11
VIBRATIONS AND WAVES

The world around us is filled with waves. Some of them we can see or hear, but many more our senses of sight or hearing cannot detect. In the submicroscopic world, atoms and molecules are made up of electrons, protons, neutrons, and mesons that move around as waves within their boundaries. Appropriately stimulated, these same atoms and molecules emit waves we call γ rays, X rays, light waves, heat waves, and radio waves.

In our world of macroscopic bodies, water waves and sound waves are produced by moving masses of considerable size. Earthquakes produce waves as the result of sudden shifts in land masses. Water waves are produced by the wind or ships as they pass by. Sound waves are the result of quick movements of objects in the air.

Any motion that repeats itself in equal intervals of time is called *periodic motion.* The swinging of a clock pendulum, the vibrations of the prongs of a tuning fork, and a mass dancing from the lower end of a coiled spring are but three examples. These particular motions and many others like them that occur in nature are referred to as *simple harmonic motion* (SHM).

11.1 SIMPLE HARMONIC MOTION

Simple harmonic motion is defined as the projection on any diameter of a graph point moving in a circle with uniform speed. The motion is illustrated in Fig. 11A. The *graph point p* moves around the circle of radius a with a uniform speed v. If at every instant of time a normal is drawn to the diameter AB, the intercept P, called the *mass point*, moves with SHM.

Moving back and forth along the line AB, the mass point is continually changing speed v_x. Starting from rest at the end points A or B, the speed increases until it reaches C. From there it slows down again coming to rest at the other end of its path. The return of the mass point is a repetition of this motion in reverse.

The displacement of an object undergoing SHM is defined as the distance from its equilibrium position C to the point P. It will be seen in Fig. 11A that the displacement x varies in magnitude from zero up to its maximum value a, which is the radius of the *circle of reference.*

The maximum displacement a is called the *amplitude*, and the time required to make one complete vibration is called the *period.* If a vibration starts at B, it is completed when the mass point P moves across to A and back again to B. If it starts at C and moves to B and back to C, only half a vibration has been completed. The amplitude a is measured in meters, or a fraction thereof, while the period is measured in seconds.

The frequency of vibration is defined as the number of complete vibrations per second. If a particular vibrating body completes one vibration in $\frac{1}{3}$ s, the period $T = \frac{1}{3}$ s and it will make three complete vibrations in 1 s. If a body makes 10 vibrations in 1 s, its period will be $T = \frac{1}{10}$ s. In other words, the frequency of vibration ν and the period T are reciprocals of each other:

$$\text{frequency} = \frac{1}{\text{period}} \qquad \text{period} = \frac{1}{\text{frequency}}$$

In algebraic symbols,

$$\nu = \frac{1}{T} \qquad T = \frac{1}{\nu} \tag{11a}$$

If the vibration of a body is described in terms of the graph point p, moving in a circle, the frequency is given by the number of *revolutions per second*, or *cycles per second*

$$1 \text{ cycle/second} = 1 \text{ vibration/second} \tag{11b}$$

now called the hertz* (Hz)

$$1 \text{ vib/s} = 1 \text{ Hz} \tag{11c}$$

* Heinrich Rudolf Hertz (1857–1894), German physicist, was born at Hamburg. He studied physics under Helmholtz in Berlin, at whose suggestion he first became interested in Maxwell's electromagnetic theory. His researches with electromagnetic waves which made his name famous were carried out at Karlsruhe Polytechnic between 1885 and 1889. As professor of physics at the University of Bonn, after 1889, he experimented with electrical discharges through gases and narrowly missed the discovery of X rays described by Röntgen a few years later. By his premature death, science lost one of its most promising disciples.

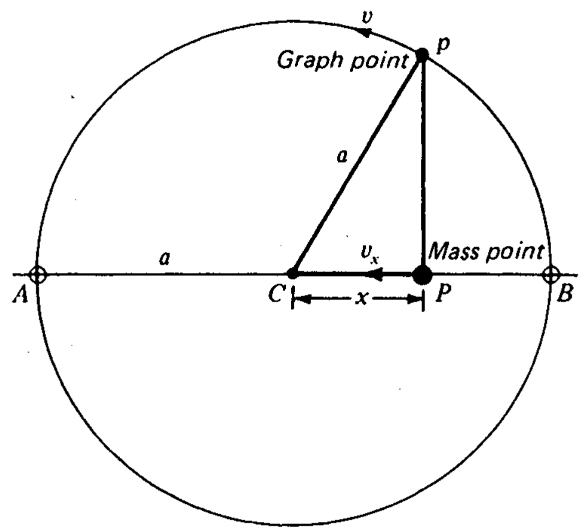

FIGURE 11A
Simple harmonic motion along a straight line *AB*.

11.2 THE THEORY OF SIMPLE HARMONIC MOTION

At this point we present the theory of SHM and derive an equation for the period of vibrating bodies. In Fig. 11B we see that the displacement x is given by

$$x = a \cos \theta$$

As the graph point p moves with constant speed v, the radius vector a rotates with constant angular speed ω, so that the angle θ changes at a constant rate

$$x = a \cos \omega t \tag{11d}$$

The graph point p, moving with a speed v, travels once around the circle of reference, a distance equal to $2\pi a$, in the time of one period T. We now use the relation in mechanics that time equals distance divided by speed, and obtain

$$T = \frac{2\pi a}{v} \tag{11e}$$

To obtain the angular speed ω of the graph point in terms of the period, we have

$$T = \frac{2\pi}{\omega} \qquad \text{or} \qquad \omega = \frac{2\pi}{T} \tag{11f}$$

An object moving in a circle with uniform speed v has a *centripetal acceleration* toward the center, given by

$$a_c = \frac{v^2}{a} \tag{11g}$$

Since this acceleration a_c continually changes the direction of the motion, its component a_x along the diameter, or x axis, changes in magnitude and is given by $a_x = a_c \cos \theta$. Substituting in Eq. (11g), we find

$$a_x = \frac{v^2}{a} \cos \theta$$

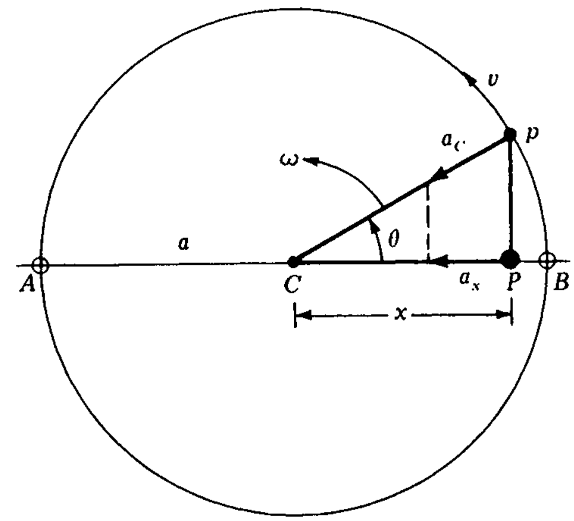

FIGURE 11B
The acceleration a_x of any mass moving with simple harmonic motion is toward a position of equilibrium C.

From the right triangle CPp, $\cos\theta = x/a$, direct substitution gives

$$a_x = \frac{v^2}{a}\frac{x}{a} \quad \text{or} \quad a_x = \frac{v^2}{a^2}x$$

We now multiply both sides of the equation by $a^2/a_x v^2$, take the square root of both sides of the equation, and obtain

$$\frac{a^2}{v^2} = \frac{x}{a_x} \quad \text{and} \quad \frac{a}{v} = \sqrt{\frac{x}{a_x}}$$

For a/v in Eq. (11e) we now substitute $\sqrt{x/a_x}$ and obtain for the period of any SHM the relation

$$T = 2\pi\sqrt{\frac{x}{a_x}} \qquad \text{(11h)}$$

If the displacement is to the right of C, its value is $+x$, and if the acceleration is to the left, its value is $-a_x$. Conversely, when the displacement is to left of C, we have $-x$, and the acceleration is to the right, or $+a_x$. This is the reason for writing

•

$$T = 2\pi\sqrt{-\frac{x}{a_x}} \qquad \text{(11i)}$$

11.3 STRETCHING OF A COILED SPRING

As an illustration of the relationships generally applied to vibrating sources, we consider in some detail the stretching of a coiled spring, followed by its vibration with SHM when the stretching force is suddenly released (see Fig. 11C).

As a laboratory experiment, one end of a meterstick is placed at marker Q. A force of 2.0 newtons (N) is applied to the spring, stretching it a distance of 1.25 cm.

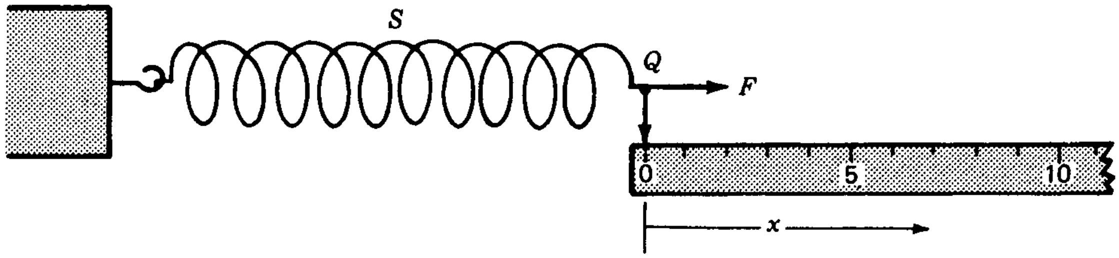

FIGURE 11C
An experiment for measuring the distance x a coiled spring S stretches for different values of the applied force.

When a total force of 4.0 N is applied, the total stretch is 2.50 cm. By applying forces of 6.0, 8.0, and 10.0 N, respectively, the total distances recorded are as shown in Table 11A.

