Lagrange Multiplier Algorithm

Assume that $f(x,y)$ is a differentiable function and $g \in C^1$. To find the maximum value and minimum value of f subject to the constraint $g(x,y)=k$, evaluate $f(x,y)$ at all points (a,b) which satisfy one of the following conditions.

level curve

1. $\nabla f(a,b) = \lambda \nabla g(a,b)$ and $g(a,b)=k$
2. $\nabla g(a,b) = (0,0)$ and $g(a,b)=k$
3. (a,b) is an end point of the curve $g(x,y)=k$

The maximum/minimum value of $f(x,y)$ is the largest/smallest value of f obtained at the points found in conditions 1-3.

Ex. Find the maximum value of $6x+4y-7$ on the ellipse $3x^2+y^2=28$.

(1) Get gradients

$$\nabla f(x,y) = (6, 4) \qquad \nabla g(x,y) = (6x, 2y)$$

(2) Solve for points (a,b) that satisfy condition 1:

$$\nabla f(x,y) = \lambda \nabla g(x,y) \quad \text{and} \quad g(a,b) = 28$$

$$\Rightarrow \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \lambda \begin{bmatrix} 6a \\ 2b \end{bmatrix} \quad \text{and} \quad 3a^2+b^2=28$$

$$\Rightarrow \begin{array}{ll} * & 6a\lambda = 6 \\ ☆ & 2b\lambda = 4 \\ \square & 3a^2+b^2=28 \end{array}$$

3 equations
3 unknowns

Eliminate λ: (*) $\lambda = \frac{1}{a}$ since $a \neq 0$ by (*)

$\Rightarrow 2b = 4a$

$\Rightarrow b = 2a$ (☆)

$\Rightarrow 3a^2 + 4a^2 = 28$ ($\square$)

$\Rightarrow (2,4)$; $(-2,-4)$

For this course there is usually no need to solve for lambda

(3) Solve for points (a,b) that satisfy condition 2:

$$\nabla g(a,b) = (0,0) \quad ; \quad g(a,b) = 28$$

$$\Rightarrow \begin{bmatrix} 6a \\ 2b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } 3a^2 + b^2 = 28$$

$\Rightarrow$ None

(4) Condition 3: endpoints

- g(x,y) is an ellipse, which has no endpoints

(5) Evaluate function at all points (a,b) calculated earlier

- We only have two points: (2,4) and (-2,-4)
- f(2,4) = 21
- f(-2,-4) = -35

Thus, the maximum point is (2,4,21)

And the minimum point is (-2,-4,-35)

Example 4

Find the point on the sphere $x^2 + y^2 + z^2 = 1$ which is closest to the point $(1, 2, 2)$.

Solution:

We want to minimize the distance between the point $(1, 2, 2)$ and a point (x, y, z) on the given sphere. To simplify things, we consider the square of this distance, which is given by the function

$$f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-2)^2$$

The constraint is $g(x, y, z) = x^2 + y^2 + z^2 = 1$.

Get minimum (|f(x,y,z)| closest to 0) value