university of groningen / faculty of science and engineering

Lecture 8: Laplace Transform

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Signals and Systems
1B - 2024/2025

Preliminaries

- The deadline for Lab 3 is on Friday, January 17, at 17:30.
- The LSP for Lab 3 runs until Wednesday, January 22, at 23:59.
- Remember to submit your **code in Themis and documentation in Brightspace.**
- We will hold an **online Q&A session on Friday**, January 17, at 9:00. The link will be posted on Brightspace. As a remark, this is not a lecture but a Q&A session, so **prepare questions beforehand.**
- Remember to evaluate the course on Brightspace.
- Recall the final grade calculation:

$$\text{Final Grade} = \begin{cases} \min(ASM, WE), & \text{if ASM} < 5 \text{ or WE} < 5, \\ 0.25(ASM) + 0.75(WE), & \text{otherwise,} \end{cases}$$

  where ASM is the average grade on assignaments and WE is the written final exam grade.

Preliminaries

- Material to prepare for the Final Exam is in the Useful Resources tab on Brightspace.
- You can bring an A4 with notes on both sides to the Final Exam.
- Use it wisely! We will provide useful formulas as well.
- Please check **Rooster** for the date and time of the Final Exam.

And finally, remember to make
the final exam for the **course Signals and Systems (for AI)!**

Some tips for the Final Exam:

- Use the formulas and table as partial hints to verify your process.
- If you encounter a challenging operation or an angle not listed in the table, assign a variable to it and proceed with your process, e.g., $a = \sin(\pi/12)$ or $b = \frac{\sqrt{5}-2}{\sqrt{7}-1}$. Remember, both the final answer and **the process** are key components of your grade.
- If the process seems overly complicated, consider exploring a **simpler strategy**. Aim to keep your approach straightforward, especially for questions worth fewer than 5 points. As shown in the answer sheet, the expected process for most questions is relatively short.

(Cont.) Some tips for the Final Exam:

- There is no need to solve the questions in order. In fact, it is recommended to start with the ones you feel most confident about. **Read the whole exam first!**
- Use the practice exam to familiarize yourself with the types of questions and develop a strategy.
- Consider that the evaluator (in singular) will grade a large volume of questions (2400+) within a short time-frame (2 weeks). **Please make your responses easy to understand and grade.**
- And the most important tip: **Stay calm and focused!**


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NATIONALE STUDENTEN ENQUETE

Financial ML Reading Group

Figure: Financial ML Reading Group - The sign-ups for the reading group will open on the 6th of January and close on the 24th. We look forward to seeing you at the first meeting!

Overview

1. Recap
2. The Continuous Time
3. The Laplace Transform
4. ODEs and the Laplace Transform
5. Stability Analysis in the Continuous Domain
6. Applications of the Laplace Transform
7. Closing Remarks

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Recap

The z-transform is a generalization of the frequency response in discrete time.

- Represents signals and systems as a polynomial for the complex variable z.
- The z-transform of a length L-length signal $x[n]$ is

$$X(z) = \sum_{k=0}^{L-1} x[k] z^{-k}, \tag{1}$$

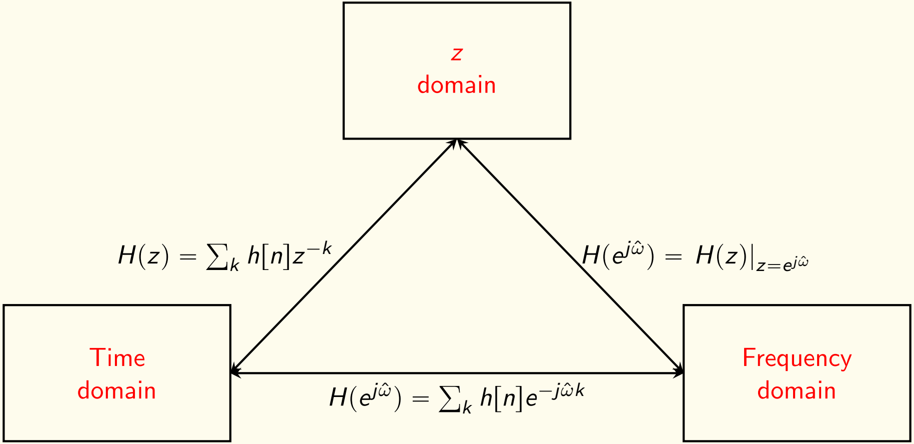
- The system function is the z-transform of a system

