

Suppose we have a function $\langle , \rangle : V \times V \rightarrow \mathbb{R}$

Dot product is an example of this class of functions

Definition: Inner product

We say that \langle , \rangle is an inner product if:

- $\langle x, y \rangle = \langle y, x \rangle$
- $\langle x, x \rangle \geq 0$ if $x \neq 0$
- $\langle x, x \rangle = 0$ if and only if $x = 0$
- $\langle cx + y, z \rangle = c\langle x, z \rangle + \langle y, z \rangle$

symmetry

non-negativity

linearity in 1st argument

In this case, (V, \langle , \rangle) is a real inner product space

The first and fourth points together imply linearity in the second argument

$$\text{Ex. } \langle \vec{x}, \vec{y} \rangle = x_1 y_1 + \underbrace{3x_2 y_2 + 5x_3 y_3}_{\text{not symmetric}}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\text{Ex. } \langle \vec{x}, \vec{y} \rangle = x_1 y_1 - 3x_2 y_2 + 5x_3 y_3 : \text{ fails non-negativity}$$

$$\text{Ex. } \langle \vec{x}, \vec{y} \rangle = (x_1 y_1)^2 - 3x_2 y_2 + 5x_3 y_3$$

Square \rightarrow nonlinear

$$\text{Ex. } \langle \vec{x}, \vec{y} \rangle = x_1 y_1 + 3x_2 y_2 + 5x_3 y_3 : \checkmark$$

Proving symmetry:

$$\begin{aligned} \langle \vec{y}, \vec{x} \rangle &= y_1 x_1 + 3y_2 x_2 + 5y_3 x_3 \\ &= x_1 y_1 + 3x_2 y_2 + 5x_3 y_3 = \langle \vec{x}, \vec{y} \rangle \end{aligned}$$

Proving non-negativity:

$$\langle \vec{x}, \vec{x} \rangle = x_1^2 + 3x_2^2 + 5x_3^2 \geq 0$$

Proving linearity:

$$\alpha \vec{x} + \vec{y} = \alpha x_1 + y_1 + \alpha x_2 + y_2 + \alpha x_3 + y_3$$

$$\begin{aligned} \langle \alpha \vec{x} + \vec{y}, \vec{z} \rangle &= (\alpha x_1 + y_1)z_1 + 3(\alpha x_2 + y_2)z_2 + 5(\alpha x_3 + y_3)z_3 \\ &= (\alpha x_1 z_1 + 3\alpha x_2 z_2 + 5\alpha x_3 z_3) + (y_1 z_1 + 3y_2 z_2 + 5y_3 z_3) \\ &= \alpha \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle \end{aligned}$$

Ex. $V = P_2(\mathbb{R})$

$$\langle p, q \rangle = a_0 b_0 + 2a_1 b_1 + 7a_2 b_2 \quad : \text{linear}$$

Ex. $\langle p, q \rangle = a_0 b_0 + 2a_1 b_1 + 3a_2 b_2 - 2a_1 b_2 - 2a_2 b_1$

$$\langle p, p \rangle = a_0^2 + 2a_1^2 + 3a_2^2 - 4a_1 a_2$$

$$\begin{aligned} 2a_1^2 - 4a_1 a_2 &= 2(a_1^2 - 2a_1 a_2) \\ &= (a_1 - a_2)^2 - 2a_2^2 \quad \text{complete square} \end{aligned}$$

$$\begin{aligned} &= a_0^2 + 3a_2^2 + (a_1 - a_2)^2 - 2a_2^2 \\ &= a_0^2 + a_2^2 + (a_1 - a_2)^2 \geq 0 \end{aligned}$$

Ex. For continuous functions on $[a, b]$, $\int_a^b f(x)g(x) dx$ is an IP

Ex. $V = M_{m \times n}(\mathbb{R})$

$$\langle A, B \rangle = \sum_{i=1}^m \sum_{j=1}^n a_{ij} b_{ij} \quad \text{is an IP}$$

$$= \text{tr}(B^T A) = \text{tr}(A^T B)$$

Complex Inner Products

Properties:

- $\langle x, y \rangle = \overline{\langle y, x \rangle}$ — conjugate symmetry
- Non-negativity
- $\langle cx + y, z \rangle = c \langle x, z \rangle + \langle y, z \rangle$ — linearity in the first argument

But unlike real inner products, we do not get linearity in the second argument for free

