In one variable:

If $g'(a)$ exists, then

$$\lim_{x\to a}\frac{|R_{1,a}(x)|}{|x-a|}=0$$

where $R_{1,a}(x) = g(x) - L_a(x)$: error → dist. to TL

$$= g(x) - g(a) - g'(a)(x-a)$$

$R_{1,a}(x) \to 0$ faster than $|x-a| \to 0$

In two variables:

Definition: Differentiable

A function $f(x,y)$ is **differentiable** at (a,b) if

$$\lim_{(x,y)\to(a,b)}\frac{|R_{1,(a,b)}(x,y)|}{\|(x,y)-(a,b)\|}=0$$

where

$$R_{1,(a,b)}(x,y)=f(x,y)-L_{(a,b)}(x,y)$$

Theorem 2

If a function $f(x,y)$ satisfies

$$\lim_{(x,y)\to(a,b)}\frac{|f(x,y)-f(a,b)-c(x-a)-d(y-b)|}{\|(x,y)-(a,b)\|}=0$$

for some constants c and d then $c=f_x(a,b)$ and $d=f_y(a,b)$.

Proof: Since L=0, f approaches 0 from any path, including y=b. So we can reduce this to

$$\lim_{x\to a}\frac{|f(x,b)-f(a,b)-c(x-a)-\cancel{d(b-b)}|}{\underbrace{\|(x,b)-(a,b)\|}_{\hookrightarrow\ |x-a|}}=0$$

$$= \lim_{x \to a} \left| \frac{f(x, b) - f(a, b)}{x - a} - c \right|$$

$h = x - a$
$x = h + a$

$$= f_x(a, b) - c = 0$$
$$\Rightarrow f_x(a, b) = c$$

Ex. $f(x, y) = |x|^4 \cdot |y|^{1/3}$ diff. at $(0, 0)$?

TP: $\frac{\partial f}{\partial x} = 4|x|^3 \cdot \frac{x}{|x|} = 4x \cdot |x|^2 \cdot |y|^{1/3}$
$= 4x^3 \cdot |y|^{1/3} \to 0$

$\frac{\partial f}{\partial y} = |x|^4 \cdot \frac{1}{3}|y|^{-2/3} \cdot \frac{y}{|y|} \to 0$ using limits $(h \to 0)$

$$\Rightarrow \frac{|x|^4 \cdot |y|^{1/3}}{\sqrt{x^2 + y^2}} \to 0?$$

Along $x = y$:

$$\frac{x^{13/3}}{x^2} = x^{7/3} \to 0$$

So using limits: $(L = 0)$

$$x^2 \leq x^2 + y^2$$
$$\Rightarrow x \leq (x^2 + y^2)^{1/2}$$

$$\Rightarrow \frac{|x|^4 \cdot |y|^{1/3}}{\sqrt{x^2 + y^2}} \leq \frac{[(x^2 + y^2)^{1/2}]^4 \cdot |y|^{1/3}}{(x^2 + y^2)^{1/2}}$$

$$= (x^2 + y^2)^{3/2} \cdot |y|^{1/3} \to 0 \checkmark$$

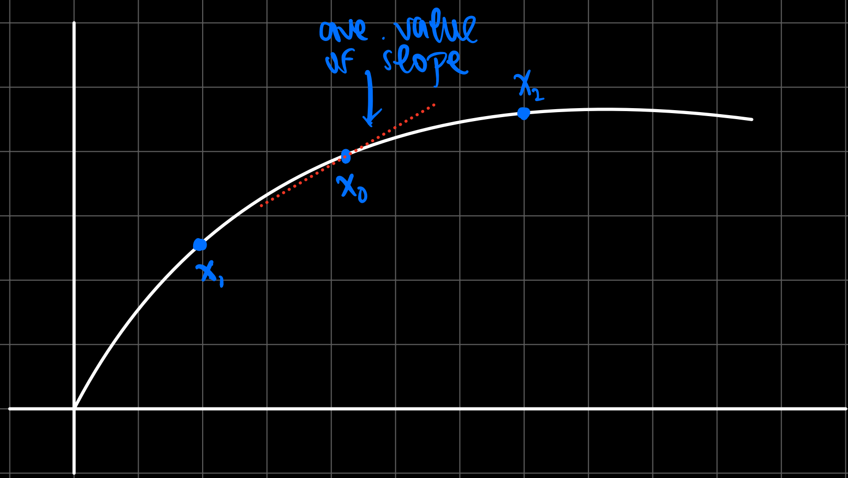
If f(x,y) is differentiable at (a,b), then it is also continuous at (a,b)

Continuous Partial Derivatives and Differentiability

Theorem 1: The Mean Value Theorem

If $f(x)$ is continuous on the closed interval $[x_1, x_2]$ and f is differentiable on the open interval (x_1, x_2), then there exists $x_0 \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(x_0)(x_2 - x_1)$$



Theorem 2

If the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are both continuous at (a, b), then $f(x, y)$ is differentiable at (a, b).

Ex. **Example 1**

Determine at which points $f(x, y) = (x^2 + y^2)^{2/3}$ is differentiable.

By Theorem 2, f(x,y) are differentiable at the points (a,b) where its partial derivatives with respect to both x and y are continuous

$$\frac{\partial f}{\partial x} = \frac{2}{3}(x^2 + y^2)^{-1/3} \cdot 2x$$

$$= \frac{4x}{3 \cdot (x^2 + y^2)^{1/3}}$$

This is not continuous at (0,0); the same applies to the partial derivative with respect to y

As such, f is differentiable for all (x,y) ≠ (0,0)

Note that THE CONVERSE IS NOT TRUE:

- If differentiable at (a,b), its partial derivatives are not necessarily continuous
- If some of its partial derivatives are not continuous at (a,b), it may still be differentiable
- However, if all are continuous at (a,b), the function is differentiable

Steps for checking differentiability at (a,b):

- Calculate partial derivatives
- If all continuous at (a,b), stop -> it is differentiable
- If not, calculate this limit

Definition: Differentiable

A function $f(x, y)$ is **differentiable** at (a, b) if

$$\lim_{(x,y)\to(a,b)} \frac{|R_{1,(a,b)}(x, y)|}{\|(x, y) - (a, b)\|} = 0$$

where

$$R_{1,(a,b)}(x, y) = f(x, y) - L_{(a,b)}(x, y)$$