Tutorial 2 – Spectrum Representation, Periodic Signals, and the Fourier Series

1. *(p.111, ex. P-3.2)* A signal composed of sinusoids is given by the equation

$$x(t) = 16\cos(500\pi t + \pi/4) + 9\cos(1000\pi t - \pi/3) - 5\cos(750\pi t).$$

(a) Sketch the spectrum of this signal, indicating the complex amplitude of each frequency component

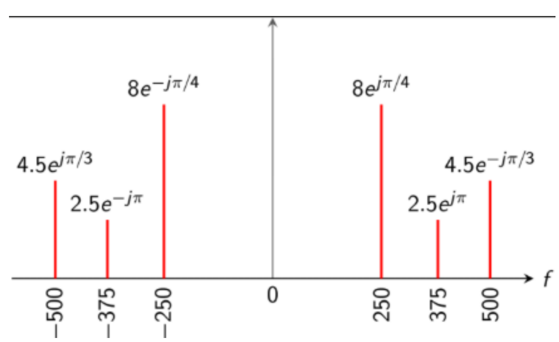
Solution: Using $\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$ we can rewrite the signal as

$$\begin{aligned} x(t) &= 8e^{j\pi/4}e^{j2\pi 250t} + 8e^{-j\pi/4}e^{-j2\pi 250t} \\ &+4.5e^{-j\pi/3}e^{j2\pi 500t} + 4.5e^{j\pi/3}e^{-j2\pi 500t} \\ &-2.5e^{j2\pi 375t} - 2.5e^{-j2\pi 375t}. \end{aligned}$$

By using that $-\cos\theta = \cos(\theta-\pi) = \cos(\theta+\pi)$ we obtain

$$\begin{aligned} x(t) &= 8e^{j\pi/4}e^{j2\pi 250t} + 8e^{-j\pi/4}e^{-j2\pi 250t} \\ &+4.5e^{-j\pi/3}e^{j2\pi 500t} + 4.5e^{j\pi/3}e^{-j2\pi 500t} \\ &+2.5e^{j\pi}e^{j2\pi 375t} + 2.5e^{-j\pi}e^{-j2\pi 375t}. \end{aligned}$$

We sketch the spectrum as



(b) Is $x(t)$ periodic? If so, what is the period?

Solution:

$$T_0 = 1/\gcd(250, 375, 500) = 1/125 = 0.008\text{s}$$

(c) Now consider a new signal defined as

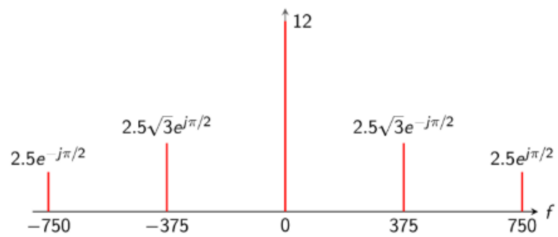
$$y(t) = 12 - 5\sqrt{3}\cos(750\pi t + \pi/2) + 5\cos(1500\pi t + \pi/2).$$

Draw the spectrum of this new signal. Is y(t) periodic? If so, what is the period?

Solution:

$$y(t) = 12 + 2.5\sqrt{3}e^{-j\pi/2}e^{j2\pi 375t} + 2.5\sqrt{3}e^{j\pi/2}e^{-j2\pi 375t} \\ +2.5e^{j\pi/2}e^{j2\pi 750t} + 2.5e^{-j\pi/2}e^{-j2\pi 750t},$$

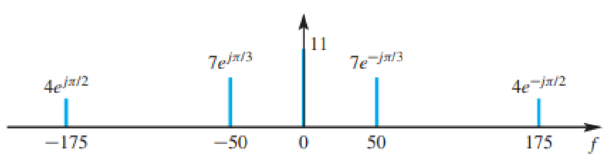
where again we performed a shift by π so that the minus is removed and the phase remains in $(-\pi, \pi]$.



