Definition: Orthogonality

Let (V, < , >) be an inner product space.

We say that two vectors v and w in the IPS are *orthogonal* if <v, w> = 0.

Ex. $p(x) = 1 + 2x + 3x^2$

In $P_2([-1, 1])$, $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$

Find all polynomials q(x) that are orthogonal to p(x).

$[p(x)]_S = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $[q(x)]_S = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Recall: $\langle \underline{v}, \underline{w} \rangle = \underline{a}^T G_B \underline{b}$

where $\underline{a} = [\underline{v}]_B$, $\underline{b} = [\underline{w}]_B$

$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 2/5 \end{bmatrix} \begin{bmatrix} \bar{a} \\ \bar{b} \\ \bar{c} \end{bmatrix} = 0$ ; solve

Theorem: Real Pythagorean Theorem

Let (V, < , >) be a real inner product space and let v, w be in V.

If v and w are orthogonal, then ||v + w||^2 = ||v||^2 + ||w||^2

Theorem: Complex Pythagorean Theorem

Let (V, < , >) be a real inner product space and let v, w be in V.

If v and w are orthogonal, then ||v + w||^2 = ||v||^2 + ||w||^2

and Re(<v, w>) = 0 (*).

Proof: Consider $\|\underline{v} + \underline{w}\|^2 = \|\underline{v}\|^2 + \|\underline{w}\|^2$

$= \langle \underline{v} + \underline{w}, \underline{v} + \underline{w} \rangle$

$= \langle \underline{v}, \underline{v} \rangle + \langle \underline{v}, \underline{w} \rangle + \langle \underline{w}, \underline{v} \rangle + \langle \underline{w}, \underline{w} \rangle$

by linearity in 1st arg;

conjugate linearity in 2nd

Note that $\|\underline{v}+\underline{w}\|^2 = \|\underline{v}\|^2 + \|\underline{w}\|^2 + 2\mathrm{Re}(\langle \underline{v}, \underline{w}\rangle)$
since orthogonal

$\Rightarrow 2\mathrm{Re}(\langle \underline{v}, \underline{w}\rangle) = 0$

Ex. Let (V, < , >) be an inner product space.

Let v and w be linearly independent vectors in V. Let S = Span({$\underline{v}$, $\underline{w}$}).

Find a basis for S in which the basis vectors are orthogonal.

Approach: Keep v. Modify w, set $\underline{z} = \underline{w} + c\underline{v}$ for some constant c.

Make v and z orthogonal:

$$0 = \langle \underline{v}, \underline{z}\rangle = \langle \underline{v}, \underline{w} + c\underline{v}\rangle$$
$$= \langle \underline{v}, \underline{w}\rangle + \langle \underline{v}, c\underline{v}\rangle$$

$\vdots$

$$\underline{z} = \underline{w} - \left(\frac{\langle \underline{w}, \underline{v}\rangle}{\langle \underline{v}, \underline{v}\rangle}\right)\underline{v}$$

↳ take off multiple of $\underline{v}$;
this constant is the component of $\underline{w}$ in the direction of $\underline{v}$

<u>Definition:</u> Orthogonal set

Let (V, < , >) be an inner product space.

Let S be a subset of V.

We say that S is an orthogonal set to mean that for every $\underline{x} \neq \underline{y}$ in S, $\langle \underline{x}, \underline{y}\rangle = 0$.

- Every vector is orthogonal to all the others

Ex. $V = \mathbb{R}^3$; standard IP (dot product)

$$S = \left\{ \underset{\vec{v}_1}{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \underset{\vec{v}_4}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} \right\}$$

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0 \checkmark$$

check all other combinations

Ex. $P_2([-1, 1])$, $\quad \langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$

$$S = \{1 - x + x^2, 1 + 4x\}$$

$$\langle 1 - x + x^2, 1 + 4x \rangle = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \frac{2}{15} \begin{bmatrix} 15 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} = 0$$

Definition: Kronecker delta symbol

In n dimensions, we use the symbol

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

Where $1 \leq i, j \leq n$

$$\delta_{ij} = (I_n)_{ij}$$

Let (V, < , >) be an inner product space.

Let S be an orthogonal subset of V (all vectors in S are nonzero)

Then S is linearly independent.

Proof: let $\underline{v}_1, \ldots, v_p \in S$; let $a_1, \ldots, a_p \in \mathbb{F}$

Consider $\sum_{j=1}^{p} a_i \underline{v}_j$. Let $\underline{v}_j \in S$

$$0 = \langle \underline{v}_j, \sum_{j=1}^{p} a_i \underline{v}_j \rangle$$

$$= a_1 \langle \underline{v}_j, \underline{v}_1 \rangle + \cdots + a_j \langle \underline{v}_j, \underline{v}_j \rangle + a_p \langle \underline{v}_j, \underline{v}_p \rangle$$

all IPs $= 0$ except $\langle \underline{v}_j, \underline{v}_j \rangle$

so all $a = 0 \rightarrow$ LI

<u>Definition:</u> Orthogonal basis

Let (V, < ,>) be an inner product space. Let B = {v1, …, vn} be a basis for V.