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \sum_{k=0}^{M} h[k] z^{-k} \tag{2}$$

- The relationship between the frequency domain and the z-domain is given by

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}} \tag{3}$$

Recap



z
domain
$H(z) = \sum_k h[n] z^{-k}$
$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$
Time
domain
Frequency
domain
$H(e^{j\hat{\omega}}) = \sum_k h[n] e^{-j\hat{\omega} k}$

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The Continuous Fourier Transform

Consider a continuous-time signal $x(t)$. The Continuous Fourier transform (CDT) for $x(t)$ is defined as

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt. \tag{4}$$

The resulting $X(\omega)$ is the representation of the signal in the (continuous) frequency domain. The inverse CFT of $X(\omega)$ is given by

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega. \tag{5}$$

A relevant interpretation for (5) is that it tells us the weight importance represented by $X(\omega)$ of each frequency ω in the original signal $x(t)$.

Convolution in the Continuous Time

The convolution of two continuous time signals $x(t)$ and filter $h(t)$ is defined as

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau. \tag{6}$$

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$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau. \tag{6}$$

In the continuous-time domain, convolution is equivalent to multiplication in the frequency domain. In particular, the Fourier transform of the convolution of two signals equals the product of their Fourier transforms, i.e.,

$$y(t) = h(t) * x(t) \quad \longleftrightarrow Y(\omega) = H(\omega)X(\omega),$$

where $Y(\omega)$, $X(\omega)$, and $H(\omega)$ are the Fourier transforms of $y(t)$, $x(t)$, and $h(t)$, respectively.

LTI Systems and ODEs

- In discrete time, we saw that LTI systems can be represented by difference equations.
- In continuous time, Causal LTI systems can be represented by ordinary differential equations (ODEs).
- The general form of a causal LTI system in continuous time as an ODE is given by

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k},$$

  where $x(t)$ is the input, $y(t)$ is the output, and a_k and b_k are the system coefficients.
- The system above needs initial conditions to be fully defined!

LTI Systems and ODEs - Example

Consider the following LTI system in continuous time:

$$a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t).$$

The solution for the system above is given by

$$y(t) = \frac{b_0}{a_1} \int_{-\infty}^{t} x(\tau) e^{\frac{a_0}{a_1}(t-\tau)} d\tau.$$

LTI Systems and ODEs - Example

Consider the following LTI system in continuous time:

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The solution for the system above is given by

$$y(t) = \frac{b_0}{a_1} \int_{-\infty}^{t} x(\tau) e^{\frac{a_0}{a_1}(t-\tau)} d\tau.$$

Now, let us compute the impulse response of the system, i.e., $x(t) = \delta(t)$,

$$y(t) = \frac{b_0}{a_1} e^{-\frac{a_0}{a_1} t} u(t).$$

Does it look familiar from the discrete-time case?

Under what conditions the system is stable (bounded output)?

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The Laplace Transform

The Laplace transform is a generalization of the Fourier transform. It is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt, \tag{7}$$

where s is a complex number.

The Laplace transform is a generalization of the continuous Fourier transform. One can simplify the definition from above by looking at the **one-side Laplace transform**, which is defined as

$$X(s) = \int_{0}^{\infty} x(t) e^{-st} dt, \tag{8}$$

under the assumption that $x(t) = 0$ for $t < 0$. We denote the one-side Laplace transform as $X(s) = \mathcal{L}\{x(t)\}$.

The inverse Laplace Transform

The inverse Laplace transform is defined as

$$x(t) = \lim_{T\to\infty} \frac{1}{2\pi j} \int_{\gamma - jT}^{\gamma + jT} X(s) e^{st} ds, \tag{9}$$

where γ is a real number such that the integral converges. In practice, we use already-known Laplace transforms to find the inverse Laplace transform. We denote the inverse Laplace transform as $x(t) = \mathcal{L}^{-1}\{X(s)\}$.

Relation between the Laplace and Fourier Transforms

The Fourier transform is a special case of the Laplace transform when $s = j\omega$. That is, the Fourier transform of a signal $x(t)$ is given by

$$X(j\omega) = X(s)|_{s=j\omega}$$

Relation between the Laplace and Fourier Transforms

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Recall: In the discrete time, the z variable is related to the frequency domain by $X(e^{j\hat{\omega}}) = X(z)|_{z=e^{j\hat{\omega}}}$. This slightly different relation will have relevant implications for the poles and zeros of the Laplace transform.

Let us take a closer look at this relation. Consider the Laplace transform of a signal $x(t)$, $X(s)$, and recall that the complex variable $s = \sigma + j\omega$. Then,

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt \\ &= \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt \\ &= \int_{-\infty}^{\infty} (x(t)e^{-\sigma t})e^{-j\omega t}dt \\ &= \mathcal{F}\{x(t)e^{-\sigma t}\}. \end{aligned}$$

Let us take a closer look at this relation. Consider the Laplace transform of a signal $x(t)$, $X(s)$, and recall that the complex variable $s = \sigma + j\omega$. Then,

$$
\begin{aligned}
X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt \\
&= \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt \\
&= \int_{-\infty}^{\infty} (x(t)e^{-\sigma t})e^{-j\omega t}dt \\
&= \mathcal{F}\{x(t)e^{-\sigma t}\}.
\end{aligned}
$$

The Laplace transform of a signal $x(t)$ can be seen as the Fourier transform of the signal $x(t)e^{-\sigma t}$.

Properties of the Laplace Transform

Linearity

$$\mathcal{L}\{ax_1(t) + bx_2(t)\} = aX_1(s) + bX_2(s)$$

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Time Shifting

$$\mathcal{L}\{x(t - t_0)\} = e^{-st_0}X(s)$$

Frequency Shifting

$$\mathcal{L}\{e^{at}x(t)\} = X(s - a)$$

Note: The proof of these properties comes from the definition of the Laplace transform and the properties of the exponential function.

Properties of the Laplace Transform

Convolution

Furthermore, a convolution in the time domain is equivalent to a multiplication in the Laplace domain. That is,

$$y(t) = x(t) * h(t) \quad \longleftrightarrow Y(s) = X(s)H(s),$$

Properties of the Laplace Transform

Derivative

The Laplace transform of a derivative of a signal is given by

$$\mathcal{L}\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s) - s^{n-1}x(0) - s^{n-2}\frac{dx(0)}{dt} - \ldots - \frac{d^{n-1}x(0)}{dt^{n-1}},$$

with initial conditions $x(0), \frac{dx(0)}{dt}, \ldots, \frac{d^{n-1}x(0)}{dt^{n-1}}$.

Properties of the Laplace Transform

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with initial conditions $x(0), \frac{dx(0)}{dt}, \ldots, \frac{d^{n-1}x(0)}{dt^{n-1}}$.

Integral

The Laplace transform of an integral of a signal is given by

$$\mathcal{L}\left\{\int_0^t x(\tau)d\tau\right\} = \frac{1}{s}X(s).$$

Function	**Time Domain**	**s-Domain**
Unit pulse	$\delta(t)$	1
Delayed pulse	$\delta(t-\tau)$	$e^{-\tau s}$
Unit step	$u(t)$	$\frac{1}{s}$
Ramp	$tu(t)$	$\frac{1}{s^2}$
Exponential decay	$e^{-at}u(t)$	$\frac{1}{s+a}$
Sine	$\sin(\omega t)u(t)$	$\frac{\omega}{s^2+\omega^2}$
Cosine	$\cos(\omega t)u(t)$	$\frac{s}{s^2+\omega^2}$
Exponentially Decaying Sine	$e^{-at}\sin(\omega t)u(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
Exponentially Decaying Cosine	$e^{-at}\cos(\omega t)u(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Table: Relevant Laplace Transform Pairs

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ODEs and the Laplace transform

As discussed earlier, an ODE (with well-defined initial conditions)

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k},$$

in the s-domain is a polynomial equation in s of the form

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s),$$

where $X(s)$ and $Y(s)$ are the Laplace transforms of $x(t)$ and $y(t)$, respectively. And $\{a_k\}_{k=0}^{N}$ and $\{b_k\}_{k=0}^{M}$ are the system coefficients.

