university of groningen / faculty of science and engineering

Lecture 2: Spectrum representation

Juan Diego Cardenas-Cartagena, M.Sc.
(j.d.cardenas.cartagena@rug.nl)

Signals and Systems
1B - 2024/2025

Preliminaries

- The first lab assignment is available now. And its deadline is on Friday, December 6, at 17:30.
- We will upload a video and slides with the answers to the tutorial problems on Friday. However, it is strongly recommended that you attend the tutorials, as the TAs will provide close support.

Preliminaries

- The first lab assignment is available now. And its deadline is on Friday, December 6, at 17:30.
- We will upload a video and slides with the answers to the tutorial problems on Friday. However, it is strongly recommended that you attend the tutorials, as the TAs will provide close support.

AI & CCS/CS Programme Committees event

On **November 20**, the programme committees of CS, AI, and CCS will hold an event where they will introduce themselves and explain their roles. You will have the opportunity to meet your representatives, ask questions, raise any issues you have encountered in your studies, and recommend outstanding lecturers for this year's Teacher of the Year Award.
The event will begin at 12:00 in **BB 5161.0116**, and you are welcome to stop by until 16:00. Free pizza and drinks will be provided.

Overview

1. Recap
2. Spectrum
3. Harmonics
4. Fourier Series
5. Closing Remarks

Table of Contents

Sinusoidal signals

A **sinusoidal signal** is a continuous time signal of the form

$$x(t) = A\cos(\omega_0 t + \varphi) \tag{1}$$

Sinusoidal signals

A **sinusoidal signal** is a continuous time signal of the form

$$x(t) = A\cos(\omega_0 t + \varphi) \tag{1}$$

Where,

- A is the **amplitude**

Sinusoidal signals

A **sinusoidal signal** is a continuous time signal of the form

$$x(t) = A\cos(\omega_0 t + \varphi) \tag{1}$$

Where,

- A is the **amplitude**
- ω_0 is the **radian frequency**.
    - A **cyclic frequency** of f Hertz (Hz) corresponds to

$$\omega_0 = 2\pi f \tag{2}$$

    - **Period** in seconds

$$T = \frac{1}{f} = \frac{2\pi}{\omega_0} \tag{3}$$

Sinusoidal signals

A **sinusoidal signal** is a continuous time signal of the form

$$x(t) = A\cos(\omega_0 t + \varphi) \tag{1}$$

Where,

- A is the **amplitude**
- ω_0 is the **radian frequency**.
  - A **cyclic frequency** of f Hertz (Hz) corresponds to

    $$\omega_0 = 2\pi f \tag{2}$$

  - **Period** in seconds

    $$T = \frac{1}{f} = \frac{2\pi}{\omega_0} \tag{3}$$

- And, φ is the **phase**

Euler's formula

Euler's formula relates sinusoids to the complex exponential

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{4}$$

Euler's formula

Euler's formula relates sinusoids to the complex exponential

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{4}$$

The inverse Euler's formulas switch the relation around

$$\begin{aligned} \cos(\theta) &= \frac{e^{j\theta} + e^{-j\theta}}{2} \\ \sin(\theta) &= \frac{e^{j\theta} - e^{-j\theta}}{2j} \end{aligned}$$

Phasor addition rule

To add signals with the same frequency

$$x(t) = \sum_{k=1}^{N} A_k \cos(\omega_0 t + \varphi_k) = A \cos(\omega_0 t + \varphi) \tag{5}$$

Phasor addition rule

To add signals with the same frequency

$$x(t) = \sum_{k=1}^{N} A_k \cos(\omega_0 t + \varphi_k) = A\cos(\omega_0 t + \varphi) \quad (5)$$

1. Obtain the phasor representations $X_k = A_k e^{j\varphi_k}$ for each term
2. Convert polar to Cartesian coordinates $X_k = a_k + jb_k$
3. Calculate the sum of phasors $X = (\sum a_k) + j(\sum b_k)$
4. Convert Cartesian to polar coordinates $X = Ae^{j\varphi}$
5. Obtain the sinusoid $A\cos(\omega_0 t + \varphi)$

Table of Contents

Time-domain representation

Suppose we have a signal with **time-domain representation**

$$x(t) = A\cos(2\pi f_0 t + \varphi) \tag{6}$$

Time-domain representation

Suppose we have a signal with **time-domain representation**

$$x(t) = A\cos(2\pi f_0 t + \varphi) \tag{6}$$

Using inverse Euler relation $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$ we obtain

$$x(t) = \frac{A}{2}\left(e^{j(2\pi f_0 t+\varphi)} + e^{-j(2\pi f_0 t+\varphi)}\right)$$

Time-domain representation

Suppose we have a signal with **time-domain representation**

$$x(t) = A\cos(2\pi f_0 t + \varphi) \tag{6}$$

Using inverse Euler relation $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$ we obtain

$$\begin{aligned} x(t) &= \frac{A}{2}\left(e^{j(2\pi f_0 t + \varphi)} + e^{-j(2\pi f_0 t + \varphi)}\right) \\ &= \frac{1}{2}Ae^{j\varphi}e^{j2\pi f_0 t} + \frac{1}{2}Ae^{-j\varphi}e^{-j2\pi f_0 t} \end{aligned}$$

Frequency-domain representation

$$
\begin{aligned}
x(t) &= A\cos(2\pi f_0 t + \varphi) \\
&= \frac{1}{2}Ae^{j\varphi}e^{j2\pi f_0 t} + \frac{1}{2}Ae^{-j\varphi}e^{-j2\pi f_0 t}
\end{aligned}
$$

This signal is described by $\left\{(f_0, \frac{1}{2}Ae^{j\varphi}), (-f_0, \frac{1}{2}Ae^{-j\varphi})\right\}$

- This **frequency-domain representation** is independent of t

Frequency-domain representation

$$
\begin{aligned}
x(t) &= A\cos(2\pi f_0 t + \varphi) \\
&= \frac{1}{2}Ae^{j\varphi}e^{j2\pi f_0 t} + \frac{1}{2}Ae^{-j\varphi}e^{-j2\pi f_0 t}
\end{aligned}
$$

This signal is described by $\left\{(f_0, \frac{1}{2}Ae^{j\varphi}), (-f_0, \frac{1}{2}Ae^{-j\varphi})\right\}$

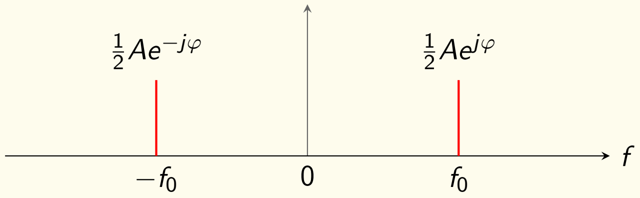
- This **frequency-domain representation** is independent of t
- This representation is instead in terms of frequency f

Frequency-domain representation

$$
\begin{aligned}
x(t) &= A\cos(2\pi f_0 t + \varphi) \\
&= \frac{1}{2}Ae^{j\varphi}e^{j2\pi f_0 t} + \frac{1}{2}Ae^{-j\varphi}e^{-j2\pi f_0 t}
\end{aligned}
$$

This signal is described by $\left\{(f_0, \frac{1}{2}Ae^{j\varphi}), (-f_0, \frac{1}{2}Ae^{-j\varphi})\right\}$

- This **frequency-domain representation** is independent of t
- This representation is instead in terms of frequency f

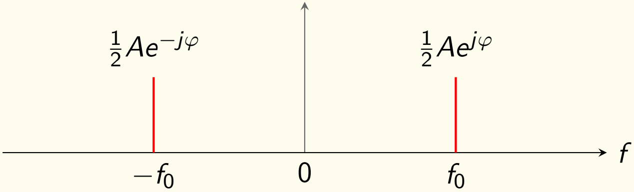


Frequency-domain representation

$$
\begin{aligned}
x(t) &= 10\cos(2\pi(100)t + \pi/2) \\
&= \frac{1}{2}Ae^{j\varphi}e^{j2\pi f_0 t} + \frac{1}{2}Ae^{-j\varphi}e^{-j2\pi f_0 t}
\end{aligned}
$$

This signal is described by $\left\{(f_0, \frac{1}{2}Ae^{j\varphi}), (-f_0, \frac{1}{2}Ae^{-j\varphi})\right\}$

- This **frequency-domain representation** is independent of t
- This representation is instead in terms of frequency f

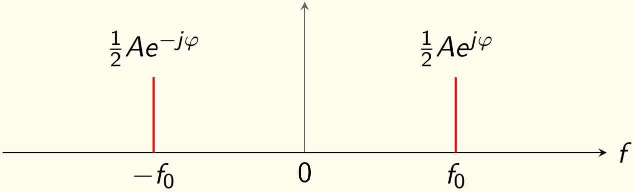


Frequency-domain representation

$$
\begin{aligned}
x(t) &= 10\cos(2\pi(100)t + \pi/2) \\
&= \frac{1}{2}10e^{j\pi/2}e^{j2\pi(100)t} + \frac{1}{2}Ae^{-j\varphi}e^{-j2\pi f_0 t}
\end{aligned}
$$

This signal is described by $\left\{(f_0, \frac{1}{2}Ae^{j\varphi}), (-f_0, \frac{1}{2}Ae^{-j\varphi})\right\}$

- This **frequency-domain representation** is independent of t
- This representation is instead in terms of frequency f

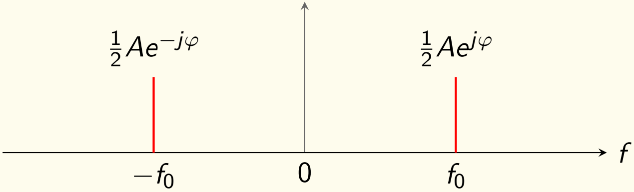


Frequency-domain representation

$$
\begin{aligned}
x(t) &= 10\cos(2\pi(100)t + \pi/2) \\
&= \frac{1}{2}10e^{j\pi/2}e^{j2\pi(100)t} + \frac{1}{2}10e^{-j\pi/2}e^{-j2\pi(100)t}
\end{aligned}
$$

This signal is described by $\left\{(f_0, \frac{1}{2}Ae^{j\varphi}), (-f_0, \frac{1}{2}Ae^{-j\varphi})\right\}$

- This **frequency-domain representation** is independent of t
- This representation is instead in terms of frequency f

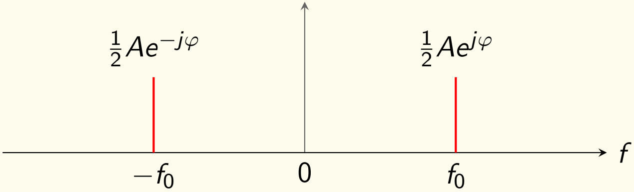


Frequency-domain representation

$$\begin{aligned} x(t) &= 10\cos(2\pi(100)t + \pi/2) \\ &= \frac{1}{2}10e^{j\pi/2}e^{j2\pi(100)t} + \frac{1}{2}10e^{-j\pi/2}e^{-j2\pi(100)t} \end{aligned}$$

This signal is described by $\left\{(100, 5e^{j\pi/2}), (-100, 5e^{-j\pi/2})\right\}$

- This **frequency-domain representation** is independent of t
- This representation is instead in terms of frequency f

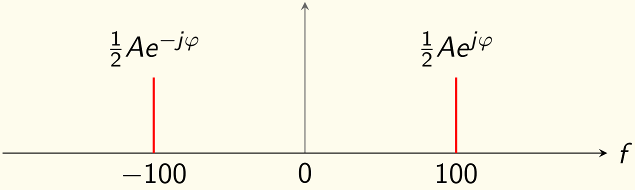


Frequency-domain representation

$$\begin{aligned} x(t) &= 10\cos(2\pi(100)t + \pi/2) \\ &= \frac{1}{2}10e^{j\pi/2}e^{j2\pi(100)t} + \frac{1}{2}10e^{-j\pi/2}e^{-j2\pi(100)t} \end{aligned}$$

This signal is described by $\left\{(100, 5e^{j\pi/2}), (-100, 5e^{-j\pi/2})\right\}$

- This **frequency-domain representation** is independent of t
- This representation is instead in terms of frequency f

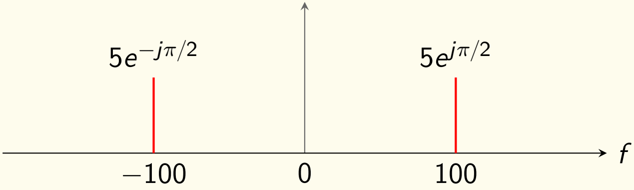


Frequency-domain representation

$$
\begin{aligned}
x(t) &= 10\cos(2\pi(100)t + \pi/2) \\
&= \frac{1}{2}10e^{j\pi/2}e^{j2\pi(100)t} + \frac{1}{2}10e^{-j\pi/2}e^{-j2\pi(100)t}
\end{aligned}
$$

This signal is described by $\left\{(100, 5e^{j\pi/2}), (-100, 5e^{-j\pi/2})\right\}$

- This **frequency-domain representation** is independent of t
- This representation is instead in terms of frequency f



Frequency-domain representation

Question

Consider the signal $x(t) = 3\cos(2\pi)$, i.e., $f = 0$. What do you expect to see in the frequency-domain representation? (Hint: Use the inverse Euler's formula)

Adding signals with different frequencies

We can also create a **spectrum** for compound signals with varying amplitudes A_k, phase φ_k, and frequencies f_k. Consider the following signal:

$$x(t) = 10 + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2) \tag{7}$$

Adding signals with different frequencies

We can also create a **spectrum** for compound signals with varying amplitudes A_k, phase φ_k, and frequencies f_k. Consider the following signal:

$$x(t) = 10\cos(0t) + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2) \tag{7}$$

Creating the frequency-domain representation gives us

$$\Big\{$$

Adding signals with different frequencies

We can also create a **spectrum** for compound signals with varying amplitudes A_k, phase φ_k, and frequencies f_k. Consider the following signal:

$$x(t) = 10\cos(0t) + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2) \tag{7}$$

Creating the frequency-domain representation gives us

$$\Big\{(0, 5), (-0, 5),$$

Adding signals with different frequencies

We can also create a **spectrum** for compound signals with varying amplitudes A_k, phase φ_k, and frequencies f_k. Consider the following signal:

$$x(t) = 10\cos(0t) + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2) \tag{7}$$

Creating the frequency-domain representation gives us

$$\Big\{(0, 10),$$

Adding signals with different frequencies

We can also create a **spectrum** for compound signals with varying amplitudes A_k, phase φ_k, and frequencies f_k. Consider the following signal:

$$x(t) = 10\cos(0t) + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2) \tag{7}$$