Plotting these data on graph paper produces a straight line, as shown in Fig. 11D. Properly interpreted, this graph means that the applied force F and the displacement of the spring x are directly proportional to each other, and we can write

$$F \propto x \qquad \text{or} \qquad F = kx$$

The proportionality constant k is the slope of the straight line and is a direct measure of the stiffness of the spring. The experimental value of k in this experiment is calculated as follows:

$$k = \frac{F}{x} = \frac{10 \text{ N}}{0.0625 \text{ m}} = 160 \text{ N/m} \tag{11j}$$

The stiffer the spring, the larger its *stretch constant* k.

Within the limits of this experiment, the spring exerts an equal and opposite force $-F$, as the reaction to the applied force $+F$. For the spring, $-F = kx$, and we can write

$$F = -kx \tag{11k}$$

Table 11A RECORDED DATA FOR STRETCHING A COILED SPRING

$\frac{F}{\text{N}}$	$\frac{x}{\text{m}}$
0	0
2	0.0125
4	0.0250
6	0.0375
8	0.0500
10	0.0625

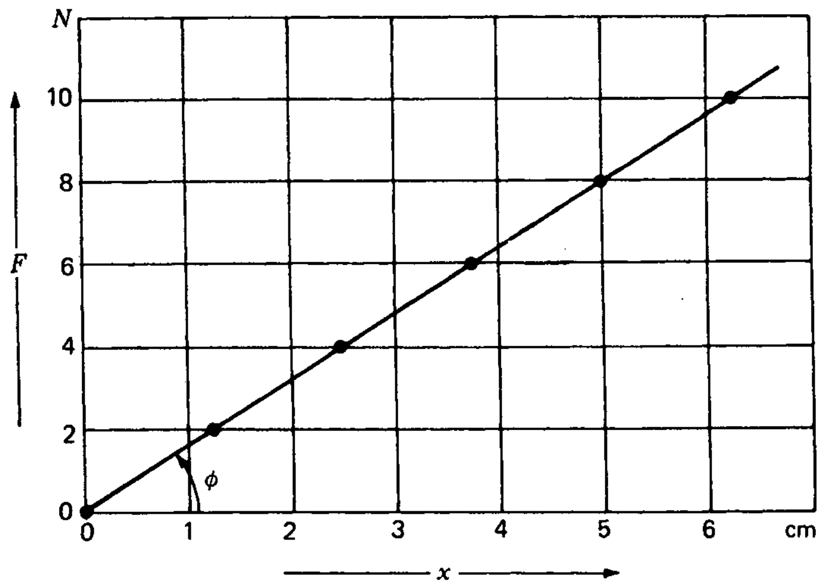

FIGURE 11D
Experimental results on the stretching of a coiled spring as shown in Fig. 11C. This is a demonstration of Hooke's law.

The fact that we obtain a straight line graph in Fig. 11D shows that the stretching of a spring obeys *Hooke's law.** This is typical of nearly all elastic bodies as long as the body is not permanently deformed, indicating that the forces applied had been carried beyond the *elastic limit.*

Since the *work done* in stretching the spring is given by the *force* multiplied by the *distance* and the force here varies linearly with the distance,

$$\text{Work} = \int F\,dx \qquad (11l)$$

As can be seen in Fig. 11E, the *average force* is given by $\frac{1}{2}F$. This, multiplied by the distance x through which it acts, gives the area under the curve, which is the work done†

$$W = \tfrac{1}{2}Fx \qquad (11m)$$

If we now replace F by its equivalent value kx from Eq. (11j), we obtain

$$W = \tfrac{1}{2}kx^2 \qquad (11n)$$

* Robert Hooke (1635–1703), English experimental physicist, is known principally for his contributions to the wave theory of light, universal gravitation, and atmospheric pressure. He originated many physical ideas but perfected few of them. Hooke's scientific achievements would undoubtedly have received greater acclaim if his efforts had been confined to fewer subjects. He had an irritable temper and made many virulent attacks on Newton and other men of science, claiming that work published by them was due to him.

† In most elementary physics texts it is shown that the area under the curve of a graph, where F is plotted against x, is equal to the total work done.

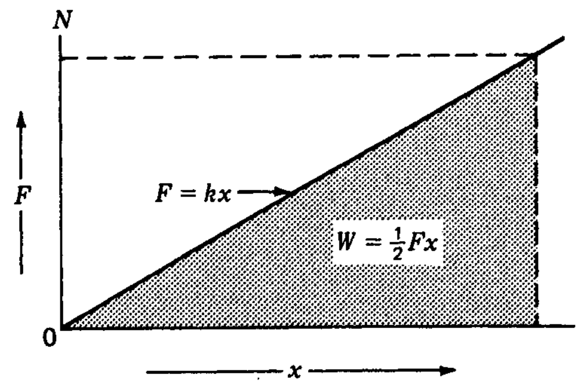

FIGURE 11E
The work done and the energy stored in stretching a spring are given by the area under the graph line $F = kx$.

This relation shows that if the stretch of a spring increases twofold, the energy required, or stored, is increased fourfold, and increasing the displacement threefold increases the energy ninefold.

11.4 VIBRATING SPRING

All bodies in nature are elastic, some more so than others. If a distorting force is applied to change the shape of a body and its shape is not permanently altered, upon release of the force it will be set in vibration.

This property is demonstrated in Fig. 11F by a mass m suspended from the lower end of a spring. In diagram (*a*) a force F has been applied to stretch the spring a distance a. Upon release, the mass moves up and down with SHM. In diagram (*c*), m is at its highest point and the spring is shown compressed. The amplitude of the vibration is determined by the distance the spring is stretched from its equilibrium position, and the period of vibration T is given by

• $$T = 2\pi\sqrt{\frac{m}{k}} \qquad (11o)$$

where k is the stiffness of the spring and m is the mass of the vibrating body. This equation shows that if a stiffer spring is used, k being in the denominator, the period is decreased and the vibration frequency is increased. If the mass m is increased, the period is increased and the frequency is decreased.

Since the stretching of the spring obeys Hooke's law, we can apply Eq. (11k). Using the *force equation* from mechanics,

$$F = ma$$

and replacing F in Eq. (11k) by ma, we obtain

$$ma = -kx \qquad \text{or} \qquad \frac{-x}{a} = \frac{m}{k} \qquad (11p)$$

Hence by the replacement of $-x/a$ by m/k in Eq. (11i) we obtain Eq. (11o).

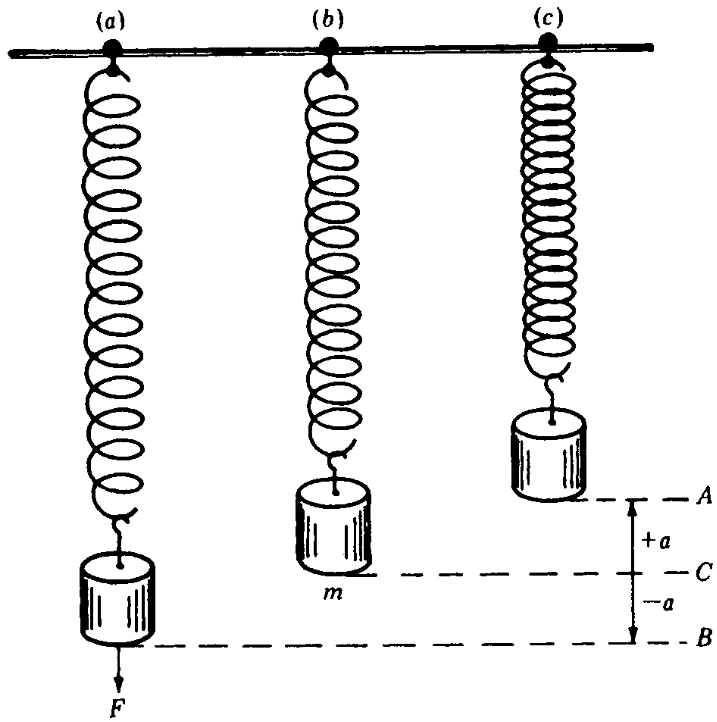

FIGURE 11F
A mass m suspended from a coiled spring is shown in three positions as it vibrates up and down with simple harmonic motion.

EXAMPLE 1 If a 4.0-kg mass is suspended from the lower end of a coiled spring, as shown in Fig. 11F, it stretches a distance of 18.0 cm. If the spring is then extended farther and released, it will be set vibrating up and down with SHM. Find (*a*) the spring constant k, (*b*) the period T, (*c*) the frequency ν, and (*d*) the *total energy* stored in the vibrating system.

SOLUTION The given quantities in the mks system of units are $m = 4.0$ kg, $x = 0.180$ m; the acceleration due to gravity is $g = 9.80\ \text{m/s}^2$.

(*a*) We can use Eq. (11k), solve for the value of k, and substitute the appropriate values:

$$k = \frac{-F}{x} = \frac{4.0 \times 9.80}{0.180} = 217.8\ \text{N/m}$$

(*b*) We can use Eq. (11o), and upon direct substitution of the known values obtain

$$T = 2\pi\sqrt{\frac{m}{k}} \qquad T = 2\pi\sqrt{\frac{4.0\ \text{kg}}{217.8\ \text{N/m}}}$$

$$T = 0.852\ \text{s}$$

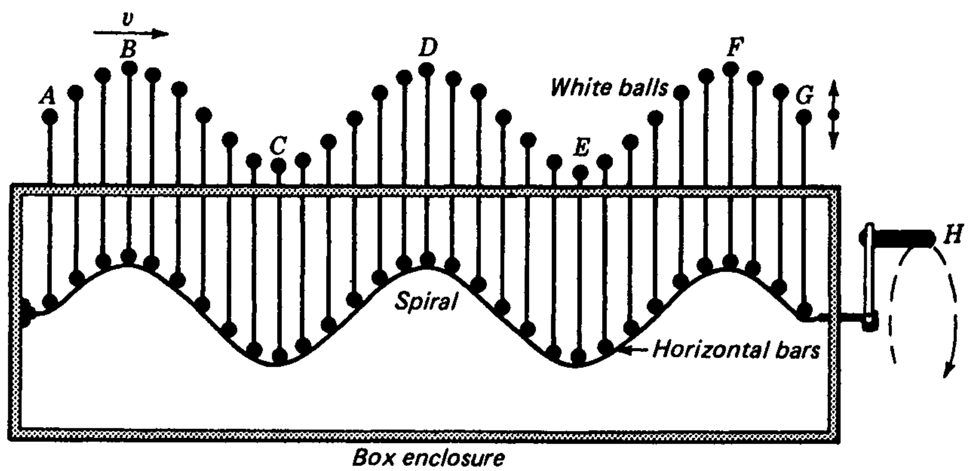

FIGURE 11G
Machine for demonstrating transverse waves.

