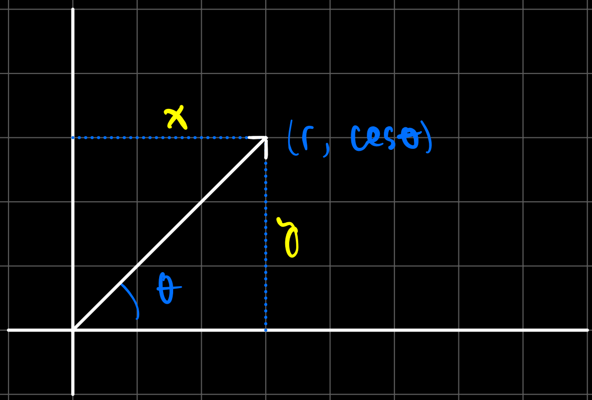
Polar Coordinates



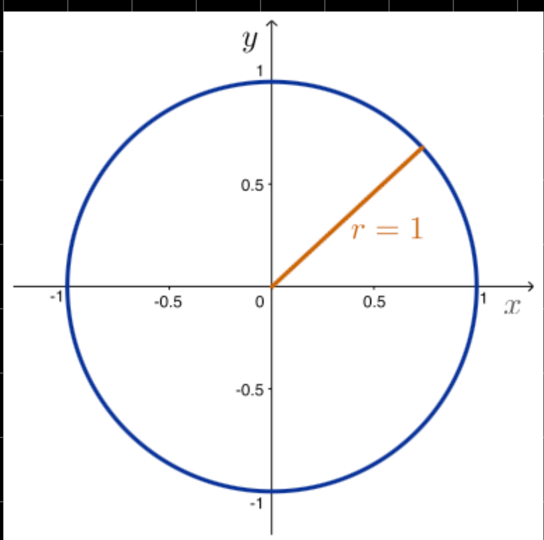
$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$
$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$
$$x^2 + y^2 = r^2$$

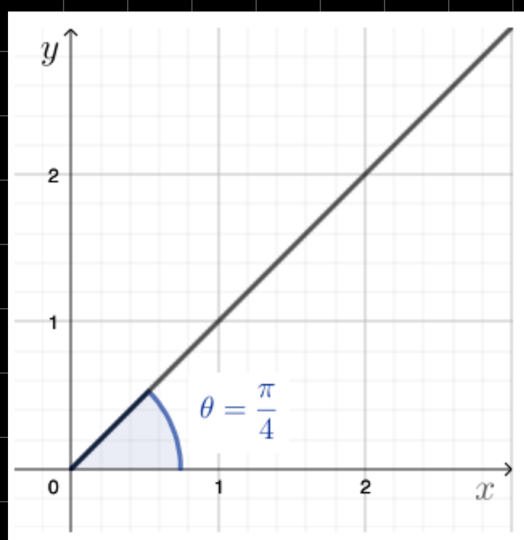
Cartesian to polar:

1. Use formula for r (distance formula)
2. Calculate angle using the formula for either x or y

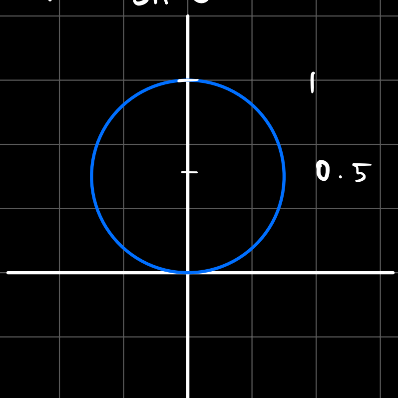
Polar to Cartesian is extremely trivial

<u>Graphs in polar coordinates</u>

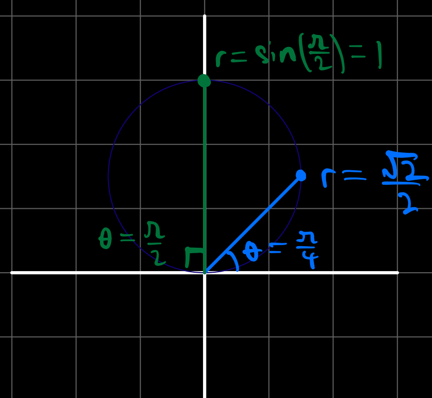




Ex. $r = \sin\theta$



$\rightarrow$



Equations of polar coordinates

Example 9

Convert the equation of the curve $(x^2+y^2)^{3/2}=2xy$ to polar coordinates.

