Recall: <u>Riemann sum</u>

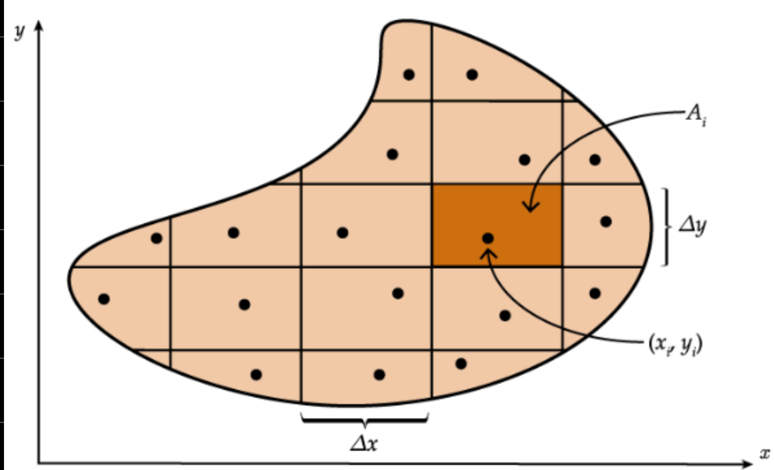
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

As n -> ∞, the accuracy of the Riemann sum gradually approaches the actual value

In three dimensions:

- Let D be a closed (contains all its boundary points) and bounded set in R2 whose boundary is a piecewise smooth closed curve
- Let f(x,y) be a function that is bounded on D:
  - There exists some real number M such that |f(x,y)| ≤ M for all (x,y) in D
  - f(x,y) exists in *three dimensions* but all (x,y) points are contained within D
- Subdividing D into rectangles:

$$\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i = \sum_{i=1}^{n} f(x_i, y_i)\Delta x_i \Delta y_i$$



Definition: Integrable function

Let $D \subset \mathbb{R}^2$ be closed and bounded. Let P be a partition of D as described above, and let $|\Delta P|$ denote the length of the longest side of all rectangles in the partition P. A function $f(x,y)$ which is bounded on D is **integrable** on D if all Riemann sums approach the same value as $|\Delta P| \to 0$.

ΔP -> 0 : rectangles get infinitely smaller; same principle as in single-variable calculus

Definition: Double Integral

If $f(x,y)$ is integrable on a closed bounded set D, then we define the **double integral** of f on D as

$$\iint_D f(x,y)\,dA = \lim_{\Delta P\to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$$

Interpretations of the Double Integral

The double integral can be used to compute the area of a set D:

$$A(D) = \iint_D 1 \, dA$$

↳ all points have the same height

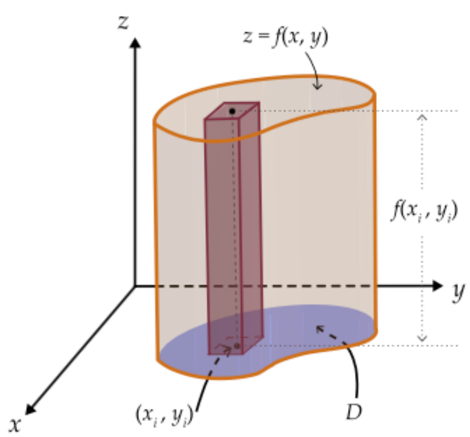
⇒ volume of plane = area

It can also be used to calculate the volume of a function f within a set D:

$$S = \left\{ (x, y, z) \mid 0 \leq z \leq f(x, y), (x, y) \in D \right\}$$

$$V(S) = \iint_D f(x, y) \, dA$$

$$V(S) \approx \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i$$



Properties of the Double Integral

Theorem 1: Linearity

If $D \subset \mathbb{R}^2$ is a closed and bounded set and f and g are two integrable functions on D, then for any constant c:

$$\iint_D (f + g) \, dA = \iint_D f \, dA + \iint_D g \, dA$$

and

$$\iint_D cf \, dA = c \iint_D f \, dA$$

Theorem 2: Basic Inequality

If $D \subset \mathbb{R}^2$ is a closed and bounded set and f and g are two integrable functions on D such that $f(x, y) \leq g(x, y)$ for all $(x, y) \in D$, then

