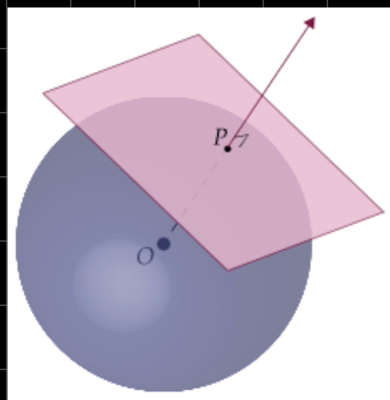
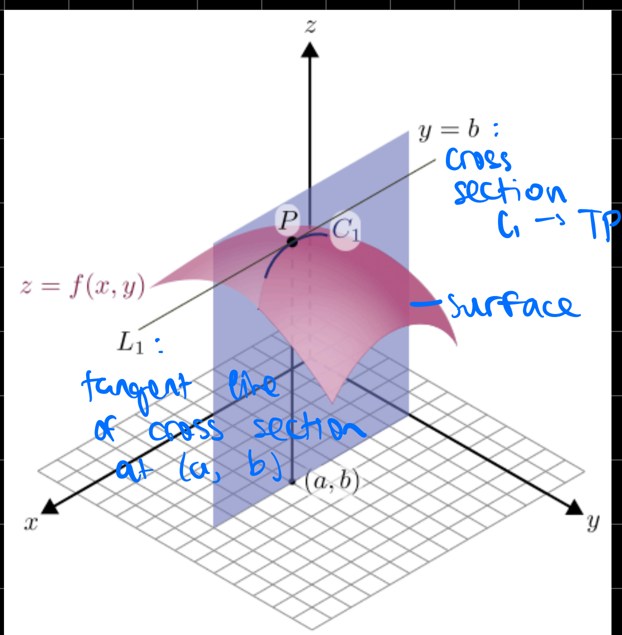
Suppose we have a sphere defined in R3, with centre O.

The tangent plane P:

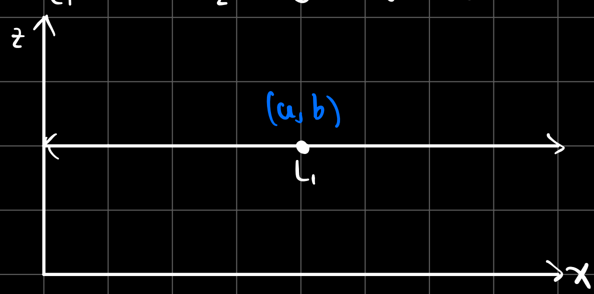
- Approximates the surface of the sphere near P
- Is orthogonal (perpendicular) to the line connecting P and O



This can be generalized to a surface of the form z = f(x,y)



L_1 in 2 dimensions:



As such, the slope of L_1 over $(a, b, f(a,b))$ is

$$\frac{\partial f}{\partial x} f(a,b)$$

Similarly, a cross-section C2 of f over x=a is given by z = f(a,y). If L2 is a tangent line of C2 over the point (a, b, f(a,b)), its slope is given by

$$\frac{\partial f}{\partial y} f(a,b)$$

Definition: Tangent Plane

The **tangent plane** to $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$z = f(a,b) + \underbrace{\frac{\partial f}{\partial x}(a,b)(x-a)}_{\substack{\text{slope of TL} \\ \text{along TP} \\ x=a}} + \underbrace{\frac{\partial f}{\partial y}(a,b)(y-b)}_{\substack{\text{slope of TL} \\ \text{along TP} \\ y=b}}$$

Ex. Let $f(x, y) = \dfrac{x^2 y}{y^2+1}$.

a. Find the equation of the tangent plane of f at $(1, 2)$.

