《现代控制理论》MOOC课程

6.2.2　无约束条件的变分问题(1)

引理： 如果函数$F(t)$在区间$t\in[t_0,t_f]$上是连续的，而且对于只满足某些一般条件的任意选定的函数$\eta(t)$有$\int_{t_0}^{t_f}F(t)\eta(t)dt=0$，则在区间$t\in[t_0,t_f]$上有： $F(t)\equiv 0$

一 欧拉方程

讨论一个固定端点时间，固定端点状态的无约束条件变分问题。

问题： 考虑泛函为 $J[x(t)]=\int_{t_0}^{t_f}L[x(t),\dot{x}(t),t]dt$

式中$x(t)$在$t\in[t_0,t_f]$上连续，$L[x(t),\dot{x}(t),t]$连续，二阶可微，求使$J[x(t)]$取极值，且满足给定边界条件：$x(t_0)=x_0$，$x(t_f)=x_f$的函数$x^*(t)$。

解：根据泛函极值定理,在极值曲线$x^*(t)$上，必有$\delta J[x^*(t)]=0$

而 $\delta J[x(t)]=\frac{\partial}{\partial\alpha}J[x(t)+\alpha\delta x(t)]\Big|_{\alpha=0}=\int_{t_0}^{t_f}\frac{\partial}{\partial\alpha}L[x(t)+\alpha\delta x(t),\dot{x}(t)+\alpha\delta\dot{x}(t),t]\Big|_{\alpha=0}dt$

$$= \int_{t_0}^{t_f} \left\{ \frac{\partial}{\partial(\boldsymbol{x}(\boldsymbol{t}) + \alpha\delta\boldsymbol{x}(\boldsymbol{t}))} \boldsymbol{L}[\boldsymbol{x}(\boldsymbol{t}) + \alpha\delta\boldsymbol{x}(\boldsymbol{t}), \dot{\boldsymbol{x}}(\boldsymbol{t}) + \alpha\delta\dot{\boldsymbol{x}}(\boldsymbol{t}), \boldsymbol{t}] \frac{\partial(\boldsymbol{x}(\boldsymbol{t}) + \alpha\delta\boldsymbol{x}(\boldsymbol{t}))}{\partial\alpha} \right.$$

$$\left. \left. + \frac{\partial}{\partial(\dot{\boldsymbol{x}}(\boldsymbol{t}) + \alpha\delta\dot{\boldsymbol{x}}(\boldsymbol{t}))} \boldsymbol{L}[\boldsymbol{x}(\boldsymbol{t}) + \alpha\delta\boldsymbol{x}(\boldsymbol{t}), \dot{\boldsymbol{x}}(\boldsymbol{t}) + \alpha\delta\dot{\boldsymbol{x}}(\boldsymbol{t}), \boldsymbol{t}] \frac{\partial\left(\dot{\boldsymbol{x}}(\boldsymbol{t}) + \alpha\delta\dot{\boldsymbol{x}}(\boldsymbol{t})\right)}{\partial\alpha} \right\} \right|_{\alpha=0} \boldsymbol{d}t$$

$$= \int_{t_0}^{t_f} \left\{ \frac{\partial \boldsymbol{L}[\boldsymbol{x}(\boldsymbol{t}), \dot{\boldsymbol{x}}(\boldsymbol{t}), \boldsymbol{t}]}{\partial \boldsymbol{x}(\boldsymbol{t})} \delta\boldsymbol{x}(\boldsymbol{t}) + \frac{\partial \boldsymbol{L}[\boldsymbol{x}(\boldsymbol{t}), \dot{\boldsymbol{x}}(\boldsymbol{t}), \boldsymbol{t}]}{\partial \dot{\boldsymbol{x}}(\boldsymbol{t})} \delta\dot{\boldsymbol{x}}(\boldsymbol{t}) \right\} \boldsymbol{d}t$$

$$\delta J=\int_{t_0}^{t_f}\left(\frac{\partial L}{\partial x}\delta x+\frac{\partial L}{\partial \dot{x}}\delta \dot{x}\right)dt$$

右边第二项：$\displaystyle\int_{t_0}^{t_f}\frac{\partial L}{\partial \dot{x}}\delta\dot{x}dt=\int_{t_0}^{t_f}\frac{\partial L}{\partial \dot{x}}\delta dx=\int_{t_0}^{t_f}\frac{\partial L}{\partial \dot{x}}d(\delta x)=\frac{\partial L}{\partial \dot{x}}\delta x\Big|_{t_0}^{t_f}-\int_{t_0}^{t_f}d\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x$

$$=\frac{\partial L}{\partial \dot{x}}\delta x\Big|_{t_0}^{t_f}-\int_{t_0}^{t_f}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x dt$$

故： $\displaystyle\delta J=\frac{\partial L}{\partial \dot{x}}\delta x\Big|_{t_0}^{t_f}+\int_{t_0}^{t_f}\left(\frac{\partial L}{\partial x}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right)\delta x dt$

考虑端点固定，故有：$\delta x(t_0)=\delta x(t_f)=0$

所以： $$\delta J=\int_{t_0}^{t_f}\left(\frac{\partial L}{\partial x}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right)\delta x dt$$

在极值曲线$x^*(t)$ 上，必有 $\delta J[x^*(t)]=0$

即： $$\delta J[x^*(t)]=\int_{t_0}^{t_f}\left(\frac{\partial L}{\partial x}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right)\Big|_{x=x^*}\delta x dt=0$$

故： $$\frac{\partial L}{\partial x}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=0$$

该式称为**欧拉方程**，是泛函极值的必要条件。

由于

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=\frac{d}{dt}\left(\frac{\partial L[x,\dot{x},t]}{\partial \dot{x}}\right)=\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \dot{x}}\right)\frac{dx}{dt}+\frac{\partial}{\partial \dot{x}}\left(\frac{\partial L}{\partial \dot{x}}\right)\frac{d\dot{x}}{dt}+\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right)=\frac{\partial^2 L}{\partial \dot{x}\partial x}\dot{x}+\frac{\partial^2 L}{\partial \dot{x}^2}\ddot{x}+\frac{\partial^2 L}{\partial \dot{x}\partial t}$$

欧拉方程可进一步表示为：

$$\frac{\partial L}{\partial x}-\frac{\partial^2 L}{\partial \dot{x}\partial x}\dot{x}-\frac{\partial^2 L}{\partial \dot{x}^2}\ddot{x}-\frac{\partial^2 L}{\partial \dot{x}\partial t}=0$$

写成简洁的形式：$L_x-L_{\dot{x}x}\dot{x}-L_{\dot{x}\dot{x}}\ddot{x}-L_{\dot{x}t}=0$

这样无约束泛函极值问题就归结为求解欧拉方程问题。

例：求泛函 $J=\int_0^{\frac{\pi}{2}}(\dot{x}^2-x^2)\,dt$

在边界条件： $x(0)=1, x\left(\frac{\pi}{2}\right)=2$ 下的极值曲线。

解：$L=\dot{x}^2-x^2$　　$L_x=-2x$　　$L_{\dot{x}}=2\dot{x}$　　$L_{\dot{x}\dot{x}}=2$　　$L_{\dot{x}x}=L_{\dot{x}t}=0$

由欧拉方程：$L_x-L_{\dot{x}x}\dot{x}-L_{\dot{x}\dot{x}}\ddot{x}-L_{\dot{x}t}=0$　　可得：$-2x-2\ddot{x}=0$

解得：$x^*(t)=cos(t)+2sin(t)$