATM swaptions is then necessary to store all the parameters estimated as the procedure goes by. We refer to the collection of the parameters estimated up to the current stage as $Stack\hat{\Phi}$, which stores, in sequence, all the relevant time-dependent optimal parameters. The minimization of (4.1) for a single ATM swaption results in the estimation of a $\hat{\Phi}'$ that is added to the parameters collection $Stack\hat{\Phi}$.

Table 4.1 shows how the parameters set increases in each step of the procedure. Here Φ defines an unknown(not stochastic) quantity, while $\hat{\Phi}$ is its estimate. In Algorithm 1 we report the logical implementation of the overall calibration. The following points are the basis of the procedure:

1. Computation of the relevant discounting and forwarding curves
2. Closed-form formula (e.g. Black76[2] or Bachelier[22]) applied on the Bloomberg volatility cube to derive swaptions' prices
3. Optimization procedure for parameters estimation
4. Valuation of model efficiency

As far as market volatilities are concerned we have collected them from the volatility cube (VCUB) listed by Bloomberg. Where we have chosen quantities computed on 3-month tenor products using OIS as discount curve. Moreover the provider gives the user the optionality to chose between lognormal and normal volatilities.

We have then applied Black76 to lognormal quantities and Bachelier for the normal ones in order to compute swaptions' quotes. Indeed, while no problems arise in scenario of

Algorithm 1 Pseudo-code Full Calibration

procedure CALIBRATION

volatilitiesMkt ← collects relevant market volatilities

swpnQuotes ← computes swaption quotes Black/Bachelier

9x1 Smile Calibration:

non linear least square: swpnQuotes9x1 - modelQuotes9x1

return *optimalParameters*

parametersStack = add(optimalParameters)

optimal9x1Quotes ← model swaption quotes given *optimalParameters*

optimal9x1Volatilities ← *implied volatilities* from *optimal9x1Quotes*

plot *optimal9x1Volatilities* vs *market9x1Volatilities*

Co-terminal Calibration:

ATMSwaptions ← ATM swaptions' quotes Black/Bachelier

for each *anAtmSwaption* **in** *ATMSwaptions* **do**

{

non linear least square: anATMSwaption - anATMmodelquote

return *optimalParameters*

parametersStack = add(optimalParameters)

anOptimalATMPrice ← ATM swaption quotes using *parametersStack*

anATMvolatility ← compute implied volatility for *anOptimalATMPrice*

optimalATMVolatilities = add(anATMvolatility)

}

plot *optimalATMVolatilities* vs *marketATMVolatilities*

	Parameters Estimates in Time			
Step	Objective index n	Input Parameters	Estimates	Stack
9x1 Smile	9	Φ_1	$\hat{\Phi}_1$	$Stack\hat{\Phi} = \hat{\Phi}_1$
ATMs:				
8x2	1	$Stack\hat{\Phi} + \Phi_2$	$\hat{\Phi}_2$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_2$
7x3	1	$Stack\hat{\Phi} + \Phi_3$	$\hat{\Phi}_3$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_3$
6x4	1	$Stack\hat{\Phi} + \Phi_4$	$\hat{\Phi}_4$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_4$
5x5	1	$Stack\hat{\Phi} + \Phi_5$	$\hat{\Phi}_5$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_5$
4x6	1	$Stack\hat{\Phi} + \Phi_6$	$\hat{\Phi}_6$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_6$
3x7	1	$Stack\hat{\Phi} + \Phi_7$	$\hat{\Phi}_7$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_7$
2x8	1	$Stack\hat{\Phi} + \Phi_8$	$\hat{\Phi}_8$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_8$
1x9	1	$Stack\hat{\Phi} + \Phi_9$	$\hat{\Phi}_9$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_9$

Table 4.1: Time-evolution of the parameters stack

positive interest rates, in our second valuation period, the -200bps option shows a negative strike, making the logarithmic evaluation impossible.

The functions that derive swaptions' quotes from the relevant volatilities have been named as “priceBlack” and “priceNormal”and reported in the Appendix. Before moving forward, we have positively checked that the quotes computed by our Bachelier and Black functions were consistent not only with each other, when possible, but also with the ones reported by Bloomberg's swaption pricer.

Figure 4.1 visually summarizes the procedure. In case of normal volatilities, Black Formula needs to be substituted with Bachelier.

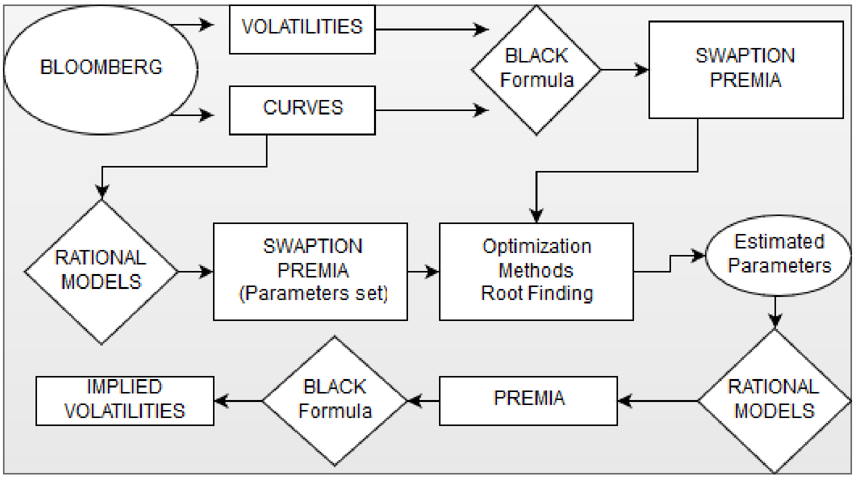


Figure 4.1: Calibration Procedure

4.2 Positive Interest Rates

4.2.1 One-factor Lognormal

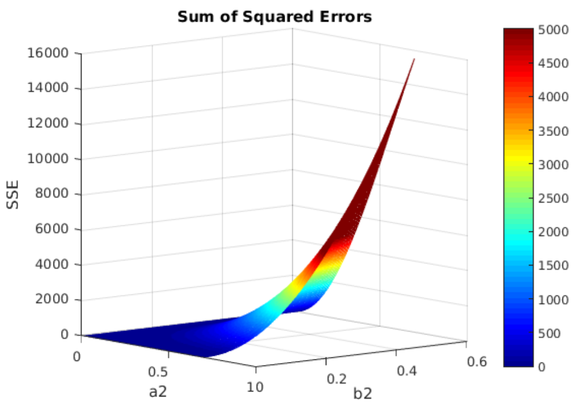
In this scenario the valuation date is the 4^{th} of January 2011.

We have chosen this specific period, not only for the positive interest rates setting but also for a comparison with Crépey(2015). In this way we could have a benchmark to evaluate the consistency between our and his used forwarding curves. Indeed, the author in his work reports $L(0, 9.75, 10) = 0.0328$, the same quantity in our case takes a value of 0.032827. To estimate the relevant quantities of equation (3.17) we have applied the procedure in Algorithm 1. In Table 4.2 we have reported the evolution of the parameters stack for the one-factor framework as already done for the general case exhibited in Table 4.1. In this case, the red superscripts imply that the values differ in time.

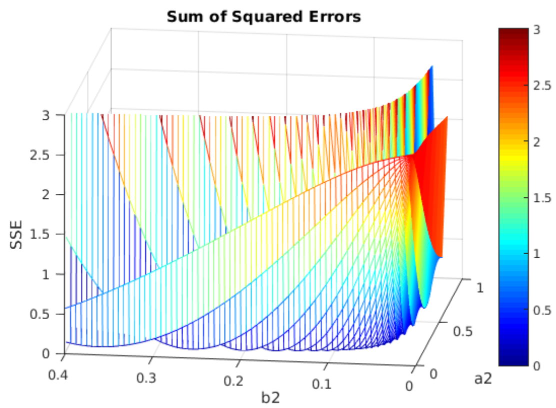
Following Algorithm 1, the first step to be performed is the smile calibration. In this case the model prices in (4.1) are computed implementing the one-factor pricing formula in (3.17). Before lunching the optimization procedure we have tried to find meaningful starting points and constraints to be chosen for our algorithms. To this aim, we have performed some analysis in order to understand how the objective function behaves by changing the parameters. That is why, in Figure 4.2a we have plotted, for the 9x1 smile the evolution of the sum of squared errors against the parameters. In Figure 4.2b we have zoomed the results around the origin for a better understanding of the problem. We have then run different scenarios analysis in which couples of the driving parameters have been taken into account in order to choose the right bounds. Furthermore, the

	Parameters Estimates in Time, one-factor			
Step	Objective index n	Input Parameters	Estimates	Stack
9x1 Smile	9	a_2, b_2^{9x1}	$\hat{\Phi}_1 = [\hat{a}_2, b_2^{9x1}]$	$Stack\hat{\Phi} = \hat{\Phi}_1$
ATMs:				
8x2	1	$Stack\hat{\Phi} + b_2^{8x2}$	$\hat{\Phi}_2 = \hat{b}_2^{8x2}$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_2$
7x3	1	$Stack\hat{\Phi} + b_2^{7x3}$	$\hat{\Phi}_3 = \hat{b}_2^{7x3}$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_3$
6x4	1	$Stack\hat{\Phi} + b_2^{6x4}$	$\hat{\Phi}_4 = \hat{b}_2^{6x4}$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_4$
5x5	1	$Stack\hat{\Phi} + b_2^{5x5}$	$\hat{\Phi}_5 = \hat{b}_2^{5x5}$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_5$
4x6	1	$Stack\hat{\Phi} + b_2^{4x6}$	$\hat{\Phi}_6 = \hat{b}_2^{4x6}$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_6$
3x7	1	$Stack\hat{\Phi} + b_2^{3x7}$	$\hat{\Phi}_7 = \hat{b}_2^{3x7}$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_7$
2x8	1	$Stack\hat{\Phi} + b_2^{2x8}$	$\hat{\Phi}_8 = \hat{b}_2^{2x8}$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_8$
1x9	1	$Stack\hat{\Phi} + b_2^{1x9}$	$\hat{\Phi}_9 = \hat{b}_2^{1x9}$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_9$

Table 4.2: Time-evolution of the parameters stack, one-factor



(a) SSE as function of a_2, b_2



(b) Zoom SSE

Figure 4.2: SSE of 9x1 smile objective function

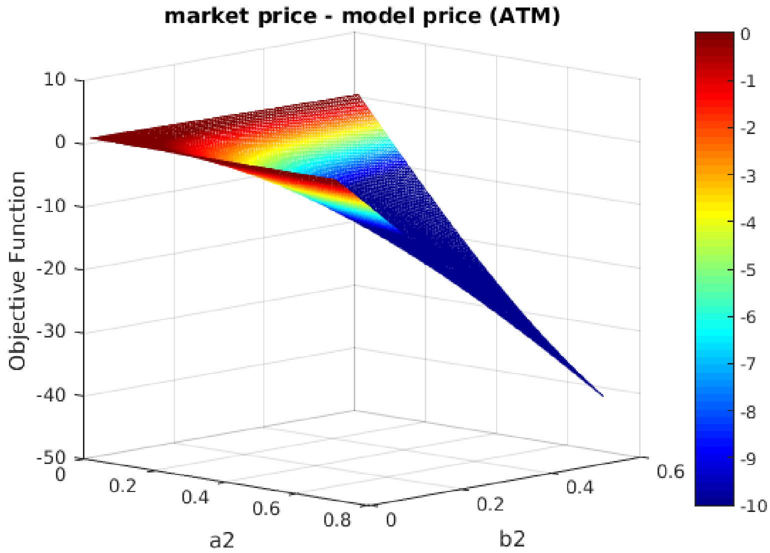


Figure 4.3: ATM market VS model

choice of the conditions imposed for the calibration procedure has been enforced by the analysis of Figure 4.3. Here we have plotted the difference between market and model 9x1 ATM prices with respect to different parameters combinations. As far as the algorithms implemented are concerned we have used the "lsqnonlin " Matlab function for solving a non-linear least square problem. The options, the solver engine, the starting points and the constraints chosen have been reported in Appendix D.3. We have opted to use the 'PlotFcns' option to have a clear view on how the objective function reacts to changes into the set of parameters. This is a very useful tool since helps us to debug visually the optimization procedure. We have estimated the following values of interest:

$$a_2 = 0.05, \quad b_2 = 0.11974 \quad SSE = 0.007$$

The magnitude of the sum of squared errors(SSE) cannot be statistically ignored. There is, indeed, a non-negligible amount of discrepancy. But being the one factor model driven by a single risky factor this result is relatively satisfactory.

After the smile calibration, according to Algorithm 1, we have used the estimated parameters to price the swaptions for the relevant strikes and compare market and model implied volatilities. Our results are plotted in Figure 4.4.

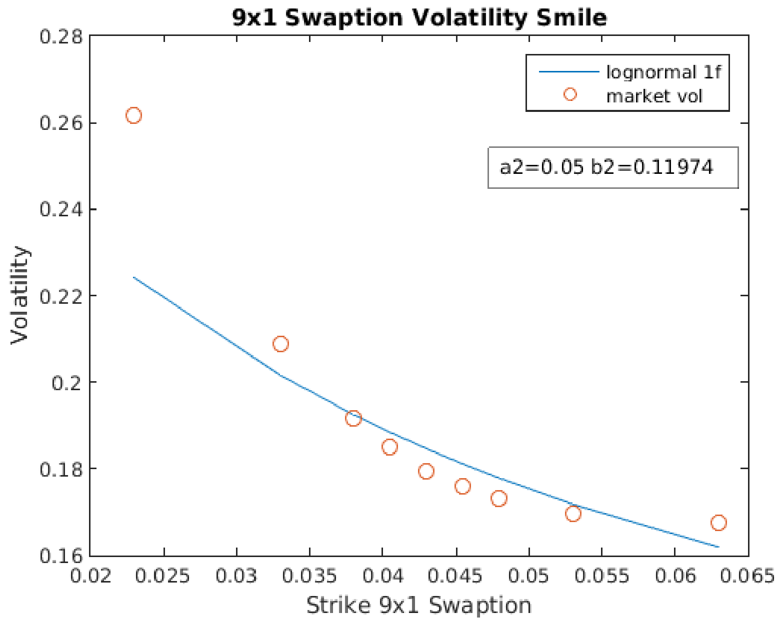


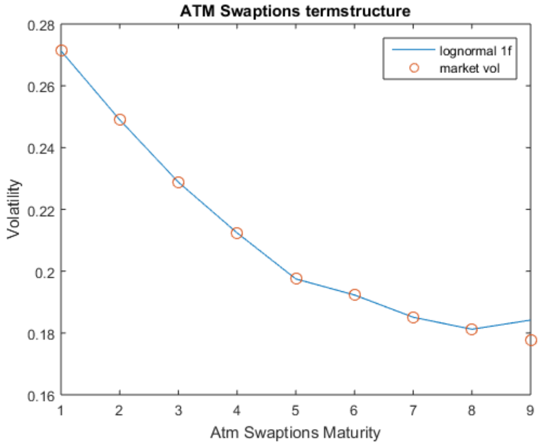
Figure 4.4: 9x1 Swaption Smile, one-factor

We have coded a Newton algorithm to compute the implied volatility by numerical inversion. Along with Newton, to test our code's efficiency, the fzero Matlab function has been used for solving the same root-finding problem.

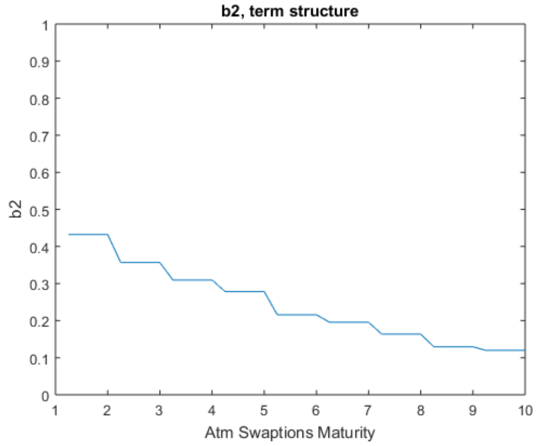
The value found for a_2, once derived in this first step with the 9x1 swaptions, has been kept constant for all the remaining calibration phases concerning the co-terminal ATM products. While in the case of b_2, this first step has just resulted in an estimate for the time span between 9 and 10 years from the valuation date. Being a 3-month tenor product we have four b_2 terms between 9 and 10 due to the quarterly payments of the floating leg. All the remaining b_2 parameters are derived by mean of a bootstrapping procedure using the ATM swaptions.

Therefore, according to the co-terminal procedure showed in Algorithm 1, having a_2 fixed and starting from the estimates of the 9x1 swaption we have moved backwards to the 8x2 product, calibrating the b_2s between 8 and 9. With these and the previous ones we have then considered the ATM 7x3 computing the parameters between 7 and 8 and so on, back to the 1x9 ATM swaption. Such an evolution of the parameters is clear from the previously exhibited Table 4.2. The ATM term structure resulting from the calibration of the one-factor lognormal rational model has been reported in Figure 4.5a.

Although the worsening in the 9x1 case, the performance of the model in pricing ATM swaptions is impressive. Even more if we take a cost-benefit approach, due to its simple theoretical specification and its non-demanding computational implementation. The convergence is, indeed, very fast even in a mediocre machine. In Figure 4.5b we have reported the term structure of the coefficient of the driving martingale.



(a) ATM Swaptions, one-factor



(b) b2 term structure

Figure 4.5: Co-terminal ATM swaptions

In Table 4.3 we report a summary of the results of the overall calibration procedure for the one-factor lognormal model. The improvement for the ATM swaptions is evident from the analysis of the evolution of the sum of squared errors in the fourth column.

	Calibration Summary One-factor, positive rates			
Step	a_2	b_2	SSE	$\#b_2$
9x1 Smile	0.05	0.11974	0.007	4
ATMs:				
8x2	0.05	0.1293	1.3151e-19	8
7x3	0.05	0.1635	2.2067e-14	12
6x4	0.05	0.1957	3.1989e-23	16
5x5	0.05	0.2158	1.0010e-20	20
4x6	0.05	0.2784	5.6815e-15	24
3x7	0.05	0.3095	6.0658e-13	28
2x8	0.05	0.3568	8.8056e-20	32
1x9	0.05	0.4321	3.1285e-13	36

Table 4.3: Summary one-factor calibration under positive rates

	Parameters Estimates in Time, two-factor			
Step	Objective index n	Input Parameters	Estimates	Stack
9x1 Smile	9	$a_1, a_2, a_3, b_1,$ $b_2^{9x1}, \rho, b_3^{9x1}$	$\hat{\Phi}_1 = [\hat{a}_1, \hat{a}_2, \hat{a}_3,$ $\hat{b}_1, \hat{b}_2^{9x1}, \hat{\rho}, b_3^{*,9x1}]$	$Stack\hat{\Phi} = \hat{\Phi}_1$
ATMs:				
8x2	1	$Stack\hat{\Phi} + b_2^{8x2}, b_3^{8x2}$	$\hat{\Phi}_2 = [b_2^{*,8x2}, \hat{b}_3^{8x2}]$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_2$
7x3	1	$Stack\hat{\Phi} + b_2^{7x3}, b_3^{7x3}$	$\hat{\Phi}_3 = [b_2^{*,3x7}, \hat{b}_7^{3x2}]$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_3$
6x4	1	$Stack\hat{\Phi} + b_2^{6x4}, b_3^{6x4}$	$\hat{\Phi}_4 = [b_2^{*,4x6}, \hat{b}_3^{4x6}]$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_4$
5x5	1	$Stack\hat{\Phi} + b_2^{5x5}, b_3^{5x5}$	$\hat{\Phi}_5 = [b_2^{*,5x5}, \hat{b}_3^{5x5}]$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_5$
4x6	1	$Stack\hat{\Phi} + b_2^{4x6}, b_3^{4x6}$	$\hat{\Phi}_6 = [b_2^{*,6x4}, \hat{b}_3^{6x4}]$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_6$
3x7	1	$Stack\hat{\Phi} + b_2^{3x7}, b_3^{3x7}$	$\hat{\Phi}_7 = [b_2^{*,7x3}, \hat{b}_3^{7x3}]$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_7$
2x8	1	$Stack\hat{\Phi} + b_2^{2x8}, b_3^{2x8}$	$\hat{\Phi}_8 = [b_2^{*,2x8}, \hat{b}_3^{2x8}]$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_8$
1x9	1	$Stack\hat{\Phi} + b_2^{1x9}, b_3^{1x9}$	$\hat{\Phi}_9 = [b_2^{*,9x1}, \hat{b}_3^{9x1}]$	$Stack\hat{\Phi} = Stack\hat{\Phi} + \hat{\Phi}_9$

NOTE: x^* means that the value of x is assumed and not estimated by optimization procedures

Table 4.4: Time-evolution of the parameters stack, two-factor

4.2.2 Two-factor Lognormal

In this case, model prices in (4.1) are computed by mean of (3.21). As already done for the one-factor model, we have applied the algorithmic procedure showed in Algorithm 1, considering the relevant quantities and formulae.

The calibration of the two-factor lognormal model, as expected, has resulted in a better fit to the market smile of the 9x1 swaptions. Indeed, a gain in the flexibility of the model has been experienced by the introduction of a second stochastic process. But the presence of two risky factors has required, along with the parameters of the driving martingales, also the estimation of a correlation coefficient.

The high model parameterization is evident in Table 4.4 where we have reported the evolution of the parameters stack for the two-factor framework. To ensure smoothness we have just assumed that the parameters b_3 between year 9 and 10 are equal to the period forward rates times a scaling factor of 0.15. In other words:

$$b_3\left(T, T+0.25\right) = 0.15 L\left(0, T, T+0.25\right) \quad T \in [9, 9.75]$$

In the calibration of the smile, due to a lower speed of convergence, the option 'PlotFcns'

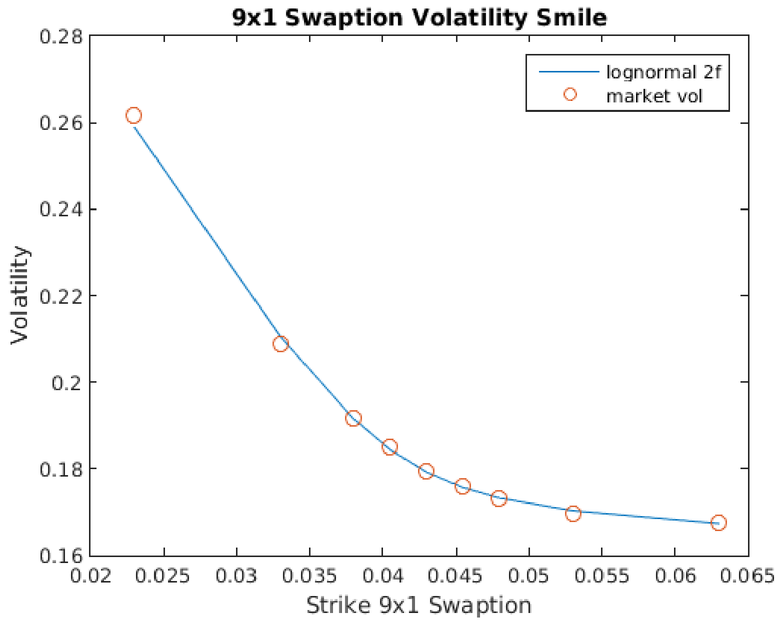


Figure 4.6: 9x1 Swaption Smile, two-factor

for plotting the objective during the iterations has been much more useful than in the one-factor model for choosing both the starting points and the constraints. In this step we have decided to not make any assumptions beyond the one on b_3.

Although the high parametrization of the model, the Matlab Optimset toolbox has been of great help, especially thanks to the possibility of visually debugging during the iterations. The following estimates have been derived:

$a_2 = 0.1733;\ b_1 = 0.4294\ ;\ b_2 = 0.0250;\ \rho = 0.6915; a_1 = 0.8986\ ; a_3 = 1.0504\ ; SSE = 7.5019e - 05$

The value of the sum of the squared errors highlights the better statistical power of the two-factor with respect to the one-factor framework, where SSE was equal to 0.007. The higher goodness of fit to the market smile is evident from Figure 4.6. Notwithstanding, this framework is much more computational intensive than the previous one and the convergence takes more time.

As far as the co-terminal swaptions are considered, for the periods $T \in [1, 9]$, we have kept constant the previously estimated values for a_2, b_1, ρ, a_1, a_3. Furthermore, we have assumed that:

$$b_2\left(T, T+0.25\right) = 0.55L\left(0, T, T+0.25\right) \quad T \in [1, 8.75]$$

and for b_3 we have assumed a piecewise process such that for $T \in [1, 8]$:

$$b_3\left(T, T+0.25\right) = b_3\left(T+0.25, T+0.5\right) = b_3\left(T+0.5, T+0.75\right) = b_3\left(T+0.75, T+1\right)$$

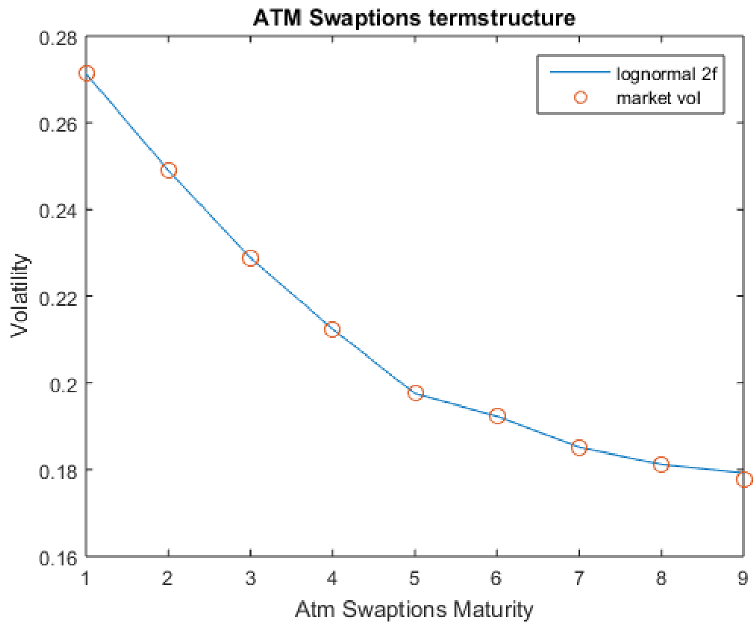


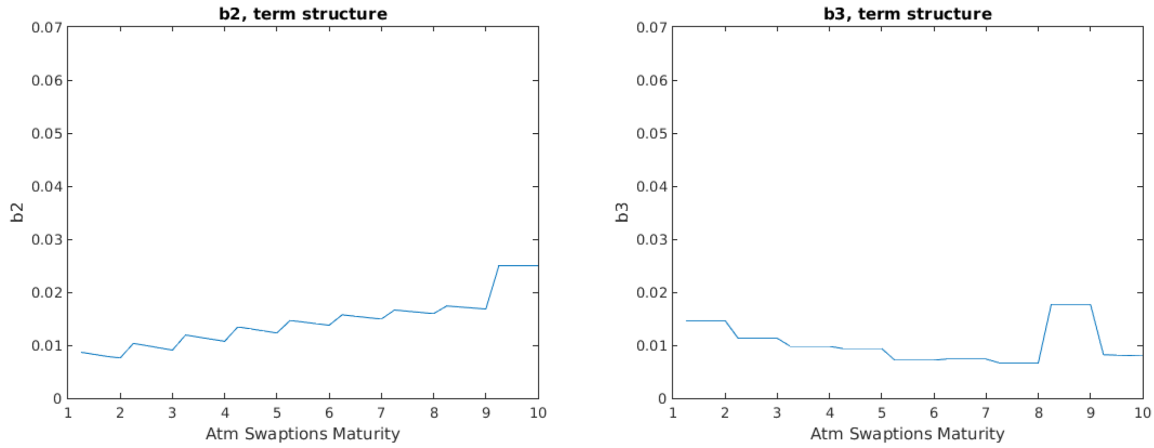
Figure 4.7: Co-terminal ATM Swaptions, two-factor

The results of this step are reported in Figure 4.7. With the term structures of b_2 and b_3 displayed in Figure 4.8.

In table 4.5 the summary of the main quantities of interest resulting from the calibration procedure have been reported. Since, according to our assumptions, b_2 and b_3 are not piece-wise constant for every period, their average per year is showed.

	Calibration Summary Two-factor, positive rates									
Step	a_1	a_2	a_3	b_1	$\bar{b_2}$	$\bar{b_3}$	ρ	SSE	$\#b_1$	$\#b_2\&\#b_3$
9x1 Smile	0.8986	0.1733	1.0504	0.4294	0.0250	0.0081	0.6915	7.5019e-05	1	8
ATMs:										
8x2	0.8986	0.1733	1.0504	0.4294	0.0171	0.0177	0.6915	8.7731e-15	2	8
7x3	0.8986	0.1733	1.0504	0.4294	0.0163	0.0067	0.6915	4.6195e-20	3	12
6x4	0.8986	0.1733	1.0504	0.4294	0.0153	0.0074	0.6915	7.3621e-23	4	16
5x5	0.8986	0.1733	1.0504	0.4294	0.0142	0.0072	0.6915	1.5871e-22	5	20
4x6	0.8986	0.1733	1.0504	0.4294	0.0129	0.0094	0.6915	1.1917e-15	6	24
3x7	0.8986	0.1733	1.0504	0.4294	0.0113	0.0098	0.6915	1.6201e-16	7	28
2x8	0.8986	0.1733	1.0504	0.4294	0.0097	0.0113	0.6915	2.2705e-19	8	32
1x9	0.8986	0.1733	1.0504	0.4294	0.0081	0.0146	0.6915	1.0816e-13	9	36

Table 4.5: Parameters and errors, calibration two-factor, positive rates



(a) b2 term structure (b) b3 term structure

Figure 4.8: Parameters term structures two-factor

4.3 Negative Rates

Theoretically, as seen in section 3.2, there is no limitation to the application of the class of rational models to negative interest rates. Intrigued by their potentials, we have decided to study the behavior of both the one-factor and the two-factor version at a time when negative rates were a consolidate reality.

We have chosen our valuation date to be the 4$^{\text{th}}$ of January 2016. The logical algorithmic procedure is the same as the one performed for the positive rates scenario and reported in Algorithm 1. A point needs to be stressed before presenting our results. At our valuation date the 9x1 forward swap rate, that is the rate of a future swap spot-starting 9 years from now and with a 1-year tenor is equal to 1.9276%. Being it also the ATM strike of a 9x1 swaption it can be easily seen how applying the [-200 +200]bps smile we could end up in a negative region. Therefore, Black76 cannot be used if no modifications are applied. Other models, such as Bachelier or Displaced Diffusion, should be chosen.

The former assumes the normality of the underlying, the latter, instead, deals with shifted lognormals. In the Displaced Diffusion, although the computation of the shift is not trivial, we are still dealing with lognormal quantities so that the rational models previously presented should be expected to keep their performance, after the necessary considerations for the sign of interest rates.

More problems arise in case we decide to calibrate on quotes derived from normal standard deviations by mean of the Bachelier formula. Here the usual skew of the lognormal volatilities is no more displayed. However, even if with a different shape, we still refer to the plot of volatilities using the word smile.