The period is

$$T_0 = 1/\text{gcd}(375, 750) = 1/375\text{s}.$$

2. *(p. 110, ex. P-3.1)* A signal $x(t)$ has the two-sided spectrum representation shown in the figure below.



(a) Write an equation for $x(t)$ as a sum of cosines.

Solution: The equation can be read from the figure as

$$x(t) = 11 + 14\cos(2\pi 50t - \pi/3) + 8\cos(2\pi 175t - \pi/2).$$

(b) Is $x(t)$ a periodic signal? If so, determine its fundamental period and its fundamental frequency.

Solution: It is periodic with

$$F_0 = \text{gcd}(50, 175) = 25Hz \iff T_0 = 1/25 = 0.04\text{s}$$

(c) Explain why negative frequencies are needed in the spectrum.

Solution: The negative frequencies are a direct result of Euler's formula, and are needed to eliminate the imaginary part of the signal. For complex signals, the plot is not mirrored around the vertical axis.

3. *(p.113, ex. P-3.11)* A periodic signal is given by the equation

$$x(t) = 2 + 4\cos(40\pi t - \pi/5) + 3\sin(60\pi t) + 4\cos(120\pi t - \pi/3)\,.$$

(a) Determine the fundamental frequency ω_0, the fundamental period T_0, the number of terms N, and the coefficients a_k in the finite Fourier representation for the signal $x(t)$ above. It is possible to do this without evaluating any integrals.

Solution:

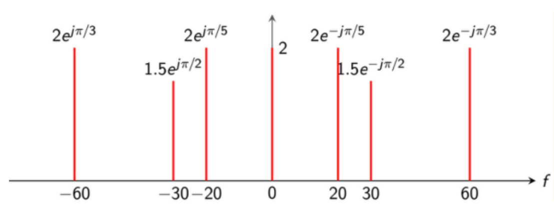
$$\omega_0 = \gcd(40\pi, 60\pi, 120\pi) = 20\pi \iff T_0 = \frac{2\pi}{\omega_0} = 0.1\text{s}.$$

The coefficients can be directly seen as

- $a_0 = 2$
- $a_{\pm 2} = 2e^{\mp j\pi/5}$
- $a_{\pm 3} = 1.5e^{\mp j\pi/2}$ as $\sin\theta = cos(\theta - \pi/2)$, also correct with sign flipped.
- $a_{\pm 6} = 2e^{\mp \pi/3}$

(b) Sketch the spectrum of this signal indicating the complex amplitude of each frequency component.

Solution:



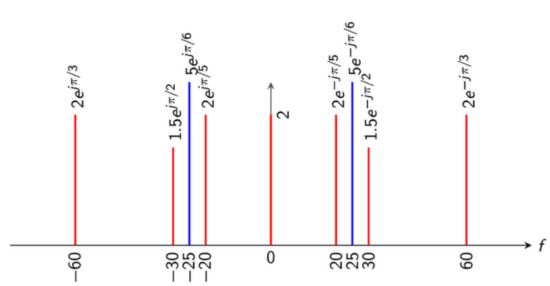
(c) Now, consider the signal

$$y(t) = x(t) + 10\cos(50\pi t - \pi/6).$$

How is the spectrum changed? Is $y(t)$ still periodic? If so, what is the fundamental period? Plot it.

Solution:

$$\omega_0 = \gcd(20\pi, 50\pi) = 10\pi \iff T_0 = \frac{2\pi}{\omega_0} = 0.2\text{s}.$$



4. *(p. 88 ex. 3.5)* Show that one possible period of the complex exponential signal

$$v_k(t) = e^{j2\pi k F_0 t}$$

is $T_0 = \frac{1}{F_0}$ and that the fundamental period is $\frac{1}{kF_0}$.

