This section deals with finding the *population mean* — μ = E(x) — of various probability distributions

Binomial Distribution

If X ~ Bin(n,p), then μ=E(X) = np

Proof:

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$E(x) = \sum_{\forall x} x \cdot f(x)$$

$$= \sum_{\forall x} x \cdot \binom{n}{x} p^x (1-p)^{n-x}$$

$$\vdots$$

$$= np$$

Alternate proof: Let X_i be a Bernoulli random variable.

$$X_i = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } 1-p \end{cases}$$ results of trials

If $X \sim Bin(n, p)$, then

$$X = x_1 + x_2 + \cdots + x_n$$

$$\Rightarrow E(X_i) = 1(p) + 0(1-p) = p$$: each trial occurs with probability p

$$\Rightarrow E(X) = E(x_1 + x_2 + \cdots + x_n)$$
$$= p + \cdots + p$$
$$= np$$

Poisson Distribution

Let X ~ Pois(λt) $\lambda t = \mu$

λ - positive samples per interval t

$$f(x) = \frac{e^{-\mu}\mu^{x}}{x!}$$

$$\Rightarrow E(X) = \sum_{\forall x} x \cdot \frac{e^{-\lambda t}(\lambda t)^{x}}{x!}$$

$$= \lambda t \text{ eventually}$$

Variability

Some potential measures for variability:

- E(X-μ)
  - x-μ for all x — how far from the mean?
  - By linearity, E(X-μ) = E(X) - μ = μ - μ = 0. So this sucks.
- E(|X-μ|) — not a bad idea, but absolute values don't have *nice* mathematical properties
- Better one:

$$\text{Variance: } \sigma^2 = Var(x) = E[(x-\mu)^2]$$

summed over all x

But these are in squared units

$$\text{Standard deviation: } \sqrt{E[(x-\mu)^2]}$$

Alternate Form:

$$Var(x) = E[(x-\mu)^2] = E(x^2 - 2\mu x + \mu^2)$$

$$= E(x^2) - 2\mu \underbrace{E(x)}_{\mu} + \mu^2$$ ↳ constant

$$= E(x^2) - \mu^2$$

More variance → data moves away from the mean

The variance of a binomial distribution is $np(1-p)$

Proof : $\text{Var}(X) = E[X(X-1)] + \mu - \mu^2$

$$E[X(X-1)] = \sum_{\forall x} \underbrace{(x)(x-1)}_{g(x)} \underbrace{\binom{n}{x} p^x (1-p)^x}_{f(x)}$$

$$= n(n-1)p^2$$

$$\Rightarrow n(n-1)p^2 + np - n^2p^2$$

$$= np(1-p)$$

The variance of a Poisson distribution is μ

3. A person plays a game in which a fair coin is tossed until the first tail occurs. The person wins $\$2^x$ if x tosses are needed for $x = 1, 2, 3, 4, 5$ but loses $\$256$ if $x > 5$.

(a) Determine the expected winnings.

(b) Determine the variance of the winnings.

(a) $P(X \leq 5) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$

$$= \frac{16 + 8 + 4 + 2 + 1}{32} = \frac{31}{32}$$

$\Rightarrow P(X>5) = 1 - \frac{31}{32} = \frac{1}{32}$

$\Rightarrow$ Expected winnings : $\left(\frac{1}{2}\right) \cdot 2 + \left(\frac{1}{4}\right) \cdot 2^2 + \left(\frac{1}{8}\right) \cdot 2^3 \cdots + \frac{1}{32} \cdot (-256)$

$= 5 - \frac{256}{32}$

$= -3???$