- $\langle x, cy \rangle = \bar{c} \langle x, y \rangle$
↳ conjugate

“Prototype” complex inner product from MATH 136:

$$W = \mathbb{C}^n : \bar{x}, \bar{y} \in \mathbb{C}^n$$

$$\langle \bar{x}, \bar{y} \rangle = \sum_{i=1}^n x_i \bar{y}_i$$

$$= \underline{x}^T \bar{\underline{y}} = (\underline{x}^T \bar{\underline{y}})^T = \underline{x} \bar{\underline{y}}^T$$

↳ Hermitian conjugate of \bar{y}

Ex. $W = \mathbb{C}^2$

$$\langle \underline{x}, \underline{y} \rangle = 6x_1 \bar{y}_1 + 8x_2 \bar{y}_2 + \underbrace{2ix_1 \bar{y}_2 + 2ix_2 \bar{y}_1}_{\text{red flag}}$$

$$\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rangle = 0 + 0 + 2i + 0 = 2i$$

$$\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle = 2i : \text{not complex conjugates}$$

Definition: Hermitian conjugate

The Hermitian conjugate of a matrix A is

$$A^* = \bar{A}^T = \overline{(A^T)}$$

$$\text{Ex. } \langle A, B \rangle = \sum_{i=1}^m \sum_{j=1}^n a_{ij} \bar{b}_{ij} \\ = \text{tr}(AB^*)$$

Definition: Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space, and let $B = \{v_1, \dots, v_n\}$ be a basis for V

The matrix representation of this inner product space in basis B

$$G_B : (G_B)_{ij} = \langle \bar{v}_i, \bar{v}_j \rangle$$

$$\text{Ex. } \langle \vec{x}, \vec{y} \rangle = x_1 y_1 + 3x_2 y_2 + 5x_3 y_3$$

$V = \mathbb{R}^3 \rightarrow B$: standard basis $(1, 0, 0), \dots$

$$G_B : \begin{bmatrix} \langle \bar{v}_1, \bar{v}_1 \rangle & \langle \bar{v}_1, \bar{v}_2 \rangle & \dots \\ \vdots & \langle \bar{v}_2, \bar{v}_2 \rangle & \vdots \\ \dots & \dots & \dots \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$\hookrightarrow 3 : x_2 = y_2 = 1, \text{ rest are } 0$

$\text{Ex. } V = P_2([-1, 1])$: polynomials defined from $[-1, 1]$

$$\langle \bar{p}, \bar{q} \rangle = \int_{-1}^1 p(x)q(x) dx$$

$$B = \{1, x, x^2\}$$

IPs:

- $\langle 1, 1 \rangle, \langle 1, x \rangle, \langle 1, x^2 \rangle$
- $\langle x, 1 \rangle, \langle x, x \rangle, \langle x, x^2 \rangle$
- $\langle x^2, 1 \rangle, \langle x^2, x \rangle, \langle x^2, x^2 \rangle$

Note that this IP is symmetric, so we don't have to do all 9 calculations

$$\int_{-1}^1 1 \, dx = 2$$

$$\int_{-1}^1 x^3 \, dx = 0 \quad : \text{ odd function, bounds are inverses } \rightarrow 0$$

$$\int_{-1}^1 x \, dx = 0$$

$$\int_{-1}^1 x^4 \, dx = \frac{2}{5}$$

$$\int_{-1}^1 x^2 \, dx = \frac{2}{3}$$

$$\begin{bmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 2/5 \end{bmatrix}$$

Lemma 1

Let $(V, \langle \cdot, \cdot \rangle)$ be an n -dimensional inner product space and let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for V , and let G_B be the matrix representation of the inner product (in the basis B). If $\mathbf{v}, \mathbf{w} \in V$, with $[\mathbf{v}]_B = \mathbf{a} \in \mathbb{F}^n$, and $[\mathbf{w}]_B = \mathbf{b} \in \mathbb{F}^n$, then

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{a}^T G_B \bar{\mathbf{b}}.$$

Ex. $p(x) = 1 - 2x + 3x^2$

$$\Rightarrow \langle p(x), p(x) \rangle = [1 \ -2 \ 3] \begin{bmatrix} 2 & 0 & 2/3 \\ 0 & 2/3 & 0 \\ 2/3 & 0 & 2/5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

Definition: Hermitian matrix

If $A = A^*$, it is a Hermitian matrix, where

$$A^* = (\bar{A})^T = \overline{(A^T)}$$

Notes:

- The matrix representation of a real inner product is always symmetric (and consequently Hermitian)
- The matrix representation of a complex inner product is ALWAYS Hermitian.

As such, for $\langle \cdot, \cdot \rangle$ to be a valid inner product:

- M must be Hermitian — proves symmetry / conjugate symmetry
- M must be positive definite — proves non-negativity
- (We get linearity in the first argument for free)