Let V and W be vector spaces *over the same field.*

Let L: V -> W be a function.

L is a *linear transformation* if it has:

- Linearity over addition — L(x+y) = L(x) + L(y)
- Linearity over scalar multiplication — L(ax) = a • L(x)
- $L(\vec{0}) = \vec{0}$

Ex. $L: P_1(\mathbb{R}) \to P_1(\mathbb{R})$ ; $L(p(x)) = 2p(x)$

$$\Rightarrow L(p_1(x) + p_2(x)) = 2(p_1(x) + p_2(x))$$

$$= 2p_1(x) + 2p_2(x)$$ multiplication works since $P_1(\mathbb{R})$ is a vector space

$$= L(p_1(x)) + L(p_2(x))$$

As such, L is closed under addition

Ex. $L: P_1(\mathbb{R}) \to P_1(\mathbb{R})$

$L(a+bx) = b + a^2x$

$L(1) = L(1+0x) = x$  $L(2) = 4x$  $L(3) = 9x$

For this to be closed under addition, L(1+2) = L(3) must be equal to L(1) + L(2) = 5x

L(3) = 9x

L(1) + L(2) = 5x

9x ≠ 5x

The derivative is a linear transformation:

- Let L(f(x)) = f'(x)
- If f(x) = sin(x) + 2x, f'(x) = cos(x) + 2 = (sin(x))' + (2x)' = L(sin(x)) + L(2x)

Ex. Let V be a vector space with B = {v1, …, v_n}.

Then, $[\ ]_B : V \to F^n$ is a linear transformation.

Proof: Let v and w be in V.

$$\vec{v} = a_1\vec{v}_1 + \cdots a_n\vec{v}_n \qquad \vec{w} = b_1\vec{v}_1 + \cdots + b_n\vec{v}_n$$

$$\text{Then } \vec{v} + \vec{w} = \sum_{i=1}^{n} (a_i + b_i)\vec{v}_i$$

$$\Rightarrow [\vec{v} + \vec{w}]_B = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} \in \mathbb{F}^n$$

<u>Lemmas</u>

Let $V$ and $W$ be two vector spaces.

Let $L : V \to W$ be a linear transformation, then

(a) $L(\mathbf{0}_V) = \mathbf{0}_W$

(b) $L\left(\sum_{i=1}^{n} a_i \mathbf{v}_i\right) = \sum_{i=1}^{n} a_i L(\mathbf{v}_i)$, where $a_i \in \mathbb{F}$ and $\mathbf{v}_i \in V$, $i = 1, \ldots, n$.

Proof of (a):

$$\text{Suppose } L(\vec{0}_v) = q \in W$$

Since W is a vector space, there exists a -q in W such that q + (-q) = $\vec{0}_W$

Proof (1):

$$\vec{0}_v + \vec{0}_v = \vec{0}_v$$
$$\Rightarrow L(\vec{0}_v + \vec{0}_v) = L(\vec{0}_v) = \vec{q}$$
$$\Rightarrow L(\vec{0}_v) + L(\vec{0}_v) = \vec{q} \qquad \text{by linearity}$$
$$\Rightarrow \vec{q} + \vec{q} = \vec{q}$$

Adding the additive inverse -q to both sides, we have $\vec{q} = \vec{0}_w$

Proof (2):

$$
\begin{aligned}
L(\mathbf{0}_V) &= L(0\,(\mathbf{0}_V)) && \text{(property of the scalar 0)} \\
&= 0\,(L(\mathbf{0}_V)) && (L \text{ is linear}) \\
&= \mathbf{0}_W && \text{(property of the scalar 0).}
\end{aligned}
$$

# Range

Definition: Range

Let T: V -> W be a linear transformation. Then, the range of T is

$$R(T) = \{T(\vec{v}) : \vec{v} \in V\}$$

Lemma: R(T) is a subspace.

Proof:

First, $T(0_V) = 0_W$, so R(T) is not empty

Next, let w1 and w2 be in R(T).

$$\vec{w}_1 \in R(T) \rightarrow \exists \vec{v}_1 \in V : T(\vec{v}_1) = \vec{w}_1$$
$$\vec{w}_2 \in R(T) \rightarrow \exists \vec{v}_2 \in V : T(\vec{v}_2) = \vec{w}_1$$

$$\text{Then, } \vec{w}_1 + \vec{w}_2 = T(\vec{v}_1) + T(\vec{v}_2) = T(\vec{v}_1 + \vec{v}_2)$$

v1+v2 is in V since V is a vector space

So, T(v1+v2) is in R(T) since it satisfies

$$R(T) = \{T(\vec{v}) : \vec{v} \in V\}$$

Proof of scalar multiplication is similar; this completes the proof

Definition: Rank of a linear transformation

If the range of T is finite-dimensional, we define Rank(T) as the dimension of the range of T

- The number of vectors in the basis of R(T)

Lemma: Let T: V -> W be a linear transformation.

