Problem 0.1. *For a set S of positive integers and a positive integer n, consider the game of (n, S)-nim, which is as follows. A pile starts with n watermelons. Two players, Deric and Erek, alternate turns eating watermelons from the pile, with Deric going first. On any turn, the number of watermelons eaten must be an element of S. The last player to move wins. Let $f(S)$ denote the set of positive integers n for which Deric has a winning strategy in (n, S)-nim.*

In this article I will show how I solved this problem, not just the answer itself. I will first give an example, then discuss how to prove.

And I should point out $T = \emptyset$ is trivial, so now we assume $T \neq \emptyset$.

1 Basic observations

It's easily to define recursively what means that Deric has a winning strategy in (n, S)-nim.

Definition 1.1. *We define P-position and N-position as follows. P-position means n that Deric has a winning startegy, and conversely N-position means not. Then obviously it is equivalent to say:*

1. *Every integer $n \in S$ is in P-position.*
2. *$n \in \mathbb{Z}_{>0}$ and $n \notin S$. n is in N-position if and only if for all m in S, either $n - m < 0$, or $n - m$ in P-position. If not then n is in P-position.*

$f(S)$ gathers all positive integers in P-position.

Since this definition has some degradation property, so it's easily to see that it's well-defined. Then it's obvious that

Lemma 1.2. *For all T we have*

$$T \subset f(T).$$

From this definition we can not directly see what means for $n \in f(S)$ (in fact it is truly the case). So let's see an example.

2 Example

When I do this problem, I first try the case that $T = \{5, 7, 8, 10, 11\}$. Let's first calculate what $f(T)$ is.

For $n = 1, 2, 3, 4$, it's easily to see that n is in N-position. Because they are all less than the smallest number in T, which is 5. So we can deduce that

Lemma 2.1. *For any nonempty set T, it has a smallest number m. If $m > 1$, then $1, 2, \cdots, m-1$ is in N-position. And therefore we can easily see that for $f(T)$ this number m does not change.*

Then for 5, it is in P-position. For $6, 7, 8, 9, 10, 11, 12, 13, 14, 15$ it is also the case. For example, $15 - 11 = 4$, but $11 \in T$, and 4 is in N-position.

But for 16 it's not the case again. We can check that 16 is in N-position, so for $17, 18, 19, 20$.

Another but for 21 then. Now although we can not deduce to the case $1, 2, 3, 4$, we can deduce 21 to 16, because $21 - 5 = 16$.But we have show 16 is in N-position, so 21 is in P-position. Similarly for 22 to 31, actually they are all in P-position.

Then for $32, 33, 34, 35, 36$ it's not the case again. We can not deduce to any smaller case for P-position now, so they are all in N-position. For 37 to 47 we again can deduce to the case for $32, 33, 34, 35, 36$. You see we seem to get some kind of periodic change from N-position to P-position. You may think it is a natural phenomenon. However,since we pick T to be such a finite set,so we have no more "start number". For example, I can add 32 to T, then 32 is not a number in N-position anymore. So the periodic phenomenon in some sense depends on my choice of T, it's not the general case.The properties for sets in this time do not behave well.

Observation 2.2. *For $T = \{5, 7, 8, 10, 11\}$, all elements in N-position or equivalent to say not in $f(T)$ is $\{1, 2, 3, 4, 16, 17, 18, 19, 20, 32, 33, 34, 35, 36, \cdots, 16j, 16j + 1, 16j + 2, 16j + 3, 16j + 4, \cdots\}$.*

And what for $f(f(T))$? We want this chain finally stops, so we want to know how far we can go.

Now we have more elements in the start. Now we only need to check what elements is kicked off from N-position.

Now since $16 - 15 = 1, 17 - 15 = 2, 18 - 15 = 3, 19 - 15 = 4$, and 15 is now in $f(T)$, $1, 2, 3, 4$ are still in N-position. So $16, 17, 18, 19$ is in P-position now. How about 20? Well, we could do anymore for it, it's still in N-position. Similarly for 32 we have $32 - 31 = 1$ and so on.

Observation 2.3. *For $T = \{5, 7, 8, 10, 11\}$, all elements not in $f(f(T))$ is $\{1, 2, 3, 4, 20, 36, 52, \cdots, 16j+ 4, \cdots\}$.*

For the third time, we finally can eliminate 20. And we can see that

Observation 2.4. *For $T = \{5, 7, 8, 10, 11\}, f(f(f(T))) = \mathbb{Z}_{>0}/\{1, 2, 3, 4\}$.*

Then it becomes constant.

Now let's start our proof.

3 Case 1 — $m > 1$

We can see that the number m is important. We will see why soon.

In definition, the way we can deduce n is in $f(T)$ is that it is a sum of a number not in $f(T)$ and a number in T. If $m > 1$, we can choose the number not in $f(T)$ to be $1, 2, \cdots, m - 1$. So we deduce that

Lemma 3.1 (A stronger version of the Lemma 1.2)**.**

$$f(T) \supset (T \cup (T + 1) \cup \cdots \cup (T + (m - 1))).$$

Here m is the number defined in Lemma 2.1.

Now let us assume $m > 1$.

We may try to cover all numbers in $\mathbb{Z}_{>0}/\{1, 2, \cdots, m - 1\}$ by shifting T a little and get a bigger set just using the previous lemma. However the set may behave bad. So we could not directly do that. However we can do it with a sentinel value. But since $\{1, 2, \cdots, m - 1\}$ will always be in N-position, we can shift them again and again, and we could get a weak result:

Proposition 3.2.

$$f^{\infty}(T) = \cup_{t=0}^{\infty} f^t(T) = \mathbb{Z}_{>0}/\{1, 2, \cdots, m - 1\}.$$

Proof. We can see from the lemma that $f^k(T) \supset \{m, m+1, \cdots, (k+1)m-1\}$ for $k > 1$. $\square$

In the previous example, we eliminate $16, 17, 18, 19$ from 15 such a number in P-position less than them but close to them. We get some inspirations from that. Now we are ready to prove our main theorem.

Theorem 3.3 (Main Theorem 1)**.** *For $m > 1$, $f(f(f(T))) = f^{\infty}(T)$.*

Proof. We need a lemma first.

Lemma 3.4. *If n is not in $f(T)$, then $n+m$ is. Therefore any continual sequence of numbers not in $f(T)$ has length less or equal to m. This is also true for $m = 1$.*

The lemma is obvious. Now let's recall what happened in $f(f(T))$. Inspired by the example before, let's first pick any continual sequence of numbers not in $f(T)$. By the lemma we can extend it to the longest such that the next term of the last term and the previous term of the first term is in $f(T)$. We choose the previous term of the first term (it plays a role like 15 do), denote it by r.

By Lemma 3.1, $r+1, r+2, \cdots r+m-1$ must all be in $f(f(T))$, since the length of this sequence should be less or equal to m, then the only case that $f(f(T))$ does not include the whole sequence is that the length of the sequence is r, in which case that $r+m$ is the only probable survivor (just like 20). But in the third irratation it must be eliminated. It completes the proof. $\square$

The 3 irratation times can not be improved, since we have the example.

4 Case 2 — $m = 1$, Part I: Introduction to the number s

Now we need to deal with the case $m = 1$, however this time the result may not be the whole $\mathbb{Z}_{>0}$. We have the following example.

Example 4.1. *Take $T = \{1, 3, 5, 7, 9, \cdots, 2j-1, \cdots\}$, then $T = f(T)$.*

So it's a little different from the case $m = 1$. If we use Lemma 3.4 we can only get we can't have two numbers both in N-position but next to each other, which has no help to our proof.

So let's try to control this set using a different number, the first positive integer which is not in T, let's call it s. If s does not exist, or say, $T = \mathbb{Z}_{>0}$, it is trivial. Then we can repeat the process of choosing an obvious number in $f(T)$.

Lemma 4.2. *When $m = 1$, the definition of s is as above. And for $f(T)$ this number is also s. We can see $f(T) \supset (T \cup (T+s))$ from that.*

The lemma is obvious.

Now since $\{1, 2, \cdots s-1\}$ is in T, so we can deduce a weak result as before:

Proposition 4.3 (Analogue for Proposition 3.2)**.**

$$f^{\infty}(T) \supset \mathbb{Z}_{>0, \not\equiv 0 \pmod{s}}.$$

The proof is just like that for Proposition 3.2.

Then we want to ask what about the case $n \equiv 0 \pmod{s}$?

5 Case 2 — $m = 1$, Part II: Introduction to the number g and the proof of the main theorem in this case

We use another number g to describe this case.

First, if T doesn't contain any number n such that $n \equiv 0 \pmod{s}$, then we can easily show that

Proposition 5.1. *If T doesn't contain any number n such that $n \equiv 0 \pmod{s}$, then*

$$f(T) = f^{\infty}(T) = \mathbb{Z}_{>0, \not\equiv 0 \pmod{s}}.$$

Proof. We just need to use Lemma 4.2 once, then we complete the proof. □

If T contains a number n such that s divides n, we can pick the smallest n for that. Let's denote this number gs, where $g \geq 2$ is a positiv e integer by definition, and we can easily check that

Proposition 5.2 (Analogue for Proposition 5.1 with the number g)**.** *If g, s are defined as above, then $\{s, 2s, \cdots, (g-1)s\}$ will not be contained in $f(T)$, and all number less than gs and can't be divided by s will be in $f(T)$. Therefore for $f(T)$ the value for g does not change.*

Now we are going to prove our main theorem as before.

Theorem 5.3 (Main Theorem 2)**.** *In the case $m = 1$ with g, s,*

$$f(f(f(T))) = f^{\infty}(T) = \mathbb{Z}_{>0}/\{s, 2s, \cdots, (g-1)s\}.$$

Proof. We also need a lemma first.

Lemma 5.4.

$$f(T) \supset (T \cup (T + s) \cup (T + 2s) \cup \cdots \cup (T + (g-1)s)).$$

To prove it we just need to use the previous proposition.

Now we look at the subsequence $\mathbb{Z}_{>0, \equiv d \pmod{s}}$.

By the lemma, similarly we have that the length of a continual subsequence of the intersection of this subsequence and $\mathbb{Z}_{>0}/f(T)$ must be less or equal to g. We extend this subsequence to the longest. Similarly first time we can kill at most $g-1$ in $f(f(T))$. And in $f(f(f(T)))$ we can kill the last term. So we only need 3 times as before. It completes the proof. □