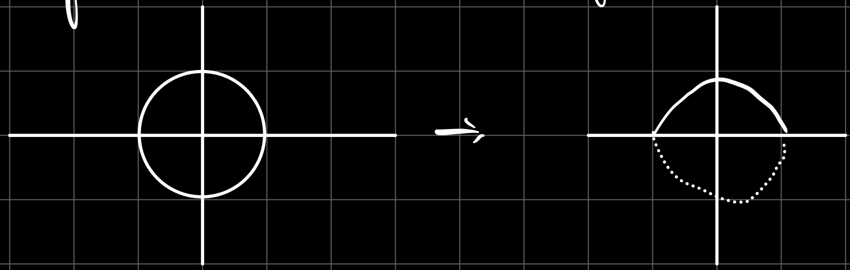
Get the derivative of:

Ex. $x^2 + y^2 = 1$ $\qquad$ $y = \sqrt{1-x^2}$ and $y = -\sqrt{1-x^2}$



Instead of doing this, we can use the *chain rule* to get the derivative in terms of x and y. This is called *implicit differentiation*.

$$x^2 + y^2 = 1 \rightarrow 2x + 2yy' = 0$$

$$\therefore y' = \frac{-2x}{2y} = \frac{-x}{y}$$

$f(g(x)) = y^2$

$f(x) = y^2$; $g(x) = y$

$2(y) \cdot y' = 2yy'$

Ex. 2. Find y' if $3x^3y^3 + x^2y + 13x = 12$

$$\Rightarrow \underbrace{9x^2y^3 + 9x^3y^2y'}_{\text{product rule}} + \underbrace{2xy + x^2y'}_{\text{product rule}} + 13 = 0$$

Ex. 3. $x^2 + y^2 + 1 = 0$

This is not defined anywhere, so the derivative does not exist

Ex.4. 2x = x

This is only defined at one point (x=0), so since derivatives are really limits, it doesn't make sense to talk about derivatives.

Logarithmic Differentiation

We can use implicit differentiation to differentiate functions of the form f(x)^g(x)

Ex. $y = (\ln x)^{\sin x}$ $\qquad x > 1$

$$\ln(y) = \sin x \ln(\ln(x))$$

Note that $\frac{d}{dy}\ln y = \frac{1}{y}$, so by the chain rule this becomes $\frac{y'}{y}$

$\frac{y'}{y} = \cos x \ln(\ln(x)) + \frac{\sin x}{x \ln x}$ (product rule)

$y = (\ln x)^{\sin x}$, so $y' = (\ln x)^{\sin x}\left[\cos x \ln(\ln(x)) + \frac{\sin x}{x \ln x}\right]$

Ex. 2. Find y' if $y = x^{\arctan x}$ $\quad x > 0$
(solve later)

$\ln(y) = \arctan x \cdot \ln(x)$

Note that $\frac{d}{dy}\ln(y) = \frac{1}{y}$

So by the chain rule, $\frac{y'}{y} = \frac{d}{dx}\arctan x \cdot \ln(x)$

$= \frac{\ln x}{x^2+1} + \frac{\arctan x}{x}$

$\Rightarrow y = \left(\frac{\ln x}{x^2+1} + \frac{\arctan x}{x}\right)\left(x^{\arctan x}\right)$

1. Find $\frac{dy}{dx}$ for $\arctan(xy) + y = x$

$= \frac{dy}{dx}\arctan(xy) + y'$

$\frac{dy}{dx}\arctan(xy) = \frac{1}{x^2y^2+1} \cdot$

$\arcsin(x^2y) + xy = 1$

$\Rightarrow \frac{d}{dx}[\arcsin(x^2y) + xy] = \frac{d}{dx} 1$

$\Rightarrow \frac{d}{dx}\arcsin(x^2y) + (y + xy') = 0$

Solving for $\frac{d}{dx}\arcsin(x^2y)$ using the chain rule:

$\Rightarrow \frac{1}{\sqrt{1-(x^2y)^2}} \cdot (x^2y)'$

$\Rightarrow \frac{1}{\sqrt{1-x^4y^2}} \cdot (2xy + x^2y') =$

$\frac{2xy + x^2y'}{\sqrt{1-x^4y^2}} + xy' = -y$

$\overline{-x\sqrt{1-x^4y^2}}$

Let $u = \sqrt{1-x^4y^2}$ (for convenience)

$\frac{2xy + x^2y'}{u} + xy' = -y$

$2xy + x^2y' + uxy' = -yu$

$y'(x^2 + ux) = -yu - 2xy$

$y' = \frac{-yu - 2xy}{x^2 + ux} = \frac{-y\sqrt{1-x^4y^2} - 2xy}{x^2 + x\sqrt{1-x^4y^2}}$

$xy + y^3 = \arctan(x) - 1$

$\Rightarrow \frac{d}{dx}(xy+y^3) = \frac{d}{dx}[\arctan(x) - 1]$

$\Rightarrow y + xy' + 3y^2 \cdot y' = \frac{1}{x^2+1} - 1$

$xy' + 3y^2 \cdot y' = \frac{1}{x^2+1} - 1 - y$

$y'(x+3y^2) = \frac{1}{x^2+1} - 1 - y$

$y' = \frac{dy}{dx} = \frac{\frac{1}{x^2+1} - 1 - y}{x+3y^2}$

At $(0,-1)$, $\frac{dy}{dx} = \frac{\frac{1}{0+1} - 1 + 1}{0+3} = \frac{2}{3}$

$y - y_0 = \frac{2}{3}(x - x_0)$

$y+1 = \frac{2}{3}(x-0) \Rightarrow y = -1 + \frac{2}{3}x$