Creating the frequency-domain representation gives us

$$\Big\{(0, 10), (100, 7e^{-j\pi/3}), (-100, 7e^{j\pi/3}),$$

Adding signals with different frequencies

We can also create a **spectrum** for compound signals with varying amplitudes A_k, phase φ_k, and frequencies f_k. Consider the following signal:

$$x(t) = 10\cos(0t) + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2) \tag{7}$$

Creating the frequency-domain representation gives us

$$\left\{(0, 10), (100, 7e^{-j\pi/3}), (-100, 7e^{j\pi/3}), (250, 4e^{j\pi/2}), (-250, 4e^{-j\pi/2})\right\} \tag{8}$$

Adding signals with different frequencies

We can also create a **spectrum** for compound signals with varying amplitudes A_k, phase φ_k, and frequencies f_k. Consider the following signal:

$$x(t) = 10\cos(0t) + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2) \tag{7}$$

Creating the frequency-domain representation gives us

$$\left\{(0, 10), (100, 7e^{-j\pi/3}), (-100, 7e^{j\pi/3}), (250, 4e^{j\pi/2}), (-250, 4e^{-j\pi/2})\right\} \tag{8}$$

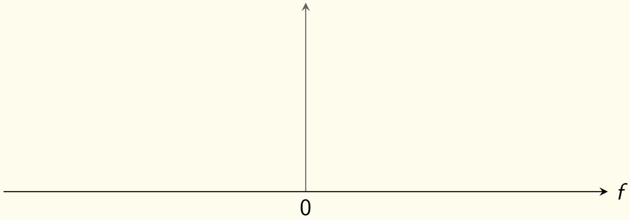


Figure: Frequency-domain representation for signal in (7)

Adding signals with different frequencies

We can also create a **spectrum** for compound signals with varying amplitudes A_k, phase φ_k, and frequencies f_k. Consider the following signal:

$$x(t) = 10\cos(0t) + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2) \tag{7}$$

Creating the frequency-domain representation gives us

$$\left\{(0, 10), (100, 7e^{-j\pi/3}), (-100, 7e^{j\pi/3}), (250, 4e^{j\pi/2}), (-250, 4e^{-j\pi/2})\right\} \tag{8}$$

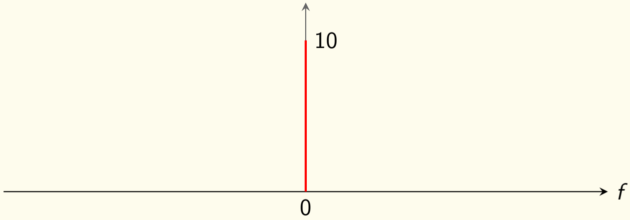


Figure: Frequency-domain representation for signal in (7)

Adding signals with different frequencies

We can also create a **spectrum** for compound signals with varying amplitudes A_k, phase φ_k, and frequencies f_k. Consider the following signal:

$$x(t) = 10\cos(0t) + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2) \tag{7}$$

Creating the frequency-domain representation gives us

$$\left\{(0, 10), (100, 7e^{-j\pi/3}), (-100, 7e^{j\pi/3}), (250, 4e^{j\pi/2}), (-250, 4e^{-j\pi/2})\right\} \tag{8}$$

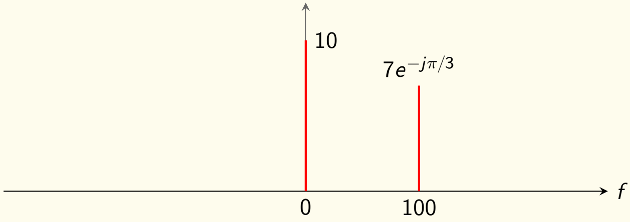


Figure: Frequency-domain representation for signal in (7)

Adding signals with different frequencies

We can also create a **spectrum** for compound signals with varying amplitudes A_k, phase φ_k, and frequencies f_k. Consider the following signal:

$$x(t) = 10\cos(0t) + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2) \tag{7}$$

Creating the frequency-domain representation gives us

$$\left\{(0, 10), (100, 7e^{-j\pi/3}), (-100, 7e^{j\pi/3}), (250, 4e^{j\pi/2}), (-250, 4e^{-j\pi/2})\right\} \tag{8}$$

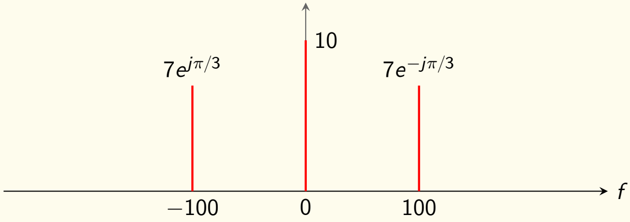


Figure: Frequency-domain representation for signal in (7)

Adding signals with different frequencies

We can also create a **spectrum** for compound signals with varying amplitudes A_k, phase φ_k, and frequencies f_k. Consider the following signal:

$$x(t) = 10\cos(0t) + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2) \tag{7}$$

Creating the frequency-domain representation gives us

$$\left\{(0, 10), (100, 7e^{-j\pi/3}), (-100, 7e^{j\pi/3}), (250, 4e^{j\pi/2}), (-250, 4e^{-j\pi/2})\right\} \tag{8}$$

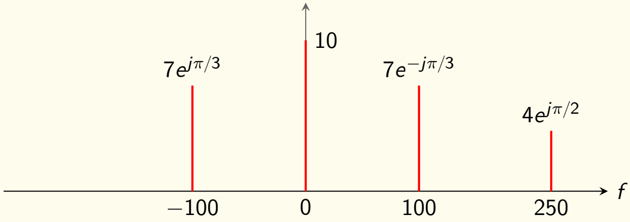


Figure: Frequency-domain representation for signal in (7)

Adding signals with different frequencies

We can also create a **spectrum** for compound signals with varying amplitudes A_k, phase φ_k, and frequencies f_k. Consider the following signal:

$$x(t) = 10\cos(0t) + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2) \tag{7}$$

Creating the frequency-domain representation gives us

$$\left\{(0, 10), (100, 7e^{-j\pi/3}), (-100, 7e^{j\pi/3}), (250, 4e^{j\pi/2}), (-250, 4e^{-j\pi/2})\right\} \tag{8}$$

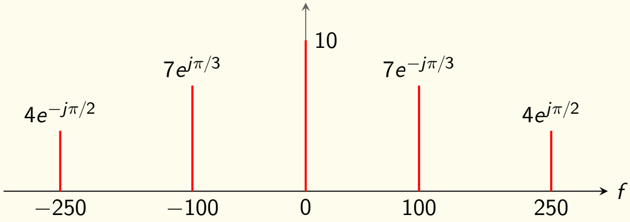


Figure: Frequency-domain representation for signal in (7)

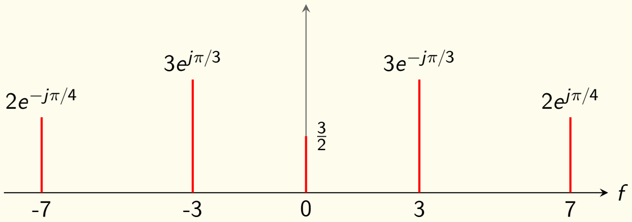
Adding signals with different frequencies

Question

What is the meaning behind the positive-negative frequencies pair?

