The limit of a single variable function f(x) is defined at a if:

- f(a) is defined
- Limit exists (and approach the same value) when approached from both sides
- The limit of f as x approaches a is equal to f(a)

Remark

Just like in single variable calculus, there are three requirements in this definition:

1. $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists,
2. f is defined at (a,b), and
3. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

Example 2

Prove that $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ is not continuous at $(0,0)$.

Solution:

To prove that f is not continuous at $(0,0)$, we need to prove that the limit

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

does not equal 0. Therefore, if we can find one path such that the limit does not equal 0, then, since the value of a limit must be unique, this will prove that the limit cannot be equal to 0.

Approaching the limit along the line $y = x$ gives

$$\lim_{x\to0} \frac{x^2}{x^2+x^2} = \lim_{x\to0} \frac{x^2}{2x^2} = \frac{1}{2} \neq 0$$

Thus, the limit cannot equal 0, so f is not continuous at $(0,0)$.