PDF: f(x, y) = P(X=x, Y=y)

$$= P(\{X = x\} \cap \{Y = y\})$$

Example: Consider the following joint pf given in table form for the r.v.'s X and Y

			x	
	f(x,y)	1	2	3
	1	0.09	0.12	0.13
y	2	0.12	0.11	0.11
	3	0.13	0.10	0.09

Should all sum to 1

Now, suppose that we are only interested in the random variable X. Then, we could need to calculate *marginal* probabilities for X

To get this, sum down the columns:

f_X(1) = f(1,1) + f(1,2) + f(1,3)

$$\Rightarrow f_X(x) = \sum_{\forall y} f(x, y)$$

<u>Independent Random Variables</u>

Recall: Events A and B are independent if P(A ∩ B) = P(A) • P(B)

Then, two *random variables* are independent if

$$f(x, y) = \underbrace{F_X(x) \cdot F_Y(y)}_{\text{product of marginals}} \qquad \text{for all } x, y$$

Ex.

		x			
	$f(x,y)$	1	2	3	$f_Y(y)$
	1	0.09	0.12	0.13	0.34
y	2	0.12	0.11	0.11	0.34
	3	0.13	0.10	0.09	0.32
	$f_X(x)$	0.34	0.33	0.33	1

$$f(1,1) \stackrel{?}{=} f_X(1)\cdot f_Y(1)$$
$$\stackrel{?}{=} (0.34)(0.34)$$
$$= 0.1156$$
$$\neq 0.09 \quad \times$$

Conditional Probability

The conditional probability function of X given Y=y is

$$f(x|y) = P(X=x|Y=y) = \frac{P(\{X=x\}\cap\{Y=y\})}{P(Y=y)} = \frac{f(x,y)}{f_Y(y)}$$

If X and Y are independent random variables:

$$f(X=x|Y=y) = \frac{f(x,y)}{F_Y(y)} = \frac{F_X(x)F_Y(y)}{F_Y(y)} \quad \text{since independent}$$
$$= F_X(x)$$

Similarly, $F(Y=y|X=x) = F_Y(y)$

Functions of random variables

Let U = X - Y. Then, the probability function of U is a function of X and Y.

		x		
	u=x-y	1	2	3
	1	0	1	2
y	2	-1	0	1
	3	-2	-1	0

$$f_u(-2) = f(1,3) = 0.13$$
$$f_u(-1) = f(1,2) + f(2,3) = 0.22$$

If we have T = X + Y, then

$$f_T(t) = \sum_{\substack{\text{all } (x,y) \\ \text{with } x+y=t}} f(x,y)$$

But y = t - x, so this can instead be written as

$$f_T(t) = P(T = t) = \sum_{\text{all } x} f(x, t-x)$$

$$= \sum_{\text{all } x} P(X = x, Y = t-x)$$

Ex. Let X and Y be independent random variables having Poisson distributions with expected values μ1 and μ2. If T = X + Y, find the probability function of T.

Claim: T = X + Y ~ Poisson(μ1 + μ2)

Joint probability function:

$$f(x, y) = F_X(x) \cdot F_Y(y) \quad \text{since independent}$$

$$= \frac{e^{-\lambda_1} \cdot \lambda_1^x}{x!} \cdot \frac{e^{-\lambda_2} \cdot \lambda_2^y}{y!}$$

Since $y = t - x$:

$$P(T = t) = \sum_{\text{all } x} P(X = x, Y = t-x)$$

$$= \sum_{x=0}^{t} \frac{e^{-\lambda_1} \cdot \lambda_2^x}{x!} \cdot \frac{e^{-\lambda_2} \cdot \lambda_2^{t-x}}{(t-x)!}$$

$\therefore$ binomial theorem; multiply by $\frac{t!}{t!}$

$$= \frac{(\mu_1 + \mu_2)^t e^{-(\mu_1 + \mu_2)}}{t!}$$

So $T \sim \text{Poisson}(\mu_1 + \mu_2)$