Suppose we have a linear transformation, $T \in \mathcal{L}(V, W)$. We say that the function S is an **inverse** for T when $S: W \to V$ satisfies $S \circ T = I_V$ and $T \circ S = I_W$, where I_V and I_W denote the identity operators on V and W, respectively.

If $T \in \mathcal{L}(V, W)$, T is invertible $\to$ T^{-1} is linear

T invertible $\Leftrightarrow$ $\dim(V) = \dim(V)$

A function has an inverse if and only if it is both one-to-one and onto.

Proof: (<=)

If T: V -> W is both one-to-one and onto, then there exists two distinct vectors v1 and v2 in V such that T(v1) = w1 and T(v2) = w2, for any two vectors w1 and w2 in W. Then:

$$\begin{aligned} T^{-1}(c\vec{w}_1 + c\vec{w}_2) &= T^{-1}(cT(\vec{v}_1) + T(\vec{v}_2)) \\ &= T^{-1}(T(c\vec{v}_1 + \vec{v}_2)) \\ &= c\vec{v}_1 + \vec{v}_2 \\ &= cT^{-1}(\vec{w}_1) + cT^{-1}(\vec{w}_2) \end{aligned}$$

Lemma 2

Let $T \in \mathcal{L}(V, W)$ be a linear transformation from the finite-dimensional vector space V to the finite dimensional vector space W.
Let B_V and B_W be bases for V and W, respectively. Then

$$T \text{ is invertible } \textbf{ iff } {}_{B_W}[T]_{B_1} \text{ is invertible.}$$

Also, if T is invertible, then ${}_{B_V}[T^{-1}]_{B_W} = ({}_{B_W}[T]_{B_V})^{-1}$.

↳ square matrix

Ex. $T(a + bx) = \begin{bmatrix} a+2b \\ 3a+4b \end{bmatrix}$. Find T^{-1}

$T(1) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\quad T(x) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$\Rightarrow {}_{S_{\mathbb{R}^2}}[T]_{S_{\mathbb{R}}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Inverse: $\frac{-1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \to T^{-1}(\vec{x}) = \frac{-1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} \to P_1(\mathbb{R})$

$= -2a + b + \frac{3}{2}a - \frac{1}{2}b$

Isomorphism

Vector spaces V and W are isomorphic if there exists an invertible linear transformation T: V -> W.

<u>Lemma</u>: V and W are isomorphic if and only if dim(V) = dim(W).

As such, if V has n dimensions, then V is isomorphic to F^n since they have the same dimension

This is a linear transformation

$[\ ]_{B_V} : V \to \mathbb{F}^n$ (B_V is a basis of V)

(Coordinates of some vector V in B_V)

<u>Definition: Isomorphism</u>

An isomorphism is an invertible linear transformation from V to W.

To prove that T is an isomorphism, we must prove that it is invertible and linear.