The focus of this subsection is understanding how the quotes of lognormal rational models can be calibrated to prices when interest rates are negative. Although not the perfect choice, we have decided to calibrate the models without any modifications to the theoretical distribution of the underlyings but just working on the parameters set. This is done to evaluate the flexibility of the proposed frameworks. Indeed, in case of satisfactory results, this will strengthen even further the potentials of rational models. We would like to stress that the possible presence of some negative coefficients of the driving martingales is physiological. We have already seen that the value of these parameters depends on the forward curve being used. These, especially for periods near the valuation date, can show negative signs.

We still have used Bloomberg as provider for the market volatilities, but in the VCUB function we have asked for normal quantities and no more lognormal as in the case of the previous calibrations. These normal standard deviations are quoted in basis points and are provided as well for negative strikes, since there is no bound set for the underlying's distribution. For the computation of the implied volatilities the function "impliedVolatilityNewtonNormalModel" has been used, in which we have written our own Newton algorithm for root-finding.

4.3.1 One-factor Lognormal

In this subsection we calibrate the one-factor model to market data under a negative interest rates scenario. As far as the procedure for the 9x1 smile and the co-terminal ATM swaptions is considered we have still implemented Algorithm 1. Therefore, the specific time evolution of the parameters stack is still consistent with Table 4.3.

As done for the calibration in the positive interest rates scenario, in order to check how the objective function behaves with respect to the parameter sets we have performed some visual analysis. Among these we have decided to report in Figure 4.9 the combination of parameters that make the objective function for a 9x1 ATM swaptions to take values between -3 and 0. The region in which the objective reaches the null point has been highlighted with an ellipse and it is colored in dark red, according to the legend. It is visually clear that the solution for the 9x1 smile root-finding problem can be expected

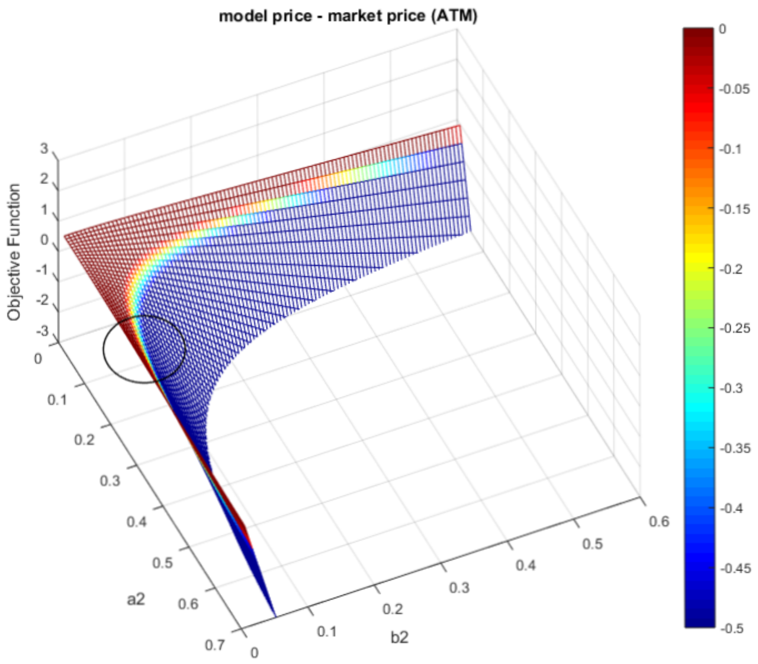


Figure 4.9: 9x1 ATM, market VS model, negative rates

to lie near values of a_2 and b_2 of 0.1 and 0.05 respectively. Actually this is not a full picture of the problem, but it is just one case out of the nine composing the 9x1 smile. Notwithstanding, Figure 4.9 helps in the choice of starting points. In fact, this finding can be used in conjunction with the analysis of the evolution of the sum of squared errors of the overall 9x1 smile. The results of the latter are reported in Figure 4.10, where we have decided to plot values very near the origin to highlight how the coefficients differ in magnitudes with respect to what found under positive rates.

From the calibration of the one-factor lognormal model we have found the following estimates:

$$a_2 = 0.1221, \; b_2 = 0.0515, \; SSE = 0.0003$$

The resulting smile is displayed in Figure 4.11.

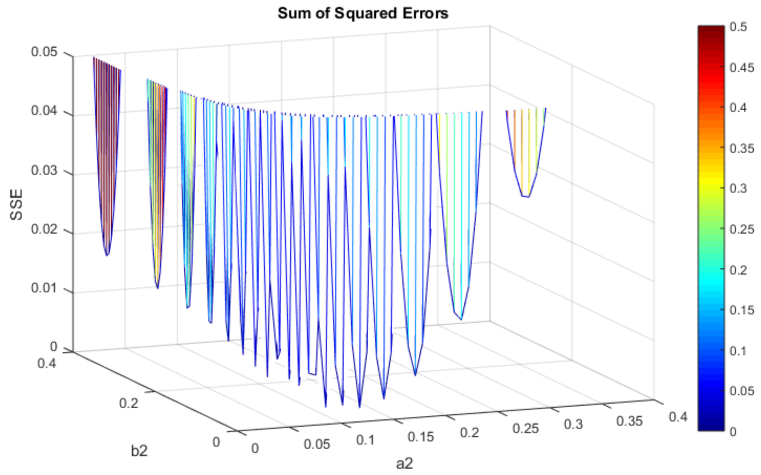


Figure 4.10: 9x1 smile, zoom SSE, negative rates

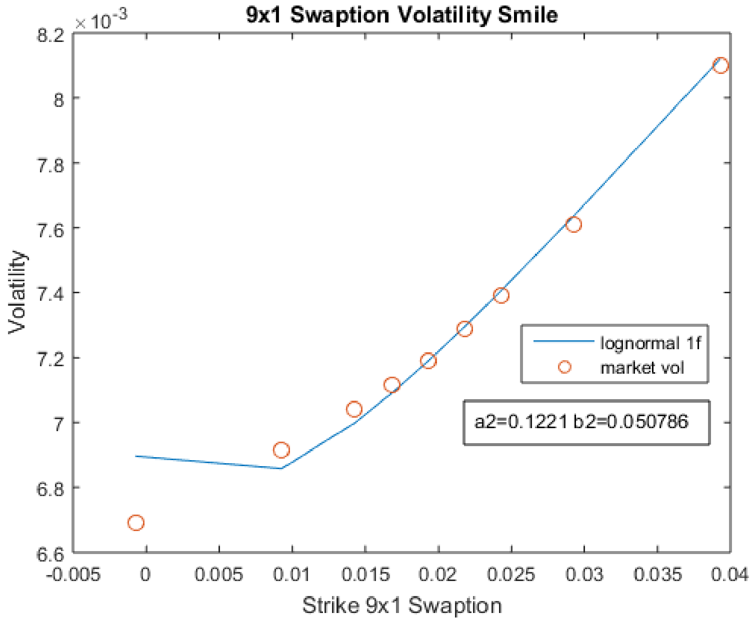


Figure 4.11: 9x1 Swaption Smile, one-factor, negative rates

It is visually clear from the smile how the one-factor lognormal framework misses the quote of the swaption with negative strike. Despite of this problem, the performance of the model is quite good. Actually, the issues with the negative strikes were not totally unexpected. Indeed, here we are forcing a lognormal process whose pricing formula is (3.17). Being c_0 defined as:

$$c_0 = \sum_{i=1}^{l} L\left(0, T_{i-1}, T_i\right) \delta_i^{Float} - \sum_{j=1}^{m} K P_{0T_j'} \delta_j^{Fix}$$

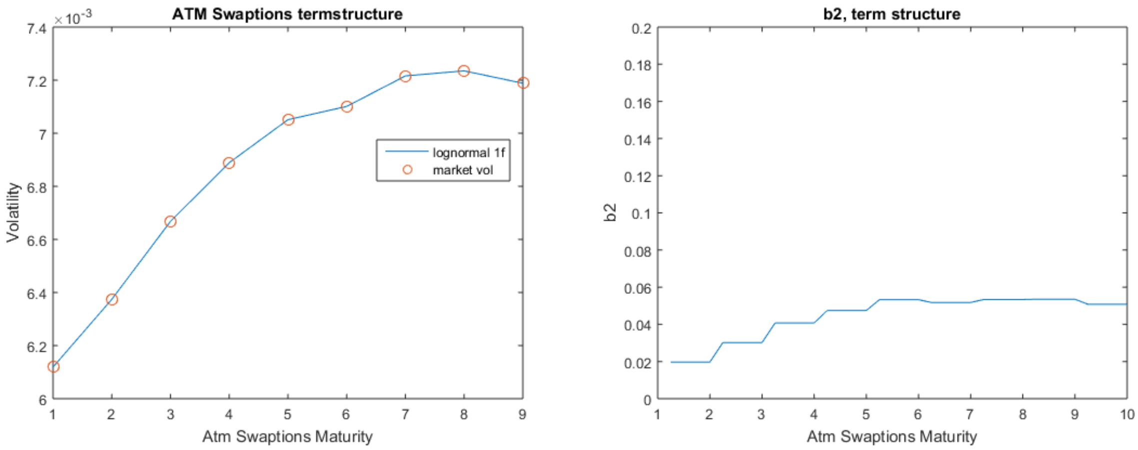
it can be seen how a negative value of K tends to increase c_0, with this violating the inverse relation between these two quantities theoretically expected by the one-factor lognormal model. However, c_0 is still lower than c_2 so that $\tilde{c_0}$ takes a negative sign and (3.17) can be fully applied. Being $\tilde{c_0}$ less negative than what the one-factor model would have expected, due to the negative sign of the strike, there is a too much positive impact on the price. This comes from the smaller logarithm in (3.17) that is inversely related to the price. Coeteris paribus, a higher quote implies a higher volatility, pushing the model quantity consistently above the market one, thus explaining the elbow shape in the left tail of the smile. Notwithstanding, the resulting smile in Figure 4.11 is a proof of the flexibility of the one-factor lognormal model.

Moreover, comparing the magnitudes of the estimates for a_2 and b_2 between the setup of positive and negative interest rates rises interesting reasonings. Indeed, from Appendix C.1, where we derive the solution to the one-factor lognormal model, it can be seen that the parameter a_2 is the instantaneous volatility of the process $S_t = c_2 e^{\left(a_2 X_t^{(2)} - \frac{1}{2} a_2^2 t\right)}$. Recalling that in section 4.2.1 the estimates were $a_2 = 0.05$ and $b_2 = 0.11974$, while now are $a_2 = 0.1221$ and $b_2 = 0.0515$, we can speculate on how the one-factor model, in order to deal with negative rates and without changing the building assumptions, put a greater emphasis on the volatility of the driving $\mathbb{M}$-martingale at expenses of the coefficient b_2.

Thanks to this flexibility, the performance of the model remains statistically good. Indeed, the 9x1 ATM quote is perfectly replicated although only one single risky factor drives the model. The results of the co-terminal procedure is reported in Figure 4.12a, where the great performance of the one-factor model in pricing ATM swaptions is evident. The term structure of b_2 is shown in Figure 4.12b. We have then summarized all the results of the full calibration procedure in Table 4.6.

	Calibration Summary One-factor			
Step	a_2	b_2	SSE	$\#b_2$
9x1 Smile	0.1221	0.0508	0.0003	4
ATMs:				
8x2	0.1221	0.0535	3.0382e-26	8
7x3	0.1221	0.0534	2.2928e-13	12
6x4	0.1221	0.0518	1.1990e-15	16
5x5	0.1221	0.0533	1.2591e-13	20
4x6	0.1221	0.0475	1.7030e-21	24
3x7	0.1221	0.0408	9.3763e-13	28
2x8	0.1221	0.0302	1.1946e-13	32
1x9	0.1221	0.0197	5.8179e-14	36

Table 4.6: Summary of the procedure, one-factor, negative rates



(a) ATM one-factor, negative rates (b) b2 term structure

Figure 4.12: Co-terminal ATM swaptions

4.3.2 Two-factor Lognormal

We report in this subsection the results of the implementation of Algorithm 1 in case of the two-factor framework under negative rates. We have found that the introduction of a second driving martingale, as shown by the smile in Figure 4.13, has increased the performance of the model in dealing with negative strikes. But such a result has come with the cost of missing the 9x1 ATM quote as it is clear from the plot. Therefore, for this date there is a kind of trade-off between the ATM options and the ones with negative strikes. The parameters estimated in the first sub-procedure of our calibration algorithm are:

$a_2 = 0.1901;\ b_1 = 0.3299\ ;\ b_2 = 0.0178;\ \rho = 0.3601; a_1 = 0.8629\ ; a_3 = 1.0504,\ SSE = 5.7172e{-}05$

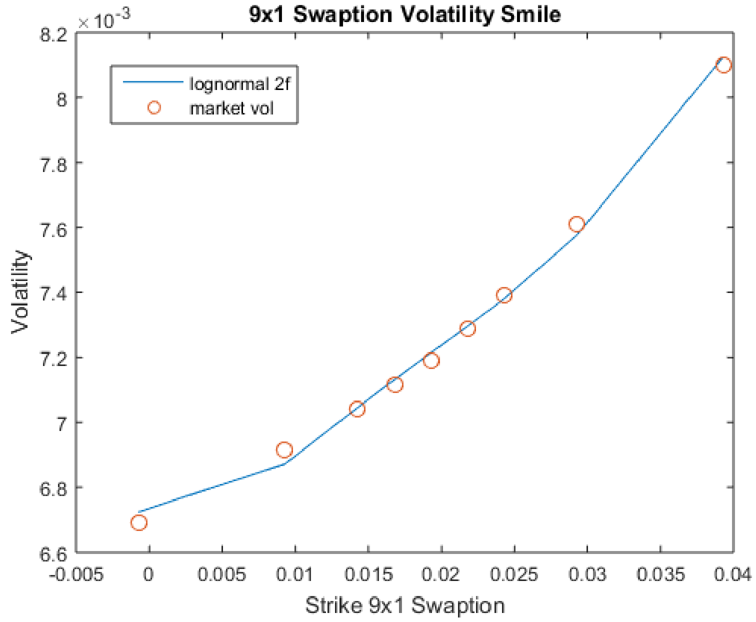


Figure 4.13: 9x1 Swaption Smile, two-factor, negative rates

The results for the co-terminal procedure on the ATM swaptions is reported in Figure 4.14, with the respective term structures show in Figure 4.15. Being the coefficients dependent from the forward curve, under a scenario of negative rates, their positivity is not ensured. We have summarized all the relevant estimates in Table 4.7.

