If a sample space S contains n equally likely events, the likelihood of an event A, containing r points in S, is r/n

Basic counting arguments:

- Addition rule: If Job 1 can be done in p ways, and Job 2 can be done in q ways, then we can do either Job 1 or Job 2 in p+q ways.
- Multiplication rule: We can do *both* Job 1 *and* Job 2 in p • q ways.
  - Reasoning: For each way p Job 1 can be done, Job 2 can be done q ways.

Sampling:

- *With* replacement: every time an object is selected, we put it back in the pool of possible objects
  - What we get on the first selection does not affect the second
- Without: do not put back
  - Each selection affects the next

Ex. A bag contains 3 blue and 5 red marbles.

A) What is the probability of selecting two blue marbles if the selection process is done with replacement?

$P(B) = \frac{3}{8}$

Next one: $\frac{3}{8} \cdot \frac{3}{8} = \frac{9}{27}$

B) Without replacement:

$\frac{3}{8} \rightarrow \frac{2}{7} \Rightarrow \frac{3}{8} \cdot \frac{2}{7} = \frac{6}{56}$

Ex.1. Suppose a fair die is tossed 3 times. What is the probability that only one of the tosses produced a number greater than 4?

$|S| = 6^3$ 6 options, 3 tries

We are solving for:

$T_1 > 4$ and $T_2 \leq 4$ and $T_3 \leq 4$ OR (a)

2 ways, 4 ways, 4 ways, $2 \cdot 4 \cdot 4$

$T_1 \leq 4$ and $T_2 > 4$ and $T_3 \leq 4$ OR (b)

$T_1 \leq 4$ and $T_2 \leq 4$ and $T_3 > 4$ (c)

$$\Rightarrow \frac{\overset{a}{(2)(4)(4)} + \overset{b}{(4)(2)(4)} + \overset{c}{(4)(4)(2)}}{216} = \frac{96}{216}$$

Ex. 3. How many different ordered arrangements of the letters a, b, and c are possible if the letters are randomly selected without replacement?

$S = \{abc, acb, bac, bca, cab, cba\}$

6 permutations

First pick: 3 choices

Second pick: 2 choices *for each* of the three earlier -> 3 • 2

Third pick: 1 choice for *each* of the six earlier -> 3 • 2 • 1

Therefore:

$$\#\text{permutations} = n(n-1)(n-2)\cdots = n!$$

Given permutations of n objects taken k objects at a time, the formula is:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Ex. A PIN number of length 4 is formed by randomly selecting digits with replacement. Find the probability that:

A) The PIN is even

$|S| = 10^4$ (possible combinations)

$$P(A) = \frac{|A|}{|S|} = \frac{(10)(10)(10)(5)}{10^4} = \frac{1}{2}$$

Or: the first three don't matter, and the chance of getting an even number in the last digit is 1/2

B) The PIN contains at least one 1

The probability of an event A *not* happening is 1 - P(A)

It is easier to calculate this here:

$$1 - \frac{(9)(9)(9)(9)}{10^4}$$ 9 ways to get something that is NOT a 1

Ex.2. Five separate awards are to be presented to selected students from a pool of size 30. How many possible outcomes are possible if:

A) A student can receive any number of awards?

This means with replacement

$|S| = 30^5$

B) Each student can receive at most one award?

This means without replacement

30 students can receive the first award

29 can receive the second

… 26

$\Rightarrow (30)(29)\cdots(26)$

or $\binom{30}{5} \rightarrow$ number of possible 5-element subsets from 30 elements

or $30^{(5)}$

Suppose that I have 2 nephews and 1 niece in the pool of 30 students. What is the probability that each of my relatives wins *exactly one award* under assumption (a)

Say that we have 5 awards A,B,C,D,E. There are 10 ways to give all three relatives exactly one award:

- A,B,C $-$ 3! permutations of this $\rightarrow 10 = \binom{5}{3}$
- A,C,E $\rightarrow 10 \cdot 3!$ total permutations
- A,B,D
- …

So the calculation is

$$\frac{10 \cdot 3! \cdot 27 \cdot 27}{30^5}$$

Now under assumption (b):

$$\frac{10 \cdot 3! \cdot \overbrace{(27)(26)}^{\binom{27}{2}}}{30^{(5)}}$$

$\rightarrow$ not 30^5 since without replacement $\rightarrow (30)\cdot(29)\cdot 28 \cdots (26)$ possible combinations

Ex.3. Suppose we have 3 books, b1, b2, and b3. We choose two of the books to read. In how many ways can the two books be read if:

A) Order matters

- Three possible choices for the first one
- Two for the second one
- So 6 = 3!

