Definition: Linear dependence

Let V be a vector space and let S be a subset of V.

S is *linearly dependent* if there exists a linear combination

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0} \qquad \text{(scalars not all } 0; \; S = \{\vec{v}_1, \ldots \vec{v}_n\})$$

Otherwise, S is *linearly independent*

- For a linear combination to be equal to the zero vector, c1 = c2 = … = c_n = 0

Lemma

A set S is linearly dependent if there exists a vector in S that can be written as a linear combination of some other vectors in S.

Definition: Basis

Let V be a vector space, and let B be a subset of V.

B is a *basis* for V if:

- V = Span(B)
- B is linearly independent

Ex. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis of $M_{2\times 2}(\mathbb{R})$

Let V be a vector space; let B be a basis of V with n vectors.

Then, *every basis of V has exactly n vectors*.

Definition: Dimension

In the case where B has n vectors, V is finite-dimensional, and dim(V) = n.

Otherwise, V is infinitely-dimensional.

Ex. dim(R3) = 3, since a basis of R3 has three vectors.

The basis of the zero vector space {0} is the empty set. The empty set is linearly independent, and a linear combination of all vectors in the empty set is equal to the zero vector (since there are no vectors)

Theorem: Unique representation theorem

Let V be a vector space with basis B = {v1, v2, …, v_n}.

For any vector v in V, there exist unique scalars c1, …, c_n in the field that satisfy

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$

Unique: only one set of scalars can do this.

Definition: components and coordinates

We refer to the vector (c_1, …, c_n) as the *component vector* or *coordinate vector* of v.

$$[\vec{v}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

This essentially converts a vector v into another vector (c_1, …, c_n) -> transformation

$$\Rightarrow [\ ]_B : V \to \mathbb{F}^n \qquad n = |B|$$

Proof of unique representation theorem:

Recall definition of basis:

- If B is a basis of V, then V = Span(B).

$$\vec{v} \in V \to \vec{v} \in \mathrm{Span}(B)$$

$$\Rightarrow \exists c_1, \ldots c_n : \vec{v} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$$

This satisfies the "representation" part. Now, we need to prove that this representation is unique.

Suppose there is another vector v_2 such that

$$\vec{v}_2 = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = \vec{v}$$

Then, $\vec{v}_2 - \vec{v} = (a_1\vec{v}_1 - c_1\vec{v}_1) + \cdots + (a_n\vec{v}_n - c_n\vec{v}_n)$

$$= (a_1 - c_1)\vec{v}_1 + \cdots + (a_n - c_n)\vec{v}_n = \vec{0}$$

This is a linear combination of vectors in B.

B is linearly independent, so the only scalars that make the above linear combination equal to the zero vector are 0. In that regard:

$$a_1 - c_1 = 0 \quad \cdots \quad a_n - c_n = 0 \to \forall a_i = c_i$$

As such, the representation of v is unique.

Reducing bases

Suppose B is linearly dependent and has 5 vectors. It is a basis for the vector space V, which only requires 3 vectors in its basis.

Since B is linearly dependent, two vectors in B can be written as a linear combination of the others. We can safely remove these while still having B be a basis of V.

In practice: (Replacement Theorem)

1. Put each vector in B in a matrix.
2. Row reduce.
3. Pick the vectors corresponding to pivot columns. For example, if RREF(M) contains pivots in columns 1, 2, and 5, keep v_1, v_2, and v_5.

This is really solving for a solution of

$$[\vec{v}_1 \cdots \vec{v}_n \mid \vec{0}]$$

The remaining columns are linearly independent.

Change of basis

Ex. In $P_2(\mathbb{R})$:

$$B_1 = \{\underbrace{1}_{\vec{a}_1}, \underbrace{1+x}_{\vec{a}_2}, \underbrace{1+x+x^2}_{\vec{a}_3}\}$$

$$B_2 = \{\underbrace{3+2x+x^2}_{\vec{b}_1}, \underbrace{1+x^2}_{\vec{b}_2}, \underbrace{-2-3x-x^2}_{\vec{b}_3}\}$$

$$(1, 1, 1) \qquad (1, -1, 1) \qquad (-1, -1, 1)$$

$$\vec{b}_1 = \vec{a}_3 + \vec{a}_2 + \vec{a}_1 \qquad \vec{b}_2 = \vec{a}_3 - \vec{a}_2 + \vec{a}_1 \qquad \vec{b}_3 = -\vec{a}_3 - 2\vec{a}_2 + \vec{a}_1$$

$$\Rightarrow {}_{B_1}[I]_{B_2} = \left[[\vec{b}_1]_{B_2} \quad [\vec{b}_2]_{B_2} \quad [\vec{b}_3]_{B_2} \right] = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$${}_{B_2}[I]_{B_1} = \left({}_{B_1}[I]_{B_2} \right)^{-1}$$

or: $${}_{B_1}[I]_{B_2} = {}_{B_1}[I]_S \; {}_S[I]_{B_2}$$

inverse of something given for free

given for free