university of groningen / faculty of science and engineering

Lecture 5: Frequency response of FIR filters

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Signals and Systems

1B - 2024/2025

Preliminaries

- Lab session 2 is on Brightspace. The deadline for reports and code is on **Friday, December 20, at 17:30.**
- Remember to submit your **code in Themis and documentation in Brightspace.**
- Lab 3 will be released next week.
- Office hours on Friday, December 20, are online –Google Meet.
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Let us talk about the Lab 1:

- Thumbs up in clean code practices :D
- Chatbots and documentation not following the template or the page limit D:

You are responsible of your work!

Overview

1. Recap
2. Frequency Response
3. Properties of Frequency Response
4. Running Sum Filter
5. Image Smoothing
6. Closing Remarks

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Recap

The general form of causal FIR filters is

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \tag{1}$$

Recap

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Quick questions!

- Does this equation seem familiar from other courses?
- How can we interpret $y[n]$?
- Indeed, how do you choose b_k?

Recap

The convolution between a discrete signal x and filter h is

$$y[n] = x[n] \star h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]. \tag{2}$$

Indeed, the response of an **LTI filter** h is $y[n] = x[n] \star h[n]$.

Recap

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Indeed, the response of an **LTI filter** h is $y[n] = x[n] \star h[n]$.

Another quick question!

Why are LTI filters special?

Recap

We need to talk...

About convolutional neural networks and recurrent neural networks. They are not LTI systems, but...

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Sinusoidal response of FIR systems

Suppose the system faces a (discrete-time) complex exponential

$$x[n] = Ae^{j\varphi}e^{j\hat{\omega}n} \tag{3}$$

Let us compute the FIR filter for such a signal.

Sinusoidal response of FIR systems

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

Sinusoidal response of FIR systems

$$
\begin{aligned}
y[n] &= \sum_{k=0}^{M} b_k x[n-k] \\
&= \sum_{k=0}^{M} b_k A e^{j\varphi} e^{j\hat{\omega}(n-k)}
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If $x[n]$ is complex exponential, $y[n]$ is complex exponential

- $x[n]$ and $y[n]$ have the same radian frequency $\hat{\omega}$

Sinusoidal response of FIR systems

Let us expand the $H(e^{j\hat{\omega}})$ component.

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

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Frequency response has the following components: a **magnitude** $|H(e^{j\hat{\omega}})|$ and an **angle** $e^{j\angle H(e^{j\hat{\omega}})}$. The response $H(e^{j\hat{\omega}})$ is only meaningful for sinusoidal input

Teacher's comment

Avoid writing $y[n] = H(e^{j\hat{\omega}})x[n]$. This expression mixes time and frequency domains!

Time domain and frequency domain

Impulse response $h[k]$ is the time-domain representation of frequency response $H(e^{j\hat{\omega}})$.

Example: Suppose $\{b_k\} = \{-1, 3, -1\}$, for $k = 0, 1, 2$, then,

$$\begin{aligned} h[n] &= \sum_{k=0}^{M} b_k \delta[n-k] \\ H(e^{j\hat{\omega}}) &= \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} \end{aligned}$$

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$$\begin{aligned} h[n] &= \sum_{k=0}^{M} b_k \delta[n-k] = -\delta[n] + 3\delta[n-1] - \delta[n-2] \\ H(e^{j\hat{\omega}}) &= \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} = -1 + 3e^{-j\hat{\omega}} - e^{-j2\hat{\omega}} \end{aligned}$$

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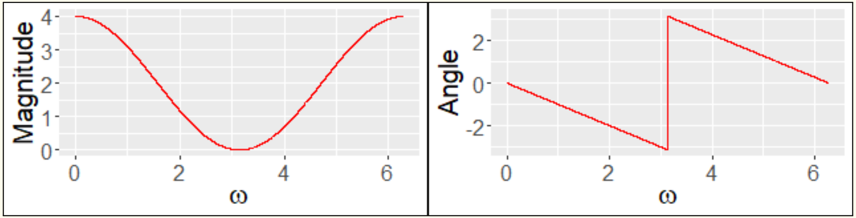

Figure: Magnitude and angle components in $H(e^{j\hat{\omega}})$ with $\{b_k\} = \{1, 2, 1\}$ for $k = 0, 1, 2$

Example - FIR complex exponential input

Consider the FIR filter $\{b_k\} = \{1, 2, 1\}$ for $k = 0, 1, 2$ and input signal

$$x[n] = 2e^{j\pi/4}e^{j(\pi/3)n} \tag{4}$$

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The output signal is

$$\begin{aligned} y[n] &= 3e^{-j\pi/3} \cdot 2e^{j\pi/4}e^{j(\pi/3)n} \\ &= 6e^{j\pi/4 - j\pi/3}e^{j(\pi/3)n} \\ &= 6e^{-j\pi/12}e^{j(\pi/3)n} \end{aligned}$$

Example - FIR sinusoidal input

Consider the FIR filter $\{b_k\} = \{1, 2, 1\}$ for $k = 0, 1, 2$ and input signal

$$x[n] = 2\cos((\pi/3)n - \pi/2) = e^{-j((\pi/3)n - \pi/2)} + e^{j((\pi/3)n - \pi/2)} \tag{5}$$

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$$\begin{aligned} y[n] &= 3e^{j\pi/3}e^{-j((\pi/3)n-\pi/2)} + 3e^{-j\pi/3}e^{j((\pi/3)n-\pi/2)} \\ &= 3e^{-j((\pi/3)n-5\pi/6)} + 3e^{j((\pi/3)n-5\pi/6)} \end{aligned}$$

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Real-valued FIR

When a filter with real-valued coefficients b_k, i.e. $b_k = \Re\{b_k\}$, processes sinusoidal input $x[n] = A\cos(\hat{\omega}n + \varphi)$,

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- The phase φ is increased by $\angle H(e^{j\hat{\omega}})$

That is,

$$y[n] = |H(e^{j\hat{\omega}})| \cdot A\cos\left(\hat{\omega}n + \varphi + \angle H(e^{j\hat{\omega}})\right) \tag{6}$$

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Comments

- **The gain of a filter cannot be negative!**
- It is called a gain even if $|H(e^{j\hat{\omega}})| < 1$
- The angle $\angle H(e^{j\hat{\omega}}) \in (-\pi, \pi]$ by convention.

