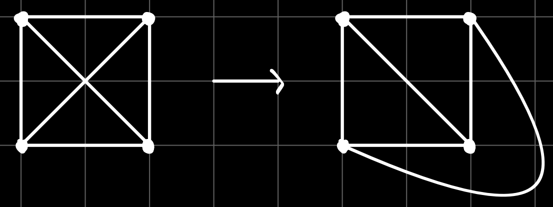
A planar graph can be drawn in R2 (two dimensions) without intersecting edges

- Actual drawing is called a *planar embedding*
- All trees are planar

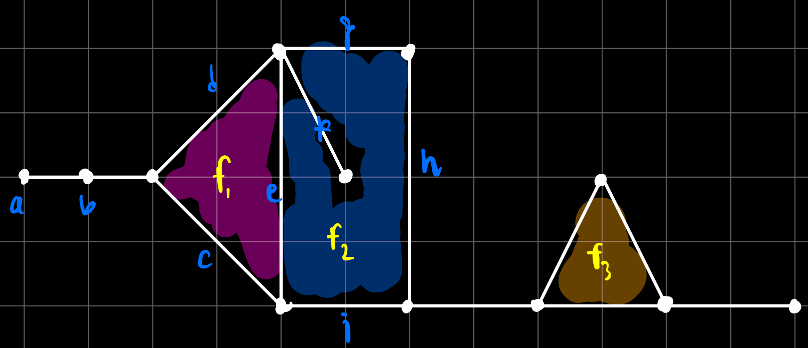
K_4 :



If a graph is not planar, then there does not exist a planar embedding

Faces: connected regions

Boundary walk: minimal closed walk of edges in face



f_2 : h, i, e, f, f, g
(closed walk)

f_4 : outer face :
everything except f

Boundary walk is either clockwise or counterclockwise

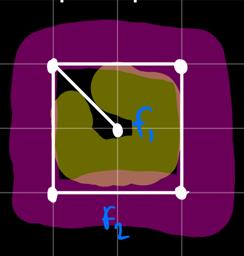
The degree of a face is the number of edges in its boundary walk

Since each edge appears twice among all boundary walks:

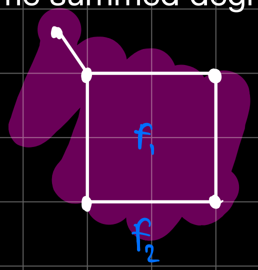
<u>Faceshaking Lemma</u>

$$\sum_{\forall f} \deg(f) = 2|E(G)|$$

Isomorphic planar embeddings have the same summed degrees of faces:

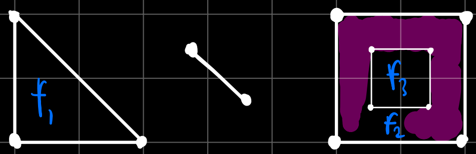


$\deg(f_1) = 6$
$\deg(f_2) = 4$



$\deg(f_1) = 4$
$\deg(f_2) = 6$

If a graph is disconnected, the boundary walk is the union of closed walks



$\deg(f_1) = 3$
$\deg(f_2) = 8$
$\deg(f_3) = 4$

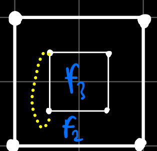
↳ The shaded region touches 8 edges

$\deg(f_4) = 9 : 3 + 2 + 4$

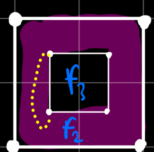
shaded region covering entire graph

Note that shaded regions cannot intersect edges, so the outer face includes everything except f3 (because we would have to shade over the outer edges of f2)

More rigorous definition:



The shaded region including this path is



Faces: all possible shaded regions from all possible paths

<u>Jordan Curve Theorem</u>

Any cycle separates a plane into 2 connected regions

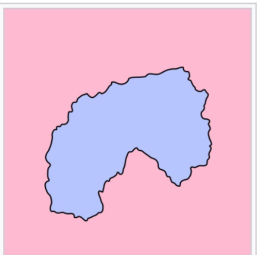
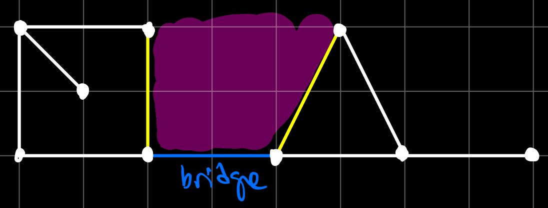


Illustration of the Jordan curve theorem. A Jordan curve (drawn in black) divides the plane into an "inside" region (light blue) and an "outside" region (pink).

