Similar to the binomial distribution, but instead of 2 outcomes per trial, we have k outcomes per trial

- k outcomes
- Independent trials
- Multiple trials
- Same probability of success in each trial

Probability of each outcome is denoted by p1, p2, …, p_k

Let X_i = number of times that the i-th outcome occurs

$$X_k = n - \underbrace{X_1 - \cdots - X_{k-1}}_{\text{how many times every other outcome happens}}$$

n: number of trials

(X1, …, X_k) follows a multinomial distribution

The number of ways to arrange all k outcomes in n trials is

$$\frac{n!}{\underbrace{x_1!\,x_2!\cdots x_k!}_{\text{repetition of sorts}}}$$

similar to the $\binom{n}{x}$ coefficient in the binomial distribution

$$\text{Where } \sum_{i=1}^{k} X_i = n$$

→ total # of trials

X_i ↳ how many trials resulted in outcome i

Each of these arrangements has probability

$$p_1^{x_1} \cdot p_2^{x_2} \cdot \ldots \cdot p_k^{x_k}$$

<u>Marginal and Joint Probability Functions</u>

If we are interested in finding the marginal distribution of one r.v., say X_2, in the multinomial distribution, we can use:

1. Mathematical Approach:
For this, we would fix the value of x_2 and then sum the joint p.f. over all the other $k-1$ variables, thereby leading to

$$f_2(x_2) = \sum_{all\ x_1, x_3, \dots, x_k} f(x_1, \quad, x_k)$$

for each $x_2 = 0, 1, \dots, n$.
This can be algebraically challenging.

Intuitive and simple approach

Let X2 = the number of occurrences of outcome 2. Clearly, every multinomial distribution will have at least two outcomes.

- Each trial will either see a type 2 outcome or not
- The probability of not getting outcome 2 is 1 - p2
- So, X2 ~ Bin(n, p2)

If we want the distribution of T = X1 + X2, we can treat either outcome 1 or outcome 2 as a “success”, and everything else as a “failure”, then model this using a binomial distribution

T ~ Bin(n, p1 + p2)

Ex. The probabilities that a certain electronic component will last less than 50 hours, last between 50 and 90 hours, or last more than 90 hours, are 0.2, 0.5, and 0.3, respectively. The time to failure of eight such components is recorded.

a) What is the probability that one will last less than 50 hours, five will last between 50 and 90 hours, and two will last more than 90 hours?

Let X1 = number of components that last ≤ 50 hours

X2 = number of components that last between 50 and 90 hours

X3 = number of components that last more than 90 hours

So (X1, X2, X3) ~ Multinomial(n=8, p1=0.2, p2=0.5, p3=0.3)

We want P(X1=1, X2=5, X3=2)

This is equal to

$$\left(\frac{8!}{1!5!2!}\right) \cdot 0.2^1 \cdot 0.5^5 \cdot 0.3^2 = 0.0945$$

b) What is the probability that at least 3 components will last between 50 and 90 hours?

X2 ~ Bin(n=8, p2=0.5)

We want P(X2 ≥ 3) = 1 - P(X2 < 3) = 1 - P(X2=0) - P(X2=1) - P(X2=2) = 0.855469

Ex. Grades in a Stats class are categorized as either A, B, C, D, or F

The probability of getting any of those grades is 0.1, 0.4, 0.3, 0.15, and 0.05, respectively

Consider a sample of n=25 randomly chosen students

a) What is the probability that A=2, B=10, C=6, D=3, F=2?

$$\frac{25!}{4!10!6!3!2!}(0.1)^4 \cdot \cdot \cdot$$

b) What is the probability that A=4 and B=10?

$$\frac{25!}{4!10!11!} \cdot (0.1)^4 (0.4)^{10} (0.5)^{11}$$

↳ everything else

c) What is P(A=4, B=10, C=6 | F=2)?

- Implies that there are 3 D's

$P(\bar{F}) = 0.95$

↳ $P(A|\bar{F}) = \frac{10}{95}$ ⋮

$P(B|\bar{F}) = \frac{40}{95}$

$$\Rightarrow \frac{23!}{4!10!6!3!} \cdot \left(\frac{10}{95}\right)^4 \cdot \left(\frac{40}{95}\right)^{10} \cdots$$

Better approach:

Let X = event of getting 4 A's, 10 B's, 6 C's, and 3 D's. We want

$$P(X \mid 2F\text{'s}) = \frac{P(X \cap 2\ F\text{'s})}{P(2\ F\text{'s})}$$

$$= \frac{P(4\ A\text{'s},\ 10\ B\text{'s},\ 6\ C\text{'s},\ 3\ D\text{'s},\ 2\ F\text{'s})}{P(2\ F\text{'s})}$$