First-difference filter

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$$H(e^{j\hat{\omega}}) \quad = \quad 1 - e^{-j\hat{\omega}} = 1 - (\cos(-\hat{\omega}) + j\sin(-\hat{\omega})) = (1 - \cos(\hat{\omega})) + j\sin(\hat{\omega})$$

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$$|H(e^{j\hat{\omega}})| = \sqrt{(1 - \cos(\hat{\omega}))^2 + \sin^2(\hat{\omega})}$$

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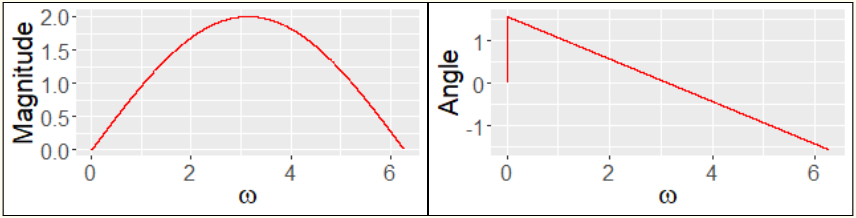


Figure: Magnitude and angle components in example system

First-difference filter

Consider again the first-difference system $y[n] = x[n] - x[n-1]$. Let us compute the magnitude and angle of the FIR filter response in another way.

$$H(e^{j\hat{\omega}}) \quad = \quad 1 - e^{-j\hat{\omega}} = e^{-j\hat{\omega}/2}(e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2}) = e^{-j\hat{\omega}/2}(2\sin(\hat{\omega}/2))$$

First-difference filter

Consider again the first-difference system $y[n] = x[n] - x[n-1]$. Let us compute the magnitude and angle of the FIR filter response in another way.

$$H(e^{j\hat{\omega}}) \quad = \quad 1 - e^{-j\hat{\omega}} = e^{-j\hat{\omega}/2}(e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2}) = e^{-j\hat{\omega}/2}(2\sin(\hat{\omega}/2))$$

The magnitude and angle are

$$\begin{aligned} |H(e^{j\hat{\omega}})| &= |2\sin(\hat{\omega}/2)| \\ \angle H(e^{j\hat{\omega}}) &= -\hat{\omega}/2 \qquad \text{for } 0 < \hat{\omega} \leqslant 2\pi \end{aligned}$$

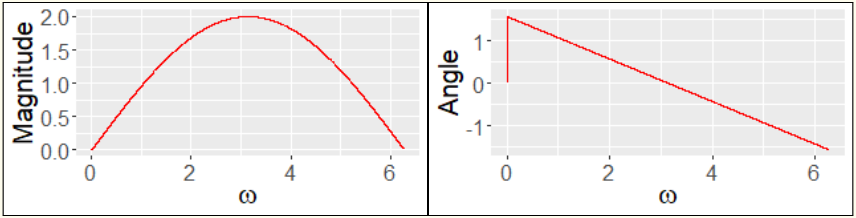


Figure: Magnitude and angle components in example system

First-difference filter

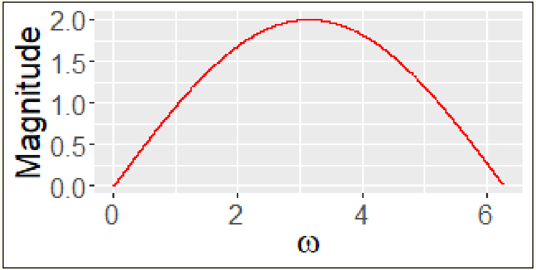
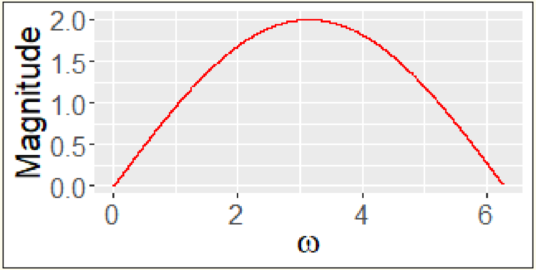


Figure: Magnitude components in example system

Note that the magnitude is zero for $\hat{\omega} = 0 + 2\pi k$.

First-difference filter



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- That is, any frequency $\hat{\omega}$ that is a multiple of 2π is eliminated.

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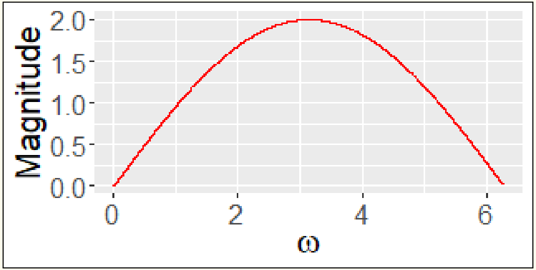

Figure: Magnitude components in example system

Note that the magnitude is zero for $\hat{\omega} = 0 + 2\pi k$.

- That is, any frequency $\hat{\omega}$ that is a multiple of 2π is eliminated.
- In particular, the DC component ($\hat{\omega} = 0$) is removed.

This behaviour makes sense for a first-difference filter...Why?

Eliminating a single frequency

Consider the FIR filter $\{b_k\} = \{1, \alpha, 1\}$ for $k = 0, 1, 2$.

Eliminating a single frequency

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The magnitude in the FIR filter is

$$|H(e^{j\hat{\omega}})| = |\alpha + 2\cos(\hat{\omega})|$$

By setting $|H(e^{j\hat{\omega}})| = 0$, we note this filter eliminates $\hat{\omega} = \arccos(-\alpha/2)$. E.g. $\alpha = -1$ eliminates $\hat{\omega} = 1/3\pi$.

