A *list* of a set S contains all the elements exactly once each, in some order.

For example, some lists of the set {1,2,3} are:

- {1,2,3}
- {2,1,3}
- {1,3,2}…

A *permutation* is a list of the set {1,2, … n}

The number of lists of an n-element set S is given by

$$p_n = n \cdot p_{n-1}$$

This can easily be proved using induction.

If n=1, you can only make one list, so p_n = 1

Next, to construct a list of S with more than one element, we take one element out, add it to the list, and keep going

If S has two elements, we first remove one element from the list. Taking out the next element, there are two places we can possibly put it; i.e.

$$p_2 = 2 \cdot p_{n-1} = 2 \cdot 1 = 2$$

Theorems

For every n ≥ 1, the number of lists of an n-element set S is

n(n-1)(n-2) … • 3 • 2 • 1

For every n ≥ 0, the number of subsets of an n-element set is 2^n

From, here it follows that for n, k ≥ 0, the number of partial lists of length k of an n-element set is

n(n-1)(n-2) … (n-k+2)(n-k+1)

(just subtract k from previous result, kind of)

Since there are:

- n choices for the first element
- n-1 choices for the next one
- (n-(k-1)) = (n-k+1) choices for the last one

Note that

$$n(n-1)\cdots(n-k+2)(n-k+1) = \frac{n!}{(n-k)!}$$

$$= \underbrace{\binom{n}{k}}_{\#\text{ subsets}} \cdot k! = \underbrace{\frac{n!}{(n-k)!}}_{\#\text{ lists}}$$

More lists because lists contain different orders of the subsets

Where $\binom{n}{k}$ is the number of k-element subsets of an n-element set S

$$\Rightarrow \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Since there are 2^n subsets:

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

Multisets

Suppose we pull 11 marbles out of a bag containing red, blue, and green marbles. All possible outcomes are given by the *multiset* (r,g,b); for example (4,5,2)

Definition

A multiset of size n with elements of t types is a sequence of nonnegative integers:

$$(m_1, m_2, \dots m_t) \to m_1 + m_2 + \cdots + m_t = n$$

n is the length of the multiset - we can assume infinite marbles in the bag

Theorem

For any n ≥ 0 and t ≥ 1, the number of n-element multisets with elements of t types is

$$\binom{n+t-1}{t-1}$$

or:

$$\left(\!\binom{n}{t}\!\right) = \binom{n+t-1}{t-1}$$

Ex. Suppose $k_1 + \cdots + k_t = n$ $(k_1, \dots, k_t)$

How many ways have k1 = 2?

$$2 + k_2 + \cdots + k_t = n$$
$$k_2 + \cdots + k_t = n - 2$$

$$\Rightarrow \left(\!\binom{n-2}{t-1}\!\right)$$

How many ways have each k_i ≥ 2?

Set $l_i = k_i - 2$
$\Rightarrow l_i \geq 0$

$k_1 + \cdots + k_t = n$

$$\Rightarrow \underbrace{(k_1 - 2) + (k_2 - 2) + \cdots + (k_t - 2)} = n - 2t$$

subtracting 2 "t" times
Hence the $-2t$

$$\Rightarrow l_1 + \cdots + l_t = n - 2t$$

$$\Rightarrow \left(\!\!\binom{n-2t}{t}\!\!\right)$$

Ex.2. If we are dealt a 5-card hand, what is the probability of getting a full house?

(Full house: 3 cards of the same number and a pair of the same number)

P = (number of ways something can happen) / (total number of possible outcomes)

$\binom{52}{5}$

Pick two distinct numbers, one for the pair, one for the three

$$13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}$$

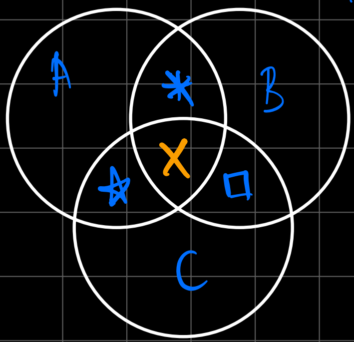
What is the probability of getting two distinct pairs?

$$\frac{13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2} \cdot 44 \cdot \frac{1}{2}}{\binom{52}{5}}$$

<u>Inclusion/Exclusion</u> - count everything *exactly once*

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = \underbrace{|A| + |B| + |C|}_{\text{counts } \times \text{ 3 times}} - \underbrace{|A \cap B| - |A \cap C| - |B \cap C|}_{\text{subtracts } \times \text{ 3 times since all the intersections include } \times} + \underbrace{|A \cap B \cap C|}_{\text{adds back } \times}$$



Ex.2. How many ways can we pick 3 numbers from 1 to 6 without repetition?

$\binom{6}{3}$ or $\left(\!\binom{3}{6}\!\right)$

↳ 3 element multiset with elements of 6 types