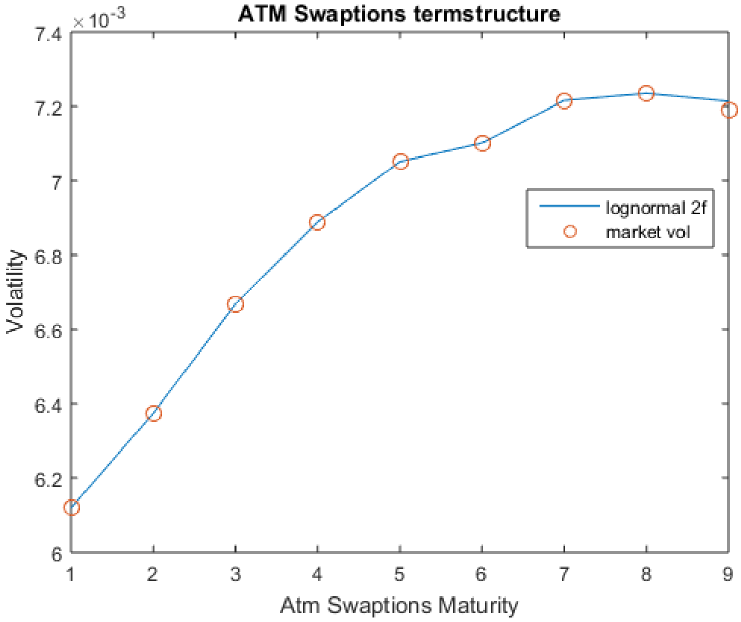
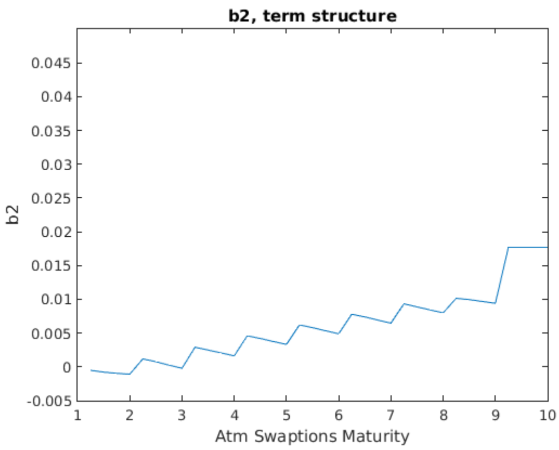
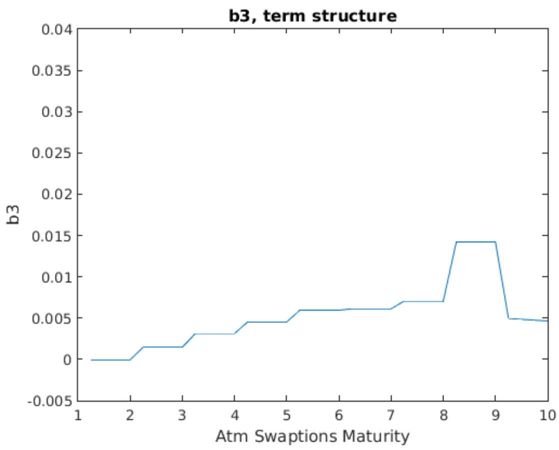


Figure 4.14: Co-terminal ATM Swaptions, two-factor, negative rates



(a) b2 term structure



(b) b3 term structure

Figure 4.15: Parameters term structures two-factor, negative rates

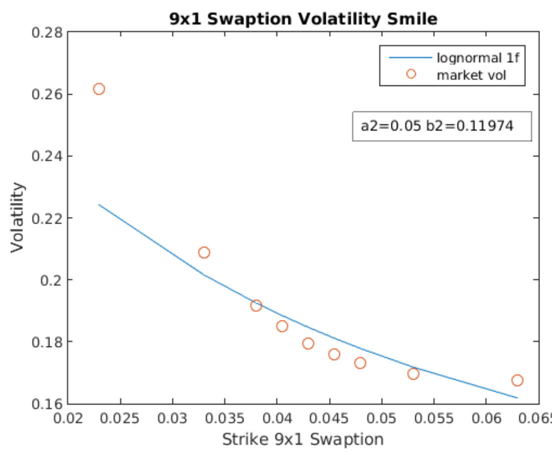
4.4 Summary

To summarize our findings, we highlight the fact that the one-factor and the two-factor model have shown a similar performance for the ATM swaptions.

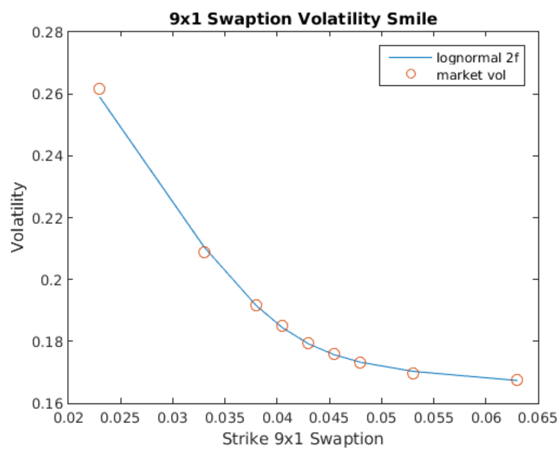
While, for the 9x1 smile, adding a second driving process has consistently increased the statistical goodness of the fit. This is evident in Figure 4.16, where we report, along with the SSEs, all the 9x1 smiles founded in our research.

	Calibration Summary Two-factor									
Step	a_1	a_2	a_3	b_1	$\bar{b}_2$	$\bar{b}_3$	ρ	SSE	$\#b_1$	$\#b_2\&\#b_3$
9x1 Smile	0.8629	0.1901	1.0504	0.3299	0.0178	0.0029	0.3601	5.7172e-05	1	8
ATMs:										
8x2	0.8629	0.1901	1.0504	0.3299	0.0098	0.0163	0.3601	3.2524e-24	2	8
7x3	0.8629	0.1901	1.0504	0.3299	0.0087	0.0073	0.3601	3.3140e-20	3	12
6x4	0.8629	0.1901	1.0504	0.3299	0.0071	0.0063	0.3601	2.3676e-20	4	16
5x5	0.8629	0.1901	1.0504	0.3299	0.0056	0.0062	0.3601	7.8125e-21	5	20
4x6	0.8629	0.1901	1.0504	0.3299	0.004	0.0047	0.3601	1.9419e-20	6	24
3x7	0.8629	0.1901	1.0504	0.3299	0.0023	0.0033	0.3601	5.3424e-20	7	28
2x8	0.8629	0.1901	1.0504	0.3299	0.0005	0.0016	0.3601	1.9869e-19	8	32
1x9	0.8629	0.1901	1.0504	0.3299	-0.0008	0.000042	0.3601	5.6386e-19	9	36

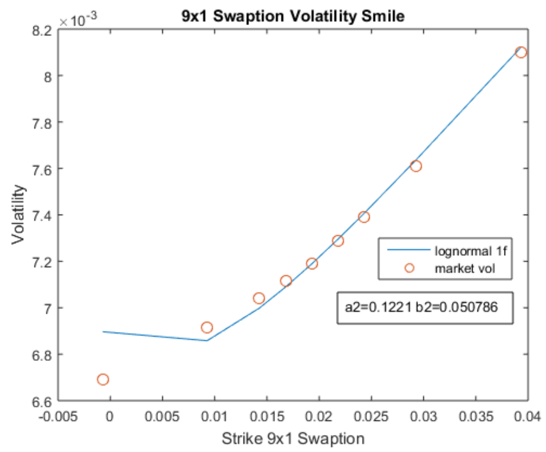
Table 4.7: Parameters and errors of the procedure



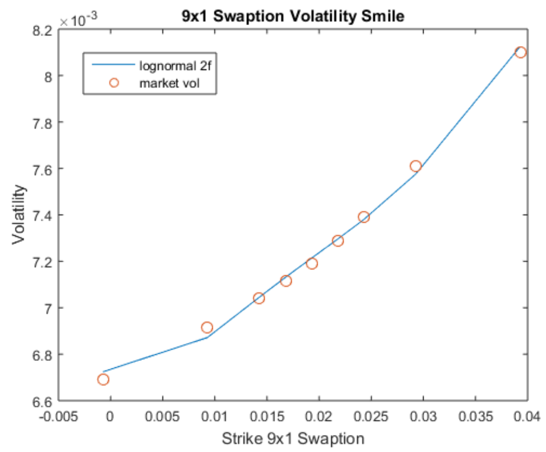
(a) One-factor, positive rates, SSE = 0.007



(b) Two-factor, positive rates, SSE = 7.5019e-05



(c) One-factor, negative rates, SSE = 0.0003



(d) Two-factor, negative rates, SSE = 5.7172e-05

Figure 4.16: 9x1 smiles, summary and SSE

5 Conclusions

In this thesis, we have performed a systematic analysis of the calibration of two rational models to the smile and term structure of vanilla swaptions, with the goal of studying their ability to capture the issues emerged after the crisis, namely the occurrence of a multi-curve setting and of negative interest rates.

We have found that, although being originally introduced for positive interest rates[9], these models prove to be extremely flexible and able to adapt both to the multi-curve framework and to negative rates. In detail, after having introduced and re-derived the properties of general rational models, we have focused on a one- and two-factor lognormal rational models, and performed the calibration at two different dates, before (January, 4 2011) and after (January, 4 2016) the onset of negative interest rates and for two sets of market instruments: a 9x1 swaption with different strikes to capture the smile and a stream of co-terminal ATM swaptions (from 8x2 to 1x9) to test the term structure.

Both calibrations have been carried out on premia, calculated starting from market quotes of implied volatilities. On January, 4 2011, we have used a Black formula (Black76) to transform lognormal volatilities into premia, while on January, 4 2016, due to the presence of negative rates, a Bachelier formula have been employed to convert normal volatilities to premia. In general, we have found that the one-factor model displays a small, although non-negligible, statistical error in the calibration of the smile on both dates, while being enough accurate to replicate the term structure of ATM co-terminal swaptions. The two-factor model, while performing well on the term structure, improves consistently the calibration to the 9x1 smile. We have also verified that, in the presence of negative rates, due to their flexible nature, rational models preserve the same degree of explanatory power as in the case of positive interest rates. This is a remarkable feature, highly desirable for any interest rate model.

Giving the promising results obtained in this project, we conclude by pointing out some further directions which can become the object of future studies. Due to their ability to describe the multi-curve framework, rational models can be applied to the valuation of more complex derivatives, e.g. swaptions having as underlying basis swaps, which by construction depend on two curves. Or, pursuing the goal of calculating and managing risk exposures, we can study in more depth how to work in different risk measures (see Macrina,Crépey[1]) .

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Appendices

A Single-curve, basics and forward rates replication

A.1 Basic Quantities

In few words, present values of future cash flows are computed by using discount factors, which count the worthiness of one unit of money received at a future date. With forward rates we indicate the time value of money, in terms of interest, between two future dates. Considering continuous compounding, we tend to refer to discount factors as the following quantity: $P(t,T) = e^{-r(T-t)}$ where r can be either quoted directly by the market or, in case, bootstrapped. Actually, $P(t,T)$ differs from what a discount factor theoretically is. $P(t,T)$ is the price in t of a unitary amount paid in T. In reality, a discount factor, here denoted by D, should be a stochastic quantity that needs to be modeled and its relation with P is:

$$P(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[D(t,T)\right] \tag{A.1}$$

Since $D(t,T) = \int_t^T r_s ds$ with r_s typically being a stochastic process, known as short rate, we can either model directly D or its kernel r. For simplicity, in our work, we will, sometimes, refer to P as discount factor, but the difference between P and D will be clear from the context.

Recalling that D is random is vitally important, particularly for the fixed income market, that is the setting we are going to deal with. We can have several "collections" of $\{r_t\}$ that differ between each other for the type of rate. For example, consider two yield curves, $\{x_{1t}\}$ and $\{x_{2t}\}$, that are used to compute respectively discounts and forward rates. Whatever the compounding used, we define with $\{P^{x_{1t}}(t,T)\}_{0\leq t\leq T}$ the sets of zcb prices computed from the curve $\{x_{1t}\}$ and used to discount future cash flows. For $t < T_1 < T_2$ forward rates are, instead, defined as:

$$F^{x_{2t}}(t,T_1,T_2) = \frac{1}{(T_2 - T_1)}\left(\frac{P^{x_{2t}}(t,T_1)}{P^{x_{2t}}(t,T_2)} - 1\right) \tag{A.2}$$

where $\{P^{x_{2t}}(t,T)\}_{0\leq t\leq T}$ is the set of discounts computed from the $\{x_{2t}\}$ curve and are usually referred to as pseudo-discounts. If $\{x_{1t}\} = \{x_{2t}\}$ there exists a unique curve and all the relevant computations are based on the same rates. This is the single-curve scenario. Conversely, we call multi-curve the setting in which these curves differ and $\{x_{1t}\}$ is the collection of risk-free rates, either Eonia or OIS. While $\{x_{2t}\}$ varies according on the frequency of interest.

A.2 Replicating a FRA under single-curve

For the sake of understanding, we consider the example of the Libor Rate, that, in few words, is the rate at which some of the world's leading banks agree to set overnight loans. Our aim is to investigate what were the main assumptions and problems of the single-curve approach for the computation of the forward rates. In order to replicate (A.2) consider three different dates: $t < T_1 < T_2$. Suppose that each of them is far from the previous one by an amount Δ. Practitioner refers to this quantity as tenor or accrual factor, that is the fraction of a year between two dates of interest.

Without accounting for all the market conventions, we deal with FRA contracts as being one period swaps. Where in a FRA the investor agrees to exchange two different cash flows at a specific future date, one flow is fixed at the inception of the contract while the other at a future reset date. In our setting, we consider the case of an agreement to exchange at time T_2 the difference between the Libor rate, valid for the period between T_1 and T_2, and a fixed amount K. Everything proportional to the right accrual factors and with a unitary notional. In symbols, we have that the T_2-payoff of such an agreement, setting the floating rate at T_1 and paying at T_2 is given by: $\left(L\left(T_1, T_2\right) - K\right)\Delta$. Our interest is to find the value of K that makes the contract fair. We stress the fact that with fairness it is meant that the agreement must have null price at inception, since in perfect market conditions all the agents are supposed to be exposed to the same set of information. One way to compute this optimal K, which we will refer to as K*, is trying to replicate the above payoff using available market instruments. Assuming simple compounding, a replicating strategy is implemented by the following steps:

1. Invest at t a quantity equal to $\frac{1}{1+L(t,T_1)\Delta}$ up to T_1
2. Invest a unitary amount between T_1 and T_2 at $L\left(T_1, T_2\right)$
3. Borrow at time t an amount of $\frac{1+K\Delta}{1+L(t,T_2)\Delta}$ to be repaid in T_2

The stategy CFs are summarized in Table A.1. We can see that the overall price to be paid

	t	T_1	T_2
Step 1	$-\frac{1}{1+L(t,T_1)\Delta}$	$+1$	
Step 2		-1	$1+L\left(T_1,T_2\right)\Delta)$
Step 3	$\frac{1+K\Delta}{1+L(t,T_2)\Delta}$		$-\left(1+K\Delta\right)$
net	$=-\frac{1}{1+L(t,T_1)\Delta}+\frac{1+K\Delta}{1+L(t,T_2)\Delta}$	/	$=\left(L\left(T_1,T_2\right)-K\right)\Delta$

Table A.1: Summary of the procedure, one-factor, Normal volatilities

at inception for implementing the strategy is equal to: $\frac{1}{1+L(t,T_1)\Delta}-\frac{1+K\Delta}{1+L(t,T_2)\Delta}=P\left(t,T_1\right)-\left(1+K\Delta\right)P\left(t,T_2\right)$. By setting this quantity equal to zero we obtain the equilibrium forward rate K*:

$$K^*=F\left(t,T_1,T_2\right)=\frac{1}{\Delta}\left(\frac{P\left(t,T_1\right)}{P\left(t,T_2\right)}-1\right) \tag{A.3}$$

Since FRAs are traded quantity, market quotes can be compared to equilibrium forward rates, expecting to see just negligible differences.

This was the case before the 2007 financial crisis, but, then the spread skyrocketed, leading to the conclusion that wrong assumptions had been made in the valuation models. But what were the market practitioners missing? Had the arbitrage free assumption becomes obsolete all of a sudden? The answer to the last question is a well sounded: no. The roots of the problem lie in what since then has become one of the main focuses of researches: counterparty credit risk. Liquidity issues play an important role as well. As far as our replicating strategy is concerned, agents' creditworthiness has a huge impact on point 2, that is when a unitary amount is invested at Libor rate between $T_1$and T_2. To understand this point we recall that Libor is, by definition, a particular average computed from a panel of large banks.

