Checking continuity

Ex. Find intervals where

$$F(x) = \begin{cases} \frac{x^2-4}{x^2-x+6} \cdot \cos(x^2) & \text{if } x \neq -3, 2 \\ 0 & \text{if } x = -3, 2 \end{cases}$$ is continuous

1. Find possible discontinuities

$$\frac{x^2-4}{x^2-x+6} = \frac{(x+2)(x-2)}{(x+3)(x-2)} = \frac{x+2}{x+3}$$

look at denominator !!

F is possibly discontinuous when $x^2-x+6=0$, or when $x=2$ or $x=-3$.

2. Check limits

$$\lim_{x\to-3} \frac{x^2-4}{x^2-x+6} \cdot \cos x^2$$

As $x \to -3$: $x^2-4 \to 5$
$x^2-x+6 \to 0$
$\cos x^2 \to \cos 9$

DNE

↳ discontinuity

$$\lim_{x\to 2} \frac{x^2-4}{x^2-x+6} \cdot \cos x^2 = \frac{x+2}{x+3} \cdot \cos x^2 = \frac{4}{5}\cos 4$$

However $\frac{4}{5}\cos 4 \neq F(2) = 0 \to$ discontinuity

Therefore, the continuous intervals a
$\{x \mid x \in \mathbb{R} \setminus \{-3, 2\}\}$
or $(-\infty, -3) \cup (-3, 2) \cup (2, +\infty)$

Finding value to assure continuity

Ex. Let $f(x) = \begin{cases} cx^2 - 2x & \text{if } x > 2 \\ x^3 - cx & \text{if } x \le 2 \end{cases}$

1. Narrow the search. Find points where $f(x)$ might be discontinuous.

- Since both are polynomials, they are continuous over their entire domains. So, we only need to worry about $x = 2$.

f is continuous at $x=2$ if

$\lim_{x\to 2} f(x)$ exists and $\lim_{x \to 2} f(x) = f(2)$

For $\lim_{x\to 2} f(x)$ to exist and $\lim_{x\to 2} f(x) = f(2)$, we need

$$\lim_{x\to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

$$\Leftrightarrow c \cdot 2^2 - 2(2) = 2^3 - 2c$$

$$4c - 4 = 8 - 2c$$

$$c = \frac{2}{3}$$

Prove that

If function is continuous at $x=0$ and satisfies $f(x+y) = f(x)f(y)$, it is continuous everywhere

We must show that

$$\lim_{x\to a} f(x) = f(a) = \lim_{h\to 0} f(a+h)$$

$$\lim_{h\to 0} f(a+h) = \lim_{h\to 0} f(a) f(h)$$

$$= f(a)\lim_{h\to 0} f(h)$$
$$= f(a)f(0)$$
$$= f(a+0)$$
$$= f(a)$$

We need to show that $\lim_{x\to a} f(x) = f(a) = \lim_{h\to 0} f(a+h)$

Since $f(x+y) = f(x) + f(y)$:

$$\lim_{h\to 0} f(a+h) = \lim_{h\to 0} f(a) + f(h)$$
$$= f(a) + f(0)$$
$$= f(a+0)$$
$$= f(a)$$