B) Order doesn't matter

All six possible pairs are (b1,b2), (b2,b1), (b1,b3), (b3,b1), (b2,b3), (b3,b2)

In this scenario, (b1,b2) can be treated as equal to (b2,b1) since order doesn't matter

So we divide this by 2!, since there are 2! permutations of each pair

So 3 possible ways

We can now expand this logic to a larger example.

Ex.4. Suppose we randomly select a subset of 3 digits from 0 to 9, where order doesn't matter. How many ways are there to do this?

S = {(0,1,2), (0,1,3), … (7,8,9)}

If order matters, the total number of ways is

$$n^{(k)} = \frac{n!}{(n-k)!} = 10 \cdot 9 \cdot 8$$

$nPr(n, k)$
"n to k factors"

However many of these subsets are just permutations of each other (for example, (0,1,2) and (0,2,1)).

There are 3! ways to arrange a group of 3 elements, so we divide by 3!

$$\Rightarrow \frac{n!}{k!(n-k)!} = \binom{10}{3} = 120$$

So $\binom{n}{k}$ is the number of k-element subsets chosen from n elements, where order doesn't matter

Ex.5. Find the probability that all digits in the selected subset are even.

$$P(A) = \frac{\binom{5}{3}}{\binom{10}{3}}$$

- #ways to pick 3 from 5 even digits
- #ways to pick 3 from 10 digits

(We're doing this in the case where order doesn't matter, but if calculating where order matters, we get the same number since 3! will just cancel out)

Ex.6. Find the probability that at least one digit in the selected subset is less than or equal to 5.

Tip: in general, if the question asks for “at least one”, calculate 1 - P(A); in this case:

1 - P(no digit ≤5)

= 1 - P(no digit ≥ 6)

The opposite of at least one is less than 5 is no digit is less than 5

So the numbers we’re picking can only be from the set {6,7,8,9}, which has 4 elements

So we’re picking 3 from a set of 4 elements

$$\Rightarrow 1 - P(A) = 1 - \frac{\binom{4}{3}}{\binom{10}{3}}$$

Ex.7. A forest contains 30 moose, of which 6 are captured, tagged, and released. Some time later, 5 of the 30 moose are randomly captured.

- 6 tagged
- 24 untagged

A) How many samples of size 5 are possible? - (30 choose 5)

B) How many samples of size 5 include two of the tagged moose?

→ 6 tagged moose

$$\binom{6}{2}\binom{24}{3}$$

↳ choose two

C) If the five captured moose represent a simple random sample drawn from the 30 moose, find the probability that:

(i) Two of the five captured moose are tagged

$$\frac{\binom{6}{2}\binom{24}{3}}{\binom{30}{5}}$$

(ii) None of the five captured moose are tagged

$$\frac{\binom{24}{5}}{\binom{30}{5}} \quad \text{or} \quad \frac{\binom{6}{0}\binom{24}{5}}{\binom{30}{5}}$$

only one way to select 0 moose

Properties of $\binom{n}{k}$

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof:

$$\binom{n}{n-k} = \frac{n!}{(n-k)!\,[n-(n-k)!]} = \frac{n!}{k!\,(n-k)!}$$

$k \to n-k$ $\quad n-k \to n-(n-k)$

sub in $n-k$ to k

Also, n must be a nonnegative integer greater than k

$$\binom{n}{0} = \binom{n}{n} = 1$$

1 3 3 1

1 4 6 4 1

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

1 3 3 1

1 4 6 4 1

$n=4$

$k=2$

Entry in Pascal's Δ is the sum of two entries above it

Next: counting with repetition

How do we determine the number of ordered arrangements when symbols are repeated?

Ex.1. Suppose the letters of the word STATISTICS are arranged at random. Find the probability of the event G that the arrangement begins and ends with S.

