Composition – finite sequence of *positive* integers

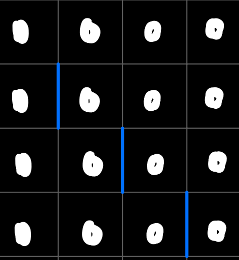
$$\gamma = (c_1, \ldots c_k)$$

length: $\ell(\gamma) = k$
Size: $|\gamma| = c_1 + \cdots + c_k$ → this is also the standard weight function $w: \mathcal{C} \to \mathbb{N}$

There are 0 compositions of size 0

1 composition of size 1

There are 2^{n-1} ways to make a composition of size n



(4)
(1, 3)
(2, 2)
(3, 1)

The number of compositions with weight n is given by the generating series

$$\Phi_{\mathcal{C}}^{w}(x) = \sum_{n \geq 0} x^n = \left\{ \begin{matrix} (c_1, \ldots, c_k) \in \mathcal{C} \\ c_1 + \cdots + c_k = n \end{matrix} \right\}$$

$$= \sum_{n \geq 0} 2^{n-1} x^n + 1$$

↳ 2^{n-1} ways to divide composition into groups $c_1, \ldots, c_k$

$$= \frac{1-x}{1-2x}$$

Alternate way to compute:

$\mathcal{C} = \mathbb{Z}_{>0}^{*}$ → string lemma

Ex. How many compositions of 100 are there?

$$[x^{100}]\Phi_e(x) = [x^{100}]\frac{1-x}{1-2x}$$

Ex. How many compositions are there such that:

(a) All elements are odd

$$P = \{1, 3, 5, \dots\}$$

$$\Phi_P(x) = 1x^1 + 1x^3 + 1x^5 + \cdots$$
$$= x(1 + x^2 + x^4 + \cdots)$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$\Rightarrow x(1 + x^2 + x^4 + \cdots) = x\sum_{n=0}^{\infty} x^{2n} = \frac{x}{1-x^2}$$

$$\Phi_{P^*}(x) = \frac{1}{1-\frac{x}{1-x^2}} \quad \text{(string lemma)}$$

(b) All elements are congruent to 2 mod 3

$$P = \{2+3j : j \geq 0\} = \{2, 5, 8, \dots\}$$

$$\Rightarrow x^2 + x^5 + x^8 + \cdots$$
$$= \sum_{j=0}^{\infty} x^{2+3j}$$

Ex. Generating series if the two parts are even, and the remaining parts are odd

E = {2j : j ≥ 1}

O = {1+2j : j ≥ 0}

$O \cup E = \mathbb{Z}_{>0}$

$\Phi_E = \dfrac{x^2}{1-x^2}$

$\Phi_O = \dfrac{x}{1-x^2}$

→ any number of even parts → as long as first 2 are even

$\Rightarrow S = (\varepsilon \cup E) \cup (E^2 O^*)$

↳ empty: no odds → any number of odds

By the lemmas:

$$S = 1 + \Phi_E + (\Phi_E)^2 \left(\frac{1}{1-\Phi_O}\right)^*$$

ε E E^2 O^*

Ex. How many compositions with odd parts of 100 are there?

$\Phi_P(x) = \dfrac{x}{1-x^2}$ (from earlier)

$\Phi_{P^*}(x) = \dfrac{1-x^2}{1-x-x^2}$ (string lemma)

$$\Rightarrow [x^{100}] \underbrace{\frac{1-x^2}{1-x-x^2}} = F(x) = \sum_{n \geq 0} a_n x^n$$

$$1 - x^2 = (1-x-x^2)F(x)$$

$$= \sum_{n \geq 0} a_n x^n - \sum_{n \geq 0} a_n x^{n+1}$$

Substring: contiguous sequence contained in string

If s and t are strings, st is the *concatenation* of s and t

- If s = 10 and t = 110, st = 10110, ts = 11010, and s^2 = 1010
  - (Or, st = 10 • 110 or (10)(110))

Definition

Let S and T be sets of binary strings. Then

$$ST = \{st : (s, t) \in S, T\}$$

Each element of S concatenated with each element of T

Ex. If S = {0,1} and T = {01, 11, E}

S x T = {(0,01), (0,11), (0,E), (1,01), (1,11), (1,E)}

ST = {001, 011, 0, 101, 111, 1} – *bijection*

Ex.2. S = {10,100}, T = {01,1}

S x T = {(10,01), (10,1), (100,01), (100,1)

ST = {1001, 101, 10001, 1001}

not a bijection

Weight function is, by default, the length of a string

w*(s,t) = w(s) + w(t) = |s| + |t| = |st|

As such, the weights of *Cartesian products* is the same as the weight of concatenations (|st|)

w* is Cartesian products

From here, it follows that

all possible strings from an alphabet S

$$S^* = \varepsilon \cup S \cup S^2 \cup S^3 \cup \cdots$$

all length 2 strings from an alphabet S since S^2 does:

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for i in S:
    for j in S:
        ij
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$S^k \to k$ nested loops

Ex. $S_1 = \{1\}\{1\}^*$
$= \{1, 11, 111, \ldots\}$

$$\Rightarrow \Phi_S(x) = x + x^2 + x^3 + \cdots$$
$$= x\underbrace{(1 + x + x^2 + \cdots)}_{\frac{1}{1-x}}$$
$$= \frac{x}{1-x}$$

Theorem

Let S and T be sets of strings. Then:

$$\Phi_{ST}(x) = \Phi_S(x)\Phi_T(x) \quad \text{if} \quad \underset{\text{(bijection exists)}}{ST \rightleftharpoons S \times T}$$

Ex. $S_1 = \underset{\hookrightarrow T}{\{1\}}S$

let $(t, s) \in T \times S$, and let $f(t, s) = ts$
To show bijection, find an inverse:

let $\sigma \in S_1$. let $g(\sigma) = (\sigma_1, \underbrace{\sigma_2\sigma_3 \cdots \sigma_k}_{\in S})$

$\hookrightarrow$ always $1 \rightarrow \sigma_1 \in T$

So $g(\sigma) \in T \times S$

Inverse exists $\rightarrow$ bijective

Theorem
If $S^k \rightleftharpoons \underbrace{S \times S \times \cdots \times S}_{k \text{ times}}$

Then $$\Phi_{S^*}(x) = \frac{1}{1 - \Phi_S(x)}$$