A point (a,b) is a *local minimum* if $f(x,y) \geq f(a,b)$ for all (x,y) in some neighborhood of (a,b)

A point (a,b) is a *local maximum* if $f(x,y) \leq f(a,b)$ for all (x,y) in some neighborhood of (a,b)

A *local extremum* is a local minimum / maximum

Thinking geometrically:

- If a point (a,b) in the domain of f is a local extremum, then it is a local extremum of all the cross-sections that pass through (a,b). In particular:
    - It is a local extremum for the cross-section f(x,b), which is a function in one variable that we will call g(x). Since x=a is a local extremum of f(x,b) = g(x), either g'(a) = 0 or g'(a) does not exist (MATH 137). From here, it follows that $f_x(a,b) = 0$ or DNE.
    - The same argument can be applied to the cross-section f(a,y).
- Thus, both partial derivatives are either equal to 0 or DNE.
- Also, the tangent plane is z = f(a,b), which is horizontal (partial derivative terms cancel out)

Theorem 1:

If (a, b) is a local maximum or minimum point of f, then each partial derivative is either equal to zero or does not exist.

Definition: Critical Point

A point (a, b) in the domain of $f(x, y)$ is called a **critical point** of f if

$$\frac{\partial f}{\partial x}(a,b) = 0 \text{ or } \frac{\partial f}{\partial x}(a,b) \text{ does not exist,}$$

and

$$\frac{\partial f}{\partial y}(a,b) = 0 \text{ or } \frac{\partial f}{\partial y}(a,b) \text{ does not exist.}$$

converse not true : if CP $\not\Rightarrow$ derivatives are 0 or DNE

All local extrema are critical points, but not all critical points are local extrema

How to calculate critical points:

1. Get partial derivatives
2. Solve for values x=a that make BOTH partial derivatives equal to 0 or DNE (using systems of equations or trial and error)
3. For each of those cases, solve for values y=b

Example 4

Find all critical points of $f(x, y) = x^2y + 3xy^2 + xy$.

Solution:

Finding the partial derivatives, we get

$$\frac{\partial f}{\partial x}(x,y) = 2xy + 3y^2 + y, \qquad \frac{\partial f}{\partial y}(x,y) = x^2 + 6xy + x$$

In this type of problem, it is helpful to take out common factors in the expressions. To find the critical points of f we will have to solve the following system of two equations

$$2xy + 3y^2 + y = 0 \Rightarrow y(2x + 3y + 1) = 0 \qquad (*)$$
$$x^2 + 6xy + x = 0 \Rightarrow x(x + 6y + 1) = 0 \qquad (**)$$

Observe that $(*)$ implies that either $y = 0$ or $2x + 3y + 1 = 0$.
We consider these two cases separately:

Case 1: $y = 0$.
Putting $y = 0$ into $(**)$ we get

$$\begin{aligned} x(x + 6y + 1) = 0 &\Rightarrow x(x + 6(0) + 1) = 0 \\ &\Rightarrow x(x + 1) = 0 \\ &\Rightarrow x = 0, \quad x = -1 \end{aligned}$$

The resulting two x values together with the case $y = 0$, gives us two critical points $(0, 0)$ and $(-1, 0)$.

Case 2: $2x + 3y + 1 = 0$.

Rearranging, we have $y = \dfrac{-2x - 1}{3}$.

Putting $y = \dfrac{-2x - 1}{3}$ into $(**)$ we get

$$\begin{aligned} x(x + 6y + 1) = 0 &\Rightarrow x\left(x + 6\left(\frac{-2x - 1}{3}\right) + 1\right) = 0 \\ &\Rightarrow x(-3x - 1) = 0 \\ &\Rightarrow x = 0, \quad x = -1/3 \end{aligned}$$

giving two values $x = 0$ and $x = -1/3$.
To find the corresponding y values we put these into $3y = -2x - 1$ to find $y = -1/3$ and $y = -1/9$.
Thus, we get two more critical points: $(0, -1/3)$ and $(-1/3, -1/9)$.

The critical points of $f(x, y)$ are therefore $(0, 0)$, $(0, -1/3)$, $(-1, 0)$, and $(-1/3, -1/9)$.

Ex. Find all critical points of $f(x, y) = x^2y + 36\,x^2 + 16\,y^2 + 2$.

$f_x : 2xy + 72x \rightarrow 2x(y + 36) = 0$

$x = 0$

$y + 36 = 0 \rightarrow y = -36$

$f_y : x^2 + 32y$

Case 1: $x = 0 \rightarrow 32y = 0$

$y = 0$

Case 2: $y = -36$

$x^2 + 32(-36) = 0$

$x^2 = 1152$

$x = \pm 24\sqrt{2}$

$\Rightarrow$ CPs: $(0, 0)$

$(24\sqrt{2}, -36)$

$(-24\sqrt{2}, -36)$