《现代控制理论》MOOC课程

6.2.3 有约束条件的变分问题 (1)

- 在本节中假定，控制$u(t)$是无约束的，且是连续的。

一. 终端时间固定、状态自由，等式约束条件下的变分问题

受控系统的状态方程为 $\dot{x}(t) = f[x(t), u(t), t]$

初始状态为$x(t_0) = x_0$

终端时间t_f固定，终端状态$x(t_f)$自由

寻求最优控制$u(t)$使性能指标 $J = \Phi\left[x\left(t_f\right), t_f\right] + \int_{t_0}^{t_f} L[x(t), u(t), t]dt$ 取得极小值。

将状态方程改写成：$f[x(t), u(t), t] - \dot{x}(t) = 0$

引入拉格朗日乘子函数$\lambda(t)$，原等式约束问题，转化为如下无约束优化问题：

$$J' = \Phi\left[x\left(t_f\right), t_f\right] + \int_{t_0}^{t_f} \{L[x(t), u(t), t] + \lambda^T(t)[f[x(t), u(t), t] - \dot{x}(t)]\}dt$$

定义标量函数 $H[\boldsymbol{x}(t),\boldsymbol{u}(t),\boldsymbol{\lambda}(\boldsymbol{t}),t]=\boldsymbol{L}[\boldsymbol{x}(t),\boldsymbol{u}(t),t]+\boldsymbol{\lambda}^T(\boldsymbol{t})\boldsymbol{f}[\boldsymbol{x}(\boldsymbol{t}),\boldsymbol{u}(\boldsymbol{t}),\boldsymbol{t}]$

称$H[\boldsymbol{x}(t),\boldsymbol{u}(t),\boldsymbol{\lambda}(\boldsymbol{t}),t]$为哈密尔顿函数。则

$$J'=\Phi[x(t_f),t_f]+\int_{t_0}^{t_f}\{H[x(t),u(t),\lambda(t),t]-\lambda^T(t)\,\dot{x}\,(t)\}dt=\Phi[x(t_f),t_f]+\int_{t_0}^{t_f}[H-\lambda^T(t)\,\dot{x}\,(t)]dt$$

泛函 J' 取得极值的必要条件为：$\delta J'=0$

$$0=\delta J'=\left(\frac{\partial\Phi[x(t_f),t_f]}{\partial x(t_f)}\right)^T\delta x(t_f)+\int_{t_0}^{t_f}\left\{\left(\frac{\partial H}{\partial x}\right)^T\delta x+\left(\frac{\partial H}{\partial u}\right)^T\delta u+\left(\frac{\partial H}{\partial \lambda}\right)^T\delta\lambda-\dot{x}^T\,\delta\lambda-\lambda^T\,\delta\,\dot{x}\right\}dt$$

而 $\displaystyle\int_{t_0}^{t_f}\lambda^T\delta\,\dot{x}\,dt=\int_{t_0}^{t_f}\lambda^T\delta dx=\int_{t_0}^{t_f}\lambda^T d(\delta x)=\lambda^T\delta x\Big|_{t_0}^{t_f}-\int_{t_0}^{t_f}d\lambda^T\delta x=\lambda^T\,\delta x\Big|_{t_0}^{t_f}-\int_{t_0}^{t_f}\dot{\lambda}^T\delta x dt$

$$0=\delta J'=\left(\frac{\partial\Phi\left[x(t_f),t_f\right]}{\partial x(t_f)}\right)^T\delta x(t_f)-\lambda^T\delta x\Big|_{t_0}^{t_f}+\int_{t_0}^{t_f}\left\{\left(\frac{\partial H}{\partial x}\right)^T\delta x+\left(\frac{\partial H}{\partial u}\right)^T\delta u+\left(\frac{\partial H}{\partial \lambda}\right)^T\delta\lambda-\dot{x}^T\delta\lambda+\dot{\lambda}^T\delta x\right\}dt$$

$$0=\delta J'=\left(\frac{\partial\Phi\left[x(t_f),t_f\right]}{\partial x(t_f)}-\lambda(t_f)\right)^T\delta x(t_f)+\lambda^T(t_0)\delta x(t_0)+\int_{t_0}^{t_f}\left\{\left(\frac{\partial H}{\partial x}+\dot{\lambda}\right)^T\delta x+\left(\frac{\partial H}{\partial u}\right)^T\delta u+\left(\frac{\partial H}{\partial \lambda}-\dot{x}\right)^T\delta\lambda\right\}dt$$

由于$x(t_0)$固定，故$\delta x(t_0)=0$

$$0=\delta J'=\left(\frac{\partial\Phi\left[x(t_f),t_f\right]}{\partial x(t_f)}-\lambda(t_f)\right)^T\delta x(t_f)+\int_{t_0}^{t_f}\left\{\left(\frac{\partial H}{\partial x}+\dot{\lambda}\right)^T\delta x+\left(\frac{\partial H}{\partial u}\right)^T\delta u+\left(\frac{\partial H}{\partial \lambda}-\dot{x}\right)^T\delta\lambda\right\}dt$$

由于$x(t),u(t),\lambda(t),x(t_f)$不受限制，故$\delta x(t),\delta u(t),\delta\lambda(t),\delta x(t_f)$ 任意

要使$\delta J'=0$ 必有：$\dfrac{\partial H}{\partial \lambda}-\dot{x}=0$ 即有 $\dot{x}=\dfrac{\partial H}{\partial \lambda}=f(x,u,t)$

$$\dot{\lambda}=-\frac{\partial H}{\partial x}$$

$$\lambda(t_f)=\frac{\partial\Phi\left[x(t_f),t_f\right]}{\partial x(t_f)}$$

$$\frac{\partial H}{\partial u}=0$$

综上，终端时间固定、状态自由，等式约束条件下的性能指标取极值的必要条件为：

状态方程：

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial \lambda} = f(x, u, t) \\ x(t_0) = x_0 \end{cases}$$

协状态方程：

$$\begin{cases} \dot{\lambda} = -\dfrac{\partial H}{\partial x} \\ \lambda(t_f) = \dfrac{\partial \Phi[x(t_f), t_f]}{\partial x(t_f)} \end{cases}$$

控制方程：

$$\frac{\partial H}{\partial u} = 0$$