Reciprocity for Generalized Dedekind Sums from a Naive Point of View

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The reciprocity of Dedekind sums is very famous and there are many beautiful generalizations of the Dedekind sums, with various reciprocity formulae. Dedekind earliest introduced these sums to express functional equation of Dedekind eta-function, and used the functional equation to prove the reciprocity. Now the reciprocity has been given various formulations in a more modern language, with some striking results from automorphic forms, algebraic geometry and topology.

Our purpose in this note is to go another way, without any theoretical knowledge, to introduce the generalized Dedekind sum and its reciprocity law as simply as possible. We show that the reciprocity can be seen as a natural consequence of certain cancellations. Using this view, we can give a much simpler proof for the reciprocity. At the end, we will talk about the relations between the classical reciprocity and modular forms, and the explanation of an extra 1/4 from the two sides.

1 Introduction

Definition 1.1 (Generalized Dedekind sums)**.**

$$S_{m,n}(a,b;c) = \sum_{0\le h<c} B_m\left(\frac{ah}{c}\right) B_n\left(\frac{bh}{c}\right). \tag{1}$$

B_n is the n'th periodic Bernoulli polynomial:

$$\sum_{r=0}^{\infty} b_n(x)\frac{y^n}{n!} = \frac{ye^{xy}}{e^y-1}, \quad B_n(x) = b_n(x) \text{ when } 0\le x<1, \quad B_n(x+1) = B_n(x) \text{ when } n\neq 1. \tag{2}$$

When $n=1$:

$$B_1(x) = b_1(x) \text{ when } 0<x<1, B_1(0)=0, \quad B_1(x+1) = B_1(x). \tag{3}$$

Because $b_n(0) = b_n(1)$ for all $n\neq 1$ but $b_1(0) = -1/2 \neq 1/2 = b_1(1)$.

The classical reciprocity for Dedekind sums is that taking a,b,c to be pairwise coprime positive integers we have this equality (first proved in [1]):

$$S_{1,1}(a,b;c) + S_{1,1}(b,c;a) + S_{1,1}(c,a;b) = \frac{a^2+b^2+c^2}{12abc} - \frac{1}{4}. \tag{4}$$

To motivate the reciprocity for generalized Dedekind sums, we first give a heuristic but flawed proof for the classical reciprocity. We will fix its gap at the end.

2 Motivation: "proof" for classical reciprocity

Periodic Bernoulli polynomial has a well-known Fourier expansion.

$$B_n(x) = -\frac{n!}{(2\pi i)^n}\sum_{k\neq 0}\frac{e^{2\pi ikx}}{k^n}. \tag{5}$$

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Assume that a, b, c are pairwise coprime positive integers.

$$S_{1,1}(a,b;c) = \sum_{0\le h<c} B_1\left(\frac{ah}{c}\right) B_1\left(\frac{bh}{c}\right) \tag{6}$$

$$= -\frac{1}{4\pi^2} \sum_{0\le h<c} \sum_{kl\ne 0} \frac{e^{2\pi i \frac{h(ak+bl)}{c}}}{kl} \tag{7}$$

$$= -\frac{c}{4\pi^2} \sum_{kl\ne 0, c|(ak+bl)} \frac{1}{kl} \tag{8}$$

$$= -\frac{c}{4\pi^2} \left(\sum_{jkl\ne 0, ak+bl+cj=0} \frac{1}{kl} + \sum_{kl\ne 0, ak+bl=0} \frac{1}{kl} \right) \tag{9}$$

$$= -\frac{c}{4\pi^2} \left(\sum_{jkl\ne 0, ak+bl+cj=0} \frac{1}{kl} - \frac{1}{ab} \sum_{s\ne 0} \frac{1}{s^2} \right) \tag{10}$$

$$= \frac{c}{12ab} - \frac{1}{4\pi^2} \sum_{jkl\ne 0, ak+bl+cj=0} \frac{cj}{jkl}. \tag{11}$$