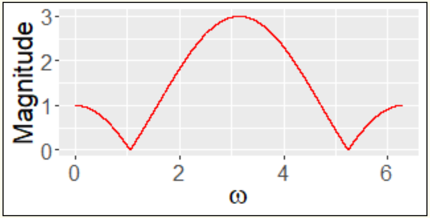

Figure: Magnitude and angle components in example system

Example - Eliminating a single frequency

Consider the FIR filter $\{b_k\} = \{1, -1, 1\}$ for $k = 0, 1, 2$, and the input signal

$$x[n] = 4 + 3\cos\left(\frac{1}{3}\pi n - \frac{\pi}{2}\right) + 3\cos\left(\frac{7}{8}\pi n\right) \tag{7}$$

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$$\begin{aligned} H(e^{j0}) &= e^{-j0}(-1 + 2\cos(0)) \\ H(e^{j\frac{1}{3}\pi}) &= e^{-j\frac{1}{3}\pi}\left(-1 + 2\cos\left(\frac{1}{3}\pi\right)\right) \\ H(e^{j\frac{7}{8}\pi}) &= e^{-j\frac{7}{8}\pi}\left(-1 + 2\cos\left(\frac{7}{8}\pi\right)\right) \end{aligned}$$

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H(e^{j0}) &= e^{-j0}(-1 + 2\cos(0)) = 1 \\
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H(e^{j\frac{7}{8}\pi}) &= e^{-j\frac{7}{8}\pi}\left(-1 + 2\cos\left(\frac{7}{8}\pi\right)\right)
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Example - Eliminating a single frequency

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The output is

$$y[n] = 1 \cdot 4 + 0 \cdot 3\cos\left(\frac{1}{3}\pi n - \frac{\pi}{2}\right) + 2.8477 \cdot 3\cos\left(\frac{7}{8}\pi n + \frac{1}{8}\pi\right)$$

Example - Eliminating a single frequency

Consider the FIR filter $\{b_k\} = \{1, -1, 1\}$ for $k = 0, 1, 2$, and the input signal

$$x[n] = 4 + 3\cos\left(\frac{1}{3}\pi n - \frac{\pi}{2}\right) + 3\cos\left(\frac{7}{8}\pi n\right) \tag{7}$$

The input x consists of three frequencies 0, $\frac{1}{3}\pi$, and $\frac{7}{8}\pi$. The FIR filter response is

$$\begin{aligned}
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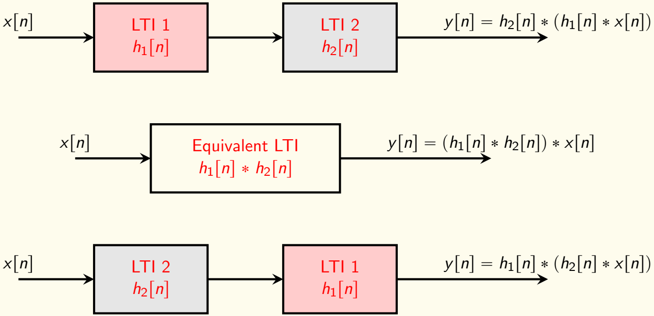
The output is

$$\begin{aligned}
y[n] &= 1 \cdot 4 + 0 \cdot 3\cos\left(\frac{1}{3}\pi n - \frac{\pi}{2}\right) + 2.8477 \cdot 3\cos\left(\frac{7}{8}\pi n + \frac{1}{8}\pi\right) \\
&= 4 + 8.5433\cos\left(\frac{7}{8}\pi n + \frac{1}{8}\pi\right)
\end{aligned}$$

Cascaded LTI systems

LTI cascades connect multiple LTI systems together

- The output of one system is the input for the next system
- The order of systems does not matter



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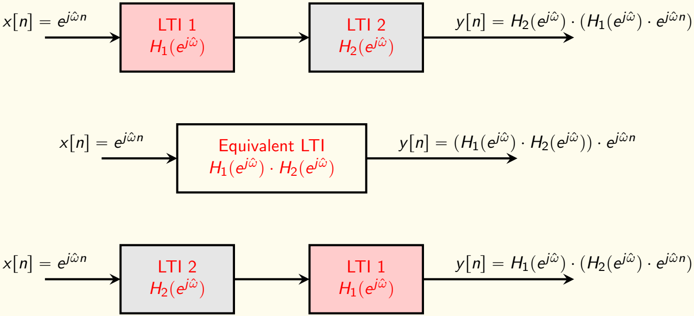

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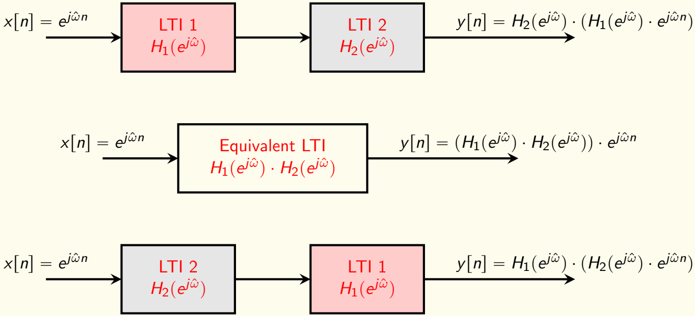

Convolution in the time domain is multiplication in the frequency domain.

Cascaded LTI systems

Convolving FIR filters equals multiplying their frequency responses. Consider the following FIR filters in time-domain,

$$\begin{aligned} h_1[n] &= \{2, 1, 2\}, \\ h_2[n] &= \{0, 3, 0, -3\} \end{aligned}$$

The respective frequency-domain response is

$$\begin{aligned} H_1(e^{j\hat{\omega}}) &= 2 + e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} \\ H_2(e^{j\hat{\omega}}) &= 3e^{-j\hat{\omega}} - 3e^{-j3\hat{\omega}} \end{aligned}$$

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The equivalent cascade system in frequency and time domain is given by

$$H(e^{j\hat{\omega}}) = H_1(e^{j\hat{\omega}}) \cdot H_2(e^{j\hat{\omega}})$$

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&= 6e^{-j\hat{\omega}} - 6e^{-j3\hat{\omega}} + 3e^{-j2\hat{\omega}} - 3e^{-j4\hat{\omega}} + 6e^{-j3\hat{\omega}} - 6e^{-j5\hat{\omega}}
\end{aligned}
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&= 6e^{-j\hat{\omega}} - 6e^{-j3\hat{\omega}} + 3e^{-j2\hat{\omega}} - 3e^{-j4\hat{\omega}} + 6e^{-j3\hat{\omega}} - 6e^{-j5\hat{\omega}} \\
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Break!

