Definition

f(x) is continuous at x=a if:

$$\lim_{x\to a} f(x) \text{ exists}$$

$$\lim_{x\to a} f(x) = f(a)$$

If f is not continuous at x=a, then x=a is a point of discontinuity of f

Epsilon-delta definition

f is continuous at $x=a$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x-a| < \delta$, then

$$|f(x) - f(a)| < \varepsilon$$

Sequential characterization definition

f is continuous at $x=a$ if whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} x_n = a$, then

$$\lim_{n\to\infty} f(x_n) = f(a)$$

Ex. Is $f(x) = |x|$ continuous as $x \to 0$?

$f(0) = |0| = 0$

Yes

$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$

$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} -x = 0$

Ex. 2. Is $f(x) = \frac{1}{x}$ continuous at $x=0$?

$\lim_{x\to 0^+} \frac{1}{x} = \infty \to$ DNE

$f(0)$ is undefined $\to$ NO

Types of discontinuities:

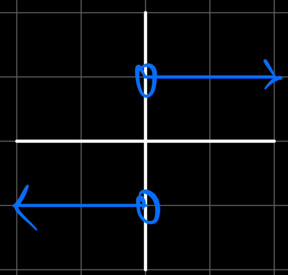
- *Removable discontinuity* - limit as x->a exists, but isn't equal to x=a

Example: hole in the graph

Ex. $\frac{x-2}{x-2} \to$ hole at $x=2$

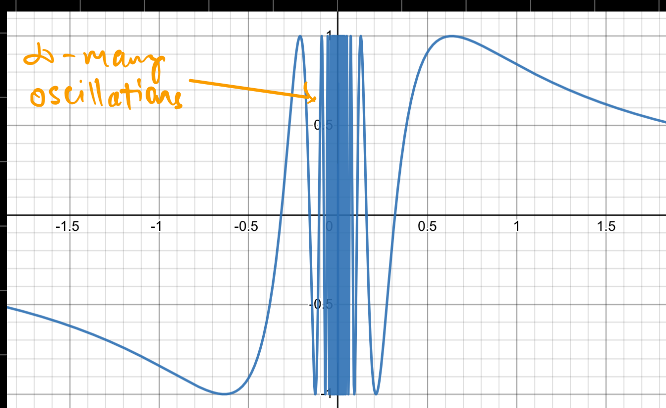
- *Jump discontinuity* - left hand and right hand limits exist but aren't equal; however there exists a FINITE gap between them

Ex. $\frac{|x|}{x}$ at $x=0$



- *Infinite discontinuity* - basically vertical asymptote. Limit as x->a = ±∞
- *Oscillatory discontinuity* - f(x) is bounded near x=a but oscillates infinitely many times

Ex. $\sin\left(\frac{1}{x}\right)$



Polynomials and rational functions (except where denominator can equal 0) are continuous over all real numbers x

Sine and cosine

$$\left.\begin{array}{l}\lim_{x\to 0} \sin x = 0 = \sin(0) \\ \lim_{x\to 0} \cos x = 1 = \cos(0)\end{array}\right\} \text{both continuous at } x=0$$

Proof: $\sin(x)$ is continuous over $\mathbb{R}$

Prove $\lim_{x \to a} \sin x = \sin(a)$ for all $a \in \mathbb{R}$

by showing $\lim_{h \to 0} \sin(a+h) = \sin(a)$

$$\lim_{h \to 0} \sin(a+h) = \lim_{h \to 0} [\sin(a)\cos(h) + \cos(a)\sin(h)]$$

$\cos(0) = 1$; $\sin(0) = 0$

$$\Leftrightarrow \sin(a)(1) + \cos(a)(0)$$
$$= \sin(a) \; \square$$

Euler's number (e^x)

e^x is continuous at x=0:

$$\lim_{x \to 0} e^x = e^0 = 1$$

Proof: Let $a \in \mathbb{R}$, $a \neq 0$

$$\lim_{h \to 0} e^{a+h} = \lim_{h \to 0} e^a \cdot e^h$$
$$= e^a \cdot e^0$$
$$= e^a$$

Therefore, e^x is continuous for all $x \in \mathbb{R}$

Continuity of Divergences

If f is invertible and continuous at x=a, and f(a) = b, then f^-1(x) is continuous at x=b

Geometric Proof

To take the graph of f^-1, we take the graph of f and reflect it over the line y=x. So, if f is continuous, so is f^-1, since reflecting something doesn't produce any holes or discontinuities nshit

Therefore, ln(x) - the inverse of e^x - is continuous over its domain (0, +∞)

Arithmetic Rules for Continuity

Assume f and g are both continuous at x=a. Then:

1. f+g is continuous at x=a
2. f • g is continuous at x=a
3. f/g is continuous at x=a, as long as g(a) ≠ 0

These are true because they follow directly from limit arithmetic laws

Ex. $f(x) = \dfrac{(x-1)(x+2)}{(x-1)(x-3)}$

f is not continuous at x=1 and x=3 since these are undefined

However, f is continuous at all other real numbers by arithmetic rules

Note: If we defined f(1) = -3/2, then f would be continuous at x=1

Compositions and Continuity

Notation: $g \circ f(x) = g(f(x))$

Theorem: If f(x) is continuous at x=a, and g(x) is continuous at x=f(a), then g(f(x)) is continuous at x=a

Proof: Assume f is continuous at x=a, g is continuous at x=f(a), and h = g(f(x))

Using sequential characterization, we can use any random sequence whose limit is x=a:

$$\lim_{n\to\infty} x_n = a$$

Since f is continuous at x=a,

$$\lim_{n\to\infty} f(x_n) = f(a)$$

but $f(x_n)$ is a sequence whose limit is $f(a)$

Therefore, since g is continuous at $f(a)$,

$$\lim_{n\to\infty} g(f(x_n)) = g(f(a))$$

$$\therefore \lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} g(f(x_n))$$

$$= g(f(a)) = h(a)$$

Therefore, h is continuous at $x=a$

Ex. $f(x) = \cos(e^{x^3})$ is continuous since $\cos(x)$, e^x, and x^3 are all continuous

Ex. 2. $\lim_{x\to 0} \ln\left(\frac{\sin x}{x}\right) = \ln\left(\lim_{x\to 0} \frac{\sin x}{x}\right)$

$= \ln(1)$

$= 0$

Fundamental Trig Limit:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Continuity on Intervals

f is continuous on (a,b) if f is continuous at x=c for all c in (a,b) f is continuous on [a,b] if:

1. f is continuous on (a,b)
2. $\lim_{x \to a^+} f(x) = f(a)$
3. $\lim_{x \to b^-} f(x) = f(b)$