(*c*) Since the frequency is the reciprocal of the period,

$$\nu = \frac{1}{T} = \frac{1}{0.852} = 1.174\ \text{Hz}$$

(*d*) The total energy stored in the vibrating system is given by Eq. (11n). By substitution of the given quantities we obtain

$$W = \tfrac{1}{2}kx^2 = \tfrac{1}{2}[(217.8)(0.180)^2] = 3.528\ \text{N m} = 3.528\ \text{J}$$

This answer is read three point five two eight joules.

11.5 TRANSVERSE WAVES

All light waves are classified as *transverse waves.* Transverse waves are those in which each small part of the wave vibrates along a line perpendicular to the direction of propagation and all parts are vibrating in the same plane. A wave machine for demonstrating transverse waves is shown in Fig. 11G. When the handle H is turned clockwise the small white balls at the top of the vertical rods move up and down with SHM. As each ball moves along a vertical line, the wave form *ABCDEFG* moves to the right. When the handle is turned counterclockwise, the wave form moves to the left. In either case each ball performs the exact same motion along its line of vibration, the difference being that each ball is slightly behind or ahead of its neighbor.

When a source vibrates with SHM and sends out transverse waves through a homogeneous medium, they have the general appearance of the waves shown in Fig. 11H. The distance between two similar points of any two consecutive wave forms is called the *wavelength* λ. One wavelength, for example, is equal to the distance between two *wave crests* or two *wave troughs.*

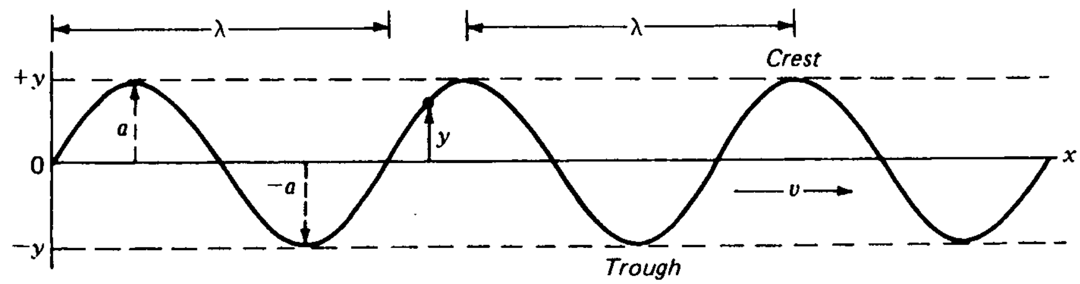

FIGURE 11H
Diagram of a transverse wave, vibrating in the plane of the page, showing the wavelength λ, the amplitude a, the displacement y, and the speed v.

The displacement y of any given point along a wave, at any given instant in time, is given by the vertical distance of that point from its equilibrium position. The value is continually changing from + to − to +, etc. The amplitude of any wave is given by the letter a in Fig. 11H, and is defined as the *maximum value of the displacement y*.

The frequency of a train of waves is given by the number of waves passing by, or arriving at, any given point per second, and is specified in *hertz*, or in vibrations per second. From the definition of frequency ν and the wavelength λ, the speed of the waves v is given by the wave equation

$$v = \nu\lambda \qquad (11q)$$

The length of one wave times the number of waves per second equals the distance the waves will travel in 1 s.

11.6 SINE WAVES

The simplest kind of wave train is that for which the motions of all points along the wave have displacements y given by the *sine* or *cosine* of some uniformly increasing function. This in effect describes what we have called SHM.

Consider transverse waves in which the motions of all parts are perpendicular to the direction of propagation. The displacement y of any point on the wave is then given by

$$y = a \sin \frac{2\pi x}{\lambda} \qquad (11r)$$

A graph of this equation is shown in Fig. 11I, and the significance of the constants a and λ is clear. To make the wave move to the right with a velocity v, we introduce the time t as follows:

$$y = a \sin \frac{2\pi}{\lambda}(x - vt) \qquad (11s)$$

Any particle of the wave, such as P in the diagram, will carry out SHM and will occupy successive positions P, P', P'', P''', etc., as the wave moves.

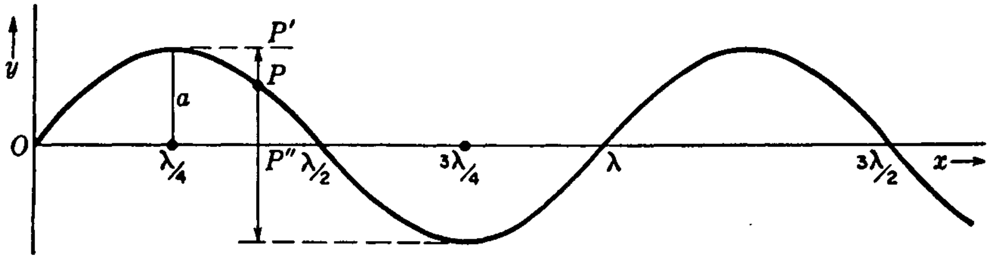

FIGURE 11I
Contour of a sine wave at time $t = 0$.

The time for one complete vibration of any one point is the same as any other point. Furthermore, the period T and its reciprocal the frequency ν are given by the wave equation (11q):

$$v = \nu\lambda = \frac{\lambda}{T} \qquad (11t)$$

If we substitute several of these variables in Eq. (11s), we can obtain useful equations for wave motion in general:

$$y = a \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda}\right)$$
$$y = a \sin \frac{2\pi}{T}\left(t - \frac{x}{v}\right) \qquad (11u)$$
$$y = a \sin 2\pi\nu \left(t - \frac{x}{v}\right)$$

11.7 PHASE ANGLES

In wave motion the instantaneous displacement and direction of propagation are described by specifying the position of the graph point on the circle of reference (see Fig. 11J). The angle θ, measured counterclockwise from the $+x$ axis, specifying the position is called the *phase angle*. As an example consider a point moving up and down along the y axis, as shown in Fig. 11J. The position of the mass point P is given by the projection of the graph point p on the y axis. From the right triangle PpC on the diagram

$$y = a \sin \theta \qquad (11v)$$

With the graph point moving at constant speed v, the angular speed ω is constant, and we can write for any angle θ

$$\theta = \omega t$$

Substitution in Eq. (11v) gives

$$y = a \sin \omega t \qquad (11w)$$

At time $t = 0$ the graph point is at $+p_0$ and the mass point is at P_0. At some later time t when the mass point is at P, the graph point is at p and we must modify Eq. (11w) by adding the angle α as follows:

$$y = a \sin (\omega t + \alpha) \qquad (11x)$$

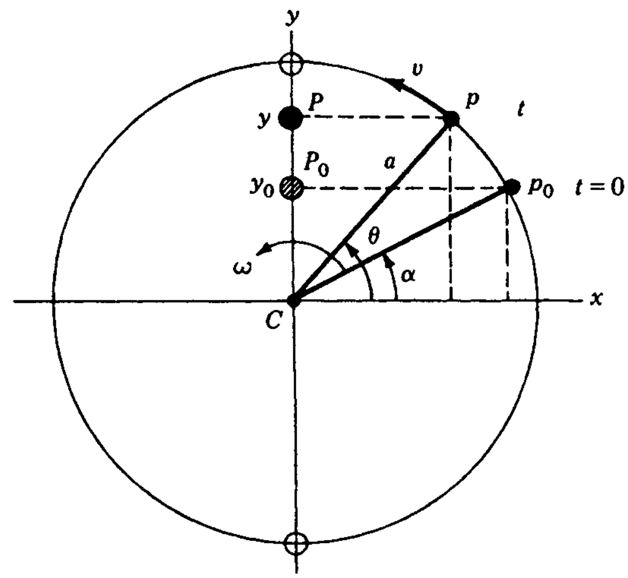

FIGURE 11J
Simple harmonic motion along the y axis, showing the circle of reference, the initial phase angle α, the angular speed ω, and the point P_0 at time $t = 0$.

The angle α is a constant and is called the *initial phase angle*. As the point p moves around the circle, the angle ωt increases at a uniform rate and is always measured from the starting angle α. The total quantity in parentheses is the total angle measured from the $+x$ axis.

It is customary to express all angles in *radian measure* rather than in degrees.

EXAMPLE 2 A given point is vibrating with SHM with a period of 5.0 s and an amplitude of 3.0 cm. If the initial phase angle is $\pi/3$ rad, (60°), find (*a*) the initial displacement and (*b*) the displacement after 12.0 s. (*c*) Make a graph.

SOLUTION (*a*) Since the graph point makes one revolution in 5.0 s, the angular speed ω is 2π rad in 5.0 s, or $2\pi/5$ rad/s [see Eq. (11f)]. At the time $t = 0$, direct substitution in Eq. (11x) gives

$$y = 3 \sin \left(\frac{2\pi}{5} 0 + \frac{\pi}{3} \right)$$

(*b*) After 12.0 s, substitution in Eq. (11x) gives

$$y = 3 \sin \left(\frac{2\pi}{5} 12 + \frac{\pi}{3} \right)$$

$$= 3 \sin \left(4.8\,\pi + \frac{\pi}{3} \right)$$

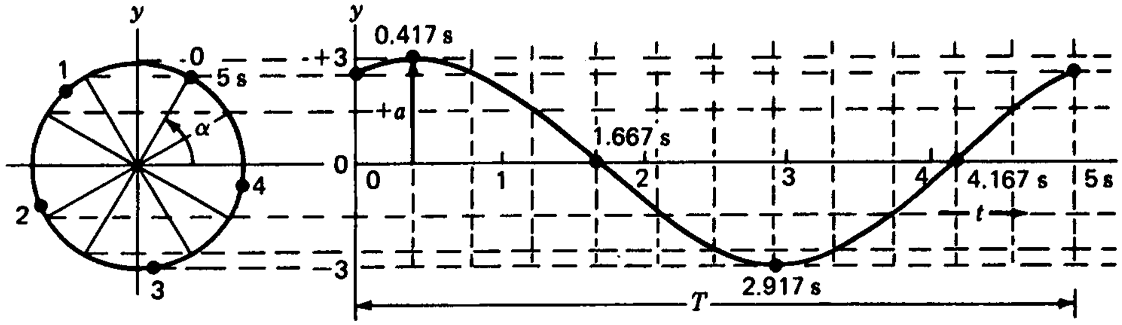

FIGURE 11K
Graph for the example $T = 5.0$ s, $a = 3.0$ cm, and $\alpha = \pi/3$ rad.

The total phase angle of $4.8\pi + \pi/3$ is equivalent to $864° + 60°$, or $924°$, and measured from the $+x$ axis places the graph point 24° below the $-x$ axis on the circle of reference. This angle gives

$$\sin 24° = 0.407$$

and

$$y = 3\,(-0.407)$$

or

$$y = -1.220\,\text{cm}$$

A plot of this example is shown in Fig. 11K. The time T is plotted horizontally, and the displacement is plotted vertically for the first complete vibration, or 5.0 s. The up-and-down motion is traced out to show the starting point and initial phase angle and the time when the motion reaches its first maximum and minimum displacement and when the displacement is zero. The amplitude $a = 3.0$ cm is seen near the left side and is equal to the radius of the circle of reference.

A useful and concise way of expressing the equation for simple harmonic waves is in terms of the *angular frequency* $\omega = 2\pi\nu$ and the *propagation number* $k = 2\pi/\lambda$. Equation (11u) then becomes

$$y = a \sin (kx - \omega t) = a \sin (\omega t - kx + \pi)$$
$$= a \cos \left(\omega t - kx + \frac{\pi}{2}\right)$$

The addition of a constant to the quantity in parentheses is of little physical significance, since such a constant can be eliminated by suitably adjusting the zero of the time scale. Thus the equations when written

$$y = a \cos (\omega t - kx) \qquad \text{and} \qquad y = a \sin (\omega t - kx) \tag{11y}$$

will describe the wave of Fig. 11I, if the curve applies at times $t = T/4$ and $T/2$, respectively, instead of at $t = 0$.

11.8 PHASE VELOCITY AND WAVE VELOCITY

It is now possible to state more precisely what actually moves with a wave. The discussion given in connection with Fig. 11K may be summed up by saying that a wave constitutes the progression of a condition of constant phase. This condition might be, for instance, the crest of the wave, where the phase is such as to yield the maximum upward displacement. The speed with which a crest moves along is usually called the wave velocity, although the more specific term phase velocity is sometimes used. That it is identical with the quantity v in our previous equations is shown by evaluating the rate of change of the x coordinate under the condition that the phase remain constant. When the form of the phase in Eq. (11y) is used, the latter requirement becomes

$$\omega t - kx = \text{const}$$

and the wave velocity

$$v = \frac{dx}{dt} = \frac{\omega}{k} \tag{11z}$$

Substitution of $\omega = 2\pi\nu$ and $k = 2\pi/\lambda$ gives agreement with Eq. (11q). For a wave traveling toward $-x$, the constant phase takes the form $\omega t + kx$, and the corresponding $v = -\omega/k$.

The ratio ω/k for a given kind of wave depends on the physical properties of the medium in which the waves travel and also, in general, on the frequency ω itself. For transverse elastic waves involving distortions small enough for the forces to obey Hooke's law, the wave velocity is independent of frequency and is given simply as

$$v = \sqrt{\frac{N}{\rho}} \tag{11za}$$

N being the shear modulus and ρ the density. The proof of this relation is not difficult. From Fig. 11L it will be seen that the sheet of small thickness δx is sheared through the angle α. The shear modulus is the constant ratio of stress to strain. The strain is measured by tan α, so that

$$\text{Strain} = \frac{\delta f}{\delta x}$$

where f is the function giving the shape of the wave at a particular instant. The stress is the tangential force F per unit area acting on the surface of the sheet, and this by Hooke's law must equal the product of the shear modulus and the strain, so that

$$\text{Stress} = F_x = N\frac{\delta f}{\delta x}$$

Because of the curvature of the wave, the stress will vary with x, and the force acting on the left side of the sheet will not be exactly balanced by the force acting on its right side. The net force per unit area is

$$F_x - F_{x+\delta x} = \frac{\partial F}{\partial x}\,\delta x = N\frac{\partial^2 f}{\partial x^2}\,\delta x$$

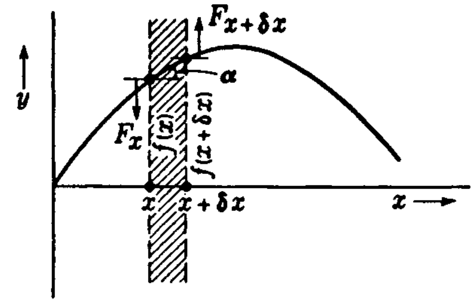

FIGURE 11L
The geometry and mechanics for the shear involved in a transverse wave.

We now apply Newton's second law of motion, equating this force to the product of the mass and the acceleration of unit area of the sheet:

$$N \frac{\partial^2 f}{\partial x^2} \delta x = \rho \, \delta x \frac{\partial^2 f}{\partial t^2}$$

From the fact that they can be polarized (Chap. 24), light waves are known to be transverse waves. Measurements show that their velocity in a vacuum is approximately 3×10^{10} cm/s. If one assumes them to be elastic waves, as was commonly done in the nineteenth century, the question arises: What medium transmits them? In the early *elastic-solid theory*, a medium called the "ether", having the property of a high ratio of rigidity to density, was assumed to occupy all space. Its density was supposed to increase in material substances to account for the lower velocity. There are obvious objections to such a hypothesis. For example, in spite of its resistance to shear, which had to be postulated because light waves are transverse, the ether produces no detectable effects on the motions of astronomical bodies. All the difficulties disappeared when Maxwell developed the present *electromagnetic theory* of light (Chap. 20). Here the mechanical displacement of an element of the medium is replaced by a variation of the electric field (or more generally of the dielectric displacement) at the corresponding point.

The elastic-solid theory was successful in explaining a number of properties of light. There are many parallelisms in the two theories, and much of the mathematics of the earlier theory can be rewritten in electromagnetic terms without difficulty. Consequently, we shall frequently find mechanical analogies useful in understanding the behavior of light. In fact, for the material presented in the next seven chapters, it is immaterial what type of waves are assumed.

11.9 AMPLITUDE AND INTENSITY

Waves transport energy, and the amount of it that flows per second across unit area perpendicular to the direction of travel is called the *intensity* of the wave. If the wave flows continuously with the velocity v, there is a definite energy density, or total energy per unit volume. All the energy contained in a column of the medium of unit cross section and of length v will pass through the unit of area in 1 s. Thus the intensity is given by the product of v and the energy density. Either the energy density or the intensity is proportional to the square of the amplitude and to the square of the frequency. To prove this proposition for sine waves in an elastic medium, it is necessary only to determine the vibrational energy of a single particle executing simple harmonic motion.

Consider for example the particle P in Fig. 11I. At the time for which the figure is drawn, it is moving upward and possesses both kinetic and potential energy. A little later it will have the position P'. Here it is instantaneously at rest, with zero kinetic energy and the maximum potential energy. As it subsequently moves downward, it gains kinetic energy, while the potential energy decreases in such a way that the total energy stays constant. When it reaches the center, at P'', the energy is all kinetic. Hence we may find the total energy either from the maximum potential energy at P' or from the maximum kinetic energy at P''. The latter procedure gives the desired result most easily.

According to Eq. (11y), the displacement of a particular particle varies with time according to the relation

$$y = a \sin (\omega t - \alpha)$$

where α is the value of kx for that particle. The velocity of the particle is

$$\frac{dy}{dt} = \omega a \cos (\omega t - \alpha)$$

When $y = 0$, the sine vanishes and the cosine has its maximum value. Then the velocity becomes $-\omega a$, and the maximum kinetic energy

$$\tfrac{1}{2}m\left[\frac{dy}{dt}\right]^2_{\max} = \tfrac{1}{2}m\omega^2a^2$$

Since this is also the total energy of the particle and is proportional to the energy per unit volume, it follows that

● $$\text{Energy density} \approx \omega^2 a^2 \qquad (11zb)$$

The intensity, v times this quantity, will then also be proportional to ω^2 and a^2.

In spherical waves, the intensity decreases as the inverse square of the distance from the source. This law follows directly from the fact that, provided there is no conversion of the energy into other forms, the same amount must pass through any sphere with the source as its center. Since the area of a sphere increases as the square of its radius, the energy per unit area at a distance r from the source, or the intensity, will vary as $1/r^2$. The amplitude must then vary as $1/r$, and one may write the equation of a spherical wave as

$$y = \frac{a}{r} \sin (\omega t - kr) \qquad (11zc)$$

Here a means the amplitude at unit distance from the source.