See you at ______________________________

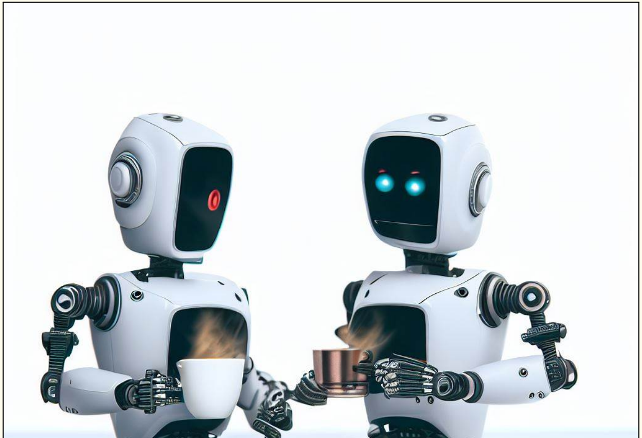

Table of Contents

Running sum

The L-point running sum filter is a common LTI system

$$y[n] = \sum_{k=0}^{L-1} x[n-k] = x[n] + x[n-1] + \cdots + x[n-L+1] \tag{8}$$

What is the frequency response?

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What is the frequency response?

$$H(e^{j\hat{\omega}}) \quad = \quad 1 \quad + \quad e^{-j\hat{\omega}} \quad + \quad e^{-j2\hat{\omega}} \quad + \cdots + \quad e^{-j(L-1)\hat{\omega}}$$

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$$\begin{aligned} H(e^{j\hat{\omega}}) &= 1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} + \cdots + e^{-j(L-1)\hat{\omega}} \\ -e^{-j\hat{\omega}}H(e^{j\hat{\omega}}) &= \quad\;\; -e^{-j\hat{\omega}} - e^{-j2\hat{\omega}} - \cdots - e^{-j(L-1)\hat{\omega}} - e^{-jL\hat{\omega}} \end{aligned}$$

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$$y[n] = \sum_{k=0}^{L-1} x[n-k] = x[n] + x[n-1] + \cdots + x[n-L+1] \tag{8}$$

What is the frequency response?

$$\begin{array}{rcccccccccc}
H(e^{j\hat{\omega}}) & = & 1 & + & e^{-j\hat{\omega}} & + & e^{-j2\hat{\omega}} & + \cdots + & e^{-j(L-1)\hat{\omega}} & & \\
-\, e^{-j\hat{\omega}} H(e^{j\hat{\omega}}) & = & & & -e^{-j\hat{\omega}} & - & e^{-j2\hat{\omega}} & - \cdots - & e^{-j(L-1)\hat{\omega}} & - & e^{-jL\hat{\omega}} \\
\hline
(1 - e^{-j\hat{\omega}}) H(e^{j\hat{\omega}}) & = & 1 & + & 0 & + & 0 & + \cdots + & 0 & - & e^{-jL\hat{\omega}}
\end{array}$$

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By solving the equation above, we have

$$H(e^{j\hat{\omega}}) = \frac{1 - e^{-jL\hat{\omega}}}{1 - e^{-j\hat{\omega}}} \tag{9}$$

Running sum

$$H(e^{j\hat{\omega}}) = \frac{1 - e^{-jL\hat{\omega}}}{1 - e^{-j\hat{\omega}}}$$

Running sum

$$
\begin{aligned}
H(e^{j\hat{\omega}}) &= \frac{1 - e^{-jL\hat{\omega}}}{1 - e^{-j\hat{\omega}}} \\
&= \frac{e^{-j\hat{\omega}L/2}}{e^{-j\hat{\omega}/2}} \cdot \left(\frac{e^{j\hat{\omega}L/2} - e^{-j\hat{\omega}L/2}}{e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2}} \right)
\end{aligned}
$$

Running sum

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\begin{aligned}
H(e^{j\hat{\omega}}) &= \frac{1 - e^{-jL\hat{\omega}}}{1 - e^{-j\hat{\omega}}} \\
&= \frac{e^{-j\hat{\omega}L/2}}{e^{-j\hat{\omega}/2}} \cdot \left(\frac{e^{j\hat{\omega}L/2} - e^{-j\hat{\omega}L/2}}{e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2}} \right) \\
&= e^{-j\hat{\omega}(L-1)/2} \cdot \left(\frac{\sin(\hat{\omega}L/2)}{\sin(\hat{\omega}/2)} \right)
\end{aligned}
$$

Running sum

$$
\begin{aligned}
H(e^{j\hat{\omega}}) &= \frac{1 - e^{-jL\hat{\omega}}}{1 - e^{-j\hat{\omega}}} \\
&= \frac{e^{-j\hat{\omega}L/2}}{e^{-j\hat{\omega}/2}} \cdot \left(\frac{e^{j\hat{\omega}L/2} - e^{-j\hat{\omega}L/2}}{e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2}} \right) \\
&= e^{-j\hat{\omega}(L-1)/2} \cdot \left(\frac{\sin(\hat{\omega}L/2)}{\sin(\hat{\omega}/2)} \right) \\
&= e^{-j\hat{\omega}(L-1)/2} \cdot D_L(\hat{\omega})
\end{aligned}
$$

$$
D_L(\hat{\omega}) = \frac{\sin(\hat{\omega}L/2)}{\sin(\hat{\omega}/2)}
$$

The function $D_L(\hat{\omega})$ is called the Dirichlet form

Dirichlet form

Consider an L-point running sum filter with $L = 11$.

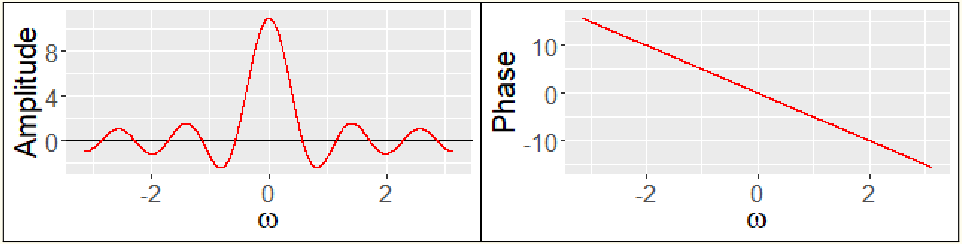


Figure: Amplitude and phase for a running sum filter with $L = 11$.

