Limits as x -> ±∞ - horizontal asymptotes

Limits as f(x) -> ±∞ as x->a - vertical asymptotes

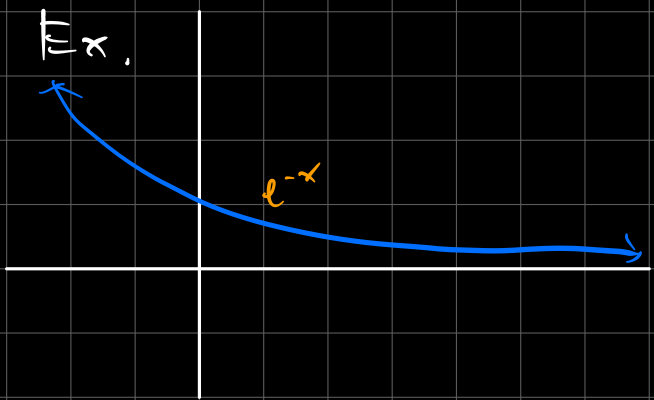
Definition: $\lim_{x \to \infty} f(x) = L$

If for all $\varepsilon > 0$ there exists an $N \in \mathbb{R}$ such that it if $x > N$ then $|f(x) - L| < \varepsilon$

$\lim_{x \to -\infty} f(x) = L$

If for all $\varepsilon > 0$ there exists an $N \in \mathbb{R}$ such that it if $x \leq N$ then $|f(x) - L| < \varepsilon$

Ex.



$$\lim_{x \to \infty} e^{-x} = 0$$

$$= \infty$$

Asymptotes

If $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$,

y = L is a horizontal asymptote of f(x)

Infinite limits

$$\lim_{x \to \infty} f(x) = \infty$$

if for all m > 0 there exists a real number N such that if x > N then f(x) > m

Ex. $\lim\limits_{x \to \infty} \dfrac{x^3 + 3x^2 + 2x}{3x^3 + x + 1}$

$$= \lim_{x \to \infty} \frac{1 + \overset{0}{\frac{3}{x}} + \overset{0}{\frac{2}{x^2}}}{3 + \underset{0}{\frac{1}{x^2}} + \underset{0}{\frac{1}{x^3}}} = \frac{1}{3}$$

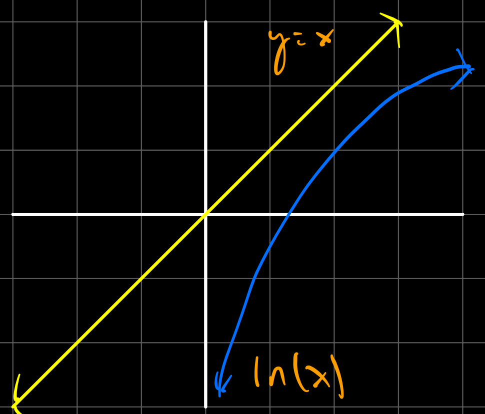
In general:

$$\lim_{x \to \pm\infty} \frac{a_n + a_1 x + a_2 x^2 + \cdots + a_m x^m}{b_0 + b_1 x + b_2 x^2 + \cdots + b_n x_n}$$

$$= \begin{cases} \dfrac{a_m}{b_n} & m = n \\ 0 & m < n \\ \text{DNE} & m > n \end{cases}$$

Fundamental Log Limit

Claim: $\lim\limits_{x \to \infty} \dfrac{\ln x}{x} = 0$



Clearly for $x > 0$,

$\ln x \leq x \to \dfrac{\ln x}{x} \leq 1$

Since $x \to \infty$, say $x > 1$

$\therefore \dfrac{\ln x}{x} \geq 0$

Also:

$$\frac{\ln x}{x} = \frac{\ln[(\sqrt{x})^2]}{\sqrt{x}\cdot\sqrt{x}} = \frac{2\ln(\sqrt{x})}{\sqrt{x}\cdot\sqrt{x}}$$

$$= \frac{2}{\sqrt{x}}\cdot\frac{\ln(\sqrt{x})}{\sqrt{x}} \le \frac{2}{\sqrt{x}}$$

$$\therefore 0 \le \frac{\ln x}{x} \le \frac{2}{\sqrt{x}}$$

For $p > 0$, $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x^p} = 0 \quad (\star)$

$$\lim_{x\to\infty}\frac{\ln(x^p)}{x} = 0$$

Ex. $\displaystyle\lim_{x\to\infty}\frac{(\ln x)^{3000}}{x} = \lim_{x\to\infty}\left(\frac{\ln x}{x^{1/3000}}\right)^{3000} = 0^{3000} \quad (\star)$

Ex. $\displaystyle\lim_{x\to 0^+} x^p \ln x = 0$ for $p > 0$

Let $u = \frac{1}{x} \to \displaystyle\lim_{u\to\infty}\frac{\ln\left(\frac{1}{u}\right)}{u^p}$

$$= \lim_{u\to\infty}\frac{-\ln|u|}{u^p} = 0 \quad (\star)$$

Ex. 2. $\lim_{x \to \infty} \frac{x^p}{e^x} = 0$ for $p > 0$

Let $u = e^x \rightarrow x = \ln(u)$

$$\Rightarrow \lim_{u \to \infty} \frac{(\ln(u))^p}{u} = 0$$

Therefore

$$(\ln x)^p << x^p << e^x$$

Vertical asymptote - x approaches a finite point, function approaches infinity

Definition:

$\lim_{x \to a^+} f(x) = \infty$ if for all $M > 0$ there exists a $\delta > 0$ such that if $a < x < a + \delta$ then $f(x) > M$

$\lim_{x \to a^-} f(x) = \infty$ if for all $M > 0$ there exists a $\delta > 0$ such that if $a - \delta < x < a$ then $f(x) > M$

$\lim_{x \to a} f(x) = \infty$

If left hand limit and right hand limit exist

(Note: saying a limit = ∞ means it doesn't exist and gets infinitely large)

Definition of vertical asymptotes

If $\lim_{x \to a^{\pm}} f(x) = \pm\infty$, then $x = a$ is a vertical asymptote of f

Ex. $\lim_{x \to 3^+} \frac{(x+7)(x-7)}{(x-3)(x-2)} = -\infty$

approaches -4

Since as $x \to 3^+$, $\frac{\overset{+}{(x+7)}\overset{-}{(x-7)}}{\underset{+}{(x-3)(x-2)}} < 0$

Ex. 2. $f(x) = \frac{e^x}{e^x - 1}$

Vertical asymptotes

If $x = 0$, denominator is undefined

$\therefore \lim_{x \to 0^+} f(x) = \infty$

VA: $x = 0$

Horizontal asymptotes

$\lim_{x \to \infty} \frac{e^x}{e^x - 1} = 1$

(Kinda intuitively based on ratio of numerator : denominator)

$\lim_{x \to -\infty} \frac{e^x}{e^x - 1} = 0$