Reading a spectrum

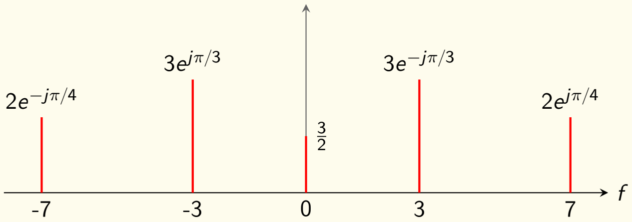
What time-domain representation belongs to this spectrum?



$x(t) =$

Reading a spectrum

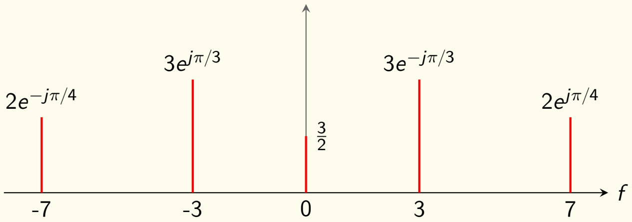
What time-domain representation belongs to this spectrum?



$$x(t) = \frac{3}{2}$$

Reading a spectrum

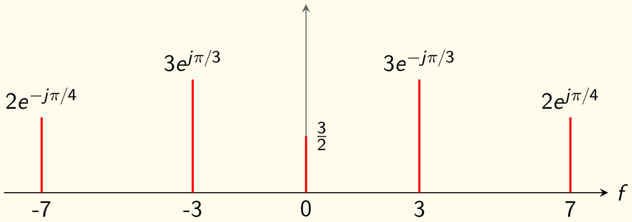
What time-domain representation belongs to this spectrum?



$$x(t) = \frac{3}{2} + 6\cos(2\pi(3)t - \pi/3)$$

Reading a spectrum

What time-domain representation belongs to this spectrum?



$$x(t) = \frac{3}{2} + 6\cos(2\pi(3)t - \pi/3) + 4\cos(2\pi(7)t + \pi/4) \tag{9}$$

Table of Contents

Synthetic vowel

f_k	$\frac{1}{2}Ae^{j\varphi}$
200	$0.1223e^{j1.508}$
400	$0.2942e^{j1.877}$
500	$0.4884e^{-j0.185}$
1600	$0.1362e^{-j1.449}$
1700	0.04724

Table: Approximation of the vowel sound "ah"[1]

[1] http://dspfirst.gatech.edu/...

Synthetic vowel

f_k	$\frac{1}{2}Ae^{j\varphi}$
200	$0.1223e^{j1.508}$
400	$0.2942e^{j1.877}$
500	$0.4884e^{-j0.185}$
1600	$0.1362e^{-j1.449}$
1700	0.04724

Table: Approximation of the vowel sound "ah"[1]

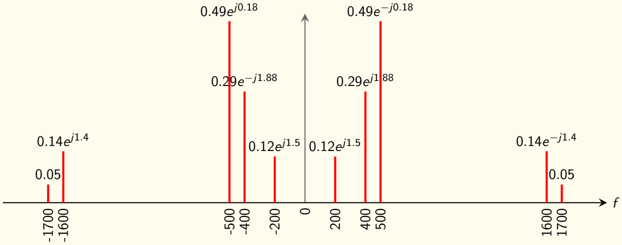


Figure: Frequency spectrum of the vowel sound "ah"

[1] http://dspfirst.gatech.edu/...

Synthetic vowel

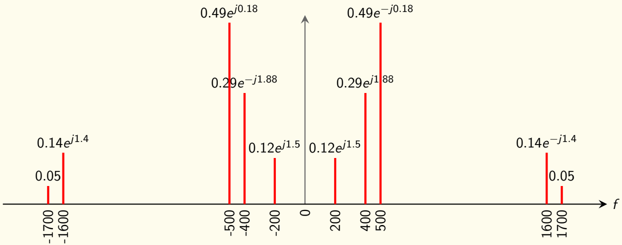


Figure: Frequency spectrum of the vowel sound "ah"

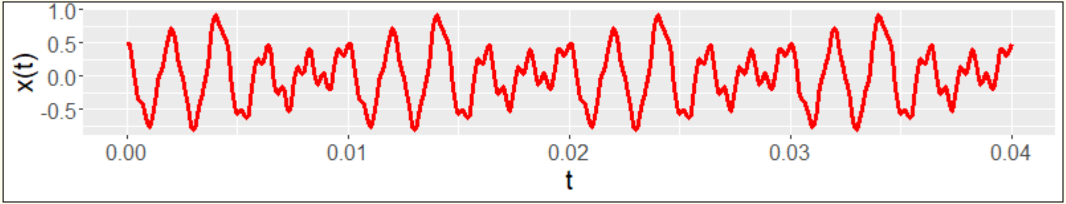


Figure: Time representation of the vowel sound "ah"

Periodic signals

Definition

A signal $x(t)$ is **periodic** with a period $T_0 > 0$ if it satisfies

$$x(t + T_0) = x(t), \text{ for all } t \tag{10}$$

The smallest period of a signal is its **fundamental period**.

- The **fundamental frequency** $F_0 = \frac{1}{T_0}$.
- $f_k = kF_0$ is the kth **harmonic** of F_0.

Periodic signals

Definition

A signal $x(t)$ is **periodic** with a period $T_0 > 0$ if it satisfies

$$x(t + T_0) = x(t), \text{ for all } t \tag{10}$$

The smallest period of a signal is its **fundamental period**.

- The **fundamental frequency** $F_0 = \frac{1}{T_0}$.
- $f_k = kF_0$ is the kth **harmonic** of F_0.

If the sum of sinusoids is a periodic signal, their frequencies are harmonically related. I.e., individual f_k are multiples of a fundamental frequency F_0. And

$$F_0 = \gcd\{f_k\}, \ k = 1, 2, \ldots. \tag{11}$$

Example of a periodic signal

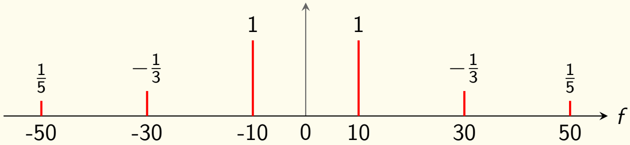
Consider the signal

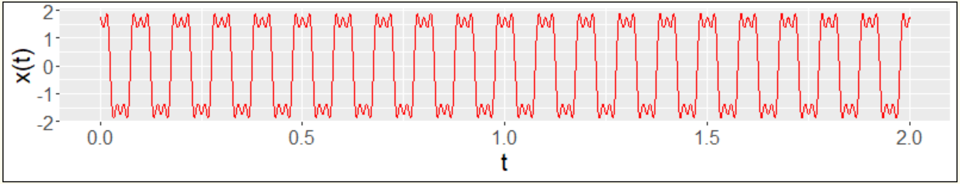
$$x(t) = 2\cos(2\pi(10)t) - \frac{2}{3}\cos(2\pi(30)t) + \frac{2}{5}\cos(2\pi(50)t) \tag{12}$$

Example of a periodic signal

Consider the signal

$$x(t) = 2\cos(2\pi(10)t) - \frac{2}{3}\cos(2\pi(30)t) + \frac{2}{5}\cos(2\pi(50)t) \tag{12}$$

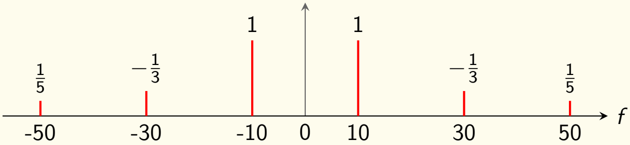


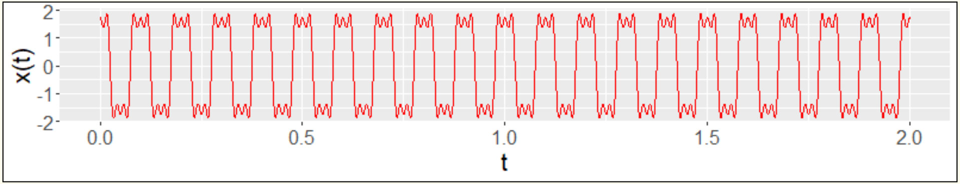


Example of a periodic signal

Consider the signal

$$x(t) = 2\cos(2\pi(10)t) - \frac{2}{3}\cos(2\pi(30)t) + \frac{2}{5}\cos(2\pi(50)t) \qquad (12)$$



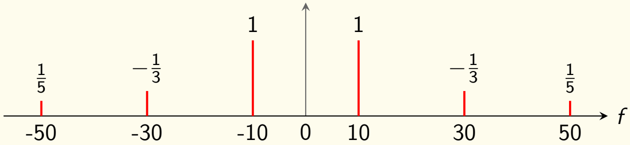


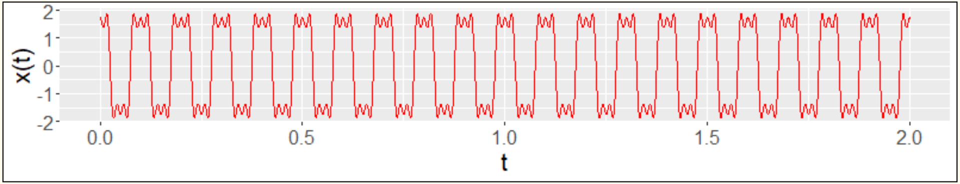
$x(t)$ is a periodic signal and its **fundamental frequency** $F_0 = \gcd(10, 30, 50) =$

Example of a periodic signal

Consider the signal

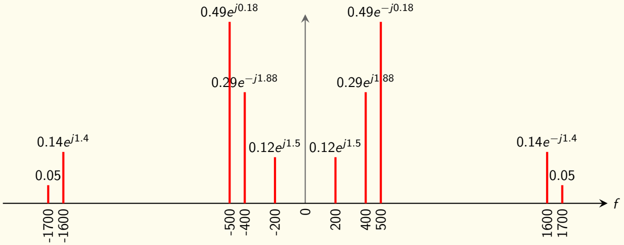
$$x(t) = 2\cos(2\pi(10)t) - \frac{2}{3}\cos(2\pi(30)t) + \frac{2}{5}\cos(2\pi(50)t) \tag{12}$$

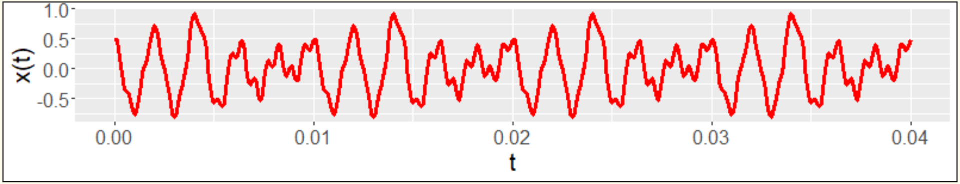




$x(t)$ is a periodic signal and its **fundamental frequency** $F_0 = \gcd(10, 30, 50) = 10$ Hz

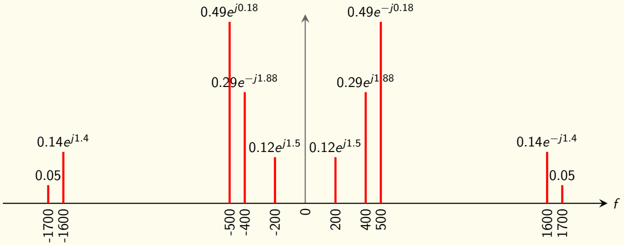
Synthetic vowel

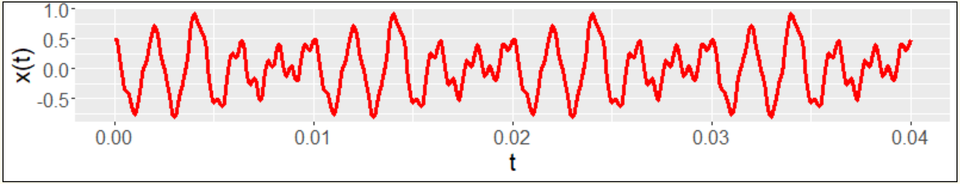




The synthetic vowel "ah" is a periodic signal and its **fundamental frequency** $F_0 = \gcd(200, 400, 500, 1600, 1700) =$

Synthetic vowel



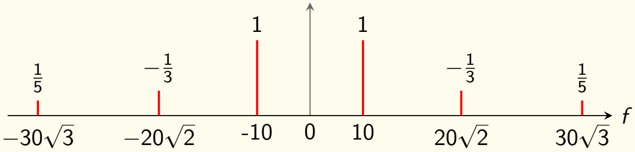


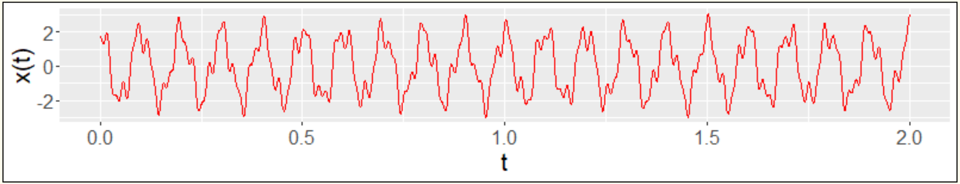
The synthetic vowel "ah" is a periodic signal and its **fundamental frequency** $F_0 = \gcd(200, 400, 500, 1600, 1700) = 100$ Hz

Example of a non-periodic signal

Consider the signal

$$x(t) = 2\cos(2\pi(10)t) - \frac{2}{3}\cos(2\pi(20\sqrt{2})t) + \frac{2}{5}\cos(2\pi(30\sqrt{3})t) \tag{13}$$

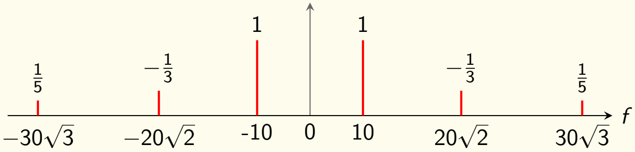


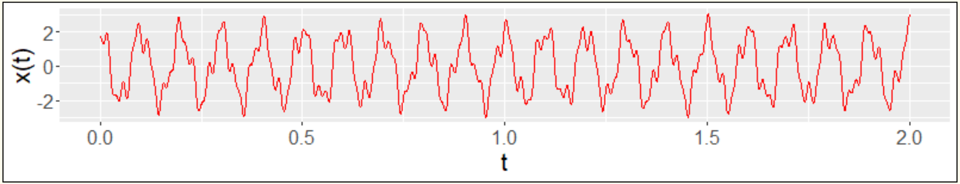


Example of a non-periodic signal

Consider the signal

$$x(t) = 2\cos(2\pi(10)t) - \frac{2}{3}\cos(2\pi(20\sqrt{2})t) + \frac{2}{5}\cos(2\pi(30\sqrt{3})t) \tag{13}$$





$x(t)$ is not a periodic signal.

