Canonical Gauss-Jordan

Steps:

1. Find the first pivot (first nonzero entry that appears)
2. Divide/multiply row to make the pivot 1.
3. Clear out everything below it by subtracting that row.

General rules:

- Only skip to next row if there are no pivots
- Always *scale rows first*

Solution 2

We will now solve this system using the Canonical Gauss-Jordan Algorithm.

There is already a pivot in the $(1, 1)$ position: let us scale it to unity.

$$R_1 \to \frac{1}{3}R_1$$

$$\left(\begin{array}{cccc|c} 1 & \frac{-4}{3} & \frac{-1}{3} & \frac{-19}{3} & \frac{-8}{3} \\ 2 & -3 & 1 & -22 & -1 \\ 1 & 2 & -1 & 7 & 2 \\ 6 & -12 & 2 & -70 & -12 \end{array}\right)$$

Use the leading one in the $(1, 1)$ position to obtain entries of zero in column 1, after row 1.

$$\begin{aligned} R_2 &\to R_2 - 2R_1 \quad \text{and} \\ R_3 &\to R_3 - R_1 \quad \text{and} \\ R_4 &\to R_4 - 6R_1 \quad \text{give} \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & \frac{-4}{3} & \frac{-1}{3} & \frac{-19}{3} & \frac{-8}{3} \\ 0 & \frac{-1}{3} & \frac{5}{3} & \frac{-28}{3} & \frac{13}{3} \\ 0 & \frac{10}{3} & \frac{-2}{3} & \frac{40}{3} & \frac{14}{3} \\ 0 & -4 & 4 & -32 & 4 \end{array}\right).$$

There is already a pivot in the $(2, 2)$ position: let us scale it to unity.

$$R_2 \to -3R_2$$

Consistent system test: a system of equations $Ax = b$ has solutions if and only if the rank of the matrix A is equal to the rank of its augmented matrix $A \mid b$

Notation

a_1, a_2, …, a_n — columns

Rows:

$$A = \begin{bmatrix} \vec{A}_1 \\ \vdots \\ \vec{A}_m \end{bmatrix} \to \vec{A}_j \in M_{1 \times n}$$

Matrix/Vector Multiplication

Multiplying a matrix by a vector produces a linear combination of the matrix's columns

$$A\vec{x} = x_1\vec{a}_1 + \cdots x_n\vec{a}_n$$

Multiplying two matrices:

$$AB = [A\vec{b}_1 \quad \cdots \quad A\vec{b}_n] \longrightarrow \text{hard to do in practice}$$

if $C = AB$, $c_{ij} = A_i \cdot B_j$

$$\Rightarrow A\vec{b}_j = A\vec{b}_{j1} + \cdots + A\vec{b}_{jn}$$

↳ each column

Here, each column of C = AB is a linear combination of the columns of A

$$(AB)_i = C_i = A_i B = (A_i)_1\vec{b}_1 + \cdots + (A_i)_n\vec{b}_n$$

???

This is a linear combination of the rows of B

Linear Transformations

Basic properties:

- T(x+y) = T(x) + T(y)
- T(cx) = c • T(x)

Thus, T(cx+y) = c • T(x) + T(y)

One basic linear transformation: If A is an m x n matrix, then

$$T_A(\vec{x}) : \mathbb{F}^n \rightarrow \mathbb{F}^m = A\vec{x}$$: vector with m rows

Inverse Functions

A linear transformation has an inverse if it is:

- One-to-one
- Onto: range and codomain of T are equal. For every y in the range of T, y = T(x) for some x in the domain of T

A transformation from n to m is one-to-one if the rank of its standard matrix = n, or if:

- Ker(T) = {}
- Null([T]) = {} — nullspace of standard matrix is empty

From here, it follows that if m ≠ n, the function is not invertible

Inverse is unique and also a linear transformation.

$$T^{-1}\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \underbrace{[T]_{\mathcal{E}}^{-1}}_{\text{inverse of standard matrix}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Also, $T(\vec{x}) = [T]_{\mathcal{E}}\vec{x}$