Recall: $R(\theta) = \dfrac{L(\theta)}{L(\hat{\theta})}$

Ex. $\dfrac{L(\theta_1)}{L(\theta_2)} = 0.8 \rightarrow$ θ_2 more likely

$\max(R(\theta)) = 1$ for any θ

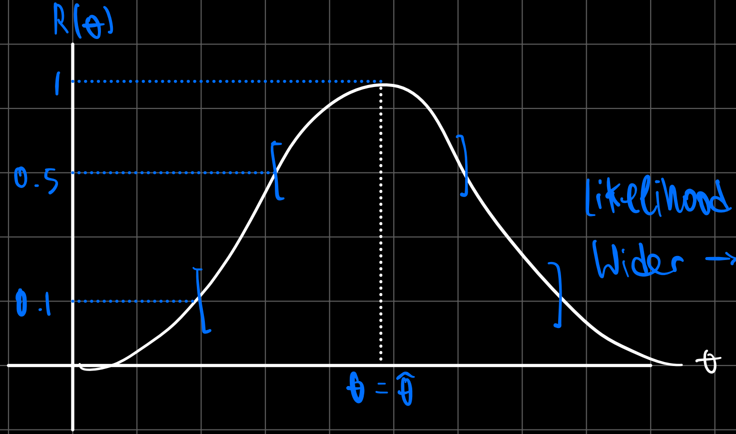
A 100p% confidence interval for θ is the set {θ: R(θ) > p}

For example, if p = 0.90, we are looking at all values such that R(θ) ≥ 0.9 — within a pretty close threshold of the M

- Excludes values of θ whose probability of occurring is < (0.9) • P(MLE)
- Naturally, decreasing p produces wider intervals

Example: Coin Tossing

- Suppose you observe 10 heads in 100 coin tosses
  - The MLE for the probability of heads is $\hat{\theta} = 0.10$
  - $\theta = 0.50$ wouldn't be very likely (not a fair coin)
  - Values *close* to 0.10 would be more plausible than 0.50 and are still close to $\hat{\theta}$



Likelihood intervals get wider as $p \rightarrow 0$

Wider → more likely to contain true mean

⇒ low LI ≈ High CI

1% LI = 99.8% CI

Log Relative Likelihood Functions

- We can also consider the log relative likelihoo function:

$$r(\theta) = \log(R(\theta)) = \ell(\theta) - \ell(\hat{\theta})$$

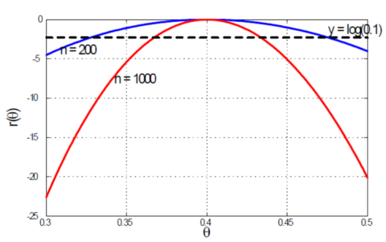


Figure 4.4: Log relative likelihood function for two polls with different sample sizes

Likelihood Ratio Statistic

$$\Lambda(\theta) = -2\log\left[\frac{L(\theta)}{L(\tilde{\theta})}\right]$$

Derivation:

The likelihood interval is defined as the set

$$\{\theta : R(\theta) > p\} = \{\theta : -2\log(R(\theta)) > -2\log(p)\}$$

$$= \left\{\theta : -2\log\left[\frac{L(\theta)}{L(\hat{\theta})}\right] > -2\log(p)\right\}$$

Then, using the estimator $\tilde{\theta} = \hat{\theta}$:

$$= \left\{\theta : -2\log\left[\frac{L(\theta)}{L(\tilde{\theta})}\right] > -2\log(p)\right\}$$

$$= \{\theta : \Lambda(\theta) > -2\log(p)\}$$

Note that $\Lambda(\theta) \sim \chi^2(1)$

$$\Rightarrow P(\underbrace{\lambda(\theta)}_{\sim \chi^2_{(1)}} \le -2\log p)$$

$$\Rightarrow P(\underbrace{|Z|}_{\sim G(0,1)} \le \sqrt{-2\log p}) \quad \leftarrow \text{pivotal quantity} - \text{in terms of } \tilde{\theta}$$

Ex. Show that a 1% likelihood interval is equal to a 99.8% confidence interval.

Setting p = 0.01:

$$P(|Z| \le \sqrt{-2\log(0.01)}) = 0.998$$

$\mapsto$ equal to $P(|Z| \le a)$

Theorem 35 *If a is a value such that $p = 2P(Z \le a) - 1$ where $Z \sim N(0,1)$, then the likelihood interval $\left\{\theta : R(\theta) \ge e^{-a^2/2}\right\}$ is an approximate $100p\%$ confidence interval.*

Ex. Show that a 95% confidence interval is equal to a 15% likelihood interval.

Then, the value a that satisfies $P(|Z| \leq a) = 2P(Z \leq a) - 1$ is a=1.96.

Substituting a=1.96 into

$$R(\theta) \geq e^{-a^2/2} \rightarrow 0.15$$

Question: which is better — likelihood interval or confidence interval?

- If MLE is close to 0.5 or n is large, then the likelihood interval would be fairly symmetric around the MLE, and would not be extremely useful as it would just match the Gaussian (??)
- If the MLE is close to 0 or 1 or n is small, then the likelihood function is not symmetric around the MLE and would be useful

<u>R commands</u>

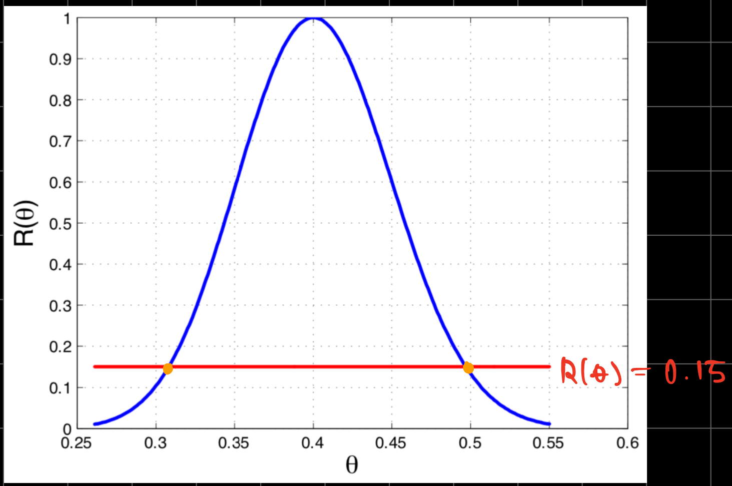
Finding value of a:

- qnorm(0.95,0,1) = 1.64
- qnorm(0.05,0,1) = -1.64

qnorm – "normal quantile"
$\mu = 0$; $\sigma = 1$

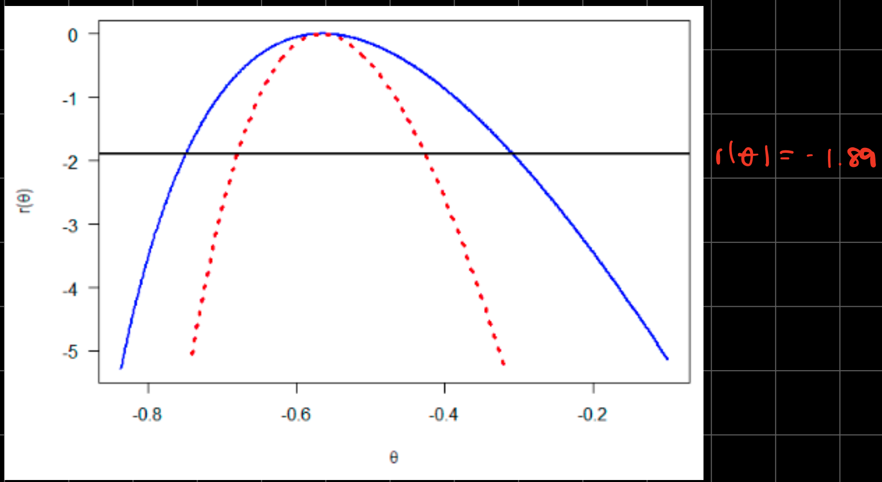
Then $P(-1.64 \leq Z \leq 1.64) = 0.90$

In this course likelihood intervals are usually calculated by staring at a graph of $R(\theta)$



An approximate 95% CI for θ is [0.31, 0.5]

If given relative log likelihood r(θ):



Suppose we want to find a 15% likelihood interval.

Note that $r(\theta) = \ln(R(\theta))$, so we would draw the line at $r(\theta) = \ln(0.15) = -1.89$.