“Order” notation: $g(\Delta t) = o(\Delta t)$ as $\Delta t \to 0$

As Δt approaches 0, $g(\Delta t)$ approaches 0 faster

Ex. $g(x^2)$

$\Delta t = 1 \to g(\Delta t) = 1$
$\Delta t = 0.5 \to g(\Delta t) = 0.25$
$\Delta t = 0.1 \to g(\Delta t) = 0.01$

g shrinks faster than Δt

Physical setup: Assume that a certain type of event occurs at random points in time (or space) and satisfies the following conditions:

- Independence: occurrences in non-overlapping intervals are independent event
- Individuality: P(2 or more events in $(t, t + \Delta t)$) = $o(\Delta t)$ as $\Delta t=0$
    - Probability of two events happening at the exact same time approaches 0 as the interval in which they *both* occurred gets smaller
- Homogeneity/Uniformity: events occur at a homogenous rate λ per time interval t

If X = the number of events occurring in a time period of length t, $\mu = \lambda t$

Ex. Suppose earthquakes recorded in Ontario each year follow a Poisson process with an average of 6 per year. What is the probability that 7 will be recorded in a 2-year period?

$\lambda = 6$

μ = (6 earthquakes)(2 years) = 12

$$P(X = x) = f(x) = \frac{e^{-12} 12^x}{x!}$$

We want $P(X = 7)$

Poisson process also applies to *space* (replace time with volume/area)

Ex. In the manufacturing process of commercial carpet, small faults occur at random in the carpet according to a Poisson process at an average rate of 0.95 per 20 m2. One of the rooms of a new office block has an area of 80 m2 and has been carpeted using the same commercial carpet described above. What is the probability that the carpet in that room contains at least 4 faults?

$$\mu = \lambda a = (0.95 \text{ faults}/20\text{ m}^2)(4 \text{ 20 m}^2/80\text{ m}^2)$$
$$= 3.8 \text{ faults}$$

$$P(X=x) = \frac{e^{-3.8}\, 3.8^x}{x!}$$

$$P(X \geq 4) = 1 - P(x \leq 3)$$

Using R: 1 - ppois(3,3.8)

<u>R aside</u>

- ppois(x, μ) — P(X≤x)
- dpois(x, μ) — P(X=x)

Ex. Suppose that emergency calls to 911 follow a Poisson process with an average of 3 calls per minute. Find the probability there will be:

(a) 6 calls in a period of 2.5 minutes — dpois(6, 7.5) = 0.137; trivial

(b) 2 calls in the first minute of a 2.5 minute period, given that 6 calls occurred in the entire period

Let A = the event that 2 calls occur in the first minute

Let B = the event that 6 calls occur in the entire 2.5 minute period

We want $P(A|B)$:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\overbrace{\left(\frac{e^{-3}3^2}{2!}\right)}^{\substack{1 \text{ min} \\ \mu=(1)(3)}} \overbrace{\left(\frac{e^{-4.5}4.5^4}{4!}\right)}^{\substack{1.5 \text{ mins} \\ \mu = (3 \text{ calls/min})(1.5 \text{ mins}) = 4.5}}}{0.137}$$

4 calls in the remaining period

Server requests come in according to a Poisson process with a rate of 100 requests per minute. A second is defined as "quiet" if it has no requests.

Let X = number of requests in one second

μ = (100 requests per minute)/60 = 5/3 requests per second

(a) Probability of getting a quiet second (no requests)

This is P(X=0) = 0.189 $\rightarrow e^{-5/3}$

Using R: dpois(0, 5/3) (??)

(b) Find the probability of getting 10 quiet seconds in 1 minute

This follows a binomial distribution: 60 trials, 10 successes, P(X=0) probability of success each time

$$P(Y=y) = \binom{60}{y}(e^{-5/3})^y(1-e^{-5/3})^{60-y}$$

We want $P(Y=10)$

(c) Find the probability that we have to wait 30 seconds to get 2 "quiet" seconds

- Second quiet second occurs on the 30th second
- So on the first 29 seconds, there were 28 non-quiet seconds and 1 quiet second

This follows a negative binomial distribution: we are repeating the experiment until the k=2 success

$$\Rightarrow \binom{29}{28}(1-e^{-5/3})^{28}(e^{-5/3})^2$$

(d) If 10 quiet seconds occur in 60 seconds, what is the probability that exactly 2 occurred among the first 20 seconds?

$$= \frac{P(2 \text{ in } 20) \cdot P(8 \text{ in } 40)}{P(10 \text{ in } 60)}$$

$$= \frac{\binom{20}{2} \cancel{(1-e^{-s/3})^{18}} \cancel{(e^{-s/3})^{2}} \cdot \binom{40}{8} \cancel{(1-e^{-s/3})^{32}} \cancel{(e^{-s/3})^{8}}}{\binom{60}{10} \underset{32+18=50}{\cancel{(1-e^{-s/3})^{50}}} \underset{8+2=10}{\cancel{(e^{-s/3})^{10}}}}$$

$$= \frac{\binom{20}{2}\binom{40}{8}}{\binom{60}{10}}$$: hypergeometric