Assuming the interbank market to be no more free of default risk makes the above replication invalid because the creditworthiness of the counterparty, which the strategy is implemented with, can differ from the one of the Libor panel. In other words, the replication fails because it could be the case that: $L^{Counterpaty}\left(T_1,T_2\right)\neq L\left(T_1,T_2\right)$.

A.3 Forward Swap Rates Tables

We report here the comparison between the forward swap rates quoted by Bloomberg on the 4th of January 2011 and the ones computed according to both the single and multi-curve setting.

	Forward Swap Rates 3m Multicurve			
Maturity	Fwd Swap rate	Bloomberg	Absolute Diff.	Relative Diff
9	0.042759055	0.042994	0.000234945	0.55%
8	0.04195243	0.042263	0.00031057	0.73%
7	0.041447271	0.041828	0.000380729	0.91%
6	0.04091661	0.041152	0.00023539	0.57%
5	0.040094054	0.040234	0.000139946	0.35%
4	0.039019927	0.039061	4.11E-05	0.11%
3	0.037576006	0.037463	0.000113006	0.30%
2	0.035742573	0.035488	0.000254573	0.72%
1	0.033544738	0.03309	0.000454738	1.37%

Table A.2: Forward swap rates 3M multicurve

	Forward Swap Rates 6m Multicurve			
Maturity	Fwd Swap rate	Bloomberg	Absolute Diff.	Relative Diff
9	0.044206623	0.043702	0.000504623	1.15%
8	0.043189747	0.043197	7.25E-06	0.02%
7	0.042552637	0.042738	0.000185363	0.43%
6	0.041979909	0.04205	7.01E-05	0.17%
5	0.041147236	0.041109	3.82E-05	0.09%
4	0.040176555	0.040132	4.46E-05	0.11%
3	0.038895903	0.038588	0.000307903	0.80%
2	0.037194253	0.036683	0.000511253	1.39%
1	0.035177992	0.034368	0.000809992	2.36%

Table A.3: Forward swap rates 6M multicurve

	Forward Swap Rates 3m Singlecurve			
Maturity	Fwd Swap rate	Bloomberg	Absolute Diff.	Relative Diff
9	0.030558453	0.042994	0.012435547	28.92%
8	0.035097417	0.042263	0.007165583	16.95%
7	0.036217872	0.041828	0.005610128	13.41%
6	0.036325378	0.041152	0.004826622	11.73%
5	0.035858767	0.040234	0.004375233	10.87%
4	0.035008378	0.039061	0.004052622	10.38%
3	0.033681198	0.037463	0.003781802	10.09%
2	0.031938906	0.035488	0.003549094	10.00%
1	0.029795665	0.03309	0.003294335	9.96%

Table A.4: Forward swap rates 3M singlecurve

	Forward Swap Rates 6m Singlecurve			
Maturity	Fwd Swap rate	Bloomberg	Absolute Diff.	Relative Diff
9	0.030714779	0.043702	0.012987221	29.72%
8	0.035274461	0.043197	0.007922539	18.34%
7	0.036397724	0.042738	0.006340276	14.84%
6	0.036502552	0.04205	0.005547448	13.19%
5	0.036029053	0.041109	0.005079947	12.36%
4	0.035169175	0.040132	0.004962825	12.37%
3	0.033828872	0.038588	0.004759128	12.33%
2	0.032070796	0.036683	0.004612204	12.57%
1	0.029909784	0.034368	0.004458216	12.97%

Table A.5: Forward swap rates 6M singlecurve

B Change of measure, Numeraire and Black

In this Appendix, along with with Black Model, we have decided to present one of the most important and astonishing results of quantitative finance: the change of measure. This helps us to move between different probability measures, which differ in the specification of their numeraire. This is defined as a traded instrument such that all the quantities written relative to it are martingales under the specific measure.

In case of a generic probability $\mathbb{M}$ the numeraire, denoted by num, is such that:

$$\frac{v(t)}{num(t)} = \mathbb{E}_t^{\mathbb{M}}\left[\frac{v(T)}{num(T)}\right] \tag{B.1}$$

We stress the fact that to be valid the numeraire needs to be a traded quantity, or, at least, bootstrapped.

B.1 Black Model and Changing of Measure

Here we focus on how the Black Model and the change of measure technique are used for pricing purposes. Although at a first sight this Appendix could appear redundant to the reader, it is actually our work's foundation, with its findings being greatly useful to derive the solutions of the models reported in Appendix C. Therefore, we would like to give a clear view on our starting points.

We stress the peculiarity of interest rate markets highlighting the divergences between pricing options on interest quantities and stocks. These two settings show, indeed, a consistent difference. Just consider the consolidated no-arbitrage pricing technique under the risk-neutral world, where we have expected value of discounted payoffs. Recalling that interest rates drive both the stochastic discount factors and the payoff of derivatives written on fixed income quantities, it is inconsistent to assume that those two quantities are independent. This is why in section 2 we have highlighted the difference between zcb prices and discount factors. This is to say that while for the stock market (B.2) can be reasonably assumed to hold, this is by no means true for interest rate derivatives.

$$\mathbb{E}_t^{\mathbb{Q}}\left[\widetilde{discount} \times \widetilde{payoff}\right] = \mathbb{E}_t^{\mathbb{Q}}\left[\widetilde{discount}\right] \times \mathbb{E}_t^{\mathbb{Q}}\left[\widetilde{payoff}\right] \tag{B.2}$$

Where the tilde denotes a stochastic process and the result stems from the possible independence between these two quantities.

As said above, this could be a viable assumption for derivatives on stocks, even if may go against the work of economists, such as James Tobin[6], that spent their entire life investigating on the interconnection between interest and capital markets. The marginal error of ϵ-magnitude is worth the avoidance of computational and theoretical problems for sticking to formality. But for interest rate derivatives, the error would be no more of an infinitesimal quantity ϵ but $\frac{1}{\varepsilon}$. In fixed income markets both the processes inside the expected value in (B.2) are driven by common risk factors, not by assumptions but by definition. Fortunately all the troubles arising from this consideration will be, in a way, solved by mean of the change of measure and numeraire technique.

B.2 Caplets and Floorlets

We start with considering the case of plain vanilla options on a single Libor rate with non-linear payoff, named as caplets and floorlets, for which we will try to derive a closed form formula for their prices.

A caplet gives the right to receive at maturity an interest equal to the rate between two future dates while paying a pre-agreed amount K. A floorlet has the reverse structure. The payoff of a caplet that expires at time T_{i-1} and pays at T_i is:

$$\left(L\left(T_{i-1}, T_i\right) - K\right)^+ \Delta_i N \tag{B.3}$$

To price the payoff in (B.3), it is useful to consider the equivalence between future spot rates and forward with future inception and immediate resetting, in symbol : $L\left(T_{i-1}, T_i\right) = F\left(T_{i-1}, T_{i-1}, T_i\right)$. Being all the others real quantities, this forward rate is the only one to inject riskiness into (B.3) and for this reason needs to be modelled.

Modelling implies assumptions on its behavior in time. From the standard application of Black in the case of options on futures, the pricing could be simplified if the martingality and the lognormality assumptions holded in the risk-neutral measure also for the forward rates. But not being this the case, everything depends on the type of probability we decide to work with. Indeed, there exist different measures from $\mathbb{Q}$ that can be easily reached by mean of the change of measure through the Radon-Nikodym's derivative. Even if in the risk-neutral world, with the necessary adjustments, there is no problem in assuming the lognormality for forward rates, the martingality is by any mean out of reach. This is due to the fact that changing the measure impacts only the deterministic shift in the dynamic of a stochastic process leaving unmodified its diffusion coefficient.

The fact that forward rates cannot be assumed to be lognormal martingale under $\mathbb{Q}$ should not be discouraging. A probability measure where forward rates are both lognormal and martingales needs to be found. We show now how this technique could be implemented in the pricing of a caplet. Writing the forward rate as:

$$F\left(t, T_{i-1}, T_i\right) = \left(\frac{P\left(t, T_{i-1}\right) - P\left(t, T_i\right)}{P\left(t, T_i\right)}\right) \frac{1}{\Delta_i}$$

and recalling equation (B.1) it is easy to see what a possible numeraire could be. Indeed, from noting the presence at the denominator of the t-price of a traded zero-coupon bond with maturity T_i, we can conclude that the numeraire is the T_i-zcb. We name the relevant measure as T_i-forward measure. Indeed, we can show that forward rates are T_i-martingales:

$$F\left(t, T_{i-1}, T_i\right) = \mathbb{E}_t^{T_i}\left[F\left(T_{i-1}, T_{i-1}, T_i\right)\right]$$

We return to the pricing problem of the caplet payoff displayed in equation (B.3) , its value in t is:

$$v\left(t\right) = \mathbb{E}_t^{\mathbb{Q}}\left[D\left(t, T_i\right)\left(F\left(T_{i-1}, T_{i-1}, T_i\right) - K\right)^{+}\Delta_i N\right] = P\left(t, T_i\right)\mathbb{E}_t^{T_i}\left[\frac{\left(F\left(T_{i-1}, T_{i-1}, T_i\right) - K\right)^{+}\Delta_i N}{P\left(T_i, T_i\right)}\right] \tag{B.4}$$

with $P\left(T_i, T_i\right) = 1$ and where P is deterministic while D stochastic, even if, as previously seen, they both are usually named as discount factors. In the last passage of (B.4), where we have implemented the change of measure, the presence of $P\left(t, T_i\right)$ before the expected value is due to the fact that represents the numeraire. Having an object such that:

$$\mathbb{E}_t^{T_i}\left[\left(F\left(T_{i-1}, T_{i-1}, T_i\right) - K\right)^{+}\Delta_i N\right]$$

should immediately remind the Black procedure to evaluate it. Even if under a different measure if $F\left(T_{i-1}, T_{i-1}, T_i\right)$ respects the assumption of lognormality and martingality we can still use the Black framework. But up to now, we have built only one of the two necessary pillars, and this is the martingality assumption.

We have not made any hypothesis about the distribution of the forward rate under the T_i-forward measure. If we consider such a distribution to be lognormal we have that the dynamics of the process, under this measure is:

$$dF\left(t, T_{i-1}, T_i\right) = 0dt + \sigma^f F\left(t, T_{i-1}, T_i\right) dW^{T_i} \tag{B.5}$$

where σ^f is the instantaneous volatility of the forward rate and W^{T_i} is a standard Brownian motion under the T_i-forward measure.

The lognormality comes from the proportionality between the level and the volatility inside the diffusion coefficient, $\sigma^f F(t, T_{i-1}, T_i)$. The martingality, instead, is due to the presence of a null drift, in fact being $\mathbb{E}_t^{T_i}\left[dW^{T_i}\right] = 0$, we have that the only term affecting the expected value is the drift, the quantity proportional to dt. Knowing that the forward rates are martingales only under their specific measure it is clear why they cannot have the same property under the risk-neutral one. Having that $F(t, T_{i-1}, T_i)$ evolves as a lognormal martingale under the T_i-forward measure we can compute the price of the caplet using Black:

$$caplet\,(t) = P\,(t, T_i)\,(\ F\,(t, T_{i-1}, T_i)\,\Phi\,(d1) - K\Phi\,(d2))\,\Delta_i N \tag{B.6}$$

where Φ stands for the cumulative standard normal distribution. The quantities d_1 and d_2 are computed according to Black as:

$$d_1 = \frac{\ln\frac{F_0}{K} + \frac{1}{2}\int_t^{T_{i-1}} \sigma_f^2\,(s)\,ds}{\sqrt{\int_t^{T_{i-1}} \sigma_f^2\,(s)\,ds}} \quad d_2 = d_1 - \sqrt{\int_t^{T_{i-1}} \sigma_f^2\,(s)\,ds}$$

Caps and Floors are respectively collection of caplets and floorlets, their price is therefore equal to the sum of the values of the single components.

B.3 Swaptions

After the application to caplets we move towards the pricing of swaptions by using the change of measure.

We display in (B.7) the T_k-payoff of a European plain vanilla payer swaption with maturity T_k written on a spot starting swap with tenor $T_n - T_k$ and unitary notional:

$$\Pi_{T_k}^{swpn} = \ (\ S(T_k, T_k, T_n) - K)^+ \times \sum_{i=k+1}^{n} P\,(T_k, T_i)\,\Delta_i \tag{B.7}$$

where $\sum_{i=k+1}^{n} P\,(T_k, T_i)\,\Delta_i = A\,(T_k)$ is the annuity term. For simplicity, we assume here that the fixed and the floating legs are paid at the same dates. The general case is considered in section 3. The investor at T_k exercises if the spot-starting swap rate is greater than the strike K and from thereafter there is no more uncertain into the payoff. This means that at time T_k the options buyer checks in the market what is the equilibrium swap rate of a contract with equal characteristics to those of the underlying. If the market quotes a swap rate higher than the strike the investor will exercise the swaption.

To understand why there will be no more uncertainty for the payoff after the exercise, consider the following strategy. Define with S_2 the equilibrium spot starting swap rate quoted by the market at the options maturity and with K the agreed strike at inception. If at maturity $S_2 > K$ the owner of the swaption will exercise the contract entering in a payer swap. At the same time, the investor could take a specular contract freezing a positive cash flow for all the future payment dates. In fact, it is convenient for him to enter in a receiver swap as soon as the option is exercised. From the payer swap he will pay K and receive L_x, where the latter is the floating rate. While from the receiver one he will get S_2 and pay L_x. The overall result is $L_x - \mathrm{K} + S_2 - L_x = S_2 - K$.

This is why if the swaption is exercised what the investor will receive is a fixed amount ($S(T_k, T_k, T_n) - K$) paid during the entire swap-tenor according to the frequencies. Here should be clear to the reader the difference between swaptions and caps/floors. If exercised, the former will pay at any relevant date a positive amount, while the latter leaves the optionality to exercise or not the single caplets/floorlets.
This is why, usually, swaptions and caps/floors take respectively the names of correlation and volatility products. Indeed, swap rates, that are the underlying of swaptions, are defined as linear combinations of forward rates. This means that correlation needs to be taken into account and that plays an important role. Being the swap rate a weighted average of forward rates we cannot use a T-like-forward measure due to the fact that a linear combination of lognormals it is not, in general, lognormal. While, being the expected value a first order operator, linear combinations of martingales are themselves martingales.

But in order to use Black, lognormality is required. In theory, by applying (B.1) we know that the following holds:

$$swpn_t = num\,(t)\,\mathbb{E}_t^{num}\left[\frac{(\ S(T_k, T_k, T_n) - K)^+ \times \sum_{i=k+1}^{n} P\,(T_k, T_i)\,\Delta_i}{num\,(T_k)}\right] \tag{B.8}$$

Now the aim is finding the traded quantity that working as numeraire allows to move to a measure where Black can be used. We would like to simplify as much as possible the quantity inside the expectation in (B.8). If we assume that the annuity, which is a combination of legitimate traded assets, plays the role of numeraire, the following holds:

$$
\begin{aligned}
swpn_t = \sum_{i=k+1}^{n} P\left(t, T_i\right) \Delta_i \mathbb{E}_t^{\mathbb{S}} \left[\frac{\left(S(T_k, T_k, T_n) - K\right)^{+} \times \sum_{i=k+1}^{n} P\left(T_k, T_i\right) \Delta_i}{\sum_{i=k+1}^{n} P\left(T_k, T_i\right) \Delta_i} \right] = \\
= A\left(t\right)\ \mathbb{E}_t^{\mathbb{S}} \left[\left(S(T_k, T_k, T_n) - K\right)^{+} \right]
\end{aligned}
\tag{B.9}
$$

the result is a new probability measure $\mathbb{S}$, known as swap-measure. We now have:

$$
\mathbb{E}_t^{\mathbb{S}} \left[\left(S(T_k, T_k, T_n) - K\right)^{+} \right]
$$

that looks similar to the expected values solved by Black.

