Vector Spaces

To prove whether or not V is a vector space:

1. Find the zero vector using V's addition laws.
2. Check if the zero vector is in V.
3. Check if V is closed under addition and scalar multiplication — this can be done in one step

If these have all been satisfied, V is a vector space. But we still need to suck off Conrad's math rules boner and prove the following EIGHT axioms:

- *Commutativity*: v + w = w + v
- *Associativity*: a + (b + c) = (a + b) + c
- v + 0 = v (already proven)
- v + (-v) = 0
- *Distributivity*: c(v + w) = (c • v) + (c • w) → c is scalar
- (c + d)(v) = (c • v) + (d • v) } c, d are scalar
- (cd)(v) = c • (d • v)
- 1v = v

Example proof of c(v + w) = (c • v) + (c • w) in the vector space V = R2 with normal laws:

$$c\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} m \\ n \end{bmatrix}\right) = c\begin{bmatrix} x+m \\ y+n \end{bmatrix} = \begin{bmatrix} cx+cm \\ cy+cn \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix} + \begin{bmatrix} cm \\ cn \end{bmatrix} = c\begin{bmatrix} x \\ y \end{bmatrix} + c\begin{bmatrix} m \\ n \end{bmatrix} \checkmark$$

Example 5

Consider the vector space $P_2(\mathbb{R})$ over $\mathbb{R}$, with the usual addition of polynomials and the usual scalar multiplication.

Is the vector $p(x) = 1 + 2x + 3x^2 \in Span(\{(1 - 2x + 3x^2), (9 - 5x + x^2)\})$?

Solution

The questions is thus whether we can find constants $a, b \in \mathbb{R}$, such that

$$1 + 2x + 3x^2 = a(1 - 2x + 3x^2) + b(9 - 5x + x^2) \;\Leftrightarrow$$
$$1 + 2x + 3x^2 = (1a + 9b) + (-2a - 5b)x + (3a + b)x^2.$$

Comparing coefficients yields the following system of three equations:

$$\begin{cases} a + 9b = 1 \\ -2a - 5b = 2 \;. \\ 3a + b = 3 \end{cases}$$

Bases and Subspaces

Example 7

Let $W = \{1+2x+3x^2, 3+2x+x^2, 6+6x+6x^2, 6-6x^2, x^2\}$.

Reduce W to a basis for $P_2(\mathbb{R})$.

We want

$$a(1+2x+3x^2) + b(3+2x+x^2) + c(6+6x+6x^2) + d(6-6x^2) + e(x^2) = 0$$

$$\Rightarrow \begin{aligned} a + 3b + 6c + 6d + e &= 0 \\ 2a + 2b + 6c + 0d + 0e &= 0 \\ 3a + b + 6c - 6d + e &= 0 \end{aligned}$$

coefficient of x^0: make all constant terms equal to 0

coeff. of x^1

of x^2

Then:

1. Put in matrix form and RREF
2. Take columns with pivots
3. Leave only elements of W corresponding to those pivots. This is now linearly independent.

If we have a pivot for each row, the columns with pivots form a basis.

If not, the remaining columns are linearly independent but do not form a basis.

Ex. Find a basis for $W = \text{Span}\{\cos(2x), \sin^2(x), \cos^2(x)\}$

Note that $\cos(2x) = \cos^2(x) - \sin^2(x) \rightarrow$ we can already remove this

Then:

$$a(\sin^2 x) + b(\cos^2 x) = 0$$

Using $x = 0$, we have $b \cdot \cos^2(0) = 0 \rightarrow b = 0$

Using $x = \frac{\pi}{2}$, ... $\rightarrow a = 0$

So $(a, b) = (0, 0)$ is the only solution $\rightarrow$ LI $\rightarrow$ basis ✓

Linear Transformations

T is linear *if and only if* $T(cv_1 + v_2) = cT(v_1) + T(v_2)$.

Range

Let T: V -> W. The range of T is

$$R(T) = \{T(\vec{v}) : \vec{v} \in V\} = \text{Span}\{T(\vec{s}) : s \in S\}$$

where S is a basis of V.

Use this to calculate the range of a linear transformation.

T is *onto* if and only if R(T) = W.

Nullspace

To find the nullspace, solve $T(v) = 0_W$.

Just let v be the default form of a vector in V. For example, if V = R2, then set v = (x,y) and solve.

T is *one-to-one* if and only if its nullspace only contains the zero vector.