Claim. The following three equations

① $9x^4+4y^4=z^2 \quad (3x^2,2y^2)=1, \quad x,y,z\in\mathbb{Z}^+$

② $x^4+36y^4=z^2 \quad (x^2,6y^2)=1, \quad x,y,z\in\mathbb{Z}^+$

③ $x^4-y^4=3z^2 \quad (x,y)=1. \quad x,y,z\in\mathbb{Z}^+$

has no solution!

pf. Assume the contrast. Choose a solution $(x,y,z)\in\mathbb{Z}^+$ s.t. z minimum.

Case 1. $9x^4+4y^4=z^2 \Rightarrow (3x^2)^2+(2y^2)^2=z^2$

$(3x^2,2y^2)=1 \Rightarrow \exists (u,v)=1.\ u,v\in\mathbb{Z}^+ \quad \begin{cases} 3x^2=u^2-v^2 \\ 2y^2=2uv \\ z=u^2+v^2 \end{cases}$

$y^2=uv \quad (u,v)=1 \Rightarrow u=a^2, v=b^2 \quad a,b\in\mathbb{Z}^+ \Rightarrow 3x^2=a^4-b^4$ ③ $x<z$ $\times$

Case 2. $x^4+36y^4=z^2 \Rightarrow (x^2)^2+(6y^2)^2=z^2$

$(x^2,6y^2)=1 \Rightarrow (u,v)=1 \quad \begin{cases} x^2=u^2-v^2 \\ 6y^2=2uv \\ z=u^2+v^2 \end{cases}$

$uv=3y^2 \quad u^2=x^2+v^2 \quad (u,v)=1 \Rightarrow (x,v)=1 \Rightarrow 3\nmid u$ (otherwise $3|u, 3|x, 3|v$)

$\Rightarrow u=a^2 \quad v=3b^2 \quad a,b\in\mathbb{Z}^+$

$u^2=x^2+v^2 \quad (x,v)=1 \quad 2\nmid x \Rightarrow \begin{cases} x=c^2-d^2 \\ v=2cd \\ u=c^2+d^2 \end{cases} \quad (c,d)=1$

$a^2=c^2+d^2, \quad 3b^2=2cd \Rightarrow 2|b \quad cd=6\left(\frac{b}{2}\right)^2$.

$(c,d)=1 \Rightarrow a^2=c_1^4+36d_1^4$ ② or $a^2=9c_1^4+4d_1^4$ ③ $a<z$ $\times$

Case 3 $x^4-y^4=3z^2 \Rightarrow 3z^2=(x+y)(x-y)(x^2+y^2)$

$(x,y)=1 \Rightarrow 3\nmid x^2+y^2$.

If $2\nmid z$, $x+y, x-y, x^2+y^2$ are pairwise coprime $\Rightarrow x^2+y^2$ is a square

$3 \mid x+y$ or $3 \mid x-y \Rightarrow x^2+y^2 \equiv 2x^2 \equiv 2 \mod 3$ impossible

So $2 \mid z, \Rightarrow 2 \nmid x, 2 \nmid y \Rightarrow 4 \mid z$. $\quad 6\left(\frac{z}{4}\right)^2 = \frac{x+y}{2} \frac{x-y}{2} \frac{x^2+y^2}{2}$

$2 \nmid \frac{x^2+y^2}{2} \quad 3 \nmid \frac{x^2+y^2}{2} \Rightarrow \frac{x^2+y^2}{2} = u^2$. $\quad \left\{\frac{x+y}{2}, \frac{x-y}{2}\right\} = \{v^2, 6w^2\}$ or $\{3v^2, 2w^2\}$

$\frac{x^2+y^2}{2} = \left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2 \Rightarrow u^2 = v^4 + 36w^4$ (2) or $u^2 = 9v^4 + 4w^4$ (1)

Since $\left(\frac{x+y}{2}, \frac{x-y}{2}\right) = 1$, $(v^2, 6w^2) / (3v^2, 2w^2) = 1$ $\quad u < z$ $\quad$ X

Claim is proved

Now if $a_n = \frac{1}{2}\left((2+\sqrt{3})^n + (2-\sqrt{3})^n\right) = u^2 \quad u \in \mathbb{Z}^+, n > 0$

let $v = \frac{1}{2\sqrt{3}}\left((2+\sqrt{3})^n - (2-\sqrt{3})^n\right) \in \mathbb{Z}^+$.

Then $u^4 - 3v^2 = a_n^2 - 3v^2 = 1 \Rightarrow u^4 - 1^4 = 3v^2$ (3) $\quad$ X