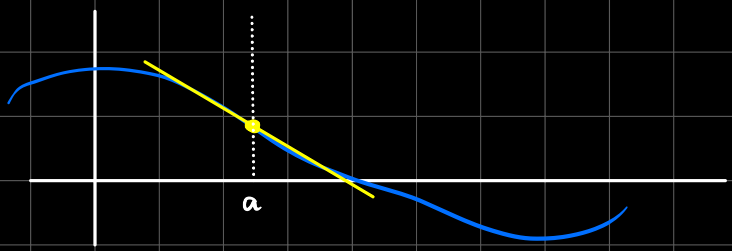
Assume f(x) is differentiable at x=a. We want a linear function h(x) such that h(a) = f(a) and h’(a) = f’(a)



Since f’(a) exists:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

So, if f is close to a:

$$\frac{f(x) - f(a)}{x - a} \approx f'(a)$$

$$L_4^f(x) = f(4) + f'(4)(x - 4)$$

L^f_a over or underestimates f based on the curvature of f

Error in Linear Approximations

$$|\text{error}| = |f(x) - L_a^f(x)$$

The two factors that affect the error are:

- The distance from x to a
- How *curved* f is (|f”|)

The more curved f is, the faster the slopes of the tangent lines change, meaning that f’ changes faster and |f”| is larger.

<u>Theorem</u>: Error in Linear Approximations

If f satisfies |f”(x)| ≤ 1 for all x in an open interval I containing x=a, then

$$|\text{error}| = |f(x) - L_a^f(x)| \leq \frac{M}{2}(x - a)^2$$

Ex. Find an upper bound on the error in using L^f_4(x) to approximate

$f(x) = \sqrt{x}$ if $x \in [1, 6]$

Solution: $f'(x) = \frac{1}{2\sqrt{x}}$

$f''(x) = \frac{-1}{4} x^{-3/2}$

So $|f''(x)| = \left| \frac{-1}{4} x^{-3/2} \right| \leq \frac{1}{4}$

$\left| \frac{-1}{4} x^{-3/2} \right|$ is decreasing, so use $x = 1$, $m = \frac{1}{4}$:

$|f(x) - L^f_4| \leq \frac{m}{2}(x-4)^2 = \frac{\left(\frac{1}{4}\right)}{2}(1-4)^2 = \frac{9}{8}$

So $|\text{error}| \leq \frac{9}{8}$

Estimating Change

We want to approximate the change in f(x) as x goes from a to b.

I.e. We want to approximate Δf = f(b)-f(a) if we know Δx = b-a

$$
\begin{aligned}
\Delta f = f(b) - f(a) &\approx L^f_a(b) - f(a) \\
&= f(a) + f'(a)(b-a) - f(a) \\
&= f'(a)(b-a) \\
&= f'(a)\Delta x \\
&\therefore \Delta f \approx f'(a)\Delta x
\end{aligned}
$$

Ex. You are inflating a spherical balloon. At some point, the radius is 20 m. If you exhale once and the radius increases to 20.01 m, estimate the change in volume.

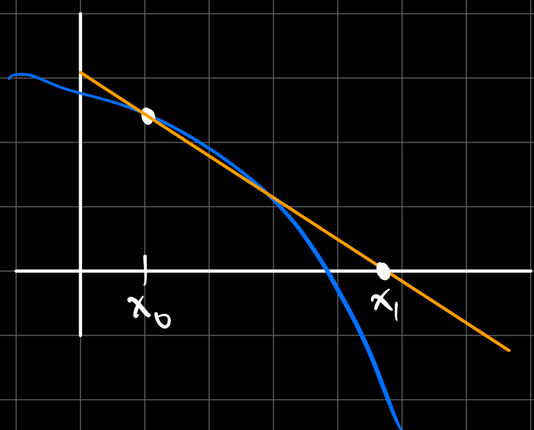
$$V(r) = \frac{4}{3}\pi r^3$$

$$\Delta V = V'(20)\Delta r$$
$$V'(r) = 4\pi r^2 \rightarrow V'(20) = 1600\pi$$
$$\therefore \Delta V \approx (1600\pi)(20.01 - 20)$$
$$= 16\pi \text{ m}^3$$

Newton's Method (Preview)

Suppose we want to estimate the root of a function f.

First, pick a random point x0, and get the tangent line of f at x0.



Then get the intersection of the tangent line with the x-axis x1, get the tangent line of f at x1, and repeat until you get the root: