In one variable, a point x=c is an absolute maximum of f over an interval I if f(x) ≤ f(c) for all points x in I. The value f(c), then, is the absolute maximum value.

Extreme Value Theorem

If a function f(x) is continuous on a finite closed interval I, then there exist points c1 and c2 in I such that f(c1) ≤ f(x) ≤ f(c2) for all x in I.

- Every finite, closed interval has an absolute minimum and maximum

Definition: Absolute Maximum and Minimum

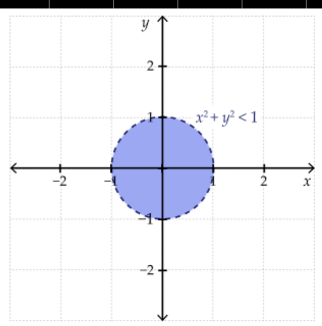
Given a function $f(x,y)$ and a set $S \subseteq \mathbb{R}^2$,

1. a point $(a,b) \in S$ is an **absolute maximum point** of f on S if
$$f(x,y) \leq f(a,b) \quad \text{for all } (x,y) \in S$$
The value $f(a,b)$ is called the **absolute maximum value** of f on S.
2. a point $(a,b) \in S$ is an **absolute minimum point** of f on S if
$$f(x,y) \geq f(a,b) \quad \text{for all } (x,y) \in S$$
The value $f(a,b)$ is called the **absolute minimum value** of f on S.

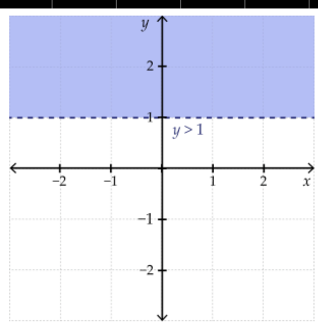
Topological Aspects of R2

A set S in R2 is *bounded* if and only if it is contained within some neighborhood of the origin.

Ex.



The set of points $\{(x,y) \mid x^2+y^2<1\}$ is an example of a bounded set since it can be contained in the neighbourhood of radius 2 around the origin.

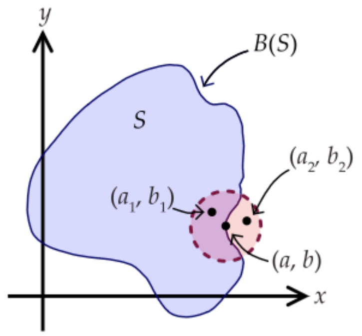


The set of points $\{(x,y) \mid y>1\}$ is an example of an unbounded set. It cannot be contained in a neighbourhood of finite radius around the origin.

- The set $S_0 = \{(x,y) \in \mathbb{R}^2 \quad | \quad -2 \leq x \leq 3\}$ is unbounded since it cannot be contained in a neighbourhood of finite radius around the origin.
- The set $S_1 = \{(x,y) \in \mathbb{R}^2 \quad | \quad x<y\}$ is unbounded since it cannot be contained in a neighbourhood of finite radius around the origin.
- The triangular region S_2 with vertices $(0,0)$, $(0,2)$, and $(2,0)$ is bounded since it can be contained in the neighbourhood of radius 3 around the origin.
- The set $S_3 = \{(x,y) \in \mathbb{R}^2 | \quad 0 \leq \|(x,y)\| \leq 1\}$ is bounded since it can be contained in the neighbourhood of radius 2 around the origin.

Definition: Boundary Point

Given a set $S \subset \mathbb{R}^2$, a point $(a, b) \in \mathbb{R}^2$ is said to be a **boundary point** of S if and only if every neighbourhood of (a, b) contains at least one point in S and one point not in S.



Recall: neighborhoods are defined in terms of radii around a certain point. So, if a point (a,b) exists on the boundary of a set, there is guaranteed to be at least one point on either side of the boundary

The *boundary* of S, B(S), is the set of all boundary points of S

Example 1
Consider $S = (0, 1) \subset \mathbb{R}$.
0 and 1 are the boundary points of S.

one dimension

Example 2
Consider $S = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\}$.
The boundary of S is the y-axis which is in S.

