Union of strings (???)

Recursive definition:

- E, 0, and 1 are regular expressions.
- If R and S are regular expressions, then so is R U S and so is RS.
- If R is a regular expression, then so is R^k.

Some examples of regular expressions:

- R = {1, 11, 111}
- R = 0*
- R = (0 U 00)*

these produce the same thing
⇒ ambiguous

R is *unambiguous* if every element can be produced in a *unique* way. More specifically:

- RS is unambiguous if there exists a bijection between RS and R x S
- R U S is unambiguous if R and S are disjoint; that is, R and S have no intersection

Unambiguous -> easy to compute generating series

Ex.

- 0* = {E, 0, 00, …} -> unambiguous
- 00*: concatenation of 0 with every element of 0* -> {0, 00, 000, …} -> unambiguous
- (0 U E)00* – first letter of each string is either 0 or empty set

↳ for i in 00*:

set.append(0 + i) 0
set.append(i) ε

$\} 0 \cup \varepsilon$

$000 = (0)(0)(0)$

OR $(\varepsilon)(0)(00)$

- 0(000)* -> unambiguous

$$0(000)^* = (0)\{\varepsilon, 000, 000000\} = \{0^{3k+1}, k \geq 0\}$$

- (E U 0 U 00)* -> ambiguous

$$00 = (\varepsilon)(0)(0)$$
$$\text{OR } (\varepsilon)(00)$$
$$\text{OR } (00)(\varepsilon)(\varepsilon)\cdots(\varepsilon)$$

Infinite amount of ways to make 00

If the empty set is included within the star, the regular expression is ambiguous

<u>Block Decomposition</u>

Block: nonempty maximal subsequence of consecutive equal bits

$$\underbrace{1\,1\,1}\,\underbrace{0\,0\,0}\,\underbrace{1}\,\underbrace{0}\,\underbrace{1}\,\underbrace{0\,0}\,\underbrace{1}$$

Regular expressions for block decomposition:

$$0^*(11^*00^*)^*1^* \to (\varepsilon)[(1)(11)(0)(00) \quad \cdots$$
$$1^*(00^*11^*)^*0^*$$

<u>Ex.</u> S = {all blocks are odd}

$$(\varepsilon \cup 0(00^*))(1(11)^*0(00^*))^*(\varepsilon \cup 1(11)^*)$$

Ex.2. S = {all blocks are even}

$$1^*(00^*11^*)^*0^* \qquad \begin{aligned} 1^* &\to (11)^* \\ 00^* &\to (00)(00)^* \\ 11^* &\to (11)(11)^* \\ 0^* &\to (00)^* \end{aligned}$$

$$\Rightarrow \underbrace{(11)^*}_{\frac{1}{1-x^2}}\Big[\underbrace{(00)(00)^*}_{\left(\frac{x^2}{1-x^2}\right.}\;\cdot\;\underbrace{(11)(11)^*}_{\left.\frac{x^2}{1-x^2}\right)^*}\Big]\underbrace{(00)^*}_{\frac{1}{1-x^2}} \longrightarrow \Phi_S(x) = \frac{1}{1-2x^2}$$

The generating series for $\{0, 1\}^*$ is $\frac{1}{1-2x}$

This looks similar to $\Phi_S(x) = \frac{1}{1-2x^2}$

$\Rightarrow$ bijection: $x = x^2$; $\{0, 1\}^* \to \{00, 11\}^*$

Thus, $\{00, 11\}^*$ represents S (all even binary blocks)

Ex. $R = 0^*(11^*00^*)^*1^*$

$$\Phi_R^w(x) = \frac{1-x+x^2}{1-x-x^2}$$

Find how many generating series have weight 6.

$$\frac{1-x+x^2}{1-x-x^2} = \sum_{n\geq 0} f_n x^n$$

$$\Rightarrow 1-x+x^2 = \underbrace{(1-x-x^2)\sum_{n\geq 0} f_n x^n}_{\text{distribute}}$$

$$1-x+x^2 = \sum_{n\geq 0} f_n x^n - \sum_{n\geq 0} f_n x^{n+1} - \sum_{n\geq 0} f_n x^{n+2}$$

$$= \sum_{n\geq 0} f_n x^n - \sum_{n\geq 1} f_{n-1} x^n - \sum_{n\geq 2} f_{n-2} x^n$$

$$= f_0 x^0 + f_1 x^1 - f_0 x^1 + \sum_{n\geq 2} (f_n - f_{n-1} - f_{n-2}) x^n$$

$[x^0]$ of $1-x+x^2 = 1$: