Epsilon shitfuckery

Formal (ε) definition of limits

$\lim_{n\to\infty} a_n = L$ if for every $\varepsilon > 0$, there exists an $n \geq N$ $(n \in \mathbb{N})$ such that $|a_n - L| < \varepsilon$

ε-δ definition

$\lim_{x\to a} f(x) = L$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$

Monotone Convergence Theorem

If a sequence is monotonic and bounded, it converges

Ex. Prove $a_{n+1} = \frac{7+a_n}{5}$ converges, if $a_1 = 1$.

Inductive hypothesis: $1 \leq a_n \leq a_{n+1} \leq 5$

$\Rightarrow 8 \leq 7+a_n \leq 7+a_{n+1} \leq 12$

$\Rightarrow \frac{8}{5} \leq \frac{7+a_n}{5} \leq \frac{7+a_{n+1}}{5} \leq \frac{12}{5}$

$\Rightarrow \frac{8}{5} \leq a_{n+1} \leq a_{n+2} \leq \frac{12}{5}$

By induction, this is true for all $n \in \mathbb{N}$.

To find the bounds, substitute $a_n = a_{n+1} = L$:

$a_{n+1} = L \Rightarrow \frac{7+a_n}{5} = L \Rightarrow \frac{7+L}{5} = L$

Limit Solving Strategies

$\lim_{x \to \infty} \frac{\sin x}{x} = 0$ $\lim_{x \to \infty} \frac{\cos x}{x} = 0$ $\lim_{x \to \infty} \frac{\ln(x)}{x^p} = 0$ $(p > 0)$

$\lim_{x \to \infty} \frac{x^p}{e^x} = 0$ $(p > 0)$ $\lim_{x \to 0} \frac{\sin x}{x} = 1$ $\lim_{x \to 0} \frac{\cos x}{x}$ DNE

L'Hopital's rule only applies to $\frac{0}{0}$ and $\frac{\infty}{\infty}$ indeterminate forms and only works if both numerator and denominator exist near limit

Minima / Maxima

Local min/max - min/max point over some subinterval. CANNOT be endpoint.

Global min/max - min/max point over entire interval. Can be endpoint.

To get global min/max, check CPs ($f'(x) = 0$) and endpoints

Taylor Polynomials

$$T_{n,a} = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Taylor Remainder

$R_{n,a}(x) = f(x) - T_{n,a}(x)$ error $= |R_{n,a}(x)|$

Taylor's Theorem

There exists a point c between x and a such that

$$R_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

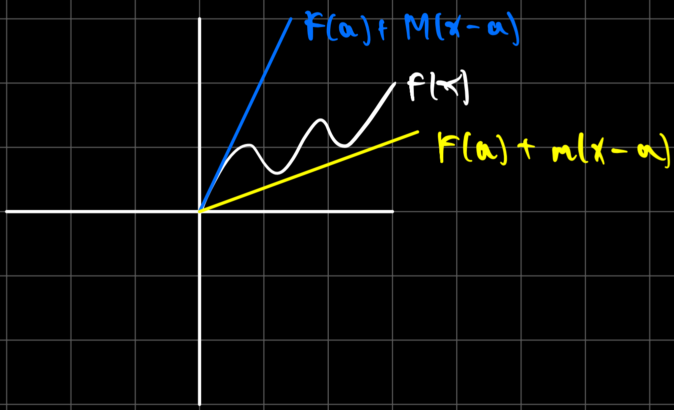
Ex. Approximate $\sqrt{1.1}$.

$$\text{error} = |f(x) - L_a^f(x)| = \frac{M}{2}(x-a)^2$$

Bounded Derivative Theorem

If f is differentiable on $[a, b]$ and $m \leq f'(x) \leq M$, then

$$\underbrace{f(a) + m(x-a)}_{\text{tangent line with lower slope}} \leq f(x) \leq \underbrace{f(a) + M(x-a)}_{\text{tangent line with higher slope}}$$



When finding HAs, get $\lim_{x \to \infty} f(x)$ AND $\lim_{x \to -\infty} f(x)$ $(-\infty)$!!!
don't forget

$0 < |x-3| < 8$

The quick brown fox jumps over the lazy dog

Genna