university of groningen

faculty of science and engineering

Lecture 7: Z-Transform

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Signals and Systems

1B - 2024/2025

Preliminaries

- The deadline for Lab 3 is Friday, January 17, at 17:30.
- Remember to submit your **code in Themis and documentation in Brightspace.**
- We will hold an **online Q&A session next Wednesday**, January 17, at 9:00. The link will be posted on Brightspace. As a remark, this is not a lecture but a Q&A session, so **prepare questions beforehand.**
- Remember to evaluate the course on Brightspace.

- There is a practice exam available on Brightspace.
- Material to prepare for the Final Exam is in the Useful Resources tab on Brightspace.
- You can bring an A4 **summary sheet** to the Final Exam. Use it wisely! We will provide useful formulas as well.
- Calculators or any other electronics are not allowed during the exam.
- Please check **Rooster** for the date and time of the Final Exam.

And finally, remember to make
the final exam for the **course Signals and Systems (for AI)!**

Financial ML Reading Group

Figure: Financial ML Reading Group - The sign-ups for the reading group will open on the 6th of January and close on the 24th. We look forward to seeing you at the first meeting!

Overview

1. Recap
2. z-transform
3. Zeros of the system function
4. z-transform in IIR filters
5. Closing Remarks

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Recap

Discrete Fourier transform

The Discrete Fourier transform (DFT) of sequence $x[n]$ is

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}, \quad \text{for } k = 0, \ldots, N-1 \tag{1}$$

- DFT transforms N samples from the time domain into N samples in the frequency domain.
- Convolution, $y[n] = h[n] * x[n]$, in the time domain is a multiplication, $Y[k] = H[k]X[k]$, in the frequency domain.

Recap

Consider the L-length signal $x[n] = \{x[0], x[1], \ldots, x[L-1]\}$. Recall that we can represent such a signal as

$$x[n] = \sum_{k=0}^{L-1} x[k]\delta[n-k] \tag{2}$$

where $\delta[n]$ is the unit impulse.

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z-transform

The z-transform of a length L-length signal $x[n]$ is

$$X(z) = \sum_{k=0}^{L-1} x[k] z^{-k}$$

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The z-transform of a length L-length signal $x[n]$ is

$$\begin{aligned} X(z) &= \sum_{k=0}^{L-1} x[k] z^{-k} \\ &= \sum_{k=0}^{L-1} x[k] (z^{-1})^{k}, \end{aligned} \tag{3}$$

where z is a complex number.

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Interpretation

The z-transform represents a signal $x[n]$ as a polynomial $X(z)$ of degree $L-1$ in the variable z^{-1}. The z-transform is a helpful tool for analyzing discrete signals and systems.

A simple example

Consider the signal $x[n] = \delta[n - n_0]$.

A simple example

Consider the signal $x[n] = \delta[n - n_0]$. The z-transform of $x[n]$ is $X(z) = z^{-n_0}$.

Example

Consider the following signal in the time domain,

$$x[n] = 2\delta[n] + 4\delta[n-1] + 6\delta[n-2] + 4\delta[n-3] + 2\delta[n-4] \tag{4}$$

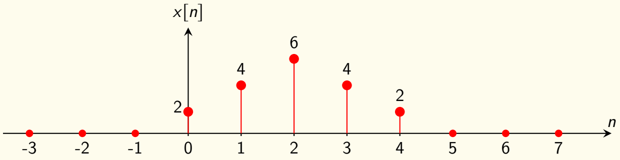

Figure: Time-domain plot for $x[n]$

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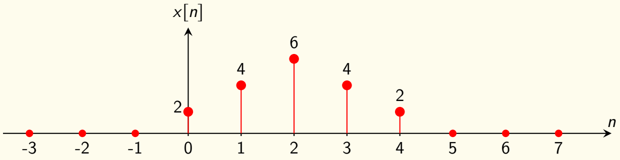

Figure: Time-domain plot for $x[n]$

The z-transform for $x[n]$ is

$$X(z) = 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4} \tag{5}$$

z-transform of FIR filter

A FIR filter is completely characterized by the impulse response

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] \tag{6}$$

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The **system function** is obtained by taking the z-transform

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \sum_{k=0}^{M} h[k] z^{-k} \tag{7}$$

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The system function also completely characterizes the filter.

Example - Unit delay

Consider the unit delay filter described by the difference equation

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Example - Unit delay

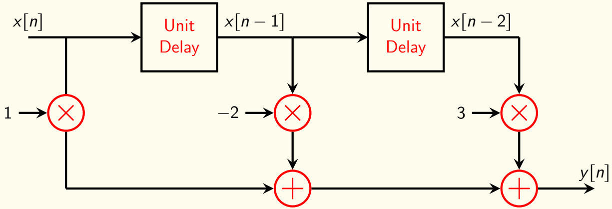
Remark

Convolution, $y[n] = h[n] * x[n]$, in the time domain is a multiplication, $Y[z] = H[z]X[z]$, in the z-domain.

Block diagrams and z^{-1} notation

Consider the system

$$y[n] = h[n] * x[n] = \sum_k h[k]x[n-k] = x[n] - 2x[n-1] + 3x[n-2] \qquad (9)$$



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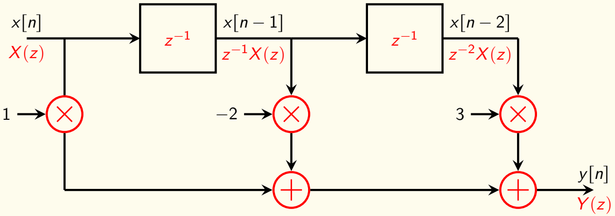

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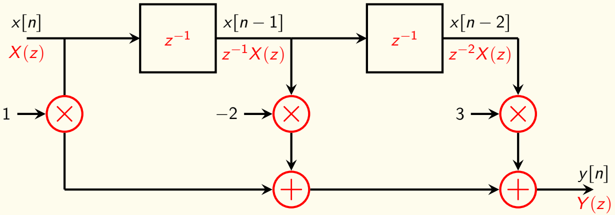

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$$Y(z) \quad = \quad \sum_k h[k](z^{-k}X(z))$$

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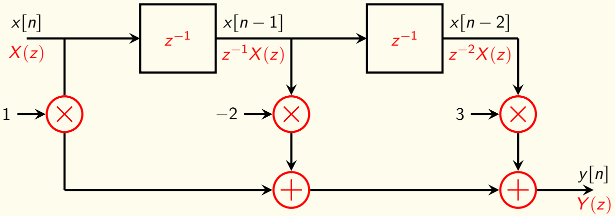
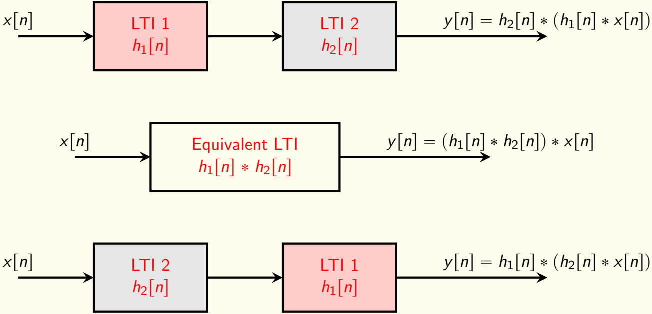


A unit delay can be represented as multiplication by z^{-1}

$$Y(z) \;=\; \sum_k h[k](z^{-k}X(z)) = \left(\sum_k h[k]z^{-k}\right)X(z) = H(z)X(z)$$

Cascaded LTI systems

- LTI cascades connect multiple LTI systems
  - The output of one system is the input for the next system
  - The order of systems does not matter



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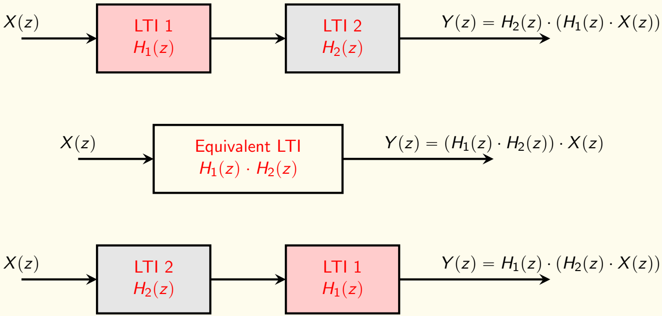

Example - Cascade

Consider the cascade of two LTI systems described by

$$\begin{aligned} w[n] &= 3x[n] - x[n-1] \\ y[n] &= 2w[n] - w[n-1] \end{aligned}$$

Find the equivalent LTI.

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$$
\begin{aligned}
H_1(z) &= 3 - z^{-1} \\
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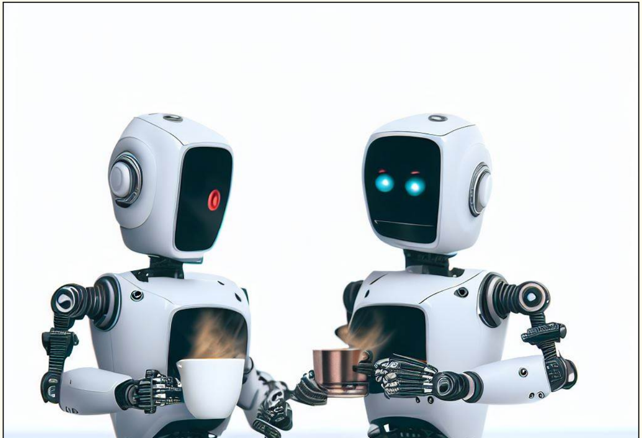
Break!

See you at ________________________________



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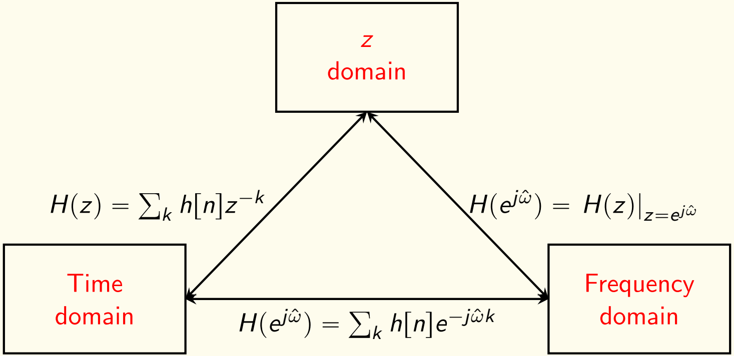
Three domains



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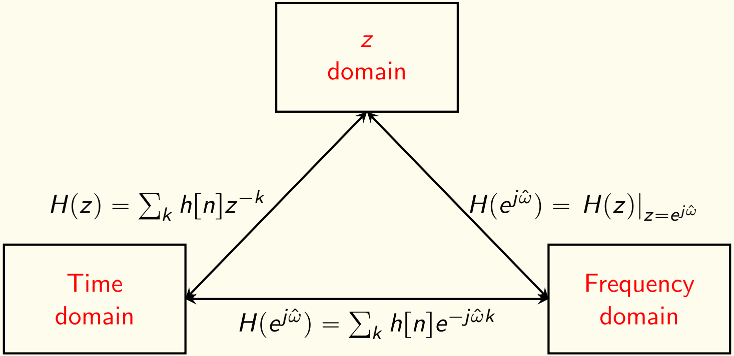

Each of the three domains can completely describe an FIR filter

- In the time domain, $h[n] = \sum_k b_k \delta[n-k]$

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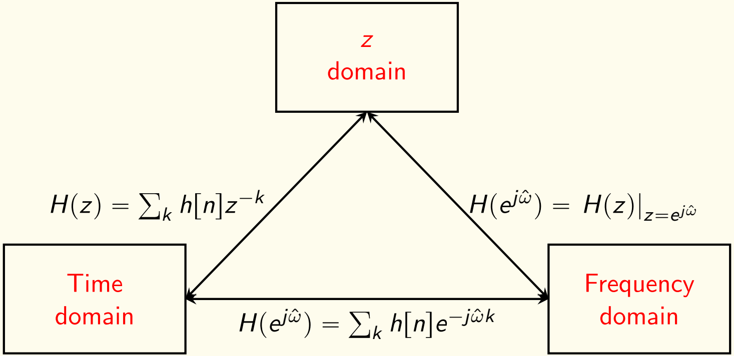

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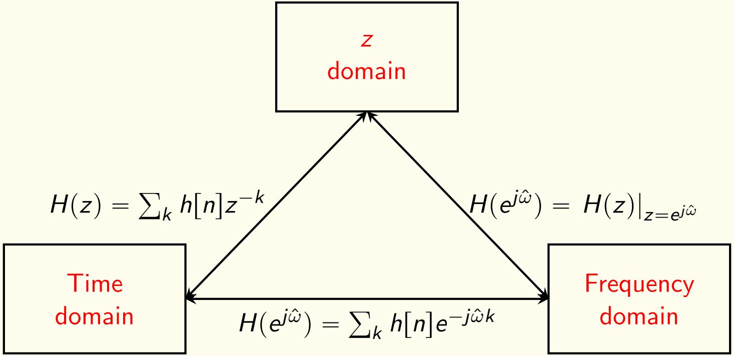

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- In the time domain, $h[n] = \sum_k b_k \delta[n-k]$
- In the frequency domain, $H(e^{j\hat{\omega}}) = \sum_k b_k e^{-j\hat{\omega}k}$
- In the z domain, $H(z) = \sum_k b_k z^{-k}$

Factoring z-transforms

Comment

A key advantage of the z-transform is that it allows us to **factor polynomials.** Indeed, **the roots**, $H(z) = 0$, of the polynomials tell us a lot about the filter.

Factoring z-transforms

The polynomial representation of the z-transform makes it easy to perform factoring

$$H(z) = 1 - 2z^{-1} + z^{-2}$$

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- The filter $h[n] = \{1, -2, 1\}$ is equivalent to applying the first-difference filter $h_1[n] = \{1, -1\}$ twice
- The filter is **zero** exactly when $z = 1$

Roots z-transforms

The z-transform of a filter can be represented in the form

$$H(z) = G \prod_{k=1}^{M} (1 - z_k z^{-1}) = G \prod_{k=1}^{M} \left(\frac{z - z_k}{z} \right), \tag{10}$$

where G is a constant.

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where G is a constant.

- The filter eliminates any input for which $z = z_k$
- The filter is defined, up to constant G, by the inputs it eliminates
  - z_k are also known as the **zeros of the system function**
  - A FIR filter of length L has $M = L - 1$ roots

Zeros of the system function

Suppose we have a system function

$$H(z) = \sum_k b_k z^{-k} = \frac{z - z_0}{z} \tag{11}$$

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$$\begin{aligned} y[n] &= \sum_k b_k x[n-k] = \sum_k b_k z_0^{n-k} \\ &= \left(\sum_k b_k z_0^{-k} \right) z_0^n \end{aligned}$$

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$$\begin{aligned} y[n] &= \sum_k b_k x[n-k] = \sum_k b_k z_0^{n-k} \\ &= \left(\sum_k b_k z_0^{-k} \right) z_0^n = H(z_0) z_0^n = 0 \end{aligned}$$

Example - Roots z-transforms

Find the zeros of the system function

$$H(z) = 1 - z^{-1} + z^{-2}$$

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$$H(z) = 1 - z^{-1} + z^{-2} = \frac{1}{z^2}\left(z^2 - z + 1\right)$$