- $D_{11}(\hat{\omega})$ is symmetric around the vertical axis
- $D_{11}(\hat{\omega})$ has period 2π
- $D_{11}(\hat{\omega})$ has maximum 11 for $\hat{\omega} = 0$
- $D_{11}(\hat{\omega})$ is zero for $\hat{\omega} = \frac{2\pi k}{11}$ for integers $k \neq 0$

Dirichlet form

Quick question!

Do you notice something strange in the amplitude and phase?

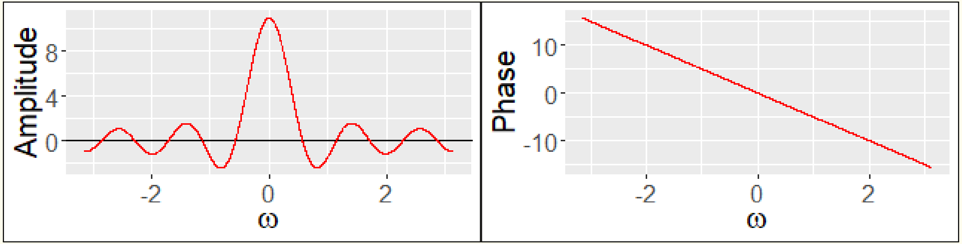


Figure: Amplitude and phase for a running sum filter with $L = 11$.

Running sum

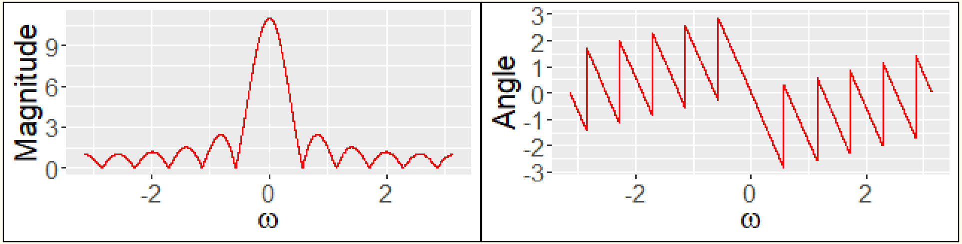


Figure: Magnitude and angle for a running sum filter with $L = 11$.

- The magnitude $|H(e^{j\hat{\omega}})| = |D_{11}(\hat{\omega})|$ cannot be negative
- Instead, $-1 = e^{j\pi}$ is a phase rotation of π radians
  - The angle is depicted in principal form $-\pi < \angle H(e^{j\hat{\omega}}) \leqslant \pi$

Running sum

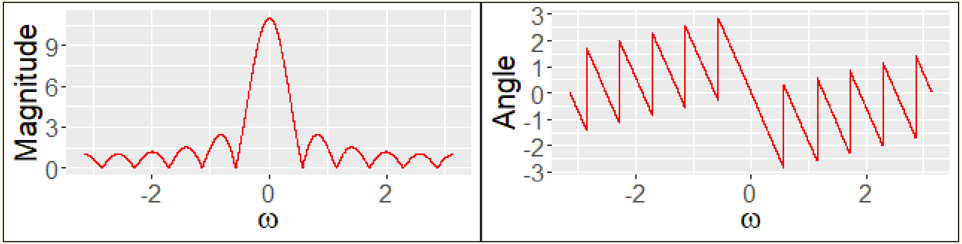
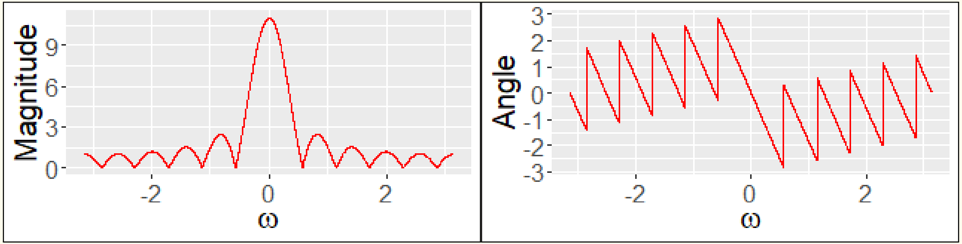


Figure: Magnitude and angle for a running sum filter with $L = 11$.

- The magnitude $|H(e^{j\hat{\omega}})| = |D_{11}(\hat{\omega})|$ cannot be negative
- Instead, $-1 = e^{j\pi}$ is a phase rotation of π radians
  - The angle is depicted in principal form $-\pi < \angle H(e^{j\hat{\omega}}) \leqslant \pi$
- The running sum behaves as a **low-pass filter**
  - Low frequencies are multiplied with a high magnitude
  - High frequencies are multiplied with a low magnitude

Continuous and discrete signals



- The frequency response $H(e^{j\hat{\omega}})$ is defined for any frequency $\hat{\omega}$
- In practice, frequencies outside the range $-\pi < \hat{\omega} \leqslant \pi$ cannot be reconstructed by an ideal D-to-C converter

Continuous and discrete signals

Consider the signal $x(t) = \cos(2\pi(25)t) + \sin(2\pi(250)t)$ passing through an **11-point averager** (not sum!) with sampling frequency is $f_s = 1000$.

Continuous and discrete signals

Consider the signal $x(t) = \cos(2\pi(25)t) + \sin(2\pi(250)t)$ passing through an **11-point averager** (not sum!) with sampling frequency is $f_s = 1000$.

There are two frequency components $\hat{\omega}_1 = \frac{2\pi(25)}{1000}$ and $\hat{\omega}_2 = \frac{2\pi(250)}{1000}$.

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$$H(e^{j\frac{2\pi(25)}{1000}}) \quad =$$

Continuous and discrete signals

Consider the signal $x(t) = \cos(2\pi(25)t) + \sin(2\pi(250)t)$ passing through an **11-point averager** (not sum!) with sampling frequency is $f_s = 1000$.

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$$H(e^{j\frac{2\pi(25)}{1000}}) = \mathbf{\frac{1}{11}} e^{-j5\frac{2\pi(25)}{1000}} \left(\frac{\sin(11\pi(25)/1000)}{\sin(\pi(25)/1000)} \right)$$

Continuous and discrete signals

Consider the signal $x(t) = \cos(2\pi(25)t) + \sin(2\pi(250)t)$ passing through an **11-point averager** (not sum!) with sampling frequency is $f_s = 1000$.

