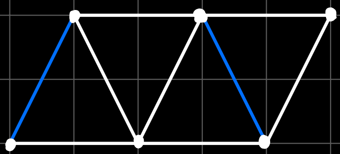
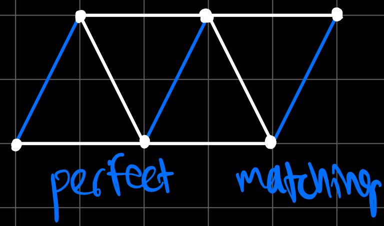
A matching is a set of edges M such that each vertex in G is incident to ≤1 edge in M

- No two edges in M have a common end
- In the spanning subgraph of G with edge set M, each vertex has degree 1
- Vertices in G are *saturated* by M if they are incident to an edge in M
- *Perfect matching:* saturates every vertex





The largest matching is

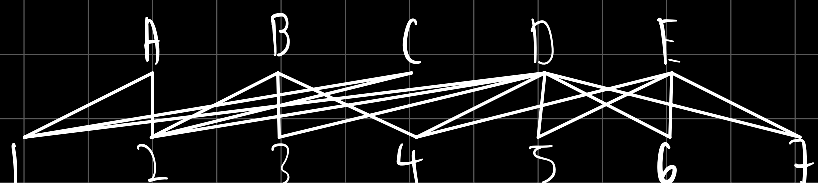
$$\left\lfloor \frac{|V(G)|}{2} \right\rfloor$$

An application of matchings is *job assignment*

- A can do jobs 1 and 2
- B can do jobs 2, 3, and 4
- C can do jobs 1, 2 …

The most jobs that can be done by 5 people A, B, C, D, and E corresponds to the maximal matching in a bipartite graph partitioned into:

- X: A, B, C, D, E
- Y: Jobs 1, 2, 3, … n



Suppose a matching is M = {A1, B2, D4, E7}

An *alternating path* is a path with edges alternating in and out of M

- 1A 2B 4D
    - 1A is in M
    - A2 is not
    - 2B is in M
    - B4 is not

An *augmenting path* is an alternating path that starts and ends at a non-saturated vertex

- 6E 7D 4B 2C
  - 6 is not saturated by M; 6E is not in M
  - 7E is in M

A larger matching M’ is M + {6E, 7D, 4B, 2C} - {E7, D4, B2}

not in M in M

|M’| = |M| + 1

now includes nonsaturated vertices

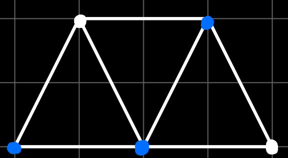
Lemma

If M has an augmenting path, it is not maximal

- Because we can replace the edges in M with edges not in M

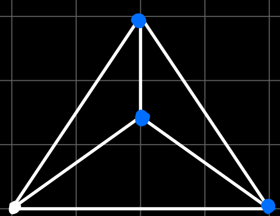
<u>Cover</u>

A cover is a set of vertices C such that for every edge xy in G, x or y (or both) is in C



Minimal cover: 3 vertices

For K4, the minimal cover has 3 vertices



For any K_n, |C| = n-1

Proof:

Inductive hypothesis: above

K_n contains K_{n-1}, and K_{n+1} contains one additional vertex. This vertex will be connected to

- All n-1 vertices in C
- 1 vertex not in C — draw an edge between the new vertex and this vertex

So |C| = n

Lemma

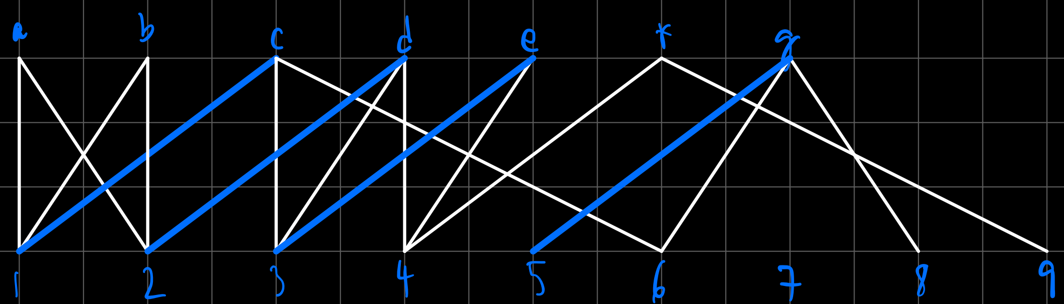
|M| ≤ C for any cover and any matching

- max |M| ≤ min |C|
- If |M|=|C| then M is max and C is min

<u>Konig's Theorem</u>

If G is bipartite, there exist matchings M and C such that |M| = |C| (max M, min C)

Proof:



(1) XY-construction - for some matching:

X = {unsaturated vertices in A} = {a,b,f}

Y = {neighbors of X in B} = {1,2,4,9}

If X is empty, there is no augmenting path, which means that we have found a maximal matching

(2) If there are any unsaturated vertices in Y, there is an augmenting path.

Here, 4 and 9 are unsaturated. So add their corresponding vertex (f) to the alternating path

Now repeat step 1:

X = {a,b}

Y = {1,2}

Now, there are no unsaturated vertices in Y

So:

X = X U {vertices in A reached via edges in the matching from Y} = {a,b} U {c,d}

connect to {1, 2}

Y = Y U {neighbors of X in B} \ Y = {1,2} U {3,4,6}

or, neighbors of newly added edges

neighbors of c and d

Keep repeating step 2 until either X or Y is empty

Then, a cover the size of the matching is C = Y U (A \ X)

Why is this a minimal cover?

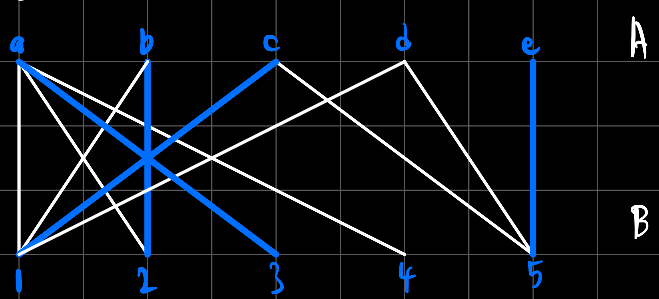
- Every edge in Y corresponds to something in X
- So adding the vertices in A \ X to the cover means that every edge connects to something in the cover
- Also, every edge connecting to A \ X, by definition, is saturated, meaning that it is part of the matching. And, clearly, every edge between vertices in X and Y are part of the matching.

As such, the cover only includes edges in the matching, so |M| = |C|.

Also, for any M and C in G, |M| ≤ |C|

So M is maximal and C is minimal

<u>Finding a minimal cover</u>



(1) Let X = {d} <- unsaturated vertex

Let Y = N(X). This is the set of all neighbors of all vertices in X.

Y = {1,5}

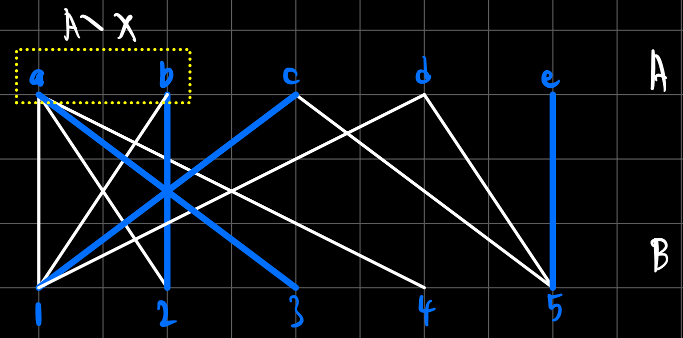
(2) Let X = X U {vertices in A matched by M with a vertex in Y}

X = {c,d,e}

Y = {1,5}

Y is the same -> *done*

Then, a minimal cover is C = Y U (A \ X)



This is a cover because:

- All vertices in X are incident to something in Y
- We then add all vertices in A, but not in X

Moreover, |C| = |M| because:

- All vertices in Y are saturated
- All vertices in A \ X are saturated — we constructed X as a set of unsaturated vertices
- Y and A \ X do not share edges — proof:

$$|M| \geq |Y| + |A \setminus X| = |C|$$

But $|M| \leq |C|$

So $|M| = |C|$

Ex. Suppose G = (A, B) — bipartite graph

Where |A| = |B| = 5 and |E(G)| = 20

Show that there is a matching of size ≥4 and a cover of size ≤4.

Suppose that the maximal matching has size 3. Then, we have 3 vertices saturating 20 edges, and by König's Theorem, there exists a cover with size 3.

A bipartite graph cannot have a vertex with degree >5

For the above graph to be possible, we need at least one vertex with degree at least 7 -> contradiction

Moreover, the most number of edges saturated by 3 vertices is 15

(Can also use Handshaking Lemma over vertices in cover)

The size of a matching is limited by the size of the smaller bipartition

- If $|A| > |B|$, we cannot have a matching that saturates every vertex in A

<u>Hall's Theorem</u>

Let $G = (A, B)$. There exists a matching saturating A *if and only if* for all subgraphs D of A, $|N(D)| \geq |D|$

- Neighbor set of D *cannot be smaller* than D

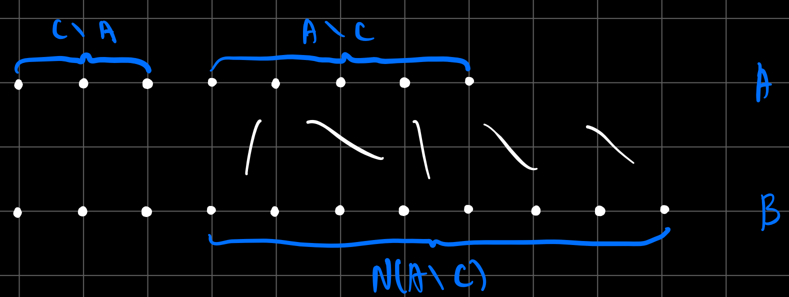
Proof:

(=>) If there exists a subgraph D of A such that $|N(D)| < D$, then D cannot be saturated. Thus, A can also not be saturated.

(<=) *Contrapositive:* If there is no matching M that saturates A, then there exists a subgraph D of A such that $|N(D)| < D$

Suppose M is a maximal matching and C is a cover

If M doesn't saturate A, then $|M| < |A|$ and $|C| < |A|$



…too complicated

<u>Theorem</u>

If $G = (A,B)$ is k-regular for all $k \geq 1$, then there is a perfect matching

Proof:

For there to be a perfect matching, we need $|A| = |B|$

$$2|E(G)| = \sum_{v \in A} \deg(v) = \sum_{v \in B} \deg(v) \quad \text{(since bipartite)}$$

$$\Rightarrow k|A| = k|B|$$

$$\Rightarrow |A| = |B|$$

Now we must show that it satisfies Hall's Theorem

Show $|N(D)| \geq |D|$

$$\sum_{v \in A} \deg(v) = \sum_{v \in N(D)} \deg(v)$$

$$k|D| \leq k|N(D)|$$