Recurrence Relations

Recurrence relation → explicit formula:

1. Get characteristic polynomial

Ex. $a_n - 3a_{n-2} + 2a_{n-3} = 0 \rightarrow 1 - 3x^2 + 2x^3$

2. Factor: $(1 - \underset{\lambda_1}{x})^2 (1 + \underset{\lambda_2}{2x}) \rightarrow \lambda_1 = 1, \quad \lambda_2 = -2$
3.

$$a_n = p_1(\lambda_1)^n + p_2(\lambda_2)^n + \cdots + p_n(\lambda_n)^n$$

where $p_1, \ldots p_n$ are polynomials and $\deg(p_i) <$ multiplicity of λ_i

So, if $\deg(p_i) < 2$, $p_i = A + Bn$

4. Use initial values (usually given) to solve for constants $A, B, \ldots$
5. Plug back into a_n

Rational Function → recurrence relation:

1. $A(x) = \frac{P(x)}{Q(x)} \rightarrow Q(x)A(x) = P(x)$

2. $Q(x)\underbrace{(a_0 + a_1 x + \cdots + a_n x^n)}_{A(x)} = P(x)$

3. Calculate $a_0, \ldots, a_n$ using coefficients of $P(x)$

Ex. $(1-3x^2-2x^3)(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = 6 - x + 5x^2$

Constant term 6 can only be formed by $(1)(a_0)$
So $a_0 = 6$

x^1 : $(a_1)(1) = -1$ since $[x^1]\,P(x) = -1$

x^2 : $\underbrace{(a_0)(-3)}_{-3x^2} + \underbrace{(a_2)(1)}_{a_2 x^2} = \underbrace{5}_{5x^2}$ since $[x^2]\,P(x) = 5$

x^n : $(1)(a_n) + (-3)(a_{n-2}) + (-2)(a_{n-3}) = 0$

Regular Expressions

General form: $0^*(11^*00^*)^*1^*$

Usually don't remove components - only extend

$0 \rightarrow 0^n$ $\quad 0^* \rightarrow (0^n)^*$ $\quad (n > 0)$
$1 \rightarrow 1^n$ $\quad 1^* \rightarrow (1^n)^*$

unless problem says something like "cannot end w/ 1"

Generating series

Calculating coefficients

Ex. $[x^{11}]\, x^2 \left(1-x^3\right)^{-5}\left(3-9x^2\right)^{-1}$

1. Remove positive powers of x

$$\Rightarrow [x^9]\left(1-x^3\right)^{-5}\left(3-9x^2\right)^{-1}$$

2. Factor things out to leave everything in the form $(1-x^n)^{-m}$

$$\Rightarrow [x^9]\left(3^{-1}\right)\underbrace{\left(1-x^3\right)^{-5}}_{(a)}\underbrace{\left(1-3x^2\right)^{-1}}_{(b)}$$

3. Break into cases. x^9 can be formed by either

(i) Taking $[x^9]$ from (a); $[x^0]$ from (b)

(ii) Taking $\underset{n=1}{[x^3]}$ from (a); $\underset{n=3}{[x^6]}$ from (b) $\quad : x^6 = (x^2)^3$

By the NBT, the coefficient for case (ii) is

$$\left(3^{-1}\right) \qquad \left(1-x^3\right)^{-5} \qquad \left(1-3x^2\right)^{-1}$$

$$\left(3^{-1}\right)\underset{\substack{n=1\\ t=5}}{\binom{1+5-1}{5-1}}\left(1^1\right)\underset{\substack{n=3\\ t=1}}{\binom{3+1-1}{1-1}}\left(3^3\right)$$

↳ coefficient of x term; raised to the n-th power

4. Add results from cases

Excluded Substrings

Ex. Find a generating series for all strings without 10101.

Let A: set of all strings without 10101

B: all strings with ONE instance of 10101, at the end

1. Find overlap:

$$\begin{array}{ccccccccc}
1 & 0 & 1 & 0 & 1 & & & & \\
 & 1 & 0 & 1 & 0 & 1 & & & \\
 & & \boxed{1} & \boxed{0} & \boxed{1} & 0 & 1 & & \text{leaves } 01 \\
 & & & 1 & 0 & 1 & 0 & 1 & \\
 & & & & \boxed{1} & 0 & 1 & 0 & 1 \quad \text{leaves } 0101
\end{array}$$

2. $A(10101) = B \cup B(01) \cup B(0101)$

$= B(\varepsilon \cup 01 \cup 0101)$

$1 + x^2 + x^4 \rightarrow C(x) = x^2 + x^4$

3.

$$A(x) = \frac{1 + x^2 + x^4}{(1-2x)(1+x^2+x^4) + \underbrace{x^5}_{x^n}} \quad \text{or} \quad \frac{1 + C(x)}{(1-2x)(1+C(x)) + x^n}$$

x^n: n = length of excluded substring

Alternate approach: $A \cup B = \varepsilon \cup A(0 \cup 1)$

Use other expression to find $A(x)$

Random combinatorial rules

$$\binom{a+b}{a} = \sum_{j=0}^{b} \binom{a+j-1}{a-1}$$

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Using $a=2$, $b=3$:

$$\binom{2+3}{2} = \binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \binom{4}{1}$$

The number of compositions of size n; k parts is

$$\binom{n-1}{k-1}$$

The number of compositions of size n is 2^{n-1}

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$: just the pattern of Pascal's Triangle

Recursive Decompositions

All binary strings are:

$$S = \varepsilon \cup S(0 \cup 1)$$

Graph proofs

Try using:
- Minimal walk / path
- Shortest walk / path

Isomorphisms

Count:
- Degrees of vertices
- Lengths of cycles

Series

$$1+x^2+x^4+\cdots=\frac{1}{1-x^2}$$

$$(1+x)^{-t}=\sum_{n=0}^{\infty}\binom{n+t-1}{t-1}$$

$$(1+x)^{n}=\sum_{k=0}^{n}\binom{n}{k}$$