Break!

See you at ______________________________

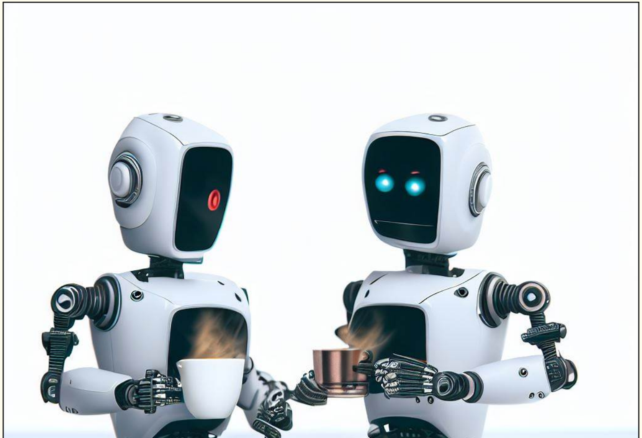


Table of Contents

Change in notation

We will use the following notation for convenience,

$$a_k = \begin{cases} \frac{A_k}{2} e^{-j\varphi}, & k < 0, \\ A_0, & k = 0, \\ \frac{A_k}{2} e^{j\varphi}, & k > 0. \end{cases} \tag{14}$$

A_0, also known as the **DC component**, is the amplitude associated with frequency 0

Fourier synthesis summation

Any periodic function can be expressed as a (possibly infinite) sum of harmonically related sinusoids

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{j2\pi F_0 kt} \\ &= \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt} \end{aligned} \tag{15}$$

Fourier synthesis summation

Any periodic function can be expressed as a (possibly infinite) sum of harmonically related sinusoids

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{j2\pi F_0 kt} \\ &= \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt} \end{aligned} \tag{15}$$

For any periodic function $x(t)$ with fundamental frequency F_0, there are a_k so that the **Fourier series** equals $x(t)$.

Fourier synthesis summation

Any periodic function can be expressed as a (possibly infinite) sum of harmonically related sinusoids

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{j2\pi F_0 kt} \\ &= \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt} \end{aligned} \tag{15}$$

For any periodic function $x(t)$ with fundamental frequency F_0, there are a_k so that the **Fourier series** equals $x(t)$.

- If a_k and F_0 are known, it is easy to generate $x(t)$

Fourier synthesis summation

Any periodic function can be expressed as a (possibly infinite) sum of harmonically related sinusoids

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{j2\pi F_0 kt} \\ &= \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt} \end{aligned} \tag{15}$$

For any periodic function $x(t)$ with fundamental frequency F_0, there are a_k so that the **Fourier series** equals $x(t)$.

- If a_k and F_0 are known, it is easy to generate $x(t)$
- If $x(t)$ is known, it is not straightforward to determine a_k

Fourier analysis

Definition

For a **periodic signal** with fundamental period $T_0 = \frac{1}{F_0}, F_0 > 0$, the **Fourier coefficients** are

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt, \text{ for all } k \in \mathbb{Z} \tag{16}$$

Example - square wave

Suppose we want to find the spectrum representation of the square wave with $T_0 = 2$.

$$x(t) = \begin{cases} 1, & 0 \leqslant t < T_0/2, \\ 0, & T_0/2 \leqslant t < T_0. \end{cases} \tag{17}$$

Let us compute the Fourier coefficients for this signal.

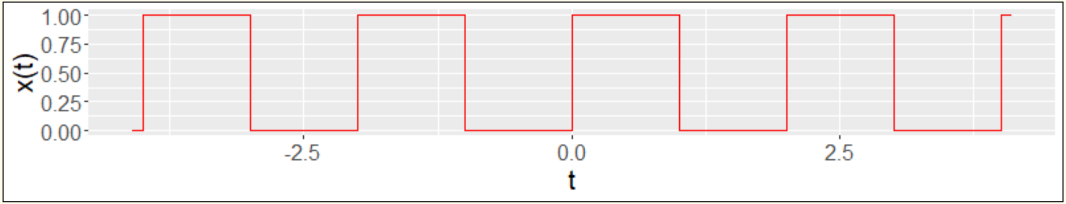


Figure: Time representation of example signal, $T_0 = 2$.

Example - square wave

$$x(t) = \begin{cases} 1, & 0 \leqslant t < T_0/2, \\ 0, & T_0/2 \leqslant t < T_0. \end{cases} \tag{18}$$

Using the Fourier series

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt$$

Example - square wave

$$x(t) = \begin{cases} 1, & 0 \leqslant t < T_0/2, \\ 0, & T_0/2 \leqslant t < T_0. \end{cases} \tag{18}$$

Using the Fourier series

$$\begin{aligned} a_k &= \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \\ &= \frac{1}{2}\int_0^{2} x(t)e^{-j(2\pi/2)kt}dt \end{aligned}$$

Example - square wave

$$x(t) = \begin{cases} 1, & 0 \leqslant t < T_0/2, \\ 0, & T_0/2 \leqslant t < T_0. \end{cases} \tag{18}$$

Using the Fourier series

$$\begin{aligned} a_k &= \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \\ &= \frac{1}{2}\int_0^{2} x(t)e^{-j(2\pi/2)kt}dt \\ &= \frac{1}{2}\int_0^{1} e^{-j\pi kt}dt \end{aligned}$$

For all $k \in \mathbb{Z}$.

Example - square wave

$$a_k = \frac{1}{2}\int_0^1 e^{-j\pi kt}dt \tag{19}$$

We need to analyze all the $k \in \mathbb{Z}$. So, let us divide the work in $k = 0$ and $k \neq 0$ as starting point.

Example - square wave

$$a_k = \frac{1}{2}\int_0^1 e^{-j\pi k t} dt \tag{19}$$

We need to analyze all the $k \in \mathbb{Z}$. So, let us divide the work in $k = 0$ and $k \neq 0$ as starting point. For $k = 0$, we have

$$a_0 = \frac{1}{2}\int_0^1 e^{-j\pi 0 t} dt$$

Example - square wave

$$a_k = \frac{1}{2}\int_0^1 e^{-j\pi k t} dt \tag{19}$$

We need to analyze all the $k \in \mathbb{Z}$. So, let us divide the work in $k = 0$ and $k \neq 0$ as starting point. For $k = 0$, we have

$$a_0 = \frac{1}{2}\int_0^1 e^{-j\pi 0 t} dt = \frac{1}{2}\int_0^1 1 dt$$

Example - square wave

$$a_k = \frac{1}{2}\int_0^1 e^{-j\pi kt}dt \tag{19}$$

We need to analyze all the $k \in \mathbb{Z}$. So, let us divide the work in $k = 0$ and $k \neq 0$ as starting point. For $k = 0$, we have

$$\begin{aligned} a_0 &= \frac{1}{2}\int_0^1 e^{-j\pi 0t}dt = \frac{1}{2}\int_0^1 1dt \\ &= \frac{1}{2}\, t\Big|_0^1 \end{aligned}$$

Example - square wave

$$a_k = \frac{1}{2}\int_0^1 e^{-j\pi kt}dt \tag{19}$$

We need to analyze all the $k \in \mathbb{Z}$. So, let us divide the work in $k = 0$ and $k \neq 0$ as starting point. For $k = 0$, we have

$$\begin{aligned} a_0 &= \frac{1}{2}\int_0^1 e^{-j\pi 0t}dt = \frac{1}{2}\int_0^1 1dt \\ &= \frac{1}{2}\left.t\right|_0^1 = \frac{1}{2}(1-0) = \frac{1}{2} \end{aligned}$$

The amplitude for the DC component in this signal is $\frac{1}{2}$.