Solution:

Since $x=r\cos\theta$ and $y=r\sin\theta$ we get

$$\begin{aligned}(x^2+y^2)^{3/2}&=2xy\\ r^3&=2(r\cos\theta)(r\sin\theta)\\ r^3&=r^2\sin 2\theta\\ r&=\sin 2\theta\end{aligned}$$

all $x \to r\cos\theta$
all $y \to r\sin\theta$

Notice that the last simplification is only valid since the pole, $r=0$, is still included in the graph (the case where $\theta=\pi$).

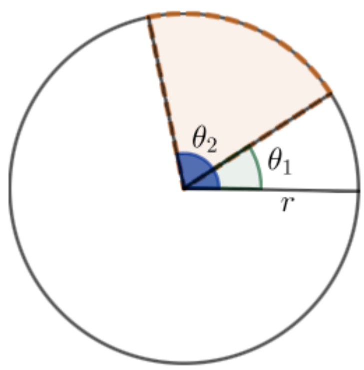
Observe that since we have the restriction $r\geq 0$ we must also have $\sin 2\theta\geq 0$. Hence, we find that a domain of the function is

$$0\leq\theta\leq\frac{\pi}{2},\quad \pi\leq\theta\leq\frac{3\pi}{2}$$

Area in polar coordinates

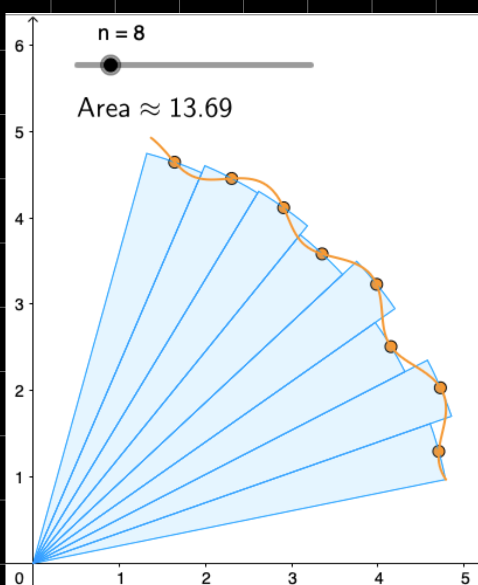
The area of the sector of a circle is given by

$$\text{Area of sector}=\frac{\theta_2-\theta_1}{2\pi}\pi r^2=\frac{1}{2}r^2(\theta_2-\theta_1)$$



Through Riemann sum black magic fuckery, the area under a polar curve is given by

$$\frac{1}{2}\int_a^b [f(\theta)]^2\, d\theta$$



Ex. Find the area inside the curve $r = 2\sqrt{\sin 2\theta}$.

By default, use 0 ≤ θ ≤ 2π, since values beyond that bound will produce the same values

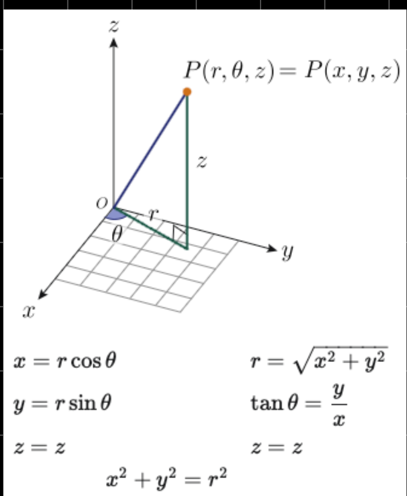
However, in this case, sin(2θ) cannot be negative since it is under a square root

