Mean Value Theorem

If:

- f is continuous on [a,b]
- f is differentiable on [a,b]

Then there exists a c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

<u>Proof</u>. Assume f is continuous and differentiable on [a,b]. Let:

$$h(x) = f(x) - f(a) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a)$$

Since f is continuous on [a,b], so is k.

Since f is differentiable on (a,b), so is h.

$$h(a) = f(a) - f(a) - \left[\frac{f(b) - f(a)}{b - a}\right](a - a) = 0$$

$$h(b) = f(b) - f(a) - \left[\frac{f(b) - f(a)}{\cancel{b - a}}\right]\cancel{(b - a)} = 0$$

Rolle's theorem states that if a function f is continuous on the closed interval [a, b] and differentiable on the open interval (a, b) such that f(a) = f(b), then f′(x) = 0 for some x with a ≤ x ≤ b.

Therefore, h satisfies Rolle's theorem, and there exists a c in (a,b) such that h'(c) = 0.

$$\text{But } h'(c) = \frac{d}{dc} f(c) - \underbrace{f(a)}_{\substack{\text{constant}\\ d = 0}} - \left[\frac{f(b) - f(a)}{b - a}\right]\underbrace{(c - a)}_{\substack{\text{constant}\\ d = 0}}$$

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a} \rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

MVT example

If f(x) = x^2 + 2x + 1, find the c's that come from the MVT on [1,2]

$a = 1$ Polynomial → continuous and differentiable

$b = 2$

$$\frac{f(2) - f(1)}{2 - 1} = 5$$

Antiderivatives

F(x) is an antiderivative of f(x) is F'(x) = f(x)

Antiderivatives are NOT unique since they take the form of *(antiderivative)* + C

Two antiderivatives of the same function will always differ by a constant

<u>Constant Function Theorem</u>

If f'(x) = 0 for all x over an interval I, then there exists a real number α such that f(x) = α for all x in I

constant function - rate of change is 0 not necessarily continuous

<u>Proof</u>. Assume f'(x) = 0 for all x in I; let x1 be in I.

Say f(x1) = α. Let x2 be in I and x2 ≠ x1.

Since f' exists on I, we can apply MVT to f between x1 and x2. As such, there exists a c between x1 and x2 such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

But $f'(c) = 0$, so

$$0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$f(x_2) - f(x_1) = 0$$

$f(x_1) = \alpha$, so $f(x_2) = \alpha$

Since x2 was arbitrary, f(x) = α for all x in I

Recall definition - QED

<u>Antiderivative Theorem</u>

If f'(x) = g'(x) for all x in I, then f(x) = g(x) + α for some real number α

<u>Proof</u>. Assume f'(x) = g'(x). So f'(x) - g'(x) = 0

Consider $h(x) = f(x) - g(x)$.
So $h'(x) = f'(x) - g'(x) = 0$

By the Constant Function Theorem, $h(x) = \alpha$ for some $\alpha \in \mathbb{R}$

$\therefore f(x) - g(x) = \alpha$
$\Rightarrow f(x) = g(x) + \alpha$