Here in (10) we use the fact that a, b are coprime, deducing that

$$ak + bl = 0 \Leftrightarrow k = bs, l = -as. \quad (s \in \mathbf{Z}) \tag{12}$$

In (11) we use the fact that

$$\sum_{s\ne 0} \frac{1}{s^2} = \frac{\pi^2}{3}. \tag{13}$$

So

$$S_{1,1}(a,b;c) + S_{1,1}(b,c;a) + S_{1,1}(c,a;b) = \frac{c}{12ab} + \frac{a}{12bc} + \frac{b}{12ca} - \frac{1}{4\pi^2} \sum_{jkl\ne 0, ak+bl+cj=0} \frac{ak+bl+cj}{jkl} \tag{14}$$

$$= \frac{a^2+b^2+c^2}{12abc} \quad ? \tag{15}$$

The result has lost a 1/4 because of our arbitrary exchange of terms in a not absolutely convergent series. Even though the proof is illegal, we can get the idea. In the first step we write the original Dedekind sum as a sum over a lattice on a plane in 3-dim space ($ak + bl + cj = 0$). In the second step, we separate a part of the sum over a principal line ($ak + bl = j = 0$). In the third step, the most important step, we combine the remaining parts of three Dedekind sums to get a cancellation. For generalized Dedekind sum we can do the same thing.

3 Generalized Dedekind sum

Similarly we have

$$S_{m,n}(a,b;c) = \sum_{0\le h<c} B_m\left(\frac{ah}{c}\right) B_n\left(\frac{bh}{c}\right) \tag{16}$$

$$= \frac{m!n!}{(2\pi i)^{m+n}} \sum_{0\le h<c} \sum_{kl\ne 0} \frac{e^{2\pi i \frac{h(ak+bl)}{c}}}{k^m l^n} \tag{17}$$

$$= \frac{m!n!}{(2\pi i)^{m+n}} \sum_{kl\ne 0, c|(ak+bl)} \frac{c}{k^m l^n} \tag{18}$$

$$= \frac{m!n!}{(2\pi i)^{m+n}} \left(\sum_{jkl\ne 0, ak+bl+cj=0} \frac{c}{k^m l^n} + \sum_{kl\ne 0, ak+bl=0} \frac{c}{k^m l^n} \right) \tag{19}$$

$$= \frac{m!n!}{(2\pi i)^{m+n}} \left(\sum_{jkl\ne 0, ak+bl+cj=0} \frac{c}{k^m l^n} + \frac{c}{b^m(-a)^n} \sum_{s\ne 0} \frac{1}{s^{m+n}} \right). \tag{20}$$

If $m+n$ is even, then

$$S_{m,n}(a,b;c)=\frac{m!n!}{(2\pi i)^{m+n}}\left(\sum_{jkl\neq 0,ak+bl+cj=0}\frac{c}{k^m l^n}+\frac{(-1)^n c}{b^m a^n}\cdot 2\zeta\left(m+n\right)\right). \tag{21}$$

And if n is even, it is well-known that

$$\zeta(n)=(-1)^{\frac{n}{2}+1}\frac{(2\pi)^n}{2\cdot n!}B_n. \tag{22}$$

Here B_n is the Bernoulli number:

$$B_n=B_n(0). \tag{23}$$

Finally we get this property.

Proposition 3.1. When a,b,c are positive integers and a,b are coprime, $(mn\neq 0,1)$

$$\frac{S_{m,n}(a,b;c)}{m!n!}=\frac{1}{(2\pi i)^{m+n}}\sum_{jkl\neq 0,ak+bl+cj=0}\frac{c}{k^m l^n}-\frac{(-1)^n\cdot B_{m+n}}{(m+n)!}\cdot\frac{c}{b^m a^n}. \tag{24}$$

For odd $m+n$ this also holds trivially, since both sides are 0.
If $n=0$, a,c coprime and positive, we can directly get:

$$S_{m,0}(a,b;c)=\sum_{0\leq h<c}B_m\left(\frac{ah}{c}\right)B_0\left(\frac{bh}{c}\right)=\sum_{0\leq h<c}B_m\left(\frac{h}{c}\right)=\frac{1}{c^{m-1}}B_m. \tag{25}$$

Here we use the multiplication formula for (periodic) Bernoulli polynomials:

$$B_n(mx)=m^{n-1}\sum_{0\leq k<m}B_n\left(x+\frac{k}{m}\right). \tag{26}$$

So it is trivial to talk about reciprocity about $S_{m,0}$. In the next section we will compute several non-trivial examples to motivate the correct form of the reciprocity.