Let B = {v1, …, v_n} be a basis for V.

$$\text{Then } R(T) = \text{Span}(\{T(\vec{v}_1), \dots T(\vec{v}_n)\})$$

But the above span is NOT always a basis for R(T)

Proof:

(a) Show $S = \text{Span}(\{T(\vec{v}_1), \dots T(\vec{v}_n)\}) \subseteq R(T)$

$$\begin{aligned} \text{Let } \vec{w} &= c_1 T(\vec{v}_1) + \dots + c_n T(\vec{v}_n) \\ &= T(c_1\vec{v}_1 + \dots + c_n\vec{v}_n) \quad \text{by linearity} \\ &\in R(T) \end{aligned}$$

(b) Show that $R(T) \subseteq S$

Let $\vec{w} \in R(T) \rightarrow \exists \vec{v} \in V : T(\vec{v}) = w$

$\vec{v} \in V \rightarrow \vec{v} = a_1\vec{v}_1 + \dots + a_n\vec{v}_n$

$$\begin{aligned} \text{So } \vec{w} = T(\vec{v}) &= a_1 T(\vec{v}_1) + \dots + a_n T(\vec{v}_n) \quad \text{linearity of vector space } V \\ &\subseteq \text{Span}(\{T(\vec{v}_1), \dots T(\vec{v}_n)\}) \end{aligned}$$

Definition: onto

Let T : V -> W

If R(T) = W, then T is onto

Whether a function is onto depends on the codomain:

- The function T(x): R -> R, T(x) = sin(x) is *not* onto
- But the function T(x): R -> [-1,1] is onto

**Example 1**

Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $L(x, y) = (x, x)$. Then

$$\mathcal{R}(L) = \{L(x, y) : (x, y) \in \mathbb{R}^2\} = \{(x, x) : x \in \mathbb{R}\}.$$

Thus the range is the straight line $y = x$, which is a one-dimensional subspace of $\mathbb{R}^2$.

# Nullspace

Definition: Nullspace

Let T: V -> W. The nullspace of T, N(T) (or Ker(T)) is the set of all vectors v in V such that T(v) = 0

$$\Rightarrow N(T) = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}$$

Lemma: N(T) is a subspace of V

If N(T) is finite-dimensional, we can refer to the *dimension* of the nullspace as the *nullity* of T

Proof:

First, $T(0_V) = 0_W$

Next, let v1 and v2 be in the nullspace of V

Then, T(v1) + T(v2) = $0_W$

Is $cv_1 + v_2$ in N(T)? If so, linear -> subspace

$$\begin{aligned} T(c\vec{v}_1 + \vec{v}_2) &= T(c\vec{v}_1) + T(\vec{v}_2) \\ &= c\underbrace{T(\vec{v}_1)}_{\vec{0}_W} + \underbrace{T(\vec{v}_2)}_{\vec{0}_W} \qquad \text{linearity} \\ &= \vec{0}_W \end{aligned}$$

Definition: One-to-one

Let T: V -> W, and let $v_1$, $v_2$ be in V.

If $T(v_1) = T(v_2)$, then $v_1 = v_2$.

Lemma: T is one-to-one if and only if $N(T) = \{\vec{0}\}$.

Lemma: T maps a linearly independent set onto another linearly independent set

- The *image* of a linearly independent set under T is also linearly independent

**Proof**

Let $\mathbf{w}_i \in L(S),\ i = 1, 2, \ldots, m$, and suppose that $\sum_{i=1}^{m} a_i \mathbf{w}_i = \mathbf{0}_W$ for some $a_i \in \mathbb{F}$.

We first note that each $\mathbf{w}_i = L(\mathbf{v}_i)$ for some $\mathbf{v}_i \in S$.

Thus $\sum_{i=1}^{m} a_i\, \mathbf{w}_i = \mathbf{0}_W$ can be written as $\sum_{i=1}^{m} a_i\, L(\mathbf{v}_i) = \mathbf{0}_W$.

Since $L$ is linear, we have

$$L\left(\sum_{i=1}^{m} a_i\, \mathbf{v}_i\right) = \mathbf{0}_W,$$

that is,

$$\sum_{i=1}^{m} a_i\, \mathbf{v}_i \in \mathcal{N}(L).$$

However, $L$ is one-to-one and so $\mathcal{N}(L) = \mathbf{0}_V$.

So we have $\sum_{i=1}^{m} a_i \mathbf{v}_i = \mathbf{0}_V$.

Since $S$ is linearly independent and $\mathbf{v}_i \in S, \forall i = 1, \ldots, m$, then $a_1 = a_2 = \cdots = a_m = 0$.

We conclude that $L(S)$ is linearly independent. ■.