Monotone Convergence Theorem (MCT)

If {a_n} is non-increasing/non-decreasing and bounded below/above, then it converges to either the greatest lower bound or greatest upper bound of sequence

$a_n \leq a_{n+1} \to$ non-decreasing

$a_n < a_{n+1} \to$ increasing

Ex. Prove that $a_{n+1} = \frac{7+a_n}{5}$ converges

Substituting in values: a1 = 1, a2 = 8/5… therefore, the sequence seems to be *increasing*

Claim: $1 \leq a_n \leq a_{n+1} \leq 5$ for all n

Proof

Base case

$a_1 = 1, \; a_2 = \frac{8}{5} \ldots$

Inductive hypothesis

Suppose $1 \leq a_k \leq a_{k+1} \leq 5$ for some $k \in \mathbb{N}$

Then

$$7 \leq 7 + a_k \leq 7 + a_{k+1} \leq 12$$

$$1 \leq \frac{7}{5} \leq \frac{7+a_k}{5} \leq \frac{7+a_{k+1}}{5} \leq \frac{12}{5} \leq 5$$

$\therefore 1 \leq a_n \leq a_{n+1} \leq 5$ for all n

Since {a_n} is monotonic and bounded, it converges by the MCT

Finding the limit:

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$$

$$\therefore \lim_{n\to\infty} \frac{7+a_n}{5} = \frac{7+L}{5}$$

$$\Leftrightarrow L = \frac{7+L}{5} \rightarrow L = \frac{7}{4}$$