Vector-Valued Function

A function whose domain is a subset of R^n and whose codomain is R^m

A *mapping* is a vector-valued function whose domain and codomain are both subsets of R^n (this is also called a *transformation*)

The Geometry of Mappings

Suppose we have two equations u = f(x,y) and v = g(x,y)

Then, the mapping (u,v) is a vector-valued function F(x,y) = (f(x,y), g(x,y)) that maps each point (x,y) in R2 to another point (f(x,y), g(x,y)) in R2

Ex. Consider the mapping defined by $(u,v) = F(x,y) = \left(\frac{1}{2}(x+y), \frac{1}{2}(-x+y)\right)$.

a. Find the images of the lines $x = k$ and $y = \ell$ under F.

$$u = \frac{1}{2}(x+y) \rightarrow 2u = x+y$$

$$v = \frac{1}{2}(-x+y) \rightarrow 2v = -x+y$$

$$\Rightarrow 2u + 2v = 2y$$

$$\Rightarrow y = u+v$$

$$y = \ell \rightarrow \ell = u+v$$

From here, we can easily get x = k = u - v

These lines can now easily be graphed in the u-v plane, with u corresponding to x and v -> y

Ex. Find the image of $D = \{(x,y) \mid -1 \le x \le 3, 0 \le y \le 2\}$ under the mapping

$$(u,v) = T(x,y) = (x^2 - y^2, xy)$$

Go over every line in the boundary of D

Case 1: x = -1, 0 ≤ y ≤ 2

$u = 1 - y^2$
$v = -y$

We want everything in terms of u and v:

$\Rightarrow u = 1 - (-v)^2$
$= 1 - v^2$

$0 \le y \le 2 \to 0 \le -v \le 2$
$\Rightarrow -2 \le v \le 0$

Case 2: x = 3, 0 ≤ y ≤ 2

Recalculate equations in terms of u and v, then recalculate bounds

Case 3: y = 0, -1 ≤ x ≤ 3

Case 4: y = 2, -1 ≤ x ≤ 3

These are relatively straightforward

Linear Approximations of Mappings

Consider a mapping F defined by u = f(x,y) and v = g(x,y)

This mapping maps points (a,b) in the x-y plane to points c = f(a,b) and d = g(a,b) in the u-v plane

Suppose we want a nearby point (a+Δx, b+Δy)

We can somewhat get this by approximating the image (c+Δu, d+Δv), where:

$$\Delta u \approx \frac{\partial f}{\partial x}(a,b)\Delta x + \frac{\partial f}{\partial y}(a,b)\Delta y$$
$$\Delta v \approx \frac{\partial g}{\partial x}(a,b)\Delta x + \frac{\partial g}{\partial y}(a,b)\Delta y$$

This can be written in matrix form as:

$$\begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} \approx \underbrace{\begin{bmatrix} \frac{\partial f}{\partial x}(a,b) & \frac{\partial f}{\partial y}(a,b) \\ \frac{\partial g}{\partial x}(a,b) & \frac{\partial g}{\partial y}(a,b) \end{bmatrix}}_{\substack{\text{derivative matrix} \\ DF}} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

Suppose, given a linear approximation F, that we want to estimate the image of the point (3.02, 3.99).

1. Pick a nearby point that's easy to work with — in this case, (3,4).
2. Evaluate DF(3,4).
3. Calculate (Δu, Δv) by getting the matrix product DF(3,4) • (Δx, Δy), where (Δx, Δy) = (0.02, -0.01).
4. F(3.02, 3.99) = F(3,4) + (Δu, Δv)

Composite Mappings

Suppose we have two mappings F and G of R2 into R2

$$F: \begin{cases} p = p(u,v) \\ q = q(u,v) \end{cases} \qquad G: \begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

G maps the x-y plane onto the u-v plane

F maps the u-v plane onto the p-q plane

Then:

The composite mapping $F \circ G$, defined by

$$\begin{cases} p = p\big(u(x,y), v(x,y)\big) \\ q = q\big(u(x,y), v(x,y)\big) \end{cases}$$

maps the xy-plane directly into the pq-plane.

Theorem 1: Chain Rule in Matrix Form for mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$

Let F and G be mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$. If G has continuous partial derivatives at (x,y) and F has continuous partial derivatives at $(u,v) = G(x,y)$, then the composite mapping $F \circ G$ has continuous partial derivatives at (x,y) and

$$D(F \circ G)(x,y) = DF(u,v)DG(x,y)$$

Proof: just multiply the matrices

Ex. Let $(u,v) = F(x,y) = (x\ln(-x^3+y), 4y^2+2x)$. Suppose that $G(u,v)$ has continuous partial derivatives with $G(0,100) = (0,3)$ and $DG(0,100) = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}$.

Use the linear approximation to find an approximation, (a,b), for $(G \circ F)(-0.1, 5.1)$, where a and b are numbers to be determined. Enter your results correct to at least 2 decimal places.

$f(x) = x \cdot \ln(-x^3 + y)$

$f_x : x \cdot \dfrac{-3x^2}{-x^3+y} + \ln(-x^3+y)$

$= \dfrac{3x^3}{x^3 - y} + \ln(-x^3+y)$

$g(x) = 4y^2 + 2x$

$g_x : 2$

$g_y : 8y$

$f_y: \frac{x}{y-x^3}$

$$\Rightarrow DF(x,y) = \begin{bmatrix} \frac{3x^3}{x^3-y} + \ln(-x^3+y) & \frac{x}{y-x^3} \\ 2 & 8y \end{bmatrix}$$

$$DF(0,5) = \begin{bmatrix} \ln(5) & 0 \\ 2 & 40 \end{bmatrix}$$

$$(u,v) = F(0,5) = (0,100)$$

$$\Rightarrow D(G \circ F)(0,5) = \underbrace{\begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}}_{DG(u,v)} \begin{bmatrix} \ln(5) & 0 \\ 2 & 40 \end{bmatrix}$$

$$= \begin{bmatrix} 4\ln(5) & 0 \\ \ln(5)+10 & 200 \end{bmatrix}$$

An approximation for $(G \circ F)(-0.1, 5.1)$ is

$$(G \circ F)(0,5) + D(G \circ F)(0,5) \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix} \begin{matrix} \Delta x \\ \Delta y \end{matrix}$$

$F(0,5) = (0,100)$

$G(0,100) = \underbrace{(0,3)}_{\text{given}}$

$$\Rightarrow \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 4\ln(5) & 0 \\ \ln(5)+10 & 200 \end{bmatrix} \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}$$

$$= (-0.4\ln(5), -0.1\ln(5) + 22)$$ somehow a good approximation