Lecture Notes
Homological Algebra Seminar

University of Science and Technology of China

February 16, 2024

1 Tor for Abelian Groups

1.1 Introduction

The importance of Tor functor was first noticed by Cĕch. We will not go through the details in this lecture. We first study Tor for Abelian groups, in other words, $\mathbb{Z}$-modules.

We can choose a free resolution for $\mathbb{Z}$-module A.

$$0 \to \mathbb{Z}^m \to \mathbb{Z}^n \to A \to 0.$$

Then we tensor with B.

$$0 \to \mathrm{Tor}(A, B) \to B^m \to B^n \to A \otimes B \to 0.$$

We use $\mathrm{Tor}(A, B)$ to fix the exactness.

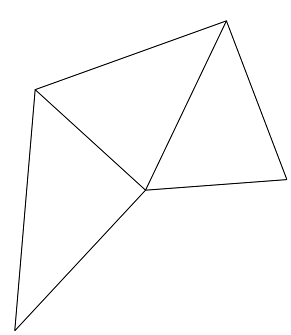
There are some questions for this.

1. Where does the definition come from?
2. It seems to depend on resolution of A?
3. Why is it called Tor?
4. How to compute it?

We will try to answer these four questions.

1.2 Motivation for the Definition

It actually comes from algebraic topology.



We choose a manifold M and suppose we can triangulate it.

We can form a chain complex like this

$$\cdots \xrightarrow{d} C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_0.$$

C_i has a basis of the i-dim simplexes. And d is the boundary map.

We can compute the homology of M to be the homology groups of this complex.

$$H_i(M) = \frac{\operatorname{Ker} C_i \to C_{i-1}}{\operatorname{Im} C_{i+1} \to C_i}.$$

We can also compute homology of M with coefficients in a group G, $H_i(M, G)$. It is defined as the homology of the following complex.

$$\cdots \to C_2 \otimes G \to C_1 \otimes G \to C_0 \otimes G.$$

This is exactly what we are doing in the definition of Tor.

1.3 Well-Defined

Now we want to check if $\operatorname{Tor}(A, B)$ is well-defined. Again the motivation comes from the algebraic topology. For M we can have two different triangulations. The problem is that do we get the same homology groups? It is a tricky question. You can ask a slightly easier question.

Suppose we have got two maps from M to N, f and g. Do these induce the same map from the homology groups of M to the homology groups of N. The answer is yes if the two maps are **homotopic**.

$$\begin{tikzcd}
{} \arrow[r] & C_3 \arrow[r, "d"] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & C_2 \arrow[r, "d"] \arrow[dl, "s"'] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & C_1 \arrow[r, "d"] \arrow[dl, "s"'] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & C_0 \arrow[dl, "s"'] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] \\
{} \arrow[r] & D_3 \arrow[r, "d"] & D_2 \arrow[r, "d"] & D_1 \arrow[r, "d"] & D_0
\end{tikzcd}$$

We can talk about two chain complexes to be homotopic.

So what does homotopic mean? It means that we can create maps denoted by s in the diagram, such that we have

$$\boxed{sd + ds = f - g.}$$

And this is the algebraic analogue of two maps being homotopic in algebraic topology. It turns out this idea can be used to show Tor is well-defined.

1.4 Name

The third question is why is this group called Tor?

The answer is that $\operatorname{Tor}(A, B)$ for A, B f.g. Abelian groups, depends only on **torsion** (elements of finite order) subgroups of A, B.

1.5 Computation

We want to compute $\operatorname{Tor}(A, B)$ for A, B f.g. abelian groups.

First we notice that

$$\operatorname{Tor}(A \oplus B, C) \simeq \operatorname{Tor}(A, C) \oplus \operatorname{Tor}(B, C),$$

$$\mathrm{Tor}(A, B \oplus C) \simeq \mathrm{Tor}(A, B) \oplus \mathrm{Tor}(A, C).$$

It is just an easy exercise.

So just need to compute $\mathrm{Tor}(A, B)$ for A, B **cyclic**.

First, let's calculate $\mathrm{Tor}(\mathbb{Z}, G)$.

$$0 \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0.$$

Now we tensor with G. $\mathrm{Tor}(\mathbb{Z}, G)$ is just the kernel of the map

$$\begin{aligned} 0 \otimes G &\longrightarrow \mathbb{Z} \otimes G \\ = 0 &\longrightarrow G. \end{aligned}$$

Well it is obvious that the kernel is just 0.

$$\boxed{\mathrm{Tor}(\mathbb{Z}, G) = 0.}$$

Now we study $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, G)$. Now it's more interesting. We have got this sequence.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

Now we tensor with G

$$\underbrace{\mathbb{Z} \otimes G \xrightarrow{\times n} \mathbb{Z} \otimes G}_{G \xrightarrow{\times n} G} \longrightarrow \mathbb{Z}/n\mathbb{Z} \otimes G \longrightarrow 0.$$

So we have

$$0 \longrightarrow \underbrace{\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, G)}_{\text{Elements of order } n \text{ in } G} \longrightarrow G \xrightarrow{\times n} G.$$

So $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ will be the elements of order n in $\mathbb{Z}/m\mathbb{Z}$. If you think about it, it is $\mathbb{Z}/(n, m)\mathbb{Z}$. And $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = 0$.

You notice that $\mathrm{Tor}(A, B)$ depends only on torsion of A, B.

Note that it seems $\mathrm{Tor}(A, B) \simeq \mathrm{Tor}(B, A)$. This turns out to be true in general. In fact it is true in arbitrary rings although this takes a little bit work to prove. But it is not obvious, since our definition is not symmetric.

We can also note that for A, B finite, $\mathrm{Tor}(A, B) \simeq A \otimes B$. However it is a rather **bad** isomorphism. The point is no natural isomorphism between the two in general. But $\mathrm{Tor}(A, B) \simeq \mathrm{Tor}(B, A)$ is natural. However $\mathrm{Tor}(A, B) \simeq A \otimes B$ depends on choice of generators of A, B.

2 Properties of Tor

Assume A, B to be Abelian groups. We need to check some basic properties.

1. Well-defined.

2. $\mathrm{Tor}(A, B)$ is a **functor** in A, B.

$$B \longrightarrow C \implies \mathrm{Tor}(A, B) \longrightarrow \mathrm{Tor}(A, C).$$

3.
$$\mathrm{Tor}(A, B) \simeq \mathrm{Tor}(B, A).$$

4. Long exact sequence.

$$\begin{aligned} &0 \to A \to B \to C \to 0 \\ \implies &0 \to \mathrm{Tor}(A, M) \to \mathrm{Tor}(B, M) \to \mathrm{Tor}(C, M) \to A \otimes M \to B \otimes M \to C \otimes M \to 0. \end{aligned}$$

2.1 Well-Defined

To define Tor, we need to take a resolution of A.

$$0 \longrightarrow \mathbb{Z}^{m_1} \xrightarrow{d} \mathbb{Z}^{n_1} \xrightarrow{d} A \longrightarrow 0.$$

And we may take the second resolution of A.

$$0 \longrightarrow \mathbb{Z}^{m_2} \xrightarrow{d} \mathbb{Z}^{n_2} \xrightarrow{d} A \longrightarrow 0.$$

We want to compare these two sequence.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^{m_1} \arrow[r] \arrow[d, dashed] & \mathbb{Z}^{n_1} \arrow[r] \arrow[d, dashed] & A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{Z}^{m_2} \arrow[r] & \mathbb{Z}^{n_2} \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

First we map A to itself. Then we can lift because it's a free module. We can continue to do this trick.