From the previous paragraphs we have a toolset with which we are able to compute such an expectation but only if the stochastic process has a lognormal martingale dynamics. Does this happen for the swap rate under the measure $\mathbb{S}$? Checking the martingale property just means looking at the definition of numeraire. Having the annuity at the denominator of swap rates implies that they are relative quantities with respect to the numeraire and, by consequence, martingales. What about the lognormality? This has to come exogenously, we need to assume that the swap rate has a lognormal dynamic under $\mathbb{S}$. See Brigo&Mercurio[3] for a theoretical validation of such an assumption.

Being a lognormal martingale means that:

$$
dS\left(t, T_k, T_n\right) = \sigma^{s} S\left(t, T_k, T_n\right) dW^{\mathbb{S}} \tag{B.10}
$$

where σ^{s} is the instantaneous volatilities of the swap rate. Therefore, we are able to use Black to derive the price of the swaption:

$$
swpn_t = A\left(t\right) \left(S\left(t, T_k, T_n\right) \Phi\left(d1\right) - K \Phi\left(d2\right)\right) \tag{B.11}
$$

C Solving Models' formulae

Here we derive the closed-form solutions to the rational models presented in section 3. We make use of the Black approach presented in Appendix B. In the case of a two-factor model we have also used Appendix E of Brigo&Mercurio[3] in order to validate our results. The calibration in section 4 has been performed implementing the formulae displayed here. Black model is very powerful, but as can be seen in Appendix B is strictly dependent on the assumptions of lognormality and martingality of the underlying. We now show all the passages implemented to derive our solutions. All the steps will be explained with exhaustive details.

C.1 One-factor Lognormal

Under this framework the pricing problem is defined as:

$$Swn_{0T_k} = N\mathbb{E}^{\mathbb{M}}\left[\left(c_2 A_{T_k}^{(2)} + c_0\right)^+\right]; \quad A_{T_k}^{(2)} = e^{\left(a_2 X_{T_k}^{(2)} - \frac{1}{2}a_2^2 T_k\right)} - 1; \quad X_{T_k}^{(2)} \overset{\mathbb{M}}{\sim} N\left(0, T_k\right)$$

We focus now only on the $\mathbb{M}$ expectation, recalling that $\widetilde{c_0} = c_0 - c_2$:

$$\mathbb{E}^{\mathbb{M}}\left[\left(c_2 A_{T_k}^{(2)} + c_0\right)^+\right] = \mathbb{E}^{\mathbb{M}}\left[\left(c_2 e^{\left(a_2 X_{T_k}^{(2)} - \frac{1}{2}a_2^2 T_k\right)} - c_2 + c_0\right)^+\right] = \mathbb{E}^{\mathbb{M}}\left[\left(c_2 e^{\left(a_2 X_{T_k}^{(2)} - \frac{1}{2}a_2^2 T_k\right)} + \widetilde{c_0}\right)^+\right]$$

The last term looks familiar to the expectations seen in Black, where, for a generic measure $\mathbb{G}$, we have: $\mathbb{E}^{\mathbb{G}}\left[(F_{T_k} - K)^+\right]$ with F_{T_k} stochastic and K a real value. In our case, we have: $F_{T_k} = c_2 e^{\left(a_2 X_{T_k}^{(2)} - \frac{1}{2}a_2^2 T_k\right)}$ and $K = -\widetilde{c_0}$. We recall that in order for Black to be applied it is necessary that the process $\{F_t\}_{0 \le t \le T}$ is a log-normal martingale under the specific measure.

From $A_t^{(i)} = S_t^{(i)} - 1$ with $S_t^{(i)}$ being a $\mathbb{M}$-martingale we can see how the martingality is preserved also for F_t. In fact, being c_2 a real value, multiplying it by a null drift does not have any impact on average. On the other hand, to show that also the lognormality holds, it is sufficient to note that F_t is defined as the exponential of a normal distribution, that is, indeed, a lognormal. Having found that F_t evolves as a $\mathbb{M}$ lognormal martingale we can then apply Black.

Back to the generic measure $\mathbb{G}$ we have that at time $t = 0$:

$$\mathbb{E}^{\mathbb{G}}\left[(F_{T_k} - K)^+\right] = F_0 \Phi^{\mathbb{G}}\left(d_1\right) - K\Phi^{\mathbb{G}}\left(d_2\right)$$

where $\Phi^{\mathbb{G}}$ is the standard normal cumulative function, with $\Phi^{\mathbb{G}}$ meaning that d_1 and d_2 are computed according to the dynamics of the stochastic process under the relevant measure. In particular, we stress the point that $\Phi^{\mathbb{G}}(d_2)$ is the $\mathbb{G}$-probability for the option to be in the money at maturity.

In our problem, we have that such a probability under $\mathbb{M}$ is:

$$P^{\mathbb{M}}\left(F_{T_k}-K\geq 0\right)=P^{\mathbb{M}}\left(c_2e^{\left(a_2X^{(2)}_{T_k}-\frac{1}{2}a_2^2T_k\right)}+\widetilde{c_0}\geq 0\right)=P^{\mathbb{M}}\left(X^{(2)}_{T_k}\geq\frac{\ln\frac{-\widetilde{c_0}}{c_2}+\frac{1}{2}a_2^2T_k}{a_2}\right)$$

For solving the last term, knowing that $X^{(2)}_{T_k}$ is normal, we shift it towards the origin applying the standardization procedure:

$$P^{\mathbb{M}}\left(\frac{X^{(2)}_{T_k}-\mu_X}{\sigma_X}\geq\frac{\frac{\ln\frac{-\widetilde{c_0}}{c_2}+\frac{1}{2}a_2^2T_k}{a_2}-\widehat{\mu_X}}{\widehat{\sigma_X}}\right)$$

where, for coherence, the hats mean that the RHS is a realization while the LHS is stochastic. Being $X^{(2)}_{T_k}$ a standard Brownian motion we have $\widehat{\mu_X}=0$ and $\widehat{\sigma_X}=\sqrt{T_k}$, we can rewrite it, denoting with Z the standard normal distribution, as:

$$P^{\mathbb{M}}\left(\frac{X^{(2)}_{T_k}-\mu_X}{\sigma_X}\geq\frac{\frac{\ln\frac{-\widetilde{c_0}}{c_2}+\frac{1}{2}a_2^2T_k}{a_2}-0}{\sqrt{T_k}}\right)=P^{\mathbb{M}}\left(\mathrm{Z}_{T_k}\geq\frac{\frac{\ln\frac{-\widetilde{c_0}}{c_2}+\frac{1}{2}a_2^2T_k}{a_2}-0}{\sqrt{T_k}}\right)=P^{\mathbb{M}}\left(\mathrm{Z}_{T_k}\geq\frac{\ln\frac{-\widetilde{c_0}}{c_2}+\frac{1}{2}a_2^2T_k}{a_2\sqrt{T_k}}\right)=P^{\mathbb{M}}\left(\mathrm{Z}_{T_k}\geq z^*\right)$$

From the symmetry of the normal we have that:

$$P^{\mathbb{M}}\left(\mathrm{Z}_{T_k}\geq z^*\right)=P^{\mathbb{M}}\left(\mathrm{Z}_{T_k}\leq -z^*\right)=\Phi^{\mathbb{M}}\left(-z^*\right)$$

Therefore, we have that $\Phi^{\mathbb{M}}(-z^*)$ is the $\mathbb{M}$-probability for the option to be in the money at maturity. But the theory reminds us that such a probability is $\Phi^{\mathbb{M}}(d_2)$. This implies:

$$d_2=-z^*=\frac{-\ln\frac{-\widetilde{c_0}}{c_2}-\frac{1}{2}a_2^2T_k}{a_2\sqrt{T_k}}$$

The general definition of d_2, for pricing at time $t=0$, is:

$$d_2=\frac{\ln\frac{F_0}{K}-\frac{1}{2}\sigma^2T_k}{\sigma\sqrt{T_k}}$$

where σ is the instantaneous volatility of the process $\{F_t\}$.Here, for simplicity, we are assuming that σ evolves constantly through time. For a more consistent notation we should have:

$$d_2=\frac{\ln\frac{F_0}{K}-\frac{1}{2}\int_0^{T_k}\sigma^2(s)\,ds}{\sqrt{\int_0^{T_k}\sigma^2(s)\,ds}}$$

From the Appendix on Black model we have $d_2 = d_1 - \sigma\sqrt{T_k}$ and so $d_1 = d_2 + \sigma\sqrt{T_k}$. We now need to understand what value of σ is implied by our value of d_2. It is important to remember that F is an $\mathbb{M}$-martingale implying $F_0 = \mathbb{E}^{\mathbb{M}}\left[F_{T_K}\right]$, where the absence of subscript to $\mathbb{E}$ means that it is computed at time t=0.

Recalling the properties of the standard Brownian motion, with the mean and variance equal, respectively, to zero and the time step, we have:

$$F_0 = \mathbb{E}^{\mathbb{M}}\left[c_2 e^{\left(a_2 X^{(2)}_{T_k} - \frac{1}{2}a_2^2 T_k\right)}\right] = c_2 e^{-\frac{1}{2}a_2^2 T_k}\mathbb{E}^{\mathbb{M}}\left[e^{a_2 X^{(2)}_{T_k}}\right] = c_2 e^{-\frac{1}{2}a_2^2 T_k} e^{\frac{1}{2}a_2^2 T_k} = c_2$$

Where for the last passage we have used the relation between the first moments of a lognormal and a normal distribution. Indeed, if X has a normal distribution with $\mathbb{E}\left[X\right] = \mu$ and $Var\left(X\right) = \sigma^2$, for any real value α we have that $e^{\alpha X}$ is lognormal with $\mathbb{E}\left[e^{\alpha X}\right] = e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}$, where the term $e^{\frac{1}{2}\alpha^2\sigma^2}$ is there to take into account the convexity of the logarithm.

Having $F_0 = c_2$ allows us to compute the implied volatility after some adjustments to d_2:

$$d_2 = \frac{-\ln\frac{-\widetilde{c_0}}{c_2} - \frac{1}{2}a_2^2 T_k}{a_2\sqrt{T_k}} = \frac{\ln\frac{c_2}{-\widetilde{c_0}} - \frac{1}{2}a_2^2 T_k}{a_2\sqrt{T_k}}$$

where we can see that $\sigma = a_2$. With this information we can compute d_1:

$$d_1 = d_2 + \sigma\sqrt{T_k} = \frac{\ln\frac{c_2}{-\widetilde{c_0}} + \frac{1}{2}a_2^2 T_k}{a_2\sqrt{T_k}}$$

And finally applying Black under the measure $\mathbb{M}$ we have:

$$Swn_{0T_k} = N\mathbb{E}^{\mathbb{M}}\left[\left(c_2 A^{(2)}_{T_k} + c_0\right)^+\right] = N\left(c_2\Phi^{\mathbb{M}}\left(\frac{\ln\frac{c_2}{-\widetilde{c_0}} + \frac{1}{2}a_2^2 T_k}{a_2\sqrt{T_k}}\right) + \widetilde{c_0}\Phi^{\mathbb{M}}\left(\frac{\ln\frac{c_2}{-\widetilde{c_0}} - \frac{1}{2}a_2^2 T_k}{a_2\sqrt{T_k}}\right)\right)$$

C.2 Two-factor Lognormal

In solving equation (3.19), our starting point is the computation of the following expected value:

$$\mathbb{E}^{\mathbb{M}}\left[\left(c_2 e^{\left(a_2 X\sqrt{T_k} - \frac{1}{2}a_2^2 T_k\right)} + c_3 e^{\left(a_3 Y\sqrt{T_k} - \frac{1}{2}a_3^2 T_k\right)} - c_1 e^{\left(a_1 Y\sqrt{T_k} - \frac{1}{2}a_1^2 T_k\right)} + \widetilde{c_0}\right)^+\right]$$

with $\widetilde{c_0} = c_0 + c_1 - c_2 - c_3$

Knowing that:

$$K\left(y\right) = c_1\left(e^{\left(a_1 y\sqrt{T_k} - \frac{1}{2}a_1^2 T_k\right)} - 1\right) - c_3\left(e^{\left(a_3 y\sqrt{T_k} - \frac{1}{2}a_3^2 T_k\right)} - 1\right) - c_0 + c_2$$

We can rewrite:

$$\mathbb{E}^{\mathbb{M}}\left[\left(\ c_2 e^{\left(a_2 X\sqrt{T_k}-\frac{1}{2}a_2^2 T_k\right)} -\ K\left(y\right)\ \right)^+\right] \tag{C.1}$$

This looks similar to the expectations seen in the Black model. It is easy to see that $c_2 e^{\left(a_2 X\sqrt{T_k}-\frac{1}{2}a_2^2 T_k\right)}$ is a lognormal martingale due to the normality of X. But, being the term K inside the expectation stochastic and not a real value, Black cannot be straightly applied.

Defining the jointly distribution as proportional to the conditional and the marginal plays here a major role. Indeed we know that $f\left(x,y\right)=f\left(x|y\right)f\left(y\right)$. We want to apply this identity in order to bring everything back to known problems, where we could be able to use an approach à la Black.

$$\begin{aligned}
&\mathbb{E}^{\mathbb{M}}\left[\left(\ c_2 e^{\left(a_2 X\sqrt{T_k}-\frac{1}{2}a_2^2 T_k\right)} -\ K\left(y\right)\ \right)^+\right] = N\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\left(\ c_2 e^{\left(a_2 x\sqrt{T_k}-\frac{1}{2}a_2^2 T_k\right)} - K\left(y\right)\right)^+ f\left(x|y\right)f\left(y\right)dxdy = \\
&= \int\limits_{K(y)>0}\left(\int\limits_{-\infty}^{\infty}\left(\ c_2 e^{\left(a_2 x\sqrt{T_k}-\frac{1}{2}a_2^2 T_k\right)} - K\left(y\right)\right)^+ f\left(x|y\right)dx\ \right)f\left(y\right)dy + \\
&+ \int\limits_{K(y)<0}\left(\int\limits_{-\infty}^{\infty}\left(\ c_2 e^{\left(a_2 x\sqrt{T_k}-\frac{1}{2}a_2^2 T_k\right)} - K\left(y\right)\right) f\left(x|y\right)dx\ \right)f\left(y\right)dy = \\
&= \int\limits_{K(y)>0}\mathbb{E}^{\mathbb{M}}{}_{X|Y}\left[\left(\ c_2 e^{\left(a_2 x\sqrt{T_k}-\frac{1}{2}a_2^2 T_k\right)} - K\left(y\right)\right)^+\right]f\left(y\right)dy + \int\limits_{K(y)<0}\mathbb{E}^{\mathbb{M}}{}_{X|Y}\left[\left(\ c_2 e^{\left(a_2 x\sqrt{T_k}-\frac{1}{2}a_2^2 T_k\right)} - K\left(y\right)\right)\ \right]f\left(y\right)dy
\end{aligned} \tag{C.2}$$

We focus our attention on the first integral in the last passage of equation C.2:

$$\mathbb{E}^{\mathbb{M}}{}_{X|Y}\left[\left(\ c_2 e^{\left(a_2 x\sqrt{T_k}-\frac{1}{2}a_2^2 T_k\right)} - K\left(y\right)\right)^+\right] \tag{C.3}$$

Although this could look similar to equation (C.1), it, actually, differs for being a conditional expectation, testified by the presence of the subscript $X|Y$. This is to say that we are in a case where y is known. This makes $K\left(y\right)$ a realization and no more a stochastic quantity. In this way we are now able to use Black to find a closed form solution.