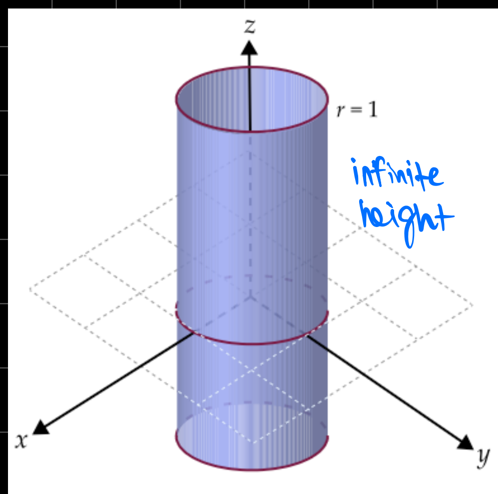
So θ must only range from [0, π/2] and [3π/2, 2π]

$$\Rightarrow A = 2\int_0^{\pi/2} \frac{1}{2}\left[2\sqrt{\sin(2\theta)}\,\right]^2 d\theta = 4$$

Cylindrical Coordinates

Graphs look identical to polar in the x-y plane, but now have a height z



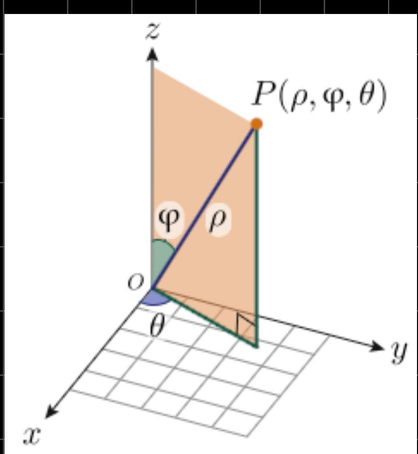


Equations of Cylindrical Coordinates

Ex. Find the equation of $z = \frac{y}{\sqrt{x^2+y^2}}$ in cylindrical coordinates assuming $r \neq 0$.

$$z = \frac{r\sin\theta}{r} = \sin\theta$$

Spherical Coordinates



ρ: length

θ: angle in $\mathbb{R}^2$ (same)

φ: angle to z-axis

$$x = \rho \sin\varphi \cos\theta \qquad \rho = \sqrt{x^2 + y^2 + z^2}$$

$$y = \rho \sin\varphi \sin\theta \qquad \tan\theta = \frac{y}{x}$$

$$z = \rho \cos\varphi \qquad \cos\varphi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$x^2 + y^2 + z^2 = \rho^2$$

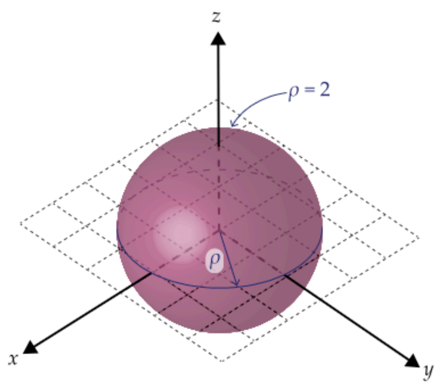
Graphs of Spherical Coordinates

Example 3

Sketch $\rho = 2$.

Solution:

Observe that this is the graph with all points 2 units from the origin. Hence, it is a sphere of radius 2.

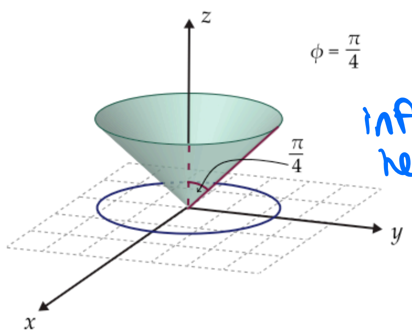


Example 4

Sketch $\varphi = \frac{\pi}{4}$.

Solution:

First, imagine a line that makes a $\frac{\pi}{4}$ angle with the positive z-axis. Since there is no restriction on θ, the graph of the surface will be this line rotated around the positive z-axis. Hence, we get a cone.



infinite height

Conversion

Ex. $z = x^2 + y^2$

$\Rightarrow \rho\cos\varphi = \rho^2 \sin^2\varphi \cos^2\theta + \rho^2 \sin^2\varphi \sin^2\theta$

$\Rightarrow \rho\cos\varphi = \rho^2 \sin^2\varphi (\cos^2\theta + \sin^2\theta)$

$\Rightarrow \cos\varphi = \rho \sin^2\varphi \quad : \quad 0 \leq \varphi \leq \frac{\pi}{2}$ since $0 \leq \sin^2(x) \leq 1 \ \forall x$

$\Rightarrow x \in \left[0, \frac{\pi}{2}\right]$