Inverse Mappings

Definition: Invertible Mapping and Inverse Mapping

Let F be a mapping from a set D_{xy} onto a set D_{uv}. If there exists a mapping F^{-1}, called the **inverse of F** which maps D_{uv} onto D_{xy} such that

$$(x, y) = F^{-1}(u, v) \quad \text{if and only if} \quad (u, v) = F(x, y)$$

then F is **invertible** on D_{xy}.

Recall: One-to-one

A mapping is one-to-one (or invective) if when F(a,b) = F(c,d), then (a,b) = (c,d)

Theorem

If a mapping F is one-to-one, it is also invertible

Theorem 2: Inverse of the Derivative Matrix

Consider a mapping F which maps D_{xy} onto D_{uv}.

If F has continuous partial derivatives at $\vec{x} \in D_{xy}$ and there exists an inverse mapping F^{-1} of F which has continuous partial derivatives at $\vec{u} = F(\vec{x}) \in D_{uv}$, then

$$DF^{-1}(\vec{u})DF(\vec{x}) = I$$

Proof:

By the Chain Rule in Matrix Form we get

$$DF^{-1}(\vec{u})DF(\vec{x}) = D(F^{-1} \circ F)(\vec{x})$$

Then, by equation $(*)$ we have

$$D(F^{-1} \circ F)(\vec{x}) = D\vec{x} = \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

as required.

Solving for inverse mappings

Ex. $(u, v) = F(x, y) = (y + x^2, x)$

$u = y + x^2 \qquad v = x$

$\Rightarrow u = y + v^2 \rightarrow y = u - v^2$

$\rightarrow F^{-1}(u, v) = (\underset{x}{v}, \underset{y}{u - v^2})$

The Jacobian

Definition: The Jacobian

The **Jacobian** of a mapping

$$(u, v) = F(x, y) = (u(x, y), v(x, y))$$

is denoted $\frac{\partial(u,v)}{\partial(x,y)}$, and is defined by

$$\frac{\partial(u,v)}{\partial(x,y)} = \det[DF(x,y)] = \det\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

Ex. Calculate the Jacobian $\frac{\partial(x,y)}{\partial(r,\theta)}$ of the mapping F given by

$$(x, y) = F(r, \theta) = (r\cos\theta, r\sin\theta)$$

1. Calculate the derivative matrix F(r,θ)
2. Calculate the Jacobian by getting the determinant

(1) $f(r,\theta) = r\cos\theta$ $\qquad$ $g(r,\theta) = r\sin\theta$

$f_r : \cos\theta$ $\qquad$ $g_r : \sin\theta$

$f_\theta : -r\sin\theta$ $\qquad$ $g_\theta : r\cos\theta$

$$\therefore \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

(2)
$$\begin{aligned} \det(DF(r,\theta)) &= r\cos^2\theta - (-r\sin^2\theta) \\ &= r(\cos^2\theta + \sin^2\theta) \\ &= r \end{aligned}$$

Corollary

If F has an inverse mapping with continuous partial derivatives on D_uv, then its Jacobian is nonzero

Corollary 4: Inverse Property of the Jacobian

Consider a mapping F which maps D_{xy} onto D_{uv}. If F has continuous partial derivatives at $\vec{x} \in D_{xy}$ and there exists an inverse mapping F^{-1} of F which has continuous partial derivatives at $\vec{u} = F(\vec{x}) \in D_{uv}$, then

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \det[dF^{-1}(u,v)]$$

Example 2

Consider the mapping defined by

$$(u, v) = F(x, y) = (e^x \cos y, e^x \sin y)$$

Show that $\frac{\partial(u,v)}{\partial(x,y)} \neq 0$ on $\mathbb{R}^2$, but that F^{-1} does not exist on $\mathbb{R}^2$.

Solution:

Observe that

$$\frac{\partial(u,v)}{\partial(x,y)} = e^{2x} > 0 \quad \text{for all } (x, y) \in \mathbb{R}^2$$

contrapositive of corollary says zero ∴ no inverse mapping, so this means F may still have one

However, F is not one-to-one on $\mathbb{R}^2$, since, for example

$$F(0,0) = F(0, 2\pi) = (1, 0)$$

Thus, F^{-1} does not exist on $\mathbb{R}^2$.

assumes the converse???

The Inverse Mapping Theorem

If a mapping $(u, v) = F(x, y)$ has continuous partial derivatives in some neighborhood of (a, b) and $\frac{\partial(u,v)}{\partial(x,y)} \neq 0$ at (a, b), then there is a neighborhood of (a, b) in which F has an inverse mapping $(x, y) = F^{-1}(u, v)$ which has continuous partial derivatives.

continuous partial derivatives AND nonzero Jacobian near (a, b) ⇒ invertible near (a, b)

Ex. Consider the mapping defined by

$$(u, v) = F(x, y) = (xy - x^2, x + y)$$

Show that F has an inverse mapping in a neighborhood of $(1, -2)$.

The Jacobian of F is y-3x

At (1,-2), this is equal to 7, which is nonzero.

Meanwhile, the partial derivatives of F (all four) are continuous

Thus, by the inverse mapping theorem, there exists some neighborhood of (1,-2) in which F has an inverse mapping

Example 1

Calculate the approximate area of the image of a small rectangle of area $\Delta x \Delta y$, located at the point $(3, 4)$, under the mapping F defined by

$$(u, v) = F(x, y) = \left(-x + \sqrt{x^2 + y^2}, \quad x + \sqrt{x^2 + y^2}\right)$$

Solution:

Differentiation and evaluation at $(3, 4)$ give the derivative matrix at $(3, 4)$:

$$DF(3,4) = \begin{bmatrix} -\frac{2}{5} & \frac{4}{5} \\ \frac{8}{5} & \frac{4}{5} \end{bmatrix}$$

At $(3, 4)$ the Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = \det \begin{bmatrix} -\frac{2}{5} & \frac{4}{5} \\ \frac{8}{5} & \frac{4}{5} \end{bmatrix} = -\frac{8}{5}$$

Therefore, the area of the image is approximately

$$\Delta A_{uv} \approx \left|\frac{\partial(u,v)}{\partial(x,y)}\right| \Delta A_{xy} \approx \frac{8}{5} \Delta A_{xy}$$