Then we tensor with B, and this induce a map from $\mathrm{Tor}(A, B)$, to another $\mathrm{Tor}(A, B)$, let's call this $\mathrm{Tor}'(A, B)$. We want to show these two groups are isomorphic. Also, the maps from $\mathbb{Z}^{m_i}$ to $\mathbb{Z}^{n_i}$ is **not unique**. We want to show this induce map is unique. How can we show that?

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tor}(A, B) \arrow[r] \arrow[d] & B^{m_1} \arrow[r] \arrow[d] & B^{n_1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Tor}'(A, B) \arrow[r] & B^{m_2} \arrow[r] & B^{n_2} \arrow[r] & 0 \end{tikzcd}$$

Suppose we have got two different maps. I can define a new map s. s is going to have the following property. If we first do s then d we can make $ds = f - g$. This exists because $d(f - g) = 0$. So the difference of f and g must be in the image of the former map. So we can lift this map.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^{m_1} \arrow[r, "d"] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & \mathbb{Z}^{n_1} \arrow[r, "d"] \arrow[dl, "s"'] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & A \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{Z}^{m_2} \arrow[r, "d"] & \mathbb{Z}^{n_2} \arrow[r, "d"] & A \arrow[r] & 0 \end{tikzcd}$$

Now we look at another triangle. We notice that we have $f - g = sd$. This follows because $d(f - g) = (f - g)d = dsd$. Since d is injective they must be the same. We can combine these two identities by saying

$$\boxed{f - g = sd + ds.}$$

That's if we extend s to be 0 here. We say f, g to be **homotopic**. It is very closely related to the condition in algebraic topology that two maps are homotopic.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^{m_1} \arrow[r, "d"] \arrow[dl, "s=0"'] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & \mathbb{Z}^{n_1} \arrow[r, "d"] \arrow[dl, "s"'] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & A \arrow[r] \arrow[dl, "s=0"'] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}^{m_2} \arrow[r, "d"] & \mathbb{Z}^{n_2} \arrow[r, "d"] & A \arrow[r] & 0
\end{tikzcd}$$

We can now use this to show that these two groups are isomorphic. If we got two maps. They still satisfy the identities before, so we know

$$f - g = sd.$$

So f, g is **same** on the kernel of d. And $\text{Tor}(A, B)$ is in the kernel of d, so induces the same map on Tor groups. We get **well-defined map**

$$\text{Tor}(A, B) \leftrightarrows \text{Tor}'(A, B).$$

We can do the same thing with the second resolution of A, so we can define two maps between them. Now we notice the composition of them is the **identity** map. So these groups are canonically isomorphic.

$$\begin{tikzcd}
0 \arrow[r] & \text{Tor}(A, B) \arrow[r] \arrow[d, "f=g"'] & B^{m_1} \arrow[r] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & B^{n_1} \arrow[r] \arrow[dl, "s"] \arrow[d, bend right, "f"'] \arrow[d, bend left, "g"] & 0 \\
0 \arrow[r] & \text{Tor}'(A, B) \arrow[r] & B^{m_2} \arrow[r] & B^{n_2} \arrow[r] & 0
\end{tikzcd}$$

2.2 Functor

Suppose we have got a map from $B \to C$. We want to get a map from $\text{Tor}(A, B) \to \text{Tor}(A, C)$. Suppose we have got a resolution of A.

$$0 \to \mathbb{Z}^m \to \mathbb{Z}^n \to A \to 0.$$

Then tensor with B and C.

$$\begin{tikzcd}
0 \arrow[r] & \text{Tor}(A, B) \arrow[r] \arrow[d, dashed] & B^m \arrow[r] \arrow[d] & B^n \arrow[d] \\
0 \arrow[r] & \text{Tor}(A, C) \arrow[r] & C^m \arrow[r] & C^n
\end{tikzcd}$$

As before you can check the map is independent of the choice od resolutions. So we get a well-defined map. You can check Tor is also a functor in its second argument. Since we need to show it's symmetric, we don't really need to do that.

2.3 Symmetry

$$\text{Tor}(A, B) \simeq \text{Tor}(B, A)$$

It's not obvious they are the same.

Let's take resolutions of A and B.

$$0 \to \mathbb{Z}^m \to \mathbb{Z}^n \to A \to 0$$

$$0 \to \mathbb{Z}^s \to \mathbb{Z}^t \to B \to 0$$

How do we compare these Tors? The idea is we tensor these resolutions together. That's getting a huge diagram.

$$\begin{tikzcd}
& & & & 0 \arrow[d] & \\
& & 0 \arrow[d] & 0 \arrow[d] & \mathrm{Tor}(B,A) \arrow[d] & \\
& 0 \arrow[r] & \mathbb{Z}^m \otimes \mathbb{Z}^s \arrow[r] \arrow[d] & \mathbb{Z}^n \otimes \mathbb{Z}^s \arrow[r] \arrow[d] & A \otimes \mathbb{Z}^s \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & \mathbb{Z}^m \otimes \mathbb{Z}^t \arrow[r] \arrow[d] & \mathbb{Z}^m \otimes \mathbb{Z}^t \arrow[r] \arrow[d] & A \otimes \mathbb{Z}^t \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Tor}(A,B) \arrow[r] & \mathbb{Z}^m \otimes B \arrow[r] \arrow[d] & \mathbb{Z}^n \otimes B \arrow[r] \arrow[d] & A \otimes B \arrow[r] \arrow[d] & 0 \\
& & 0 & 0 & 0 &
\end{tikzcd}$$

How do we get from one group to the other? We get it by doing a sort of zigzag.

We start with $\mathrm{Tor}(A,B)$ and then take image in $\mathbb{Z}^m \otimes B$. Since the map from $\mathbb{Z}^m \otimes \mathbb{Z}^t \to \mathbb{Z}^m \otimes B$ is onto, we can lift the element up to $\mathbb{Z}^m \otimes \mathbb{Z}^t$. Then we can take its image in $\mathbb{Z}^n \otimes \mathbb{Z}^t$. Its image in $\mathbb{Z}^n \otimes B$ is 0, because it is the image of $\mathrm{Tor}(A,B)$, so we can lift to $\mathbb{Z}^n \otimes \mathbb{Z}^s$, then take image in $A \otimes \mathbb{Z}^s$, and that finally gives us an element in $\mathrm{Tor}(B,A)$. And that gives us a homomorphism from $\mathrm{Tor}(A,B)$ to $\mathrm{Tor}(B,A)$, or does it?