$$\iint_D f \, dA \leq \iint_D g \, dA$$

Theorem 3: Absolute Value Inequality

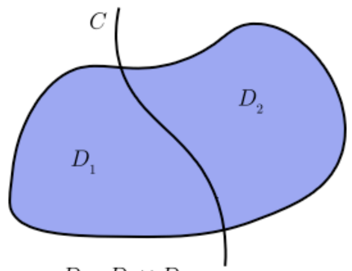
If $D \subset \mathbb{R}^2$ is a closed and bounded set and f is an integrable function on D, then

$$\left| \iint_D f\, dA \right| \le \iint_D |f|\, dA$$

Theorem 4: Decomposition

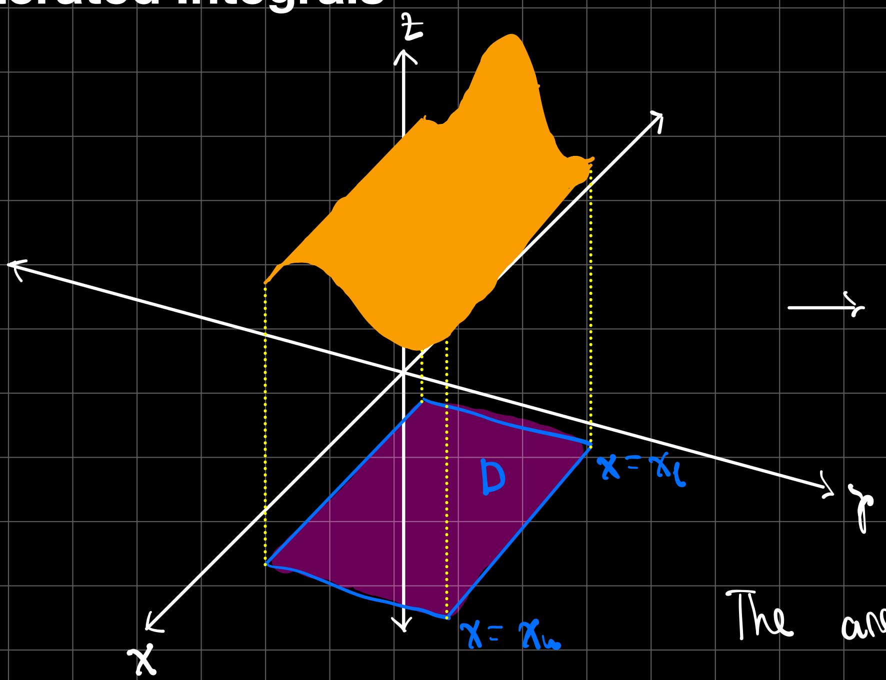
Let $D \subset \mathbb{R}^2$ be a closed and bounded set and let f be an integrable function on D. If D is decomposed into two closed and bounded subsets D_1 and D_2 by a piecewise smooth curve C, then

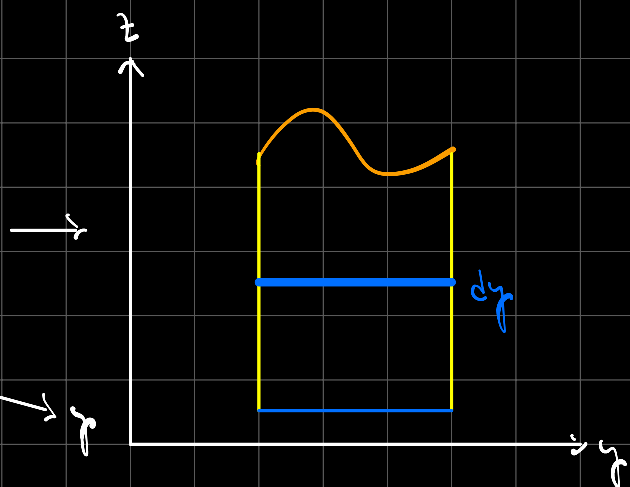
$$\iint_D f\, dA = \iint_{D_1} f\, dA + \iint_{D_2} f\, dA$$



