Definition

Consider a discrete random variable X with probability function f(x)

The moment generating function of X is

$$M(t) = E(e^{tx}) = \sum_{\forall x} e^{tx} \cdot f(x)$$

MGF "uniquely identities" a distribution

We can use it to obtain the "moments" of a random variable X

$E[X^r]$: r^{th} moment of X

The mean μ is the first moment, E(X), of X

$$\mathrm{Var}(X) = \underbrace{E(X^2)}_{\text{2nd moment}} - \underbrace{E(X)^2}_{\text{1st moment}}$$

Theorem

Let a random variable X have the moment generating function M_x(t). Then:

$$E[X^r] = M^{(r)}(0)$$

r-th derivative of M, evaluated at t=0

Binomial Distribution

$$M_X(t) = E[e^{tx}] = \sum_{x=0}^{n} e^{tx} \cdot \binom{n}{x} p^x (1-p)^{n-x}$$

$$\vdots$$

$$= (pe^t + 1 - p)^n$$

Then, $E(X) = M_X'(0)$

$= (pe^t + 1 - p)^{n-1} \cdot npe^t$ chain rule

$t = 0 \rightarrow np$

Example: Suppose we are given the following pf in table form:

x	0	1	2	3	4
f(x) = P(X = x)	0.1	0.2	0.2	0.3	0.2

Determine the mgf of X.

The mgf of X is given by $M_X(t) = E[e^{tX}]$.

In this case, $E[e^{tX}] = \sum_{x=0}^{4} e^{tX} * f(x)$

$E[e^{tX}] = (0.1)e^{t(0)} + (0.2)e^{t(1)} + (0.2)e^{t(2)} + (0.3)e^{t(3)} + (0.2)e^{t(4)}$

So, mgf is given by:
$M_X(t) = E[e^{tX}] = 0.1 + 0.2e^{t} + 0.2e^{2t} + 0.3e^{3t} + 0.2e^{4t}$

Exercise: Calculate E(X) using the pf explicitly and show that E(X) = 2.3. Then, use the mgf to verify this result!

Over a sum of random variables:

Let Z = X + Y

$\Rightarrow M_Z(t) = E[e^{tZ}] = E[e^{t(X+Y)}]$
$= E(e^{tX}) \cdot E(e^{tY})$
$= M_X(t) \cdot M_Y(t)$

As such, if X ~ Bin(n, p) and Y ~ Bin(m, p):

$Z = X + Y \sim Bin(n+m, p)$