There is a bit of a problem here. We don't know if the map is well-defined. When we lifted this element in $\mathbb{Z}^m \otimes B$, it is not unique. So we don't actually get a homomorphism from $\mathrm{Tor}(A,B)$ to this group because there's an ambiguity. We can change the element lifted to $\mathbb{Z}^m \otimes \mathbb{Z}^t$. But if we change it, the elements in $\mathbb{Z}^m \otimes \mathbb{Z}^s$ will be different. But the image of an elements in $\mathbb{Z}^m \otimes \mathbb{Z}^s$ in $A \otimes \mathbb{Z}^s$ is 0 so although the maps from $\mathrm{Tor}(A,B)$ to some intermediate groups are not well-defined, the map from $\mathrm{Tor}(A,B)$ to $\mathrm{Tor}(B,A)$ is well-defined.

So we get a homomorphism. And we can do the backwards. You can check they are inverses of each other. So it is an isomorphism.

2.4 Long Exact Sequence

If we get $0 \to A \to B \to C \to 0$, we want to get a nice long exact sequence $0 \to \mathrm{Tor}(A,M) \to \mathrm{Tor}(B,M) \to \mathrm{Tor}(C,M) \to A \otimes M \to B \otimes M \to C \otimes M \to 0$. How do we get this?

We want to tensor this sequence with M.

$$0 \to \mathbb{Z}^m \to \mathbb{Z}^n \to M \to 0.$$

And we get this.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Tor}(A,M) \arrow[r] \arrow[d] & \mathrm{Tor}(B,M) \arrow[r] \arrow[d] & \mathrm{Tor}(C,M) \arrow[d] & \\
0 \arrow[r] & A^m \arrow[r] \arrow[d] & B^m \arrow[r] \arrow[d] & C^m \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A^n \arrow[r] \arrow[d] & B^n \arrow[r] \arrow[d] & C^n \arrow[r] \arrow[d] & 0 \\
 & A \otimes M \arrow[r] \arrow[d] & B \otimes M \arrow[r] \arrow[d] & C \otimes M \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

There we still don't put two arrows. Because the map $\mathrm{Tor}(B.M) \to \mathrm{Tor}(C,M)$ need not to be surjective and $A \otimes M \to B \otimes M$ need not to be injective. What we want to do is to define a map that goes all the way from this corner down to this corner. This is sometimes called the **snake map**. We want to show this "snake line" is **exact**.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{Tor}(A,M) \arrow[r] \arrow[d] & \mathrm{Tor}(B,M) \arrow[r] \arrow[d] & \mathrm{Tor}(C,M) \arrow[d] \arrow[dddll] & \\
0 \arrow[r] & A^m \arrow[r] \arrow[d] & B^m \arrow[r] \arrow[d] & C^m \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A^n \arrow[r] \arrow[d] & B^n \arrow[r] \arrow[d] & C^n \arrow[r] \arrow[d] & 0 \\
 & A \otimes M \arrow[r] \arrow[d] & B \otimes M \arrow[r] \arrow[d] & C \otimes M \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

We want to define a map from $\mathrm{Tor}(C,M)$ to $A \otimes M$. We do it by using a zigzag just like before. We have the similar problem and argument. We can check this map is **well-defined**. Now we can check it makes this big sequence exact. The only problem is that we want to show if an element in $A \otimes M$ whose image in $B \otimes M$ is 0, then it's the image of $\mathrm{Tor}(C,M)$.

First you can lift it to something in A^n, then you can take the image in B^n. Now comes the key point. Since its image in $B \otimes M$ is 0, you can lift it to B^m. Now you can continue all the way backup to $\mathrm{Tor}(C,M)$.

As an application of the long exact sequence, we might use it to find $\mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$. Note that $\mathrm{Tor}(\mathbb{Q}, A) = 0$. This is because $\mathbb{Q}$ is **flat**.

Now we can take the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0.$$

Then we tensor with A.

$$\to \underbrace{\mathrm{Tor}(A,\mathbb{Q})}_{0} \to \mathrm{Tor}(A,\mathbb{Q}/\mathbb{Z}) \to \underbrace{\mathbb{Z}\otimes A}_{A} \to \mathbb{Q}\otimes A \to$$

I haven't written out the whole exact sequence.

So $\mathrm{Tor}(A,\mathbb{Q}/\mathbb{Z})$ is just the kernel of $A \to \mathbb{Q}\otimes A$. And the kernel is just the torsion of A. So $\mathrm{Tor}(\mathbb{Q}/\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \simeq \mathbb{Q}/\mathbb{Z}$.

So far we have talked about Tor modules over the ring of integers, next we'll be discussing for modules over more complicated rings where things get considerably more complicated.

3 Tor over Rings

3.1 Definition

Now we talk about

$$\mathrm{Tor}_i^R(A,B),$$

where R is a ring and A,B are R-modules. Before we discussed the case R is the ring of integers. In the case of integers these groups are non-zero only for i equals 0 or 1, so you generally miss out the subscript i.

So we first give the definition. This is similar to the definition for integers. In the integers if we take a free module, then any submodule of the free module is also free, so if we want we can stop the resolution here. For more general rings we have to keep going if you want this sequence to be **exact**, so that we probably have an infinite sequence.

$$\cdots \to R^{n_3} \to R^{n_2} \to R^{n_1} \to R^{n_0} \to A \to 0.$$

It is the first step.

Step two will be tensor with B.

$$\cdots \longrightarrow B^{n_3} \xrightarrow{d} B^{n_2} \xrightarrow{d} B^{n_1} \xrightarrow{d} B^{n_0} \longrightarrow 0.$$

We drop out the A and put a zero there.

The third step is we take the homology group H_* of the sequence, which means we look at

$$\frac{\mathrm{Ker}(B^{n_i} \to B^{n_{i-1}})}{\mathrm{Im}(B^{n_{i+1}} \to B^{n_i})} = \mathrm{Tor}_i^R(A,B).$$

So just like the integers, we have several questions to ask.

1. Well-defined. (independent of choice of resolution)
2. Functor in A,B.
3. Symmetric. $(\mathrm{Tor}_i(A,B) \simeq \mathrm{Tor}_i(B,A))$
4. Long exact sequence.

$$\begin{gathered} 0 \to A \to B \to C \implies \\ \cdots \to \mathrm{Tor}_2(C,M) \to \mathrm{Tor}_1(A,M) \to \mathrm{Tor}_1(B,M) \to \\ \mathrm{Tor}_1(C,M) \to A\otimes M \to B\otimes M \to C\otimes M \to 0. \end{gathered}$$

We leave these questions to next part.

3.2 Examples

Example 1: Serre's definition of intersection multiplicities.
length $(\sum_i (-1)^i \operatorname{Tor}_i^R(R/I, R/J))$, where R is a local ring, I and J are ideals.
Example 2: Homology of group G.
Eilenberg-Maclane defined it. Their original definition turns out to be the same as

$$H_i(G, M) = \operatorname{Tor}_i^{\mathbb{Z}[G]}(\mathbb{Z}, M),$$

where $\mathbb{Z}[G]$ is the **group ring**.
Example 3: Homology of Lie algebra A.

$$H_i(A, M) = \operatorname{Tor}_i^{U(G)}(\mathbb{R}, M),$$

where $U(G)$ is the universal enveloping algebra of A.
Example 4: Hochschild cohomology.

$$HH_i(A, M) = \operatorname{Tor}_i^{A \otimes A^{\mathrm{op}}}(A, M).$$

If you look at this you'll see that these three examples were originally defined independently in different rather harder ways and it was afterwards noticed that they were all special cases of Tor over modules over a suitable algebra. So Tor unifies three apparently different theories.