Transfer Function

Let us define the transfer function of the system as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}.$$

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Then, solving the ODE in the s-domain is equivalent to solving a polynomial equation in s. The solution of the ODE in the s-domain is given by

$$Y(s) = H(s)X(s).$$

And come back to the time domain by taking the inverse Laplace transform of $Y(s)$.

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And come back to the time domain by taking the inverse Laplace transform of $Y(s)$.

Note: In practice, computing the inverse Laplace transform involves using tables (along with techniques such as partial fraction decomposition) or software with symbolic computation capabilities, e.g., MATLAB, Mathematica, or Python-sympy.

Example

Consider again the LTI system

$$a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_0 x(t),$$

with initial conditions $y(0) = 0$ and $x(0) = 0$.

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The transfer function of the system is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_0}{a_1 s + a_0}.$$

Example

The output of the system is given by

$$Y(s) = H(s)X(s) = \frac{b_0}{a_1 s + a_0} X(s).$$

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$$Y(s) = H(s)X(s) = \frac{b_0}{a_1 s + a_0} X(s).$$

Let $x(t) = \delta(t)$. Then, $X(s) = 1$, and

$$Y(s) = \left(\frac{b_0}{a_1}\right) \left(\frac{1}{s + a_0/a_1}\right).$$

Example

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$$Y(s) = \left(\frac{b_0}{a_1}\right)\left(\frac{1}{s + a_0/a_1}\right).$$

Taking the inverse Laplace transform of $Y(s)$ (look at the tables from slide 23), we have

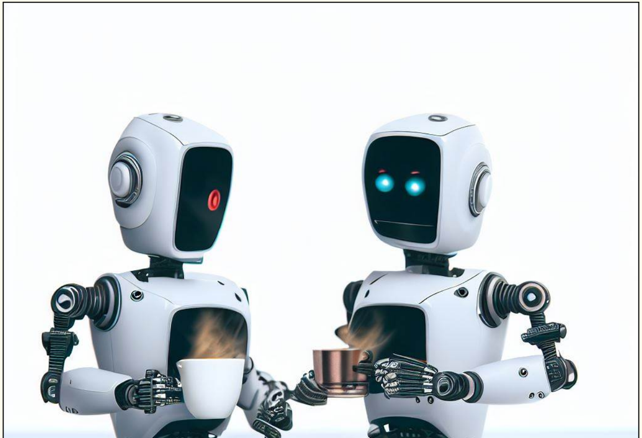
$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\left(\frac{b_0}{a_1}\right)\left(\frac{1}{s + a_0/a_1}\right)\right\} = \frac{b_0}{a_1} e^{-\frac{a_0}{a_1}t} u(t).$$

Question

What are the conditions in the s-domain for the system to be stable?

Break!

See you at ______________________________



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Initial Value Theorem

If $x(t) = 0$ for $t < 0$, and $x(t)$ does not have any impulses at $t = 0$, then

$$\lim_{t \to 0^+} x(t) = \lim_{s \to \infty} sX(s).$$

Initial Value Theorem

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Final Value Theorem

If $x(t) = 0$ for $t < 0$, and $x(t)$ has a finite limit as $t \to \infty$, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s).$$

Poles and Zeroes

We can rewrite the transfer function as

$$H(s) = G\frac{\prod_{m=0}^{M}(s - z_m)}{\prod_{n=0}^{N}(s - p_n)},$$

where G is a constant, the roots of the numerator, z_i, are the zeroes of the transfer function, and the roots of the denominator, p_i, are the poles of the transfer function.

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where G is a constant, the roots of the numerator, z_i, are the zeroes of the transfer function, and the roots of the denominator, p_i, are the poles of the transfer function.

For a system $H(s)$, the poles should be in the left half of the complex $s-$plane for the system to be stable, i.e., $\Re\{p_i\} < 0$.

Example

Consider again the transfer function

$$H(s) = \frac{Y(s)}{X(s)} = \left(\frac{b_0}{a_1}\right)\left(\frac{1}{s + a_0/a_1}\right).$$

There are no zeroes for $b_0 \neq 0$. The poles of the system are given by the roots of the denominator, i.e., $s = -a_0/a_1$. Thus, the system is stable if the ratio $a_0/a_1 > 0$.

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Applications of the Laplace Transform

We will discuss three applications of the Laplace Transform as a tool for modelling and analysis of systems:

- Mechanical spring damper system
- Armature-controlled DCmotor.
- Immune system in humans

Mechanical spring damper system

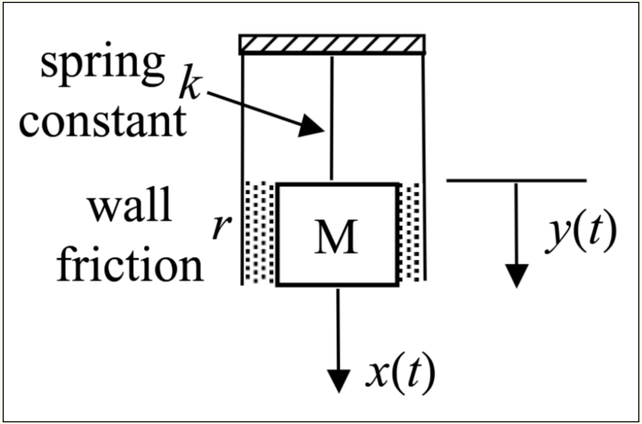

Figure: Mechanical spring damper system. From [1].

Let us consider a mechanical spring damper system as shown in the figure, with mass M, spring constant k, and friction coefficient r. The equation of motion for the system is given by

$$M\frac{d^2y(t)}{dt^2} + r\frac{dy(t)}{dt} + ky(t) = x(t),$$

where $x(t)$ is the input force and $y(t)$ is the output displacement. Assume that the initial conditions are $y(0) = 0$ and $\left.\frac{dy(t)}{dt}\right|_{t=0} = 0$.

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where $x(t)$ is the input force and $y(t)$ is the output displacement. Assume that the initial conditions are $y(0) = 0$ and $\frac{dy(t)}{dt}\Big|_{t=0} = 0$.

The Laplace transform of the equation of motion is given by

$$Ms^2Y(s) + rsY(s) + kY(s) = X(s).$$

The transfer function of the system is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{Ms^2 + rs + k} = \frac{1/M}{s^2 + 2\zeta_n\omega_n s + \omega_n^2},$$

where $\zeta_n = r/(2\sqrt{Mk})$, $\omega_n = \sqrt{k/M}$.

The transfer function of the system is

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where $\zeta_n = r/(2\sqrt{Mk})$, $\omega_n = \sqrt{k/M}$.

The system does not have zeroes, and the poles are given by

$$s_{1,2} = -\zeta_n\omega_n \pm \omega_n\sqrt{\zeta_n^2 - 1}.$$

Let $\zeta_n = 1$, then the transfer function of the system is

$$H(s) = \frac{1/M}{s^2 + 2\omega_n s + \omega_n^2} = \frac{1/M}{(s + \omega_n)^2}.$$

The poles of the system are $s_{1,2} = -\omega_n$. The impulse response of the system is given by

$$h(t) = \frac{1}{M} t e^{-\omega_n t} u(t).$$

Armature-controlled DCmotor

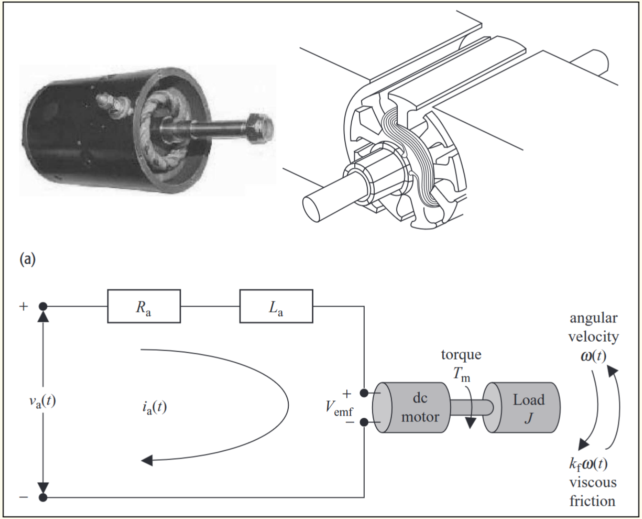

Figure: Armature-controlled DCmotor. From [1].