$D = D_1 \cup D_2$

Iterated Integrals





The area of this region is:

$$A(x) = \int_{y_l(x)}^{y_u(x)} f(x, y)\, dy$$

Notice that in each of these cross-sectional areas, x stays constant

If D is contained within the vertical lines x=x_l and x=x_u, we can calculate the volume of this solid by summing over all the cross-sectional areas from x_l to x_u:

$$V = \int_{x_\ell}^{x_u} A(x)\, dx \rightarrow V = \int_{x_\ell}^{x_u} \left(\int_{y_\ell(x)}^{y_u(x)} f(x,y)\, dy \right) dx$$

Theorem 1: Iterated Integrals

Let $D \subset \mathbb{R}^2$ be defined by

$$y_\ell(x) \le y \le y_u(x), \quad \text{and} \quad x_\ell \le x \le x_u$$

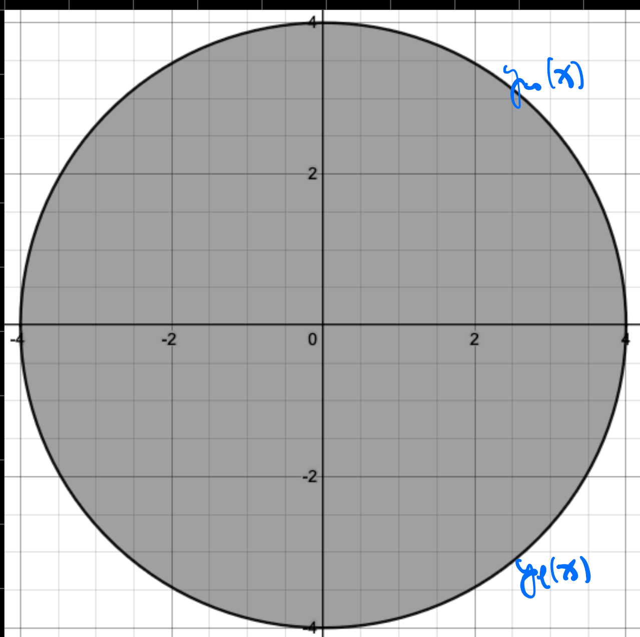
where $y_\ell(x)$ and $y_u(x)$ are continuous for $x_\ell \le x \le x_u$. If $f(x,y)$ is continuous on D, then

$$\iint_D f(x,y)\, dA = \int_{x_\ell}^{x_u} \int_{y_\ell(x)}^{y_u(x)} f(x,y)\, dy\, dx$$

These can be solved using partial integration

Ex. Let D be the unit disc $x^2 + y^2 \le 16$. Evaluate $\iint_D e^{x^2+y^2}\, dA =$

volume of the solid $e^{x^2+y^2}$ within D



Boundary: $y^2 = 16 - x^2$

$\Rightarrow y_u(x) = (16 - x^2)^{1/2}$

$y_\ell(x) = -(16 - x^2)^{1/2}$

Meanwhile, $-4 \le x \le 4$

$$\Rightarrow \int_{-4}^{4} \int_{-(16-x^2)^{1/2}}^{(16-x^2)^{1/2}} e^{x^2+y^2}\, dy\, dx$$

$$= \int_{-4}^{4} \underline{e^{y^2}} \cdot \int_{-(16-x^2)^{1/2}}^{(16-x^2)^{1/2}} e^{x^2}\, dy\, dx$$ factor out y

$$= \int_{-4}^{4} e^{y^2} \cdot \left(y e^{x^2} \right) \Big|_{-(16-x^2)^{1/2}}^{(16-x^2)^{1/2}} dx$$ integrate x expression wrt y

$$= \int_{-4}^{4} \left[e^{16-x^2} \cdot (16-x^2) \cdot e^{x^2} \right] - \left[e^{16-x^2} \cdot -(16-x^2) \cdot e^{x^2} \right]$$

plug in y = x bounds

$$= \int_{-4}^{4} 2e^{16-x^2} \cdot (16-x^2) \cdot e^{x^2}\, dx$$

$$= \int_{-4}^{4} 2e^{16} \cdot (16 - x^2) \, dx$$

somehow $\pi(e^{16} - 1)$ → convert to polar

Ex. Evaluate the following double integral. Give an exact value.