3.3 Calculations

If you look at the definition of Tor, it looks kind of really complicated and roundabout and difficult to use, but in fact, in practice, it's usually rather easy to calculate Tor groups.

Let's look at some examples.

$R = k[x]/(x^2)$. It has base $1, x$. $M = k = R/(x)$. We want to calculate $\operatorname{Tor}_i^R(k, k)$. We make k into a R-module with $x \cdot k = 0$.

We have to take a resolution of k.

$$\xrightarrow{\times x} R \xrightarrow{\times x} R \xrightarrow{\times x} R \xrightarrow{1 \to 1} k \longrightarrow 0.$$

Then we tensor with k.

$$\xrightarrow{0} k \xrightarrow{0} k \xrightarrow{0} k.$$

Then we take the homology groups. Since these are all zero maps, the homology is obvious.

k	k	k
$\operatorname{Tor}_2(k, k)$	$\operatorname{Tor}_1(k, k)$	$\operatorname{Tor}_0(k, k)$

So we can see $\operatorname{Tor}_i(k, k) = k$ for all $k \geq 0$. In particular it shows that there's no finite resolution of k.

Now let's look at another example.

$R = k[x, y]$, $k = k_{00} = R/(x, y)$. Let's find $\operatorname{Tor}_i(k_{00}, k_{00})$. We need to be careful since there are different ways to make something into a module. We take the notation $k_{ab} = R/(x - a, y - b)$. We can make k_{ab} into a module for each point of the two-dimensional affine space over R.

We need to find a resolution of k_{00}.

The easiest way to do it is to draw a picture. Now let's draw a picture for R.

$$\begin{array}{lllll}
\cdots & & & & \\
y^3 & \cdots & & & \\
y^2 & xy^2 & \cdots & & \\
y & xy & x^2y & \cdots & \\
1 & x & x^2 & x^3 & \cdots
\end{array}$$

And now we can map R onto k. And the kernel will be generated by x, y so we can find two free R-modules (x) and (y) (Restricted to my level of using TikZ I can not draw a picture here), both 1-dimension. These two modules, maps onto R, will have a common kernel, which is (xy). Let's write it in an algebraic way.

$$0 \longrightarrow R \overset{1\to(y,-x)}{\longrightarrow} R \oplus R \overset{(1,0)\to x,(0,1)\to y}{\longrightarrow} R \longrightarrow k \to 0.$$

Here we've got a resolution of k of length 3. Now we can tensor this with k_{00}.

$$0 \to k \to k^2 \to k \to 0.$$

As usual we cross the last k. We have to figure out what these maps are. Since x, y are 0 in k, so all maps are 0. So if we take the homology we just get

$$\begin{array}{lll}
k & k^2 & k \\
\mathrm{Tor}_2 & \mathrm{Tor}_1 & \mathrm{Tor}_0
\end{array}$$

The fact Tor_2 is not 0 shows that we couldn't find a resolution of length less than this.

The other thing we can do is to calculate $\mathrm{Tor}_i(k_{00}, k_{ab})$.

Now we need to tensor the exact sequence with k_{ab}, say $k_{0,1}$ just to be explicit.

$$k \to k^2 \to k$$

However these maps here are no longer zero. Now the first map takes $1 \to (1,0)$, and the second map takes $(1,0) \to 0, (0,1) \to 1$. So we see this sequence is **exact**. So if we take homology we get

$$\begin{array}{lll}
0 & 0 & 0 \\
\mathrm{Tor}_2 & \mathrm{Tor}_1 & \mathrm{Tor}_0
\end{array}$$

So $\mathrm{Tor}_i(k_{00}, k_{01}) = 0$ for all values i. In fact you see if we take two of these modules corresponding to points in the plane, then the dimensions of Tors will be 1,2,1 if these are the same points and will all be zero if the points are different.

As an exercise, you can do the same thing to $k[x, y, z]$. You should find a Tor_3 that is non-zero.

For the last example, I am going to calculate the homology of a cyclic group of order n with coefficients in, say $\mathbb{Z}$. We remember it is defined to be

$$H_i(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \mathrm{Tor}_i^{\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}).$$

The first thing is to figure out what the group ring is. Let's write the group in multiplication and the ring has basis $1, g, g^2, \cdots g^{n-1}$, with $g^n = 1$.

Now we have to find a resolution of $\mathbb{Z}$ over this group ring. First we map $\mathbb{Z}[G]$ onto $\mathbb{Z}$ which is easy and we can just map 1 to 1, and g^i also maps to 1, because in case I didn't say so, I am taking the group G to be acting trivially on $\mathbb{Z}$. (Maybe to -1 is possible but we ignore that now)

Now we need to figure out the kernel. It is obvious the kernel is generated by $1-g$. So we can make this sequence exact by mapping something onto $(1-g)$. Let's just take 1 and map it to $1-g$. We can see the kernel is generated by $1+g+g^2+\cdots+g^{n-1}$. So we can make this exact by taking another copy of $\mathbb{Z}[G]$ and mapping 1 to $1+g+g^2+\cdots+g^{n-1}$. And the kernel is again generated by $1-g$. We just continue. You see we get into a circle. So this sequence just extends indefinitely to the left with period 2.

$$\longrightarrow \mathbb{Z}[G] \xrightarrow{1\to 1-g} \mathbb{Z}[G] \xrightarrow{1\to 1+g+g^2+\cdots+g^{n-1}} \mathbb{Z}[G] \xrightarrow{1\to 1-g} \mathbb{Z}[G] \xrightarrow{1\to 1} \mathbb{Z} \longrightarrow 0.$$

The next step is to tensor this sequence with $\mathbb{Z}$.

$$\longrightarrow \mathbb{Z} \xrightarrow{1\to 1-g=0} \mathbb{Z} \xrightarrow{1\to 1+g+g^2+\cdots+g^{n-1}=n} \mathbb{Z} \xrightarrow{1\to 1-g=0} \mathbb{Z} \longrightarrow 0.$$

So our map is

$$\xrightarrow{\times n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0.$$

Now we take the homology.

0	$\mathbb{Z}/n\mathbb{Z}$	0	$\mathbb{Z}/n\mathbb{Z}$	$\mathbb{Z}$
H_4	H_3	H_2	H_1	H_0

So the homology group of cyclic group is periodic apart from the first term which is a little bit odd.

4 Properties of Tor over Rings

Now we check the basic properties.

1. Well-defined.
2. Functorial.
3. Symmetry: $\mathrm{Tor}_i(A,B) = \mathrm{Tor}_i(B,A)$.
4. Long exact sequence.

4.1 Well-Defined

The problem is we choose a resolution of A, but there are lots and lots of different resolutions and at first sight it looks like what we get here is going to depend on which resolution we choose.