> **Solution:** Need to show that $v_k(t) = v_k(t + T_0)$. For $T_0 = \frac{1}{F_0}$ we have
>
> $$v_k(t+T_0) = e^{j2\pi k F_0 (t+T_0)} = e^{j2\pi k F_0 t} e^{j2\pi k F_0 T_0} = e^{j2\pi k F_0 t} e^{j2\pi k} = e^{j2\pi k F_0 T_0} = e^{j2\pi k F_0 t},$$
>
> where we used that $F_0 T_0 = 1$ and $e^{j2\pi k} = (e^{j2\pi})^k = 1^k = 1$. So, T_0 is a possible period.
> Suppose that $T_0 = \frac{1}{kF_0}$ is not the fundamental period, meaning there exists another period T_1 such that T_0 is a multiple of T_1 for some $n \in \mathbb{N}$. Hence, $T_1 = \frac{1}{nT_0} = \frac{1}{nkF_0}$. Now consider
>
> $$v_k(t+T_1) = e^{j2\pi k F_0 (t+T_0/n)} = e^{j2\pi k F_0 t} e^{j2\pi k F_0 T_0/n} = e^{j2\pi k F_0 t} e^{j2\pi/n}.$$
>
> We require
>
> $$e^{j2\pi/n} = 1 \iff \cos(2\pi/n) + j\sin(2\pi/n) = 1 \iff n = 1.$$
>
> So, $T_1 = \frac{T_0}{1} = T_0$. Hence, T_0 is the fundamental period.

5. *(p.113, P-3.10)* Consider a signal $x(t)$ such that

$$x(t) = 2\cos(\omega_1 t) cos(\omega_2 t) = cos[(\omega_2 + \omega_1)t] + cos[(\omega_2 - \omega_1)t],$$

for $0 < \omega_1 < \omega_2$.

(a) What is the general condition that must be satisfied by $\omega_2 - \omega_1$ and $\omega_2 + \omega_1$ so that $x(t)$ is periodic with period T_0?

> **Solution:** The signal is periodic if $\gcd(\omega_2 + \omega_1, \omega_2 - \omega_1)$ exists.

(b) What does the result of (a) imply about ω_1 and ω_2? For example, must ω_2 be an integer multiple of ω_1?

> **Solution:** The signal is periodic with frequency ω_0 when there are integers $n > m > 0$ such that
>
> $$\omega_2 - \omega_1 = m\omega_0 \text{ and } \omega_2 + \omega_1 = n\omega_0.$$

6. *(p. 111, ex. P-3.4)* Define $x(t)$ as

$$x(t) = \sin^3(54\pi t) + \sin^2(36\pi t)$$

(a) Determine a formula for $x(t)$ as the real part of a sum of complex exponentials.

Solution:

$$\sin^3(54\pi t) + \sin^2(36\pi t) = \cos^3(54\pi t - \pi/2) + \cos^2(36\pi t - \pi/2) =$$
$$\frac{1}{8}\left(e^{-j\pi/2}e^{j54\pi t} + e^{j\pi/2}e^{-j54\pi t}\right)^3 + \frac{1}{4}\left(e^{-j\pi/2}e^{j36\pi t} + e^{j\pi/2}e^{-j36\pi t}\right)^2.$$

After many rows of unnecessary computations and expanding powers,

$$x(t) = \frac{1}{8}e^{j\pi/2}e^{j162\pi t} + \frac{3}{8}e^{-j\pi/2}e^{j54\pi t} + \frac{3}{8}e^{j\pi/2}e^{-54\pi t} + \frac{1}{8}e^{-j\pi/2}e^{-j162\pi t} +$$
$$\frac{1}{4}e^{j\pi}e^{j72\pi t} + \frac{1}{4}e^{j\pi}e^{-j72\pi t} + \frac{1}{2}.$$

(b) Determine the fundamental period for $x(t)$.

Solution:

$$T_0 = 1/\gcd(27, 36, 81) = 1/9\text{s}$$

(c) Plot the spectrum for $x(t)$.

Solution: