Tutorial 3 – Sampling and Aliasing

1. *Warm-up* Prove the identity

$$\cos(\theta)\cos(\phi) = \frac{1}{2}(\cos(\theta+\phi) + \cos(\theta-\phi)).$$

> **Solution:** During the first tutorial we showed
>
> $$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta).$$
>
> .
>
> Writing the respective equations for $\cos(\alpha+\beta)$ and $\cos(\alpha-\beta)$ and summing them, we obtain the desired equality.

2. *p. 159, ex. P-4.11* Consider the wave

$$x(t) = 3\cos(800\pi t),$$

sampled at a rate f_s, after which we obtain the discrete-time signal

$$x[n] = x(n/f_s) = 3\cos(800\pi n/f_s)$$

for $-\infty < n < \infty$. In the following parts, assume that $f_s = 3600$ Hz.

(a) Determine how many samples are taken in one period of the cosine wave $x(t)$. This answer is the average number of samples per period, which is an integer in this case.

> **Solution:** The frequency of $x(t)$ is 400 Hz. This means that there are $3600/400 = 9$ samples per period of $x(t)$.

(b) Now consider another cosine waveform $y(t)$ with a different frequency ω_0:

$$y(t) = 3\cos(\omega_0 t).$$

Find a value for ω_0, between 7000π and 9999π rad/s, such that the signal samples are identical to $x[n]$ above, that is, $y[n] = y(n/f_s) = x(n/f_s)$ for all n.

> **Solution:** We want $\omega_0 \in (7000\pi, 9999\pi)$ such that
>
> $$y[n] = 3\cos((\omega_0/3600)n) = x[n] = 3\cos((800/3600)\pi n).$$
>
> Since cosine waves are periodic with period $2k\pi$ for $k \in \mathbb{Z}$, we have
>
> $$(\omega_0/3600)n = (800/3600)\pi n + 2k\pi n \iff w_0 = 800\pi + 2k(3600)\pi.$$
>
> The required condition is satisfied for $k = 1$, resulting in
>
> $$\omega_0 = 800\pi + 7200\pi = 8000\pi.$$

(c) For the frequency found in b), determine the average number of samples taken in one period of $y(t)$.

> **Solution:** Since $\omega_0 = 8000\pi$, $f_0 = 4000$. Then, there are 3600/4000 samples per one period.

3. *p. 155, ex. P-4.1* Let

$$x(t) = 10\cos(9\pi t - \pi/5).$$

A discrete time signal $x[n]$ is obtained by sampling $x(t)$ at a rate f_s samples/s. For each part, write the general form of $x[n]$

$$x[n] = A\cos(\omega_0 n + \phi).$$

and state whether the signal has been undersampled or oversampled. Where does folding occur?

(a) $f_s = 11$ samples/s

> **Solution:** The general form is
>
> $$x[n] = A\cos((9/f_s)\pi n + \phi),$$
>
> where $A = 10$, $\phi = -\pi/5$, and f_s corresponds to subquestions a), b), or c). For $f_s = 11$, the signal is oversampled ($f_0 = 4.5$ and $11 > 2f_0$).

(b) $f_s = 7$ samples/s

> **Solution:** $7 < 2f_0 = 9$, so the signal is undersampled. Folding occurs.

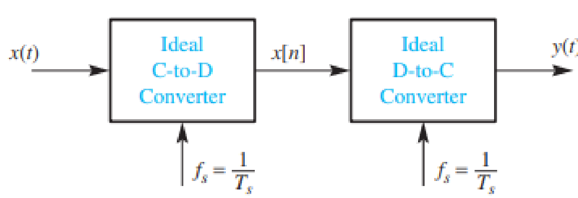
(c) $f_s = 4$ samples/s

> **Solution:** $4 < 2f_0 = 9$, so the signal is undersampled. Folding occurs.

4. *p. 160, ex. P-4.14* An amplitude-modulated (AM) cosine wave is represented by the formula

$$x(t) = [4 + \sin(6600t)]\cos(2000\pi t).$$

Sketch the two-sided spectrum of this signal. Is the signal periodic? What relation must the sampling rate f_s satisfy so that it is an ideal C-to-D-to-C converter in the figure below?



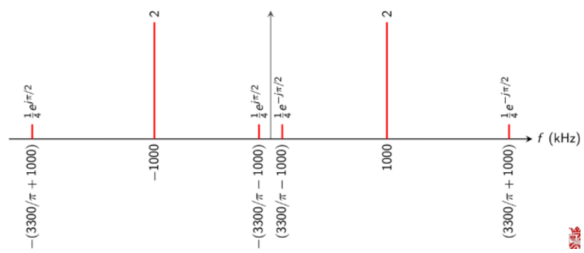
Solution: Using the identities

$$\sin(\theta) = \cos(\theta - \pi/2)$$

$$\cos(\theta)\cos(\phi) = \frac{1}{2}(\cos(\theta + \phi) + \cos(\theta - \phi)),$$

we obtain that

$$\begin{aligned} x(t) = [4 + \sin(6600t)]\cos(2000\pi t) = \\ 4\cos(2000\pi t) + \frac{1}{2}\cos((6600 + 2000\pi)t - \pi/2) + \frac{1}{2}\cos((6600 - 2000\pi)t - \pi/2) = \\ 4\cos(2\pi(1000)t) + \frac{1}{2}\cos(2\pi(3300/\pi + 1000)t - \pi/2) + \frac{1}{2}\cos(2\pi(3300/\pi - 1000)t - \pi/2) \end{aligned}$$



The signal is not periodic as $\gcd(3300/\pi - 1000, 3300/\pi + 1000)$ does not exist.

To avoid aliasing any of the components, we need $f_s > 2f_{max} = 6600 + 2000\pi$ Hz.

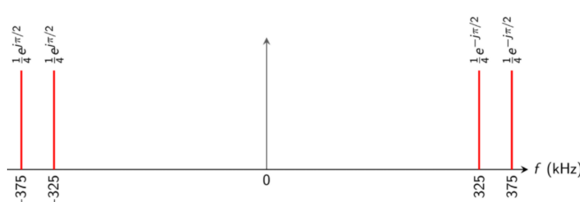
5. *P.48* Draw and sketch the spectrum of the signal

$$x(t) = \cos(50\pi t)\sin(700\pi t).$$

Is the signal periodic? Determine the minimum sampling rate to sample $x(t)$ without aliasing any of the components.

Solution:

$$\begin{aligned} x(t) = \cos(2\pi(25)t)\cos(2\pi(350)t - \pi/2) = \\ \frac{1}{2}\cos(2\pi(325)t - \pi/2) + \frac{1}{2}\cos(2\pi(375)\pi t - \pi/2) \end{aligned}$$



It is periodic with $f_0 = 25$. f_s should be at least $2 \times 375 = 750$.

6. A chirp signal is one that sweeps in frequency from $\omega_1 = 2\pi f_1$ to $\omega_2 = 2\pi f_2$ as time goes from $t = 0$ to $t = T_2$. The general formula of a chirp signal is

$$x(t) = A\cos(\alpha t^2 + \beta t + \phi) = A\cos(\psi(t)).$$

The instantaneous frequency $w_i(t)$ is the derivative of $\psi(t)$, which is also heard if the frequencies are in the audible range.

(a) Given an chirp arbitrary signal,

$$x(t) = \Re\left\{e^{j(\alpha t^2+\beta t+\phi)}\right\},$$

determine its instantaneous frequency in terms of time.

Solution:

$$\nabla\psi(t) = 2\alpha t + \beta.$$

(b) Determine formulas for ω_1 and ω_2 in terms of α, β, and T_2.

Solution: We have that

$$\omega_1 = \omega_i(0) = \beta$$
$$\omega_2 = \omega_i(T_2) = 2\alpha T_2 + \beta.$$

(c) For the signal

$$x(t) = \Re\left\{e^{j(40t^2+27t+14)}\right\},$$

plot the instantaneous frequency in Hz versus time over the range $0 \leq t \leq 1$.

Solution: By substituting $\alpha = 40$ and $\beta = 27$, from a) and b), we get that it is just a straight line with some slope and bias, namely

$$\omega_i = 80t + 27.$$

Using what we learned in kindergarten, we obtain the plot.