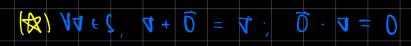
Recall: a set S is a vector space defined under F if:

- Zero vector is in S
- For all vectors x and y in S, x+y is also in S (closed under addition)
- For all vectors x and constants c in F, cx is in S (closed under scalar multiplication)

Ya, BEF, LA+ OXES



Ex. A weird vector space

Consider the set R^2, defined under R, where

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 9 \\ C \end{bmatrix} = \begin{bmatrix} 0+9+1 \\ 0+C \end{bmatrix}$$

$$= \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix}$$

$$2eta testal, is (0 -1) \in C$$

$$3eta testal, is (0 -1) \in C$$

More vector spaces:

- (1) Field: real numbers; set: R^n, zero vector is zero vector, addition and multiplication are standard
- (2) Field: (m,n) in R2; set: all m x n matrices; zero vector is m x n zero matrix, addition and multiplication are both standard

(3)
$$T = IR$$
 $S = P_1(IR)$: all polynomials with order $\leq n$
 $+ / \times$ are standard

 Zero vector : $f(x) = 0$ since $f(x) + 0 = f(x)$ $\forall f(x) \in S$

- (4) Field: R; Set: all functions from R->R that are infinitely differentiable
- Are equal to their Taylor series
- Zero vector: y = 0
- · Addition and multiplication are standard

[7. Is
$$W = \{g(x) \in V : g(0) + g(1) = 1\}$$
 a vector space?

(1) Find the zero vector

$$0: J(x) = 0$$
 Since $J(x) + 0 = J(x)$ $J(x) \in M$

However, notice that g(x) = 0 cannot be part of W, since $g(0) + g(1) = 0 \neq 1$.

Also, this set is not closed under addition:

$$\mathcal{J}_{3}(0) + \mathcal{J}_{3}(1) = \mathcal{J}_{1}(0) + \mathcal{J}_{2}(0) + \mathcal{J}_{1}(1) + \mathcal{J}_{2}(1)$$

$$= g_1(0) + g_1(1) + g_2(0) + g_2(1) = 2 \mp 1$$

Lemma

For all vector spaces:

- The zero vector is unique
- The additive inverse is unique

If cv = 0, then either c=0 (constant) or v is the zero vector

However, if the matrix product AB is equal to the zero matrix, that does not necessarily mean that one of A or B is the zero matrix

Definition: Linear combination

Let V be a vector space.

Let v_1, ..., v_n be members of V, and let c_1, ..., c_n be scalars.

Then not massarly wetors

Definition: Span

If S is a subset of V, then Span(S) is the set of all linear combinations of S.

$$[x] = \{ \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 \end{bmatrix} \}$$

Is
$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
 in Span(S)?

4 equations, 3 unknowns; one equation for each element of S:

$$\alpha = c$$
, (1) + $c_3(i)$ + $c_3(2i)$ (simple matrix addition)

Definition: Subspace

Let V be a vector space, and let W be a subset of V.

W is a subspace if it is itself a vector space.

Examples:

- The line y=x (or any straight line through the origin) is a subspace of R2
 - \circ (1,1) + c(2,2) is obviously also on y=x
 - \circ (0,0), the zero vector, is on y=x
- Quadratic forms (everything not linear) is not.

Subspaces are linear subsets

Also, since they are vector spaces, they are closed under addition and scalar multiplication

When checking if W is a subspace of V, try the following (in order):

- Check if the zero vector is in W. If not, then either:
 - W is empty
 - \circ For some vector w in W, 0w = $\vec{0}$. But the zero vector is not in W, so W is not closed under multiplication and thus W is not a vector space.
- Check if closed under addition

Ex. Let V be the set of all 2x2 matrices.

Some subsets of V:

$$\mathbf{U}_{1} = \left\{ \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix} : \mathbf{a} + 2\mathbf{b} + 4\mathbf{c} + 3\mathbf{d} = 5 \right\}$$

Zero matrix not in W1 -> not a subspace

<u>Theorem</u>

If V is a vector space and S is a subset of V, then Span(S) is a vector subspace

- Spans are closed under addition and scalar multiplication
- · Closed under scalar multiplication implies that the zero vector is in S

Also, spanning the empty set gives us the zero vector, which is itself a subspace

V (itself) and {0} are trivial subspaces of V

Efficient definition of subspaces:

- Let x and y be in S; let a and b be in a field F
- If ax + by is in S, then S is a subspace

Since subspaces are closed, when working with subspaces, no need to worry about the set that it is part of. R2 is a subspace of R3, etc...