Definition: Norm

Let (V, < , >) be an inner product space.

The length, or norm, of a vector v in V is given by

$$\|\underline{v}\| = \sqrt{\langle \underline{v}, \underline{v} \rangle}$$

Ex. $V = P_2([-1, 1])$ $\qquad \langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$

$\underline{v} = 1 - 2x + 3x^2$

$$\|\underline{v}\| = \left(\frac{184}{15}\right)^{1/2}$$

Lemma: Let (V, < , >) be an inner product space. Then:

- For all vectors $\underline{v}$ in V, $\|\underline{v}\| \geq 0$ and $\|\underline{v}\| = 0$ if and only if $\underline{v} = \underline{0}$
- For all constants c, $\|c\underline{v}\| = |c| \cdot \|\underline{v}\|$

Definition: Distance

Let (V, < , >) be an inner product space, and let $\underline{v}$ and $\underline{w}$ be in V.

Then, the *distance* from $\underline{v}$ to $\underline{w}$ is $\text{dist}(\underline{v}, \underline{w}) = \|\underline{v} - \underline{w}\|$

The distance from v to w is the same as the distance from w to v.

Lemma: Cauchy-Schwartz Inequality

If v and w are multiples of each other, $\langle \underline{v}, \underline{w} \rangle \leq \|\underline{v}\| \, \|\underline{w}\|$

Triangle Inequality: Let (V, < , >) be an inner product space

$\|\underline{v} + \underline{w}\| \leq \|\underline{v}\| + \|\underline{w}\|$

Proof:

Recall that if z = a+bi, the modulus of the complex number

$$|z| = \sqrt{a^2 + b^2} \geq |a| = |\text{Re}(z)|$$

$\langle \underline{v}+\underline{w}, \underline{v}+\underline{w}\rangle = \langle \underline{v}, \underline{v}+\underline{w}\rangle + \langle \underline{w}, \underline{v}+\underline{w}\rangle$ linearity in 1st ar

$= \langle \underline{v}, \underline{v}\rangle + \langle \underline{v}, \underline{w}\rangle + \langle \underline{w}, \underline{v}\rangle, \langle \underline{w}, \underline{w}\rangle$

$\|\underline{v}+\underline{w}\|^2 = \|\underline{v}\|^2 + 2\|\underline{v}\|\,\|\underline{w}\| + \|\underline{w}\|^2$

$\vdots$

Use Cauchy-Schwartz

Definition: Unit vector

$\|\underline{v}\| = 1$

Definition: Normalization

Let (V, < , >) be an inner product space. Let w be a nonzero vector in V. Then:

$$\hat{\underline{w}} = \frac{\underline{w}}{\|\underline{w}\|}$$

Is a unit vector in the direction of $\underline{w}$.

Ex. In $P_2([-1, 1])$, $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$

$B = \{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$

$\underline{v}_1 = 1 \;:\; \langle 1, 1\rangle = \int_{-1}^{1} 1\,dx = 2 \quad : \hat{\underline{v}}_1 = \frac{1}{\sqrt{\langle 1, 1\rangle}} = \frac{1}{\sqrt{2}}$

$\underline{v}_2 = x \;:\; \langle x, x\rangle = \int_{-1}^{1} x^2\,dx = \frac{2}{3} \quad : \hat{\underline{v}}_2 = \frac{x}{\sqrt{2/3}}$

$\underline{v}_3 = x^2 \;:\; \langle x^2, x^2\rangle = \frac{2}{5}$