Review of single-variable Taylor polynomials

The Taylor polynomial of a function f(x) with degree n at point a is given by

$$P_{n,a}(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} \cdot (x-a)^i$$

$$= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots$$

With $P^{(n)}_{n,a}(a) = f^{(n)}(a)$

The Two Variable Case

$$P_{2,(a,b)}(x,y) = L_{(a,b)}(x,y) + A(x-a)^2 + B(x-a)(y-b) + C(y-b)^2$$

Where A, B, and C are constants

We can compute values for A, B, and C by calculating partial derivatives — in this case, second partial derivatives since that will cancel out all the variables

$$2A = \frac{\partial^2 f}{\partial x^2}(a,b), \quad B = \frac{\partial^2 f}{\partial x \partial y}(a,b), \quad 2C = \frac{\partial^2 f}{\partial y^2}(a,b)$$

$\to$ 2A because it's the coefficient of $(x-a)^2$ $\to$ power rule

Definition: 2nd degree Taylor polynomial

Let f be a function of two variables. The **second degree Taylor polynomial** $P_{2,(a,b)}$ of $f(x,y)$ at (a,b) is given by

$$P_{2,(a,b)}(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \to L_{(a,b)}(x,y)$$
$$+ \frac{1}{2}\left[f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right]$$

For such computations, it is useful to use the *Hessian matrix* Hf(x,y):

$$Hf(x,y) = \begin{bmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y \end{bmatrix}$$

Theorem 2: Taylor's Theorem for Functions of Two Variables

If $f(x,y) \in C^2$ in some neighborhood $N(a,b)$ of (a,b), then for all $(x,y) \in N(a,b)$ there exists a point (c,d) on the line segment joining (a,b) and (x,y) such that

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + R_{1,(a,b)}(x,y)$$

where

$$R_{1,(a,b)}(x,y) = \frac{1}{2}\left[f_{xx}(c,d)(x-a)^2 + 2f_{xy}(c,d)(x-a)(y-b) + f_{yy}(c,d)(y-b)^2\right]$$

Like the single-variable case, this is an *existence theorem:* it tells us that the point (c,d) exists, but not how to find it

Corollary

If $f(x,y) \in C^2$ in some closed neighborhood $\overline{N}_r(a,b)$ of (a,b), then there exists a positive constant M such that

$$|R_{1,(a,b)}(x,y)| \le M\|(x,y)-(a,b)\|^2, \quad \text{for all } (x,y) \in \overline{N}_r(a,b)$$

Ex.

Let $f(x,y) = \ln(1+x+6y)$. We can Use Taylor's Theorem to show that for $x \ge 0$, $y \ge 0$ we have

$$|R_{1,(0,0)}(x,y)| \le K(x^2+y^2)$$

Find the smallest value of K that would make the statement above true.

$R_{1,(0,0)}(x,y) = f(x,y) - L_{(0,0)}(x,y)$

$= f(x,y) - \underset{\ln(1)}{f(0,0)} - f_x(a,b)(x-a) - f_y(a,b)(y-b)$

$f_x = \dfrac{1}{1+\underset{0}{x}+6\underset{0}{y}} \to 1$; $f_y = \dfrac{6}{1+\underset{0}{x}+6\underset{0}{y}} \to 6$

$\Rightarrow f(x,y) - (x-0) - 6(y-0)$

$= \ln(1+x+6y) - x - 6y$

By Taylor's Theorem, there exists a point (c, d) on the line segment from (x, y) to $(0,0)$ such that

$$|R_{1,(0,0)}(x,y)| = \left|\frac{1}{2}\left[f_{xx}(c,d)(x-0)^2 + 2f_{xy}(c,d)(x-0)(y-0) + f_{yy}(c,d)(y-0)^2\right]\right|$$

Since we cannot find (c, d), we want to find an upper bound for this function.

$$\le \frac{-1}{(1+0+0)^2} = -1$$

$$\le -6$$

Calculating second partial derivatives:

$$f_x = \frac{1}{1+x+6y} \to f_{xx} = \frac{-1}{(1+x+6y)^2}$$

$$f_{xy} = \frac{-6}{(1+x+6y)^2}$$

$$f_y = \frac{6}{1+x+6y} \to f_{yy} = \frac{-36}{(1+x+6y)^2}$$

Using the Triangle Inequality:

$$|R_{1,(0,0)}(x,y)| \le \frac{1}{2}\left[|f_{xx}(c,d)|\,x^2 + 2\,|f_{xy}(c,d)|\,|x|\,|y| + |f_{yy}(c,d)|\,y^2\right]$$

$$\Rightarrow |R_{1,(0,0)}(x,y)| \le \frac{1}{2}\left[\left|\frac{-1}{(1+x+6y)^2}\right|x^2 + 2\left|\frac{-6}{(1+x+6y)^2}\right||x||y| + \left|\frac{-36}{(1+x+6y)^2}\right|y^2\right]$$

By Taylor's Theorem, (c,d) lies on the line segment between (x,y) and (0,0). The problem gives the condition that x ≥ 0 and y ≥ 0, which means that for (c,d) to be between (x,y) and (0,0), c ≥ 0 and d ≥ 0. Thus:

$$\left|\frac{-1}{(1+\underset{c}{x}+\underset{6d}{6y})^2}\right| \le \left|\frac{-1}{(1+0+0)^2}\right| = 1$$

0 is smallest value c and d can take on

$$\left|\frac{-6}{(1+x+6y)^2}\right| \le |-6| = 6$$

$$\left|\frac{-36}{(1+x+6y)^2}\right| \le 36$$

$$\Rightarrow |R_{1,(0,0)}(x,y)| \le \frac{1}{2}x^2 + 6|x||y| + 18y^2$$

Note that $0 \le (x-y)^2 \qquad \forall x, y \in \mathbb{R}$

$$\Rightarrow 0 \le x^2 - 2xy + y^2$$

$$\Rightarrow 2xy \le x^2 + y^2$$

$$\Rightarrow 2|x||y| \le x^2 + y^2 \quad \star$$

$$\Rightarrow |R_{1,(0,0)}(x, y)| \le \frac{1}{2}x^2 + 3(x^2 \underset{\star}{+} y^2) + 18y^2$$

$$= \frac{7}{2}x^2 + 21y^2$$

$$\le 21(x^2 + y^2)$$

$$\Rightarrow K = 21$$