$$\int_{x=0}^{1} \int_{y=0}^{1} (3x^7 + 2y^4) \, dy \, dx =$$

$$\int_{x=0}^{1} \left(3x^7 y + \frac{2}{5} y^5\right)\Big|_{y=0}^{1} \, dx$$

$$= \int_{x=0}^{1} 3x^7 + \frac{2}{5} \, dx$$

$$= \frac{3}{8} x^8 + \frac{2}{5} x \Big|_0^1 = \frac{3}{8} + \frac{2}{5} = \frac{31}{40}$$

Ex. Determine the volume enclosed between the hemispherical surface $z = f(x, y) = \sqrt{64 - x^2 - y^2}$ and the $z = 0$ plane by evaluating the double integral

$$\iint_R f(x, y) \, dA$$

in polar coordinates. Give an exact value.

$V =$

$$r = x^2 + y^2 \rightarrow z = \sqrt{64 - r^2}$$

$$0 \le r \le 8$$

$$\Rightarrow \int_0^8 \sqrt{64 - r^2} \, dr$$

⋮

Ex. Compute the following integral where D_{xy} is the region bounded by the ellipse $10x^2 + 6xy + y^2 = 2$.
$\iint_D x^2 dA =$

$$\Rightarrow x^2 + y^2 + 6xy + 9x^2 = 2$$

$$\Rightarrow x^2 + (y + 3x)^2 = 2$$ complete the square

$$\Rightarrow y + 3x = \pm(2 - x^2)^{1/2}$$

$$\Rightarrow y = -3x \pm (2 - x^2)^{1/2} \quad \Big| \; -\sqrt{2} \le x \le \sqrt{2}$$

Solve double integral with these bounds

Change of Variables Theorem

Theorem 1: Change of Variable Theorem

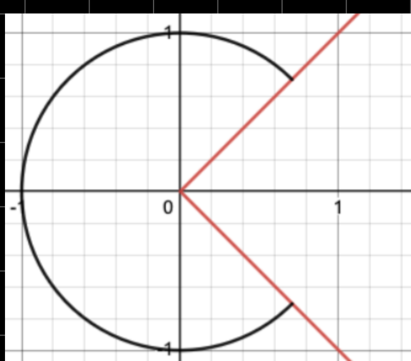
Let each of D_{uv} and D_{xy} be a closed bounded set whose boundary is a piecewise-smooth closed curve. Let

$$(x, y) = F(u, v) = (f(u, v), g(u, v))$$

be a one-to-one mapping of D_{uv} onto D_{xy}, with $f, g \in C^1$, and $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$ except for possibly on a finite collection of piecewise-smooth curves in D_{uv}. If $G(x, y)$ is continuous on D_{xy}, then

$$\iint_{D_{xy}} G(x,y)\, dx\, dy = \iint_{D_{uv}} G\Big(f(u,v), g(u,v)\Big) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

Ex.



Compute the integral $M_x = \iint_R x\; dA$.

$$= \iint_{D_{r\theta}} r\cos\theta \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \iint_{D_{r\theta}} r\cos\theta \cdot |r|$$ since $r > 0$ for it to be one-to-one and nonzero; this then satisfies the CVT

$$= \iint_{D_{r\theta}} r^2 \cos\theta \, dr \, d\theta$$

Bounds : $\frac{\pi}{4} \leq \theta \leq \frac{7\pi}{4}$; $0 \leq r \leq 1$

$$\Rightarrow \int_{\pi/4}^{7\pi/4} \int_0^1 r^2 \cos\theta \, dr \, d\theta$$

$$= \frac{1}{3} \int_{\pi/4}^{7\pi/4} r\cos\theta \Big|_0^1 \, d\theta = \frac{1}{3} \int_{\pi/4}^{7\pi/4} \cos\theta$$

$$= \frac{1}{3} \sin\theta \Big|_{\pi/4}^{7\pi/4} = \frac{-\sqrt{2}}{6} - \frac{-\sqrt{2}}{6} = \frac{-\sqrt{2}}{3}$$