Suppose we have a function < , > : V x V -> R

Dot product is an example of this class of functions

Definition: Inner product

We say that < , > is an inner product if:

- <x, y> = <y, x> symmetry
- <x,x> ≥ 0 if x ≠ 0 non-negativity
- <x,x> = 0 if and only if x = 0
- <cx + y, z> = c<x,z> + <y,z> linearity in 1st argument

In this case, (V, < , >) is a real inner product space

The first and fourth points together imply linearity in the second argument

Ex. $\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + \underbrace{3x_2 y_3 + 5x_3 y_2}_{\text{not symmetric}}$

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Ex. $\langle \vec{x}, \vec{y} \rangle = x_1 y_1 - 3x_2 y_2 + 5x_3 y_3$: Fails non-negativity

Ex. $\langle \vec{x}, \vec{y} \rangle = (x_1 y_1)^2 - 3x_2 y_2 + 5x_3 y_3$

Square -> nonlinear

Ex. $\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + 3x_2 y_2 + 5x_3 y_3$: ✓

Proving symmetry:

$$\begin{aligned} \langle \vec{y}, \vec{x} \rangle &= y_1 x_1 + 3y_2 x_2 + 5y_3 x_3 \\ &= x_1 y_1 + 3x_2 y_2 + 5x_3 y_3 = \langle \vec{x}, \vec{y} \rangle \end{aligned}$$

Proving non-negativity:

$$\langle \vec{x}, \vec{x} \rangle = x_1^2 + 3x_2^2 + 5x_3^2 \geq 0$$

Proving linearity:

$$\alpha\vec{x} + \vec{y} = \alpha x_1 + y_1 + \alpha x_2 + y_2 + \alpha x_3 + y_3$$

$$\langle \alpha\vec{x} + \vec{y}, \vec{z} \rangle = (\alpha x_1 + y_1)z_1 + 3(\alpha x_2 + y_2)z_2 + 5(\alpha x_3 + y_3)z_3$$

$$= (\alpha x_1 z_1 + 3\alpha x_2 z_2 + 5\alpha x_3 z_3) + (y_1 z_1 + 3y_2 z_2 + 5y_3 z_3)$$
$$= \alpha\langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$$

Ex. $V = P_2(\mathbb{R})$

$\langle p, q \rangle = a_0 b_0 + 2a_1 b_1 + 17 a_2 b_2$: linear

Ex. $\langle p, q \rangle = a_0 b_0 + 2a_1 b_1 + 3a_2 b_2 - 2a_1 b_2 - 2a_2 b_1$

$$\langle p, p \rangle = a_0^2 + 2a_1^2 + 3a_2^2 - 4a_1 a_2$$

$$2a_1^2 - 4a_1 a_2 = 2(a_1^2 - 2a_1 a_2)$$
$$= (a_1 - a_2)^2 - 2a_2^2 \quad \text{complete square}$$

$$= a_0^2 + 3a_2^2 + (a_1 - a_2)^2 - 2a_2^2$$
$$= a_0^2 + a_2^2 + (a_1 - a_2)^2 \geq 0$$

Ex. For continuous functions on $[a, b]$, $\int_a^b f(x)g(x)\,dx$ is an IP

Ex. $V = M_{m\times n}(\mathbb{R})$

$$\langle A, B\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ij} \quad \text{is an IP}$$

$$= tr(B^T A) = tr(A^T B)$$

Complex Inner Products

Properties:

- $\langle x, y\rangle = \overline{\langle y, x\rangle}$ — conjugate symmetry
- Non-negativity
- $\langle cx+y, z\rangle = c\langle x,z\rangle + y\langle z\rangle$ — linearity in the first argument

But unlike real inner products, we do not get linearity in the second argument for free

- $\langle x, cy\rangle = \bar{c}\langle x,y\rangle$

↳ conjugate

“Prototype” complex inner product from MATH 136:

$W = \mathbb{C}^n \ ; \ \vec{x}, \vec{y} \in \mathbb{C}^n$

$$\langle \vec{x}, \vec{y}\rangle = \sum_{i=1}^{n} x_i \bar{y}_i$$

$$= \underline{x}^T \bar{y} = (\underline{x}^T \bar{y})^T = \underline{x}\bar{y}^T$$

↳ Hermission conjugate of y

Ex. $W = \mathbb{C}^2$

$$\langle \underline{x}, y\rangle = 6x_1\bar{y}_1 + 8x_2\bar{y}_2 + \underbrace{2ix_1\bar{y}_2 + 2ix_2\bar{y}_1}_{\text{not flag}}$$

$$\left\langle \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\rangle = 0 + 0 + 2i + 0 = 2i$$

$$\left\langle \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix} \right\rangle = 2i$$: not complex conjugates

Definition: Hermitian conjugate

The Hermitian conjugate of a matrix A is

$$A^* = \bar{A}^T = \overline{(A^T)}$$

Ex. $\langle A, B \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \bar{b}_{ij}$

$$= tr(AB^*)$$

Definition: Let (V, < , >) be an inner product space, and let B = {v1, …, vn} be a basis for V

The matrix representation of this inner product space in basis B

$$G_B : (G_B)_{ij} = \langle \vec{v}_i, \vec{v}_j \}$$

Ex. $\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + 3x_2 y_2 + 5x_3 y_3$

$V = \mathbb{R}^3 \to B$: standard basis $(1, 0, 0), \ldots$

$$G_B : \begin{bmatrix} \langle \vec{v}_1, \vec{v}_1 \rangle & \overset{0}{\langle \vec{v}_1, \vec{v}_2 \rangle} & \\ \vdots & \langle \vec{v}_2, \vec{v}_2 \rangle & \vdots \\ \cdots & & \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$\hookrightarrow 3 : x_2 = y_2 = 1$, rest are 0

Ex. $V = P_2([-1, 1])$: polynomials defined from $[-1, 1]$

$$\langle \vec{p}, \vec{q} \rangle = \int_{-1}^{1} p(x)q(x)\, dx$$

$$B = \{1, x, x^2\}$$

IPs:

- <1,1>, <1,x>, <1,x^2>
- <x,1>, <x,x>, <x,x^2>
- <x^2, 1>, <x^2, x>, <x^2, x^2>

Note that this IP is symmetric, so we don't have to do all 9 calculations

$$\int_{-1}^{1} 1 \, dx = 2 \qquad \int_{-1}^{1} x^3 \, dx = 0$$: odd function, bounds are inverses $\rightarrow 0$

$$\int_{-1}^{1} x \, dx = 0 \qquad \int_{-1}^{1} x^4 \, dx = \frac{2}{5}$$

$$\int_{-1}^{1} x^2 \, dx = \frac{2}{3}$$

$$\begin{bmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 2/5 \end{bmatrix}$$

Lemma 1
Let $(V, <,>)$ be an n-dimensional inner product space and let $B = \{\mathbf{v}_1, .., \mathbf{v}_n\}$ be a basis for V, and let G_B be the matrix representation of the inner product (in the basis B). If $\mathbf{v}, \mathbf{w} \in V$, with $[\mathbf{v}]_B = \mathbf{a} \in \mathbb{F}^n$, and $[\mathbf{w}]_B = \mathbf{b} \in \mathbb{F}^n$, then

$$< \mathbf{v}, \mathbf{w} >= \mathbf{a}^T G_B \bar{\mathbf{b}}.$$

Ex. $p(x) = 1 - 2x + 3x^2$

$$\Rightarrow \langle p(x), p(x) \rangle = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 2/5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

Definition: Hermitian matrix

If A = A*, it is a Hermitian matrix, where

$$A^* = (\bar{A})^T = \overline{(A^T)}$$

Notes:

- The matrix representation of a real inner product is always symmetric (and consequently Hermitian)
- The matrix representation of a complex inner product is ALWAYS Hermitian.

As such, for < , > to be a valid inner product:

- M must be Hermitian — proves symmetry / conjugate symmetry
- M must be positive definite — proves non-negativity
- (We get linearity in the first argument for free)