Suppose we've got two resolution of A.

Step one is we can find a morphism of **complexes**. ($d^2=0$)

$$\begin{tikzcd}
{} \arrow[r] & R^{m_2} \arrow[r] \arrow[d, dashed] & R^{m_1} \arrow[r] \arrow[d, dashed] & R^{m_0} \arrow[r] \arrow[d, dashed] & A \arrow[r] \arrow[d] & 0 \\
{} \arrow[r] & R^{n_2} \arrow[r] & R^{n_1} \arrow[r] & R^{n_0} \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

At least we get a map from $\underbrace{\mathrm{Tor}_i(A,B)}_{\text{Computed with the first sequence}} \to \underbrace{\mathrm{Tor}_i(A,B)}_{\text{Computed with the second sequence}}$. At least we can compare these two Tors. Now we've got another problem. Because this morphism is far from being unique.

Step two. Suppose we take two resolutions of A, and suppose we've got two maps between these resolutions. We can compare these two maps by homotopic map s. The homotopic map satisfies

$$f - g = sd + ds.$$

What we do is to construct s step by step the same way we construct morphism. I leave it as an exercise. Then we construct **homotopy** s from f to g.

$$\begin{tikzcd}
{} \arrow[r] & R^{m_2} \arrow[r] \arrow[d, bend right] \arrow[d, bend left] & R^{m_1} \arrow[r] \arrow[d, bend right] \arrow[d, bend left] \arrow[dl, "s"'] & R^{m_0} \arrow[r] \arrow[d, bend right] \arrow[d, bend left] \arrow[dl, "s"'] & A \arrow[r] \arrow[d] & 0 \\
{} \arrow[r] & R^{n_2} \arrow[r] & R^{n_1} \arrow[r] & R^{n_0} \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Incidently you may find s isn't unique either because you've got a lot of choice. If you got two homotopies, say s_1 and s_2, they are related by a higher homotopic. This time we take a different map. We don't need them here.

The nice thing is that if f, g are **homotopic**, they induce the **same** map on **homology**.

This is easy to check. Because if $f - g = sd + ds$, then $f(x) - g(x) = ds(x)$ if $dx = 0$. If x is in the homology group then the difference of f and g is the image of d which means 0 in homology group.

You may be thinking to yourself this is kind of stupid, since the homology of the rows are 0 since we are talking about exact sequence. However if we tensor with B, the exact resolutions are no longer exact in general and the homology groups are now the torsion groups of A and B. So we showed f, g induce the **same** map from $\mathrm{Tor}(A,B) \to \mathrm{Tor}(A,B)$ (using different resolutions).

Then we can define $\mathrm{Tor}(A,B) \leftrightarrows \mathrm{Tor}(A,B)$, and you can check the composite of these maps is actually the identity map because it is homotopic to the identity map of complexes. So $\mathrm{Tor}(A,B) \simeq \mathrm{Tor}(A,B)$ canonically isomorphic.

4.2 Symmetry

We can compute $\mathrm{Tor}(A,B)$ in many different ways. We can not only choose different resolutions of A but we can either choose a resolution of A and tensor with B, or we can choose a resolution of B and then tensor with A, and then take the homology. What we want to do is to show that we get the same result whichever resolution we take.

Suppose we take resolutions

$$\to F_2 \to F_1 \to F_0 \to A \to 0$$

$$\to G_2 \to G_1 \to G_0 \to B \to 0$$

where F_i, G_i **free** and the rows are **exact**. How do we compare tension with B and tension with A. What we do is to **tensor 2 resolutions**!

Incidentally this is called a double complex (more exactly we ignore some details).

$$\begin{tikzcd}
 & & & & \cdot \arrow[d] \\
 & & & \cdot \arrow[r] \arrow[d] & \cdot \arrow[d] \\
 & & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & F_0 \otimes G_2 \arrow[d] \\
 & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & F_1 \otimes G_1 \arrow[r] \arrow[d] & F_0 \otimes G_1 \arrow[d] \\
\cdot \arrow[r] & \cdot \arrow[r] & F_2 \otimes G_0 \arrow[r] & F_1 \otimes G_0 \arrow[r] & F_0 \otimes G_0
\end{tikzcd}$$

The rows are **exact**, and the columns are **exact** now. We should just add a little bit.

$$\begin{tikzcd}
 & & & & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & 0 \\
 & & & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & 0 \\
 & & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & F_0 \otimes G_2 \arrow[r] \arrow[d] & A \otimes G_2 \arrow[r] \arrow[d] & 0 \\
 & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & F_1 \otimes G_1 \arrow[r] \arrow[d] & F_0 \otimes G_1 \arrow[r] \arrow[d] & A \otimes G_1 \arrow[r] \arrow[d] & 0 \\
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & F_2 \otimes G_0 \arrow[r] \arrow[d] & F_1 \otimes G_0 \arrow[r] \arrow[d] & F_0 \otimes G_0 \arrow[r] \arrow[d] & A \otimes G_0 \arrow[r] & 0 \\
\cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & F_2 \otimes B \arrow[r] \arrow[d] & F_1 \otimes B \arrow[r] \arrow[d] & F_0 \otimes B \arrow[d] & & \\
0 & 0 & 0 & 0 & 0 & &
\end{tikzcd}$$

Now everything is exact except that the final row and the bottom column is **not exact**. We are trying to do is compare the Tor group using the resolution of A with the Tor group using the resolution of B.

What we want to do is to get from the groups in bottom column to the groups in the final row because their homology groups are Tor groups. We do that by doing a sort of zigzag. But there are an ambiguities. The argument is just like before. So these two groups are canonically isomorphic.

4.3 Long Exact Sequence

We sketch the key points.

We have $0 \to A \to B \to C \to 0$.

Forst we choose compatible resolutions of A, B, C.

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \arrow[d] & \\
0 \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

Then we tensor everything with M.

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \arrow[d] & \\
0 \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The rows are still **exact** but the homology of column are $\mathrm{Tor}(-, M)$.

The key step is we want to get a map from $\mathrm{Tor}_{i-1}(C, M)$ to $\mathrm{Tor}_i(A, M)$. To do that we repeat the argument before. And we need to check the exactness.

5 Ext(A,B)

5.1 Definition

Now we are going to talk about Ext groups. We're going to define these groups.

$$\mathrm{Ext}^i_R(A, B)$$

They are analogues of Tor groups. They play roles in the homomorphism of modules as Tor groups in the tensor products of modules.

$$A \otimes_R B \longrightarrow \mathrm{Tor}^R_i(A, B)$$

$$\mathrm{Hom}_R(A, B) \longrightarrow \mathrm{Ext}^i_R(A, B)$$

We are going to be lazy, just recall the definition of Tor groups and say Ext's definition is similar.

