# Linear Approximations

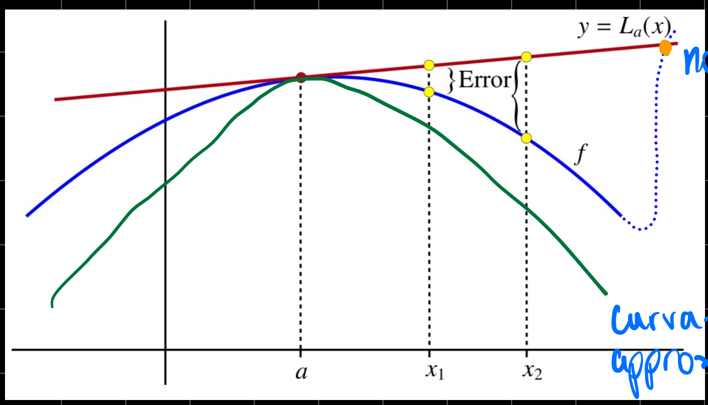
L.A. of $f$ at $x = a$ is denoted by $L_a^f$

$$L_a^f = f(a) + f'(a)(x-a)$$

Properties:
- $L_a = f(a)$
- $L_a$ is differentiable and $L_a' = f'(a)$
- $L_a^f$ is unique – the only 1st degree polynomial that satisfies the first two conditions

$$\text{error} = |f(x) - L_a(x)|$$

Distance from a $\propto$ error




no error here
so error will NOT ALWAYS increase with distance

Curvature also affects error: $L_a(x)$ approximates blue better than green

Curvature can be measured as the rate at which the slope of the tangent line changes, or $f''(x)$. This leads us to:

**Theorem:** Error in Linear Approximation

Assume $|f''(x)| \leq m$ for each $x$ in an interval $I$ s.t. $a \in I$.
Then:

$$\text{error} = |f(x) - L_a(x)| \leq \underbrace{\frac{m}{2}(x-a)^2}_{\text{upper bound on error}} \quad \text{for each } x \in I$$

Ex. Find the upper bound of the error in using $L_4^f$ to approximate $f(x) = \sqrt{x}$ if $x \in [1, 6]$

Since we assumed $|f''(x)| \le m$, we must first find $m$:

$$f'(x) = \frac{1}{2}x^{-0.5} = \frac{1}{2\sqrt{x}}$$

$$\Rightarrow L_4^f = f(4) + f'(4)(x-4) = 2 + \left(\frac{1}{2\sqrt{4}}\right)(x-4) = 2 + \frac{1}{4}x - 1 = \frac{1}{4}x + 1$$

(you can probably skip this)

$$f''(x) = \frac{d}{dx}\left(\frac{1}{2}x^{-0.5}\right) = \frac{-1}{4}x^{-3/2}$$

Since our bounds are $[1, 6]$, $|f''(x)| \le \frac{1}{4} \rightarrow m = \frac{1}{4}$ (at $x = 1$)

m is maximized when you sub 1 into $f''(x)$.
To get the lower bound, use 6

$$\text{error} \le \frac{m}{2}(x-a)^2 = \frac{1/4}{2}(1-4)^2 = \frac{9}{8}$$

## Estimating Change

$$\Delta f \approx f'(a)\Delta x$$

Ex. You are inflating a spherical balloon. At some point, the radius is 20 m. If you exhale once and the radius increases to 20.01 m, estimate the change in volume.

$$V(r) = \frac{4}{3}\pi r^3 \rightarrow V'(r) = 4\pi r^2$$

$$\Delta f = \underbrace{4\pi \cdot (\underset{r^2}{20^2})}_{f'(a)} \cdot (\underset{\Delta x}{0.01}) = 16\pi \text{ m}^3$$

Inverse Function Theorem - assume:

- $y = f(x)$ is continuous and invertible on $[c, d]$
- $f$ is differentiable on any $a \in [c, d]$
- Inverse fxn: $x = g(y)$

If $f'(a) \neq 0$, then $g$ is differentiable at $b = f(a)$, and:

$$g'(b) = \frac{1}{f'(a)} = \frac{1}{f'(g(b))}$$

$L_a^f$ is also invertible: $(L_a^f)^{-1}(x) = L_b^g(x)$

Ex. $\frac{d}{dx}\sqrt{\log_{10}(7+\sin x)}$

$$= \frac{1}{2}(\log_{10}(7+\sin x))^{-1/2} \cdot (\log_{10}(7+\sin x))'$$

$$= \frac{1}{2}(\log_{10}(7+\sin x))^{-1/2} \cdot \left(\frac{\ln(7+\sin x)}{\ln 10}\right)'$$

Note that $\log_a x = \frac{\ln x}{\ln a}$

$$\hookrightarrow \frac{1}{(7+\sin x)\ln 10} \cdot (7+\sin x)'$$

$$= \frac{\cos x}{(7+\sin x)\ln 10}$$

$$\Rightarrow \frac{1}{2\ln 10} \cdot (\log_{10}(7+\sin x))^{-1/2} \cdot \frac{\cos x}{(7+\sin x)}$$

Derivatives of Inverse Functions:

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arctan x = \frac{1}{x^2+1}$$

$$\frac{d}{dx}\arccos x = \frac{-1}{\sqrt{1-x^2}}$$

Proof of $\frac{d}{dx}\arcsin x$:

Let $f(g(x)) = \sin(\arcsin x) = x$
Note that $f(x) = g^{-1}(x)$. So:

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos(\arcsin x)}$$

Note that $\sin^2 x + \cos^2 x = 1 \Rightarrow \cos x = \sqrt{1-\sin^2 x}$

So $$g'(x) = \frac{1}{\sqrt{1-\sin^2(\arcsin x)}} = \frac{1}{\sqrt{1-x^2}}$$

Using the fact that $(\ln(x))' = \frac{1}{x}$, prove that $(\log_a(x))' = \frac{1}{x\ln(a)}$ for $a > 0$, $a \neq 1$.

$$(\log_a x)' = \left(\frac{\ln x}{\ln a}\right)' = (\ln x)' \cdot \frac{1}{\ln a} = \frac{1}{x}\cdot\frac{1}{\ln(a)} = \frac{1}{x\cdot\ln(a)} \square$$

$f(x) = \frac{6x+1}{3x+5}$ Newton's method: $x_n - \frac{f(x_n)}{f'(x_n)}$

$$f'(x) = \frac{(3x+5)6 - (6x+1)(3)}{(3x+5)^2} = \frac{18x+30-18x-3}{(3x+5)^2} = \frac{27}{(3x+5)^2}$$

$$x_{n+1} = x_n - \frac{\frac{6x_n+1}{\cancel{3x_n+5}}}{\frac{27}{(3x_n+5)^{\cancel{2}}}} = x_n - \frac{(6x_n+1)(3x_n+5)}{27}$$

We know that $x_2 < 0$. So, if $x_n$ is decreasing from $x_2$, the sequence will never converge to $0$.

Proof: Expand $x_{n+1} < x_n$

$$\cancel{x_n} - \frac{(6x_n+1)(3x_n+5)}{27} < \cancel{x_n}$$

$$\frac{(6x_n+1)(3x_n+5)}{27} > 0$$

$$(6x_n+1)(3x_n+5) > 0$$

$$x_n = \frac{-5}{3} \qquad x_n = \frac{1}{6}$$

$$x_n \in \left(-\infty, \frac{-5}{3}\right) \cup \left(\frac{1}{6}, +\infty\right)$$

$x_2 \in \left(-\infty, \frac{-5}{3}\right)$. Then, since $x_n \in \left(-\infty, \frac{-5}{3}\right)$, $x_{n+1} < x_n$. As such, the sequence is indeed decreasing, and since $0 \notin \left(-\infty, \frac{-5}{3}\right)$, the sequence never converges to $0$.

$$\frac{d}{dx} \arcsin(\tan x + x^3) e^x$$

Using the product rule:

$$\arcsin(\tan x + x^3) e^x \to \arcsin(\tan x + x^3)(e^x)' + [\arcsin(\tan x + x^3)]' e^x$$

Solving $[\arcsin(\tan x + x^3)]'$:

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx} \tan x + x^3 = \sec^2 x + 3x^2$$

$$\Rightarrow \frac{\sec^2 x + 3x^2}{\sqrt{1-(\tan x + x^3)^2}}$$

So the function becomes $\arcsin(\tan x + x^3) e^x + \frac{\sec^2 x + 3x^2}{\sqrt{1-(\tan x + x^3)^2}} \cdot e^x$

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