Ex. Find the boundary set of the set $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$

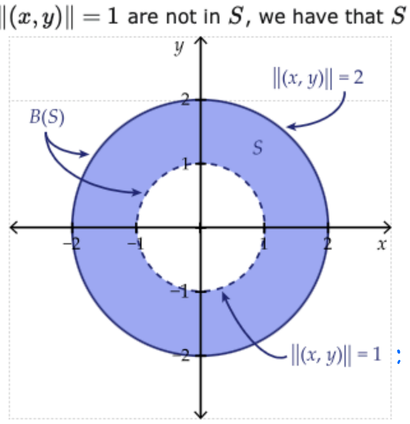
The boundary set of S is the set $B(S) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4\}$

A set is *closed* if it contains all its boundary points

Example 3
Consider $S = \{(x, y) \in \mathbb{R}^2 \mid 1 < \|(x, y)\| \leq 2\}$. The boundary of S is the set of all boundary points. So, as indicated in the diagram, the boundary of S is

$$B(S) = \{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| = 1 \text{ or } \|(x, y)\| = 2\}$$

Since the points (x, y) such that $\|(x, y)\| = 1$ are not in S, we have that S is not closed.



: all points on dashed line are boundary points, but not part of S

Theorem 2: Extreme Value Theorem (EVT) for Functions of Two Variables

If $f(x, y)$ is continuous on a closed and bounded set $S \subset \mathbb{R}^2$, then there exist points $(a, b), (c, d) \in S$ such that

$$f(a,b) \leq f(x,y) \leq f(c,d) \quad \text{for all } (x,y) \in S$$

Your Turn

Determine in which cases the EVT for functions of two variables applies.

- $S = \mathbb{R}^2$ and $f(x, y) = x^2 + y^2$ — ○ EVT applies ● EVT does not apply ✓ — not bounded
- $S = \{(x, y) \mid x^2 + y^2 < 1\}$ and $f(x, y) = x^2 + y^2$ — ○ EVT applies ● EVT does not apply ✓ — not closed
- $S = \{(x, y) \mid x^2 + y^2 \leq 1\}$ and $f(x, y) = xy$ — ● EVT applies ○ EVT does not apply ✓

Algorithm for Extreme Values

Algorithm

First, check to see if the given set $S \subset \mathbb{R}^2$ is closed and bounded.
Next, check to see if the given function $f(x, y)$ is continuous on S.
If both conditions above are satisfied, then to find the maximum and/or minimum value of a function $f(x, y)$:

1. Find all critical points of f that are contained in S.
2. Evaluate f at each such point.
3. Find the maximum and minimum values of f on the boundary $B(S)$.
4. The maximum value of f on S is the largest value of the function found in steps 2 and 3.
   The minimum value of f on S is the smallest value of the function found in steps 2 and 3.

Ex. Use the EVT to find the maximum and minimum values of the function $f(x, y) = e^{3x^2 - 4y^2}$ on the disc $x^2 + y^2 \leq 4$.

Bounded ✓

$B(S) = \{(x,y) : x^2 + y^2 = 4\}$

$\Rightarrow (0, 2), (2, 0), (-2, 0), (0, -2)$

all $\in S$

$\Rightarrow$ closed ✓

CPs:

$f_x : 6x \cdot e^{3x^2 - 4y^2} \rightarrow$ only 0 when $x = 0$

$f_y : -8y \cdot e^{3x^2 - 4y^2} \rightarrow y = 0$

$\Rightarrow (0, 0)$; $(0, 0) \in S$ ✓ $\rightarrow f(0, 0) = 1$

Solving for min/max on B(S):

- $e^{3x^2-4y^2}$ is maximized at $(\pm 2, 0) \to e^{12}$
- $e^{3x^2-4y^2}$ is minimized at $(0, \pm 2) \to e^{-16}$

Ex. Use the EVT to find the maximum and minimum values of the function $f(x,y) = -3x^2y + 3xy^2$ on the square $0 \le x \le 5$, $0 \le y \le 5$.

$3xy(y-x)$

Bounded ✓

$B(S) = \{(x,0) : 0 \le x \le 5\} \cup \{(x,5) : 0 \le x \le 5\}$

$\Rightarrow$ closed ✓

CPs:

$f_x : -6xy + 3y^2 = -3y(2x-y) \to y = 0$ or $2x - y = 0$

$f_y : -3x^2 + 6xy$

$\Rightarrow y = 2x$

Case 1: $y = 0$

$\Rightarrow 3x^2 = 0 \to x = 0$

Case 2: $y = 2x$

$\Rightarrow -3x^2 + 12x^2 = 0$

$\Rightarrow 9x^2 = 0 \to x = 0$

$y = 2x \to y = 0$

$\Rightarrow$ CPs: $(0,0)$

max when $y = 5$

$15x(5-x)$

$= 75x - 15x^2$

$\Rightarrow 75 - 30x = 0$

$30x = 75$

$$x = \frac{75}{30} = \frac{5}{2}$$

EVT vs. Lagrange Multipliers

- EVT is used for any closed, bounded set
- Lagrange multipliers are used for general constraints
    - Can be a closed/bounded set like a circle, but it must be described by some function $g(x,y) = k$.
    - Can also be used for open sets (hence the condition for endpoints)