4 Examples of reciprocities

The reciprocities we want should have the following form.

$$\text{(sum of generalized Dedekind sums)} = \text{(rational function)} + \sum_{jkl\neq 0,ak+bl+cj=0}\frac{(ak+bl+cj)\cdot(*)}{(jkl)^N} \tag{27}$$

$$= \text{(rational function)}. \tag{28}$$

$(*)$ and N in (27) determines the form of reciprocity. $(*)$ should be a homogenous polynomial in ak,bl,cj. By using the method of undetermined coeffients we can get various kinds of reciprocities. We first give a smallest non-trivial example other than the classical one (4).

Example 4.1. We want to set our reciprocity for $m+n=4$. 2 possible non-trivial choices are $S_{2,2},S_{3,1}$. By Proposition 3.1 we have:

$$\frac{S_{2,2}(a,b;c)}{2!2!}=\frac{1}{(2\pi i)^4}\sum_{jkl\neq 0,ak+bl+cj=0}\frac{c}{k^2l^2}-\frac{B_4}{4!}\cdot\frac{c}{b^2a^2} \tag{29}$$

$$=\frac{1}{16\pi^4}\cdot\frac{1}{abc^2}\sum_{jkl\neq 0,ak+bl+cj=0}\frac{(ak)(bl)(cj)^3}{j^3k^3l^3}+\frac{1}{720}\cdot\frac{c}{a^2b^2}, \tag{30}$$

$$\frac{S_{3,1}(a,b;c)}{3!1!}=\frac{1}{(2\pi i)^4}\sum_{jkl\neq 0,ak+bl+cj=0}\frac{c}{k^3l^1}+\frac{B_4}{4!}\cdot\frac{c}{b^3a^1} \tag{31}$$

$$=\frac{1}{16\pi^4}\cdot\frac{1}{b^2c^2}\sum_{jkl\neq 0,ak+bl+cj=0}\frac{(bl)^2(cj)^3}{j^3k^3l^3}-\frac{1}{720}\cdot\frac{c}{ab^3}. \tag{32}$$

Here we use the constant

$$B_4 = -\frac{1}{30}. \tag{33}$$

Doing some calculations:

$$(A+B+C)(A^2BC + AB^2C + ABC^2) = T + 2S, \tag{34}$$
$$(A+B+C)(A^2B^2 + B^2C^2 + C^2A^2) = W + S, \tag{35}$$
$$S = A^2B^2C + AB^2C^2 + A^2BC^2, \tag{36}$$
$$T = A^3BC + AB^3C + ABC^3, \tag{37}$$
$$W = A^3B^2 + A^2B^3 + B^3C^2 + B^2C^3 + C^3A^2 + C^2A^3. \tag{38}$$

So we can get the key identity:

$$(A+B+C)M = T - 2W, \tag{39}$$
$$M(A,B,C) = A^2BC + AB^2C + ABC^2 - 2A^2B^2 - 2B^2C^2 - 2A^2C^2. \tag{40}$$

Substituting $A = ak, B = bl, C = cj$, we can get the reciprocity law.

