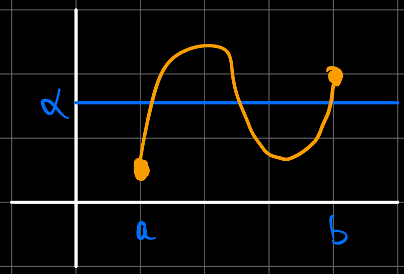
Theorem Definition

If f is continuous on [a,b] and f(a) < m < f(b) or f(a) > m > f(b), then there exists a real number c in (a,b) such that f(c) = m



Ex. Prove $f(x) = x^5 - 2x^3 - 2$ has a root on $[0, 2]$

let $\alpha = 0$

$f(a) = f(0) = -2 < \alpha$

$f(b) = f(2) = 14 > \alpha$

Since f is a polynomial, it is also continuous from $[a, b]$, which means by IVT, there exists a $c \in [0, 2]$ such that $f(c) = 0$

Ex. 2. Prove that there exists a $c \in (0, 1)$ such that $\cos(c) = c$

Proof. Consider $f(x) = \cos x - x$

f is continuous since $\cos x$ and x are continuous

Let $\alpha = 0$

Approximating Solutions

Ex. We know $f(x) = x^5 - 2x^3 - 2$ has a root on $[0, 2]$

$f(0) = -2 < 0$; $f(2) = 14 > 0$

Cut search in half at midpoint:

$f(1) = -3$

$\therefore f$ has a root on $(1, 2)$ by IVT

Since $f(1)$ is still < 0, we can raise the lower bound and cut in half again:

$f\left(\frac{3}{2}\right) = \frac{-37}{32} < 0 \rightarrow \left(\frac{3}{2}, 2\right)$ has a root

$f\left(\frac{7}{4}\right) = 3.694\ldots > 0 \rightarrow \left(\frac{3}{2}, \frac{7}{4}\right)$ has a root

The Bisection Method (formal method)

Say we want to approximate a solution to F(x) = 0 (or f(x) = g(x) using F(x) = f(x) - g(x) where F is continuous and we want the error to be less than epsilon

1. Find real numbers a0, b0 (a0 < b0) such that F(a0) and F(b0) have different signs. IVT guarantees a solution on (a0, b0)
2. Evaluate F(d) where $d = \frac{a_0 + b_0}{2}$
3. If F(d) and F(a0) have the same sign, let a1 = d and b1 = b0; otherwise let a1 = a0 and b1 = d. Either way (a1, b1) has a solution and the length of (a1,b1) is half the length of (b0-a0)
4. Repeat steps 2 and 3; get intervals [a2, b2] and [a3, b3] where each interval has a solution and the length of [ak, bk] is $\frac{1}{2^k}(b_0 - a_0)$

5. Stop when

$$\frac{1}{2^{k+1}}(b_0 - a_0) < \epsilon$$

then $d = \dfrac{a_k + b_k}{2}$ is within epsilon units of a solution

Or |d - soln| < epsilon