The linear model of the DCmotor circuit is given by

$$L_a \frac{di_a(t)}{dt} + R_a i_a(t) + k_f \omega(t) = V_a(t),$$

where $i_a(t)$ is the armature current, $\omega(t)$ is the angular velocity, $V_a(t)$ is the input voltage, L_a is the inductance, R_a is the resistance, and k_f is the feedback factor.

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where $i_a(t)$ is the armature current, $\omega(t)$ is the angular velocity, $V_a(t)$ is the input voltage, L_a is the inductance, R_a is the resistance, and k_f is the feedback factor.

The torque T_m developed by the motor is given by

$$T_m(t) = k_m i_a(t),$$

where k_m is the motor constant. The model for the net torque T_{net} is given by

$$T_{net}(t) = J \frac{d\omega(t)}{dt} = T_m(t) - r\omega(t) - T_d(t)$$

where J is the moment of inertia.

The Laplace transform DCmotor circuit is given by

$$L_a s I_a(s) + R_a I_a(s) + k_f \Omega(s) = V_a(s),$$

where $I_a(s), \Omega(s)$, and $V_a(s)$ are the Laplace transforms of $i_a(t), \omega(t)$, and $V_a(t)$, respectively. Also, using the relation between $T_m(t)$ and $i_a(t)$, we have

$$\frac{1}{k_m}(sL_a + R_a) T_m(s) + k_f \Omega(s) = V_a(s).$$

The Laplace transform DCmotor circuit is given by

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$$\frac{1}{k_m}(sL_a + R_a)T_m(s) + k_f \Omega(s) = V_a(s).$$

The Laplace transform of the net torque (with $T_d = 0$) is given by

$$T_m(s) = (Js^2 + rs)\Theta(s) = (Js + r)\Omega(s).$$

The Laplace transform DCmotor circuit is given by

$$L_a s I_a(s) + R_a I_a(s) + k_f \Omega(s) = V_a(s),$$

where $I_a(s), \Omega(s)$, and $V_a(s)$ are the Laplace transforms of $i_a(t), \omega(t)$, and $V_a(t)$, respectively. Also, using the relation between $T_m(t)$ and $i_a(t)$, we have

$$\frac{1}{k_m}(sL_a + R_a)T_m(s) + k_f \Omega(s) = V_a(s).$$

The Laplace transform of the net torque (with $T_d = 0$) is given by

$$T_m(s) = (Js^2 + rs)\Theta(s) = (Js + r)\Omega(s).$$

Thus, joining the equations above, we have

$$(sL_a + R_a)(Js + r)\Omega(s) + k_f \Omega(s) = k_m V_a(s).$$

The transfer function of the DCmotor is given by

$$H(s) = \frac{\Omega(s)}{V_a(s)} = \frac{k_m}{L_a J s^2 + [R_a J + L_a r]\, s + [R_a r + k_m k_f]}.$$

The poles of the system are given by the roots of the denominator, i.e., via ABC formula, we have,

$$s_{1,2} = \frac{-[R_a J + L_a r] \pm \sqrt{[R_a J + L_a r]^2 - 4 L_a J \,[R_a r + k_m k_f]}}{2 L_a J}.$$

Question

What are the relations between the parameters of the DCmotor for the system to be stable?

