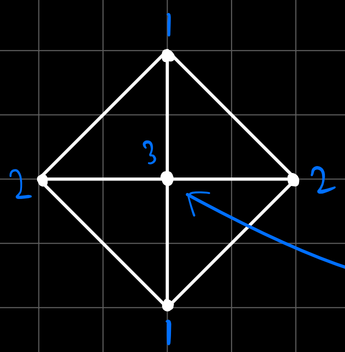
k-coloring

- Every vertex's color must be different from that of its neighbors
- The color of a vertex is defined by the following function f

$$f : V(G) \to [k] : \forall uv \in E(G),\ f(u) \neq f(v)$$

↳ colors are $1, 2, \ldots, k$



cannot be 1 or 2

⇒ NOT 2-colorable

Theorem

A graph is 2-colorable if and only if it is bipartite

(<=) If a graph is bipartite, for all edges uv, we can choose a bipartition such that u is in A and v is in B

Specific graph colorings:

- K_n is n-colorable, but not n-1 colorable.

Proof: Suppose K_n is n-1 colorable. Then two vertices, u and v, have the same color.

Since K_n is a complete graph, u and v are neighbors

This is a contradiction, so K_n cannot be n-1 colorable

Coloring is normally a very hard problem, but it is much easier on planar graphs

6-Color Theorem

All planar graphs are 6-colorable

Proof: Let G be a planar graph.

Suppose that G is 6-colorable for all |V(G)| ≤ k.

Prove that G is 6-colorable if |V(G)| = k+1

Since G is planar, then there is a vertex v in G such that deg(v) ≤ 5

Let G' = G \ {v}

By the inductive hypothesis, G' is 6-colorable

Since deg(v) ≤ 5, its neighbors cannot possibly have all six colors

So we can choose the color of v to be a different color from all its neighbors

This proves that G is 6-colorable

<u>Edge contraction</u>

Let G be a graph, and let e = uv be an edge in G

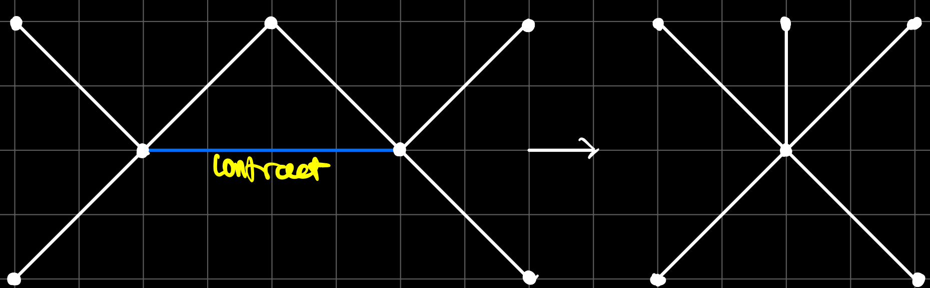
Contraction:

- V(G \ e) = V(G) \ {u}   or V(G) \ {v}
- E(G \ e) = E(G) \ {uv} U {av | au ∈ E(G)

make all the neighbors of u
neighbors of v
ignore all duplicate edges

If G is planar, then G \ {e} is planar

- G doesn't have any overlaps, so removing an edge and rearranging also doesn't produce any
- *Converse is not true*: If G \ {e} is planar, then G + e is not necessarily planar



<u>5-Color Theorem</u>

All planar graphs are 5-colorable

Proof:

Suppose |V(G)| ≤ k.

For |V(G)| = k+1, we will choose a degree v

Case 1: deg(v) ≤ 4 -> done

Case 2: deg(v) = 5

Let v have neighbors a, b, c, d, and f

In this case, we can find two neighbors of v — a and b — such that a and b are not adjacent. These will always exist because if every neighbor of v was connected to another neighbor, G would include K5 and would therefore not be planar

So, if we contract av and bv:

- H = G \ av
- K = H \ bv

Since G is planar, both H and K are planar

K includes 2 less vertices, and so |V(K)| = k-1

By the inductive hypothesis, |V(K)| is 5-colorable

We can let c have color 1, d have color 2, and f have color 3. This is valid for K.

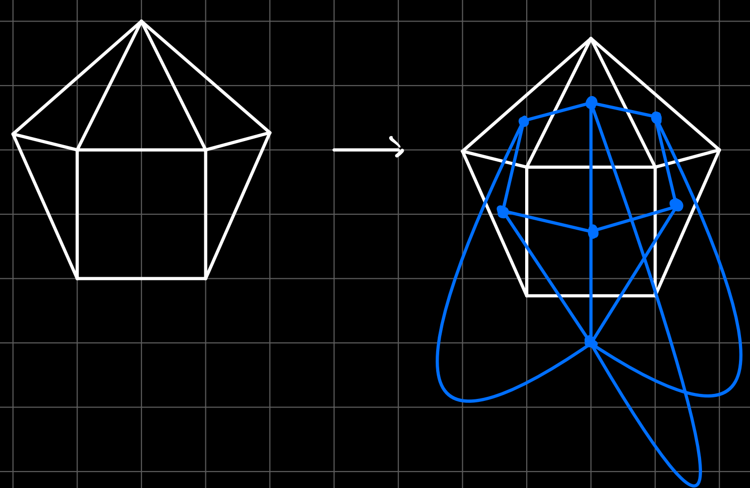
Going back to G, a and b can both be color 4 since they are not neighbors.

Then, making v have color 5 is valid in both K and G.

<u>Duals of planar graphs</u>

In the dual of a graph:

- Assign a vertex to each face
- Draw an edge between vertices if their corresponding faces share a boundary edge



Vertex degrees in the dual correspond to face degrees

Degree of vertex in a dual = number of edges in the boundary walk of its corresponding face