Fourier and its transforms

A Signals & Systems (for AI) summary by

Laura Mª Quirós Conesa

Last edited: November 29, 2024

Figure 1: Jean-Baptiste Joseph Fourier (1768 - 1830)

Contents

Important functions

- The Euler's formula:

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{1}$$

- The inverse Euler's formulas:

$$A\cos(\theta) = A\frac{e^{j\theta} + e^{-j\theta}}{2} \tag{2}$$

$$A\sin(\theta) = A\frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{3}$$

- Periodic signal:

$$x(t) = x(t + T_0) \tag{4}$$

The Unit Circle

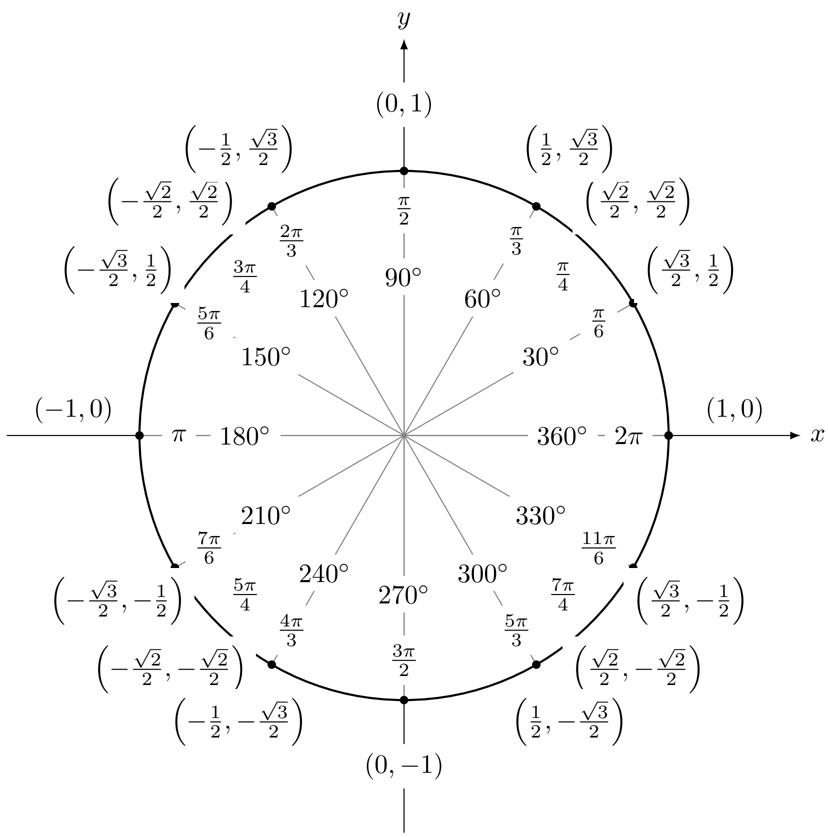

Figure 2: Relation between radians, angle degrees, and Cartesian coordinates.

1 Concept behind Fourier Transform and Series

Most of the functions below are based on a single concept:

> A complex signal can be torn apart into sinusoids of different frequency. Once we add up all these sinusoids, they will cancel each other or add each other in the amplitude resulting in our initial signal.

The more sinusoids we use, the more accurate our reconstructed signal. This concept helps us understand encoding and decoding of sounds.

2 Fourier Analysis

The Fourier Analysis of a discrete or continuous function $x(t)$ is given by:

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) \cdot e^{(\frac{-j2\pi kt}{T_0})} dt$$

Where:

- j is the imaginary unit, i.e., $j^2 = -1$,
- t is the time variable,
- $T_0 = 1/f_0$ is the fundamental period which is inversely related to the fundamental frequency,
- k is the index of the coefficient,
- a_k is what we call Fourier Coefficients.

In exams we will be asked to get a_k for different values of k, mainly $k = 0$ and $k \neq 0$ but there is also instances in which we need to identify the difference for odd k and even k. For the following integral,

$$\int_0^1 e^{-j\pi k} = -j\pi k * e^{-j\pi k} = -j\pi k \cdot (-1)^k = -j\pi * -1 = j\pi \tag{5}$$

if we have an odd k we get $j\pi$ and otherwise we get $-j\pi$. The simplification is done using equivalence **??** introduced at the beginning of the cheatsheet. Coefficient extraction can also be done manually as seen in section 2.2.

2.1 Fundamental frequency and how to get it from a graph or signal

Compound signals (signals with more than one frequency) always have a fundamental frequency $f_0 = gcd(f_1, f_2, \cdots, f_n)$ where n is the number of frequencies and `gcd` is the greatest common denominator. You can encounter the frequencies of a compound signal by looking into the overall composition of the sinusoid function

$$x(t) = A_1 \cos(2\pi f_1 t + \phi_1) + A_2 \cos(2\pi f_2 t + \phi_2) + \cdots$$

If none of the frequencies can be divided by the same number with remainder 0, i.e., there is no `gcd`, then there is no fundamental frequency and the signal is not periodic. This is the case when one or more frequencies are irrational numbers.