There are two frequency components $\hat{\omega}_1 = \frac{2\pi(25)}{1000}$ and $\hat{\omega}_2 = \frac{2\pi(250)}{1000}$.The 11-point average filter response is

$$H(e^{j\frac{2\pi(25)}{1000}}) \quad = \quad \mathbf{\frac{1}{11}} e^{-j5\frac{2\pi(25)}{1000}} \left(\frac{\sin(11\pi(25)/1000)}{\sin(\pi(25)/1000)} \right) \approx \mathbf{0.8811} e^{-\mathbf{j}\pi/\mathbf{4}}$$

Continuous and discrete signals

$$x(t) \longrightarrow \boxed{\text{A-to-D converter}} \xrightarrow{x[n]} \boxed{\text{11-point averager}} \xrightarrow{y[n]} \boxed{\text{D-to-A converter}} \xrightarrow{y(t)}$$

Consider the signal $x(t) = \cos(2\pi(25)t) + \sin(2\pi(250)t)$ passing through an **11-point averager** (not sum!) with sampling frequency is $f_s = 1000$.

There are two frequency components $\hat{\omega}_1 = \frac{2\pi(25)}{1000}$ and $\hat{\omega}_2 = \frac{2\pi(250)}{1000}$. The 11-point average filter response is

$$\begin{aligned}
H(e^{j\frac{2\pi(25)}{1000}}) &= \mathbf{\frac{1}{11}} e^{-j5\frac{2\pi(25)}{1000}} \left(\frac{\sin(11\pi(25)/1000)}{\sin(\pi(25)/1000)} \right) \approx \mathbf{0.8811} e^{-\mathbf{j}\pi/\mathbf{4}} \\
H(e^{j\frac{2\pi(250)}{1000}}) &= \mathbf{\frac{1}{11}} e^{-j5\frac{2\pi(250)}{1000}} \left(\frac{\sin(11\pi(250)/1000)}{\sin(\pi(250)/1000)} \right)
\end{aligned}$$

Continuous and discrete signals

Consider the signal $x(t) = \cos(2\pi(25)t) + \sin(2\pi(250)t)$ passing through an **11-point averager** (not sum!) with sampling frequency is $f_s = 1000$.

There are two frequency components $\hat{\omega}_1 = \frac{2\pi(25)}{1000}$ and $\hat{\omega}_2 = \frac{2\pi(250)}{1000}$.The 11-point average filter response is

$$
\begin{aligned}
H(e^{j\frac{2\pi(25)}{1000}}) &= \frac{\mathbf{1}}{\mathbf{11}} e^{-j5\frac{2\pi(25)}{1000}} \left(\frac{\sin(11\pi(25)/1000)}{\sin(\pi(25)/1000)} \right) \approx \mathbf{0.8811} e^{-\mathbf{j}\pi/\mathbf{4}} \\
H(e^{j\frac{2\pi(250)}{1000}}) &= \frac{\mathbf{1}}{\mathbf{11}} e^{-j5\frac{2\pi(250)}{1000}} \left(\frac{\sin(11\pi(250)/1000)}{\sin(\pi(250)/1000)} \right) \approx \mathbf{0.0909} e^{-\mathbf{j}\pi/\mathbf{2}}
\end{aligned}
$$

Continuous and discrete signals

Consider the signal $x(t) = \cos(2\pi(25)t) + \sin(2\pi(250)t)$ passing through an **11-point averager** (not sum!) with sampling frequency is $f_s = 1000$.

There are two frequency components $\hat{\omega}_1 = \frac{2\pi(25)}{1000}$ and $\hat{\omega}_2 = \frac{2\pi(250)}{1000}$.The 11-point average filter response is

$$
\begin{aligned}
H(e^{j\frac{2\pi(25)}{1000}}) &= \mathbf{\frac{1}{11}} e^{-j5\frac{2\pi(25)}{1000}} \left(\frac{\sin(11\pi(25)/1000)}{\sin(\pi(25)/1000)} \right) \approx \mathbf{0.8811} e^{-\mathbf{j}\pi/\mathbf{4}} \\
H(e^{j\frac{2\pi(250)}{1000}}) &= \mathbf{\frac{1}{11}} e^{-j5\frac{2\pi(250)}{1000}} \left(\frac{\sin(11\pi(250)/1000)}{\sin(\pi(250)/1000)} \right) \approx \mathbf{0.0909} e^{-\mathbf{j}\pi/\mathbf{2}}
\end{aligned}
$$

And the output signal is

$$
y(t) = \mathbf{0.8811} \cos(2\pi(25)t - \pi/\mathbf{4}) + \mathbf{0.0909} \sin(2\pi(250)t - \pi/\mathbf{2}) \tag{10}
$$

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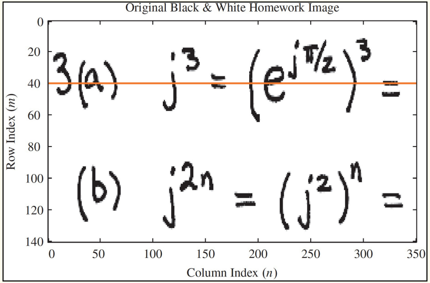
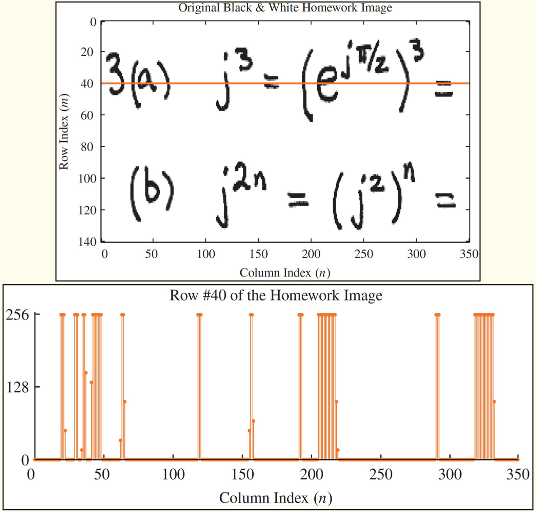
Images as signals



