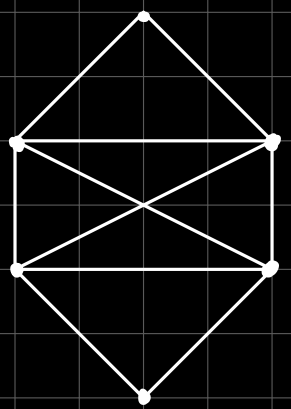
n-cycle: closed walk with n distinct vertices and n distinct edges

<u>Eulerian circuit:</u> closed walk that uses every edge in G exactly once

- *Closed walk:* must start and end at the same vertex
- Vertices may repeat



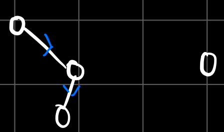
← has Eulerian circuit
all vertices have even degrees

<u>Theorem</u>

A connected graph G has a Eulerian circuit if and only if all its vertices have even degrees

- When you traverse a vertex through one edge, you need another edge to exit it

Isolated vertices are ok:



← no additional edges

<u>Proof of theorem:</u>

(=>) Each use of v contributes 2 to deg(v) for all v in V(G)

(<=) We will prove this by induction on the number of edges in G.

Base case: G has 0 edges -> has a Eulerian circuit

Inductive hypothesis:

There is an Eulerian circuit for all graphs with less than m edges and all even-degree vertices.

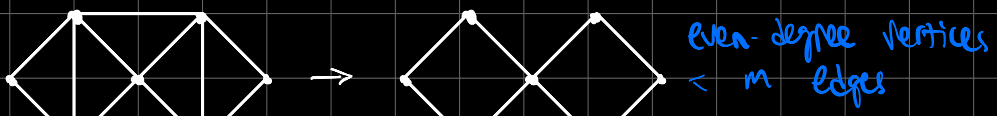
Now, suppose G has m edges, is connected, and all its vertices have even degrees.

Then, all its vertices have degree ≥ 2, which means that there is a cycle C.

Now, let G’ = G - {edges in C}

Removing a cycle yields a number of components (possibly disconnected)

Each vertex that was once in the cycle has degree -= 2, since a cycle requires one way going in and one way going out. The rest have their degrees unchanged.



Now, each component of G’ is connected, has a even-degree vertices, and < m edges.

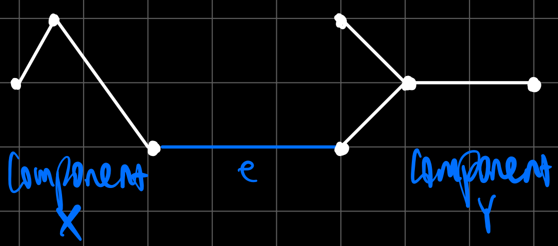
As such, by the inductive hypothesis, they each have a Eulerian circuit.

Since each component shares ≥1 vertex with C, G has a Eulerian circuit.

If the graph is disconnected, we can start and end each Eulerian circuit on a vertex that was on C, then traverse C, then move on to the other components. This yields a circuit that covers the entire graph.

<u>Bridges</u>

An edge e is a bridge if G-e has more components than G



Lemma

If e = xy is a bridge in a connected graph G, then G-e has 2 components:

- x in one component
- y in another

As such, to show that a particular edge xy is *not* a bridge, show that there exists an xy-path in G-xy

Proof. By definition, G-e has at least 2 components, with x in one and y in another.

Let H be a component of G-e including x.

Let z be a vertex not in H.

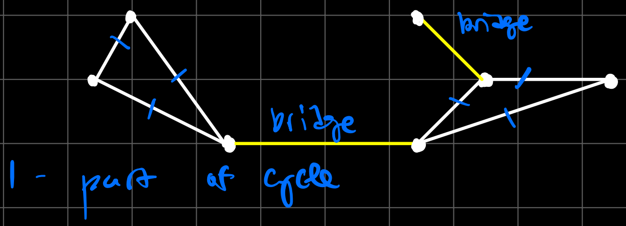
Since there is a path from x->z in G, but not in G-e, every path from x->z must contain e

Since e eventually leads to z, then there is a path from y->z

Thus, z is in the same component as y, and so G-e has 2 components, one with x and one with y

<u>Theorem</u>

An edge xy is either a bridge or part of a cycle.



Proof: Suppose an edge xy is not a bridge.

Then G-xy has an xy-path P that goes from

x -> … -> y

Since xy is not in P, then P+xy is a cycle in G. Since P+xy is

$\underbrace{x \to \dots \to y}_{P}\ \underbrace{\to x}_{xy}$

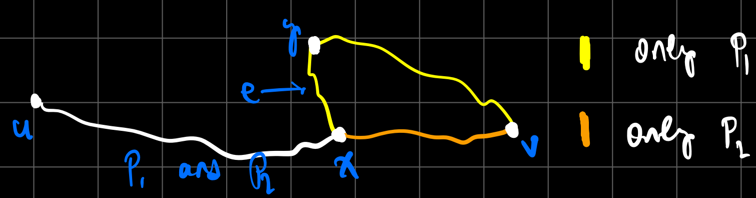
Lemma

If there are distinct paths P1, P2 between vertices u and v in G, then v has a cycle.

Proof. Let e in P1 be the first edge where P1 and P2 diverge.

Then e is not in P2.

Let e=xy, for some y in P1 but not in P2.



Since y eventually connects to v, which connects to x through P1, G-xy has an xy-path.

So e=xy is not a bridge in G

So e is in a cycle

Alternate proof:

There is a path from u->…->x->…->v (through P2)

And a path from u->…->x->y->…->v (through P2)

But v also connects back to x (v->…->x) through P2

So there is a path from x->y->…->v->…->x

This is a cycle

Ex. Can a 2-regular graph have a bridge?

- No: every edge in the graph is part of a cycle

Ex. Can a 4-regular graph have a cycle?

Suppose we have a component H, with one vertex connecting to a bridge.

If we remove this bridge, we have 1 vertex with degree 3 and k vertices with degree 4.

$$3 + 4k = 2|E(G)|$$

(odd) + (even) = (even) → impossible