Definition

$\lim_{x \to a} f(x) = L$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that if

$|f(x) - L| < \epsilon$, then

$0 < |x - a| < \delta$

Also, $\lim_{x \to a} f(x) = L$ if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

In one dimension, (x->a) we can only approach a from two directions

However, in two dimensions:

- (x,y) -> (a,b)

We can approach (a,b) in infinitely many ways

Open interval: $(-r, r) : \{x : |x| < r\}$ $\quad r \in \mathbb{R}$

r-neighborhood of $(a,b) \in \mathbb{R}^2$ is

$N_r(a,b) = \{(x,y) \in \mathbb{R}^2 : \|(x,y) - (a,b)\| < r\} : r \in \mathbb{R}$

↳ Euclidean distance

$r = 0 \to N_r = \{\}$

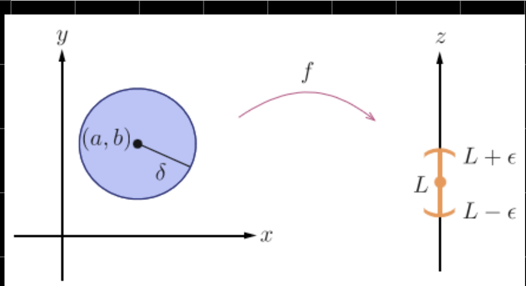
Definition: Limit

Assume $f(x,y)$ is defined in a neighbourhood of (a,b), except possibly at (a,b). If, for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$0 < \|(x,y) - (a,b)\| < \delta \quad \text{implies} \quad |f(x,y) - L| < \epsilon$$

then

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$



δ - neighborhood

Limit Theorems

Limit Theorem 1

If $\lim_{(x,y)\to(a,b)} f(x,y)$ and $\lim_{(x,y)\to(a,b)} g(x,y)$ both exist, then

a. $\lim_{(x,y)\to(a,b)} [f(x,y)+g(x,y)] = \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y)$.

b. $\lim_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = \left[\lim_{(x,y)\to(a,b)} f(x,y)\right]\left[\lim_{(x,y)\to(a,b)} g(x,y)\right]$.

c. $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)}$, provided $\lim_{(x,y)\to(a,b)} g(x,y) \neq 0$.

Proof :

(a) $\lim_{(x,y)\to(a,b)} f(x,y) = L_1$ $\qquad$ $\lim_{(x,y)\to(a,b)} g(x,y) = L_2$

So $\|(x,y)-(a,b)\| \to |f(x) - L_1| < \frac{\varepsilon}{2}$

and $|g(x) - L_2| < \frac{\varepsilon}{2}$

$\Rightarrow |f(x) + g(x) - L_1 - L_2| = |[f(x) - L_1] + [g(x) - L_2]|$

$\leq |f(x) - L_1| + |g(x) - L_2|$

(triangle inequality)

$< \varepsilon$

Limit Theorem 2

If $\lim_{(x,y)\to(a,b)} f(x,y)$ exists, then the limit is unique.

Proof : Assume not unique

Then $\lim_{(x,y)\to(a,b)} f(x,y) = L_1$ and $\lim_{(x,y)\to(a,b)} f(x,y) = L_2$

for some $L_1, L_2 \in \mathbb{R}$

$$\Rightarrow |L_1 - L_2| = \left| \lim_{(x,y)\to(a,b)} f(x,y) - \lim_{(x,y)\to(a,b)} f(x,y) \right|$$
$$= 0$$

$\Rightarrow L_1 = L_2$

$\Rightarrow$ not unique

Proving that a limit does not exist

Ex. $\dfrac{xy}{x^2+y^2}$ as $(x, y) \to (0, 0)$

Trace path along two lines:

(a) $y = 0 \to f(x, 0)$

$$\Rightarrow \lim_{x\to 0} \frac{0}{x^2} = \lim_{x\to 0} 0 = 0$$

(b) $y = x$

$$\lim_{(x,y)\to(0,0)} f(x, x) = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

$0 \neq \frac{1}{2} \to$ does not exist

Example 2

Prove that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$ does not exist.

Solution:

Approaching the limit along lines $y = mx$ we get

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{\sin(x(mx))}{x^2+(mx)^2} &= \lim_{x\to 0} \frac{\sin(mx^2)}{x^2(1+m^2)} \\
&= \lim_{x\to 0} \frac{2mx\cos(mx^2)}{2x(1+m^2)} \qquad \text{by L'Hôpital's rule} \\
&= \lim_{x\to 0} \frac{m\cos(mx^2)}{1+m^2} \\
&= \frac{m}{1+m^2}
\end{aligned}$$

Since the limit depends on m we can get different limits along different lines $y = mx$ and hence $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+y^2}$ does not exist.

Proving a limit exists

Theorem 1: Squeeze Theorem

If there exists a function $B(x,y)$ such that

$$|f(x,y) - L| \leq B(x,y), \quad \text{for all } (x,y) \neq (a,b)$$

in some neighborhood of (a,b) and $\lim_{(x,y)\to(a,b)} B(x,y) = 0$, then

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

Ex. Prove that $\lim_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^2+y^2} = 0$.

$\downarrow$ $f(x,y)$ $\quad$ $\hookrightarrow L = 0$

$$|f(x,y) - L| = |f(x,y) - 0| = \frac{x^2 \cdot |y|}{x^2+y^2}$$

Since $y^2 \geq 0$, $x^2 + y^2 \geq x^2$

$$\Rightarrow x^2 \leq x^2 + y^2$$

$$\Rightarrow \frac{x^2 \cdot |y|}{x^2+y^2} \leq \frac{(x^2+y^2)\cdot|y|}{x^2+y^2}$$

$$= |y| \to 0 \text{ as } (x,y) \to (0,0)$$