Show that there are “too many edges”

Lemma

If a planar embedding G has a cycle, then the boundary of every face has a cycle

So if G has a cycle, then it has ≥2 faces

Every face shares ≥1 edge with another face

Proof: Consider the subgraph with a face f and its boundary walk w

Suppose w contains the edge e from x->y

w contains e once

There exists a walk from x->y without e

And there is an x->y path p

So P + e is a cycle on the boundary walk of f

Lemma 1

always true for $n \geq 3$

Suppose G has $n \geq 3$ vertices and e edges. If its faces each have degree ≥3, then

$$e \leq \frac{d(n-2)}{d-2}$$

where d : min degree

<u>Proof:</u>

Case 1: G is connected. Then, by Euler’s Formula, n - e + f = 2

Rearranging, f = e - n + 2

The sum of face degrees is at least df

df ≤ sum of face degrees = 2e (Faceshaking Lemma)

So multiplying both sides of f = e - n + 2 by d, we have

df = d(e - n + 2)

But df ≤ 2e

So $2e \geq d(e - n + 2)$

Rearranging yields us the above formula

Case 2: Suppose G is not connected.

Let G' = G + {some edges}; these edges make G' a connected graph

$$|E(G')| \leq \frac{d(n-2)}{d-2}$$

$$e \leq |E(G')|$$

$$\Rightarrow e \leq \frac{d(n-2)}{d-2}$$

Lemma 2

For any planar graph with $n \geq 3$ vertices, $e \leq 3n-6$

does not satisfy inequality → not planar
satisfies inequality → not necessarily planar

<u>Proof:</u>

Case 1: G is a forest (composed of trees)

Then $e \leq n-1$

If $n \geq 3$ then clearly $e \leq 3n-6$ is also true

Case 2: G has a cycle

So every face boundary has a cycle. Any cycle must have at least 3 vertices, so the minimum face degree of G is 3

Using the formula

$$e \leq \frac{d(n-2)}{d-2} \rightarrow d = 3$$

We have $e \leq 3n - 6$

Lemma 3

If G is a bipartite planar embedding with $n \geq 3$, then $e \leq 2n - 4$

Case 1: G is a forest

Then $e \leq n - 1$

$$n - 1 \overset{?}{\leq} 2n - 4$$
$$\Rightarrow n \geq 3 \qquad \text{(true)}$$

Case 2: G has a cycle

In general, the minimum degree of a face is 3. But in a bipartite graph, all cycles are even, so d=4.

Plugging this into

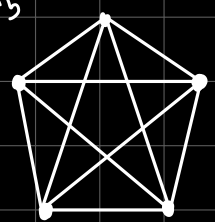
$$e \leq \frac{d(n-2)}{d-2}$$

Yields $e \leq 2n - 4$.

To check if a graph is planar, use Lemma 2 or Lemma 3

Ex.

K_5



$n = 5$
$e = \binom{5}{2} = 10$
Doesn't satisfy $e \leq 3n - 6$

If G is planar, then it has ≥1 vertex with degree ≤5.

<u>Proof:</u>

Suppose G has deg(v) ≥ 6

By the HL, 2e ≥ 6n

e ≥ 3n