Bases of Fundamental Subspaces

The four fundamental subspaces of an m x n matrix A are

- Columnspace (in R^m, since one entry for each row and there are m rows)
- Rowspace (in R^n)
- Nullspace: the set of all vectors x such that Ax = $\vec{0}$
- Left nullspace: the set of all vectors x such that A^Tx = $\vec{0}$

Recall: a set S is a <u>subspace</u> of F^n if:

recall laws of addition / multiplication

- Zero vector is in S
- For all vectors x and y in S, x+y is also in S (closed under addition)
- For all vectors x and constants c in F, cx is in S (closed under scalar multiplication)

$\forall \alpha, \beta \in \mathbb{F}, (\alpha + \beta)\vec{x} \in S$

Basically any operation in F^n you can perform on something in S would yield a vector in S

Recall: A set B = {b1,b2,…,b_k} is a <u>basis</u> of a subspace S if S = Span(B) and B is linearly independent

You can find bases by turning S into a matrix and picking the pivot columns of rref(S).

<u>Theorem</u>

Col(A) and Null(A^T) are subspaces of R^m, and Row(A) and Null(A) are subspaces of R^n

Proof seems trivial following from the definition of subspaces

<u>Theorem</u>

Let A be an $m \times n$ matrix. The columns of A which correspond to leading ones in the reduced row echelon form of A form a basis for $\mathrm{Col}(A)$. Moreover,

$$\dim \mathrm{Col}(A) = \operatorname{rank} A$$

↳ # of vectors in any basis

<u>Proof.</u> If A is the zero matrix, this is trivial, since dim Col(A) = rank(A) = 0.

So we will assume that rank A = r > 0.

(Continue this during class)

Subspaces of Linear Mappings

Recall: a linear mapping L: R^n -> R^m has domain R^n and codomain R^m.

It must satisfy the property L(sx + ty) = sL(x) + sL(y) for all vectors x,y in R^n and real numbers s and t.

The *range* of L is the set of all mappings L(x), for all x in R^n.

The *kernel* of L is the set of all vectors x in R^n such that L(x) = 0.

- Essentially, this is the nullspace of the standard matrix [L] = [L(e1) … L(e_n)], where e1,…,e_n are the standard basis vectors (column space of the identity matrix) of R^n.

Notably, L(x) = [L]x for all x in R^n.