$$f(a,b) = \frac{2}{5}$$

$$f_x(x,y) = \frac{(y^2+1)(2xy) - 0}{(y^2+1)^2} = \frac{2xy}{y^2+1}$$

$$f_y(x,y) = \frac{(y^2+1)(x^2) - (x^2y)(2y)}{(y^2+1)^2} = \frac{x^2y^2 + x^2 - 2x^2y^2}{(y^2+1)^2}$$

$$= \frac{x^2 - x^2y^2}{(y^2+1)^2}$$

$$= \frac{-x^2(y^2-1)}{(y^2+1)^2}$$

OR:

$$f(x,y) = x^2 \cdot \frac{y}{y^2+1}$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{2xy}{y^2+1}$$

$$\vdots$$

$$\Rightarrow \text{TL}: \frac{2}{5} + \frac{4}{5}(x-1) - \frac{3}{25}(y-2)$$

Linear approximations

In two dimensions, the line y = f(a) + f'(a)(x-a) approximates the function f for values of x sufficiently close to a. Meanwhile, in three dimensions:

Definition: Linearization and Linear Approximation

For a function $f(x,y)$ we define the **linearization** $L_{(a,b)}(x,y)$ of f at (a,b) by

$$L_{(a,b)}(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

We call the approximation

$$f(x,y) \approx L_{(a,b)}(x,y)$$

the **linear approximation** of $f(x,y)$ at (a,b)

Ex. Use the linear approximation to approximate $\sqrt{(0.95)^3 + (1.98)^3}$.

We will use $f(x,y) = \sqrt{x^3 + y^3}$

$$\frac{\partial f}{\partial x} = \frac{1}{2}(x^3 + y^3)^{-1/2} \cdot 3x^2$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(x^3 + y^3)^{-1/2} \cdot 3y^2$$

Using $(x,y) = (0.95, 1.98)$ and $(a,b) = (1,2)$, we get 2.935

Ex. A silo consists of a circular cylinder of radius 5 meters, and height 25 meters, capped by a hemisphere. Suppose that the radius is decreased by 5 centimeters and the height of the cylinder is increased by 10 centimeters. Use the linear approximation to estimate the change in volume. Use $\pi = 3.14$ and round your answer to 1 decimal.

$V = \pi r^2 h$

$$\frac{\partial V}{\partial r} = 2\pi r h \qquad \frac{\partial V}{\partial h} = \pi r^2$$

Using $a = 5$, $b = 25$, $r = 4.95$, $h = 25.1$:

$\Delta V = (2)(3.14)(5)(25)(4.95 - 5) + (3.14)(5^2)(25.1 - 25)$

Linear Approximations in R3

Consider a function $f(x, y, z)$. By analogy with the case of a function of two variables, we define the linearization of f at $\vec{a} = (a, b, c)$ by

$$L_{\vec{a}}(x, y, z) = f(\vec{a}) + \underbrace{f_x(\vec{a})(x-a) + f_y(\vec{a})(y-b) + f_z(\vec{a})(z-c)}$$

$$= \begin{bmatrix} x-a \\ y-b \\ z-c \end{bmatrix} \cdot \begin{bmatrix} f_x(\vec{a}) \\ f_y(\vec{a}) \\ f_z(\vec{a}) \end{bmatrix}$$

(proof: just expand)

Second vector: gradient, $\nabla f(\vec{a})$

Definition: Gradient

Suppose that $f(x, y, z)$ has partial derivatives at $\vec{a} \in \mathbb{R}^3$. The **gradient** of f at $\vec{a}$ is defined by

$$\nabla f(\vec{a}) = (f_x(\vec{a}), f_y(\vec{a}), f_z(\vec{a}))$$

Definition: Linearization and Linear Approximation

Suppose that $f(\vec{x})$, $\vec{x} \in \mathbb{R}^3$, has partial derivatives at $\vec{a} \in \mathbb{R}^3$.

The **linearization** of f at $\vec{a}$ is defined by

$$L_{\vec{a}}(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

The **linear approximation** of f at $\vec{a}$ is

$$f(\vec{x}) \approx f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$