If any of the energy is transformed to heat, i.e., if there is *absorption*, the amplitude and intensity of plane waves will not be constant but will decrease as the wave passes through the medium. Similarly with spherical waves, the loss of intensity will be more rapid than is required by the inverse-square law. For plane waves, the fraction dI/I of the intensity lost in traversing an infinitesimal thickness dx is proportional to dx, so that

$$\frac{dI}{I} = -\alpha\, dx$$

To obtain the decrease in traversing a finite thickness x, the equation is integrated to give

$$\int_0^x \frac{dI}{I} = -\alpha \int_0^x dx$$

Evaluating these definite integrals, we find

● $$I_x = I_0 e^{-\alpha x} \qquad (11zd)$$

This law, which has been attributed to both Bouguer* and Lambert,† we shall refer to as the *exponential law of absorption.* Figure 11M is a plot of the intensity against thickness according to this law for a medium having $\alpha = 0.4$ per centimeter. The wave equations can be modified to take account of absorption by multiplying the amplitude by the factor $e^{-\alpha x/2}$, since the amplitude varies with the square root of the intensity.

For light, the intensity can be expressed in joules per square meter per second. Full sunlight, for example, has an intensity in these units of about 1.4×10^3. Here it is important to realize that not all this energy flux affects the eye, and not all that does is equally efficient. Hence the intensity as defined above does not necessarily correspond to the sensation of brightness, and it is more usual to find light flux expressed in visual units.‡ The intensity and the amplitude are the purely physical quantities, however, and according to modern theory the latter must be expressed in electrical units. Thus it may be shown that according to equations to be derived in Chap. 20 the amplitude in a beam of sunlight having the above-mentioned value of the intensity represents an electric field strength of 7.3 V/cm and an accompanying magnetic field of 2.4×10^{-7} tesla (T).

The amplitude of light always decreases more or less rapidly with the distance traversed. Only for plane waves traveling in vacuum, such as the light from a star coming through outer space, is it nearly constant. The inverse-square law of intensities may be assumed to hold for a small source in air at distances greater than about 10 times the lateral dimension of the source. Then the finite size of the source causes an error of less than 0.1 percent in computing the intensity, and for laboratory distances the absorption of air may be neglected. In greater thicknesses, however, all "transparent" substances absorb an appreciable fraction of the energy. We shall take up this subject again in some detail in Chap. 22.

* Pierre Bouguer (1698–1758). Royal Professor of Hydrography at Le Havre.

† Johann Lambert (1728–1777). German physicist, astronomer, and mathematician. Worked primarily in the field of heat radiation. Another law, which is always called Lambert's law, refers to the variation with angle of the radiation from a surface.

‡ See, for example, F. W. Sears, "Principles of Physics," vol. 3, "Optics," 3d ed., chap. 13, Addison-Wesley Publishing Company, Inc., Reading, Mass., 1948.

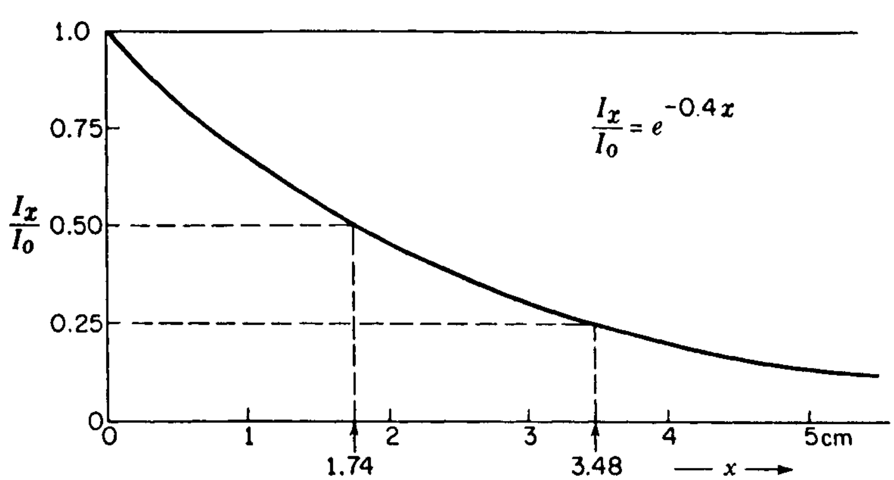

FIGURE 11M
Logarithmic decrease of intensity in an absorbing medium.

11.10 FREQUENCY AND WAVELENGTH

Any wave motion is generated by some sort of vibrating source, and the frequency of the waves is equal to that of the source. The wavelength in a given medium is then determined by the velocity in that medium and by Eq. (11t) is obtained by dividing the velocity by the frequency. Passage from one medium to another causes a change in the wavelength in the same proportion as it does in the velocity, since the frequency is not altered. If we remember that a wave front represents a surface on which the phase of motion is constant, it should be clear that, regardless of any changes of velocity, two different wave fronts are separated by a certain number of waves. That is, the length of any ray between two such surfaces is the same, provided this length is expressed in wavelengths in the appropriate media.

As applied to light, the last statement is equivalent to saying that the optical path is the same along all rays drawn between two wave fronts. For since wavelengths are proportional to velocities, we have

$$\frac{\lambda}{\lambda_m} = \frac{c}{v} = n$$

when the light passes from a vacuum, where it has wavelength λ and velocity c, into a medium where the corresponding quantities are λ_m and v. The optical path corresponding to a distance d in any medium is therefore

$$nd = \frac{\lambda}{\lambda_m} d$$

or the number of wavelengths in that distance multiplied by the wavelength in vacuum. It is customary in optics and spectroscopy to refer to the wavelength of a particular radiation, of a single spectral line, for example, as its wavelength in air under normal conditions. This we shall designate by λ (without subscript), and except in rare circumstances it may be taken as the same as the wavelength in vacuum.

The wavelengths of visible light extend between about 4×10^{-7} m or 400 nm for the extreme violet and 7.2×10^{-7} m or 700 nm for the deep red. Just as the ear becomes insensitive to sound above a certain frequency, so the eye fails to respond to light vibrations of frequencies greater than that of the extreme violet or less than that of the extreme red. The limits, of course, depend somewhat upon the individual, and there is evidence that most persons can see an image with light of wavelength as short as 300 nm, but this is a case of fluorescence in the retina. In this case the light appears to be bluish gray in color and is harmful to the eye. Radiation of wavelength shorter than that of the visible is termed *ultraviolet light* down to a wavelength of about 5 nm and beyond this we are in the region of X rays to 6×10^{-1} nm. Shorter than these, in turn, are the γ rays from radioactive substances. On the long-wavelength side of the visible lies the infrared, which may be said to merge into the radio waves at about 1×10^{6} nm. Figure 11N shows the names which have been given to the various regions of the spectrum of radiation, though we know that no real lines of demarcation exist. It is convenient to use the same units of length throughout such an enormous range. Hence wavelengths are now generally expressed in nanometers (nm) or angstroms (Å) (see Appendix VI).*

It will be seen that visible light covers an almost insignificant fraction of this range. Therefore, although all these radiations are similar in nature, differing only in wavelength, the term *light* is conventionally extended only to the adjacent portions of the spectrum, namely, the ultraviolet and infrared. Many of the results that we shall discuss for light are common to the whole range of radiation, but naturally there are qualitative differences in behavior between the very long and very short waves, which we shall occasionally point out. The divisions between the different types of radiation are purely formal and are roughly fixed by the fact that in the laboratory the different types are generated and detected in different ways. Thus the infrared is emitted copiously by hot bodies and is detected by an energy-measuring instrument such as the thermopile. The shortest radio waves are generated by electric discharges between fine metallic particles immersed in oil and are detected by electrical devices. Nichols and Tear, in 1917, produced infrared waves having wavelengths up to 4.2×10^{5} nm and radio waves down to 2.2×10^{5} nm. The two regions may therefore be said to overlap, keeping in mind, however, that the waves themselves are of the same nature for both. The same holds true for the boundaries of all the other regions of the spectrum.

In sound and other mechanical waves, a change of wavelength occurs when the source has a translational motion. The waves sent out in the direction of motion are shortened, and, in the opposite direction, lengthened. No change is produced in the velocity of the waves themselves; so a stationary observer receives a frequency which is larger or smaller than that of the source. If, on the other hand, the source is at rest and the observer in motion, a change of frequency is also observed, but for a different reason. Here there is no change of wavelength, but the frequency is altered by the change in relative velocity of the waves with respect to the observer. The two cases involve approximately the same change of frequency for the same speed of motion,

* A. J. Ångstrom (1814–1874). Professor of physics at Uppsala, Sweden. Chiefly known for his famous atlas of the solar spectrum, which was used for many years as the standard for wavelength determinations.

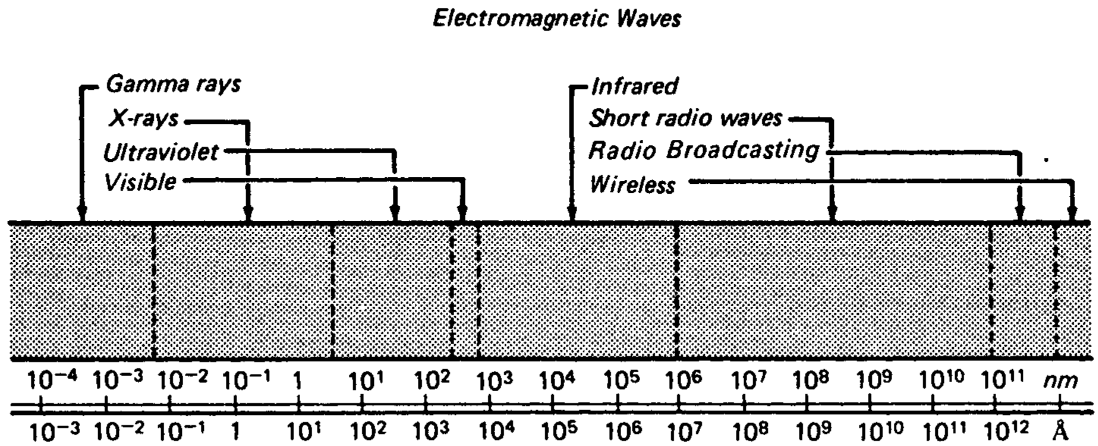

FIGURE 11N
Scale of wavelengths for the range of known electromagnetic waves.

provided this is small compared with the velocity of the waves. These phenomena are known as the *doppler effect** and are most commonly experienced in sound as changes in the acoustic pitch.