You can see Tor's definition in fact works for all right exact functor $F(A)$. It means that

$$0 \to A \to B \to C \to 0 \implies \underbrace{F(A) \to F(B)}_{\textbf{not onto}} \to F(C) \to 0.(\textbf{exact})$$

A typical example of this is $F(A) = M \otimes_R A$ for some fixed module M. We can define the so-called **derived functors** or left derived functors as follows. Let's denote them by $L_i F(A)$.

First we take resolution of A.

$$\to R^{n_1} \to R^{n_0} \to A \to 0.$$

Then instead of taking tensors, what we did for Tor, we just apply functor F. We sort of delete A. Then we get a sequence not exact in general.

$$F(R^{n_1}) \to F(R^{n_0}) \to 0.(\textbf{not exact})$$

Thirdly we take **homology**. The homology of these would be called the left derived functors of F. And there are some basic properties of these derived functors.

1. $L_0 F = F$. That follows because f is right exact.
2. L_i is well-defined. We just copy the proof that Tor is well-defined.
3. It's functorial.
4. We get a long exact sequence.

Now we notice that for this we do not need **free** modules R^{n_i}, we can use **projective** modules. In other words, we can take a resolution of A by projective modules.

$$\to P_1 \to P_0 \to A \to 0$$

The reason for that is the only property of free modules we ever used was the fact they're projective.

$$\begin{tikzcd}
& P \arrow[d] \arrow[dl, dotted] & \\
B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

Well that doesn't really gain us anything because we may as well just use free modules.

There is a dual notion of projective modules called **injective modules**, which says that if we've got a submodule C of a module B, and a map from C to I, we can always lift this map from B to I.

$$\begin{tikzcd}
& I & \\
B \arrow[ur, dotted] & C \arrow[l] \arrow[u] & 0 \arrow[l]
\end{tikzcd}$$

You see injective modules are just like projective modules except you reverse every single arrow you can think of.

Suppose we've got a **left exact functor**. It means if

$$0 \to A \to B \to C \to 0$$

is exact, then

$$0 \to F(A) \to F(B) \to F(C)$$

is exact.

There is an obvious example.

Fix a module X, put $F(A) = \mathrm{Hom}_R(X, A)$. Then F is **left exact**.

$$0 \to \mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, C)$$

You may ask if the last map is onto. We can take our standard counterexample.

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Take $X = \mathbb{Z}/2\mathbb{Z}$.

$$\begin{array}{ccc} 0 \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to & \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to & \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \\ 0 & 0 & \mathbb{Z}/2\mathbb{Z} \end{array}$$

Now we can define derived functors of this. We define right derived functors $R^iF(A)$ as follows.

1. First we take **injective** resolution of A.

$$0 \to A \to I_0 \to I_1 \to I_2 \to \cdots$$

2. Then we apply F.

$$0 \to F(I_0) \to F(I_1) \to \cdots$$

3. Thirdly we take the cohomology of this. And the cohomology groups of this complex will be $R^iF(A)$.

Our right derived functors work just as the left right derived functors we defined earlier. This has the usual properties

1. Well-defined.
2. $R^0F = F$.
3. Functorial.
4. Long exact sequence.

If $F(A) = \mathrm{Hom}(X, A)$, $R^iF(A) = \mathrm{Ext}^i_R(X, A)$.

Let's work out some examples to get familiar with it.

5.2 Examples

We'll do the simplest example. Let's take $R = \mathbb{Z}$. We need to know what are the injective modules for R. We'll study injective modules in a little more detail later. So let's just comment now.

Injective modules are **divisible** ones. (true over any P.I.D., but not true in general)

So let's calculate $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$. $(n > 0)$

So we need an injective resolution of $\mathbb{Z}$.

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0.$$

$\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are the injective ones. In order to work out Ext, we apply $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, *)$ to this and we delete $\mathbb{Z}$.

So we get

$$0 \to \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}) \to \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \to 0.$$

Now we need to take cohomology H^* of this sequence.

$$\begin{array}{rr} 0 \to \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}) \to & \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \to 0 \\ 0 = \mathrm{Ext}^0(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \to & \mathbb{Z}/n\mathbb{Z} = \mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \to 0 \end{array}$$

Now let's work out $\mathrm{Ext}(\mathbb{Z}, \mathbb{Z})$. Again we take

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0.$$

This time if we apply $\mathrm{Hom}(\mathbb{Z}, *)$ we get

$$0 \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0.$$

If we take the cohomology of this. We just take the kernel of $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ which is $\mathbb{Z}$, and $\mathbb{Q}/\mathbb{Z} \to 0$ is onto, so the cohomology is 0.

So $\mathrm{Ext}^0(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$, and $\mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}) = 0$.

Finally we should finish off by calculating $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$.

Now we need to take an injective resolution of $\mathbb{Z}/m\mathbb{Z}$.

$$0 \to \mathbb{Z}/m\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \overset{\times m}{\to} \mathbb{Q}/\mathbb{Z} \to 0.$$

Now we apply $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, *)$ as usual. Well $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ is just $\mathbb{Z}/n\mathbb{Z}$. So we get

$$0 \to \mathbb{Z}/n\mathbb{Z} \overset{\times m}{\to} \mathbb{Z}/n\mathbb{Z} \to 0.$$

Now we take the cohomology. So we get

$$\mathrm{Ext}^0(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/(m,n)\mathbb{Z}.$$

The higher Exts always vanish.

5.3 Name

The next question is why is Ext called Ext? Well, it is short for **"extensions"**. And the reason is that Ext^1 classifies extensions. More precisely, $\mathrm{Ext}^1(C, A)$ classifies extensions $0 \to A \to B \to C \to 0$.

You can divide extension into equivalence classes.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "="] & B \arrow[r] \arrow[d, "\simeq"] & C \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & A \arrow[r] & B' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

So two extensions B, B' are considered the same if there are isomorphism from B to B' like this. And it turns out that the isomorphism classes of extensions of A and C corresponds to the group Ext^1. Let's see why this is so.

Suppose you have got an extension $0 \to A \to B \to C \to 0$. Then we can look at the long exact sequence of Exts.

$$0 \to \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, B) \to \mathrm{Hom}(C, C) \to \mathrm{Ext}^1(C, A) \to \cdots$$

We don't care what happens after that. Now $\mathrm{Hom}(C, C)$ has an obvious element, identity, a sort of canonical element in it. We can just take the image here.

$$\begin{array}{ccccc} 0 \to \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, B) \to & \mathrm{Hom}(C, C) \to & \mathrm{Ext}^1(C, A) & \to \cdots \\ & \mathrm{Id} \to & x & \end{array}$$

So that gives us a way to get elements in Ext groups.

Conversely, suppose we've got an element in $\mathrm{Ext}^1(C, A)$. We can get an extension out of it as follows. We take an injective resolution of A.

$$0 \to A \to I_0 \to I_1 \to I_2 \to \cdots$$

And we take the Hom.

$$\mathrm{Hom}(C, I_0) \to \mathrm{Hom}(C, I_1) \to \mathrm{Hom}(C, I_2) \to \cdots$$

x is an element in the cohomology groups, so it means that x gives us a map from C to I_1 whose image in I_2 is zero.

