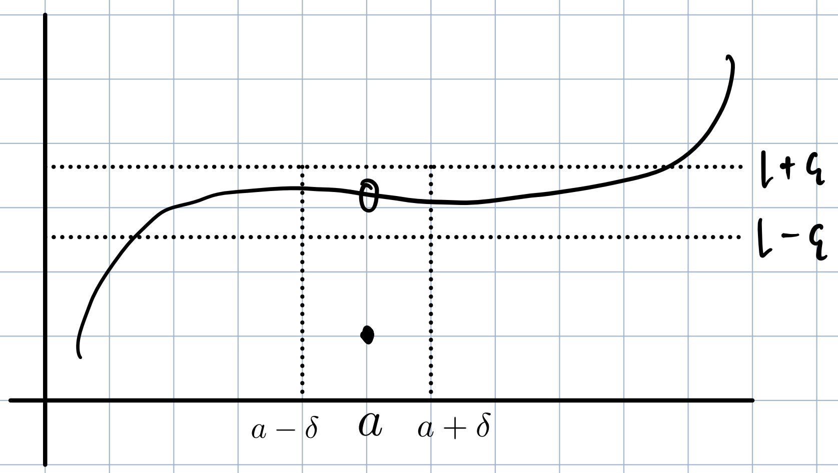
Ex. $f: \mathbb{R} \to \mathbb{R}$ (inputs $\mathbb{R}$, outputs $\mathbb{R}$)

$$\lim_{x \to a} f(x) = L$$



Definition

$\lim_{x \to a} f(x) = L$ if

For all $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$ then

$$|f(x) - L| < \varepsilon$$

Delta depends on epsilon, but epsilon does not depend on delta

Limit investigates the behavior of f as x approaches a from *both sides*

Ex. Prove $\lim_{x \to 2} 3x + 7 = 13$

Let $\varepsilon > 0$, $\delta = \frac{\varepsilon}{3}$ (leave δ blank first)

then if $0 < |x - 2| < \delta$ then

$$|3x + 7 - 13| = |3x - 6| = 3|x - 2| < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

QED

Ex. 2. Prove $\lim_{x \to 0} x^2 + 6x - 9 = -9$

Let $\varepsilon > 0$ be given, let $\delta =$
then if $0 < |x| < \delta$, we get

$$|x^2 + 6x - 9 - (-9)| = |x^2 + 6x| = |x| \cdot |x+6|$$
$$< \delta |x+6|$$

Say $\delta \le 1$

$\therefore |x| \le \delta \le 1 \to -1 \le x \le 1$

Then $5 \le x + 6 \le 7$
Since $\delta \le 1$
$\delta |x+6| \le 7\delta \le 7\left(\frac{\varepsilon}{7}\right) = \varepsilon$

QED

Ex. 3, $\lim_{x \to 3} \frac{x^2 - 6x + 20}{x - 5} = -\frac{11}{2}$

Let $\varepsilon > 0$ be given
Let $\delta = \min\{1, \frac{2\varepsilon}{13}\}$ then if

$0 < |x - 3| < \delta$ we get

$$\left| \frac{x^2 - 6x + 20}{x - 5} + \frac{-11}{2} \right| = \frac{2x^2 - x - 13}{2x - 10} = \frac{|x-3||2x+5|}{2|x-5|} < \frac{\delta|2x+5|}{2|x-5|}$$

Since $|x - 3| < \delta \le 1$, $2 \le x \le 4$
Then $9 \le 2x + 5 \le 13$
and $-3 \le x - 5 \le -1 \to 1 \le |x - 5| \le 3$

$$\therefore \frac{\delta|2x+5|}{2|x-5|} \le \frac{13\delta}{2|x-5|} \le \frac{13}{2(1)} = \frac{13\delta}{2} \le \frac{13\left(\frac{2\varepsilon}{13}\right)}{2} = \varepsilon \quad \text{QED}$$