The word "statistics" has 10 letters in total:

- 3 S's
- 3 T's
- 2 I's
- 1 A
- 1 C

First, we must look at the total number of arrangements. If all the elements were unique (no repetition), this would just be 10!, but since there's repetition, it is now

$$\frac{10!}{\underset{S}{3!}\,\underset{T}{3!}\,\underset{I}{2!}}$$

also includes A and C (1!)

Now looking at the possible arrangements for event G:

S [8 letters] S

1 S, 3 T, 2 I, 1 A, 1 C

Total number of arrangements in G:

$$\frac{8!}{\underset{T}{3!}\,\underset{I}{2!}} = \binom{8}{3}\binom{5}{2}$$

So $P(G) = \dfrac{\frac{8!}{3!2!}}{\frac{10!}{3!3!2!}} = \dfrac{8!3!\cancel{3!}\cancel{2!}}{10!\cancel{3!}\cancel{2!}} = \dfrac{3!}{90} = \dfrac{1}{15}$

The total number of arrangements $\frac{10!}{3!3!2!}$ can also be written as

$$\binom{10}{3}\binom{7}{3}\binom{4}{2}\binom{2}{1}\binom{2}{1}$$

$\rightarrow$ 3 taken out, 7 left

$\hookrightarrow 3+3=6$ taken out, 4 left

Ex.2. Find the probability that a bridge hand (13 cards randomly dealt from a standard deck of 52 cards without replacement) has:

A) At least one ace.

Recall: at least one -> 1 - P(no aces)

$$\Rightarrow 1 - P(\text{no aces}) = 1 - \frac{\binom{4}{0}\binom{48}{13}}{\binom{52}{13}} \doteq 0.696$$

<u>Technique:</u> multinomial coefficients

Here, we have n objects to arrange, with k types of objects. The number of arrangements is given by

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}$$

Since there are n1 type-1 objects (and n1! ways to arrange them), n2 type-2 objects…

B) 6 spades, 4 hearts, 2 diamonds, 1 club

There are 13 of each type

So:

$$P(A) = \frac{\binom{13}{6}\binom{13}{4}\binom{13}{2}\binom{13}{1}}{\binom{52}{13}} = 0.00196$$

C) A 6-4-2-1 split between the 4 suits.

4 ways to choose first suit

3 ways to choose second suit

2 ways to choose third suit…

$$P(A) = \frac{\binom{13}{6}\binom{13}{4}\binom{13}{2}\binom{13}{1} \cdot 4!}{\binom{52}{13}}$$

D) A 4-3-3-3 split between the 4 suits.

Here, the group sizes *are not all different*

- 3 of size 3
- 1 of size 4

$$P(A) = \frac{\binom{13}{4}\binom{13}{3} \cdot \frac{4!}{3!}}{\binom{52}{13}}$$

→ 4! ways to arrange 4 elements

→ but 3 repeat

Ex.3. A person has 8 friends, of whom 5 will be invited to a party.

A) How many choices are there if 2 friends are feuding and will not attend together?

Number of choices = (total number of choices) - (choices where feuding friends are together)

$\binom{8}{5}$ $\qquad$ $\binom{2}{2}\binom{6}{3}$

→ choose other 3 from a pool of 6

↳ choose both from the pool of two feuding friends

B) How many choices are there if 2 of the friends will only attend together?

$$\binom{2}{2}\binom{6}{3} + \binom{2}{0}\binom{6}{5}$$

→ both don't go

↳ both friends go

Ex.4. There are 5 blue beads and 4 green beads to be arranged in a row on a string. The two ends of the string are not connected. Beads with the same color are indistinguishable. Find the probability of the following events:

A) All 5 blue beads are adjacent to each other.

There are 5 ways to make 5 blue beads in a row - you can start a chain with each of the first 5 indices

Indistinguishable: don't worry about combinations of beads within the row of 5. Order doesn't matter

$$P(A) = \frac{5}{\binom{9}{5}}$$

B) No green bead is adjacent to any other green bead

To visualize this, let's put all the blue beads in a line, in slots before, between, and after

_ B _ B G B G B G B G

_ B G B G B G B G B _

G B G B G B G B _ B _

6 slots to choose from

4 greens to place

$$P(A) = \frac{\binom{6}{4}}{\binom{9}{5}}$$

Ex.5. What is the probability that there is at least one matched birthday in a group of k people?

= 1 - P(no matches)

= 1 - (365/365)(364/365)(363/365)…((365-(k-1))/365)