Let G be a graph.

- V(G) — set of vertices
- E(G) — set of edges

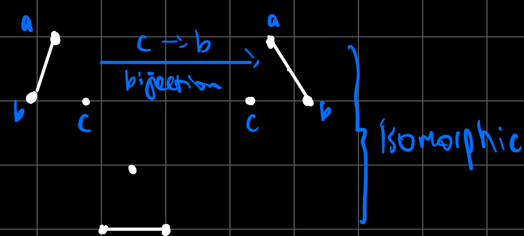
Let a and b be vertices.

- N(a) — set of neighbors
- |N(a)| — degree of a = number of neighbors
- If {a,b} is in E(G), and are joined by an edge e, then:
    - a and b are adjacent
    - e is incident to a and b

A graph is *connected* if it has a single component

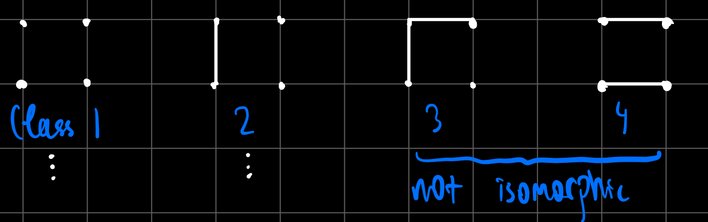
Two graphs G and H are *isomorphic* if they are the same

- Labels can be different, but vertex adjacency must be the same
- For any {a,b} in E(G), there is an equivalent {c,d} in E(H)
- There must be a *bijection* that preserves adjacency



Isomorphism class: set of graphs that are isomorphic to each other

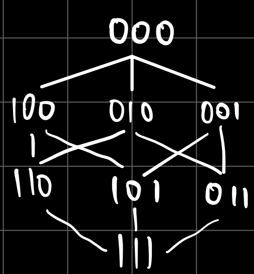
Ex. V={1,2,3,4}



Degree of vertices affects isomorphism
Can form bijection by changing labels of vertices; don't touch edges

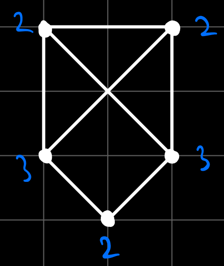
Ex. Let V(G_n) be the set of binary strings with n bits.

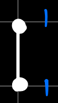
Let E(G_n) = {(a,b) such that a and b in V(G_n) differ by at most 1 bit}



$\begin{cases} 0 : \text{if} \ldots \\ 1 : \text{if} \end{cases}$ bijection applies to some graphs

Degree Sequence : $\{\deg(v) : v \in V(G)\}$





$\rightarrow \{1, 1, 2, 2, 2, 3, 3\}$

$\#\text{edges} = 7 = \frac{1}{2}(\text{Sum}(\{1, 1, 2, 2, 2, 3, 3\}))$

Handshaking lemma

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|$$

Since each edge is shared by exactly two vertices, and counting degrees counts each edge twice

“Handshaking”: a handshake is shared by two people

Counting the number of even degree vs. odd degree vertices:

$$V(G) = \underset{\text{\# of odd degrees}}{O} + E$$

$$\Rightarrow \sum_{v \in V(G)} \deg(v) = \sum_{v \in E} \overbrace{\deg(v)}^{\text{even}} + \sum_{v \in O} \overbrace{\deg(v)}^{\text{odd}} = 2|E(G)|$$

$$\text{even} + |O|(2k+1) = \text{even}$$

so $|O|$ must be even

<u>Corollary</u>: the number of vertices in a graph with odd degree is even

A *k-regular graph* has degree k for every vertex

- A 2-regular graph is a *cycle* (if everything is connected)

What is the minimum number of vertices (denoted by p) to be 3-regular?

$$\sum_{v \in V(G)} \deg(v) = 3|V(G)| = 2|E(G)|$$

$$\Rightarrow 3p = 2|E(G)|$$

Note that $E(G) \subseteq \{\text{set of all } (a, b) : a, b \in V(G)\}$
$= \{2\text{-element subsets of } V(G)\}$
size of this is $\binom{p}{2} = \frac{p(p-1)}{2}$

So $3p = 2|E(G)| \leq \frac{p(p-1)}{2}$

$\Rightarrow 3p \leq p(p-1)$
$\Rightarrow p \geq 4$

Alternatively, the maximum degree of a vertex in a graph is p-1, so by the HSL, 2(E|G|) ≤ p(p-1)

The minimum number of vertices in a k-regular graph is p ≥ k+1

<u>Complete graph:</u> every vertex is connected to every other vertex

- K_n — complete graph with n vertices

$$|E(K_n)| = \binom{n}{2}$$

K_n is the smallest possible n-regular graph

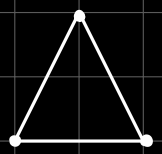
Ex. Let O_n be the graph with V(O_n) = {n-subsets of [2n+1]}

↳ 1, 2, ... 2n+1

$$E(O_n) = \left\{ AB,\ \begin{matrix} A, B \in V(O_n) \text{ and} \\ A \cap B = \emptyset \end{matrix} \right\}$$

$V(O_1) = \{1\text{-element subsets of } [3]\}$
$= \{\{1\}, \{2\}, \{3\}\}$

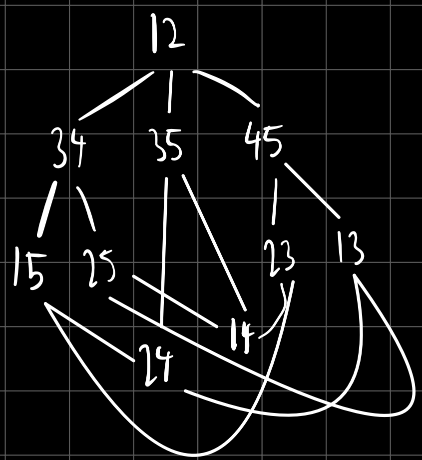
Graph:



V(O_2) = {2-subset elements of [5]} = {12, 13, 14, 15, 23, 24, 25, 34, 35, 45}

each one connects to $\binom{3}{2}$ vertices $= 3$

so this is 3-regular



Computing $\deg(A)$: $A \in V(O_n)$

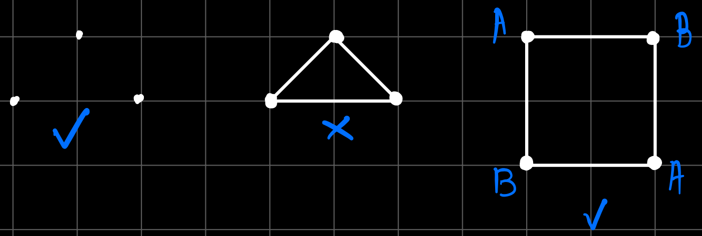
$A \in [2n+1]$; size n, $B \cap A = \emptyset$

$\Rightarrow B \in [2n+1-n] = [n+1]$; size n

$\Rightarrow \deg(v) = \binom{n+1}{n} = n+1$

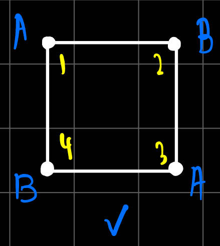
<u>Bipartite Graphs</u>

Can be defined in 2 colors

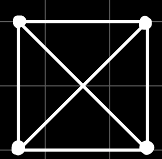


A graph is bipartite if there exists a *bipartition*(A,B) such that V(G) = A U B

Ex.



