Let X be a random variable with PDF given by

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad \text{for } -\infty < x < \infty$$

where $-\infty < \mu < \infty$ and $\sigma > 0$ are parameters.

Notation: X ~ N(μ, σ^2)

Where μ is the mean and σ^2 is the variance

We can also write X ~ G(μ, σ)

Where σ is the standard deviation

<u>Effects of changing the mean and variance</u>

Changing the mean moves the top of the curve to the left or the right

Changing the variance either makes the curve:

- Shorter at the mean, more spread out (more variance)
- Taller at the mean, less spread out (less variance)

CDF:

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy$$

But this is hard to integrate without using numerical methods, so we will use a probability table

As such, if X ~ N(μ,σ), then we can transform X as follows:

$$z = \frac{X-\mu}{\sigma}$$

This transforms X into a random variable Z, where Z ~ N(0,1)

Normal distribution can now be a distribution of z-scores

Now, the "observed value" z:

- Has no units
- Represents the number of standard deviations an observation is away from the mean

PDF:

$$f(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}z^2} \longrightarrow \int_{-\infty}^{\infty} f_z(z) = 1$$

The mean of z is equal to its median, since it is a symmetric distribution

As shown by

$$z = \frac{X - \mu}{\sigma}$$

If X > mean, it's to the right

If X < mean, it's to the left

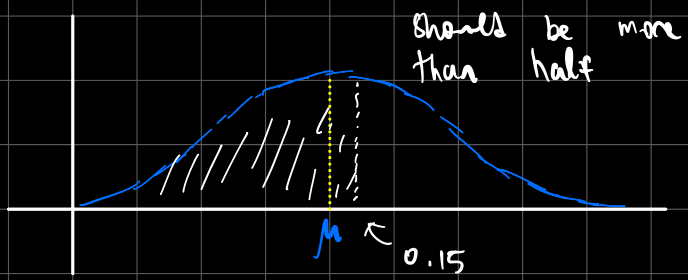
If z=0, X is the mean

68-95-99.7 rule

- 68% of the population lies within one standard deviation of the mean
- 95% lies within two standard deviations
- 99.7% lies within three standard deviations

Draw picture when calculating normal distributions/quantiles

Ex. P(Z < 0.15)

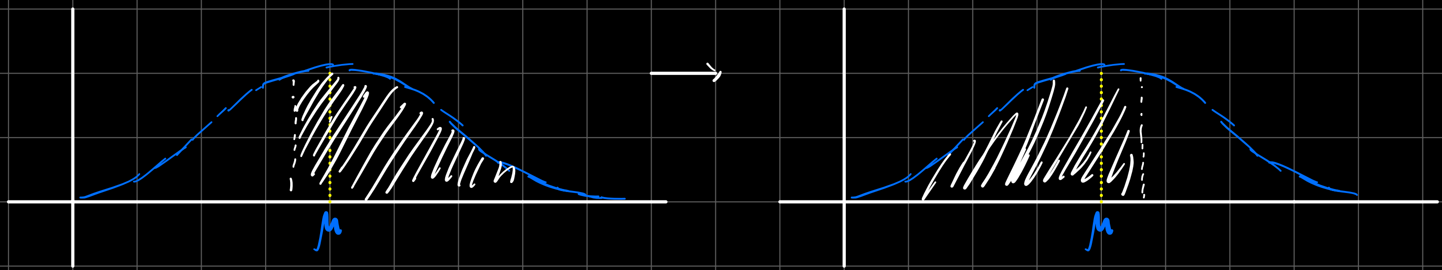


Using the table: 0.59871

Ex. $P(Z < -1.22) = P(Z > 1.22)$ since the normal distribution is symmetric

This is equal to $1 - P(Z \leq 1.22)$

Ex. $P(Z > -0.73)$

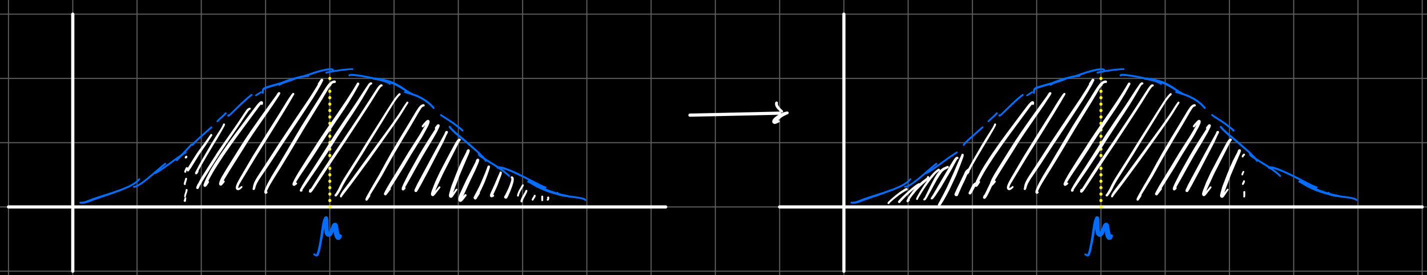


$P(Z > -0.73) = 1 - P(Z \leq -0.73) = 1 - P(Z \geq 0.73) = 1 - (1 - P(Z < 0.73)) = P(Z < 0.73)$

switch sign   symmetry   switch sign

Ex. $P(Z > d) = 0.90$. Find d.

