Continuous Uniform Distribution

- X can only take on values from [a,b]
- Same probability for every interval of a fixed length
- X ~ Uniform(a,b)

Probability density function: since all points are equally likely, f(x) = k for all $a \le x \le b$

$$\int_a^b f(x)\,dx = 1$$

$$\Rightarrow \int_a^b k\,dx = 1$$

$$\Rightarrow k = \frac{1}{b-a}$$

CDF : $F(x) = P(X \le x)$

$$= \int_a^x f(x)\,dx$$

$$= \int_a^x \frac{1}{b-a}\,dx$$

$$= \frac{x-a}{b-a}$$

Mean : $\dfrac{a+b}{2}$

Variance : $\dfrac{(b-a)^2}{12}$

Example: How to transform a r.v. with a general continuous distribution to obtain a uniformly distributed one.

Let X be a continuous r.v. with p.d.f. of the form

$$f(x) = 0.1e^{-0.1x} \text{ for } x > 0.$$

Let the new r.v. be given by

$$Y = e^{-0.1X}.$$

Show that Y has a uniform distribution on [0,1].

$$f(x) = 0.1e^{-0.1x}$$

$$F_x(x) = P(X \leq x) = \int_0^x 0.1e^{0.1u}\,du$$

$$= 1 - e^{-0.1x}$$

We know that $Y = e^{-0.1x}$

$$F_Y(y) = P(Y \leq y)$$

$$= P(e^{-0.1x} \leq y)$$

$$= P\left(X > \frac{\ln(y)}{0.1}\right)$$

$$= 1 - P\left(X \leq \frac{\ln(y)}{0.1}\right)$$

$$\vdots$$

$$f_Y(y) = 1$$

<u>Exponential Distribution</u>

In a Poisson process in which events occur at a rate λ over time, let X represent the length of time we wait for the first event occurrence.

Since X = TIME (not number of occurrences, which is discrete), exponential is *continuous*

Ex. Calls to a fire station follow a Poisson process, while the *length of time between consecutive calls* follows an exponential distribution.

Let X ~ Exponential(λ)

$$\text{PDF}: f(x) = \lambda e^{-\lambda x}$$

$$\text{CDF}: F(x) = P(X \leq x) = \int_0^x \lambda e^{-\lambda u} \, du$$

$$= 1 - e^{-\lambda x}$$

<u>Link between a Poisson process and an exponential distribution</u>

Suppose that lightning strikes occur according to a Poisson process with rate λ.

Now, let X represent the time until the first lightning strike.

F(X) = P(X≤x) = P(time until the first strike ≤ x)

Since we are calculating the probability of one strike in a time interval ≤ x, then there must be at least one strike in (0,x). The number of strikes by x has a Poisson distribution with mean μ=λx.

So F(x) = P(at least one strike by time x)

F(x) = 1 - P(no strikes by time x)

$$= 1 - \frac{e^{-\lambda x}(\lambda x)^0}{0!}$$

$$= 1 - e^{-\lambda x}$$

$$\text{PDF}: f(x) = F'(x) = \lambda e^{-\lambda x}$$

Alternate form: for θ > 0:

$f(x) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$ $\quad : \lambda = \frac{1}{\theta}$

0 otherwise

$F(x) = 1 - e^{-x/\theta}$

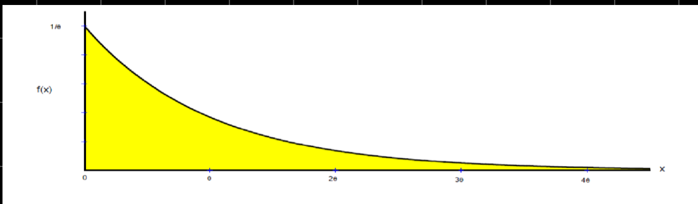

Figure 8.9: Graph of the probability density function of a *Exponential* (θ) random variable

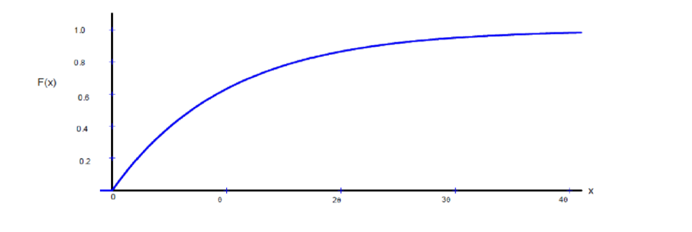

Figure 8.10: Cumulative distribution function for a *Exponential* (θ) random variable

Deriving the mean and variance

We could do this by using integration by parts

Or we could use a gamma function:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx \qquad (\alpha > 0)$$

Properties:

$$\Gamma(1) = \int_0^\infty e^{-x} \, dx = 1$$

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$$
$$= (\alpha - 1)! \quad ☆$$

$$\Gamma(1/2) = \sqrt{\pi}$$

Let X ~ Exponential(θ)

$$E(x) = \int_0^\infty x \cdot f(x) \, dx$$

$$= \int_0^\infty x \cdot \frac{1}{\theta} e^{-x/\theta}\, dx$$

Using $y = \frac{x}{\theta} \Rightarrow x = \theta y,\ dx = \theta\, dy$

$$E(X^2) = \int_0^\infty \theta^2 y^2 \cdot \frac{1}{\theta} \cdot e^{-y} \cdot \theta\, dy$$

$$= \theta^2 \int_0^\infty y^2 e^{-y}$$

$$= \theta^2 \cdot \Gamma(3)$$

$$= \theta^2 \cdot 2!$$
$$= 2\theta^2$$

The same substitution gives us $E(X) = \theta$

$$\Rightarrow Var(X) = E(X^2) - E(X)^2$$
$$= 2\theta^2 - \theta^2$$
$$= \theta^2$$

$$\Rightarrow sd(X) = \theta$$

Memoryless Property of Exponential Distributions

$$P(X > b + c \mid X > b) = \frac{P(X > b + c \cap X > b)}{P(X > b)}$$

$$= \frac{e^{-(b+c)/\theta}}{e^{-b/\theta}}$$

$$= e^{-c/\theta}$$

$$= P(X > c)$$

Ex. The amount of time in hours that a computer survives before breaking down is exponentially distributed with a mean of 100 hours.

Let C = event that if a computer survives more than 100 hours, it survives an additional 50 hours

Let A = event that computer survives more than 100 hours

B = event that a computer survives more than 150 hours

$P(C) = P(B|A) = P(\text{survives } 100\text{-}50 \text{ hours}) = P(X > 50)$

“Forgets” that it survived the last 100 hours