- Each row in the image can be interpreted as a signal

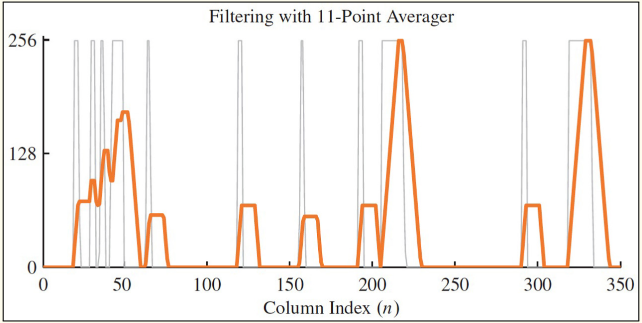
Images from McClellan, Shafer, & Yoder (2015) *DSP First*

Images as signals



Images from McClellan, Shafer, & Yoder (2015) *DSP First*

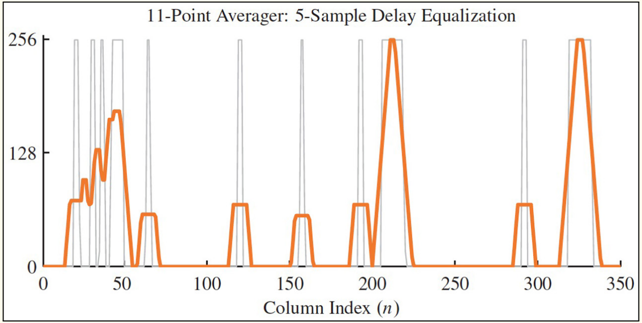
Blurring



- The 11-point running average filter blurs the signal horizontally
  - Note that causal filters cause a shift to the right

Images from McClellan, Shafer, & Yoder (2015) *DSP First*

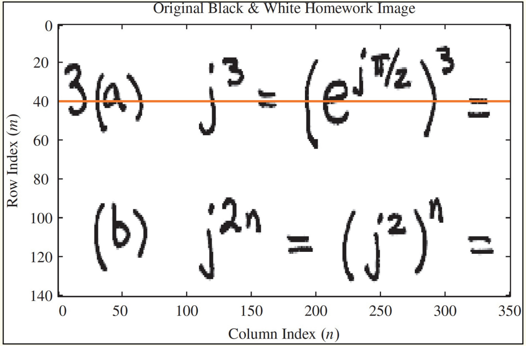
Blurring

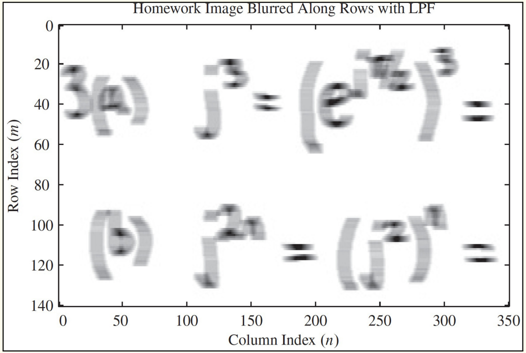


- The 11-point running average filter blurs the signal horizontally
  - Note that causal filters cause a shift to the right

Images from McClellan, Shafer, & Yoder (2015) *DSP First*

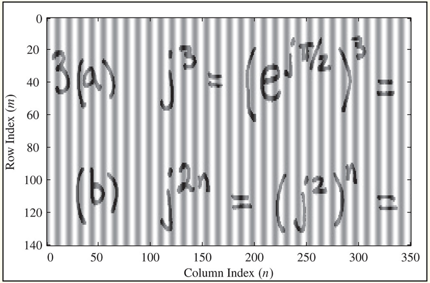
Horizontal blurring on images

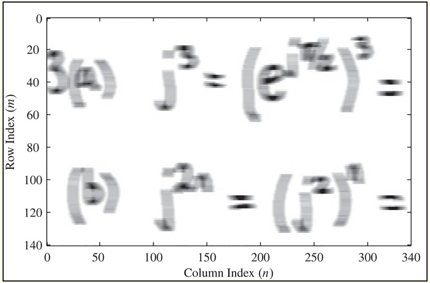




Images from McClellan, Shafer, & Yoder (2015) *DSP First*

Eliminating frequencies

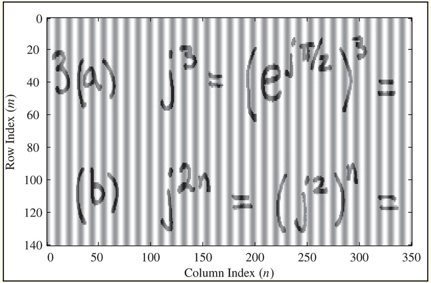


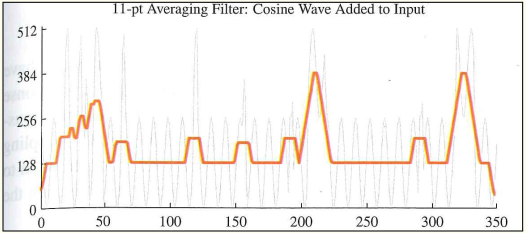


- The 11-point running average filter eliminates frequency multiples of 11
  - Cosine signal $128\cos(2\pi n/11) + 128$ is added to the signal

Images from McClellan, Shafer, & Yoder (2015) *DSP First*

Eliminating frequencies

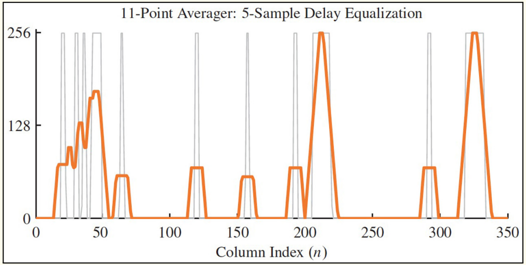


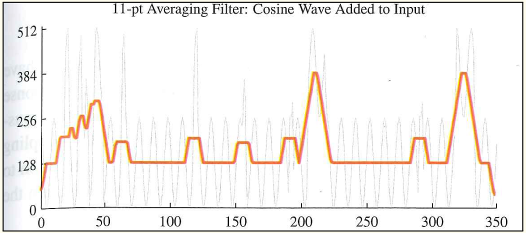


- The 11-point running average filter eliminates frequency multiples of 11
  - Cosine signal $128\cos(2\pi n/11) + 128$ is added to the signal
  - The running average divides the cosine across all samples

