这一次我们解决的问题与Erdös猜想有关。

Problem 0.1. 令$\alpha > 1$。记$F_\alpha(n) = \sum_{1\le i\le \tau(n)}((d_{i+1}/d_i) - 1)^\alpha$，其中$\{d_i\}_{i=1}^{\tau(n)}$表示$n$的所有因子的渐升列。

1. 记$\chi(n;u,v) = \begin{cases} 1, & 若 d|n \implies d \notin (u,v], \\ 0, & 若 \exists d|n, u < d \le v. \end{cases}$ 证明，

$$F_\alpha(n) = \alpha \int_1^n \int_1^v \chi(n;u,v)(v-u)^{\alpha-2}(\alpha v - u)u^{-1-\alpha} du dv.$$

2. 令n为整数。若有限序列$\{n_j : 1 \le j \le k\}$使得$n_1 = 1, n_j | n_{j+1} (1 \le j < k), n_k = n$，则称之为$n$的分解。若实数列$\{\epsilon_j : 1 < j \le k\}$使得对任意$j \in [2,k]$及任意$z \in (\sqrt{n_{j-1}}, \sqrt{n_j}]$，区间$(z, (1+\epsilon_j)z]$含有至少一个$n_j$的因子，则说它相对于$\{n_j\}$是“好的”。证明，对$n$的任意分解$\{n_j : 1 \le j \le k\}$以及任意相对于它的好的数列$\{\epsilon_j : 1 < j \le k\}$均有

$$F_\alpha(n) \le \alpha \sum_{2\le j\le k} (1+\epsilon_j)\epsilon_j^{\alpha-1} \ln(n_j/n_{j-1}).$$

1的证明.

$$\begin{aligned} &\int_1^v \chi(n;u,v)(v-u)^{\alpha-2}(\alpha v - u)u^{-1-\alpha} du \\ =&\int_{\max_{d_i\le v} d_i}^v (v-u)^{\alpha-2}(\alpha v-u)u^{-1-\alpha} du \\ \overset{d=\max_{d_i\le v} d_i}{=}& -u^{-\alpha}(v-u)^{\alpha-1}|_d^v \\ =& d^{-\alpha}(v-d)^{\alpha-1}. \end{aligned}$$

注意到这里d是关于v的分段函数。

$$\begin{aligned} &\int_1^n d^{-\alpha}(v-d)^{\alpha-1} dv \\ =&\sum_{1\le i<\tau(n)} \int_{d_i}^{d_{i+1}} d_i^{-\alpha}(v-d_i)^{\alpha-1} dv \\ =&\frac{1}{\alpha}\sum_{1\le i<\tau(n)} (v-d_i)^\alpha d_i^{-\alpha}|_{d_i}^{d_{i+1}} \\ =&\frac{1}{\alpha}\sum_{1\le i<\tau(n)} (d_{i+1}-d_i)^\alpha d_i^{-\alpha} \\ =&\frac{1}{\alpha}F_\alpha(n). \end{aligned}$$

$\square$

2的证明. 先来证明$k=2$的情形。简记此时$\epsilon_2=\epsilon$。注意到此时对任意$z\in(1,\sqrt(n)]$有$(z,(1+\epsilon)z]$内含有n的因子。不难看出d_{i+1}/d_i关于$\sqrt{n}$两侧的因子对称分布，于是我们得到对任意$z\in(1,n]$有$\chi(n;u,v)=0$对一切$v\geq(1+\epsilon)u$成立，由1自然有

$$\begin{aligned}
F_\alpha(n)\leq&\alpha\int_1^n\int 0\frac{v}{1+\epsilon}(v-u)^{\alpha-2}(\alpha v-u)u^{-1-\alpha}dudv\\
=&\alpha\int_1^n -u^{-\alpha}(v-u)^{\alpha-1}|_0^{v/(1+\epsilon)}dv\\
=&\alpha\int_1^n -(1+\epsilon)\epsilon^{\alpha-1}\frac{1}{v}dv\\
=&\alpha(1+\epsilon)\epsilon^{\alpha-1}\ln n.
\end{aligned}$$

下面证明一般情形，此时注意到$(x-1)^\alpha+(y-1)^\alpha\leq(xy-1)^\alpha$对$x,y,\alpha>1$成立。于是有

$$\begin{aligned}
F_\alpha(n)=&\sum_{1\leq i\leq\tau(n)}((d_{i+1}/d_i)-1)^\alpha\\
=&\sum_{2\leq j\leq k}\sum_{n_{j-1}\leq d_i\leq n_j}((d_{i+1}/d_i)-1)^\alpha\\
\leq&\sum_{2\leq j\leq k}\sum_{1\leq i'\leq\tau(d_j)}((d'_{i'+1}/d'_{i'})-1)^\alpha\\
\leq&\alpha\sum_{2\leq j\leq k}(1+\epsilon_j)\epsilon_j^{\alpha-1}\ln(n_j/n_{j-1}).
\end{aligned}$$

其中我们用$d'_{i'}$这个记号统一指代了d_j的因子的渐升列。 □