Goal: $\Phi_A(x)$

$$G(x) = \frac{1+7x}{1-x-6x^2} = \frac{1+7x}{(1-3x)(1+2x)} = \frac{A}{1-3x} + \frac{B}{1+2x}$$

Cross-multiplying, we have $A=2$, $B=-1$

$$\Rightarrow G(x) = \underset{A=2}{2\sum_{n=0}^{\infty}(-3x)^n} - \underset{B=-1}{\sum_{n=0}^{\infty}(2x)^n}$$

Theorem (general)

$$G(x) = \frac{P(x)}{(1-\lambda_1)^{d_1}\cdots(1-\lambda_n)^{d_n}} = \frac{d_1-1}{(1-\lambda_1)^{d_1}} + \cdots + \frac{d_n-1}{(1-\lambda_n)^{d_n}}$$

Ex. $$G(x) = \frac{1-2x}{(1+4x)^2(1-3x)(1+5x)^3}$$

$$= \frac{A}{(1+4x)^2} + \frac{B}{1+4x} + \frac{C}{1-3x} + \frac{D}{(1+5x)^3} + \frac{E}{(1+5x)^2} + \frac{F}{1+5x}$$

We can simplify this using the negative binomial theorem / geometric series:

$$\frac{A}{(1+4x)^2} \to \sum_{n=0}^{\infty}\binom{4+1-1}{2-1}(-4x)^n \quad \text{NBT}$$

$$\frac{B}{1+4x} \to \sum_{n=0}^{\infty}(-4x)^n \quad \text{geometric}$$

Ex. Generating series for the set of compositions with odd parts

$P = \{1, 3, 5, \ldots\}$

$C = P^* \rightarrow \Phi_C = \dfrac{1-x^2}{1-x-x^2}$

Same degree → do long division

$$\begin{array}{r|l} & \;1 \quad \text{rem. } x \\ -x^2 - x + 1 & -x^2 + 0x + 1 \\ & -(-x^2 - x + 1) \\ \hline & \quad x \end{array}$$

$$\frac{1-x^2}{1-x-x^2} = \frac{(1-x-x^2)(1)}{1-x-x^2} + \frac{x}{1-x-x^2}$$

$$= \frac{x}{1-x-x^2} + 1$$

Partial fractions:

$$\frac{x}{1-x-x^2} = \frac{A}{1-\lambda_1 x} + \frac{B}{1-\lambda_2 x}$$

Trick: roots for $ax^2 + bx + c$ are the roots of
$cx^2 + bx + a$

$\Rightarrow 1 - x - x^2 \rightarrow x^2 - x - 1$

Using the QF: $\lambda_1, \lambda_2 = \dfrac{1 \pm \sqrt{5}}{2}$

Solving for A, B:

$$\frac{x}{1-x-x^2} = \frac{A}{1-\lambda_1 x} + \frac{B}{1-\lambda_2 x}$$: multiply by $(1-\lambda_1 x)(1-\lambda_2 x)$

$\Rightarrow A(1-\lambda_2 x) + B(1-\lambda_1 x) = x$ Usually $B = -A$

$A = \frac{-\sqrt{5}}{5}$ $B = \frac{\sqrt{5}}{5}$

Now, note that $\frac{A}{1-\lambda_1 x}$ and $\frac{B}{1-\lambda_2 x}$ are geometric series

$\frac{1}{1-x}$ form

$$\Rightarrow G(x) = A\sum_{n=0}^{\infty}(\lambda_1 x)^n + B\sum_{n=0}^{\infty}(\lambda_2 x)^n$$

Recurrence for coefficients

$$G(x) = \sum_{n=0}^{\infty} g_n x^n = \frac{1-x^2}{1-x-x^2}$$

$\Rightarrow G(x)\cdot(1-x-x^2) = 1-x^2$ or $\underset{(a)}{1} + \underset{(b)}{0x} - x^2$

General form:

$$[x^n][G(x)\cdot(1-x-x^2)] = [x^n]G(x) - [x^n]xG(x) - [x^n]x^2G(x)$$
$$= g_n - g_{n-1} - g_{n-2}$$

$[x^0][G(x)\cdot(1-x-x^2)] = g_0 = 1$ (a)

$[x^1][G(x)\cdot(1-x-x^2)] = g_1 - g_0 = 0$ (b)

Theorem

Theorem 4.8. *Let* $\mathbf{g} = (g_0, g_1, g_2, \ldots)$ *be a sequence of complex numbers, and let* $G(x) = \sum_{n=0}^{\infty} g_n x^n$ *be the corresponding generating series. The following are equivalent.*

(a) The sequence $\mathbf{g}$ *satisfies a homogeneous linear recurrence relation*

$$g_n + a_1 g_{n-1} + \cdots + a_d g_{n-d} = 0 \text{ for all } n \geq N,$$

with initial conditions $g_0, g_1, \ldots, g_{N-1}$.

(b) The series $G(x) = P(x)/Q(x)$ *is a quotient of two polynomials. The denominator is*

$$Q(x) = 1 + a_1 x + a_2 x^2 + \cdots + a_d x^d$$

and the numerator is $P(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_{N-1} x^{N-1}$, *in which*

$$b_k = g_k + a_1 g_{k-1} + \cdots + a_d g_{k-d}$$

for all $0 \leq k \leq N-1$, *with the convention that* $g_n = 0$ *for all* $n < 0$.

Ex. $a_n = 3a_{n-2} - 2a_{n-3}$

where $a_0 = 0,\ a_1 = -5,\ a_2 = -1$

$a_n - 3a_{n-2} + 2a_{n-3} = 0$

$1 - 3x^2 + 2x^3 = 0$ CP

$\Rightarrow Q(x) = (1-x)^2(1+2x)$

Using the theorem

$Q(x) = c_0 + c_1 x + \cdots + c_k x^k = (1-\lambda_1 x)^{d_1} \cdots (1-\lambda_1 x)^{d_k}$