Images from McClellan, Shafer, & Yoder (2015) *DSP First*

Eliminating frequencies





- The 11-point running average filter eliminates frequency multiples of 11
  - Cosine signal $128\cos(2\pi n/11) + 128$ is added to the signal
  - The running average divides the cosine across all samples
  - Note the cosine causes transient effects at the start and end of the signal

Images from McClellan, Shafer, & Yoder (2015) *DSP First*

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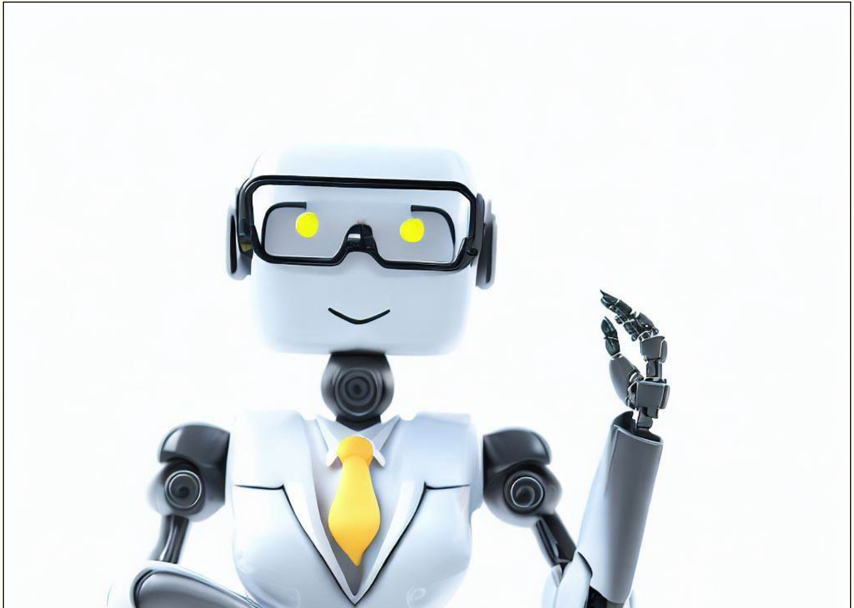
Let us wrap up the lecture!



Take-home Messages

LTI filters respond to complex exponentials in a particular way

$$\begin{aligned} x[n] &= Ae^{j\varphi}e^{j\hat{\omega}n} \\ y[n] &= \left(\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}\right) Ae^{j\varphi}e^{j\hat{\omega}n} \\ &= H(e^{j\hat{\omega}})Ae^{j\varphi}e^{j\hat{\omega}n} \end{aligned}$$

- An LTI system is characterized by impulse response $h[n]$
- An LTI system is characterized by frequency response $H(e^{j\hat{\omega}})$

Take-home Messages

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} = |H(e^{j\hat{\omega}})|e^{j\angle H(e^{j\hat{\omega}})}$$

- The frequency response consists of
  - Magnitude $|H(e^{j\hat{\omega}})| \geqslant 0$
  - Phase $-\pi < \angle H(e^{j\hat{\omega}}) \leqslant \pi$
- The input frequency is not changed by an LTI system
- Each frequency component in the input may have a different frequency response

And Finally.

A convolution in the time domain is a multiplication in the frequency domain.

Practice Questions

The following questions might appear in the final exam:

- Let us suppose that $y[n]$ is the response of a FIR filter $h[n]$ and the signal $x[n]$ whose (radian) frequency is $\hat{\omega}$. What is the (radian) frequency of $y[n]$?
- What is the magnitude and angle in a sinusoidal response of an FIR system?
- Why would you eliminate a single frequency in a signal?
- What is advantage of computing $h[n] \star x[n]$ in the frequency domain?

Tutorial exercises

During the tutorial, the exercises below will be discussed in class

- Attempt to complete the exercises **before** class starts
- As the weeks progress, more time is needed for an explanation

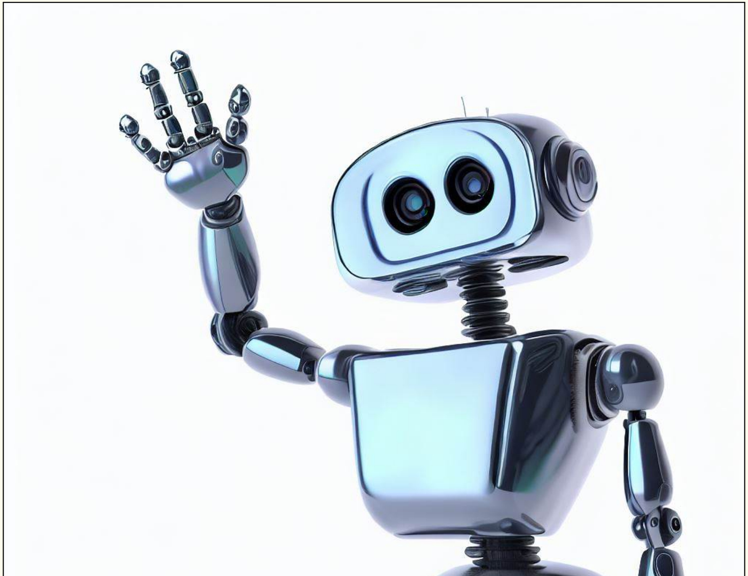
SPF	DSPF
P 6.1 (p. 157)	P 6.1 (p. 250)
P 6.4 (p. 157)	P 6.4 (p. 251)
P 6.5 (p. 158)	P 6.5 (p. 252)
P 6.7 (p. 158)	P 6.7 (p. 252)
P 6.16 (p. 160)	P 6.17 (p. 256)

Closing Remarks: Next Lecture

Let us discuss how we can transform a periodic signal from time to frequency domain.

Fourier transform

Have a nice day!



Acknowledgements

The material for this lecture series was developed by dr. Arnold Meijster and dr. Harmen de Weerd and modified by Juan Diego Cardenas-Cartagena.

Disclaimer

- Grammar was checked with Grammarly and Grammar checker GPT.
- Images without source were created with the assistance of DALL·E.