Now we construct the extension of A by C just by the product of I_0 and C over I_1, which means we take the products of elements in I_0 and C with the same image in I_1.

$$\begin{tikzcd}
 & & I_0 \times_{I_1} C \arrow[r] \arrow[d] & C \arrow[d, "x"] \arrow[dr, "0"] & & \\
0 \arrow[r] & A \arrow[r] \arrow[ur] & I_0 \arrow[r] & I_1 \arrow[r] & I_2 \arrow[r] & \cdots
\end{tikzcd}$$

The map from $A \to I_0 \times_{I_1} C$ is just defined as the map from $A \to I_0$ times zero map to C.

Now you can check we have an extension of C by A. You can check the two correspondences are inverses of each other.

We can just see a very quick example of this. Take $A = C = \mathbb{Z}/2\mathbb{Z}$, we know $\mathrm{Ext}^1(C, A) = \mathbb{Z}/2\mathbb{Z}$. It has elements 0 and 1. 0 corresponds to $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ (**split** extension), and 1 corresponds to $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ (**non-split** extension).

5.4 Balance Property

For Tor, we have this balance property

$$\mathrm{Tor}(A, B) \simeq \mathrm{Tor}(B, A).$$

This obvious analogue is certanly not true for Ext.

$$\mathrm{Ext}(A, B) \neq \mathrm{Ext}(B, A) \text{ in general.}$$

That's not even true for Ext^0.

However there is a sort of analogue of the balance property. This corresponds to the fact that we can compute Tor by taking a resolution either of A or of B. And similarly for Ext we did it by computing using a resolution of B, but we can also do it by using a resolution of A. However you have to be a little bit careful.

The problem is first of all, $\mathrm{Hom}(C, A)$ is **contravaraint** in C. That means it sort of reverses arrows.

If we got an exact sequence

$$0 \to A \to B \to C \to 0.$$

Then we got a sequence

$$\mathrm{Hom}(A, M) \overset{\textbf{not onto}}{\leftarrow} \mathrm{Hom}(B, M) \leftarrow \mathrm{Hom}(C, M) \leftarrow 0.$$

This gets a bit confusing whether this is left exact or right exact. If it is right exact we should use a projective resolution and if it is left exact we should use an injective resolution. It turns out that we could calculate $\mathrm{Ext}^1(C, A)$ by using **projective** resolution of C, and the result is same as using **injective** resolution of A.

The proof is similar so we don't go through it.

5.5 Another Example

We should have an example of $\mathrm{Ext}^i \neq 0$ for $i > 1$. For the integers Ext always vanishes if $i > 1$. That's not difficult, we can more or less copy the example we gave for Tor. We just take $R = k[x]/(x^2)$ and $A = C = k = R/(x)$.

We want to calculate $\mathrm{Ext}^i(C, A)$. Let's take a projective resolution of C.

$$\to R \to R \to k \to 0.$$

Now we apply $\mathrm{Hom}(*, A)$ to this.

$$\leftarrow \mathrm{Hom}(R, k) \leftarrow \mathrm{Hom}(R, k).$$

These are all just k and the maps between them are all just 0.

$$\overset{0}{\leftarrow} k \overset{0}{\leftarrow} k.$$

If we take the cohomology of that we just get $\mathrm{Ext}^i(k, k) \simeq k$ for all $i > 0$.

There is still one problem we didn't go into too much detail. We said we take an injective resolution of modules, however we haven't actually shown that a module has an injective resolution. So the problems ia are there enough injective modules over a ring in other words for every module can we find an injective module that it embeds into. And the answer is you can in fact there's even an almost canonical injective module that embeds into called an injective envelope. We will discuss them now.

6 Injective Modules

6.1 Introduction

Problem:

Given a module M, can we find

1. Projective $\overset{\text{onto}}{\to} M$? (**easy**, we can take the free modules)

2. $M \overset{\text{into}}{\to}$ injective? (**tricky**, what we are going to talk about)

If you solve the second problem you can easily find an injective resolution of M. You just take $0 \to M \to I_0$, and you take $0 \to I_0/\operatorname{Im} M \to I_1$, and you take $0 \to I_1/\operatorname{Im} I_1 \to I_2$ and you keep on going like this, and that gives you your injective resolution.

6.2 Case of Integers

The first problem is it's not easy to find injective modules at all. Let's look at the case $R = \mathbb{Z}$. What are injective modules?

The answer turns out to be injective $\equiv$ divisible. This definitely isn't true over the general rings.

Divisible means if $a \in M, n \in \mathbb{Z}, n \neq 0$, then $a = bn$ for some $b \in M$ (possibly not unique).

First let's show injective $\Longrightarrow$ divisible. You just look at this diagram.

$$\begin{tikzcd} 0 \arrow[r] & n\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[dl, dotted] \\ & I & \end{tikzcd}$$

Now we want to show divisible $\Longrightarrow$ injective, which is a more useful criterion. Suppose we've got this.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \\ & I & \end{tikzcd}$$

What we do is we pick an element $b \in B$. We let (n) be an ideal of integers x, $xb \in A$.

So nb has some image $i \in I$, and then we can find $j \in I$ with $nj = i$, now we can extends to a homomorphism from $< A, b >$ to I. If $n = 0$ we just maps to 0.

What we do is just keep repeating. In order to show that we can use **Zorn's lemma**. Until we run out of elements of B. So we've found a map from B to I.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[dl] \\ & I & \end{tikzcd}$$

Now we can easily see $\mathbb{Z}$ has enough injectives. We mean that for any module M we can embed it into an injective module. All we need to do is to notice that $\mathbb{Q}/\mathbb{Z}$ is divisible, so is injective.

For M is finitely generated, we just maps the free part to $\mathbb{Q}^n$, and the torsion part to $(\mathbb{Q}/\mathbb{Z})^m$. For the more general case, we take limits, again we need to use **Zorn's lemma**. And we notice that the product of injective modules are also injective.

You may get the idea from this proof that the injectives over $\mathbb{Z}$ are quite complicated, I mean, we have to take the infinite products of things. Near the end of this part I'll explain that actually over Noetherian rings at least, they are reasonably easy to classify.

6.3 General Case

Fortunately, for the general ring, we have the following trick to the case of injectives over $\mathbb{Z}$. We are going to take

$$\operatorname{Hom}_R(M, \operatorname{Hom}_{\mathbb{Z}}(R, I)),$$

where M is R-module, I is $\mathbb{Z}$-module.

We want to say that more or less

$$\operatorname{Hom}_R(M, \operatorname{Hom}_{\mathbb{Z}}(R, I)) \simeq \operatorname{Hom}_{\mathbb{Z}}(M, I).$$

From this we can now show that

$$\mathrm{Hom}_{\mathbb{Z}}(R, I)$$

is injective R-module, if I is injective $\mathbb{Z}$-module. And the reason for this is if we've got

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[dl, dotted] \\
& \mathrm{Hom}_R(R, I) &
\end{tikzcd}$$

We're trying to find map from B to $\mathrm{Hom}_{\mathbb{Z}}(R, I)$ in order to show this is injective. Here A and B are R-modules.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[dl, dotted] \\
& I &
\end{tikzcd}$$

What we do is to look at this. We now forget the R-module structures, and think A and B as $\mathbb{Z}$-modules. Then R-module A maps to $\mathrm{Hom}_R(R, I)$ is the same as $\mathbb{Z}$-module A maps to I. From I is injective, we can extends a map from B to I. That corresponds to a map from B to this R-module. So

$$\boxed{\mathrm{Hom}_R(R, I)}$$

is **injective**. This gives us lots of injective R-modules. And using these R-modules, it's very easy to show that any R-module is a submodule of an injective modules. The proof is very similar to what we do for integers.

6.4 Injective Envelope

What do injective modules look like? It turns out there's a really nice property of injective modules. That is, any module is contained in a "smallest" injective module, which is in turn contained in any other injective module.

Notice that it fails for free/projective modules. For instance if we've got $\mathbb{Z}/5\mathbb{Z}$, we can have two different ways of mapping $\mathbb{Z}$ onto it. We can map 1 to 1 of $\mathbb{Z}/5\mathbb{Z}$ or 2 of $\mathbb{Z}/5\mathbb{Z}$, but neither of them are smaller or bigger than the other in any sensible way.

$$\begin{tikzcd}
\mathbb{Z} \arrow[dr, "1\to1"] & \\
& \mathbb{Z}/5\mathbb{Z} \\
\mathbb{Z} \arrow[ur, "1\to2"] &
\end{tikzcd}$$

It can be defined more precisely.

$A \subset B$ is called an **essential** extension of A if any **nonzero** submodule of B has **nonzero** intersection with A. So if we had a submodule of B with zero intersection with A, we sort of quotient out by that and get a smaller module containing A. So an essential extension is in some sense an extension you can't make any smaller.

$0 \longrightarrow A \longrightarrow I$ is called an **injective envelope** of A if

1. I is **injective**.

2. $A \subset I$ is an **essential extension**.

What we'll show is that any module has an injective envelope and it's this injective envelope which is a sort of minimal injective module containing A.

6.5 Construction of Injective Envelope

First of all, let's do it over integers.

We pick an injective module I containing M.

$$0 \longrightarrow M \longrightarrow I$$

Then we pick E **maximal** contained in I with $E \cap M = 0$. And we should invoke **Zorn's lemma** at this point.

We just form

$$0 \to M \to I/E$$

Now this is an essential extension of M.

I/E is injective as it is divisible, because it's a quotient of a divisible module, and over $\mathbb{Z}$, divisible $\implies$ injective.

We got some examples. So $\mathbb{Z} \subset \mathbb{Q}$, and this extension is essential, $\mathbb{Q}$ is divisible, so this is the injective envelope.

What about $\mathbb{Z}/2\mathbb{Z}$. It is a little bit trickier to find. It turns out the injective envelope of $\mathbb{Z}/2\mathbb{Z}$ is

$$\mathbb{Z}_{(2)}/\mathbb{Z} \simeq \mathbb{Z}_{/}2\mathbb{Z} \subset \mathbb{Z}/4\mathbb{Z} \subset \mathbb{Z}/8\mathbb{Z} \subset \cdots,$$

where $\mathbb{Z}_{(2)}$ is just the localization of $\mathbb{Z}$ at 2. You notice neither of these groups are finitely generated. In general, injective modules tend not to be f.g. I mean they can be but they're usually not which makes them a little bit tricky to work with at times.

Now let's see how to construct an injective envelope for a general ring. We need a lemma.

Lemma 6.1. *If I has property that every essential extension $I \subset J$ is **trivial** ($I = J$), then I is **injective**.*

Let's see why. Suppose we've got an extension $0 \to I \to M$. What we do is we pick a maximal submodule E of M with $E \cap I = 0$. (**Zorn**)

Now we look at $0 \to I \to M/E$. If this is **proper** ($I \neq M/E$), then E is not maximal. So $I \simeq M/E$. Since $E \cap I = 0$ so we have a map from $M \to I$, so we have a split exact sequence $0 \to E \to M \to M/E \simeq I \to 0$, so $M \simeq I \oplus E$. So any extension of I splits $\implies$ I is **injective**. For instance you can embed I into an injective module and that splits, so I is summand of an injective module and is therefore injective.

Now we use this to construct an injective envelope.

Choose

$$0 \to M \to I$$

with I injective. We pick $E \subset I$ to be a **maximal** essential extension of M. (**Zorn**)

$$0 \to M \to E \to I$$

We want to show E is **injective**. By the lemma we just need to show E has no proper extensions. Suppose not.

Let's take $E \subset E'$ essential. Since I is injective, we can define a map f from $E' \to I$ here.

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & E \arrow[r] \arrow[d] & I \\
 & & E' \arrow[ur, dashed, "f"] &
\end{tikzcd}$$

Now we notice $\operatorname{Ker} f \neq 0$ as E is **maximal**, or E' will be a bigger essential extension. On the other hand $\operatorname{Ker} f \cap E = 0$ as $E \subset I$. So $E \subset E'$ is not essential, a contradiction.

So E is **injective**. As E was chosen as an essential extension of M, this means that E is an injective envelope.

We have shown that an injective envelope exists, now let's show the **injective envelopes are unique** up to isomorphism.

So suppose we got two essential envelopes I, J of M. We want to show there's a map from I to J, and that's an isomorphism.

We do that in four steps.

$$\begin{tikzcd}
 & I \arrow[dd, "f"] \\
M \arrow[ur] \arrow[dr] & \\
 & J
\end{tikzcd}$$

First we can find $f : I \to J$. Why can we do that? Well that's because J is **injective**.

Secondly we show that f is injective as $\operatorname{Ker} f \cap M = 0$. Now we use the fact $M \subset I$ is **essential**.

Thirdly we know $I \subset J$ so $J = I \oplus E$ for some E. This follows from the fact that I is **injective**.

Finally we notice $E = 0$ because $M \subset J$ is **essential**.

So we have four conditions and we use them in our proof as well. So f is an **isomorphism**.

I haven't prove that any injective module containing M containing its injective envelope, but let's just leave that as an exercise.

I should have a warning, any two injective envelopes are isomorphic, **but the isomorphism is not unique** in general!

I should have an example of this. For example,

$$\mathbb{Z}/2\mathbb{Z} \subset \underbrace{\mathbb{Z}_{(2)}/\mathbb{Z}}_{\text{automorphism: } \times(-1)}$$

But this automorphism is the identity on $\mathbb{Z}/2\mathbb{Z}$. So this makes the concept of "the" injective envelope a bit iffy because this isomorphism is not unique. It's kind of like the algebraic closure of a field.

Now for any M we have a canonical way of minimal injective resolution.

$$0 \to M \to I_0 \to I_1 \to \cdots$$

If we got any other resolution of M, there's a map from the minimal injective resolution to it. Of course the map may not be unique, but as we saw last time they are at least homotopic.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & I_0 & \longrightarrow & I_1 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & J_0 & \longrightarrow & J_1 & \longrightarrow & \cdots \end{array}$$