$$\frac{S_{2,2}(a,b;c)abc^2 + S_{2,2}(b,c;a)bca^2 + S_{2,2}(c,a;b)cab^2}{4} \tag{41}$$
$$- \frac{S_{3,1}(a,b;c)b^2c^2 + S_{3,1}(b,a;c)a^2c^2 + S_{3,1}(b,c;a)c^2a^2 + S_{3,1}(c,b;a)b^2a^2 + S_{3,1}(c,a;b)a^2b^2 + S_{3,1}(a,c;b)c^2b^2}{3} \tag{42}$$
$$= \frac{1}{16\pi^4} \sum_{jkl\neq 0, ak+bl+cj=0} \frac{(ak+bl+cj)M(ak,bl,cj)}{j^3k^3l^3} \tag{43}$$
$$+ \frac{1}{720}\left(\left(\frac{c^3}{ab} + \frac{a^3}{bc} + \frac{b^3}{ca}\right) - 2\left(-\frac{c^3}{ab} - \frac{c^3}{ab} - \frac{a^3}{bc} - \frac{a^3}{bc} - \frac{b^3}{ca} - \frac{b^3}{ca}\right)\right) \tag{44}$$
$$= \frac{1}{144}\left(\frac{c^3}{ab} + \frac{a^3}{bc} + \frac{b^3}{ca}\right). \tag{45}$$

We see that the reciprocity for generalized Dedekind sums mix various pairs of indices (m,n). Under this naive perspective, the necessity of combining different indices arises from the need to make the sum over lattice points on the plane $(ak + bl + cj = 0)$ – with three lines removed – vanish. Unless the summation function is odd and can cancel out symmetrically, it is preferable to have the value at each individual point vanish. Therefore, we must appropriately combine different types of Dedekind sums so that their summation yields multiples of $(ak + bl + cj)$, thereby ensuring vanishing at every point.

It is convenient to do some rescaling.

Definition 4.2.

$$\widetilde{S}_{m,n}(a,b;c) = \frac{S_{m,n}(a,b;c)}{m!n!} \cdot a^{-m}b^{-n}c^{-1}. \tag{46}$$

By Proposition 3.1,

$$\widetilde{S}_{m,n}(a,b;c) = \frac{1}{(2\pi i)^N} \cdot \sum_{jkl\neq 0, ak+bl+cj=0} \frac{(ak)^n(bl)^{N-n}(cj)^N}{(ak\cdot bl\cdot cj)^N} - \frac{(-1)^n B_N}{N!} \cdot \frac{1}{a^N b^N}. \tag{47}$$

Here we see $N = m + n$.

We can rewrite (41)-(45) as

$$P = \widetilde{S}_{2,2}(a,b;c) + \widetilde{S}_{2,2}(b,c;a) + \widetilde{S}_{2,2}(c,a;b), \tag{48}$$
$$Q = \widetilde{S}_{3,1}(a,b;c) + \widetilde{S}_{3,1}(b,a;c) + \widetilde{S}_{3,1}(b,c;a) + \widetilde{S}_{3,1}(c,b;a) + \widetilde{S}_{3,1}(c,a;b) + \widetilde{S}_{3,1}(a,c;b), \tag{49}$$
$$P - 2Q = \frac{1}{144}\left(\frac{1}{a^4b^4} + \frac{1}{b^4c^4} + \frac{1}{a^4c^4}\right). \tag{50}$$

The reciprocity in (41)-(45) is symmetric as the classical reciprocity (4). And by our method, it is the only symmetric reciprocity for $N = m + n = 4$. However, we do not require $(*)$ in (27) to be symmetric. So it is possible to get non-symmetric reciprocities.

Example 4.3. We know

$$A^3 + B^3 = (A+B)(A^2 - AB + B^2) = (A+B+C)(A^2 - AB + B^2) - C(A^2 - AB + B^2). \tag{51}$$

So

$$A^3C^2 + B^3C^2 + A^2C^3 + B^2C^3 - ABC^3 = (A+B+C)(A^2C^2 - ABC^2 + B^2C^2). \tag{52}$$

Using this identity we can immediately get

$$\widetilde{S}_{3,1}(b,c;a) + \widetilde{S}_{3,1}(a,c;b) + \widetilde{S}_{3,1}(b,a;c) + \widetilde{S}_{3,1}(a,b;c) - \widetilde{S}_{2,2}(a,b;c) \tag{53}$$

$$= -\frac{1}{720}\left(\frac{1}{b^4c^4} + \frac{1}{a^4c^4} + \frac{1}{a^4b^4} + \frac{1}{a^4b^4} + \frac{1}{a^4b^4}\right) \tag{54}$$

$$= -\frac{1}{720}\left(\frac{1}{b^4c^4} + \frac{1}{a^4c^4} + \frac{3}{a^4b^4}\right). \tag{55}$$

In fact (53)-(55) can imply (48)-(50). We will do the generalized case in the next section.

