Suppose we have a function f : A -> B

- Each a in A is associated with a unique f(a) in B called the *image* of a under f
- A is the domain of f, D(f)
- B is the *codomain* of f
- The *range* of f is a subset of B; containing f(a) for all a in A - R(f)

Scalar Function

$f(x_1, x_2, \dots x_n) : \mathbb{R}^n \to \mathbb{R}$

$R(f) \in \mathbb{R}$

Ex. Find the domain and range of $g(x, y) = \dfrac{x^2 - y^2}{|x| + |y|}$.

Domain: $|x| + |y| \neq 0$

$\Rightarrow \mathbb{R}^2 - \{(0, 0)\}$

Range:

$g(c, 0) = c \quad \forall c > 0$ $\qquad g(c, c) = 0 \quad \forall c \in \mathbb{R}$

$g(0, c) = c \quad \forall c < 0$

$\Rightarrow R(g) = \mathbb{R}$

Geometric Interpretation

The graph of $f(x, y)$ is $\forall$ $((a, b), f(a, b))$ in $\mathbb{R}^3$

such that $(a, b) \in D(f)$

In general, if $f(x, y) = c_1 x + c_2 y + c_3$ (c's are constants), its graph is a plane

Level Curves

- 2-dimensional "slices" of a graph

Set of curves $f(x, y) = k$ $\forall k \in R(f)$

Ex. Find the level curves of the function defined by $f(x, y) = 2x - 3y + 1$.

$R(f) = \mathbb{R}$

So the level curves are

$2x - 3y + 1 = k$ $\forall k \in \mathbb{R}$

Solution:

We observe that $R(f) = \mathbb{R}$.

So, the level curves of f are

$$2x - 3y + 1 = f(x, y) = k, \quad k \in \mathbb{R}$$

For $k = 0$, we get

$$2x - 3y + 1 = 0 \Rightarrow 2x - 3y = -1$$

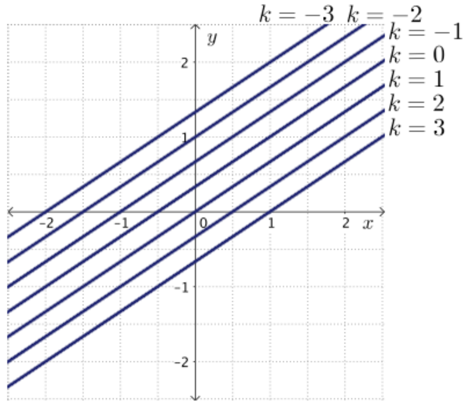
For $k = 1$, we get

$$2x - 3y + 1 = 1 \Rightarrow 2x - 3y = 0$$

For $k = -2$, we get

$$2x - 3y + 1 = -2 \Rightarrow 2x - 3y = -3$$

Sketching gives a family of parallel lines:



Setting $2x - 3y + 1 = k$ defines the line $2x - 3y + (1 - k) = 0$.

Each level curve is the *intersection* between f(x,y) and the plane defined by z=k. In that regard, they form a sort of topographic map for the graph of f in three dimensions

Ex. Sketch the level curves of $f(x, y) = x^2 + y^2$ and use them to sketch the surface $z = f(x, y)$.

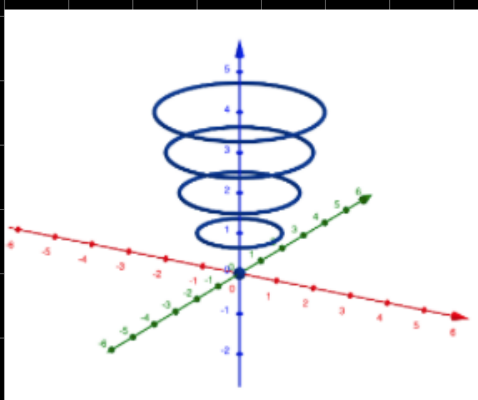
$R(f) = [0, +\infty)$

Using $k = 0$: $x^2 + y^2 = 0$
- circle centered at origin
  $r = 0$

$k = 1$: $x^2 + y^2 = 1$
- circle centered at origin
  $r = 1$

$k = 2$: $r = \sqrt{2}$

So the graph is



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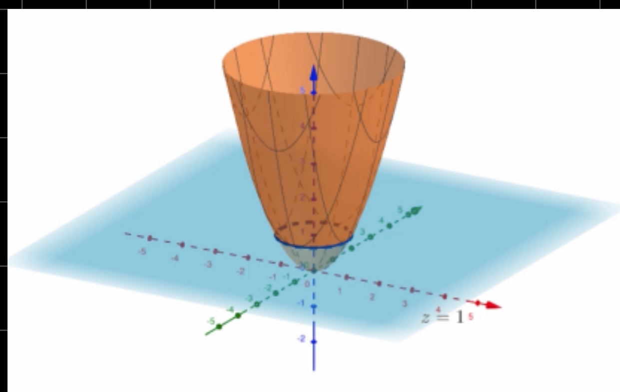

Ex. Sketch the level curves of $h(x, y) = x^2$ and use them to sketch the surface $z = h(x, y)$.

$R(h) = [0, +\infty)$

Using $k = 0$: $x^2 = 0$
$k = 1$: $x^2 = 1 \rightarrow x = \pm 1$
$\vdots$
$k = n$ $\quad x^2 = n \rightarrow x = \pm\sqrt{n}$

Using these to sketch the surface, we get a **parabolic cylinder**.