A *walk* from u to v is a finite subsequence of vertices from u to v

- All vertices must be connected by edges
- The *length* of a walk is the number of edges (# vertices - 1)

A u,v *path* is a walk from u to v without repeated vertices

Theorem

If there exists a walk between u and v, then there exists a path between u and v

Justification of sorts:

- Iterate over the walk.
- If there is a loop, remove it
- Recursively go through the rest of the walk

Proof. Suppose w = v_0 … v_n is a u,v walk.

Case 1: no repetitions -> done

Case 2: there are repetitions

$$\Rightarrow W = v_0, \ldots v_i, \underbrace{a, a_1, \ldots a_j, a,}_{\text{loop}} b_1, \ldots b_k$$

$$\Rightarrow W' = v_0, \ldots v_i, a, b_1, \ldots b_k$$

We can keep repeating this until we get to Case 1

Proof 2: Let w be a *minimal* u,v-walk (shortest path), and suppose w is not a path.

Since it is not a path, it has repeated vertices.

This means that we can write a shorter walk by removing the loop, which is a contradiction.

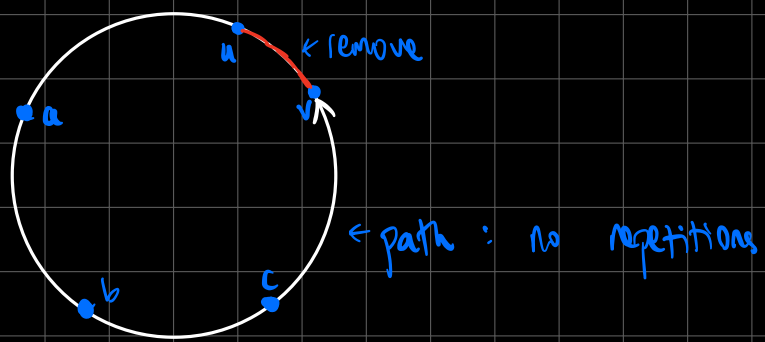
Closed walk: starts and ends on the same vertex

Cycle: closed walk with no repeated vertices (except the start and end)

$\Rightarrow v_0, v_1, \ldots v_{n-1}, v_0 \qquad v_0 \to \underbrace{c_1, c_2, \ldots}_{\text{at least 2}} \to v_0$

Distinct v_i for $0 \le i \le n-1 \le 2 \to$ at least 3 distinct vertices

If C is a cycle, and uv is an edge in C, then C-uv gives us a uv-path



Theorem

If every vertex in a graph G has degree ≥2, then G has a cycle

Proof. Suppose p = u_0 … u_n is a *longest path* in G.

Suppose u_0 has degree ≥2, meaning that it has another neighbor x.

For there to be no cycle in G, then x must not be in the path p = u_0 … u_n.

This would mean that there is a longer path than p (including x).

This is a contradiction, so there cannot be a cycle.

Girth: length of the shortest cycle in a graph G

- Denoted as g(G)
- If G has no cycle, its girth is infinite