5 General case: linear relations for generalized Dedekind sums

Now we need to classify all the possible linear relations for Dedekind sums. We still use the notation $A = ak, B = bl, C = cj$, and by our previous argument we can substitute $C = -A - B$. We will assume that N is even. All the possible generalized Dedekind sums are:

$$\widetilde{S}_{n,N-n}(a,b;c) = \frac{1}{(2\pi i)^N}\sum_{\cdots}\frac{A^nB^{N-n}(A+B)^N}{(ABC)^N} - \frac{(-1)^nB_N}{N!}\cdot\frac{1}{a^Nb^N}. \tag{56}$$

$$(-1)^n\cdot\widetilde{S}_{n,N-n}(c,b;a) = \frac{1}{(2\pi i)^N}\sum_{\cdots}\frac{(A+B)^nB^{N-n}A^N}{(ABC)^N} - \frac{B_N}{N!}\cdot\frac{1}{b^Nc^N}. \tag{57}$$

$$(-1)^n\cdot\widetilde{S}_{n,N-n}(c,a;b) = \frac{1}{(2\pi i)^N}\sum_{\cdots}\frac{(A+B)^nA^{N-n}B^N}{(ABC)^N} - \frac{B_N}{N!}\cdot\frac{1}{c^Na^N}. \tag{58}$$

We want to express $\widetilde{S}_{r,N-r}(a,b;c)$ as a linear combination of $\widetilde{S}_{n,N-n}(c,b;a), \widetilde{S}_{n,N-n}(c,a;b)$ for each $1 \le r \le N-1$. To do that, we need the binomial theorem.

$$(X+Y)^n = \sum_{0\le k\le n}\binom{n}{k}X^kY^{n-k}, \tag{59}$$

where the binomial coeffients are defined as:

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}. \tag{60}$$

Here n can be negative integers. We define that for integers $k < 0$ the binomial coeffients are 0.

Therefore,

$$(A+B)^NA^rB^{N-r} = \sum_{0\le k\le N}\binom{N}{k}A^{N+r-k}B^{N+k-r}, \tag{61}$$

$$\sum_{1\le k\le q}A^N(A+B)^kB^{N-k}\cdot(-1)^{q-k}\binom{q}{k} = A^{N+q}B^{N-q} - (-1)^q\cdot A^NB^N. \tag{62}$$

Here k is different from k in $A = ak$. We use $\binom{n}{0} = 1$ for all $n \ge 0$ in (62).

$$(A+B)^N A^r B^{N-r} = A^N B^N \cdot \binom{N}{r} + \sum_{1\le q\le r} A^{N+q} B^{N-q} \cdot \binom{N}{r-q} + \sum_{1\le q\le N-r} B^{N+q} A^{N-q} \cdot \binom{N}{r+q} \tag{63}$$

$$= A^N B^N \sum_{-r\le q\le N-r} (-1)^q \binom{N}{r+q} \tag{64}$$

$$+ \sum_{1\le q\le r} \left(A^{N+q} B^{N-q} - (-1)^q \cdot A^N B^N\right) \cdot \binom{N}{r-q} + \sum_{1\le q\le N-r} \left(B^{N+q} A^{N-q} - (-1)^q \cdot A^N B^N\right) \cdot \binom{N}{r+q} \tag{65}$$

$$= \sum_{1\le q\le r} \sum_{1\le k\le q} \binom{N}{r-q} \binom{q}{k} (-1)^{q-k} A^N (A+B)^k B^{N-k} \tag{66}$$

$$+ \sum_{1\le q\le N-r} \sum_{1\le k\le q} \binom{N}{r+q} \binom{q}{k} (-1)^{q-k} B^N (A+B)^k A^{N-k}. \tag{67}$$