Doppler mistakenly attributed the different colors of stars to their motions toward or away from the earth. Because the velocity of light is so large, an appreciable change in color would require that a star have a component of velocity in the line of sight impossibly large compared with the measured velocities at right angles to it. For most stars, the latter usually range between 10 and 30 km/s, with a few as high as 300 km/s. Since light travels at nearly 300,000 km/s, the expected shifts of frequency are small. Furthermore, it makes little difference whether one assumes that the observer or the source is in motion. Suppose that the earth were moving with a velocity u directly toward a fixed star. An observer would then receive u/λ waves in addition to the number $\nu = c/\lambda$ that would reach him if he were at rest. The apparent frequency would be

$$\nu' = \frac{c + u}{\lambda} = \nu\left(1 + \frac{u}{c}\right) \qquad \text{(11ze)}$$

With the velocities mentioned, this would differ from the true frequency by less than 1 part in 1000. A good spectroscope can, however, easily detect and permit the measurement of such a shift as a displacement of the spectrum lines. In fact, this legitimate application of Doppler's principle has become a powerful method of studying the radial velocities of stars. Figure 11O shows an example where the spectrum of μ Cassiopeiae, in the center strip, is compared with the lines of iron from a laboratory source, photographed above and below. All the iron lines also appear in the stellar spectrum as white lines (absorption lines) but are shifted toward the left, i.e., toward

* Christian Johann Doppler (1803–1853). Native of Salzburg, Austria. At the age of thirty-two, unable to secure a position, he was about to emigrate to America. However, at that time he was made professor of mathematics at the Realschule in Prague and later became professor of experimental physics at the University of Vienna.

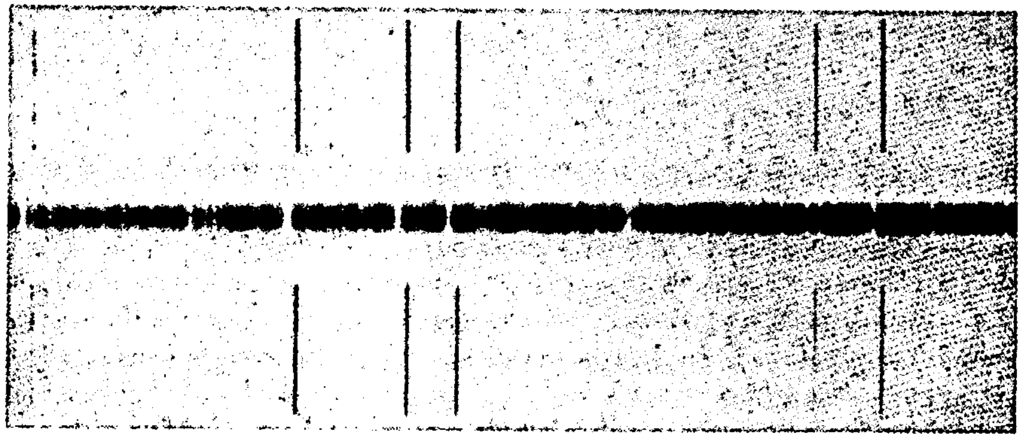

FIGURE 11O
Doppler shift of spectrum lines in a star. Both spectra are negatives. (*Courtesy of McKellar.*)

shorter wavelengths. Measurement shows that the increase of frequency corresponds to a velocity of approach of 115 km/s, which is unusually high for stars in our own galaxy. The spectra of other galaxies (spiral nebulae) all show displacements toward the red, which for the most distant ones amount to several hundred angstroms. Such values would indicate recessional velocities of tens of thousands of kilometers per second, and have been so interpreted. It is rather interesting that here there is enough reddening to change the color of the object, as postulated by Doppler, but in this case it occurs for objects far too faint to be seen by the naked eye.

In the laboratory, there have been found two ways of achieving velocities sufficient to produce detectable doppler shifts. By reflecting light from mirrors mounted on the rim of a wheel rotating at high speed, one can produce speeds of a virtual source as high as 400 m/s. Much larger values are attained by beams of atoms moving in vacuum, as will be discussed later in Sec. 19.15. There, it is also shown that with the abandonment of the material ether necessitated by relativity theory the distinction between the cases of source in motion and of observer in motion disappears. Relativity leads to an equation which is substantially Eq. (11ze) with u representing the relative velocity of approach or recession.

11.11 WAVE PACKETS

No source of waves vibrates indefinitely, as would be required for it to produce a true sine wave. More commonly the vibrations die out because of the dissipation of energy or are interrupted in some way. Then a group of waves of finite length, such as that illustrated in Fig. 11P, is produced. The mathematical representation of a wave packet of this type is rather more complex and will be briefly discussed in the next chapter. Since wave packets are of frequent occurrence, however, some features of their behavior should be mentioned here. In the first place, the wavelength is

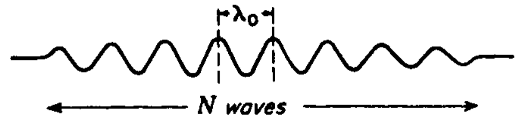

FIGURE 11P
Example of a wave packet.

not well defined. If the packet is sent through any device for measuring wavelengths, e.g., light through a diffraction grating, it will be found to yield a continuous spread over a certain range $\Delta\lambda$. The maximum intensity will occur at the value of λ_0 indicated in Fig. 11P, but energy will appear in other wavelengths, the intensity dying off more or less rapidly on either side of λ_0. The larger the number N of waves in the group, the smaller the spread $\Delta\lambda$, and in fact theory shows that $\Delta\lambda/\lambda_0$ is approximately equal to $1/N$. Hence only when N is very large may we consider the wave to have an accurately defined wavelength.

If the medium through which the packet travels is such that the velocity depends on frequency, two further phenomena will be observed. The individual wave crests will travel with a velocity different from that of the packet as a whole, and the packet will spread out as it progresses. We then have two velocities, the wave (or phase) velocity and the group velocity. The relation between these will be derived in Sec. 12.7.

In light sources, the radiating atoms emit wave trains of finite length. Usually, because of collisions or damping arising from other causes, these packets are very short. According to the theorem mentioned above, the consequence is that the spectrum lines will not be very narrow but will have an appreciable width $\Delta\lambda$. A measurement of this width will yield the effective "lifetime" of the electromagnetic oscillators in the atoms and the average length of the wave packets. A low-pressure discharge through the vapor of mercury containing the single isotope ^{198}Hg yields very sharp spectral lines, of width about 0.005 Å. Taking the wavelength of one of the brightest lines, 5461 Å, we may estimate that there are roughly 10^6 waves in a packet and that the packets themselves are some 50 cm long.

PROBLEMS

11.1 A coil spring hangs from the ceiling as shown in Fig. 11F. When a mass of 50.0 g is fastened to the lower end, the spring is stretched a distance of 15.89 cm. If the mass is now pulled down another 5 cm and released, it will vibrate up and down with SHM. Find (*a*) the spring constant, (*b*) the period of vibration, (*c*) the frequency, (*d*) the angular velocity of a graph point drawn for the vibration, (*e*) the maximum velocity of the mass, and (*f*) its maximum acceleration. (*g*) Plot a graph of the vibration for the time interval $t = 0$ to $t = 3.0$ s if the initial phase angle is 270°. (*h*) Find the time to reach the first maximum and (*i*) the total energy of vibration. (*j*) Write down an equation for the motion.

Ans. (*a*) 30.837 N/m, (*b*) 0.8001 s, (*c*) 1.2499 Hz, (*d*) 5.027 rad/s, (*e*) 0.39265 m/s, (*f*) 0.4754 m/s^2, (*g*) see Fig. P11.1, (*h*) 4.001 s, (*i*) 3.8546 J, (*j*) $y = 0.050 \sin (5.027t + 270°)$ m

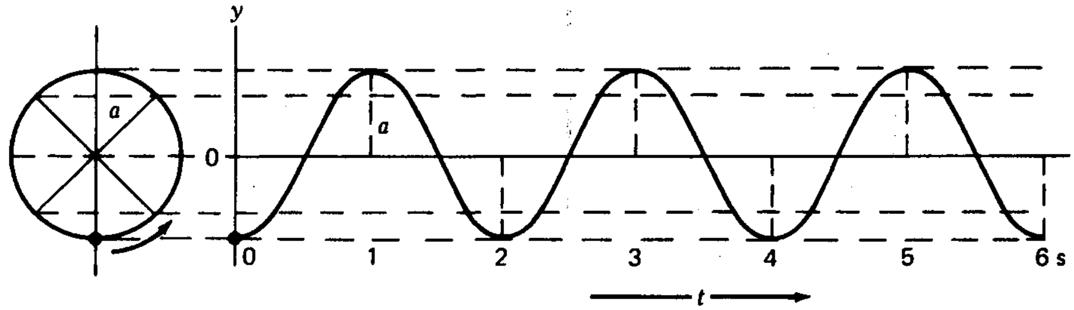

FIGURE P11.1
Graph for part (g) of Prob. 11.1

11.2 A coil spring hangs from the ceiling as shown in Fig. 11F. A mass of 1.60 kg is suspended from the lower end of the spring, stretching it a distance of 12.40 cm. The mass is now pulled down a distance of 4.0 cm more and then released to vibrate in a vertical line. Find (a) the spring constant, (b) the period of vibration, (c) the frequency, (d) the angular velocity of a graph point, (e) the maximum velocity of the mass, and (f) the maximum acceleration. (g) Plot a graph of the vibration for the time interval $t = 0$ to $t = 2.20$ s if the initial phase angle is 225°. (h) Find the time at which the mass first reaches its highest point and (i) the total energy. (j) Write down an equation for the motion.