Let G be a connected planar embedding.

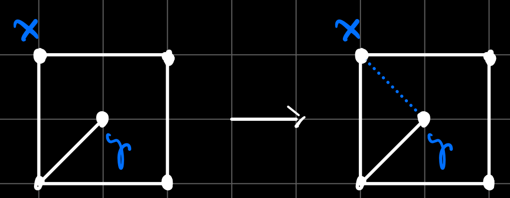
- If an edge e is a bridge, then both sides lie on boundaries of the same face
- If e is not a bridge, it is part of a cycle, and so one side is on one face and the other is on another



Suppose:

- x and y are on the boundary of a face f
- x and y are not adjacent
- There exists a path from x to y

Then, G+xy has one additional face.



Proof: adding an additional edge creates a cycle, and by the Jordan Curve Theorem, this creates an additional face.

<u>Euler's Formula</u>

If G is a connected planar embedding, then v - e + f = 2

Ex. A tree with n vertices has n-1 edges and 1 face. Then n - (n-1) + 1 = 2.

Proof. Let G be a connected planar embedding with v vertices. Performing induction on the number of edges in G:

Base case: If e = v-1 -> tree -> 1 face -> true.

Inductive hypothesis: Suppose that Euler's formula is true for all G with v≤k edges and f faces.

Consider a connected planar embedding with v vertices and k+1 edges.

k+1 > v-1, and so G is not a tree

So there exists an edge xy that is part of a cycle. By the Jordan Curve Theorem, one side of xy is in a face f1 and the other is in a face f2.

Now, let G’ = G - xy

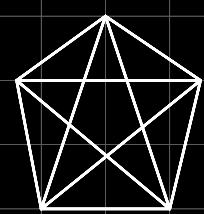
Since xy was part of a cycle in G, G’ has one less face

So G’ has v vertices, k edges, and f-1 vertices

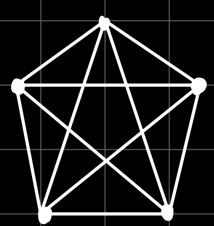
By the inductive hypothesis, v-k+(f-1) = 2

The original graph G has k+1 edges and f faces:

v - (k+1) + f = 2



Ex. K5 is not planar:



$v = 5$
$e = 10$
$5 - 10 + f = 2$
$\Rightarrow f = 7$

The minimum degree of a face in this graph is 3, since every cycle has at least 3 vertices

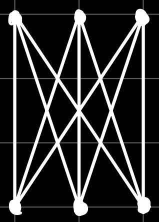
By the Faceshaking Lemma:

$$2(\#\text{edges}) = \sum_{i=1}^{7} \deg(f_i) \rightarrow \deg(f_i) \geq 3$$

$$\Rightarrow 20 = \sum_{i=1}^{7} \deg(f_i) \geq 21$$

$$\Rightarrow 20 \geq 21 \quad \times$$

Ex. The complete bipartite graph $K_{3,3}$



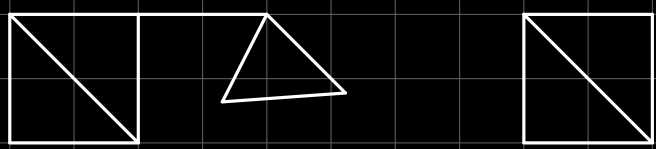
6 vertices
9 edges
5 faces : calculated using E.F.

Since the graph is bipartite, it has no odd cycles, and thus no face with degree 3

So the minimum face degree is 4

Using the Faceshaking Lemma, we eventually get $18 \geq 20$, which is a contradiction

Deriving Euler's formula for a graph with 2 components:



$f_1 = 4$ (including outer face)

$f_2 = 3$

Both subgraphs count the outer face, so we need to subtract 1:

$(v1+v2) - (e1+e2) + (f1 + f2 - 1)$

Substituting in, we get

$(v1+v2) - (e1+e2) + (f1 + f2 - 1) = 3$

If we have 3 components, the outer face is counted 3 times, so we subtract 2. So in general, if a graph has v vertices, e edges, f faces, and c components:

$v - e + f = 1 + c$

In any planar graph, there must be at least one vertex with degree ≤5