Bipartition: $(\{1, 3\}, \{2, 4\})$



Not bipartite because it contains a triangle
If a subgraph $H(G) \in V(G)$ is not bipartite, then neither is $V(G)$

Odd cycles (for example, polygons with 2k+1 sides like triangles or pentagons) cannot be bipartite

<u>Complete bipartite graph</u>

$K_{m,n}$

- $V(K_{m,n}) = \{a_1, \ldots a_m, b_1, \ldots b_n\}$
- $\underbrace{E(K_{m,n})}_{\text{size: } mn} = \{a_i b_j : 1 \le i \le m;\ 1 \le j \le n\}$

for i in m:
  for j in n:
    edge (i, j)

Two graphs are isomorphic if they have the same adjacency matrix

Ex. Adjacency matrices for K_{3,2}:

$m=3$ $n=2$

	a_1	a_2	a_3	b_1	b_2
a_1	0	0	0	1	1
a_2	0	0	0	1	1
a_3	0	0	0	1	1
b_1	1	1	1	0	0
b_2	1	1	1	0	0

	a_1	b_1	a_2	b_2	a_3
a_1		1		1	
b_1	1		1		1
a_2		1		1	
b_2	1		1		1
a_3		1		1	

Same, but this one looks weird

As such, the adjacency matrix for any complete bipartite graph is

$$A = \begin{pmatrix} 0_{m\times m} & 1_{m\times n} \\ 1_{n\times m} & 0_{n\times n} \end{pmatrix}$$

Incidence matrix

$$\begin{array}{c|ccc} & e_1 & \cdots & e_q \\ \hline v_1 & 1 & & \cdot^{\cdot^{\cdot}} \\ \vdots & & 1 & \\ v_n & \cdot^{\cdot^{\cdot}} & & \ddots \end{array}$$

Definition 4.5.2. *The* **incidence matrix** *of a graph G with vertices $v_1 \dots v_p$ and edges $e_1 \dots e_q$ is the $p \times q$ matrix $B = [b_{ij}]$ where*

$$b_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is incident with } e_j; \\ 0, & \text{otherwise.} \end{cases}$$

Each column of an incidence matrix has *exactly 2 ones,* since each edge connects two vertices'

Subgraphs

H is a subgraph of G if:

$$V(H) \subseteq V(G)$$
$$E(H) \subseteq E(G)$$

every $e \in E(H)$ contains 2 vertices in $V(H)$

H is a *proper subgraph* if H ≠ G

H is a *spanning subgraph* if V(H) = V(G) — but edges are not necessarily equal

The number of subgraphs of a complete graph K_n (no bipartitions, still complete; that is, every vertex is connected to every other vertex) is:

$$\sum_{0 \le k \le n} \binom{n}{k} \left(\text{\# subsets of edges in } K_k \right)$$

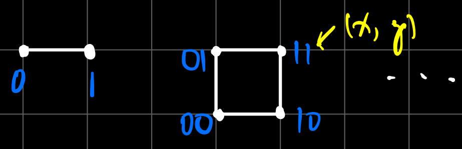
Since K_n is complete, every K_k (k≤n) is also complete

$\Rightarrow E|K_k| = \binom{k}{2}$

$$\Rightarrow \sum_{0 \le k \le n} \binom{n}{k} \left(\#\text{ subsets of edges in } K_k \right) = \sum_{0 \le k \le n} \binom{n}{k} \binom{k}{2}$$

Cubes

$n = 1$ $\qquad$ $n = 2$



neighbors differ by 1 bit

For an n-cube H_n :

$|V| = 2^n$: vertices correspond to binary strings with length n

$E = \{ab : \underbrace{|a_i - b_i| = 1}_{\text{diff. 1 bit}}\}$

By the HL, $\sum_{\forall v} \deg(v) = 2|E|$

Each vertex has n neighbors:

$$0000 \cdots 0$$

$$100 \cdots 0 \qquad 010 \cdots 0 \qquad \cdots \qquad 00 \cdots 01$$

$\Rightarrow |V|n = 2|E|$

$\Rightarrow 2^n \cdot n = 2|E|$

$\Rightarrow |E| = n \cdot 2^{n-1}$

For a general n-cube, the bipartition is

$A = \{a \in V : \text{even \# of 1's}\}$

$B = \{b \in V : \text{odd \# of 1's}\}$

$\Rightarrow \forall xy \in E(H_n)$, if $x \in A$, then $y \in B$

Proof: Let x be in A, meaning that it has an even number of 1’s. Then, y differs from x by one bit, meaning that it has an odd number of 1’s. Thus, y is in B.

Since we have proven this true for all x and y, this proves the bipartition.