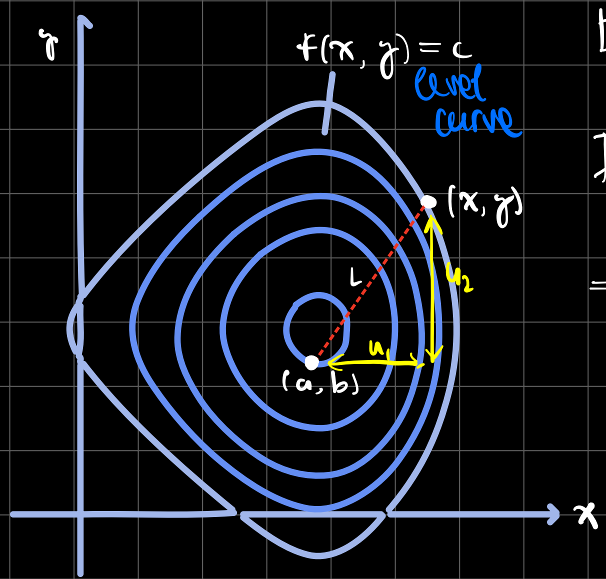
Let z = f(x,y) represent the height of a mountain at different points (x,y). If a skier is at a point (a,b), in what direction should they move in order to descend as rapidly as possible?

To solve this, we must find a point (x,y) such that the rate of change of f along the line L connecting (x,y) and (a,b) is maximized. The direction of this line can be defined by a vector u.



Direction: $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \vec{u}$

In unit vector: $s(u_1, u_2)$

$\Rightarrow \|u\| = 1$

$\Rightarrow l = (a, b) + s\vec{u}$

$= (a + su_1,\ b + su_2)$

The height of points in L is given by f(a + su1, b + su2)

Then, the rate of change of f at (a,b) in the direction of u is the derivative of f(a + su1, b + su2) with respect to s, evaluated at s=0

- Evaluated at s=0 since we are starting at (a,b), and having s=0 cancels out the vector and allows us to start at (a,b)

Definition: Directional Derivative

The **directional derivative** of $f(x, y)$ at a point (a, b) in the direction of a **unit vector** $\vec{u} = (u_1, u_2)$ where $\|\vec{u}\| = 1$ is defined by

$$D_{\vec{u}} f(a, b) = \frac{d}{ds} f(a + su_1, b + su_2)\Big|_{s=0}$$

provided that the derivative exists.

Example 1

Find the directional derivative of $f(x, y) = x^2 - y^2$ at the point $(1, 2)$ in the direction of the vector $\vec{u} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$.

Solution:

Note that $\|\vec{u}\| = \sqrt{(1/\sqrt{5})^2 + (2/\sqrt{5})^2} = 1$.

By definition, we get

$$\begin{aligned} D_{\vec{u}} f(1, 2) &= \frac{d}{ds} f\left(1 + \frac{1}{\sqrt{5}}s, 2 + \frac{2}{\sqrt{5}}s\right)\Big|_{s=0} \\ &= \frac{d}{ds}\left[\left(1 + \frac{1}{\sqrt{5}}s\right)^2 - \left(2 + \frac{2}{\sqrt{5}}s\right)^2\right]\Big|_{s=0} \\ &= \left[\frac{2}{\sqrt{5}}\left(1 + \frac{1}{\sqrt{5}}s\right) - \frac{4}{\sqrt{5}}\left(2 + \frac{2}{\sqrt{5}}s\right)\right]\Big|_{s=0} \\ &= -\frac{6}{\sqrt{5}} \end{aligned}$$

However, this method is extremely tedious; even more than everything else in this godforsaken course

Use this instead — BUT ONLY IF DIFFERENTIABLE AT (a,b)

Theorem 1: Directional Derivative (DD) Theorem

If $f(x,y)$ is differentiable at (a,b) and $\vec{u}=(u_1,u_2)$ where $\|\vec{u}\|=1$ is a **unit vector**, then

$$D_{\vec{u}}f(a,b)=\nabla f(a,b)\cdot\vec{u}$$

where $\cdot$ represents the dot product.

Ex. $f(x,y,z)=\sin(xyz)$, $(a,b,c)=\left(1,1,\frac{\pi}{4}\right)$, and $\vec{v}=(4,-\sqrt{2},4)$. (Remember to enter π as "Pi" with capital letter "P".)

a. Calculate the directional derivative of f at the point $(a,b,c)=\left(1,1,\frac{\pi}{4}\right)$ in the direction defined by $\vec{v}$. [4 points]

(1) Check if differentiable at (a,b,c) -> it is

(2) Check if v is a unit vector; if not, normalize it

$$\|\vec{v}\| = (4^2+(-\sqrt{2})^2+4^2)^{1/2} = \sqrt{34}$$

$$\Rightarrow \vec{v}^* = \frac{1}{\sqrt{34}}\vec{v}$$

(2) Calculate the gradient:

$$\frac{\partial f}{\partial x} = yz\cdot\cos(xyz) \qquad \frac{\partial f}{\partial y} = xz\cdot\cos(xyz) \qquad \frac{\partial f}{\partial z} = xy\cdot\cos(xyz)$$

$$\Rightarrow \nabla f(a,b,c) = \begin{bmatrix}\frac{\pi}{4}\cdot\cos(\frac{\pi}{4})\\ \frac{\pi}{4}\cdot\cos(\frac{\pi}{4})\\ \cos(\frac{\pi}{4})\end{bmatrix} = \begin{bmatrix}\frac{\pi\sqrt{2}}{8}\\ \frac{\pi\sqrt{2}}{8}\\ \frac{\sqrt{2}}{2}\end{bmatrix}$$

$$\nabla f(a,b,c)\cdot\vec{v}^* = \begin{bmatrix}\frac{\pi\sqrt{2}}{8}\\ \frac{\pi\sqrt{2}}{8}\\ \frac{\sqrt{2}}{2}\end{bmatrix}\cdot\frac{1}{\sqrt{34}}\begin{bmatrix}4\\ -\sqrt{2}\\ 4\end{bmatrix}$$

Greatest Rate of Change in R2

Theorem 1: The Greatest Rate of Change (GRC) Theorem

If $f(x,y)$ is differentiable at (a,b) and $\nabla f(a,b)\neq(0,0)$, then the largest value of $D_{\vec{u}}f(a,b)$ is $\|\nabla f(a,b)\|$, and occurs when $\vec{u}$ is in the direction of $\nabla f(a,b)$.

Proof: Since f(x,y) is differentiable at (a,b) and the gradient at (a,b) ≠ (0,0):

$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$$
$$= [\|\nabla f(a,b)\|][\|\vec{u}\|] \cos\theta$$

since $\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos\theta$ for any $\vec{x}, \vec{y} \in \mathbb{R}^2$

$-1 \leq \cos\theta \leq 1$, so $D_{\vec{u}} f(a,b)$ is maximized when $\cos\theta = 1$
$\Rightarrow \theta = 0$

As such, the directional derivative is maximized when the angle between u and the gradient is 0 — or, when they are pointing in the same direction. Thus, the directional derivative takes on its largest value when u is in the same direction as the gradient.

Ex. Find the largest rate of change of $f(x,y) = \ln(x+y^2)$ at the point $(0,1)$ and the direction in which it occurs.

$$\nabla f(x,y) = \left(\frac{1}{x+y^2}, \frac{2y}{x+y^2}\right)$$

LRC: $\nabla f(0,1) = (1, 2)$

Magnitude: $\sqrt{5}$

Directional derivative is 0 at points *orthogonal* to the direction with the largest rate of change

Directional derivative *decreases most* in the opposite direction as the LRC direction

This fact about orthogonality implies that the tangent plane of a surface at (a,b,c) is

$$\nabla f \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

The tangent plane is *orthogonal* to the direction with the largest rate of change; that is, directly in between the largest and smallest rates of change — so it is a good approximation

SUMMARY

Directional derivative is:

- At a point (a,b)
- In direction u = (u1,u2); this is a unit vector

The directional derivative of f at (a,b) is

$$\nabla f(a,b) \cdot \vec{u}$$

…if it is differentiable at (a,b). They probably won't give any questions on the exam where it's not

The **greatest rate of change** occurs when

$\vec{u} = \nabla f(a,b) \rightarrow$ this maximizes $\nabla f(a,b) \cdot \vec{u}$

Smallest: $\vec{u} = -\nabla f(a,b)$

No change when directional derivative is 0, or at vectors orthogonal to the gradient

$$\nabla f(a,b) \cdot \vec{u} = \vec{0}$$