Let $T \in \mathcal{L}(V, W)$

T is diagonalizable if ${}_{B_W}[T]_{B_V} = D$ is a diagonal matrix

T is diagonalizable if and only if there exists a basis B = {v1, …, vn} for V and scalars λ1, … λn such that

$$T(\vec{v}_i) = \lambda_i \vec{v}_i \quad \forall \vec{v}_i$$

<u>Definition:</u> eigenvector, eigenpair, eigenvalue

A vector v in V is an eigenvector of T if:

- $\vec{v} \neq \vec{0}$
- $\exists \lambda \in \mathbb{F} : T(\vec{v}) = \underbrace{\lambda}_{\text{eigenvalue}} \vec{v}$

$(\lambda, \vec{v})$ is an eigenpair ; $E_\lambda = \{\vec{v} : [T]_s - \lambda I = \vec{0}\}$

T is diagonalizable if and only if it has n linearly independent eigenvectors

To solve for the eigenvalues:

$$T(\vec{v}) = \lambda \vec{v}$$
$$[T(\vec{v})]_B = [\lambda \vec{v}]_B$$
$$[T(\vec{v})]_B = \lambda [v]_B$$

eigenvalue equation

Note that $[T(\vec{v})]_B = {}_B[T]_B [\vec{v}]_B$

$\vdots$

$[\vec{v}]_B = \vec{\lambda} \in \mathbb{F}^n$

Characteristic polynomial: det(A - λI)

Ex. $T : P_1(\mathbb{R}) \to P_1(\mathbb{R})$

$T(a+bx) = (a+2b) + (2a+b)x$

$${}_S[T]_S = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = A$$

$$\Delta_A(t) = (3-t)(-1-t)$$

$\lambda_1 = -1, \quad \lambda_2 = 3 \to \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \to$ diagonal $\to T$ is diagonalizable

Check: sum of eigenvalues is the trace of the matrix, product is determinant

Eigenvector for λ_1 is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 : \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\Rightarrow B^* = \{\underset{\lambda_1}{1-x}, \underset{\lambda_2}{1+x}\}$

Ex. $T(a+bx) = (a+2b) + (-2a+b)x \qquad (P_1(\mathbb{R}) \to P_1(\mathbb{R}))$

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

CP: $(1-t)^2 + 4 \geq 4 \to$ no eigenvalues

Ex. $T : P_1(\mathbb{C}) \to P_1(\mathbb{C})$

$$T(z_1 + z_2 x) = (z_1 + 2z_2) + (-2z_1 + z_2)x$$

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$\lambda_1 = 1 + 2i, \quad \lambda_2 = 1 - 2i$

Eigenvector of λ_1 is $\begin{bmatrix} 1 \\ 1-i \end{bmatrix}$

λ_2 : $\begin{bmatrix} 1 \\ 1+i \end{bmatrix}$

$B^* = \{1 + (1-i)x, \; 1 + (1+i)x\}$

$${}_{B^*}[T]_{B^*} = \begin{bmatrix} 1+2i & 0 \\ 0 & 1-2i \end{bmatrix}$$