Recall: a set S is a vector space defined under F if:

- Zero vector is in S
- For all vectors x and y in S, x+y is also in S (closed under addition)
- For all vectors x and constants c in F, cx is in S (closed under scalar multiplication)

$\forall \alpha, \beta \in \mathbb{F}, (\alpha + \beta)\vec{x} \in S$

(☆) $\forall \vec{v} \in S, \quad \vec{v} + \vec{0} = \vec{v}; \quad \vec{0} \cdot \vec{v} = 0$

Ex. A weird vector space

Consider the set R^2, defined under R, where

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d+1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} + \begin{array}{c} c \\ \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ d \end{array} = \begin{bmatrix} a \\ b \end{bmatrix}$$

"zero vector" is $(0, -1) \in S$

must satisfy $\vec{v} + \vec{0} = \vec{v}$

More vector spaces:

(1) Field: real numbers; set: R^n, zero vector is zero vector, addition and multiplication are standard

(2) Field: (m,n) in R2; set: all m x n matrices; zero vector is m x n zero matrix, addition and multiplication are both standard

(3) $\mathbb{F} = \mathbb{R}$

$S = P_n(\mathbb{R})$: all polynomials with order $\leq n$

$+ / \times$ are standard

zero vector: $f(x) = 0$ since $f(x) + 0 = f(x) \quad \forall f(x) \in S$

(4) Field: R; Set: all functions from R->R that are infinitely differentiable

- Are equal to their Taylor series
- Zero vector: y = 0
- Addition and multiplication are standard

Ex. Is $W = \{g(x) \in V : g(0) + g(1) = 1\}$ a vector space?

(1) Find the zero vector

$$\vec{0} : g(x) = 0 \quad \text{since} \quad g(x) + 0 = g(x) \quad \forall\, g(x) \in W$$

However, notice that $g(x) = 0$ cannot be part of W, since $g(0) + g(1) = 0 \neq 1$.

Also, this set is not closed under addition:

For all $g_1(x)$, $g_2(x)$, $g_1(x) + g_2(x) \in W$?

Let $g_3(x) = g_1(x) + g_2(x)$

If $g_3(0) + g_3(1) = 1$, then $g_3(x) \in W$

$$g_3(0) + g_3(1) = g_1(0) + g_2(0) + g_1(1) + g_2(1)$$

$$= \underbrace{g_1(0) + g_1(1)}_{1} + \underbrace{g_2(0) + g_2(1)}_{1} = 2 \neq 1$$

<u>Lemma</u>

For all vector spaces:

- The zero vector is unique
- The additive inverse is unique

If $cv = 0$, then either $c=0$ (constant) or v is the zero vector

However, if the matrix product AB is equal to the zero matrix, that does not necessarily mean that one of A or B is the zero matrix

<u>Definition:</u> *Linear combination*

Let V be a vector space.

Let $v_1, \dots, v_n$ be $\underbrace{\text{members of } V}_{\text{not necessarily vectors}}$, and let $c_1, \dots, c_n$ be scalars.

Then

could be matrices, polynomials, etc.

could be matrices, polynomials, etc

$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ is a linear combination

(n must be finite)

<u>Definition:</u> *Span*

If S is a subset of V, then Span(S) is the set of all linear combinations of S.

Ex. If $S = \left\{ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} i & 1+i \\ 1-i & 5 \end{bmatrix}, \begin{bmatrix} 2i & -2i \\ 1+i & 2-i \end{bmatrix} \right\}$

Is $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in Span(S)?

4 equations, 3 unknowns; one equation for each element of S:

$$a = c_1(1) + c_2(i) + c_3(2i) \quad \text{(simple matrix addition)}$$
$$\vdots$$

<u>Definition:</u> *Subspace*

Let V be a vector space, and let W be a subset of V.

W is a subspace if it is itself a vector space.

Examples:

- The line y=x (or any straight line through the origin) is a subspace of R2
  - (1,1) + c(2,2) is obviously also on y=x
  - (0,0), the zero vector, is on y=x
- Quadratic forms (everything not linear) is not.

Subspaces are *linear subsets*

Also, since they are vector spaces, they are closed under addition and scalar multiplication

When checking if W is a subspace of V, try the following (in order):

- Check if the zero vector is in W. If not, then either:
  - W is empty
  - For some vector w in W, 0w = $\vec{0}$. But the zero vector is not in W, so W is not closed under multiplication and thus W is not a vector space.
- Check if closed under addition

Ex. Let V be the set of all 2x2 matrices.

Some subsets of V:

$$W_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + 2b + 4c + 3d = 5 \right\}$$

Zero matrix not in W1 -> not a subspace

<u>Theorem</u>

If V is a vector space and S is a subset of V, then Span(S) is a vector subspace

- Spans are closed under addition and scalar multiplication
- Closed under scalar multiplication implies that the zero vector is in S

Also, spanning the empty set gives us the zero vector, which is itself a subspace

V (itself) and {0} are *trivial subspaces* of V

Efficient definition of subspaces:

- Let x and y be in S; let a and b be in a field F
- If ax + by is in S, then S is a subspace

Since subspaces are closed, when working with subspaces, no need to worry about the set that it is part of. R2 is a subspace of R3, etc…