Theorem

If y = f(x) is differentiable at x=a, and z = g(y) is differentiable at y = f(a), then

$$h(x) = g \circ f(x) = g(f(x))$$

Is differentiable at x=a, and

$$h'(a) = g' f(a) \cdot f'(a)$$

In Leibniz notation:

$$y = f(x), \quad z = g(y)$$

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

Corollary: Generalized power rule

$$[f(x)^{\alpha}]' \quad \alpha \cdot f(x)^{\alpha - 1} \cdot f'(x)$$

Chain Rule Examples

1. $f(x) = (3x^2 + 2x + 17)^{19}$

$$f(x) = 3x^2 + 2x + 17 \rightarrow 6x + 2$$
$$g(x) = x^{19} \rightarrow 19x^{18}$$

Chain rule: $19(3x^2 + 2x + 17)^{18} (6x + 2)$

2. $f(x) = \cos(x^e + e^x)$

$x^e + e^x \to ex^{e-1} + e^x$
$\cos x \to -\sin x$

Chain rule: $-\sin(x^e + e^x)(ex^{e-1} + e^x)$

3. $f(x) = a^x \quad (a > 0)$

$a^b = e^{b \cdot \ln(a)}$

$f(x) = e^{x \cdot \ln(a)}$
$f'(x) = e^{x \cdot \ln(a)} \cdot (1)(\ln a)$
$= e^{x \cdot \ln(a)} \cdot \ln(a)$

4. $f(x) = 2^{3x} + 5^{\cos x}$