For linear transformations:

- Small nullspace -> large range
- Large nullspace -> small range

Example 1

Let $L_1 : \mathbb{R}^3 \to P_2(\mathbb{R})$ be a linear transformation defined by $L_1((a,b,c)^T) = \mathbf{0}_{P_2(\mathbb{R})}$. Then

$$\mathcal{N}(L_1) = \mathbb{R}^3 \quad \text{and} \quad \mathcal{R}(L_1) = \{\mathbf{0}_{P_2(\mathbb{R})}\}.$$

"The nullspace is large and the range is small."

The relationship between the sizes of the nullspace and range is actually very precise

Theorem: Dimension Theorem

Let T: V -> W, and let V be finite-dimensional. Then:

- dim(R(T)) + dim(N(T)) = dim(V)
- Rank(T) + nullity(T) = dim(V)

Proof:

Case 1: N(T) = {0_V}. Then, T is one-to-one.

Let S = {v1, …, v_n} be a basis for V, so dim(V) = n.

Since S is a basis, it is linearly independent, and so its image under a one-to-one linear transformation is also linearly independent. Thus:

$$\underbrace{\text{Span}\{T(\vec{v}_1), \ldots T(\vec{v}_n)\}}_{\dim\ n} = R(T)$$

$$\Rightarrow \dim(R(T)) + \dim(N(T)) = n + 0 = \dim(V)$$

Case 2: R(T) is zero vector (similar proof)

Case 3: General case — dim(V) = n > 0

Let dim(V) = n.

Then, N(T) ≠ {0_V}

Let S = {v_1, …, v_k} be a basis for N(T).

Then dim(N(T)) = k = nullity(T).

By the Replacement Theorem, there exist vectors $v_{k+1}, \ldots, v_n$ in V such that

$$N(T) = \text{Span}\{T(\vec{v}_1), \ldots T(\vec{v}_k)\}$$

$$= \text{Span}\{T(\vec{v}_1), \ldots T(\vec{v}_k), T(\vec{v}_{k+1}), \ldots T(\vec{v}_n)\}$$

$$= \text{Span}\{\vec{0}_w, \ldots \vec{0}_w, T(\vec{v}_{k+1}), \ldots T(\vec{v}_n)\}$$

$$\Rightarrow N(T) = \text{Span}\{T(\vec{v}_{k+1}), \ldots T(\vec{v}_n)\}$$

$$\Rightarrow \exists c_{k+1}, \ldots, c_n : c_{k+1}T(\vec{v}_{k+1}) + \cdots + c_nT(\vec{v}_n) = \vec{0}_w$$

$$\Rightarrow c_{k+1}T(\vec{v}_{k+1}) + \cdots + c_nT(\vec{v}_n) = b_1T(\vec{v}_1) + \cdots + b_kT(\vec{v}_k)$$

$$\Rightarrow T(c_{k+1}\vec{v}_{k+1}) + T(c_n\vec{v}_n) = T(b_1\vec{v}_1) + \cdots + T(b_k\vec{v}_k)$$ by linearity

$$\Rightarrow c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n = b_1\vec{v}_1 + \cdots + b_k\vec{v}_k = \vec{0}_w$$

$$\vdots$$

$T(\vec{v}_{k+1}), \ldots T(\vec{v}_n)$ are LI $\rightarrow$ can be basis of range

$\dim(N(T)) = k$

$\dim(R(T)) = n-k$

$\Rightarrow \dim(N(T)) + \dim(R(T)) = n = \dim(V)$ ✓

Corollary: If T is both one-to-one and onto, then dim(V) = dim(W)

Corollary: If dim(V) = dim(W), then T if one-to-one if and only if T is onto.

Proof: (=>) If T is one-to-one, then $N(T) = \{0_V\}$.

Then, the Dimension Theorem says that dim(V) = dim(R(T)) + dim(N(T)). But dim(N(T)) = 0, so dim(V) = dim(R(T)).

But we have already established that dim(V) = dim(W), so now dim(W) = dim(R(T)).

Since R(T) is a subspace of W, but they have the same dimension, R(T) = W.

- Let $\{r_1, r_2, \dots, r_n\}$ be a basis for R(T).
- Since R(T) is a subspace of W, this basis is a linearly independent set of vectors in W. Moreover, it has dim(R(T)) = dim(W) vectors.
- Thus, it is also a basis of R(T).

Since R(T) and W share a basis, they must be equal. Thus, T is onto.

Matrix Representation of Linear Transformations

Definition: a new vector space

Let V and W be vector spaces over the same field. We use the notation

$$\mathcal{L}(V, W)$$

To denote the set of all linear transformations from V -> W.

Then:

$$(\mathcal{L}(V, W), \oplus, \odot) \text{ is a vector space}$$

Addition: let $T_1, T_2 \in \mathcal{L}(V, W)$

$$(T_1 \oplus T_2)(\vec{v}) = T_1(\vec{v}) + T_2(\vec{v})$$

Multiplication: $(c \odot T_1)(\vec{v}) = cT_1(\vec{v})$

Let V and W be finite-dimensional vector spaces.

Let $B1 = \{v_1, \dots, v_n\}$ be a basis for V, and let $B2 = \{w_1, \dots, w_m\}$ be a basis for W.

Then:

$$T(\vec{v}_1) = \begin{matrix} a_{11} w_1 + \\ a_{21} w_2 + \\ \vdots \\ a_{m1} w_m \end{matrix} \qquad T(\vec{v}_j) = \begin{matrix} a_{1j} w_1 \\ + \\ \vdots \\ + \\ a_{mj} w_m \end{matrix}$$

sums, not vectors

$$a_{ij} = i\text{-th component } T(\vec{v}_j)$$

<u>Definition:</u> If T: V->W, the matrix A (above) is denoted

$${}_{B_W}[T]_{B_V}$$

This is the matrix representation of T

<u>Lemma</u>

$$[T(\vec{v})]_{B_W} = {}_{B_W}[T]_{B_V}[\vec{v}]_{B_V} \quad : \vec{v} \in V$$

Proof: let $\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$

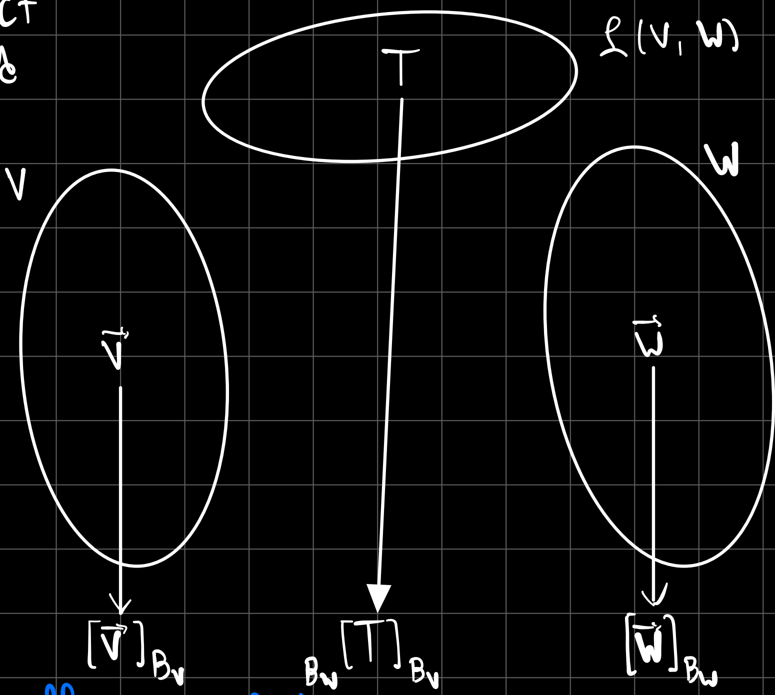
$$T(\vec{v}) = cT(\vec{v}_1) + \cdots + cT(\vec{v}_n)$$

$$[T(\vec{v})]_{B_W} = (cT(\vec{v}_1) + \cdots + cT(\vec{v}_n))\vec{w}_1 + \cdots \qquad B_W = \{\vec{w}_1, \ldots\}$$

taking coordinates

“This picture explains linear algebra”

Abstract world



Concrete

bases allow conversions into a column of numbers

Example 4

Let $T : P_1(\mathbb{R}) \to P_1(\mathbb{R})$ be defined by

$$T(a + bx) = (2a - b) + (4a - 3b)x$$

and let

$S = \{1, x\}$, be the standard basis for $P_1(\mathbb{R})$.

Use the fact that ${}_S[T]_S = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$ to evaluate $[T(-2 + 3x)]_S$.

Solution

$$\begin{pmatrix} -2 \\ 3 \end{pmatrix} = -2(1) + 3(x)$$

so that

$$\left[\begin{pmatrix} -2 \\ 3 \end{pmatrix}\right]_S = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

We know that

$$[T(\mathbf{v})]_S = {}_S[T]_S\,[\mathbf{v}]_S,$$

thus

$$\left[T\begin{pmatrix} -2 \\ 3 \end{pmatrix}\right]_S = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -7 \\ -17 \end{pmatrix}.$$

We conclude that

$$\left[T\begin{pmatrix} -2 \\ 3 \end{pmatrix}\right]_S = \begin{pmatrix} -7 \\ -17 \end{pmatrix}.$$

(1) find coords of $(-2 + 3x)$ in basis S

(2) multiply by change of basis matrix

Example 2

Let $T : \mathbb{R}^2 \to P_2(\mathbb{R})$ be a linear transformation defined by

$$T((a,b)^T) = a + 2bx + (3a+4b)x^2.$$

Using $S_{\mathbb{R}^2} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and $S_{P_2(\mathbb{R})} = \{1, x, x^2\}$, i.e. the standard bases for $\mathbb{R}^2$ and $P_2(\mathbb{R})$, respectively, find ${}_{S_{P_2(\mathbb{R})}}[T]_{S_{\mathbb{R}^2}}$.

Solution

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 + 3x^2 = \begin{array}{r} 1(1) \\ +\ 0(x) \\ +\ 3(x^2) \end{array} \quad \text{and} \quad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2x + 4x^2 = \begin{array}{r} 0(1) \\ +\ 2(x) \\ +\ 4(x^2), \end{array}$$

thus

$${}_{S_{P_2(\mathbb{R})}}[T]_{S_{\mathbb{R}^2}} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 4 \end{pmatrix}.$$

COB matrix made of coordinates of every $T(\vec{v}) : \vec{v} \in S_{\mathbb{R}^2}$

coords of $T\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

coords of $T\begin{pmatrix} 0 \\ 1 \end{pmatrix}$