Theorem 1: The Chain Rule

Let $G(t) = f(x(t), y(t))$, and let $a = x(t_0)$ and $b = y(t_0)$. If f is differentiable at (a, b) and $x'(t_0)$ and $y'(t_0)$ exist, then $G'(t_0)$ exists and is given by

$$G'(t_0) = f_x(a,b)x'(t_0) + f_y(a,b)y'(t_0)$$

In one variable:

- Let $G(t) = f(x(t))$ and $a = x(t_0)$
- If f is differentiable at a and $x'(t_0)$ exists, then $G'(t_0)$ exists and is given by

$$G'(t_0) = f'(a) \cdot x'(t_0)$$
$$= f(x(t_0)) \cdot x'(t_0)$$

Ex. Use the Chain Rule to find $\frac{df}{dt}$ for $f(x, y) = xy^3 - x^3y$ with $x(t) = t^2 + 1$ and $y(t) = t^2 - 1$ at $t_0 = 1$.

$a = x(t_0) = 2$
$b = y(t_0) = 0$

$$\frac{\partial f}{\partial x} = y^3 - 3x^2y \to 0 \qquad x'(t) = 2t$$

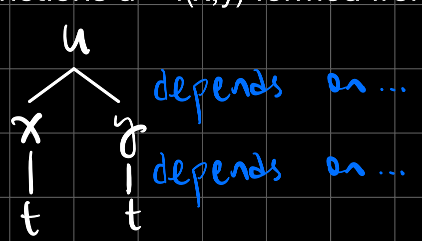
$$\frac{\partial f}{\partial y} = 3y^2x - \underset{2^3}{x^3} \to -8 \qquad y(t) = 2t$$

$$\Rightarrow \frac{df}{dt} = 0 \cdot 2t - (8)(2t)$$
$$= -16t$$

At $t = 1$, $\frac{df}{dt} = 16$

<u>Dependence Diagrams</u>

For functions u = f(x,y) formed from differentiable functions x(t) and y(t):



Here, u is the dependent variable (the thing we are measuring)

x and y are the intermediate variables

And t is the independent variable (the value we change)

Thus:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \text{rate of change wrt } x + \text{rate of change wrt } y \\ &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}\end{aligned}$$

Algorithm

To write the Chain Rule from a dependence diagram we do the following:

1. Identify all of the variables.
2. Take all possible paths from the differentiated variable to the differentiating variable.
3. For each link in a given path, differentiate the upper variable with respect to the lower variable being careful to consider if this is a derivative or a partial derivative. Multiply all such derivatives in that path.
4. Add the products from step 3 together to complete the Chain Rule.

Example 1

Let $z = f(x, y) = (x - y)^4$ where $x = st^4$ and $y = s^4 t$. Use the Chain Rule to find the first-order partial derivatives $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \underbrace{4(x-y)^3}_{\frac{\partial z}{\partial x}}\underbrace{(t^4)}_{\frac{\partial x}{\partial s}} + \underbrace{4(-1)(x-y)^3}_{\frac{\partial z}{\partial y}}\underbrace{(4s^3 t)}_{\frac{\partial y}{\partial s}}$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \underbrace{4(x-y)^3}_{\frac{\partial z}{\partial x}}\underbrace{(4st^3)}_{\frac{\partial x}{\partial t}} + \underbrace{(4)(-1)(x-y)^3}_{\frac{\partial z}{\partial y}}\underbrace{(s^4)}_{\frac{\partial y}{\partial t}}$$