11.3 A wave is specified by $y = 6 \sin 2\pi(8t - 4x + \frac{3}{4})$. Find ($a$) the amplitude, ($b$) wavelength, ($c$) frequency, ($d$) initial phase angle, and (e) the initial displacement at time $t = 0$ and $x = 0$.

11.4 A wave is specified by $y = 15 \sin 2\pi(4t - 5x + \frac{2}{3})$. Find ($a$) the amplitude, ($b$) the wavelength, ($c$) the frequency, ($d$) the initial phase angle, and (e) the displacement at time $t = 0$ and $x = 0$. *Ans.* (a) 15, (b) $\frac{1}{5}$, (c) 4, (d) 240°, (e) -13.0

12

THE SUPERPOSITION OF WAVES

When two sets of waves are made to cross each other, e.g., the waves created by dropping two stones simultaneously in a quiet pool, interesting and complicated effects are observed. In the region of crossing there are places where the disturbance is practically zero and others where it is greater than that given by either wave alone. A very simple law can be used to explain these effects, which states that the resultant displacement of any point is merely the sum of the displacements due to each wave separately. This is known as the *principle of superposition* and was first clearly stated by Young* in 1802. The truth of this principle is at once evident when we observe that after the waves have passed out of the region of crossing, they appear to have been entirely uninfluenced by the other set of waves. Amplitude, frequency, and all other characteristics are just as if they had crossed an undisturbed space. This could hold only provided the principle of superposition were true. Two different observers can

* Thomas Young (1773–1829). English physician and physicist, usually called the founder of the wave theory of light. An extremely precocious child (he had read the Bible twice through at the age of four), he developed into a brilliant investigator. His work on interference constituted the most important contribution on light since Newton. His early work proved the wave nature of light but was not taken seriously by others until it was corroborated by Fresnel.

see different objects through the same aperture with perfect clearness, whereas the light reaching the two observers had crossed in going through the aperture. The principle is therefore applicable with great precision to light, and we can use it in investigating the disturbance in regions where two or more light waves are superimposed.

12.1 ADDITION OF SIMPLE HARMONIC MOTIONS ALONG THE SAME LINE

Considering first the effect of superimposing two sine waves of the same frequency, the problem resolves itself into finding the resultant motion when a particle executes two simple harmonic motions at the same time. The displacements due to the two waves are here taken to be along the same line, which we shall call the y direction. If the amplitudes of the two waves are a_1 and a_2, these will be the amplitudes of the two periodic motions impressed on the particle, and, according to Eq. (11x) of the last chapter, we can write the separate displacements as follows:

$$\begin{aligned} y_1 &= a_1 \sin(\omega t - \alpha_1) \\ y_2 &= a_2 \sin(\omega t - \alpha_2) \end{aligned} \tag{12a}$$

Note that ω is the same for both waves, since we have assumed them to be of the same frequency. According to the principle of superposition, the resultant displacement y is merely the sum of y_1 and y_2, and we have

$$y = a_1 \sin(\omega t - \alpha_1) + a_2 \sin(\omega t - \alpha_2)$$

When the expression for the sine of the difference of two angles is used, this can be written

$$\begin{aligned} y &= a_1 \sin \omega t \cos \alpha_1 - a_1 \cos \omega t \sin \alpha_1 + a_2 \sin \omega t \cos \alpha_2 - a_2 \cos \omega t \sin \alpha_2 \\ &= (a_1 \cos \alpha_1 + a_2 \cos \alpha_2) \sin \omega t - (a_1 \sin \alpha_1 + a_2 \sin \alpha_2) \cos \omega t \end{aligned} \tag{12b}$$

Now since the a's and α's are constants, we are justified in setting

$$\begin{aligned} a_1 \cos \alpha_1 + a_2 \cos \alpha_2 &= A \cos \theta \\ a_1 \sin \alpha_1 + a_2 \sin \alpha_2 &= A \sin \theta \end{aligned} \tag{12c}$$

provided that constant values of A and θ can be found which satisfy these equations. Squaring and adding Eqs. (12c), we have

$$\begin{aligned} A^2(\cos^2 \theta + \sin^2 \theta) = a_1{}^2(\cos^2 \alpha_1 + \sin^2 \alpha_1) &+ a_2{}^2(\cos^2 \alpha_2 + \sin^2 \alpha_2) \\ &+ 2a_1a_2(\cos \alpha_1 \cos \alpha_2 + \sin \alpha_1 \sin \alpha_2) \end{aligned}$$

or

$$A^2 = a_1{}^2 + a_2{}^2 + 2a_1a_2 \cos(\alpha_1 - \alpha_2) \tag{12d}$$

Dividing the lower equation (12c) by the upper one, we obtain

$$\tan \theta = \frac{a_1 \sin \alpha_1 + a_2 \sin \alpha_2}{a_1 \cos \alpha_1 + a_2 \cos \alpha_2} \tag{12e}$$

Equation (12d) and (12e) show that values of A and θ exist which satisfy Eqs. (12c), and we can rewrite Eq. (12b), substituting the right-hand members of Eq. (12c). This gives

$$y = A \cos \theta \sin \omega t - A \sin \theta \cos \omega t$$

which has the form of the sine of the difference of two angles and can be expressed as

$$y = A \sin (\omega t - \theta) \qquad (12f)$$

This equation is the same as either of our original equations for the separate simple harmonic motions but contains a new amplitude A and a new phase constant θ. Hence we have the important result that the sum of two simple harmonic motions of the same frequency and along the same line is also a simple harmonic motion of the same frequency. The amplitude and phase constant of the resultant motion can easily be calculated from those of the component motions by Eqs. (12d) and (12e), respectively.

The addition of three or more simple harmonic motions of the same frequency will likewise give rise to a resultant motion of the same type, since the motions can be added successively, each time giving an equation of the form of Eq. (12f). Unless considerable accuracy is desired, it is usually more convenient to use the graphical method described in the following section. A knowledge of the resultant phase constant θ, given by Eq. (12e), is not of interest unless it is needed in combining the resultant motion with still another.

The resultant amplitude A depends, according to Eq. (12d), upon the amplitudes a_1 and a_2 of the component motions and upon their difference of phase $\delta = \alpha_1 - \alpha_2$. When we bring together two beams of light, as is done in the Michelson interferometer (Sec. 13.8), the intensity of the light at any point will be proportional to the square of the resultant amplitude. By Eq. (12d) we have, in the case where $a_1 = a_2$,

$$I \approx A^2 = 2a^2(1 + \cos \delta) = 4a^2 \cos^2 \frac{\delta}{2} \qquad (12g)$$

If the phase difference is such that $\delta = 0, 2\pi, 4\pi, \ldots$, this gives $4a^2$, or 4 times the intensity of either beam. If $\delta = \pi, 3\pi, 5\pi, \ldots$, the intensity is zero. For intermediate values, the intensity varies between these limits according to the square of the cosine. These modifications of intensity obtained by combining waves are referred to as *interference* effects, and we shall discuss in the next chapter several ways in which they can be brought about and used experimentally.

12.2 VECTOR ADDITION OF AMPLITUDES

A very simple geometrical construction can be used to find the resultant amplitude and phase constant of the combined motion in the above case of two simple harmonic motions along the same line. If we represent the amplitudes a_1 and a_2 by vectors making angles α_1 and α_2 with the x axis,* as in Fig. 12A(a), the resultant amplitude A

* Here we depart from the usual convention of measuring positive angles in the counterclockwise direction, because it is customary in optics to represent an advance of phase by a clockwise rotation of the amplitude vector.

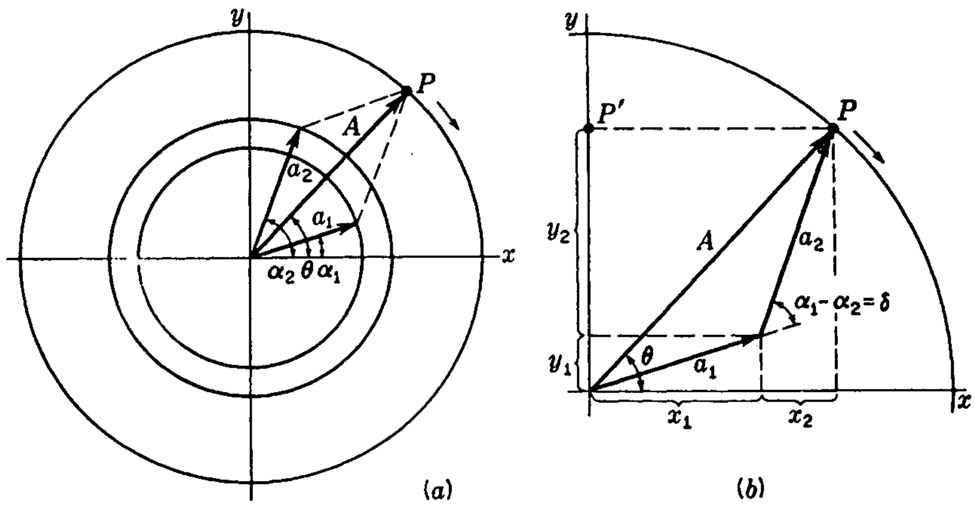

FIGURE 12A
Graphical composition of two waves of the same frequency, but different amplitude and phase.

is the vector sum of a_1 and a_2 and makes an angle θ with that axis. To prove this proposition, we first note from Fig. 12A(b) that in the triangle formed by a_1, a_2, and A the law of cosines gives

$$A^2 = a_1{}^2 + a_2{}^2 - 2a_1a_2 \cos\left[\pi - (\alpha_1 - \alpha_2)\right] \qquad \text{(12h)}$$

which readily reduces to Eq. (12d). Furthermore, Eq. (12e) is obtained at once from the fact that the tangent of the angle θ is the quotient of the sum of the projections of a_1 and a_2 on the y axis by the sum of their projections on the x axis.