Combine (63)-(67) with (56)-(58), we get the following.

$$\widetilde{S}_{r,N-r}(a,b;c) + \frac{(-1)^r B_N}{N!} \cdot \frac{1}{a^N b^N} \tag{68}$$

$$= \sum_{1\le q\le r} \sum_{1\le k\le q} \binom{N}{r-q} \binom{q}{k} (-1)^{q-k} \left((-1)^k \cdot \widetilde{S}_{k,N-k}(c,b;a) - \frac{B_N}{N!} \cdot \frac{1}{b^N c^N} \right) \tag{69}$$

$$+ \sum_{1\le q\le N-r} \sum_{1\le k\le q} \binom{N}{r+q} \binom{q}{k} (-1)^{q-k} \left((-1)^k \cdot \widetilde{S}_{k,N-k}(c,a;b) - \frac{B_N}{N!} \cdot \frac{1}{a^N c^N} \right) \tag{70}$$

$$= \sum_{1\le q\le r} \sum_{1\le k\le q} \binom{N}{r-q} \binom{q}{k} (-1)^{q} \cdot \widetilde{S}_{k,N-k}(c,b;a) + \sum_{1\le q\le N-r} \sum_{1\le k\le q} \binom{N}{r+q} \binom{q}{k} (-1)^{q} \cdot \widetilde{S}_{k,N-k}(c,a;b) \tag{71}$$

$$-\frac{B_N}{N! b^N c^N} \sum_{1\le q\le r} \sum_{1\le k\le q} \binom{N}{r-q} \binom{q}{k} (-1)^{q-k} - \frac{B_N}{N! a^N c^N} \sum_{1\le q\le N-r} \sum_{1\le k\le q} \binom{N}{r+q} \binom{q}{k} (-1)^{q-k}. \tag{72}$$

As in (62),

$$-\sum_{1\le k\le q} \binom{q}{k} (-1)^{q-k} = (-1)^q. \tag{73}$$

Therefore,

$$-\sum_{1\le q\le r} \sum_{1\le k\le q} \binom{N}{r-q} \binom{q}{k} (-1)^{q-k} = \sum_{1\le q\le r} \binom{N}{r-q} (-1)^q = (-1)^r \sum_{0\le q\le r-1} \binom{N}{q} (-1)^q. \tag{74}$$

We use the upper negation (75) and parallel summation (76) tricks for binomial coeffients. (see [3], Table 174)

$$(-1)^k \binom{r}{k} = \binom{k-r-1}{k}. \tag{75}$$

$$\sum_{k\le n} \binom{r+k}{k} = \binom{r+n+1}{n}. \tag{76}$$

Therefore,

$$-\sum_{1\le q\le r} \sum_{1\le k\le q} \binom{N}{r-q} \binom{q}{k} (-1)^{q-k} = (-1)^r \sum_{q\le r-1} \binom{q-N-1}{q} = (-1)^r \binom{r-N-1}{r-1} = -\binom{N-1}{r-1}. \tag{77}$$

For integer $n \ge 0$ the binomial coeffients have the symmetry:

$$\binom{n}{k} = \binom{n}{n-k}. \tag{78}$$

Combining with (72), similarly we have

$$-\sum_{1\le q\le N-r}\sum_{1\le k\le q}\binom{N}{r+q}\binom{q}{k}(-1)^{q-k}=\sum_{1\le q\le N-r}\binom{N}{r+q}(-1)^q=(-1)^r\sum_{r+1\le q\le N}\binom{N}{q}(-1)^q \tag{79}$$

$$=(-1)^r\sum_{0\le q\le N-r-1}\binom{N}{q}(-1)^q=(-1)^r\sum_{q\le N-r-1}\binom{q-N-1}{q}=(-1)^r\binom{-r-1}{N-r-1}=-\binom{N-1}{N-r-1}. \tag{80}$$