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$$H(z) = 1 - z^{-1} + z^{-2} = \frac{1}{z^2}\left(z^2 - z + 1\right)$$

Using the ABC-formula $(-b \pm \sqrt{b^2 - 4ac})/2a$

$$z_k \quad = \quad \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1}$$

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$$\begin{aligned} z_k &= \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} \\ &= \frac{1}{2} \pm \frac{1}{2}\sqrt{-3} \end{aligned}$$

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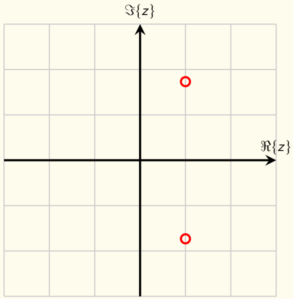
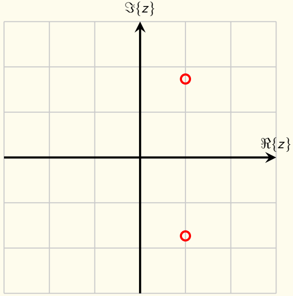
$$\begin{aligned}
z_k &= \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} \\
&= \frac{1}{2} \pm \frac{1}{2}\sqrt{-3} \\
&= \frac{1}{2} \pm j\frac{1}{2}\sqrt{3} \\
&= e^{\pm j\pi/3} \\
H(z) &= (1 - e^{-j\pi/3}z^{-1})(1 - e^{j\pi/3}z^{-1})
\end{aligned}$$

Frequency response and z-transform



Red circles indicate zeros of $H(z) = 1 - z^{-1} + z^{-2}$

Frequency response and z-transform



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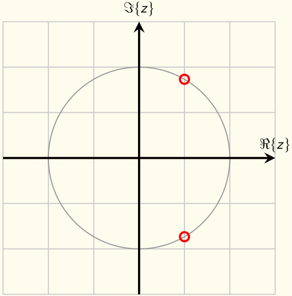

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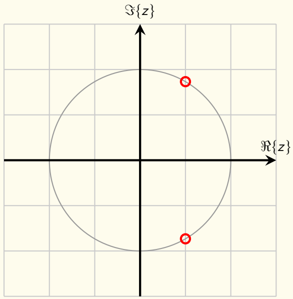

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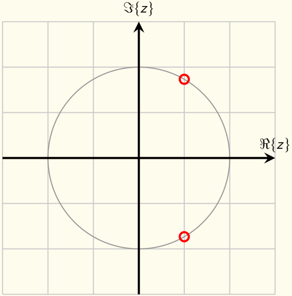

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- This plot is known as a **zeros plot**.

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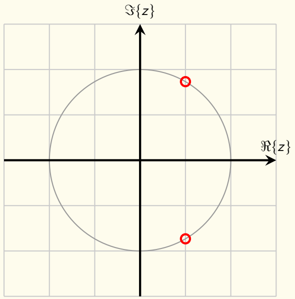

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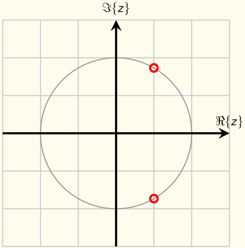
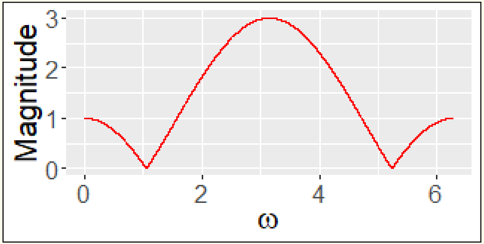
Remark

The zeros show what frequencies are eliminated from a signal

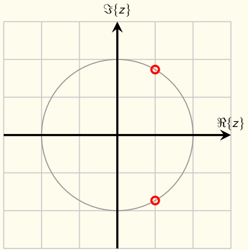
Frequency response and z-transform

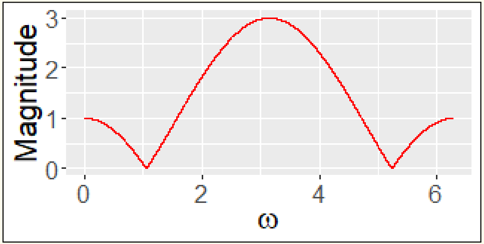




$$
\begin{aligned}
h[n] &= \delta[n] - \delta[n-1] + \delta[n-2] \\
H(e^{j\hat{\omega}}) &= e^{-j\hat{\omega}}\,(2\cos(\hat{\omega}) - 1) \\
H(z) &= (1 - e^{-j\pi/3}z^{-1})(1 - e^{j\pi/3}z^{-1})
\end{aligned}
$$

Frequency response and z-transform





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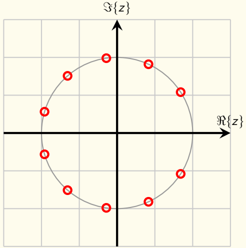
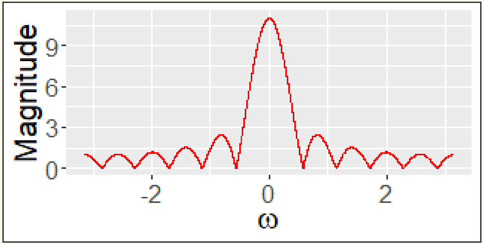
$H(e^{j\hat{\omega}})$ gives the (complex) value $H(z)$ along the unit circle

Example - 11-point running sum



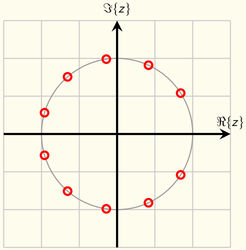


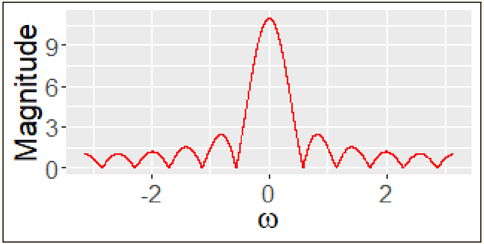
$$h[n] = \sum_{k=0}^{10} \delta[n-k]$$

$$H(e^{j\hat{\omega}}) = e^{-j5.5\hat{\omega}} \cdot \left(\frac{\sin(5.5\hat{\omega})}{\sin(\hat{\omega}/2)} \right)$$

$$H(z) = \sum_{k=0}^{10} z^{-k} = \prod_{k=1}^{10} (1 - e^{-j2\pi k/11} z^{-1})$$

Bandpass filter



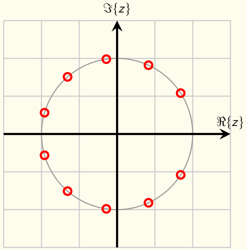


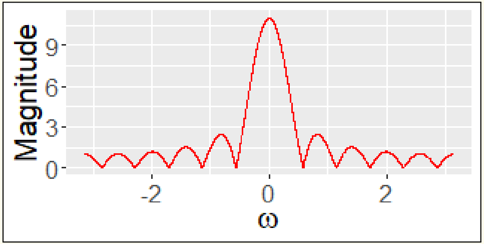
$$H(z) = \sum_{k=0}^{10} z^{-k}$$

Using the z-tranform, **we can design specific filters**

- The 11-point running sum is a lowpass filter

Bandpass filter





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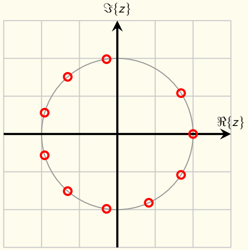
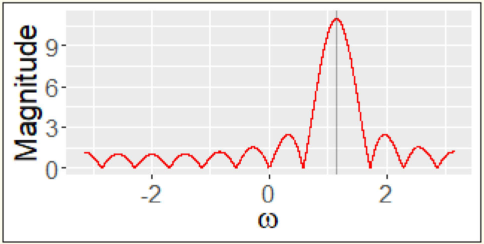
- We can create a bandpass filter by shifting the phase (ie. rotation)

Bandpass filter





$$H(z) = \sum_{k=0}^{10} \mathbf{e^{j4\pi k/11}} z^{-k}$$

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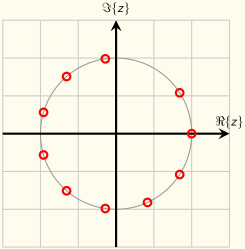

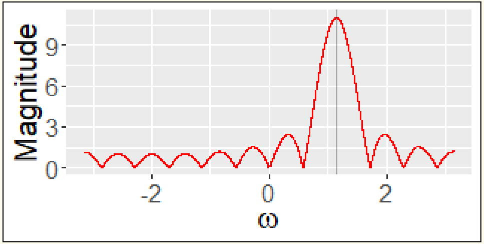

$$H(z) = \sum_{k=0}^{10} e^{j4\pi k/11} z^{-k}$$

This filter has complex-valued coefficients. This may be a problem in real systems with real-valued input and output.

Bandpass filter

What if we take the real part of the rotation?

$$H(z) = \sum_{k=0}^{10} \Re\{e^{j4\pi k/11}\} z^{-k}$$

Bandpass filter

What if we take the real part of the rotation?

$$\begin{aligned} H(z) &= \sum_{k=0}^{10} \Re\{e^{j4\pi k/11}\} z^{-k} \\ &= \sum_{k=0}^{10} \cos(4\pi k/11) z^{-k} \end{aligned}$$

Bandpass filter

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$$\begin{aligned} H(z) &= \sum_{k=0}^{10} \Re\{e^{j4\pi k/11}\} z^{-k} \\ &= \sum_{k=0}^{10} \cos(4\pi k/11) z^{-k} \\ &= \sum_{k=0}^{10} \left(\frac{1}{2} e^{4\pi k/11} + \frac{1}{2} e^{-4\pi k/11} \right) z^{-k} \end{aligned}$$

Bandpass filter

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&= \left(\sum_{k=0}^{10} \frac{1}{2} e^{4\pi k/11} z^{-k}\right) + \left(\sum_{k=0}^{10} \frac{1}{2} e^{-4\pi k/11} z^{-k}\right)
\end{aligned}
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Bandpass filter

