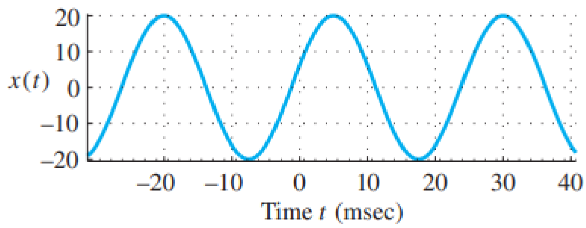
Tutorial 1 – Signals, Sinusoids, and Euler's formula

1. *(p. 60, ex. P-2.2)* Determine values for the amplitude (A), phase (ϕ), and frequency (ω_0) needed in the representation:

$$x(t) = Acos(\omega_0 t + \phi).$$

Give the answer as numerical values, including the units where applicable.



Solution:

- Amplitude is clearly 20 (vertical distance / 2)
- For ω_0, the wave completes two full oscillations from -20 to +30 msec. Then,

$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{(0.03 - (-0.02))/2} = 80\pi \frac{\text{rad}}{\text{sec}}$$

- The first positive peak in the graph is at $t_1 = 5$ msec $= 5 \times 10^{-3}$ sec, then

$$\phi = -\omega_0 \times t_1 = -0.4\pi.$$

2. *2023 Exam q. 1* Which of the following is correct?

(a) $x(t) = 2\cos(2\pi(100/6)t + \pi/6)$

(b) $x(t) = 2\cos(2\pi(100/3)t + \pi/6)$

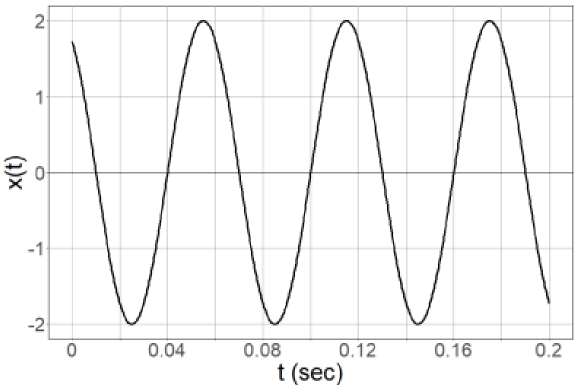
(c) $x(t) = 4\cos(2\pi(100/6)t - \pi/3)$

(d) $x(t) = 4cos(2\pi(100/3)t - \pi/3)$

Solution:

- Amplitude is clearly 2.
- The wave makes 2 full iterations from 0.04 to 0.16, so $f = 2/(0.16 - 0.04) = 100/6$ Hz

Clearly a).



3. *(p. 54, ex. 2.8)* Consider the two sinusoids

$$x_1(t) = 5\cos(2\pi(100)t + \pi/3)$$
$$x_2(t) = 4\cos(2\pi(100)t - \pi/4)$$

Prove that

$$x_1(t) + x_2(t) = 5.536\cos(2\pi(100)t + 0.2747).$$

Solution: *This exercise was explained in depth in the tutorial, this is an abstract solution that was derived at the tutorial.*

Notice that $f = 100$. We know that the real part of the sum of N arbitrary cosines with phases $\phi_1, \phi_2, \ldots, \phi_N$ and amplitudes $A_1, A_2, \ldots, A_N$ is $\sum_{i=1}^{N} A_i \cos(\phi_i)$, and the imaginary part of the sum is $\sum_{i=1}^{N} A_i \sin(\phi_i)$. Let's define

$$x = \sum_{i=1}^{N} A_i \cos(\phi_i)$$
$$y = \sum_{i=1}^{N} A_i \sin(\phi_i).$$

Then, the final amplitude is $A_{\text{final}} = \sqrt{x^2 + y^2}$ and the final phase is $\phi_{\text{final}} = \arctan(x, y) \in (-\pi, \pi]$. For the given problem,

$$x \approx 5.3284, \ y \approx 1.5017,$$

then

$$A_{\text{final}} \approx 5.536, \ \phi_{\text{final}} \approx 0.2747.$$

4. *(p. 61, ex. P-2.6)* Use Euler's formula for the complex exponential to prove DeMoivre's formula:

$$(\cos\theta + j\sin\theta)^n = \cos(n\theta) + j\sin(n\theta),$$

and use it to evaluate

$$(\frac{3}{5}+j\frac{4}{5})^{100}.$$

Use that $\tan(0.92723)=\frac{4}{3}$ (or a calculator).

Solution:

$$(\cos\theta+j\sin\theta)^n=(e^{j\theta})^n=e^{jn\theta}=\cos(n\theta)+j\sin(n\theta).$$

From kindergarten math, $\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{4}{3}\iff\theta\approx 0.92723$.
Moreover,

$$(\frac{3}{5}+j\frac{4}{5})^{100}=\cos(92.723)+j\sin(92.723),$$

which you can simplify using a calculator.