And by (see [3], Table 169) we have this summation formula:

$$\sum_{k\le l}\binom{l-k}{m}\binom{s}{k-n}(-1)^k=(-1)^{l+m}\binom{s-m-1}{l-m-n}. \tag{81}$$

Combining with (78), then we get

$$\sum_{k\le q\le r}\binom{N}{r-q}\binom{q}{k}(-1)^q=(-1)^r\binom{N-k-1}{r-k},\ \sum_{k\le q\le N-r}\binom{N}{r+q}\binom{q}{k}(-1)^q=(-1)^r\binom{N-k-1}{N-r-k}. \tag{82}$$

Therefore we get

Theorem 5.1. For all $1\le r\le N-1$, $N\ge 4$ even,

$$-\widetilde{S}_{r,N-r}(a,b;c)+\sum_{1\le k\le r}\binom{N-k-1}{r-k}(-1)^r\cdot\widetilde{S}_{k,N-k}(c,b;a)+\sum_{1\le k\le N-r}\binom{N-k-1}{N-r-k}(-1)^r\cdot\widetilde{S}_{k,N-k}(c,a;b) \tag{83}$$

$$=\frac{B_N}{N!}\left(\frac{(-1)^r}{a^Nb^N}+\frac{\binom{N-1}{r-1}}{b^Nc^N}+\frac{\binom{N-1}{N-r-1}}{a^Nc^N}\right). \tag{84}$$

This theorem is equivalent to the theorem first formulated and proved by Hall and Wilson in their 1993 paper ([2], Theorem 1). Their original proof uses residue theorem in complex analysis and involves a complicated computation.

From our proof we can see that we have exhaust all the non-trivial independent linear relations for Dedekind sums, since easily we can get that $(2N-2)$ polynomials $(A+B)^nB^{N-n}A^N, (A+B)^nA^{N-n}B^N$ – where $1\le n\le N-1$ – are linearly independent. In other words, Theorem 5.1 gives all the possible reciprocities. Again, this is showed by Hall and Wilson in their 1993 paper ([2], Theorem 3). Their original proof uses the asymptotic formula for $S_{m,n}$ to show this result.

6 Patch I: the modular form view

There is only one problem that we didn't answer – why there is a 1/4 for $S_{1,1}$. There are many explanations of this 1/4 ([1],[2],[4]). According to (6)-(15), the problem is that there are two natural ways of summing the series $S_{1,1}$, and they give different answers.

In my knowledge only one other series has this property – E_2:

$$E_2(\tau)=1-24\sum_{n\ge1}\sigma_1(n)q^n,\quad q=e^{2\pi i\tau},\quad \mathrm{Im}(\tau)>0. \tag{85}$$

Here

$$\sigma_1(n)=\sum_{d|n}d. \tag{86}$$

In modular forms, we have the series formula for E_k:

$$E_k(\tau)=\sum_{(m,n)\ne(0,0)}\frac{1}{(m\tau+n)^k}. \tag{87}$$

For odd k this is 0, so we now assume that k is even. For $k>2$, the series is absolutely convergent, so there are two natural ways of summing this series.

$$\sum_m \sum_{\substack{n,(m\neq 0)\\ n\neq 0,(m=0)}} \frac{1}{(m\tau+n)^k} = \sum_n \sum_{\substack{m,(n\neq 0)\\ m\neq 0,(n=0)}} \frac{1}{(m\tau+n)^k}. \tag{88}$$

By comparing the two sides, we can get:

$$E_k\left(\frac{-1}{\tau}\right) = \tau^k E_k(\tau). \tag{89}$$

And also it is obvious that

$$E_k(\tau+1) = E_k(\tau). \tag{90}$$

(89)-(90) are called the modularity.

