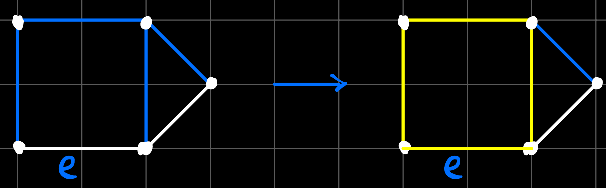
Spanning tree: touches all vertices

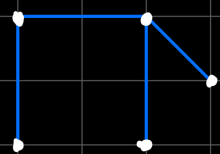
Let T be a spanning tree of a graph G, and let an edge e be in G but not in T. Then:

- T + e has *one cycle* C
- If e’ is in C, Then T + e - e’ is a spanning tree

Ex.



T+e has one cycle
Removing any of the yellow edges produces a spanning tree

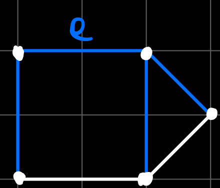


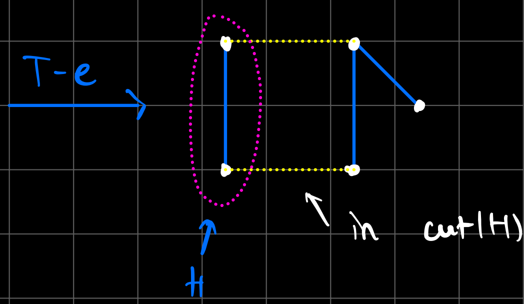
is a spanning tree

If e is an edge in T:

- T-e has two components H and H’ (since all edges e in a tree are bridges)
- For all e’ in the cut of H, T - e + e’ is a spanning tree

Ex.





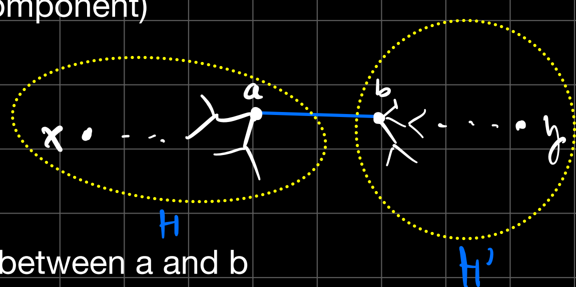
adding back one of the yellow edges produces a spanning tree

Showing that T - e + e’ is connected:

Let x be in V(H)

If y is in V(H), then there is obviously a path from x->y (same component)

If y is not in V(H):



e is a bridge between a vertex a in H and a vertex b in H’

There is a path from x->a and a path from b->y, and one edge between a and b

So there is a path from x->y

Bipartite Characterization

Lemma

Any tree T is bipartite.

- No cycles -> only one way to reach each vertex

An odd cycle is *not* bipartite:

- Suppose we color all even vertices with one color and all the odd ones with another
    - Color 1: 1, 3, 5, 7, … 1
    - This cycle starts with 1 (odd) and ends with an odd vertex, meaning that these two vertices of the same color or adjacent. So the graph cannot be bipartite.
- So any graph with an odd cycle cannot be bipartite.

Theorem

A graph G has no odd cycles if and only if it is bipartite.

(<=) If G has an odd cycle, it is *not* bipartite (proof above)

Contrapositive: If G is bipartite, then it has no odd cycles

This proves the backward direction

(=>) Contrapositive: If a graph is not bipartite, then it has an odd cycle

If G is not bipartite, then it has a component H that is not bipartite

Consider the spanning tree of H, T

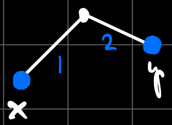
Since T is a tree, T must be bipartite

Let (A, B) be the bipartition of T

H is not bipartite, so (A,B) is not a valid bipartition of H

Which means that without loss of generality, there are two vertices x, y in A that share an edge in H

If x and y have the same color, then there must be an even number of edges between them



$x_1, x_3, x_5, \ldots, y$ have same color

(odd) - (odd) = even

So x->…->y->x is an odd cycle

This proves the contrapositive.

To show a graph is not bipartite, show there is an odd cycle (usually)

Minimal spanning tree: spanning tree with the lowest possible weight (summed weights of all edges)

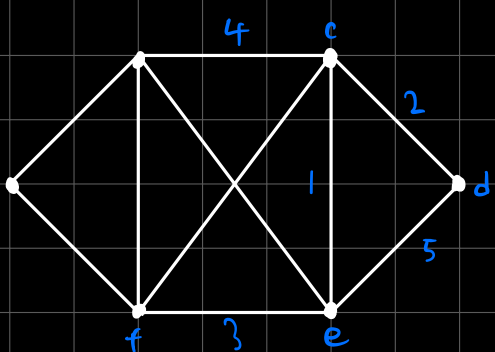
Algorithm for finding an MST:

Suppose we start at a vertex c.

$T_0 = \{c\}$

$T_{k+1} = T_k + e_k + v_k$ where $e_k \in cut_{T_k}(G)$ with minimal weight

Ex.



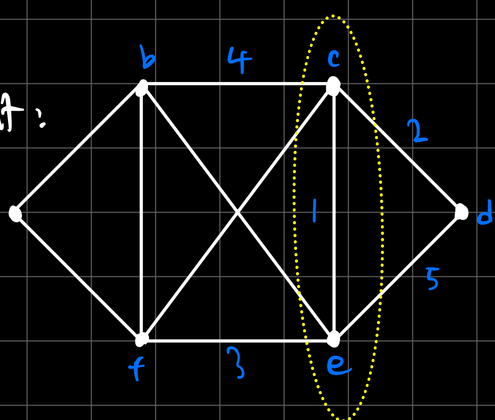
$T_0 = \{c\}$

$cut_{T_0}(G) : \{ce, cd, cf\}$

ce has min. weight

T_1 :

Cut :



$\rightarrow$ Cut : $\{cd, ed, bc, cf, ef\}$