Example - square wave

$$a_k = \frac{1}{2}\int_0^1 e^{-j\pi kt}dt \tag{20}$$

For $k \neq 0$, we have

$$a_k = \frac{1}{2}\int_0^1 e^{-j\pi kt}dt$$

Example - square wave

$$a_k = \frac{1}{2}\int_0^1 e^{-j\pi kt}dt \tag{20}$$

For $k \neq 0$, we have

$$\begin{aligned} a_k &= \frac{1}{2}\int_0^1 e^{-j\pi kt}dt \\ &= \frac{1}{2}\left.\frac{e^{-j\pi kt}}{-j\pi k}\right|_0^1 \end{aligned}$$

Since $\int e^{at}dt = \frac{1}{a}e^{at} + c$.

Example - square wave

$$a_k = \frac{1}{2}\int_0^1 e^{-j\pi kt} dt \tag{20}$$

For $k \neq 0$, we have

$$\begin{aligned} a_k &= \frac{1}{2}\int_0^1 e^{-j\pi kt} dt \\ &= \frac{1}{2}\left.\frac{e^{-j\pi kt}}{-j\pi k}\right|_0^1 = \frac{1}{2}\frac{e^{-j\pi k} - e^0}{-j\pi k} \end{aligned}$$

Example - square wave

$$a_k = \frac{1}{2}\int_0^1 e^{-j\pi kt}dt \tag{20}$$

For $k \neq 0$, we have

$$\begin{aligned} a_k &= \frac{1}{2}\int_0^1 e^{-j\pi kt}dt \\ &= \frac{1}{2}\left.\frac{e^{-j\pi kt}}{-j\pi k}\right|_0^1 = \frac{1}{2}\frac{e^{-j\pi k} - e^0}{-j\pi k} \\ &= \frac{1}{-j2\pi k}(\cos(-\pi k) + j\sin(-\pi k) - 1) \end{aligned}$$

Since $e^{j\theta} = \cos(\theta) + j\sin(\theta)$.

Example - square wave

$$a_k = \frac{1}{2} \int_0^1 e^{-j\pi kt} dt \tag{20}$$

For $k \neq 0$, we have

$$\begin{aligned}
a_k &= \frac{1}{2} \int_0^1 e^{-j\pi kt} dt \\
&= \frac{1}{2} \left. \frac{e^{-j\pi kt}}{-j\pi k} \right|_0^1 = \frac{1}{2} \frac{e^{-j\pi k} - e^0}{-j\pi k} \\
&= \frac{1}{-j2\pi k} (\cos(-\pi k) + j \sin(-\pi k) - 1) \\
&= \frac{1}{-j2\pi k} (\cos(-\pi k) - 1)
\end{aligned}$$

Example - square wave

$$a_k = \frac{1}{2}\int_0^1 e^{-j\pi kt}dt \tag{21}$$

For $k \neq 0$, we have (Continuation)

$$a_k = \frac{1}{-j2\pi k}(\cos(-\pi k) - 1)$$

Example - square wave

$$a_k = \frac{1}{2}\int_0^1 e^{-j\pi k t} dt \tag{21}$$

For $k \neq 0$, we have (Continuation)

$$\begin{aligned} a_k &= \frac{1}{-j2\pi k}(\cos(-\pi k) - 1) \\ &= \frac{j}{2\pi k}(\cos(-\pi k) - 1) \end{aligned}$$

Example - square wave

$$a_k = \frac{1}{2}\int_0^1 e^{-j\pi kt}dt \tag{21}$$

For $k \neq 0$, we have (Continuation)

$$\begin{aligned} a_k &= \frac{1}{-j2\pi k}(\cos(-\pi k) - 1) \\ &= \frac{j}{2\pi k}(\cos(-\pi k) - 1) \end{aligned}$$

Recall, $\cos(\pi k) = 1$ when k is even; and $\cos(\pi k) = -1$ when k is odd. So the Fourier coefficients are:

$$a_k = \begin{cases} \frac{1}{2}, & k = 0, \\ 0, & k \text{ is even}, \\ -\frac{j}{\pi k}, & k \text{ is odd}, \end{cases} \tag{22}$$

for all $k \in \mathbb{Z}$.

Example - square wave

The frecuency spectrum for the square wave signal

$$x(t) = \begin{cases} 1, & 0 \leqslant t < T_0/2, \\ 0, & T_0/2 \leqslant t < T_0, \end{cases} \tag{23}$$

is given by

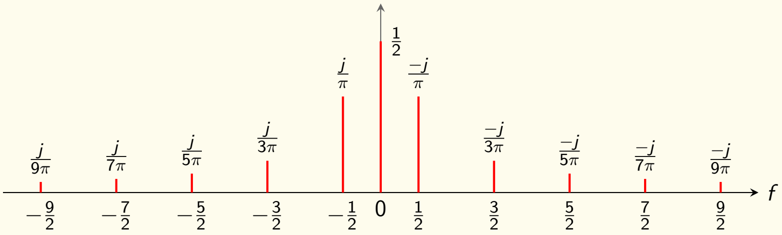


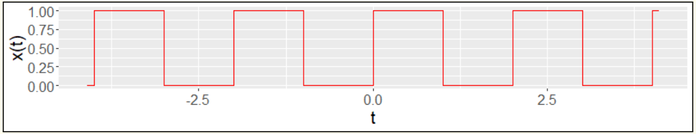
Figure: Frecuency spectrum of example signal.

Example - square wave

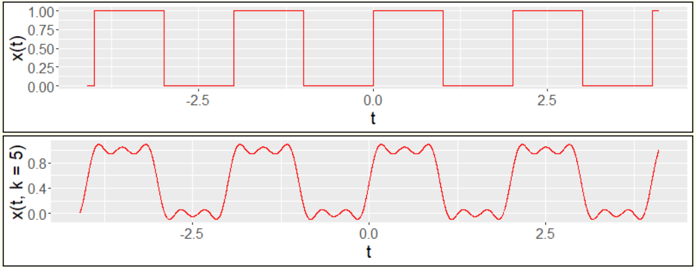
Question

Given the Fourier coefficients, we can represent the square function as a Fourier series, but our limited computational resources pose a challenge. What should we do?