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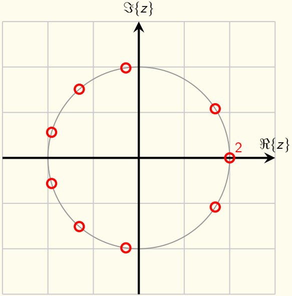
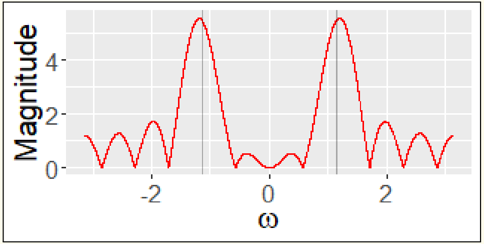
&= \left(\sum_{k=0}^{10} \frac{1}{2} e^{4\pi k/11} z^{-k} \right) + \left(\sum_{k=0}^{10} \frac{1}{2} e^{-4\pi k/11} z^{-k} \right)
\end{aligned}
$$

That is the average of rotating clockwise and anti-clockwise.

Bandpass filter





$$H(z) = \sum_{k=0}^{10} \cos(4\pi k/11) z^{-k}$$

- The peaks are slightly off-target
- The root $z_k = 1$ appears twice

Poles z-transforms

The z-transform of a filter can be represented in the form

$$H(z) = G \prod_{k=1}^{M} (1 - z_k z^{-1}) = G \prod_{k=1}^{M} \left(\frac{z - z_k}{z} \right)$$

where G is a constant

- z_k are also known as the **zeros of the system function**
  - For each zero z_k, $H(z_k) = 0$

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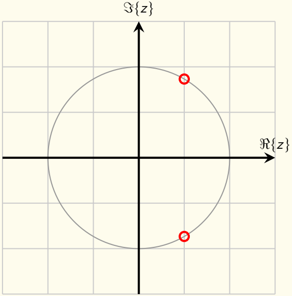
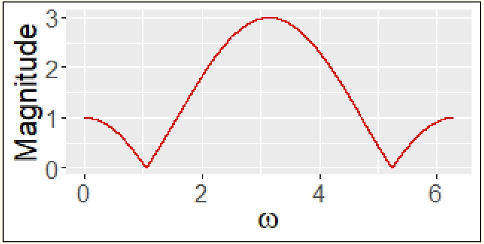
- z_k are also known as the **zeros of the system function**
  - For each zero z_k, $H(z_k) = 0$
- Note that for $z \to 0$, $H(z) \to \infty$
  - Points where $H(z) = \infty$ are the **poles** of $H(z)$
  - FIR filters of order M will have M poles at $z = 0$

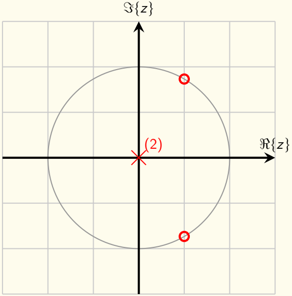
Frequency response and z-transform

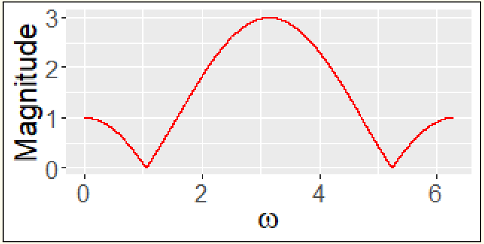




Red circles indicate zeros of $H(z) = 1 - z^{-1} + z^{-2}$

Frequency response and z-transform

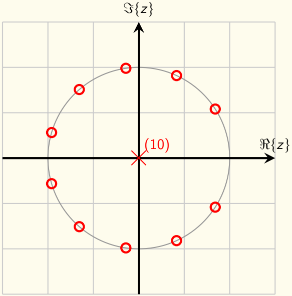


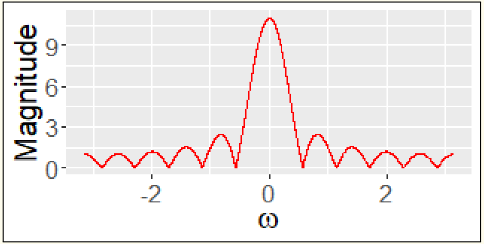


Red circles indicate zeros of $H(z) = 1 - z^{-1} + z^{-2}$

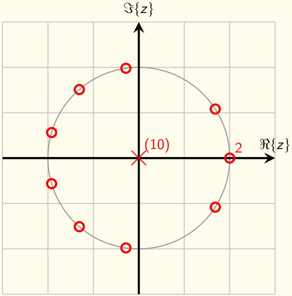
- The red cross indicates that there are 2 poles at the origin

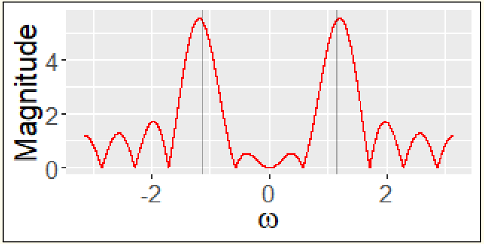
Zeros and poles plot - 11 point running sum





Zeros and poles plot - Bandpass filter





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Recall

An IIR filter is a recursive filter whose output depends on the current and previous inputs (feed-fordward) and outputs (feedback). It is characterized by a difference equation of the form

$$y[n] = \sum_{k=0}^{N} b_k x[n-k] - \sum_{k=1}^{M} a_k y[n-k]$$

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$$y[n] = \sum_{k=0}^{N} b_k x[n-k] - \sum_{k=1}^{M} a_k y[n-k]$$

- The z-transform simplifies the analysis of IIR filters as it characterize them with a **rational function**.
- The numerator polynomial is related to the feed-fordward coefficients and the denominator polynomial is related to the feedback coefficients.

$$y[n] = x[n] * h[n] \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad Y(z) = X(z)H(z)$$

Example

Consider the following first-order IIR filter

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1] \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad Y(z) = a_1 z^{-1} Y(z) + b_0 X(z) + b_1 z^{-1} X(z).$$

Let us group the output and input terms

$$Y(z)(1 - a_1 z^{-1}) = X(z)(b_0 + b_1 z^{-1}).$$

Thus, the system function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}.$$

One can generalize the system function for an IIR filter with N feed-fordward coefficients and M feedback coefficients as

$$H(z) = \frac{\sum_{k=0}^{N} b_k z^{-k}}{1 - \sum_{k=1}^{M} a_k z^{-k}} = \frac{B(z)}{A(z)} = B(z)\left(\frac{1}{A(z)}\right).$$

The last expression on the right makes explicit the cascade of the feed-fordward and feedback components of the filter.

Example 1

Let us consider the same first-order IIR and analyze the poles and zeros of the system function,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}.$$

- The zeros of the system function are the roots of the numerator polynomial $B(z) = b_0 + b_1 z^{-1}$, which are

$$z_0 = -\frac{b_0}{b_1}.$$

  This zero leads us to $H(z) = 0$ when $z = z_0$.

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- The poles of the system function are the roots of the denominator polynomial $A(z) = 1 - a_1 z^{-1}$, which are

  $$z_1 = \frac{1}{a_1}.$$

  This pole leads us to $H(z) = \infty$ when $z = z_1$.

Example 2

Recall from lecture 4

For the first order IIR filter

$$y[n] = a_1 y[n-1] + b_0 x[n],$$

its impulse response is

$$h[n] = b_0 a_1^n u[n].$$

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Let us analyze the system function of this filter in the z-domain.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{1 - a_1 z^{-1}} = b_0 z \left(\frac{1}{z - a_1} \right).$$

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- The zeros of the system function are the roots of the numerator polynomial $B(z) = b_0 z$, which are $z_0 = 0$. This zero leads us to $H(z) = 0$ when $z = z_0$.
- The poles of the system function are the roots of the denominator polynomial $A(z) = z - a_1$, which are $z_1 = a_1$. This pole leads us to $H(z) = \infty$ when $z = z_1$.

- Recall from lecture 4 that if $|a_1| < 1$, the filter is stable, which is equivalent to $|z_1| < 1$.
- Conversely, if $|a_1| > 1$, the filter is unstable, which is equivalent to $|z_1| > 1$.
- **The stability of the filter is related to the location of the poles in the z-plane. If the poles are inside the unit circle, the filter is stable.**

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- **The stability of the filter is related to the location of the poles in the z-plane. If the poles are inside the unit circle, the filter is stable.**

We can generalize this statement for any IIR filter. The stability of the filter is related to the location of the poles in the z-plane. **An IIR filter is stable if and only if all the poles are inside the unit circle of the z-plane.**

Example 2

Let us move the first-order IIR filter

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{1 - a_1 z^{-1}},$$

to the frequency domain. The frequency response of the filter when $a_1 < 1$ is

$$H(e^{j\hat{\omega}}) = \frac{b_0}{1 - a_1 e^{-j\hat{\omega}}}.$$

Consider an input signal $x[n] = e^{j\hat{\omega}_0 n}$. The output signal is

$$y[n] = H(e^{j\hat{\omega}_0}) e^{j\hat{\omega}_0 n} = \frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}} e^{j\hat{\omega}_0 n}.$$

Example 2

Now, consider the input $x[n] = e^{j\hat{\omega}_0 n} u[n]$. The z transform of this signal is

$$X(z) = \frac{1}{1 - e^{j\hat{\omega}_0} z^{-1}}.$$

The output signal is

$$Y(z) = H(z)X(z) = \left(\frac{b_0}{1 - a_1 z^{-1}} \right) \left(\frac{1}{1 - e^{j\hat{\omega}_0} z^{-1}} \right).$$

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The inverse z-transform of $Y(z)$ gives us the output signal

$$y[n] = \left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right)(a_1)^n u[n] + \left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right) e^{j\hat{\omega}_0 n} u[n]$$

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The first term on the right-hand side represents the **transient response**, while the second term represents the **steady-state response.** The transient response vanishes as $n \to \infty$ because $a_1 < 1$, whereas the steady-state response remains after the transient response has vanished.

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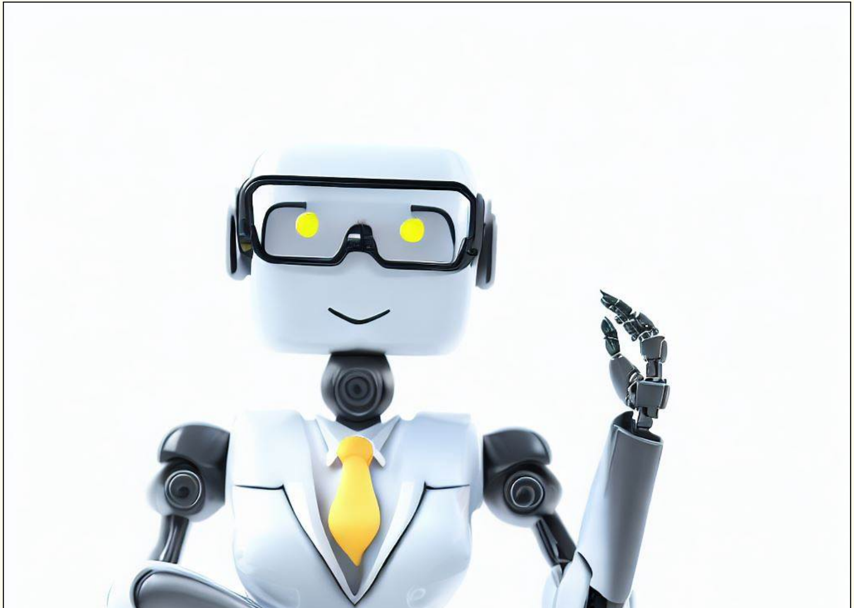
Let us wrap up the lecture!



Take-home Messages

The z-transform is a generalization of the frequency response

- Represents a system or signal as a polynomial
- The system function is the z-transform of a system

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \sum_{k=0}^{M} h[k] z^{-k} \tag{12}$$

- The relationship between the frequency domain and the z-domain is given by

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}} \tag{13}$$

Take-home Messages

The z-transform is described, up to a constant, by its roots

$$H(z) = \sum_{k=0}^{M} h[k]z^{-k} = \prod_{k=1}^{M}(1 - z_k z^{-1})$$

- The poles and zeros of $H(z)$ can give us meaningful information about a system.
- A convolution in the time domain is a multiplication in the z-domain. This property is useful for analyzing cascade systems.
- We have discussed time, frequency, and z-domain representations of signals and systems. A problem in one domain can be solved easier in another domain.

Practice Questions

The following questions might appear in the final exam:

- A particular FIR filter has a system function

$$H(z) = 1 + z^{-1} - z^{-2} - z^{-3}. \tag{14}$$

  Determine whether this filter **removes any frequencies from the input.** If so, which frequencies are removed? If not, explain why no frequencies are removed.

- Give an **impulse response function** $h[n]$ of a filter that nulls the signal

$$x[n] = 3 + 6\cos(\pi n/6). \tag{15}$$

  The trivial filter $h[n] = 0$ is not considered a valid answer.

- The system function of some FIR system is

$$H(z) = (1 + e^{-j\pi/3}z^{-1})(1 + e^{j\pi/3}z^{-1})(1 - 2z^{-1}). \tag{16}$$

  Determine **the frequency response function** and **the impulse response function** for this system.

Tutorial exercises

During the tutorial, the exercises below will be discussed in class

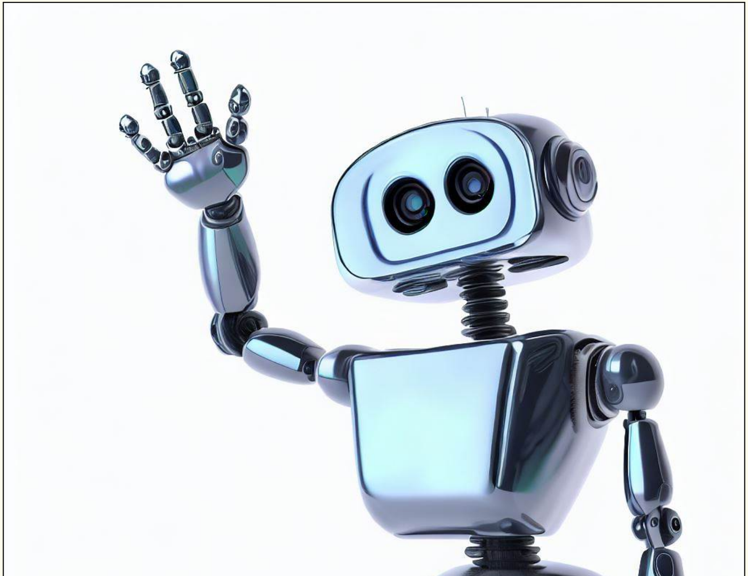
- Attempt to complete the exercises **before** class starts
- As the weeks progress, more time is needed for an explanation

SPF	DSPF
P 7.3 (p. 191)	P 9.3 (p. 407)
P 7.4 (p. 191)	P 9.4 (p. 407)
P 7.5 (p. 191)	P 9.5 (p. 408)
P 7.14 (p. 194)	P 9.13 (p. 411)
P 7.15 (p. 194)	P 9.14 (p. 411)

Let us talk again about continuous time domain...

The Laplace Transform

Have a nice day!



Acknowledgements

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Disclaimer

- Grammar was checked with Grammarly and Grammar checker GPT.
- Images without source were created with the assistance of DALL·E.