Tutorial 4 – FIR Filters

1. *P 5.1 (p. 126)* The impulse response $h[n]$ of an FIR filter is

$$h[n] = \delta[n-1] - 2\delta[n-4].$$

What is its difference equation?

Solution: The difference equation means that it should be in terms of $y[n]$ and $x[n]$. Generally, one can convert directly from an impulse response to the difference equation by just substituting h to y and δ to x. Formally, we can do it as follows:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = h[1]x[n-1] + h[4]x[n-4] = x[n-1] - 2x[n-4].$$

This is because $h = [0, 1, 0, 0, -2]$, where the index corresponds to the coefficient of δ in the impulse response. Alternatively, just write out $h = [0, 1, 0, 0, -2]$ and substitute h to y and δ to x.

2. *P 5.2 (p.126)* The running average is given by the formula

$$y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k].$$

The running average finds lots of applications when dealing with jagged graphs (e.g. in Reinforcement Learning, rewards from an optimal policy might have some variance). For a better comparison, we use the running average to smooth out the graph.

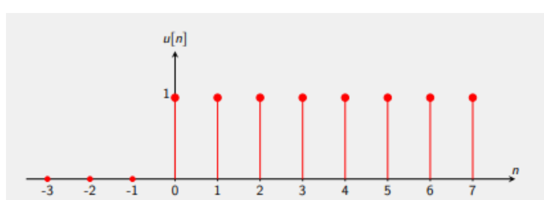
Evaluate the running average of the unit step function, namely,

$$x[n] = u[n] = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n \geq 0 \end{cases}$$

To do so, follow this order:

(a) Make a plot of $u[n]$.

Solution:



(b) For $L = 5$, compute the values of $y[n]$ for $-5 \leq n \leq 10$.

Solution:

- For $-5 \leq n < 0$, we have that $y[n] = \frac{1}{5}x[n]$, and since $-5 \leq n < 0$, $x[n] = 0$ and so $y[n] = 0$.

- For $0 \leq n \leq 10$, we have

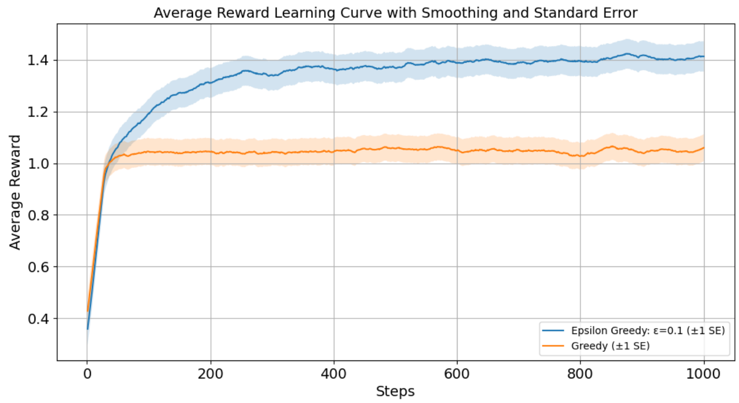
$$\begin{aligned} y[0] &= \frac{1}{5}(1+0+0+0+0) = \frac{1}{5} \\ y[1] &= \frac{1}{5}(1+1+0+0+0) = \frac{2}{5} \\ y[2] &= \frac{1}{5}(1+1+1+0+0) = \frac{3}{5}. \end{aligned}$$

  I hope you see the pattern as I will not write out all 10.

$$\begin{aligned} y[3] &= \frac{4}{5} \\ y[4] &= \frac{5}{5} = 1 \end{aligned}$$

  Hopefully, you realized that for $5 \leq n \leq 10$ it will be $y[5] = y[6] = y[7] = y[8] = y[9] = y[10] = 1$.

(This part is extra) To give a real example of a running average, compare the two graphs:



Which one looks nicer? Which one would you use when giving a presentation or submitting a report? Do the two graphs convey the same information? That is up to you to decide :)

(c) Can you derive a general formula for $y[n]$ for any $n \geq 0$ and $L \in \mathbb{N} \setminus \{0\}$? *Hint:* Yes, you can.

Solution: Notice that it is always divided by L. Moreover, when we are at n, we already have $n+1$ elements that are 1. Hence, a general formula would be

$$y[n] = \frac{\min(n+1, L)}{L}.$$

3. *P 5.3 (p.126)* An LTI system is described by the difference equation

$$y[n] = 2x[n] - 3x[n-1] + 2x[n-2].$$

The input to the system is

$$x[n] = \begin{cases} 0 & n < 0 \\ n+1 & n = 0, 1, 2 \\ 5-n & n = 3, 4 \\ 1 & n \geq 5 \end{cases}$$

Compute the output.