5. *(p. 64, ex. P-2.15)* Given

$$x(t)=5\cos(\omega t+\frac{1}{3}\pi)+7\cos(\omega t-\frac{5}{4}\pi)+3\cos(\omega t),$$

express $x(t)$ in the form $x(t)=A\cos(\omega t+\phi)$

Solution: We need to scale the phases in $(-\pi,\pi]$, so $\phi_1=\frac{1}{3}\pi$, $\phi_2=\frac{3}{4}\pi$, $\phi_3=0$.
Using the same formulas from exercise 3, we get

$$x(t)=9.3\cos(\omega t+1.51).$$

6. *(p.44, ex. 2.6)* Using Euler's formula and the properties of the exponential,

$$e^{j(\alpha+\beta)}=e^{j\alpha}e^{j\beta}$$

show that

$$\sin(\alpha\pm\beta)=\sin(\alpha)\cos(\beta)\pm\cos(\alpha)\sin(\beta),$$

and similarly,

$$\cos(\alpha\pm\beta)=\cos(\alpha)\cos(\beta)\mp\sin(\alpha)\sin(\beta).$$

Solution:

$$\begin{aligned}e^{j(\alpha+\beta)}=e^{j\alpha}e^{j\beta}=\\(\cos(\alpha)+j\sin(\alpha))\,(\cos(\beta)+j\sin(\beta))=\\\cos(\alpha)\cos(\beta)+j\cos(\alpha)\sin(\beta)+j\sin(\alpha)\cos(\beta)+j^2\sin(\alpha)\sin(\beta)=\\\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)+j\,(\cos(\alpha)\sin(\beta)+\sin(\alpha)\cos(\beta))\,.\end{aligned}$$

Moreover,

$$e^{j(\alpha+\beta)} = \cos(\alpha+\beta) + j\sin(\alpha+\beta).$$

By matching the real and the imaginary parts, we showed that

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

and

$$\sin(\alpha+\beta) = \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)$$

To get the result for rewrite $\alpha - \beta$ as $\alpha + (-\beta)$ and use what we proved:

$$\begin{aligned}\cos(\alpha-\beta) = \cos(\alpha+(-\beta)) = \cos(\alpha)\cos(-\beta) - \sin(\alpha)\sin(-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)\\ \sin(\alpha-\beta) = \sin(\alpha+(-\beta)) = \cos(\alpha)\sin(-\beta) + \sin(\alpha)\cos(-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta),\end{aligned}$$

keeping in mind that $\cos(-\beta) = \cos(\beta)$ and $\sin(-\alpha) = -\sin(\alpha)$.

7. *(p. 34 ex. 2.1)* Use the previous exercise to derive a formula for $\cos 8\theta$ in terms of $\cos\theta$, $\cos 9\theta$, and $\cos 7\theta$.

Solution: Since 9=8+1, and 7=8-1 we have,

$$\begin{aligned} cos(9\theta) = \cos(8\theta+\theta) = \cos(8\theta)\cos(\theta) - \sin(8\theta)\sin(\theta)\\ \cos(7\theta) = \cos(8\theta-\theta) = \cos(8\theta)\cos(\theta) + \sin(8\theta)\sin(\theta).\end{aligned}$$

After summing both, we have

$$\cos(9\theta) + \cos(7\theta) = 2\cos(8\theta)\cos(\theta) \iff \cos(8\theta) = \frac{\cos(9\theta)+\cos(7\theta)}{2\cos(\theta)}.$$

8. *(p. 61 ex. P-2.4)* Using Taylor series, prove Euler's formula, namely

$$e^{j\theta} = \cos\theta + j\sin(\theta).$$

Hint: expand e^x for $x = j\theta$.

The Taylor series for e^x, $\sin x$, and $\cos x$ are given below.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Solution: From the hint,

$$e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \frac{(j\theta)^5}{5!} + \dots,$$

by using $j^2 = -1$, $j^3 = -j$, and $j^4 = 1$,

$$e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{j\theta^5}{5!} + \dots,$$

then by rearranging,

$$e^{j\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right),$$

which is exactly

$$e^{j\theta} = \cos\theta + j\sin(\theta).$$

Might be useful for your cheatsheet:

- A – amplitude
- ω_0 – radian frequency
- f – cyclic frequency
- T – period
- ϕ– phase
- $T = (t_1 - t_0)/n$, n– number of iterations
- $\omega_0 = 2\pi f = \frac{2\pi}{T}$
- $T = \frac{1}{f} = \frac{2\pi}{\omega_0}$
- $\phi = -2\pi \frac{t_1}{T} = -\omega_0 t_1$, adjust in $(-\pi, \pi]$, t_1 is the first positive peak after 0.