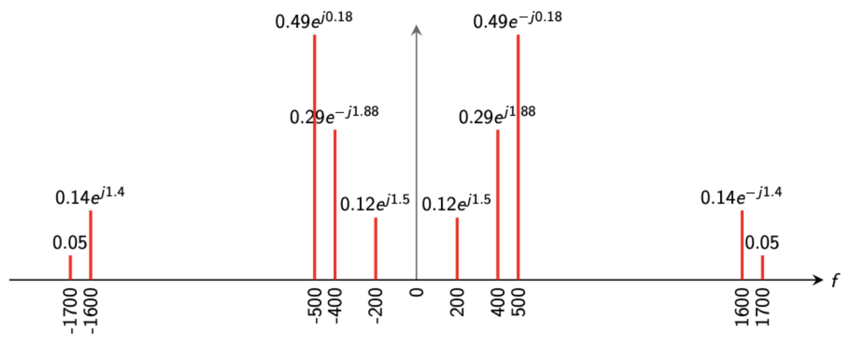

Figure 3: This is an example of an spectrum in which the frequencies are written in the x-axis. The fundamental frequency f_0 is $\text{gcd}(200, 400, 500, 1600, 1700) = 100$ Hz.

2.2 Sinusoidal sum coefficient extraction

$$x(t) = 1 + cos(2\pi 20t) + 3cos(2\pi 60t) - cos(2\pi 80t)$$

This signal is periodic with $f_0 = \text{gcd}\{20, 60, 80\} = 20Hz$, and $T_0 = \frac{1}{20}$s.
We conclude $a_0 = 1, a_{\pm 1} = 0.5, a_{\pm 3} = 1.5, a_{\pm 4} = 0.5e^{\pm j\pi}, a_{\pm k} = 0, \forall k \in \{-4, -3, -1, 0, 1, 3, 4\}$.

1. Extract the constant term
   The term 1 is a constant and represents the a_0 term in the Fourier series $a_0 = 1$

2. Extract the rest of Fourier coefficients
   Each sinusoid $Acos(2\pi ft + \phi)$ has a fourier coefficient $a_{\pm \frac{f}{f_0}} = \frac{A}{2}e^{\pm \phi}$ based off of the Euler formulas **??**.
   If the phase is subtracted, remember that $\pm$ becomes $\mp$.

   First sinusoid leads to coefficient value $\frac{A}{2} = \frac{1}{2}$ and for the a index we take the sinusoidal frequency (20) divided by the fundamental frequency ($f_0 = 20$) so $a_{\pm 1} = \frac{1}{2}$.
   Second sinusoid leads to coefficient value $\frac{3}{2}$ for sinusoidal frequency of 60 therefore index $a_{\pm \frac{f}{f_0}}$ is $3 = 60/20$ and $a_{\pm 3} = \frac{3}{2}$
   Third sinusoid leads to coefficient value $\frac{-1}{2} = \frac{1}{2}e^{\mp j\pi}$ given equivalence **??**. The index is $4 = 80/20$.

2.3 Discrete Fourier Transform (DFT)

Type of Fourier analysis that transforms a discrete signal into a sequence of coefficients of a finite combination of complex sinusoids, ordered by their frequencies.

The formula for the DFT is given by:

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{(\frac{-j2\pi k \cdot n}{N})}$$

Where:

- $X(k)$ are the DFT coefficients
- $x(n)$ is the discrete signal or a sampled continuous signal
- N is the total number of samples

- k is the frequency index
- j is the imaginary unit

3 Fourier Series

A Fourier series is a way to represent a compound signal as the sum of simpler sinusoids. The Fourier series of a periodic function can be written as:

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_k \cdot e^{-j(\frac{2\pi}{T_0})kt})$$

Where:

- $x(t)$ is the periodic function
- a_0, a_k are the Fourier Coefficients with frequency index k
- T_0 is the fundamental period
- t is the time variable

4 Time Shift in Fourier coefficients

1. Time shift of a signal by $z(t) = y(t - pT_0)$

$$\begin{aligned} b_0 &= a_0 \\ b_k &= a_k e^{-j2\pi pk} \end{aligned}$$

   adds a phase to the coefficients, representing the delay in the initial signal

2. Constant offset of a signal $z(t) = y(t) - p$

$$\begin{aligned} b_0 &= a_0 - p \\ b_k &= a_k \end{aligned}$$

   adding a unit of delay p doesn't have a frequency shift ($b_k = a_k$), but rather only affects the DC-component a_0

3. Constant offset of a signal $z(t) = p - y(t) = -(y(t) - p)$

$$\begin{aligned} b_0 &= p - a_0 \\ b_k &= -a_k \end{aligned}$$