It is easy to see that the first term inside the expectation is a lognormal martingale. Moreover, we have that $c_2 e^{\left(a_2 X\sqrt{T_k}-\frac{1}{2}a_2^2 T_k\right)} \neq c_2 e^{\left(a_2 Z^{(2)}\sqrt{T_k}-\frac{1}{2}a_2^2 T_k\right)}$ where the second one comes from the definition of $A^{(2)}$. Both X and $Z^{(2)}$ are normal distributions but they have different moments, here $X = X|Y = y \sim \mathcal{N}\left(\rho y,\left(1-\rho^2\right)\right)$ while $Z^{(2)} \sim \mathcal{N}\left(0,1\right)$. Therefore, they just differ in the magnitude of their moments, so the lognormal property is not impacted. Neither the martingality is violated.

Having lognormal martingales then we can tackle the problem with a Black like approach. The first quantity of interest is the probability under $\mathbb{M}$ that the value in the expectation in equation (C.3) will be positive at maturity. We would like to stess that in the following steps $\Phi^{\mathbb{M}}(d_2)$ is not the probability for the swaption to be in the money at T_k. d_2 is, indeed, the likelihood that $\left(c_2 e^{\left(a_2 x\sqrt{T_k}-\frac{1}{2}a_2^2 T_k\right)} - K(y)\right)$ will be positive at T_k. For its computation we have:

$$\mathrm{P}^{\mathbb{M}}\left(F_{T_k} - K(y) \geq 0\right) = \mathrm{P}^{\mathbb{M}}\left(c_2 e^{\left(a_2 X\sqrt{T_k}-\frac{1}{2}a_2^2 T_k\right)} - K(y) \geq 0\right) = \mathrm{P}^{\mathbb{M}}\left(X \geq \frac{\ln\frac{K(y)}{c_2}+\frac{1}{2}a_2^2 T_k}{a_2\sqrt{T_k}}\right)$$

Shifting the normal distribution towards the origin with the standardization procedure and recalling that $\hat{\mu} = \rho y$ and $\hat{\sigma} = \sqrt{(1-\rho^2)}$ we have:

$$\mathrm{P}^{\mathbb{M}}\left(X \geq \frac{\ln\frac{K(y)}{c_2}+\frac{1}{2}a_2^2 T_k}{a_2\sqrt{T_k}}\right) = \mathrm{P}^{\mathbb{M}}\left(\frac{X-\mu}{\sigma} \geq \frac{\frac{\ln\frac{K(y)}{c_2}+\frac{1}{2}a_2^2 T_k}{a_2\sqrt{T_k}} - \hat{\mu}}{\hat{\sigma}}\right) =$$

$$= \mathrm{P}^{\mathbb{M}}\left(\mathrm{Z} \leq \frac{-\ln\frac{K(y)}{c_2} - \frac{1}{2}a_2^2 T_k + a_2\sqrt{T_k}\rho y}{a_2\sqrt{T_k}\sqrt{(1-\rho^2)}}\right) = \Phi^{\mathbb{M}}(d_2)$$

We can arrange d_2 in a more convenient way:

$$d_2 = \frac{-\ln\frac{K(y)}{c_2} - \frac{1}{2}a_2^2 T_k + a_2\sqrt{T_k}\rho y}{a_2\sqrt{T_k}\sqrt{(1-\rho^2)}} = \frac{\ln\frac{c_2}{K(y)} - \frac{1}{2}a_2^2 T_k + a_2\sqrt{T_k}\rho y}{a_2\sqrt{T_k}\sqrt{(1-\rho^2)}}$$

Knowing that $d_2 = \frac{\ln\frac{F_0}{K} - \frac{1}{2}\sigma^2 T_k}{\sigma\sqrt{T_k}}$. We have to extract the implied volatility from our results in order to compute d_1. We have that:

$$F_0 = \mathbb{E}^{\mathbb{M}}\left[F_{T_K}\right] = \mathbb{E}^{\mathbb{M}}\left[c_2 e^{\left(a_2 X\sqrt{T_k}-\frac{1}{2}a_2^2 T_k\right)}\right] = c_2 e^{a_2\sqrt{T_k}\rho y - \frac{1}{2}a_2^2 T_k \rho^2}$$

Where for the computation of the expected value we have used the properties linking the moments of lognormal and normal distributions.

Now that we have F_0 we would like to arrange the terms in d_2 so that the implied volatility can be derived. We add and subtract $\frac{1}{2}a_2^2 T_k \rho^2$ to the numerator of d_2:

$$d_2 = \frac{\ln\frac{c_2}{K(y)} - \frac{1}{2}a_2^2 T_k + a_2\sqrt{T_k}\rho y \pm \frac{1}{2}a_2^2 T_k\rho^2}{a_2\sqrt{T_k}\sqrt{(1-\rho^2)}} = \frac{\ln\frac{c_2 e^{a_2\sqrt{T_k}\rho y - \frac{1}{2}a_2^2 T_k\rho^2}}{K(y)} - \frac{1}{2}a_2^2 T_k + \frac{1}{2}a_2^2 T_k\rho^2}{a_2\sqrt{T_k}\sqrt{(1-\rho^2)}} =$$

$$= \frac{\ln\frac{F_0}{K(y)} - \frac{1}{2}a_2^2\left(1-\rho^2\right)T_k}{a_2\sqrt{(1-\rho^2)}\sqrt{T_k}}$$

From this we can see that $\sigma = a_2\sqrt{(1-\rho^2)}$. Therefore we can now compute d_1:

$$d_1 = d_2 + \sigma\sqrt{T_k} = \frac{\ln\frac{c_2}{K(y)} - \frac{1}{2}a_2^2 T_k + a_2\sqrt{T_k}\rho y}{a_2\sqrt{T_k}\sqrt{(1-\rho^2)}} + a_2\sqrt{(1-\rho^2)}\sqrt{T_k} = \frac{\ln\frac{c_2}{K(y)} + a_2^2 T_k\left(\frac{1}{2}-\rho^2\right) + a_2\sqrt{T_k}\rho y}{a_2\sqrt{T_k}\sqrt{(1-\rho^2)}}$$

Thus, from all these findings we can use Black:

$$\mathbb{E}^{\mathbb{M}}{}_{X|Y}\left[\left(\ c_2 e^{\left(a_2 x\sqrt{T_k}-\frac{1}{2}a_2^2 T_k\right)}-K\left(y\right)\right)^+\right]=$$
$$=c_2 e^{a_2\sqrt{T_k}\rho y-\frac{1}{2}a_2^2 T_k\rho^2}\Phi^{\mathbb{M}}\left(\frac{\ln\frac{c_2}{K(y)}+a_2^2 T_k\left(\frac{1}{2}-\rho^2\right)+a_2\sqrt{T_k}\rho y}{a_2\sqrt{T_k}\sqrt{(1-\rho^2)}}\right)-K\left(y\right)\Phi^{\mathbb{M}}\left(\frac{\ln\frac{c_2}{K(y)}-\frac{1}{2}a_2^2 T_k+a_2\sqrt{T_k}\rho y}{a_2\sqrt{T_k}\sqrt{(1-\rho^2)}}\right) \tag{C.4}$$

Note that the one derived above is a closed form solution. In fact, we are conditioning on y and so there is no stochasticity in the term $K\left(y\right)$.

In case $K\left(y\right)<0$, the term inside the conditional expectation cannot be negative, by definition for the presence of the exponential and, by construction, for the assumption on c_2. So for $K\left(y\right)<0$ we just have to solve the expectation recalling the moments of the relevant distribution:

$$\mathbb{E}^{\mathbb{M}}{}_{X|Y}\left[\left(\ c_2 e^{\left(a_2 x\sqrt{T_k}-\frac{1}{2}a_2^2 T_k\right)}-K\left(y\right)\right)\ \right]=\mathbb{E}^{\mathbb{M}}{}_{X|Y}\left[\ c_2 e^{\left(a_2 x\sqrt{T_k}-\frac{1}{2}a_2^2 T_k\right)}\ \right]-K\left(y\right)=c_2 e^{a_2\sqrt{T_k}\rho y-\frac{1}{2}a_2^2 T_k\rho^2}-K\left(y\right)$$

We have now derived both the expectations displayed in the last term of (C.2). Collecting all these findings, we can finally see that the swaption's price is given by the following:

$$\begin{aligned} Swn_{0T_k} &= \int\limits_{K>0}\left[c_2 e^{a_2\sqrt{T_k}\rho y-\frac{1}{2}a_2^2 T_k\rho^2}\Phi^{\mathbb{M}}\left(d_1\right)-K\left(y\right)\Phi^{\mathbb{M}}\left(d_2\right)\right]f(y)dy \\ &+\int\limits_{K<0}\left(c_2 e^{a_2\sqrt{T_k}\rho y-\frac{1}{2}a_2^2 T_k\rho^2}-K\left(y\right)\right)f\left(y\right)dy \end{aligned} \tag{C.5}$$

C.2.1 Checking two-factor lognormal solution

In Appendix E of Brigo&Mercurio[3] a useful calculation for pricing a spread option is reported. Here we use their findings to validate our results. For $\omega\in\{-1,1\}$, $A>0$, $K>0$, $V>0$ and M being real numbers we have the following:

$$\int\limits_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}V}(\omega A e^z-\omega K)^+ e^{-\frac{1}{2}\frac{(z-M)^2}{V^2}}dz=\omega A e^{M+\frac{1}{2}V^2}\Phi\left(\omega\frac{M-\ln\frac{K}{A}+V^2}{V}\right)-\omega K\Phi\left(\omega\frac{M-\ln\frac{K}{A}}{V}\right)$$

We apply this formula to the lognormal two factor model presented in our work in order to derive the swaptions price defined in (3.20).

Particularly. the procedure reported in Brigo&Mercurio gives us a way to compute the quantity:

$$\int\limits_{-\infty}^{\infty}\left(\ c_2 e^{\left(a_2 x\sqrt{T_k}-\frac{1}{2}a_2^2 T_k\right)}-K\left(y\right)\right)^+ f\left(x|y\right)\ dx;\quad f\left(x|y\right)=\frac{1}{\sqrt{2\pi(1-\rho^2)}}e^{-\frac{(x-\rho y)^2}{2(1-\rho y)}}$$

We can arrange the integrand to apply the above formula

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left(c_2 e^{\left(a_2 x \sqrt{T_k} - \frac{1}{2} a_2^2 T_k\right)} - K(y) \right)^+ f(x|y)\ dx = \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1-\rho^2)}} \left(c_2 e^{\left(a_2 x \sqrt{T_k} - \frac{1}{2} a_2^2 T_k\right)} - K(y) \right)^+ e^{-\frac{(x-\rho y)^2}{2(1-\rho y)}}\ dx = \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(1-\rho^2)}} \left(c_2 e^{-\frac{1}{2} a_2^2 T_k} e^{\left(a_2 x \sqrt{T_k}\right)} - K(y) \right)^+ e^{-\frac{(x-\rho y)^2}{2(1-\rho y)}}\ dx
\end{aligned}$$

From this we see that for $\omega = 1$:

$$z = a_2 x \sqrt{T_k}, \quad A = c_2 e^{-\frac{1}{2} a_2^2 T_k}, \quad M = \mathbb{E}[z] = a_2 \sqrt{T_k} \rho y, \quad V = Var(z) = a_2 \sqrt{(1-\rho^2)} \sqrt{T_k} \tag{C.6}$$

Thus, we have:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left(c_2 e^{\left(a_2 x \sqrt{T_k} - \frac{1}{2} a_2^2 T_k\right)} - K(y) \right)^+ f(x|y)\ dx = \\
&= c_2 e^{-\frac{1}{2} a_2^2 T_k} e^{\left(a_2 \sqrt{T_k} \rho y + \frac{1}{2} a_2^2 \left(1-\rho^2\right) T_k\right)} \Phi\left(\frac{a_2 \sqrt{T_k} \rho y - \ln \frac{K(y)}{c_2 e^{-\frac{1}{2} a_2^2 T_k}} + a_2^2 \left(1-\rho^2\right) T_k}{a_2 \sqrt{(1-\rho^2)} \sqrt{T_k}} \right) - K(y) \Phi\left(\frac{a_2 \sqrt{T_k} \rho y - \ln \frac{K(y)}{c_2 e^{-\frac{1}{2} a_2^2 T_k}}}{a_2 \sqrt{(1-\rho^2)} \sqrt{T_k}} \right) = \\
&= c_2 e^{\left(a_2 \sqrt{T_k} \rho y - \frac{1}{2} a_2^2 T_k \rho^2\right)} \Phi\left(\frac{\ln c_2 - \ln K(y) + a_2 \sqrt{T_k} \rho y + a_2^2 \left(\frac{1}{2} - \rho^2\right) T_k}{a_2 \sqrt{(1-\rho^2)} \sqrt{T_k}} \right) - K(y) \Phi\left(\frac{\ln c_2 - \ln K(y) + a_2 \sqrt{T_k} \rho y - \frac{1}{2} a_2^2 T_k}{a_2 \sqrt{(1-\rho^2)} \sqrt{T_k}} \right)
\end{aligned}$$

We see that this result is consistent with the ons derived by using a Black like approach in C.4. The remaining quantities of C.2 can be easily derived with the already shown methodology. Therefore, our solution is validated.

D Matlab Code

D.1 Black and Bachelier

D.1.1 Lognormal Volatilities

Listing 1: Pricing Black

```
function priceBlack = priceBlack(param,priceSwp,k,T,t,delta,discount)
sigma = param;
notional = 100;
annuity = sum(delta.*discount);
    d1 = (log(priceSwp/k)+(1/2)*sigma^2*(T-t))/(sigma*sqrt(T-t));
    d2 = d1-sigma*sqrt(T-t);
    priceBlack = notional*annuity*(priceSwp*normcdf(d1)-k*normcdf(d2));
end
```

Listing 2: Implied lognormal volatility with Newton

```
function vol = impliedVolatilityNewton(discount,strikeVec,Tswpn,
   priceSwpnVec,delta)
notional = 100;
T = Tswpn;
priceSwp = 4.2994/100; %fwdswaprate
t = 0;
len = length(strikeVec);
if len >1
for j = 1:len %implied vol for each strike
    priceSwpn = priceSwpnVec(j);
    k = strikeVec(j);
    sigma = 0.3;
    sigmavec(1) = sigma;
    for i = 2:100
    d1 = (log(priceSwp/k)+(1/2)*sigma^2*(T-t))/(sigma*sqrt(T-t));
    d2 = d1-sigma*sqrt(T-t);
    annuity = (delta')*discount;
```

```
    diffpriceBlack = notional*annuity*(priceSwp*normcdf(d1)-k*normcdf(d2))
       -priceSwpn;

    %derivatives
    d11 = (sigma^2*(T-t)*sqrt(T-t)-sqrt(T-t)*(log(priceSwp/k)+(1/2)*sigma
       ^2*(T-t)))/(sigma^2*(T-t));
    d22 = d11 -sqrt(T-t);
    f1 = notional*annuity*(priceSwp*normpdf(d1)*d11-k*normpdf(d2)*d22);
    %f1=VEGA
    sigma = sigma - diffpriceBlack/f1
    sigmavec(i)=sigma;

    if (abs(sigmavec(i)-sigmavec(i-1)) < 0.00000000000001)
        iteration = i;
        break;
    end
    end
     countVecNewton(j) = iteration;
vol(j) = sigma;
end

else %in case of K scalar and not vector
    priceSwpn = priceSwpnVec;
    k = strikeVec;
    priceSwp = k;
    sigma = 0.3;
    sigmavec(1) = sigma;
    for i = 2:100
    d1 = (log(priceSwp/k)+(1/2)*sigma^2*(T-t))/(sigma*sqrt(T-t));
    d2 = d1-sigma*sqrt(T-t);
    annuity = (delta')*discount;
    diffpriceBlack = notional*annuity*(priceSwp*normcdf(d1)-k*normcdf(d2))
```

```
        -priceSwpn;
    %derivatives
    d11 = (sigma^2*(T-t)*sqrt(T-t)-sqrt(T-t)*(log(priceSwp/k)+(1/2)*sigma
        ^2*(T-t)))/(sigma^2*(T-t));
    d22 = d11 -sqrt(T-t);
    f1 = notional*annuity*(priceSwp*normpdf(d1)*d11-k*normpdf(d2)*d22);
    sigma = sigma - diffpriceBlack/f1
    sigmavec(i)=sigma;

    if (abs(sigmavec(i)-sigmavec(i-1)) < 0.00000000000001)
        iteration = i;
        break;
    end
    end
    vol = sigma
end


end
```

