Taylor's Theorem

$$f(x) - T_{n,a}(x) = \underbrace{R_{n,a}(x)}_{\text{error}}$$

n - degree

a - point

$$= \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$$

for some c between x and a

Corollary: Taylor's Inequality

$$\underbrace{|R_{n,a}(x)|}_{\text{error}} \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

where $M \ge |f^{(n+1)}(c)|$

Ex. If $f(x) = \sqrt{1+x}$, find $T_{2,0}(x)$

$$f'(x) = \frac{1}{2\sqrt{1+x}} \rightarrow f'(0) = 0.5$$

$$f''(x) = \frac{-1}{4(1+x)^{3/2}} \rightarrow f''(0) = \frac{-1}{4}$$

$$f'''(x) = \frac{3}{8(x+1)^{3/2}}$$

$$T_{2,0}(x) = \frac{f(0)}{0!} + \frac{f'(0)(x-0)}{1!} + \frac{f''(0)(x-0)^2}{2!}$$

(2) Approximate $\sqrt{1.1}$.

Since we have $f(x) = \sqrt{1+x}$, we can use $x = 0.1$ and plug this into $T_{2,0} - \frac{839}{800}$

(3) Find the error

$$|R_{n,a}| = \left|\frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}\right| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

$$f^{(n+1)}(c) = f^3(0) = \frac{3}{8}$$

$$\frac{\left(\frac{3}{8}\right)(0.1-0)^3}{6} \le \frac{M}{6}|0.1-0|^3 \rightarrow M = \frac{3}{8}$$

$$\Rightarrow \text{error} \le \frac{M}{6}|0.1-0|^3 = \frac{1}{16000}$$

Taylor's Approximation Theorem I (TAT1)

If $f^{(n+1)}(x)$ is continuous on an open interval I containing $x = a$, then there exists $N \in \mathbb{R}$ $(N > 0)$ such that

$$|f(x) - T_{n,a}(x)| \le N|x-a|^{n+1}$$

or, in squeeze theorem form,

$$-N|x-a|^{n+1} \le f(x) - T_{n,a}(x) \le N|x-a|^{n+1}$$