Definition: Quadratic Form

A function Q of the form

$$Q(u,v) = a_{11}u^2 + 2a_{12}uv + a_{22}v^2$$

where a_{11}, a_{12} and a_{22} are constants, is called a **quadratic form** on $\mathbb{R}^2$.

It is important to observe that we can use matrix notation to write

$$Q(u,v) = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

so that a quadratic form on $\mathbb{R}^2$ is determined by a 2×2 matrix.

Proof :

$$\underbrace{\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{bmatrix} a_{11}u + a_{12}v \\ a_{12}u + a_{22}v \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$= a_{11}u^2 + a_{12}vu + a_{12}uv + a_{22}v_2$$
$$= a_{11}u^2 + 2a_{12}uv + a_{22}v_2$$

Quadratic forms on $\mathbb{R}^2$ fall into four main classes:

1. If $Q(u,v) > 0$ for all $(u,v) \neq (0,0)$, then $Q(u,v)$ is **positive definite**.
2. If $Q(u,v) < 0$, for all $(u,v) \neq (0,0)$, then $Q(u,v)$ is **negative definite**.
3. If $Q(u,v) < 0$ for some (u,v) and $Q(w,z) > 0$ for some (w,z), then $Q(u,v)$ is **indefinite**.
4. If $Q(u,v)$ does not belong to classes 1) to 3), then $Q(u,v)$ is **semidefinite**. Semidefinite quadratic forms may be split into two classes:
   a. If $Q(u,v) \geq 0$ for all (u,v), then $Q(u,v)$ is **positive semidefinite.**
   b. If $Q(u,v) \leq 0$ for all (u,v), then $Q(u,v)$ is **negative semidefinite.**

These terms are also used to describe the associated symmetric matrices.

Example 1

$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ is positive definite, since the associated quadratic form $Q(u,v) = 2u^2 + 3v^2 > 0$, for all $(u,v) \neq (0,0)$.

$B = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ is indefinite, since the associated quadratic form $Q(u,v) = 2u^2 - 3v^2$, and $Q(u,0) = 2u^2 > 0$ for $u \neq 0$, and $Q(0,v) = -3v^2 < 0$ for $v \neq 0$.

$C = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ is semidefinite, since the associated quadratic form$Q(u,v) = 2u^2 \geq 0$ for all (u,v), and $Q(0,v) = 0$ for all v.

More specifically, C is positive semidefinite.

Example 2

Classify the symmetric matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$.

Solution:

The associated quadratic form is

$$Q(u, v) = u^2 + 6uv + 2v^2$$

Complete the square, obtaining

$$Q(u, v) = (u + 3v)^2 - 7v^2$$

It is now clear by inspection that A is indefinite, since

$$Q(u, 0) = u^2 > 0, \quad \text{for } u \neq 0$$

and

$$Q(-3v, v) = -7v^2 < 0, \quad \text{for } v \neq 0$$

Proposition: Determinant and Quadratic Forms

A quadratic form $Q(u, v) = a_{11}u^2 + 2a_{12}uv + a_{22}v^2$ on $\mathbb{R}^2$ is

1. Positive definite if $\det(A) > 0$ and $a_{11} > 0$
2. Negative definite if $\det(A) > 0$ and $a_{11} < 0$
3. Indefinite if $\det(A) < 0$
4. Semidefinite if $\det(A) = 0$

Theorem 1: Second Partial Derivatives Test

Suppose that $f(x, y) \in C^2$ in some neighborhood of (a, b) and that

$$f_x(a, b) = 0 = f_y(a, b)$$

1. If $Hf(a, b)$ is positive definite, then (a, b) is a local minimum point of f.
2. If $Hf(a, b)$ is negative definite, then (a, b) is a local maximum point of f.
3. If $Hf(a, b)$ is indefinite, then (a, b) is a saddle point of f.
4. If $Hf(a, b)$ is semidefinite, then the test is inconclusive.

Ex. $f(x, y) = x^2 + xy + 3y^2$

$f_x : 2x + y$ $\qquad$ $f_y : x + 6y$

$f_{xx} : 2$ $\qquad$ $f_{yy} : 6$

$f_{xy} : 1$

$\Rightarrow Hf(x, y) = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}$

$\text{Det}(Hf(x, y)) = 10 > 0$

$a_{11} = 1 > 0$

$\Rightarrow$ positive definite

$\Rightarrow$ local min $\qquad$ (all CPs are local mins)

Ex. The critical points of $f(x,y) = x^3 + 2x^2y + 16xy^2 - 16x$ are $(0,-1)$, $(0,1)$, $\left(\frac{16\sqrt{3}\sqrt{15}}{45}, -\frac{1}{45}\sqrt{3}\sqrt{15}\right)$, and $\left(-\frac{16\sqrt{3}\sqrt{15}}{45}, \frac{1}{45}\sqrt{3}\sqrt{15}\right)$.

$f_x : 3x^2 + 4xy + 16y^2 - 16$

$f_{xy} : 4x + 32y$

$f_{xx} : 6x + 4y$

$f_y : 2x^2 + 32xy$

$f_{yy} : 32x$

$(0,1) \rightarrow \begin{bmatrix} 4 & 32 \\ 32 & 0 \end{bmatrix}$ A

$\det(A) = -32^2 < 0$

$\Rightarrow$ saddle point

$(0,-1) \rightarrow \begin{bmatrix} -4 & -32 \\ -32 & 0 \end{bmatrix}$ B

$\det(B) = -32^2$