If x = (1,2,3) and S is the standard basis of R3:

$$[\vec{x}]_S = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \longrightarrow \vec{x} = \vec{e}_1 + 2\vec{e}_2 + 3\vec{e}_3$$: LC of vectors in S

$S = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

Ex. $[\vec{x}]_S = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$; find $[x]_{B_2} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ (B_2 is given)

$$\Rightarrow [\vec{x}]_S = x_1\vec{e}_1 + \cdots + x_n\vec{e}_n = \alpha\vec{b}_1 + \beta\vec{b}_2$$

$$\Rightarrow [\vec{x}]_S = \underbrace{(\vec{b}_1, \vec{b}_2)}_{\text{change of basis matrix from } S \to B_2} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$: systems of equations

This is a formula for vectors in the coordinate system of S. To get vectors in B2, get the inverse matrix:

$$[\vec{x}]_{B_2} = \overbrace{{}_{B_2}[I]_S}^{\text{inverse}} [\vec{x}]_S$$

But what if we want to convert between coordinate systems that are not the standard basis?

Ex. $[\vec{x}]_{B_1} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$; find $[\vec{x}]_{B_2}$

$$[\vec{x}]_{B_2} = {}_{B_2}[I]_S \left({}_S[I]_{B_1} [\vec{x}]_{B_1}\right)$$ where ${}_S[I]_{B_1}$ is a matrix of vectors in B_1

One useful application:

$$[x]_B = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix}$$: hard to work with

Instead, we can set B1 = {(2,4,6,8), v1, v2, v3} where v1, v2, v3 are other vectors in R4

Then, we can represent

$$[\vec{x}]_{B_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

However we are no longer dealing with standard basis vectors

Ex. What is the change of basis matrix from N to S where:

S is the standard basis for M_{2x2} and N is $\left\{ \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 4 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -3 & 4 \end{bmatrix} \right\}$.

$$S = \left\{ \underset{a}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{b}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{c}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{d}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$$

Note that ${}_S[I]_N \; (S \leftarrow N) = [\,[\vec{n}_1]_S \; [\vec{n}_2]_S \; \dots\,]$ S-coords of each $m \in N$

So, solving for S-coords:

$$\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} = -1a + 2b + 0c + 0d$$: S-coordinates

$$\begin{bmatrix} -3 & 4 \\ 0 & 0 \end{bmatrix} = -3a + 4b + 0c + 0d$$

$$\Rightarrow {}_S[I]_N = \begin{bmatrix} -1 & -3 & \vdots & \vdots \\ 2 & 4 & \vdots & \vdots \\ 0 & 0 & \vdots & \vdots \\ 0 & 0 & \vdots & \vdots \end{bmatrix}$$