That the resultant motion is also simple harmonic can be concluded if we remember that this type of motion may be represented as the projection on one of the coordinate axes of a point moving with uniform circular motion. Figure 12A is drawn for the time $t = 0$, and as time progresses, the displacements y_1 and y_2 will be given by the vertical components of the vectors a_1 and a_2, if the latter revolve clockwise with the same angular velocity ω. The resultant, A, will then have the same angular velocity, and the projection P' of its terminus P will undergo the resultant motion. If one imagines the vector triangle in part (*b*) of the figure to revolve as a rigid frame, it will be seen that the motion of P' will agree with Eq. (12f).

The graphical method is particularly useful where we have more than two motions to compound. Figure 12B shows the result of adding five motions of equal amplitudes a and having equal phase differences δ. Clearly the intensity $I = A^2$ can here vary between zero and $25a^2$, according to the phase difference δ. This is the problem which arises in finding the intensity pattern from a diffraction grating, as discussed in Chap. 17. The five equal amplitudes shown in the figure might be contributed by five apertures of a grating, an instrument which has as its primary purpose the introduction of an equal phase difference in the light from each successive pair of apertures. It will be noted that as Fig. 12B is drawn, the vibrations, starting with that at the origin, lag successively farther *behind* in phase.

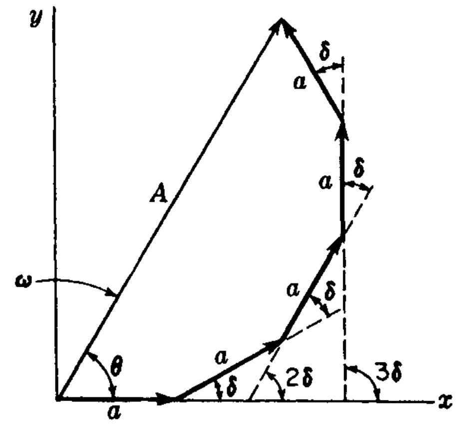

FIGURE 12B
Vector addition of five amplitudes having the same magnitude and phase difference δ.

Either the trigonometric or graphical methods for compounding vibrations may be used to find the resultant of any number of motions with given amplitudes and phases. It is even possible, as we shall see, to apply these methods to the addition of infinitesimal vibrations, so that the summations become integrations. In such cases, and especially if the amplitudes of the individual contributions vary, it is simpler to use a method of adding the amplitudes as complex numbers. We shall take up this method in Sec. 14.8, where it first becomes necessary.

12.3 SUPERPOSITION OF TWO WAVE TRAINS OF THE SAME FREQUENCY

From the preceding section we can conclude directly that the result of superimposing two trains of sine waves of the same frequency and traveling along the same line is to produce another sine wave of that frequency but having a new amplitude which is determined for given values of a_1 and a_2 by the phase difference δ between the motions imparted to any particle by the two waves. As an example, let us find the resultant wave produced by two waves of equal frequency and amplitude traveling in the same direction $+x$, but with one a distance Δ ahead of the other. The equations of the two waves, in the form of Eq. (11y), will be

$$y_1 = a \sin (\omega t - kx) \tag{12i}$$

$$y_2 = a \sin [\omega t - k(x + \Delta)] \tag{12j}$$

By the principle of superposition, the resultant displacement is the sum of the separate ones, so that

$$y = y_1 + y_2 = a\{\sin (\omega t - kx) + \sin [\omega t - k(x + \Delta)]\}$$

Applying the trigonometric formula

$$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B) \tag{12k}$$

we find

$$y = 2a \cos \frac{k\Delta}{2} \sin \left[\omega t - k\left(x + \frac{\Delta}{2}\right)\right] \tag{12l}$$

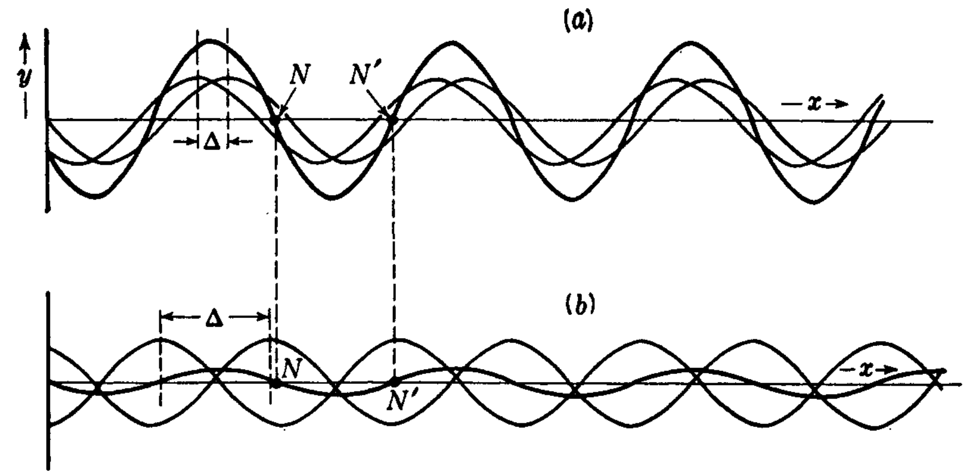

FIGURE 12C
Superposition of two wave trains (*a*) almost in phase and (*b*) almost 180° out of phase.

This corresponds to a new wave of the same frequency but with the amplitude $2a \cos (k\Delta/2) = 2a \cos (\pi\Delta/\lambda)$. When Δ is a small fraction of a wavelength, this amplitude will be nearly $2a$, while if Δ is in the neighborhood of $\frac{1}{2}\lambda$, it will be practically zero. These cases are illustrated in Fig. 12C, where the waves represented by Eqs. (12i) and (12j) (light curves) and (12l) (heavy curve) are plotted at the time $t = 0$. In these figures it will be noted that the algebraic sum of the ordinates of the light curves at any value of x equals the ordinate of the heavy curve. The student may easily verify by such graphical construction the facts that the two amplitudes need not necessarily be equal to obtain a sine wave as the resultant and that the addition of any number of waves of the same frequency and wavelength also gives a similar result. In any case, the resultant wave form will have a constant amplitude, since the component waves and their resultant all move with the same velocity and maintain the same relative position. The true state of affairs may be pictured by having all the waves in Fig. 12C move toward the right with a given velocity.

The formation of *standing waves* in a vibrating cord, giving rise to nodes and loops, is an example of the superposition of two wave trains of the same frequency and amplitude but traveling in *opposite* directions. A wave in a cord is reflected from the end, and the direct and reflected waves must be added to obtain the resultant motion of the cord. Two such waves can be represented by the equations

$$y_1 = a \sin (\omega t - kx) \qquad y_2 = a \sin (\omega t + kx)$$

By addition one obtains, in the same manner as for Eq. (12l),

$$y = 2a \cos (-kx) \sin \omega t$$

which represents the standing waves. For any value of x we have simple harmonic motion, whose amplitude varies with x between the limits $2a$ when $kx = 0, \pi, 2\pi, 3\pi, \ldots$ and zero when $kx = \pi/2, 3\pi/2, 5\pi/2, \ldots$. The latter positions correspond to the nodes and are separated by a distance $\Delta x = \pi/k = \lambda/2$. Figure 12C may also serve to illus-

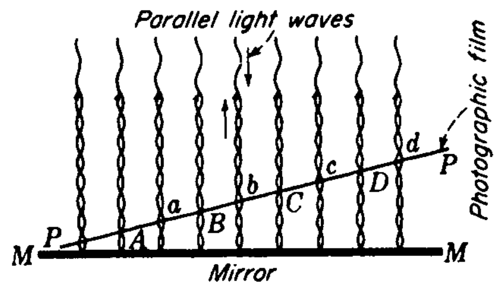

FIGURE 12D
Formation and detection of standing waves in Wiener's experiment.

trate this case if one pictures the two lightly drawn waves as moving in opposite directions. The resultant curve, instead of moving unchanged toward the right, now oscillates between a straight-line position when $\omega t = \pi/2, 3\pi/2, 5\pi/2, \ldots$ and a sine curve of amplitude $2a$ when $\omega t = 0, \pi, 2\pi, \ldots$. At the nodes, such as N and N' in the figure, the resultant displacement is zero at all times.

The standing waves produced by reflecting light at normal incidence from a polished mirror can be observed by means of an experiment due to Wiener,* which is illustrated in Fig. 12D. A specially prepared photographic film only one-thirtieth of a wavelength thick is placed in an inclined position in front of the reflecting surface so that it will cross the nodes and loops successively, as at A, a, B, b, C, c, D, $d, \ldots$. The light will affect the plate only where there is an appreciable amount of vibration and not at all at the nodes. As expected, the developed plate shows a system of dark bands separated by lines of no blackening where it crossed the nodes. Decreasing the angle of inclination of the plate with the reflecting surface causes the bands to move farther apart, since a smaller number of nodal planes are cut in a given distance. Measuring these bands establishes an important fact: the standing waves have a node at the reflecting surface. The phase relations of the direct and reflected waves at this point are therefore such that they continuously annul each other. This is analogous to the reflection of the waves in a rope from a fixed end. Other similar experiments performed by Wiener will be discussed in Sec. 25.12.

12.4 SUPERPOSITION OF MANY WAVES WITH RANDOM PHASES

Suppose that we now consider a large number of wave trains of the same frequency and amplitude to be traveling in the same direction and specify that the amount by which each train is ahead or behind any other is a matter of pure chance. From what has been said above, we can conclude that the resultant wave will be another sine wave of the same frequency, and it becomes of interest to inquire into the amplitude and intensity of this wave. Let the individual amplitudes be a, and let there be n wave trains superimposed. The amplitude of the resultant wave will be the amplitude of motion of a particle undergoing n simple harmonic motions at once, each of amplitude a. If these motions were all in the same phase, the resultant amplitude would be na and

* O. Wiener, *Ann. Phys.*, 40:203 (1890).