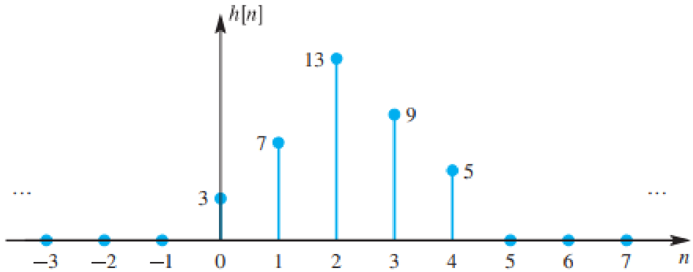
Solution: *Remark:* I changed this question slightly as it severely abuses notation.

For $0 \leq n \leq 10$, we plug it in $x[n]$ and obtain $x = [1, 2, 3, 2, 1, 1, 1, 1, 1, 1]$. We should now convolve $h = [2, -3, 2]$. Using the table method,

2	-3	2							
1	2	3	2	1	1	1	1	1	1
2	4	6	2	2	2	2	2	2	2
	-3	-6	-9	-6	-3	-3	-3	-3	-3
		2	4	6	4	2	2	2	2
2	1	2	-1	2	3	1	1	1	1

we obtain that $y[n] = [2, 1, 2, -1, 2, 3, 1, 1, 1, 1]$. Note that here the convolution is cut at 10 elements, even though there are more. But this is what the question is asking for.

4. *P 5.8 (p. 209)* The impulse response $h[n]$ is shown in the figure below. Determine the filter coefficients $\{b_k\}$ of the difference equation for the FIR filter. What is the output of the system when $x[n] = [2, 1, -1]$ is applied?



Solution: We can directly read off that $\{b_k\} = \{3, 7, 13, 9, 5\}$, hence $h = [3, 7, 13, 9, 5]$. We convolve with $x = [2, 1, -1]$ using the table method

2	1	-1				
3	7	13	9	5		
6	14	26	18	10		
	3	7	13	9	5	
		-3	-7	-13	-9	-5
6	17	30	24	6	-4	-5

and we obtain that the output $y[n] = [6, 17, 30, 24, 6, -4, 5]$.

5. *P 5.9 (p.128)* For each of the systems, determine if they are linear, time-invariant, or causal.

   (a) $y[n] = x[n]\cos(0.2\pi n)$

Solution: Let $A, B \in \mathbb{R} \setminus \{0\}$ (also for the following subquestions). A system is linear if

$$Ax_1[n] + Bx_2[n] = Ay_1[n] + By_2[n].$$

Indeed, we have that

$$Ax_1[n]\cos(0.2\pi n) + Bx_2[n]\cos(0.2\pi n) = Ay_1[n] + By_2[n],$$

so the system is linear. Linear systems can be recognized by 1) lack of constants or 2) no function applied over A or B. Of course, exceptions exist.
A system is time-invariant if $x[n-m] \implies y[n-m]$. In other words, by shifting **only** $x[n]$ by some time m, $y[n]$ has to be shifted by the same time. In this case, if we shift $y[n]$ by m,

we also shift the n within the cos, so the system won't be time-invariant if we **only** shift $x[n]$. Formally,

$$y[n-m] = x[n-m]\cos(0.2\pi n - 0.2\pi m) \neq x[n-m]\cos(0.2\pi n).$$

A system is causal if it only depends on past times. In other words, $y[n]$ does not depend on any $x[n+m]$, where $m > 0$. There are no positive indices in $y[n]$, so we conclude that it is indeed causal. Note that most systems are **obviously** causal or not causal. However, if a function is applied to n, things can get tricky. For instance, is $y[n] = x[|n-m|]$ causal? *Hint:* No.

(b) $y[n] = x[n] - x[n-1]$

Solution: Clearly linear (multiply by any constant and you can factor it out), causal (no future indices), and time-invariant (n is only within x and there are no transformations). This is an example of an LTI system.

(c) $y[n] = |x[n]|$

Solution: Not linear because we can pick a negative constant $-A$. Time invariant as n is only within x and there are no transformations and causal as no future indices.

(d) $y[n] = Ax[n] + B$, where $A, B \in \mathbb{R} \setminus \{0\}$.

Solution: Not linear because there is a constant that will get doubled, time-invariant and causal because of the same reasons as the previous.

6. *P 5.12 (p.128)* For an LTI system, when the input x_1 is the unit step, the output is $y_1[n] = \delta[n] + 2\delta[n-1] - \delta[n-2]$. What is the output when $x_2[n] = 3u[n] - 2u[n-4]$?

Solution: Since $x_1[n] = u[n]$, we know that the system is $y = [1, 2, -1]$. We convolve with $x_2 = [3, 0, 0, 0, -2]$ using the table method

1	2	-1				
3	0	0	0	-2		
3	0	0	0	-2		
	6	0	0	0	-4	
		-3	0	0	0	2
3	6	-3	0	-2	-4	2

and we obtain the output $y_2 = [3, 6, -3, 0, -2, -4, 2]$. Note that this solution in the slides is overcomplicated and is doing the same thing but longer.