We know that $d < 0$, since the probability that a certain z-score is greater than d is greater than 0.5



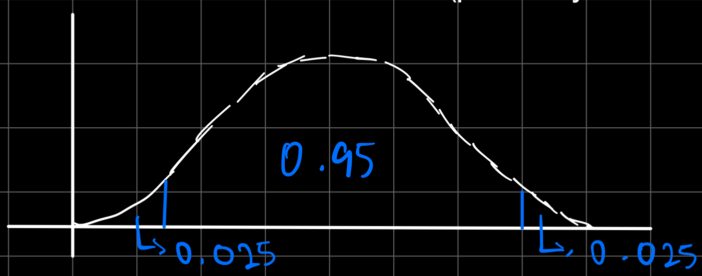
By the quantile table, $P(Z \leq 1.2816) = 0.90$

So $d = -1.2816$

So $P(Z > -1.2816) = 0.90$

Ex. Find a number b such that $P(-b < Z < b) = 0.95$.

- By the 68-95-99.7 rule, 95% of the data is within 2 standard deviations from the mean
- So the answer is around b=2 (probably a little less)



Alternate solution:

The area between b and $+\infty$ is (1-0.95)/2 = 0.025 (since the normal distribution is symmetric)

So P(Z < b) = 0.95+ 0.025 = 0.975

So we can look up the 0.975 quantile: z=1.9600

Thus, b=1.96

And P(-1.96 < Z < 1.96) = 0.95

Ex. Suppose that X ~ N(10,2). Calculate P(|X-10| ≤ 3).

μ σ

Let Y = X-10

E(Y) = E(X) - 10 = 10 - 10 = 0

Var(Y) = Var((X-10)) = Var(X) - Var(10) = Var(X) = 2

So, Y ~ N(0,2). So we now have P(|Y| ≤ 3).

Transform this into something on the tables:

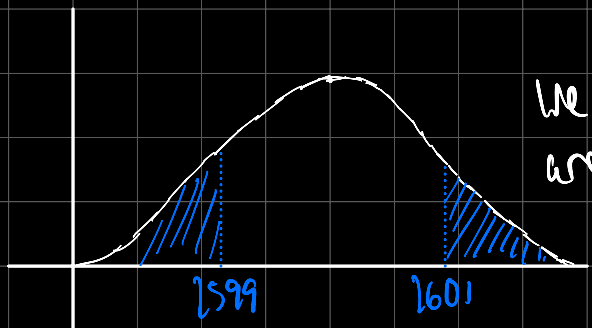
$$P\left(\underbrace{\left|\frac{Y-\mu}{\sigma}\right|}_{Z} \leq \frac{3-0}{\sqrt{2}}\right)$$

$$\begin{aligned}
&= P(|Z| \leq 2.12) \\
&= P(-2.12 < Z < 2.12) \\
&= P(Z < 2.12) - P(Z < -2.12) \\
&= P(Z < 2.12) - P(Z > 2.12) \quad \because \text{by symmetry} \\
&= P(Z < 2.12) - [1 - P(Z < 2.12)] \\
&= 2P(Z < 2.12) \\
&= 0.966
\end{aligned}$$

Ex. Suppose that a certain mechanical component produced by a company has a width that is normally distributed with mean μ = 2600 and standard deviation σ = 0.6.

A) What proportion of the components have a width outside the range of 2599 to 2601?

X ~ N(2600, 0.6^2)



We want blue areas

We want P(X < 2599) + P(X > 2601)

$$= P\left(\left|\frac{X-\mu}{\sigma}\right| < \frac{2599-2600}{0.6}\right) + P\left(\left|\frac{X-\mu}{\sigma}\right| > \frac{2601-2600}{0.6}\right)$$

= P(z < -1.67) + P(z > 1.67)

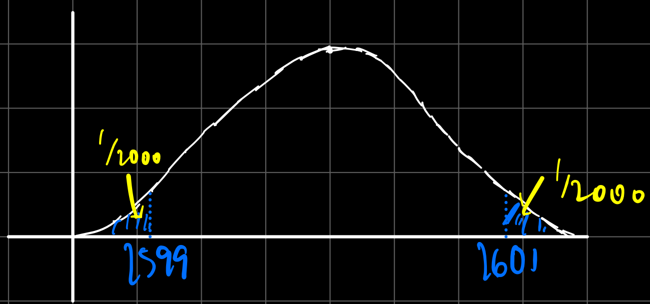
= P(z > 1.67) + P(z > 1.67) : by symmetry

= 2P(z > 1.67)

= 2[1-P(z < 1.67)]

Using the table, this is equal to 0.09492

B) If the company needs to be able to guarantee to its purchaser that no more than 1 in 1000 of its components have a width outside the range of 2599 to 2601, by how much does the value of σ need to be reduced?



To solve this, we would need to get the 0.9995 quantile (999/1000 + 1/2000)

But that's not on the table

Instead, search the normal probability table (CDF) for f(z) = 0.9995 -> z=3.29

$$3.29 = \frac{2601-2600}{\sigma} \rightarrow \sigma = 0.304$$

Fix 0.995th quantile to within 1 unit of mean