B is an orthogonal basis if it is also an orthogonal set.

Specifically:

$$\langle \underline{v}_i, \underline{v}_j \rangle = k_{ij} \delta_{ij}$$

<u>Definition:</u> Orthonormal set

Let (V, < ,>) be an inner product space and let S be a subset of V.

S is an orthonormal set if it is an orthogonal set of unit vectors.

$$\forall \underline{v}, \underline{w} \in S, \quad \langle \underline{v}, \underline{w} \rangle = 0$$
$$\langle \underline{v}, \underline{v} \rangle = 1$$

Ex. $\mathbb{R}^3$, $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\} \xrightarrow{\text{normalize}} \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$

(Working with standard IPS)

Ex. $\langle \underline{x}, \underline{y} \rangle = x_1 y_1 + 3 x_2 y_2$

$$S = \left\{ \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\} \to \left\{ \frac{1}{\sqrt{21}} \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \frac{1}{\sqrt{12}} \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}$$

Definition: Orthonormal basis

Let (V, < ,>) be an inner product space.

Let B = {v1, …, vn} be a basis for V.

We say that B is an orthonormal basis if it is also an orthonormal set

$$\langle \underline{v}_i, \underline{v}_j \rangle = \delta_{ij}$$

Ex. $S = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$ : orthogonal basis

$$\underline{v} = \alpha_1 \underline{v}_1 + \alpha_2 \underline{v}_2 + \alpha_3 \underline{v}_3$$

Find $\alpha_1, \alpha_2, \alpha_3$

Lemma: Finding components for orthogonal/orthonormal bases

Let (V, < , >) be an IPS

Let B = {v1, …, vn} be an orthogonal/orthonormal basis

let $\underline{v} \in V \to \underline{v} = \sum_{i=1}^{n} a_i \underline{v}_i$

If V is orthogonal:

$$a_j = \frac{\langle \underline{v}_i, \underline{v}_j \rangle}{\langle \underline{v}_j, \underline{v}_j \rangle}$$

$a_j$ is the coefficient of $\underline{v}$

If orthonormal:

$$a_j = \langle \underline{v}_i, \underline{v}_j \rangle$$

Proof: Consider <vi, vj>

$$\left\langle \sum_{i=1}^{n} a_i \underline{v_i},\ \underline{v_j} \right\rangle = \sum_{i=1}^{n} a_i \langle \underline{v_i}, \underline{v_j} \rangle$$ linearity in 1st

$$= \sum_{i=1}^{n} a_i k_{ij} \underbrace{\delta_{ij}}_{\text{switches all } i \to j} \qquad k_{ij} = \langle \underline{v_i}, \underline{v_j} \rangle$$

$$= \sum_{j=1}^{n} a_j k_{jj}$$

$$= \sum_{j=1}^{n} a_j \underbrace{\langle \underline{v_i}, \underline{v_j} \rangle}_{0}$$

... ?

$$\Rightarrow a_j = \langle \underline{v_i}, \underline{v_j} \rangle$$

<u>Lemma</u>: Let (V, < , >) be an IPS. Let B = {v1, …, vn} be a basis for V.

- B is orthogonal is $G_B$ is diagonal.
- B is orthonormal if $G_B = I_n$. (*)

$$I_n \ (x), \ \langle \underline{x}, \underline{y} \rangle = ([\underline{x}]_B)^T \overline{([\underline{y}]_B)}$$

Looks like the standard inner product in Rn / Cn

<u>Lemma:</u> Gram-Schmidt Algorithm

Let (V, < , >) be an inner product space.

Let S be a subspace of V with basis B1 = {v1, …, vp} (p ≥ 1)

Let B2 = $\{w_1, \dots, w_p\}$ be an orthogonal basis for S

Let B3 = $\{u_1, \dots, u_p\}$ be an orthonormal basis for S

Algorithm for finding B2 and B3:

$\underline{w}_1 = \underline{v}_1$

complex case: order matters

$$\underline{w}_2 = \underline{v}_2 - \left(\frac{\langle \underline{v}_2, \underline{w}_1 \rangle}{\langle \underline{w}_1, \underline{w}_1 \rangle}\right)\underline{w}_1$$

component of $\underline{v}_2$ in $\underline{w}_1$ direction

make orthogonal to $\underline{w}_1$

make orthogonal to $\underline{w}_2$

$$\underline{w}_3 = \underline{v}_3 - \left(\frac{\langle \underline{v}_3, \underline{w}_1 \rangle}{\langle \underline{w}_1, \underline{w}_1 \rangle}\right)\underline{w}_1 - \left(\frac{\langle \underline{v}_3, \underline{w}_2 \rangle}{\langle \underline{w}_2, \underline{w}_2 \rangle}\right)\underline{w}_2$$

normalize normalize

$\vdots$

continue

$$\underline{u}_i = \frac{\underline{w}_p}{\|\underline{w}_p\|}$$