7. *P 5.14 (p.128)*

(a) The output of an FIR filter with $h[n] = \delta[n-2]$ is $y[n] = u[n-3] - u[n-6]$. What was the input signal $x[n]$?

Solution: We have to reconstruct the unknown input $x[n]$ so that $x[n] * h[n] = y[n]$, where $y = [0,0,0,1,0,0,1]$ and $h = [0,0,1]$. We can infer this from the table method. Since we know that $|y| = |h| + |x| - 1 \iff |x| = 5$, then let's represent $x = [a,b,c,d,e]$, where $a,b,c,d,e \in \mathbb{Z}$. We do the table method,

0	0	1				
a	b	c	d	e		
0	0	0	0	0		
	0	0	0	0	0	
		a	b	c	d	e
0	0	0	1	0	0	1

We see that this holds if $b = 0$ and $e = 0$ and $a,c,d = 0$, so $x = [0,1,0,0,1]$. You already have the knowledge to express this in terms of $u[n]$ or $\delta[n]$.

(b) The output of a first-difference FIR filter is $y[n] = \delta[n] - \delta[n-4]$. Determine the input signal.

Solution: First-difference FIR filter has $h = [0,-1]$. The output $y = [1,0,0,0,-1]$ has size $|y| = 5$, the filter has size $|h| = 2$, hence from $|y| = |x| + |h| - 1$ we have that $|x| = 4$. Let $a,b,c,d \in \mathbb{Z}$ denote the elements of the input $x = [a,b,c,d]$. Again,

1	-1			
a	b	c	d	
a	b	c	d	
	$-a$	$-b$	$-c$	$-d$
1	0	0	0	-1

It is easy to see that $a = b = c = d = 1$ so that $x = [1,1,1,1]$.

(c) The output of a 4-point averager is $y[n] = -5\delta[n] - 5\delta[n-2]$. What is its input signal $x[n]$?
Hint: You can assume that $|x|$ is countably infinite.

Solution: I only added the infinite assumption so it is more clear, as otherwise you cannot infer the size of the input signal $x[n]$.

A four-point averager has $h = [1/4, 1/4, 1/4, 1/4]$. Let's assume that for $a_i \in \mathbb{Z}, i \in \mathbb{N}$ the input has the form $x[n] = [a_1, a_2, a_3, a_4, a_5, a_6, a_7...]$. We can write out the (infinite) convolution table

1/4	1/4	1/4	1/4								
a_1	a_2	a_3	a_4	a_5	a_6	a_7	$\dots$				
$a_1/4$	$a_2\ /\ 4$	$a_3/4$	$a_4/4$	$a_5/4$	$a_6/4$	$a_7/4$	$\dots$				
	$a_1/4$	$a_2/4$	$a_3/4$	$a_4/4$	$a_5/4$	$a_6/4$	$a_7/4$	$\dots$			
		$a_1/4$	$a_2/4$	$a_3/4$	$a_4/4$	$a_5/4$	$a_6/4$	$a_7/4$	$\dots$		
			$a_1/4$	$a_2/4$	$a_3/4$	$a_4/4$	$a_5/4$	$a_6/4$	$a_7/4$	$\dots$	
-5	0	-5	0	0	0	0	0	0	0	0	$\dots$

We can actually spot the pattern from here. We need that $a_1/4 = -5 \iff a_1 = -20$. Now, from the second column, we would need $a_2 = 20$. From the third column, we get that $a_3 = -20$. From the fourth column, we get that $a_4 = 20$, and so on. We conclude that $x = [-20, 20, -20, 20, -20, 20, -20, \dots]$.

8. *P 5.17 (p.128)* Three systems are connected consecutively in a cascade. The three systems are as follows:

$$\begin{aligned} \mathcal{S}_1: &\quad y_1[n] = x_1[n] - x_1[n-1] \\ \mathcal{S}_2: &\quad y_2[n] = x_2[n] + x_2[n-2] \\ \mathcal{S}_3: &\quad y_3[n] = x_3[n-1] + x_3[n-2] \end{aligned}$$

What is the impulse response $h_i[n]$ for each $\mathcal{S}_i$, $i \in \{1, 2, 3\}$? What is the overall system impulse response? What is its overall difference equation?

Solution: Substituting y to h and x to δ we can directly see that $h_1 = [1, -1]$, $h_2 = [1, 0, 1]$, and $h_3 = [0, 1, 1]$. Since they are connected consecutively, the overall system impulse response is $h = h_1 * h_2 * h_3 = (h_1 * h_2) * h_3$. We first compute $h_1 * h_2$:

1	-1		
1	0	1	
1	0	1	
	-1	0	-1
1	-1	1	-1

and then $[1, -1, 1, -1] * h_3$:

1	-1	1	-1		
0	1	1			
0	0	0	0		
	1	-1	1	-1	
		1	-1	1	-1
0	1	0	0	0	-1

The impulse response is $h = [0, 1, 0, 0, 0, -1]$. The difference equation is $y[n] = x[n-1] - x[n-5]$.