Sequence - ordered list
This course will only use *infinite sequences*

Ways to write sequences

$$\{a_1, a_2, a_3 \ldots, a_n, \ldots\} \qquad \{a_n\}$$

$$\{a_n\}_{n=1}^{\infty}$$

How to specify sequences

In the sequence

$$\left\{1, \frac{1}{2}, \frac{1}{3} \ldots \frac{1}{n}, \ldots\right\} \quad (1)$$

The nth term is given by 1/n
This can be said using

$$a_n = \frac{1}{n} \qquad \text{or} \qquad \left\{\frac{1}{n}\right\}$$

In sequences like (1), it is often useful to graph things to analyze the structure of the sequence (Graph for 1/x is very handy)

Sequences can be explicit (defined in terms of *n*) or recursive (in terms of the previous term)

Explicit

$$\left\{1, \frac{1}{2}, \frac{1}{3} \ldots \frac{1}{n}, \ldots\right\}$$

Recursive

$$a_n = a_{n-1} + a_{n-2} \quad \text{(Fibonacci)}$$

Where a1=a2=1 base cases

Subsequences and tails

Subsequence - any ordered subset of a sequence (for example, all even nos in Fibonacci)

- Has to be *in the order it was in the original sequence*

Formal definition:

$$n_1 < n_2 < n_3 \rightarrow \{a_n, a_{n+1}, a_{n+2}\}$$

is a sequence

Tail - a special subsequence of the form

$$\{a_n\}_{n-k}^{\infty}$$

Limits of sequences

Ex. $$\lim_{n\to\infty} \frac{1}{n} = 0$$

As n gets closer to infinity, 1/n gets infinitely close to 0

Formal definition of a limit:

$\lim_{n\to\infty} a_n = L$ if for all $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $n \geq N$,

$$|a_n - L| < \varepsilon$$

No matter what positive distance (epsilon) given, the distance between the function and the limit (absolute value thing) will always be smaller

Ex. *Prove with the formal definition that*

$$\lim_{n\to\infty} \frac{8n-5}{6n+2} = \frac{4}{3}$$

Let epsilon > 0
Let *N* be a natural number

$\therefore$ if $n \geq N$,

$$|a_n - L| < \varepsilon$$

$$|a_n - L| = \left|\frac{8n-5}{6n+2} - \frac{4}{3}\right|$$

$$= \left|\frac{-23}{18n+6}\right|$$

$$= \frac{23}{18n+6}$$

Since $n \geq N$

$$\frac{23}{18N+6} < \varepsilon \quad (a)$$

$$\frac{1}{18}\left(\frac{23}{\varepsilon} - 6\right) < N$$

Since N is some random natural number, we can plug it back into (a) to get epsilon = epsilon

Ex. 2. Prove that

$$\lim_{n\to\infty} \frac{4n+10}{n+11} = 4$$

(1) let $\varepsilon > 0$

let $N \in \mathbb{N} \to n \geq N$

(2) $|a_n - L| = \left|\frac{4n+10}{n+11} - 4\right|$

$$= \left|\frac{-34}{n+11}\right| = \frac{34}{n+11} \leq \frac{34}{n} \leq \frac{34}{N}$$

n is random

Since $n \geq N$, $\frac{34}{n} \geq \frac{34}{N}$

$$\frac{34}{N} \leq \varepsilon \Rightarrow \frac{34}{\varepsilon} \leq N$$

$$\therefore \frac{34}{\frac{34}{\varepsilon}} = \varepsilon \qquad \text{QED}$$

Ex.3.

If $\lim_{n\to\infty} a_n = L$, then

$$\lim_{n\to\infty}(3a_n+5)=3L+5$$

By definition:

$$|a_n - L| < \frac{\varepsilon}{3}$$

$$|(3a_n+5)-(3L+5)|$$

$$= |3a_n - 3L| + 3|a_n - L| < 3\left(\frac{\varepsilon}{3}\right) = \varepsilon$$

QED

Theorem
The following are equivalent:

(1) $\lim_{n\to\infty} a_n = L$

(2) For any $\varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ contains a tail of $\{a_n\}$

(3) Any interval (a, b) that contains L contains a tail of $\{a_n\}$

(4) For any $\varepsilon > 0$, the interval $(L-\varepsilon, L+\varepsilon)$ contains all terms $\{a_n\}$ may be

(5) Any interval (a, b) that contains L contains all terms of $\{a_n\}$ except maybe a finite number

Ex. 4. Prove

$$\lim_{n\to\infty} (-1)^n \neq 1$$

Prove

$$\lim_{n\to\infty} \frac{2n}{n+1} = 2$$

Let $\varepsilon > 0$

Let $n \geq N$

$$\left|\frac{n}{n+1} - 2\right| = \left|\frac{2n-2n+2}{n+1}\right| = \frac{2}{n+1} < \varepsilon$$

$$\therefore \frac{2}{n} < \varepsilon \to n > \frac{2}{\varepsilon}$$

Since $n \geq N$, we can let $N = \frac{2}{\varepsilon}$

Proof

Let $\varepsilon > 0$

If $N = \frac{2}{\varepsilon}$, $n > \frac{2}{\varepsilon}$ and $\varepsilon > \frac{2}{n}$

Therefore, $\left|\frac{2}{n+1} - 2\right| = \frac{2}{n+1} < \varepsilon$

QED for FUCK'S SAKE

Theorem

If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = m$

$L = m$

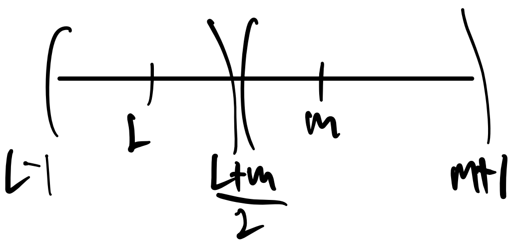
Sequence can only converge to one limit

Suppose

$\{a_n\}$ is a sequence

$\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = m$

$L \neq m$



Clearly $L \in \left(L-1, \frac{L+m}{2}\right)$

Therefore it contains a tail of a_n
Which means the other interval cannot
Therefore the limit does not converge to m

Theorem
If a_n ≥ 0 for all n, and the sequence converges to L, then L ≥ 0

Divergence to infinities

$$\lim_{n\to\infty} n \quad \text{DNE}$$

However

$$\lim_{n\to\infty} a_n = \infty$$

If for all m > 0, there exists a natural number N such that if n ≥ N then a_n > m
Similarly,

$$\lim_{n \to \infty} a_n = -\infty$$

If for all m < 0, there exists a natural number N such that if n ≥ N then a_n < m

Arithmetic Rules for Limits

If $\lim\limits_{n \to \infty} a_n = L$ and $\lim\limits_{n \to \infty} b_n = m$

and $L, m \in \mathbb{R}$

Then

$$\lim_{n \to \infty} C = C$$

$$\lim_{n \to \infty} \cancel{C} a_n = \cancel{C} L$$

$$\lim_{n \to \infty} (a_n \pm b_n) = L \pm m \quad (3)$$

$$\lim_{n \to \infty} (a_n \cdot b_n) = L \cdot m$$

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{L}{m} \quad \text{if } m \neq 0$$

If $a_n \geq 0$ for all n and $\alpha > 0$, then

$$\lim_{n \to \infty} a_n^{\alpha} = L^{\alpha}$$

Since $\lim_{n \to \infty} a_n = L$