$$S^* = \bigcup_{k=0}^{\infty} S^k$$

Disjoint union of all k-length tuples of a set S

Ex. If $B = \mathbb{N}$, $B^k = \{\text{set of multisets of } k \text{ types}\}$
$B^* = \{\text{set of multisets of all types}\}$

Computing the generating series of S*:

- The weight function of S*, w*, is inherited from the weight function w on S (bijection)

So:

$$w^*(a_1, \ldots a_k) = w(a_1) + \cdots w(a_k)$$

w* is a valid weight function *if and only if* there are no elements with weight (w, not w*) 0 in S

$$w^{-1}(0) = \emptyset$$
$$(\{c \in S : w(c) = 0\} = \emptyset)$$

Proof:

$$\{c \in S^* : w^*(c) = 0\}$$
$$= \{(a_1, \ldots a_k) \in S^* : w(a_1) + \cdots + w(a_k) = 0\}$$

Since S^* is defined for $k \to \infty$:

$$S^* = \bigcup_{k=0}^{\infty} S^k$$

There exist an infinite amount of tuples whose weight is 0

(For example, {0, 00, 000, 0000 …}

$$\Rightarrow |w^{-1}(0)| = 0$$

Then, if $w^{-1}(n) = \{(a_1, \dots a_k) : w(a_1) + \dots + w(a_k) = n, \ k \le n\}$

$|w^{-1}(n)| < \infty$

(There is a *finite* number of elements with weight n)

Also, k can be at most n since:

- The multiset has k elements
- Each element is *at least* 1

Let $w_2(a) = 1$ for all $a \in S$

Since w^* counts the size of a multiset, and in w_2, each element is 1, $w_2^*(a)$ = its length

So $|w_2^{-1}(n)| = 2^n$

Lemma 2.14 (The String Lemma.). *Let A be a set with a weight function $\omega : A \to \mathbb{N}$ such that there are no elements of A of weight zero. Then*

$$\Phi_{A^*}(x) = \frac{1}{1 - \Phi_A(x)}.$$

Proof. By the Infinite Sum and Product Lemmas 2.11 and 2.12,

$$\Phi_{A^*}(x) = \sum_{k=0}^{\infty} \Phi_{A^k}(x) = \sum_{k=0}^{\infty} (\Phi_A(x))^k = \frac{1}{1 - \Phi_A(x)}.$$

Ex. Let $w_3(a) = a + 1$ for all a

If we apply this weight function to A*, which, in this example, contains multisets of elements that are either 0 or 1, then

$w_3^*(a_1, \dots a_k) = k + \#$ of 1's

So $\Phi^{w_3}_{\{0,1\}} = x + x^2$

$\Phi^{w_2^*}_{\{0,1\}^*} = \dfrac{1}{1 - x - x^2}$ (string lemma)

Compositions

Composition – finite sequence of *positive* integers

$$\gamma = (c_1, \dots c_k)$$

Length: $\ell(\gamma) = k$

Size: $|\gamma| = c_1 + \cdots + c_k$

There are 0 compositions of size 0

1 composition of size 1

There are 2^{n-1} ways to make a composition of size n

• • • •	(4)
• | • • •	(1, 3)
• • | • •	(2, 2)
• • • | •	(3, 1)