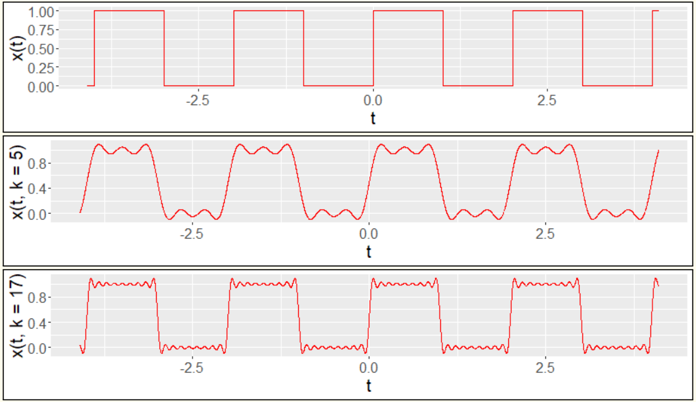
Fourier approximation - square wave



Fourier approximation - square wave



Fourier approximation - square wave



Fourier approximation - square wave

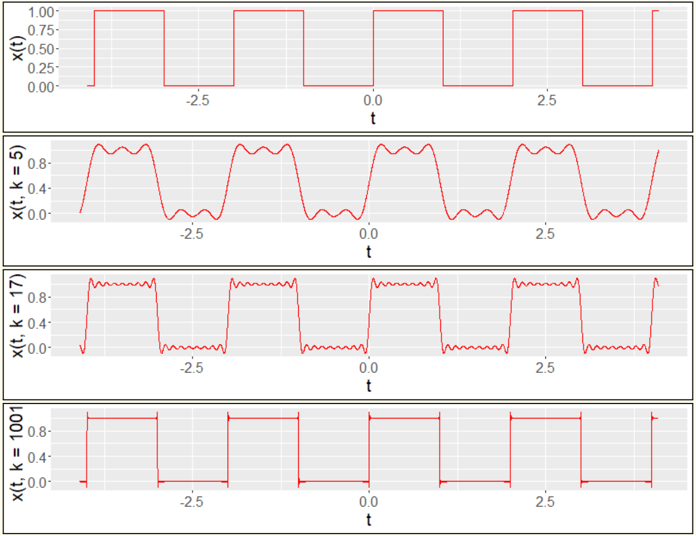


Figure: Approximation of a square wave signal with a finite number of Fourier coefficients

Fourier approximation

An approximation of a signal $x(t)$ can be obtained by taking a finite number of Fourier terms

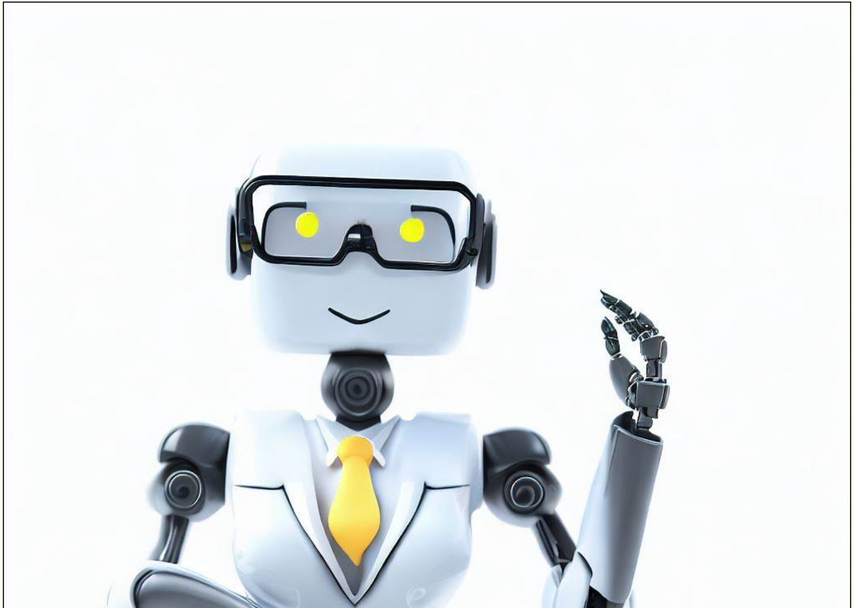
$$x_N(t) = \sum_{k=-N}^{N} a_k e^{j(2\pi/T_0)kt}, \; 0 < N < \infty \tag{24}$$

Approximations in this way may be useful for:

- signal lossy compression
- noise filtering

Table of Contents

Let us wrap up the lecture!



Take-home Messages

- A spectrum depicts a frequency-domain representation of a signal
  - Obtained using the inverse Euler relations
- If the sum of sinusoids is periodic, their frequencies are harmonically related

Take-home Messages

- Any periodic function can be expressed as a (possible infinite) sum of harmonically related sinusoids

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt} \tag{25}$$

- The values of a_k are obtained through

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{j(2\pi/T_0)kt} dt, \text{ for all } k \in \mathbb{Z} \tag{26}$$

- An approximation of the signal $x(t)$ can be obtained by taking a finite sum

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{j(2\pi/T_0)kt} \tag{27}$$

Practice Questions

The following questions might appear in the final exam:

- What is the time-domain equation for the continuous signal whose frequency-domain representation is given by (28)?

$$\left\{(0,2),(-30,e^{-j2\pi/7}),(30,e^{j2\pi/7})\right\} \tag{28}$$

- What are the Fourier coefficients of the signal in (29)?

$$x(t) = 4\sin(2\pi 20t)\cos(2\pi 35t) \tag{29}$$

- Is the signal in (30) periodic? If so, what is its fundamental period?

$$x(t) = 2\sqrt{2}\cos(2\pi 25t) + 5\sin(2\pi 20t) \tag{30}$$

Tutorial exercises

During the tutorial, the exercises below will be discussed in class

- Attempt to complete the exercises **before** class starts
- As the weeks progress, more time is needed for explanation

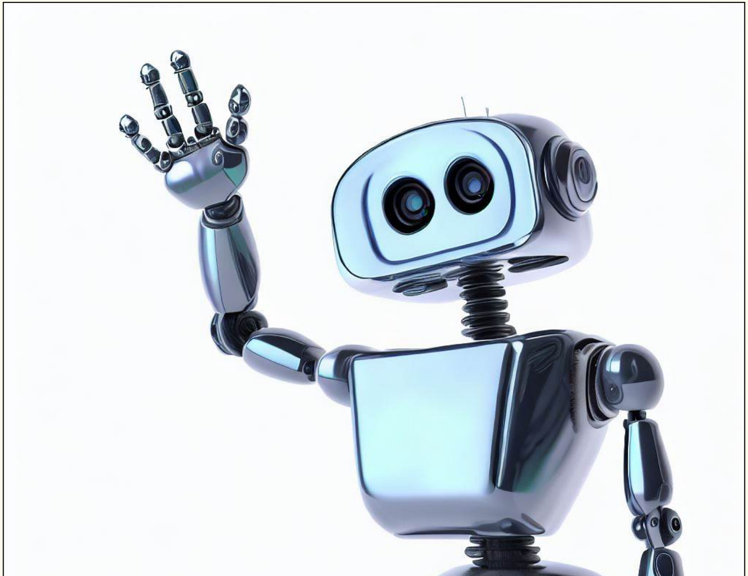
SPF	DSPF
Ex 3.3 (p. 44)	Ex 3.5 (p. 88)
P 3.1 (p. 64)	P 3.2 (p. 111)
P 3.2 (p. 64)	P 3.1 (p. 110)
P 3.3 (p. 64)	P 3.4 (p. 111)
P 3.7 (p. 65)	P 3.10 (p. 113)
P 3.8 (p. 65)	P 3.11 (p. 113)

Closing Remarks: Next Lecture

Let us discuss how to discretize a continuous signal.

Sampling and aliasing

Have a nice day!



Acknowledgements

The material for this lecture series was developed by dr. Arnold Meijster and dr. Harmen de Weerd and modified by Juan Diego Cardenas-Cartagena.

Disclaimer

- Grammar was checked with Grammarly and Grammar checker GPT.
- Images without source were created with the assistance of DALL·E.