D.1.2 Normal Volatilities

Listing 3: Pricing Bachelier

```
function priceNormal = priceNormal(param,priceSwp,k,T,t,delta,discount)


sigma = param;
notional = 100;
annuity = sum(delta.*discount);
d1 = (priceSwp-k)/(sigma*sqrt(T-t));
priceNormal =notional*annuity*sigma*sqrt(T-t)*(d1*normcdf(d1)+normpdf(d1))


end
```

Listing 4: Implied normal volatility with Newton

```
function vol = impliedVolatilityNewton(discount,strikeVec,Tswpn,
   priceSwpnVec,delta)
notional = 100;
T = Tswpn;
priceSwp = 4.2994/100; %fwdswaprate
t = 0;
len = length(strikeVec);
if len >1
for j = 1:len %implied vol for each strike
    priceSwpn = priceSwpnVec(j);
    k = strikeVec(j);
    sigma = 0.3;
    sigmavec(1) = sigma;
    for i = 2:100
    d1 = (log(priceSwp/k)+(1/2)*sigma^2*(T-t))/(sigma*sqrt(T-t));
    d2 = d1-sigma*sqrt(T-t);
    annuity = (delta')*discount;
    diffpriceBlack = notional*annuity*(priceSwp*normcdf(d1)-k*normcdf(d2))
       -priceSwpn;

    %derivatives
    d11 = (sigma^2*(T-t)*sqrt(T-t)-sqrt(T-t)*(log(priceSwp/k)+(1/2)*sigma
       ^2*(T-t)))/(sigma^2*(T-t));
    d22 = d11 -sqrt(T-t);
    f1 = notional*annuity*(priceSwp*normpdf(d1)*d11-k*normpdf(d2)*d22);
    %f1=VEGA
    sigma = sigma - diffpriceBlack/f1
    sigmavec(i)=sigma;

    if (abs(sigmavec(i)-sigmavec(i-1)) < 0.00000000000001)
        iteration = i;
```

```
        break;
    end
    end
     countVecNewton(j) = iteration;
vol(j) = sigma;
end


else %in case of K scalar and not vector
    priceSwpn = priceSwpnVec;
    k = strikeVec;
    priceSwp = k;
    sigma = 0.3;
    sigmavec(1) = sigma;
    for i = 2:100
    d1 = (log(priceSwp/k)+(1/2)*sigma^2*(T-t))/(sigma*sqrt(T-t));
    d2 = d1-sigma*sqrt(T-t);
    annuity = (delta')*discount;
    diffpriceBlack = notional*annuity*(priceSwp*normcdf(d1)-k*normcdf(d2))
       -priceSwpn;
    %derivatives
    d11 = (sigma^2*(T-t)*sqrt(T-t)-sqrt(T-t)*(log(priceSwp/k)+(1/2)*sigma
       ^2*(T-t)))/(sigma^2*(T-t));
    d22 = d11 -sqrt(T-t);
    f1 = notional*annuity*(priceSwp*normpdf(d1)*d11-k*normpdf(d2)*d22);
    sigma = sigma - diffpriceBlack/f1
    sigmavec(i)=sigma;

    if (abs(sigmavec(i)-sigmavec(i-1)) < 0.00000000000001)
        iteration = i;
        break;
    end
    end
```

```
    vol = sigma
end


end
```

D.2 Pricing Engines

D.2.1 One-factor

Listing 5: Compute 9x1 smile Swpn prices, One-factor

```
function modelprice = computeSwpnPriceOneFactor(param,discount,forward,
   strikeVec,Tswpn,delta)
N =100;
a2 = param(1);
b2 = param(2);
b2vec = b2*ones(4,1).*delta;
c2 = sum(b2vec);
len = length(strikeVec);
for i = 1:len
strike = strikeVec(i)
c0vec = (forward.*delta-strike*[0 0 0 discount(4)]');
c0 = sum(c0vec);
c0tilde = c0-c2;
if (c0tilde >=0)
    modelprice(i,1) = N*c0
else
    q1 = (0.5*a2^2*Tswpn-log(-c0tilde/c2))/(a2*sqrt(Tswpn))
    q2 = (-0.5*a2^2*Tswpn-log(-c0tilde/c2))/(a2*sqrt(Tswpn))
    modelprice(i,1) = (c2*normcdf(q1)+c0tilde*normcdf(q2))*N
end
end
end
```

Listing 6: Compute ATM Swpn prices One Factor

```
% if version=1: calibration phase --> the b2vec has to grow
%otherwise: pricing phase --> b2vec have been already increased
function atmPrice = computeSwpnPriceATMOneFactor(x,a2,discount,forward,k,
    Tswpn,b2vec,delta,version)
N =100;
if (version==1)
b2 = x;
%add b2 to the already existing vector, this only if version=1
tmp = b2vec'
tmp = [tmp b2*ones(1,4)]
b2vec = tmp'
end
c2 = sum(b2vec.*delta);
%the fixed leg has just one payment per year
%the following is to take this into account
lenDiscount = length(discount)
relevantDiscounts = zeros(lenDiscount,1)
relevantIndexes = 4:4:lenDiscount
numberIndexes = length(relevantIndexes)
for i = 1:numberIndexes
    relevantDiscounts(i*4) = discount(relevantIndexes(i))
end
c0vec = forward.*delta-k*relevantDiscounts;
c0 = sum(c0vec);
c0tilde = c0-c2;
if (c0tilde >=0)
    atmPrice = N*c0
else
    q1 = (0.5*a2^2*Tswpn-log(-c0tilde/c2))/(a2*sqrt(Tswpn))
```

```
    q2 = (-0.5*a2^2*Tswpn-log(-c0tilde/c2))/(a2*sqrt(Tswpn))
    atmPrice = (c2*normcdf(q1)+c0tilde*normcdf(q2))*N
end
end
```

D.2.2 Two-factor

Listing 7: Compute 9x1 smile Swpn prices, Two-factor

```
function swpnprice = computeSwpnPrice2Factor(param,discount,forward,
   strikeVec,Tswpn,delta)

N = 100
a2 = param(1);
b1 = param(2);
b2 = param(3);
rho = param(4);
a1 = param(5)
a3 = param(6)
b2vec = b2*ones(4,1).*delta;
b3vec = 0.15*forward;
c2 = sum(b2vec);
c3 = sum(b3vec);
%consider that the floating leg pays annualy
adjDiscount = zeros(3,1);
adjDiscount(end+1) = discount(4)
for j = 1:length(strikeVec)
strike = strikeVec(j);
c0vec = forward.*delta-strike*adjDiscount;
c0 = sum(c0vec);
c1 = strike*b1; %the floating leg is annual
```

```
myfun = @(x) (computeK(x,c1,a1,Tswpn,c3,a3,c0,c2))
%find the roots and compute the integral
root1 = fzero(myfun,0);
myPosFun = @(l) computeIntegrand1(l,rho,c1,a1,Tswpn,c3,a3,c0,a2,c2);
myNegFun = @(z) computeIntegrand2(z,rho,c1,a1,Tswpn,c3,a3,c0,a2,c2);
root2 = fzero(myfun,4);
if (isnan(root1))
    root1 = root2
end

if (abs(root1-root2)>0.000001)
    swpnprice(j,1) = (quadgk(@(x) myNegFun(x).*normpdf(x),-inf,root1)+
       quadgk(@(x) myPosFun(x).*normpdf(x),root1,root2) + quadgk(@(x)
       myNegFun(x).*normpdf(x),root2,inf))*N
else
    swpnprice(j,1) = (quadgk(@(x) myPosFun(x).*normpdf(x),-inf,root1)+
       quadgk(@(x) myNegFun(x).*normpdf(x),root1,inf))*N
end
end


end
```

Listing 8: Compute K

```
function K = computeK(y,c1,a1,Tswpn,c3,a3,c0,c2)


K = c1*(exp(a1*sqrt(Tswpn).*y-1/2*a1^2*Tswpn)-1)-c3*(exp(a3*sqrt(Tswpn).*y
   -1/2*a3^2*Tswpn)-1)-c0+c2;


end
```

Listing 9: Compute Integrand if K>0

```
function ifKpositive = computeIntegrand1(y,rho,c1,a1,Tswpn,c3,a3,c0,a2,c2)
K = computeK(y,c1,a1,Tswpn,c3,a3,c0,c2);
if (K<=0)
   ifKpositive = 0
else
q1 = (log(c2)-log(K)+a2*sqrt(Tswpn)*rho.*y+Tswpn*a2^2*(0.5-rho^2))/(sqrt(
    Tswpn)*a2*sqrt(1-rho^2))
q2 = (log(c2)-log(K)-0.5*a2^2*Tswpn+a2*sqrt(Tswpn)*rho.*y)/(sqrt(Tswpn)*a2
    *sqrt(1-rho^2))
ifKpositive = c2*exp(a2*sqrt(Tswpn)*rho.*y-0.5*a2^2*Tswpn*rho^2).*normcdf(
    q1)-K.*normcdf(q2)
logic = isfinite(ifKpositive)
ifKpositive(logic==0) = 0;
end
end
```

Listing 10: Compute Integrand if K<0

```
function ifKnegative = computeIntegrand2(y,rho,c1,a1,Tswpn,c3,a3,c0,a2,c2)

K = computeK(y,c1,a1,Tswpn,c3,a3,c0,c2);
if (K>=0) %protect against possible positive values
   ifKnegative = 0
else
ifKnegative = c2*exp(a2*sqrt(Tswpn)*rho.*(y-0.5*a2*sqrt(Tswpn)*rho))-K
logic = isfinite(ifKnegative) %avoid problems with NA
ifKnegative(logic==0) = 0;
end
end
```

Listing 11: Compute ATM Swpn prices, Two-factor

```
function atmPrice = computeSwpnPriceATM2Factor(param,a2,b1,a1,a3,rho,
    discount,forward,b2vec,b3vec,Tswpn,k,delta,version)
N = 100
if (version==1) %calibration phase
b3 = param;
tmp2 = b3vec'
tmp2 = [tmp2 b3*ones(1,4)]
b3vec = tmp2'
end
lenDiscount = length(discount)
relevantDiscounts = zeros(lenDiscount,1)
relevantIndexes = 4:4:lenDiscount
numberIndexes = length(relevantIndexes)
for i = 1:numberIndexes
    relevantDiscounts(i*4) = discount(relevantIndexes(i))
end
c0vec = forward.*delta-k*relevantDiscounts;
c2 = sum(b2vec.*delta);
c3 = sum(b3vec.*delta);
c0 = sum(c0vec);
c1 = k*b1
myfun = @(x) (computeK(x,c1,a1,Tswpn,c3,a3,c0,c2))
%root finding and integral computation
root1 = fzero(myfun,0);
myPosFun = @(l) computeIntegrand1(l,rho,c1,a1,Tswpn,c3,a3,c0,a2,c2);
myNegFun = @(z) computeIntegrand2(z,rho,c1,a1,Tswpn,c3,a3,c0,a2,c2);
root2 = fzero(myfun,4);
if (isnan(root1))
    root1 = root2
end
```

```
if (abs(root1-root2)>0.000001)
    atmPrice = (quadgk(@(x) myNegFun(x).*normpdf(x),-inf,root1)+quadgk(@(x
       ) myPosFun(x).*normpdf(x),root1,root2) + quadgk(@(x) myNegFun(x).*
       normpdf(x),root2,inf))*N
else
    atmPrice = (quadgk(@(x) myPosFun(x).*normpdf(x),-inf,root1)+quadgk(@(x
       ) myNegFun(x).*normpdf(x),root1,inf))*N
end
end
```

D.3 optimization

Listing 12: Optimization summary

```
    options = optimoptions('fmincon');
    options = optimoptions(options,'Algorithm', 'interior-point');
    options = optimoptions(options,'Display', 'iter')
    options = optimoptions(options,'MaxIter', 100000);
    options = optimoptions(options,'PlotFcns', { @optimplotfval });
    %One-factor positive rates, 9x1 smile
    x0 = [0.06 0.1];
    [output,resnorm,residual,exitflag,out] = lsqnonlin( @(param) prices -
       computeSwpnPriceOneFactor(param,discount,forward,strikeVec,Tswpn,
       delta) ,x0,[0.05 0.1],[0.07 0.2],options)
    %One-factor positive rates, co-terminal ATM
    x0 = 0.1
    [output2,resnorm2,residual2,exitflag2,out2] = lsqnonlin( @(x)
       atmBlackPrice-computeSwpnPriceATMOneFactor(x,a2,discount,forward,k,
       Tswpn,b2vec,delta,1) ,x0,[0.05],[0.6],options)
    %Two-factor positive rates, 9x1 smile
    x0 = [0.1 0.25 0.01 0.6 1 1];
    lb = [0.05 0.2 0.005 0.5 0.5 0.5];
    ub = [0.3 0.6 0.025 0.9 2 2];
```

```
[output,resnorm,residual,exitflag,out] = lsqnonlin( @(param) prices -
    computeSwpnPrice2Factor(param,discount,forward,strikeVec,Tswpn,
    delta) ,x0,lb,ub,options)
%Two-factor positive rates, co-terminal ATM
[output2,resnorm2,residual2,exitflag2,out2] = lsqnonlin( @(param)
    atmBlackPrice-computeSwpnPriceATM2Factor(param,a2,b1,a1,a3,rho,
    discount,forward,b2vec,b3vec,Tswpn,k,delta,1),0.04,0.0001,0.1,
    options)
%One-factor negative rates, 9x1 smile
x0 = [0.1 0.05];
[output,resnorm,residual,exitflag,out] = lsqnonlin( @(param) prices -
    computeSwpnPriceOneFactor(param,discount,forward,strikeVec,Tswpn,
    delta) ,x0,[0.05 0.01],[0.2 0.1],options)
%One-factor negative rates, co-terminal ATM
x0 = 0.05
[output2,resnorm2,residual2,exitflag2,out2] = lsqnonlin( @(x)
    atmNormalPrice-computeSwpnPriceATMOneFactor(x,a2,discount,forward,k
    ,Tswpn,b2vec,delta,1) ,x0,[0.005],[0.2],options)
%Two-factor negative rates, 9x1 smile
x0 = [0.2 0.25 0.01 0.6 1 1];
lb = [0.05 0.2 0.005 0.3 0.5 0.5];
ub = [0.3 0.6 0.025 0.9 2 2];
[output,resnorm,residual,exitflag,out] = lsqnonlin( @(param) prices -
    computeSwpnPrice2Factor(param,discount,forward,strikeVec,Tswpn,
    delta) ,x0,lb,ub,options)
%Two-factor negative rates, co-terminal ATM
[output2,resnorm2,residual2,exitflag2,out2] = lsqnonlin( @(param)
    atmNormalPrice-computeSwpnPriceATM2Factor(param,a2,b1,a1,a3,rho,
    discount,forward,b2vec,b3vec,Tswpn,k,delta,1),0.05,-0.05,0.1,
    options)
```