But

$$\sum_m \sum_{\substack{n,(m\neq 0)\\ n\neq 0,(m=0)}} \frac{1}{(m\tau+n)^2} \neq \sum_n \sum_{\substack{m,(n\neq 0)\\ m\neq 0,(n=0)}} \frac{1}{(m\tau+n)^2}. \tag{91}$$

So E_2 does not quite have modularity. For $k=2$, the series is not absolutely convergent. The funtional equation for E_2 is actually the following:

$$E_2\left(\frac{-1}{\tau}\right) = \tau^2 E_2(\tau) + \frac{6\tau}{\pi i}. \tag{92}$$

So it insinuates that there is a relation between E_2 and classical Dedekind sums.

$$\text{Glitch of } 1/4 \text{ for } S_{1,1} \Leftrightarrow E_2 \text{ is quasi-modular.} \tag{93}$$

In fact it was the motivation of Dedekind sums when Dedekind introduced them. This correspondence is classical and can be found on any book about Dedekind eta function and Dedekind sums (see [5][6][7]). Now we turn back to our naive point of view. We will show that, we can directly compute this difference in our naive point of view.

7 Patch II: the naive point of view

We just need to figure out the two natural ways to sum:

$$\sum_{kl\neq 0, c|(ak+bl)} \frac{1}{kl}. \tag{94}$$

In (7)-(8), we use the following summation domain:

$$I_1 = \lim_{N\to\infty} \sum_{\substack{|ak|,|bl|\leq N\\ kl\neq 0, ak+bl+cj=0}} \frac{1}{kl}. \tag{95}$$

In (14), since we need to combine three sums, their summation domains should be the same:

$$I_2 = \lim_{N\to\infty} \sum_{\substack{|ak|,|bl|,|cj|\leq N\\ kl\neq 0, ak+bl+cj=0}} \frac{1}{kl}. \tag{96}$$

Figure 1 shows their difference. I_1 sums over the green and the orange parts but I_2 only sums over the green part.

So $I_1 - I_2$ is the limit of the sum over the orange part in Figure 1. As $N\to\infty$, the limit can be computed as a Riemann sum of the following integral:

$$I_1 - I_2 = \frac{1}{c}\int_D \frac{dxdy}{xy}. \tag{97}$$

Here the integral domain is

$$D = \{(x,y)\in \mathbf{R}^2 : |x|,|y|\leq 1, |x+y|\geq 1\}. \tag{98}$$

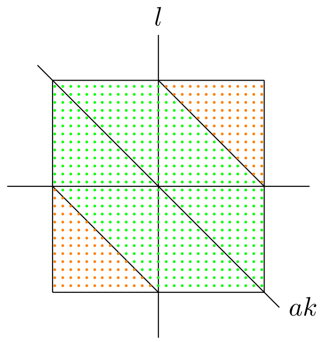

Figure 1: Summation domains of I_1, I_2.

The factor $1/c$ comes in as we only take the sum over integers k, l such that $c|(ak+bl)$.

The integral in (97) can be calculated in many ways. We give one:

$$\int_D \frac{dxdy}{xy} = 2\int_0^1 \int_{1-y}^1 \frac{dx}{x} \cdot \frac{dy}{y} = 2\int_0^1 \frac{-\log(1-y)}{y} dy = 2\,\mathrm{Li}_2(1) = \frac{\pi^2}{3}. \tag{99}$$

Adding this correction to (14) we get:

$$S_{1,1}(a,b;c) + S_{1,1}(b,c;a) + S_{1,1}(c,a;b) \tag{100}$$

$$= \frac{c}{12ab} + \frac{a}{12bc} + \frac{b}{12ca} - \frac{c}{4\pi^2}\cdot\frac{1}{c}\cdot\frac{\pi^2}{3} - \frac{a}{4\pi^2}\cdot\frac{1}{a}\cdot\frac{\pi^2}{3} - \frac{b}{4\pi^2}\cdot\frac{1}{b}\cdot\frac{\pi^2}{3} \tag{101}$$

$$= \frac{a^2+b^2+c^2}{12abc} - \frac{1}{4}. \tag{102}$$

We finally fix the flawed proof. And by the principle of (93), this argument may essentially give another proof of the functional equation of E_2 and the Dedekind eta function.

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