Armature-controlled DCmotor

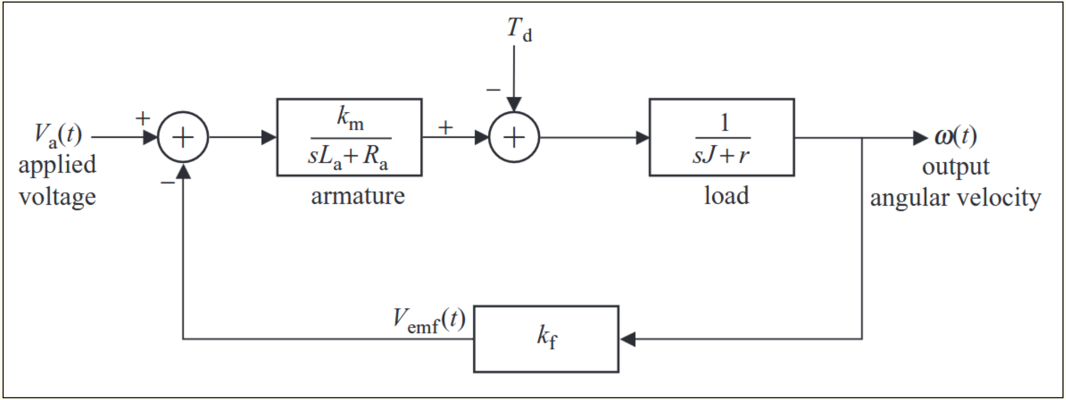

Figure: Block diagram - Armature-controlled DCmotor. From [1].

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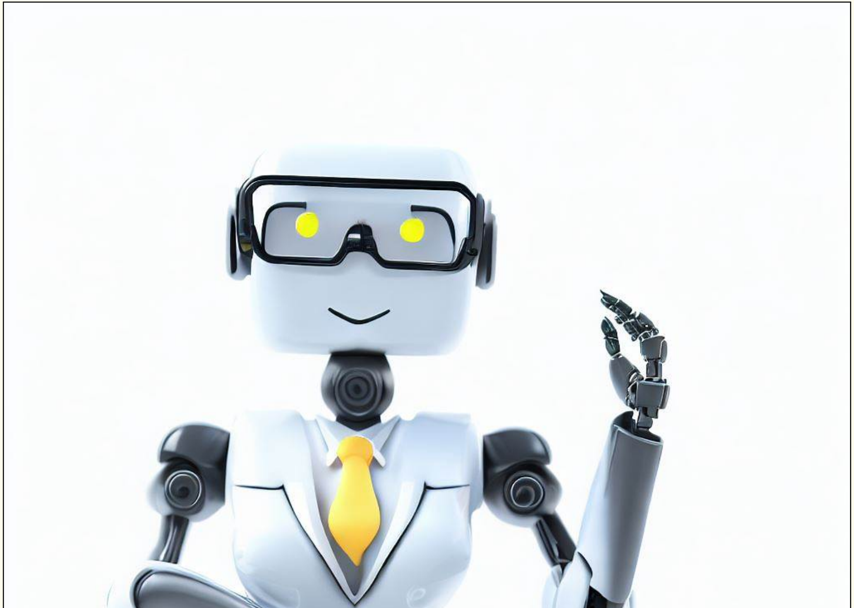
Let us wrap up the lecture!



Practice Questions

The following questions might appear in the final exam:

- Prove the following properties of the Laplace Transform: Linearity, Time Shifting, and Frequency Shifting.
- Consider the following system:

$$4\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 2x(t).$$

  where $x(t)$ is the input and $y(t)$ is the output. And initial conditions $x(0), y(0), \frac{dy(t)}{dt}\Big|_{t=0} = 0$.

  - Find the transfer function of the system, $H(s) = \frac{Y(s)}{X(s)}$.
  - Find the poles and zeros of the system.
  - Is the system stable? Justify your answer.

Do your best in the Final Exam!

(I will be cheering for you)

Bibliography

[1] Mandal, M., & Asif, A. (2008). *Continuous and discrete time signals and systems*. Cambridge University Press.

[2] Oppenheim, A. V., Willsky, A. S., & Nawab, S. H. (1997). *Signals & systems* (2nd ed). Prentice Hall.

[3] Adams, M. D. (2013). *Continuous-Time Signals and Systems*. Lecture Notes. University of Victoria, Canada.

[4] Rob van den Boomgaard, *Signal Processing*, `https://staff.fnwi.uva.nl/r.vandenboomgaard/SignalProcessing/index.html`