Tree: connected graph with no cycles

Forest: graph with no cycles

Lemma

In any tree, there is *exactly one path* between any two vertices u and v

Proof: If there are 2 distinct paths between u and v, then there must be a cycle

But trees have no cycles

So there can only be 1 distinct path from u->v

Lemma

Every edge in T is a bridge

Theorem

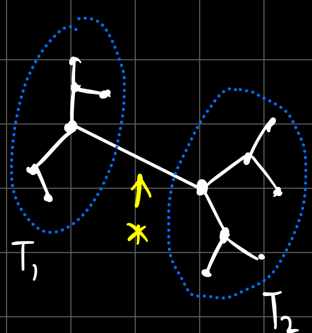
A tree T with n vertices has n-1 edges

Proof: (Induction)

Base case: n=1 vertices -> 0 edges

Inductive hypothesis: suppose that all trees with m < n vertices have m-1 edges.

Now, let T have n vertices.



$$\underbrace{|V(T_1)|}_{<n} + \underbrace{|V(T_2)|}_{<n} = |V(T)|$$

By IH, T_1 and T_2 both have $|V(T_{1 \text{ or } 2})| - 1$ edges

$$\Rightarrow |E(T)| = \underbrace{|V(T_1)| - 1}_{|E(T_1)|} + |V(T_2)| - 1 + \underbrace{1}_{= *}$$

$$|V(T_1)| + |V(T_2)| = n$$

$$\Rightarrow |E(T)| = n-2+1$$
$$= n-1$$

Lemma

A connected graph G with n vertices must have ≥n-1 edges

Proof:

Inductive hypothesis: If G has m<n edges, it has ≥m-1 edges

Suppose G is connected and has n vertices.

Let G' = G-x.

|V(G')| = n-1

So by the inductive hypothesis, |E(G')| ≥ n-2 (since n-2 < n)

Note that deg(x) ≥ 1

So |E(G)| = |E(G')| + deg(x) ≥ n-2 + 1

So |E(G)| ≥ n-1

Lemma

If G is connected with n vertices and n-1 edges, then G is a tree

Proof: If G is not a tree, then it has an edge e that is not a bridge.

Then G-e is connected

G-e has n vertices and n-2 edges

But a connected graph G with n vertices must have ≥n-1 edges

So G must be a tree

Leaves

A vertex v in a tree with degree 1 is a *leaf*

Lemma

A tree with ≥2 vertices has ≥2 leaves

Proof: Let p = u0, u1, … u_n be a longest path in T.

Since T is a tree, it has no cycles.

Both u0 and u_n must have degree 1. This is because if they connect to another node, either:

- That other node is in the longest path -> T has a cycle, which it cannot
- That other node is not in the longest path -> impossible, since p would no longer be a longest path

So T must have at least 2 leaves

Proof 2: (Handshaking Lemma)

Let p1 = #leaves (with degree 1)

p2 = vertices with degree ≥ 2

Let T have n vertices and n-1 edges.

p1 + p2 = n

p1 + p2 - 1 = |E(T)|

By the Handshaking Lemma:

$$\sum_{\forall v} \deg(v) = 2(\underbrace{p_1 + p_2 - 1}_{E(T)})$$

Also :

$$\sum_{\forall v} \deg(v) \geq p_1 + 2p_2$$

each v in p_2 has $\deg \geq 2$

$$\Rightarrow 2p_1 + 2p_2 - 2 \geq p_1 + 2p_2$$

$$\Rightarrow p_1 \geq 2$$

Ex. Suppose T has a vertex with degree 5. What is the minimum number of leaves in T?

Let p1 = number of leaves, p2 = number of vertices with degree ≥2; let x have degree 5

n = p1 + p2

|E(T)| = p1 + p2 - 1

$$2p_1 + 2p_2 - 2 = \sum_{\forall v} \deg(v)$$

$$= p_1 \cdot 1 + 5 + \overbrace{\sum_{\forall v \neq x} \deg(v)}^{\geq 2(p_2 - 1) \text{ : exclude vertex with deg. 5}}$$

$$2p_1 + 2p_2 - 2 \geq p_1 + 5 + 2p_2 - 2$$

$$\Rightarrow p_1 \geq 5$$

Alternate proof: each subtree must contribute at least 1 leaf

=> min. 5 leaves

Ex. Suppose we have a forest with 2 components and n vertices. How many edges?

One component has k vertices and k-1 edges

Other has n-k vertices and n-k-1 edges

So the total number of edges is n-k-1+k-1 = n-2

Lemma

A forest with n vertices and k components has n-k edges.